THE TWISTORIAL CONSTRUCTION OF NULL HOLOMORPHIC CURVES IN $\mathbb{C}^3$

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Introduction

The object of this paper is to elucidate a certain twistor correspondence which facilitates the study of null holomorphic curves in $\mathbb{C}^3$. This correspondence is most successfully employed when dealing with null meromorphic curves, in which case projection to $\mathbb{R}^3$ yields complete, branched minimal surfaces of finite total Gaussian curvature. There are some examples discussed in section 6; in particular we describe there a branched minimal immersion of a Klein bottle into $\mathbb{R}^3$ that has 2 ends, 6 branch points and total Gaussian curvature -4π.

We begin in section 1 by reviewing the integration of the Weierstrass representation formulae for a null curve in $\mathbb{C}^3$ and the resulting "free" formulae.

Now, a plane in $\mathbb{C}^3$ that contains a single null line is said to be null, and the collection of affine translates of such planes forms a holomorphic line bundle of degree 2 over the quadric curve, $\mathbb{Q}_1$. The affine null planes in $\mathbb{C}^3$ that pass through a fixed $z \in \mathbb{C}^3$ comprise a global holomorphic section of this line bundle and thus $\mathbb{C}^3$ is identified with the set of its global sections. The nullity, or otherwise, of z can be understood in terms of the intersection of the corresponding global section with the zero section. This is explained in section 2 where however, following Hitchin [H1], we approach this correspondence from the opposite direction, i.e. starting with T, the holomorphic tangent bundle of $\mathbf{P}_1$, we derive the conformal structure in $H^0(\mathbf{P}_1, T) \cong \mathbb{C}^3$ and interpret points of T as affine null planes there. This eases the exposition of section 3.

Section 3 is essentially an amplification of the appendix of [H1]. Viewing $\mathbb{C}^3$ as $H^0(\mathbf{P}_1, T)$, we describe there a natural lift into T of the Gauss map of a non-constant null curve in $\mathbb{C}^3$. We show that the null curve may be viewed as the collection of global sections of T that osculate this lift and thus establish a correspondence, described 3.7, between curves on T and null curves in $H^0(\mathbf{P}_1, T)$. This manifests itself locally as the Weierstrass formulae in free form.

In section 4 we explain how to "compactify" the correspondence and view it in terms of the duality between curves in $\mathbf{P}_3$ and $\mathbf{P}_3^*$: Theorem 4.9 describes the correspondence as it was understood by Lie [D], [Li]. In this context we study the behaviour of osculating sections in the vicinity of a branch point, and at the points at infinity, of an algebraic curve on T : in particular we show that this determines the asymptotic structure of the corresponding null curve.

Corollary 4.10 describes the correspondence in terms of the compactification of T to the Hirzebruch surface $S_2 = \mathbf{P}(T \oplus \mathcal{O})$. This enables us to show that the moduli spaces of null meromorphic curves in $\mathbb{C}^3$ compactify naturally to complete linear systems on S_2. In addition the numerical data associated to such a system is interpreted in terms of the geometry of the null curves thus parameterized: this is explained in section 5.

There are a number of ways in which one might hope to generalize the constructions described in this paper, we mention two. Firstly, there exists a close analogue for curves in $\mathbb{C}^4$. This is implicit in work of Eisenhart but was first made explicit by Shaw [Sh], [S1]. (In [H–S] the analogue is pursued in dimension 6, however an interesting generalization to $\mathbb{C}^n$ is not obvious.) Secondly, Hitchin's construction of Einstein–Weyl geometries as moduli of rational curves on complex surfaces provides the natural context in which to view the constructions described in this paper [H2], [S2]. An interesting example is given by the correspondence between curves on a non–singular quadric surface in $\mathbf{P}_3$ and null curves in $SL(2,\mathbb{C})$: the latter were shown by Bryant [B] to project to surfaces of constant mean curvature 1 in the hyperbolic space of curvature –1.

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§ 1. The Weierstrass Representation Formulae in Free Form

(1.1) Let M be a Riemann surface and suppose that $\Omega : M \longrightarrow \mathbb{C}^3$ is a null holomorphic curve, i.e. $(\Omega', \Omega') = 0$, where $(z,z) = z_1^2 + z_2^2 + z_3^2$ and primes denote differentiation. If Ω is non–constant then the Gauss map $\gamma_\Omega = [\Omega']$ is well–defined on M and takes values in the quadric curve $Q_1 \subset \mathbb{P}_2$. Furthermore, $\phi = \mathrm{Re}(\Omega)$ is a branched minimal immersion into $\mathbb{R}^3$. Every branched minimally immersed surface in $\mathbb{R}^3$ may be parametrized in this way. Identifying Q_1 with the unit sphere in oriented $\mathbb{R}^3$ identifies γ_Ω with the Euclidean Gauss map of ϕ. For further details see [L], [O].

(1.2) Let $g_\Omega = \chi^{-1} \circ \gamma_\Omega$, where $\chi : \mathbb{C} \cup \{\infty\} \longrightarrow Q_1$ is given by $\chi(\zeta) = [1 - \zeta^2, i(1 + \zeta^2), 2\zeta]$. Provided that γ_Ω is not the constant map taking the value $[-1,i,0]$, there exists for every $\xi_0 \in M$, a holomorphic function F, defined on a neighbourhood U of ξ_0 such that for $\xi \in U$

$$\Omega(\xi) = \frac{1}{2}\int^{\xi} F(\xi)(1 - g^2, i(1 + g^2), 2g)d\xi .$$

Away from the branch locus of g, Ω may be locally reparameterized by the 'Gauss map variable' ζ. Suppose that g^{-1} and F, as above, exist on an open set $U \subset \mathbb{P}_1$ and $g^{-1}(U)$ respectively and that $f : U \longrightarrow \mathbb{C}$ holomorphic, satisfies

$$f'''(\zeta) = f \circ g^{-1}(\zeta) \frac{dg^{-1}}{d\zeta}(\zeta) .$$

The substitution of f''' into the above, together with the change of variable to $\zeta = g(\xi)$, facilitates integration by parts over U. Correcting f up to a quadratic term, this yields the following *Weierstrass representation formulae in free form* for $\Omega \circ g^{-1}$ on U:

$$\begin{aligned}
\Omega_1 \circ g^{-1}(\zeta) &= 1/2(1 - \zeta^2)f''(\zeta) + \zeta f'(\zeta) - f(\zeta) \\
\Omega_2 \circ g^{-1}(\zeta) &= i/2(1 + \zeta^2)f''(\zeta) - i\zeta f'(\zeta) + if(\zeta) \\
\Omega_3 \circ g^{-1}(\zeta) &= \zeta f''(\zeta) - f'(\zeta) .
\end{aligned}$$

These formulae first appeared in [W] and are discussed at length in [D], see also [E], [Li] and [N].

(1.3) Remarks. (i) Substitution of any holomorphic function f into the above formulae yields a null holomorphic curve in $\mathbb{C}^3$; provided that f is not merely quadratic in ζ this projects to a branched minimal immersion into $\mathbb{R}^3$.

(ii) The collection of null curves mapping $U \subset \mathbb{C}$ into $\mathbb{C}^3$ and described by formulae of the above type has a vector space structure. This is an immediate consequence of the fact that such a curve is parameterized by its Gauss map. This is the structure that was studied in [R–T]. Note that the addition of a quadratic function to f simply translates Ω in $\mathbb{C}^3$.

(iii) If f generates a branched minimal immersion ϕ, then αf, where $\alpha \in \mathbb{C}^*$, generates a (rescaled) associate surface of ϕ.

(iv) The (branched) metric induced on M by Ω is given, with respect to the local coordinate ζ, by:

$$ds^2 = |f'''(\zeta)|^2(1 + |\zeta|^2)^2 \mathrm{Re}(d\zeta \otimes d\bar{\zeta}),$$

and the Gaussian curvature by: $K(\zeta) = -\dfrac{4}{|f'''(\zeta)|^2(1+|\zeta|^2)^4}$.

Note that ζ is a branch point of Ω iff $f'''(\zeta) = 0$.

(v) It can be shown that Ω has a Weierstrass representation in free form on some neighbourhood of $\xi_0 \in M$ precisely when any branching in the Gauss map at ξ_0 arises solely from ramification in the parameterization of Ω at ξ_0.

(1.4) Examples. (i) $f(\zeta) = \frac{1}{6}\zeta^3$ generates Enneper's surface. (It follows easily from Lemma 9.6 in [O] that a minimal surface $\phi : \mathbb{C} \longrightarrow \mathbb{R}^3$ which is complete, free of branch points and generated via 1.2 by an entire function is a scaled associate surface of Enneper.)

(ii) $f(\zeta) = \frac{1}{6}\zeta^4$ generates $\mathrm{Re}(\Omega) : \mathbb{C} \longrightarrow \mathbb{R}^3$ where $\Omega(\zeta) = (\zeta^2 - \frac{1}{2}\zeta^4, i(\zeta^2 + \frac{1}{2}\zeta^4), \frac{4}{3}\zeta^3)$. This surface is complete, has total Gaussian curvature -4π and a branch point at $\zeta = 0$. The geometry at $\zeta = 0$ is discussed in II.3 in [L].

(iii) $f(\zeta) = \frac{1}{6}(\zeta^3 + \frac{1}{4}\epsilon\zeta^4)$ generates a complete minimal surface with total Gaussian curvature -4π possessing a branch point at $-1/\epsilon$.

For further examples see the references cited above and refer to §6.

§2. Duality

(2.1) With respect to an affine coordinate ζ on $\mathbb{P}_1$ a global holomorphic section of the holomorphic tangent bundle $\pi : T \longrightarrow \mathbb{P}_1$, takes the form $(a + b\zeta + c\zeta^2)\frac{d}{d\zeta}$, where $a,b,c \in \mathbb{C}$; thus a choice of ζ permits us to identify $\mathbb{C}^3$ with $H^0 = H^0(\mathbb{P}_1,T)$. A non–zero global section of T has a double root iff $b^2 - 4ac = 0$, so the set of such sections together with the zero section comprise the null cone, $C(Q_1)$, of the conformal structure on H^0. A global section of T whose discriminant is zero is said to be *null*.

There exists a canonical identification, q, between $\mathbb{P}_1$ and the quadric, Q_1, of null directions in H^0 where $q(\zeta) = \{\sigma \in H^0 ; \sigma$ has a double root at $\zeta\}$.

(2.2) If a global section σ vanishes at ζ then it cannot possess a double root elsewhere on $\mathbb{P}_1$. Consequently the plane $\Pi_\zeta = \{\sigma \in H^0; \sigma(\zeta) = 0\}$ enjoys tangential intersection with $C(Q_1)$ along $q(\zeta)$, i.e. $\Pi_\zeta \cap C(Q_1) = q(\zeta)$. Such a plane is said to be *null* (or *isotropic*). In terms of the conformal structure, Π_ζ is simply the polar space of $q(\zeta)$, and the restriction of the conformal structure to such a plane is degenerate. Note that a null line lies on a unique null plane.

$\Pi = \bigcup_{\zeta \in \mathbb{P}_1} \Pi_\zeta$, viewed as a subbundle of the trivial bundle $\underline{H}^0$ on $\mathbb{P}_1$, is the kernel of the map $\underline{H}^0 \longrightarrow T$, $(\zeta,\sigma) \longmapsto \sigma(\zeta)$, and hence there is the following isomorphism:

$$T \cong \underline{H}^0/\Pi = \{\text{affine null planes in } H^0\} .$$

Of course, $t \in T$ corresponds with the affine plane in H^0 of sections that pass through t. Consequently t lies on the image of a global section σ iff σ lies on the affine null plane in H^0 corresponding to t.

(2.3) **Remarks** (i) By viewing $\mathbb{C}^2$ as $H^0(\mathbb{P}_1, \mathcal{O}(1))$ one obtains a similar correspondence which compactifies to the usual duality between $\mathbb{P}_2$ and $\mathbb{P}_2^*$. (Since every global section of $\mathcal{O}(1)$ has precisely one zero there is no constraint on the planes thus obtained in $H^0(P_1, \mathcal{O}(1))$.)
(ii) Identifying $\mathbb{C}^3$ with H^0 via the basis $\{-1/2(1-\zeta^2)\frac{d}{d\zeta}, -i/2(1+\zeta^2)\frac{d}{d\zeta}, -\zeta\frac{d}{d\zeta}\}$ gives the transformation $a = -1/2(z_1 + iz_2)$, $b = -z_3$, $c = 1/2(z_1 - iz_2)$ and hence the discriminant takes the form $z_1^2 + z_2^2 + z_3^2$.

The set of real sections with respect to this identification intersects $C(Q_1)$ in $\{0\}$; each affine null plane intersects the real slice in an affine line and, together with a choice of orientation on $\mathbb{R}^3$, induces an orientation thereon. T is thus identified with the collection of oriented affine lines in $\mathbb{R}^3$ and a real section may be viewed as the set of oriented affine lines that pass through the corresponding point in $\mathbb{R}^3$.
(iii) Note that the essential feature of T in the above is that it is a line bundle of degree 2 over $\mathbb{P}_1$, it is this that gives the conformal structure on H^0.
(The discriminant gives us an inner product on H^0, however this is preserved only by those bundle automorphisms induced by differentiating automorphisms of $\mathbb{P}_1$. Simply, fixing a scale in the fibres of T corresponds to fixing a scale on H^0.)

§3. The Lie–Hitchin Correspondence

(3.1) Recall that the duality alluded to in 2.3(i) leads to a duality between curves in $\mathbb{P}_2$ and P_2^*. We describe here the analogue of this correspondence for T and H^0.

(3.2) By definition, a curve $\Omega : M \longrightarrow H^0$ is null if $\frac{d\Omega}{d\xi}(\xi,\ )$ is always a null section of T. (We rewrite Ω as a map from $M \times \mathbb{P}_1$ simply for notational ease.) Suppose that Ω is non–constant and identify Q_1 with $\mathbb{P}_1$ via q in order to view γ_Ω as a map to $\mathbb{P}_1$. Thus one obtains the following characterization of nullity: Ω, non–constant, is null if for any local coordinate ξ on M and any affine coordinate ζ on $\mathbb{P}_1$, there exists a holomorphic function λ such that

$$\frac{d\Omega}{d\xi}(\xi,\zeta) = \lambda(\xi)(\zeta - \gamma_\Omega(\xi))^2 .$$

(3.3) For $\Omega : M \longrightarrow H^0$ a non–constant null curve let $\Gamma_\Omega : M \longrightarrow T$ be given by $\Gamma_\Omega(\xi) = \Omega(\xi,\gamma_\Omega(\xi))$. Γ_Ω is called the *Gauss transform* of Ω; clearly it is a globally defined lift of the Gauss map of Ω. From the duality of 2.2 observe that $\Gamma_\Omega(\xi)$ is the (unique) affine null plane with null direction $\gamma_\Omega(\xi)$ that passes through $\Omega(\xi,\ ) \in H^0$.

(3.4) We show that Ω is determined by its Gauss transform.

Theorem. If $\Omega,\Psi : M \longrightarrow H^0$ are null curves such that $\Gamma_\Omega = \Gamma_\Psi$ and γ_Ω is non–constant then $\Omega = \Psi$.

Proof. $\Gamma_\Omega = \Gamma_\Psi$ implies that $\gamma_\Omega = \gamma_\Psi = \gamma$ say. Let γ^{-1} be an inverse for γ on some open proper subset $U \subset \mathbb{P}_1$ and observe that $\Gamma_\Omega \circ \gamma^{-1} = \Gamma_\Psi \circ \gamma^{-1}$ as local section of T over U. Working in an affine coordinate on U we calculate the quadratic jet of $\Gamma_\Omega \circ \gamma^{-1}$ at $\zeta_0 \in U$ and show that it equals $\Omega(\xi_0,\zeta)$, where $\xi_0 = \gamma^{-1}(\zeta_0)$, which shows that $\Omega = \Psi$ on U and by uniqueness of analytic continuation this establishes the result. By definition $\Gamma_\Omega \circ \gamma^{-1}(\zeta) = \Omega(\gamma^{-1}(\zeta),\zeta)$, therefore

$$\frac{d}{d\zeta}\Gamma_\Omega \circ \gamma^{-1}(\zeta_0) = \frac{d\Omega}{d\xi}(\xi_0,\zeta_0)\frac{d\gamma^{-1}}{d\zeta}(\zeta_0) + \frac{d\Omega}{d\zeta}(\xi_0,\zeta_0)\,.$$

From 3.2, $\frac{d\Omega}{d\xi}(\gamma^{-1}(\zeta),\zeta) \equiv 0$ and hence

$$\frac{d^2}{d\zeta^2}\Gamma_\Omega \circ \gamma^{-1}(\zeta_0) = \frac{d^2}{d\xi d\zeta}\Omega(\xi_0,\zeta_0)\frac{d\gamma^{-1}}{d\zeta}(\zeta_0) + \frac{d^2\Omega}{d\zeta^2}(\xi_0,\zeta_0)\,.$$

Again from (3.2), $\frac{d^2}{d\zeta d\xi}\Omega(\gamma^{-1}(\zeta),\zeta) \equiv 0$, so the quadratic jet of $\Gamma_\Omega \circ \gamma^{-1}$ at ζ_0 is

$$\Omega(\xi_0,\zeta_0) + \frac{d\Omega}{d\zeta}(\xi_0,\zeta_0)(\zeta-\zeta_0) + \frac{1}{2}\frac{d^2\Omega}{d\zeta^2}(\xi_0,\zeta_0)(\zeta-\zeta_0)^2 = \Omega(\xi_0,\zeta).$$

Remark. If γ_Ω is constant, in which case the image of Ω lies on an affine null line, Γ_Ω is constant and does not determine Ω completely.

(3.5) It is clear from the proof of 3.4 that in order to recover Ω from Γ_Ω we must construct the curve in H^0 of global sections of T which each have the property that they intersect Γ_Ω with multiplicity at least 3 at some point; such sections are said to ***osculate*** Γ_Ω. A natural way to formulate this, and hence describe the process which inverts the Gauss transform, is to let Spé (T) denote the Étalé space of the sheaf of germs of sections of T and introduce the canonical map $\omega : \text{Spé}(T) \longrightarrow H^0$, which is given on stalks by the following:

$$\omega : \mathcal{O}(T)_\zeta \longrightarrow \mathcal{O}(T)_\zeta / \mathcal{J}_\zeta^3 \otimes \mathcal{O}(T)_\zeta \xrightarrow{\sim} H^0\,.$$

Here $\mathcal{J}_\zeta$ is the ideal sheaf of holomorphic function vanishing at ζ, so the intermediate object is the module of 2–jets of local sections at ζ, and the identification with H^0 says simply that a global section of T is determined by its 2–jet anywhere on $\mathbb{P}_1$ and conversely, that any 2–jet 'extends' over $\mathbb{P}_1$ to give a global section.

(3.6) It is clear that ω is holomorphic, the other relevant properties of ω are easily established. Let $\mathcal{G} \subset$ Spé (T) be the set of germs of global sections of T.

Theorem. (i) The curve $\omega : \text{Spé}(T) \longrightarrow H^0$ is null.
(ii) The Gauss map $\gamma_\omega : \text{Spé}(T) - \mathcal{G} \longrightarrow \mathbb{P}_1$ is $\gamma_\omega([\sigma]_\zeta) = \zeta$.
(iii) The Gauss curve $\Gamma_\omega : \text{Spé}(T) - \mathcal{G} \longrightarrow T$ is given by evaluation, i.e. $\Gamma_\omega([\sigma]_\zeta) = \sigma(\zeta)$.

Proof. (i) and (ii) : by definition of ω, for any $[\sigma]_{\zeta_0} \in$ Spé (T) there exists some neighbourhood of ζ_0 on which the following equation holds:

$$\sigma(\zeta) = \omega([\sigma]_{\zeta_0}, \zeta) + \mathcal{O}[(\zeta - \zeta_0)^3].$$

In the local chart $[\sigma]_{\zeta_0} \longmapsto \zeta_0$ on Spé (T), differentiation of this equation with respect to ζ_0 gives

$$\frac{d\omega}{d\zeta_0}([\sigma]_{\zeta_0}, \zeta) = \mathcal{O}[(\zeta - \zeta_0)^2],$$

so from 3.2 $\frac{d\omega}{d\zeta_0}([\sigma]_{\zeta_0}, \ )$ is a null section and hence ω is a null curve. Now (ii) follows immediately from this and the identification of $\mathbb{P}_1$ with Q_1 via q.

(iii) $$\begin{aligned}\Gamma_\omega([\sigma]_{\zeta_0}) &= \omega([\sigma]_{\zeta_0},\gamma_\omega([\sigma]_{\zeta_0}))\\ &= \omega([\sigma]_{\zeta_0},\zeta_0)\\ &= \sigma(\zeta_0) \qquad \text{from the first equation above.}\end{aligned}$$

Remark. Clearly γ_ω and Γ_ω may be defined on Spé (T). For $\sigma \in H^0$, γ_ω and Γ_ω respectively describe the $\mathbb{P}_1$ of affine null lines that pass through σ and the $\mathbb{P}_1$ of affine null planes that pass through σ.

(3.7) **Theorem.** If Ω is null, with γ_Ω non–constant, then $\Omega|_{M_*} = \omega \circ \Gamma^*_\Omega$, where $M_* = \{\xi \in M; \Gamma_\Omega(\xi)$ is transverse to the fibre of T$\}$ and $\Gamma^*_\Omega : M_* \longrightarrow$ Spé(T) is the natural lift of Γ_Ω over M_*.

Proof. It follows immediately from $\gamma_{\omega\circ\Gamma^*_\Omega} = \gamma_\omega \circ \Gamma^*_\Omega$ that $\Gamma_{\omega\circ\Gamma^*_\Omega} = \Gamma_\omega \circ \Gamma^*_\Omega$. So from Theorem 3.6 (iii), $\Gamma_{\omega\circ\Gamma^*_\Omega} = \Gamma_\Omega$ over M_* and the result follows from Theorem 3.4.

The geometric significance of the auxiliary function f in the Weierstrass formulae for Ω is now clear. If $\Omega : M \longrightarrow H^0$ has non–constant Gauss map then for $\xi_0 \in M$, not a branch point of γ_Ω, there exists a local inverse γ_Ω^{-1} and so $\Gamma_\Omega \circ \gamma_\Omega^{-1}(\zeta) = f(\zeta)\frac{d}{d\zeta}$, f a holomorphic function, on some neighbourhood of $\zeta_0 = \gamma_\Omega(\xi_0)$. ζ is an affine coordinate so the global section determined by the 2–jet at ζ_0 is obtained by Taylor expanding f at ζ_0 to order 2. It follows from Theorem 3.7 that

$$\Omega \circ \gamma_\Omega^{-1}(\zeta_0) = \omega \circ \Gamma^*_\Omega \circ \gamma_\Omega^{-1}(\zeta_0) = (a(\zeta_0) + b(\zeta_0)\zeta + c(\zeta_0)\zeta^2)\frac{d}{d\zeta},$$

where

$$a(\zeta_0) = f(\zeta_0) - \zeta_0 f'(\zeta_0) + 1/2\zeta_0^2 f''(\zeta_0)$$
$$b(\zeta_0) = f''(\zeta_0) - \zeta_0 f''(\zeta_0)$$
$$c(\zeta_0) = 1/2f''(\zeta_0).$$

Viewing ζ_0 as variable and transforming to (z_1,z_2,z_3) coordinates, as in 2.3 (ii), yields the Weierstrass formulae 1.2. Thus observe that f is a local implicit description over $\mathbb{P}_1$ of the Gauss transform Γ_Ω in T.

(3.8) For a bundle automorphism Θ of T, inducing $\theta \in PGL(2,\mathbb{C})$, the element of $GL(H^0)$ given by $\sigma \longmapsto \Theta \circ \sigma \circ \theta^{-1}$ preserves the null cone in H^0 and induces θ on $\mathbb{P}_1$ via q. Conversely, any conformal transformation of H^0 maps an affine null plane to an affine null plane and so induces an automorphism of T. Thus we obtain an isomorphism between automorphisms of T and conformal transformations of H^0. Fixing a scale in the fibres of T, (which fixes a scale on H^0), this reduces to the isomorphism $PGL(2,\mathbb{C}) \cong SO(3,\mathbb{C})$ which, in terms of the double cover $SL(2,\mathbb{C}) \longrightarrow PGL(2,\mathbb{C})$, is simply the adjoint representation. This leads to the following proposition in which the effect of coordinate transformations is characterized.

Proposition. Suppose that the local section $\eta = f(\zeta)\frac{d}{d\zeta}$ generates the null curve Ω via osculation; for $\theta \in SL(2,\mathbb{C})$ we have

$$\tilde{\eta} \circ \theta = \partial\theta \circ \eta \text{ iff } \tilde{\Omega} \circ \theta = \mathrm{Ad}(\theta)\Omega,$$

where $\tilde{\Omega}$ is generated by $\tilde{\eta}$. Writing $\theta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we have more explicitly:

$$\tilde{f}\left[\frac{a\zeta+b}{c\zeta+d}\right] = (c\zeta + d)^{-2} f(\zeta)$$

iff

$$\begin{bmatrix} \tilde{\Omega}_3 \circ \theta & -(\tilde{\Omega}_1 + i\tilde{\Omega}_2)\circ\theta \\ -(\tilde{\Omega}_1 - i\tilde{\Omega}_2)\circ\theta & -\tilde{\Omega}_3\circ\theta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \Omega_3 & -(\Omega_1 + i\Omega_2) \\ -(\Omega_1 - i\Omega_2) & -\Omega_3 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

(3.9) **Remarks.** (i) $f'''(\zeta) = 0$, in which case Ω has a branch point at ζ, iff the osculating section at $(\zeta, f(\zeta)\frac{d}{d\zeta})$ intersects with multiplicity at least 4 there, so it ***hyperosculates*** the curve.

(ii) It is not hard to see that $\mathcal{O}(n)$, the holomorphic line bundle of degree n on $\mathbb{P}_1$, may be viewed as the totality of affine hyperplanes in $H^0(\mathbb{P}_1,\mathcal{O}(n)) \cong \mathbb{C}^{n+1}$ which are translates of hyperplanes that lie tangent to K, the cone over the rational normal curve in $\mathbb{P}_n$. There exists a canonical map, analogous to ω, from the Étalé space of $\mathcal{O}(n)$ to $\mathbb{C}^{n+1}$ whose tangent lines are translates of lines on K. Correspondingly, such a map to $\mathbb{C}^{n+1}$, whose Gauss map to the rational normal curve is non–constant, possesses a Gauss transform on $\mathcal{O}(n)$ and derives from osculation of that curve. The analogous Weierstrass formulae are easily written down. We will discuss this in more detail elsewhere.

§4. Compactification

(4.1) The Lie–Hitchin correspondence is described here in terms of the duality between curves in $\mathbb{P}_3$ and $\mathbb{P}_3^*$. This change of viewpoint reveals the nature of osculation of an algebraic curve $A \subset T$ at points in the branch locus of $\pi|_A$ and at infinity; thus it elucidates the classical work of Lie and Darboux to which it is very close, see [D], [Li].

(4.2) Let $\pi^{-1}T$ denote the pullback of T over the total space of T. T embeds into $\mathbb{P}(H^0(T,\pi^{-1}T))^* \cong \mathbb{P}_3$, where it is compactified to $\mathscr{C}(Q_1)$, the projective cone over the quadric curve, Q_1—the image of the zero section of T.

(4.3) A hyperplane in $\mathbb{P}_3$ which does not pass through v, the vertex of $\mathscr{C}(Q_1)$, cuts out the image on $\mathscr{C}(Q_1)$ of a global section of T : the hyperplane H_∞, that cuts out Q_1 is thus distinguished. Hyperplanes that pass through v generically intersect $\mathscr{C}(Q_1)$ along a pair of generators.

(4.4) A hyperplane in $\mathbb{P}_3$ that intersects Q_1 at one point is said to be *null*: a null hyperplane that passes through v lies tangent to $\mathscr{C}(Q_1)$, any other cuts out the image of a null section.

Viewing $\mathscr{C}(Q_1)$ as the compactification of the null cone in $\mathbb{P}_3 - H_\infty$, (where v determines an origin), observe that since Q_1 comprises the null directions at infinity in $\mathbb{P}_3 - H_\infty$, a null hyperplane is the compactification of an affine null plane in $\mathbb{P}_3 - H_\infty$.

Thus the notion of a null hyperplane subsumes both that of a non–zero null section of T and that of an affine null plane in $\mathbb{C}^3$.

(4.5) For $p \in \mathbb{P}_3$ and a hyperplane $H \subset \mathbb{P}_3$ let p^* and H^* denote the dual hyperplane and dual point in $\mathbb{P}_3^*$ respectively. The null hyperplanes in $\mathbb{P}_3$, together with H_∞^*, comprise a dual quadric cone $\mathscr{C}(\check{Q}_1) \subset \mathbb{P}_3^*$, where the distinguished quadric curve $\check{Q}_1$ comprises the null hyperplanes through v. H_∞^* is the vertex of $\mathscr{C}(\check{Q}_1)$ and Q_1 parameterizes the hyperplanes in $\mathbb{P}_3^*$ that pass through it and are null (with respect to $\check{Q}_1$). Note that this construction is symmetric, i.e. $\mathscr{C}(Q_1)$ may be viewed as {null hyperplanes in $\mathbb{P}_3^*$} ∪ {v} .

(4.6) It follows from 4.2 that $\mathbb{P}_3^* \cong H^0(\mathbb{P}_1, T) \cup v^*$; $\mathscr{C}(\check{Q}_1)$ is thus identified with the compactification of the null cone in $H^0(\mathbb{P}_1,T)$.

(4.7) Recall that an algebraic curve $A \subset \mathbb{P}_3$ is said to be *full* if it does not lie on any hyperplane and furthermore that such curves necessarily have degree at least 3. There is a natural correspondence between full curves in $\mathbb{P}_3$ and full curves $\mathbb{P}_3^*$ which for $A \subset \mathbb{P}_3$ is given by associating to each smooth point $\alpha \in A$ the hyperplane that intersects A at α with multiplicity at least 3; that hyperplane is said to *osculate* A at α. This determines a birational map between A and a dual curve $A^* \subset \mathbb{P}_3^*$:

Proposition. Let $n : \tilde{X} \longrightarrow A \subset \mathbb{P}_3$ be the normalization and let $n^* : \tilde{X} \longrightarrow A^* \subset \mathbb{P}_3^*$ denote the map induced by osculation. Then $(\tilde{X},n^*)$ gives the normalization of A^* and $n^{**} = n$.

For further details see [G–H], [Ha].

(4.8) **Lemma.** Suppose that $A \subset \mathbb{P}_3$ is full.

(i) If A lies on $\mathscr{C}(Q_1)$ then at $\alpha \in A$, which does not lie in the branch locus of the projection to Q_1 or equal v, the osculating section is cut out by the hyperplane that osculates A at α.

(ii) Suppose that $A' = A - (A \cap v^*)$ is a null curve in $\mathbb{P}_3^* - v^*$ (with respect to $C(\tilde{Q}_1)$). For a smooth point $\alpha \in A'$, the hyperplane that osculates A' at α gives the value of the Gauss transform there viewed as a map to $\mathscr{C}(Q_1)$.

Proof. (i) is immediate. To prove (ii) let ξ be a local coordinate on $\tilde{X}$ with $n(\xi_0) = \alpha$ and recall that $\Gamma_n(\xi_0)$ is the affine null plane that passes through α with null direction

$\gamma_n(\xi_0)$. Generically, neither n' nor n'' equals zero so suppose that this is true at ξ_0. Since $(n',n') = 0$ implies that $(n'',n') = 0$, the polar $\gamma_n(\xi_0)^0 = \mathrm{span}_{\mathbb{C}}\{n'(\xi_0),n''(\xi_0)\}$. Expanding n at ξ_0 it is clear that $u(\xi) - u(\xi_0)$ intersects $\gamma_n(\xi_0)^0$ with multiplicity at least 3 and therefore the closure of $\Gamma_n(\xi_0)$ in $\mathbb{P}_3^*$ in the hyperplane that osculates A at α.

Remarks. (i) The fullness of A ensures in (i) that A does not lie on a global section or a fibre of T and in (ii) that the Gauss map is non–constant. (Recall that these are degenerate cases.)

(ii) Observe that a section hyperosculates iff the corresponding hyperplane hyperosculates.

(4.9) It is now clear that for a full curve $A \subset \mathbb{P}_3^*$, A' is null in $\mathbb{P}_3^* - v^*$ iff $A^* \subset \mathscr{C}(Q_1)$: accordingly we say that A is *null* (with respect to $\mathfrak{Q}_1$) if $A^* \subset \mathscr{C}(Q_1)$, i.e. the hyperplanes that osculate A are null.

The next result describes the Lie–Hitchin correspondence extrinsically.

Theorem. Osculation determines a correspondence between full curves on $\mathscr{C}(Q_1) \subset \mathbb{P}_3$ and full curves in $\mathbb{P}_3^*$ that are null with respect to $\mathfrak{Q}_1$. Furthermore, the obvious dual statement holds.

(4.10) We now reformulate Theorem 4.9 intrinsically; however, note that in doing so we break the inherent symmetry of the above statement.

Blowing up the vertex of $\mathscr{C}(Q_1)$ gives the Hirzebruch surface $S_2 \cong \mathbb{P}(T \oplus \mathcal{O})$, which is a rational ruled surface and the minimal smooth compactification of T : for details and notation see § 4.3 of [G–H]. $\mathbb{P}_3^*$ is thus identified with the linear system $|E_0|$ on S_2:

$|E_0| \cong H^0(\mathbb{P}_1,T) \cup \{\text{reducible divisors}\}$, where the reducible divisors are of the form $E_\infty + C_1 + C_2$ and result from blowing up hyperplane intersections of $\mathscr{C}(Q_1)$ that pass through v. Q_1 determines a distinguished irreducible element of $|E_0|$ and nullity is defined in $|E_0|$ with respect to that curve in the obvious way: note that the null divisors at infinity are divisors of the form $E_\infty + 2C$.

We say that an algebraic curve $A \subset S_2$ is *full* if it blows down to a full curve $\beta(A) \subset \mathscr{C}(Q_1)$. For a full curve $A \subset S_2$, with normalization $(\tilde{X},n)$, osculation is defined on $\tilde{X}$ via $\beta(A)$. Thus 4.9 gives

Corollary. There exists a natural correspondence between full algebraic curves on S_2 and full algebraic null curves in $|E_0|$.

Remarks. (i) Compactifying T to S_2 the Gauss transform extends over the poles of a null meromorphic curve in $\mathbb{C}^3$.

(ii) This result simply reformulates Weierstrass' observation that in 1.2, Ω is a meromorphic curve iff f is an algebraic function.

(iii) For $n : \tilde{X} \longrightarrow A \subset S_2$, the Gauss map of n^* is given by $\gamma_{n^*} = \pi|_A \circ n$, where $\pi : S_2 \longrightarrow \mathbb{P}_1$ is projection.

(4.11) We now consider, in more detail, osculation of $n : \tilde{X} \longrightarrow A \subset S_2$ at points in the branch locus of $\pi|_A$ and at points on E_∞.

A null curve in $\mathbb{P}_3^*$ intersects v^*, the hyperplane at infinity, inside $\check{Q}_1$. Consequently, if the hyperplane that osculates $\beta(A)$ at $\beta \circ n(\xi)$ passes through v it is null and hence A osculates the fibre through $n(\xi)$. In terms of divisors: $n^*(\xi) = E_\infty + 2C_{\pi \circ n(\xi)}$.

If $n(\xi)$ lies on E_∞ then $\beta \circ n(\xi) = v$ and the osculating hyperplane lies tangent to $\mathscr{C}(Q_1)$ along $\pi \circ n(\xi)$, therefore $n^*(\xi) = E_\infty + 2C_{\pi \circ n(\xi)}$ and in particular n^* is never finite on E_∞.

If $\beta \circ n(\xi)$ is a singular point on $\beta(A)$, finiteness of $n^*(\xi)$ depends on the nature of the singularity there. This can be made precise as follows. A can be described locally in S_2 in the form $n(\xi) = (\gamma_n*(\xi), h(\xi))$, where h is a meromorphic function. Suppose that ξ, a local coordinate on $U \subset \mathfrak{X}$, is centred at ξ_0 and such that $\gamma_{n*}(\xi) = \xi^q$; furthermore suppose that ζ is centred at $\gamma_{n*}(\xi_0)$. Since h is meromorphic there are Puiseux series representations of $n(U)$ in the vicinity of $n(\xi_0)$:

$$h \circ \gamma_{n*}^{-1}(\zeta) = \sum_{m=p}^{\infty} a_m \zeta^{m/q}\,, \text{ where } p \in \mathbb{Z}.$$

Write $n^* = (n_1^*, n_2^*, n_3^*) : \mathfrak{X} \longrightarrow \mathbb{C}^3 \cup v^*$. It follows from 1.2 that:

$p/q > 2$ implies that $n^*(\xi_0)$ is finite, in which case ξ_0, for $q \geq 2$, is simply a branch point of the Gauss map of the curve in $\mathbb{C}^3$;

$1 < p/q < 2$ implies that $n_1^*(\xi_0) = n_2^*(\xi_0) = \infty$, $n_3^*(\xi_0) < \infty$. In this case the corresponding end of the associated minimal surface in $\mathbb{R}^3$ is asymptotic to an affine plane;

$p/q < 1$ implies that $n_k^*(\xi_0) = \infty$, $k = 1,2,3$.

Remarks. (i) These differences may be viewed as follows. If $n(\xi_0) \in E_\infty$ then, at ξ_0, n^* osculates v^*, the hyperplane at infinity of $\mathbb{C}^3$. However, if $n(\xi_0) \notin E_\infty$ then, at ξ_0, n^* osculates a null hyperplane in $\mathbb{C}^3 \cup v^*$. For $0 < p/q < 2$, n osculates the fibre through $n(\xi_0)$, therefore n^* osculates the null hyperplane determined by $n(\xi_0)$ at infinity. $p/q > 2$ implies that $n^*(\xi_o) \in \mathbb{C}^3$ and osculation occurs there.

Clearly, the most significant difference between $0 < p/q < 1$ and $1 < p/q < 2$ concerns the second order behaviour of the curve at $n(\xi_0)$, it is this that determines the difference in the growth behaviour at the end.

(ii) It is clear that if a smooth point of A lies in the branch locus of $\pi|_A$ then A osculates the fibre there.

(iii) Points of self–intersection on A contribute to the asymptotic structure of A^* only if a component there osculates the fibre.

(iv) The above trichotomy was known in the last century, see [D].

§5. Moduli for Null Curves

(5.1) A null meromorphic curve $\Omega : M \longrightarrow \mathbb{C}^3$ extends over its poles to give a null curve $\tilde{\Omega} : \overline{M} \longrightarrow \mathbb{P}_3$. Now, $\tilde{\Omega}$ factors through $\tilde{X}$, the normalization of $A = \tilde{\Omega}(\overline{M})$ and hence the natural data associated to $\Omega(M)$, and the accompanying minimal surface $\mathrm{Re}\,\Omega(M)$, derives from $\tilde{X}$. The total Gaussian curvature, genus, branching and number of ends should be calculated there since this removes superfluous ramification in the parameterization of the image. (E.g. this gives a sharper form of Ossermann's inequality in the presence of branching.)

Corollary 4.10 gives the following diagram

$$\tilde{\Omega} : \overline{M} \longrightarrow \tilde{X} \begin{array}{l} \overset{n}{\nearrow} A \\ \underset{n^*}{\searrow} A^* \end{array}$$

where A^* is the corresponding Gauss transform on S_2. This suggests that natural moduli for null meromorphic curves in $\mathbb{C}^3$ are given, not by fixing a parameter domain M and varying Ω, but rather by the moduli of the corresponding curves on S_2 where there is no nullity constraint to satisfy.

(5.2) A^* is an irreducible algebraic curve on S_2 and is determined up to linear equivalence by the intersection numbers $A^* \cdot E_0$ and $A^* \cdot C$, which yield natural numerical data associated to A:

$A^* \cdot C = k$ gives the degree of $\pi|_{A^*}$ and therefore equals the degree of the Gauss map of n;

$A^* \cdot E_0 = c$ is the *class* of A, it counts (with multiplicity) the number of hyperplanes osculating A that pass through a point of $\mathbb{P}_3$.

A^* lies in the complete linear system $|kE_0 + (c-2k)C|$, since A^* is full it does not equal E_∞ or a fibre and hence its irreducibility implies that $k > 0$ and $c \geq 2k$.

The linear systems $|aE_0 + bC|$ on S_2, with $a > 0$, $b \geq 0$, provide natural compactifications of the moduli spaces of null meromorphic curves in $\mathbb{C}^3$. The reducible divisors in such a system correspond to (possibly degenerate) limits of sequences of null meromorphic curves, see §6 for some simple examples.

Remarks. (i) Two curves on S_2 are linearly equivalent iff they are homologous and hence these moduli spaces may be viewed as homology classes of algebraic curves on S_2, see [G–H].

(ii) There is no natural scale for the complex vector space $|E_0|$–{reducible divisors}, only a null cone. However, fixing one gives a (branched) metric, induced by n, on the complement in $\mathfrak{X}$ of the set of poles of n. The total Gaussian curvature of this metric is independent of the choice of scale and is well–known to be equal to $-4\pi \deg(\gamma_n)$, [0] . This gives another interpretation of $A \cdot C$.

(5.3) The genus of a generic null meromorphic curve of class c and with total curvature $-4\pi k$ is readily obtained: since S_2 is smooth and $|kE_0 + (c-2k)C|$ is base point free, Bertini's theorem implies that the generic element is smooth and the adjunction formula gives $g = k(c-k) - c + 1$.

Remarks. (i) In general, singularities of A^* will contribute to this formula to lower g, see [G–H].
(ii) The Gauss map of a generic null meromorphic curve does not possess branch points off the set of poles of the curve since from 4.1 the osculating hyperplane is finite at a branch of the Gauss transform only if the Gauss transform is singular there.

(5.4) On S_2 there is the linear equivalence $E_\infty \sim E_0 - 2C$. This gives $A^* \cdot E_\infty = c - 2k$: if $k = 1$, in which case A^* is simply a global meromorphic section of T, this is (6) of § 234 in [D]. Clearly, $A^* \cdot E_\infty$ measures that part of the end structure of A that arises from E_∞.

The total number of poles of A, counted with multiplicity, equals d, the degree of A as a curve in $\mathbb{P}_3$, since it gives the intersection number of A with the hyperplane at infinity. We would like to compute d from data on A^*. Blowing down A^* to $\beta(A^*) \subset \mathscr{C}(Q_1)$ there is the Plücker formula:

$$d - 2d_1 + d^* = 2g - 2 + \beta_1,$$

where d^* = degree of A^* as a curve in $\mathbb{P}_3$ and therefore equals c; d_1 = degree of the first associate curve $n_1 : \mathfrak{X} \longrightarrow G(2,\mathbb{C}^4)$; g = genus of $\mathfrak{X}$ and β_1 = total branching of n_1 : see [G–H] for details.

§6. Examples

(6.1) Let $p,q \in \mathbb{N}$ be coprime with $p+q \geq 3$. $\mathscr{C}_{p,q}$, the curve on S_2 obtained by completing the curve in $\mathbb{C}^2$ given by $\eta^q = \zeta^p$, is irreducible and rational and its normalization is given by extending $u \longmapsto (u^q,u^p)$. Osculation of $\mathscr{C}_{p,q}$ yields a non–constant null meromorphic curve in $\mathbb{C}^3$. Differentiation of $f(u^q) = u^p$ and substitution into 1.2 gives the following global formulae:

$$\Omega_1^{p,q}(u) = \frac{p}{2q}\left[\frac{p}{q} - 1\right]u^{p-2q} - \left[1 - \frac{3p}{2q} + \frac{p^2}{2q^2}\right]u^p$$

$$\Omega_2^{p,q}(u) = \frac{ip}{2q}\left[\frac{p}{q} - 1\right]u^{p-2q} + i\left[1 - \frac{3p}{2q} + \frac{p^2}{2q^2}\right]u^p$$

$$\Omega_3^{p,q}(u) = \frac{p}{q}\left[\frac{p}{q} - 2\right]u^{p-q}.$$

For $p > 2q$ this curve is defined on $\mathbb{C}$ and $\mathbb{C}^*$ otherwise. $\phi^{p,q} = \mathrm{Re}(\Omega^{p,q})$ is a complete branched minimal surface in $\mathbb{R}^3$ with total Gaussian curvature $-4\pi q$.

The branch points on $\mathscr{C}_{p,q}$ sit over 0 and ∞ on $\mathbb{P}_1$; the branch point over ∞ always corresponds to an end of $\phi^{p,q}$. If $p < 2q$ then the branch point over 0 gives an end and the surface has 2 ends. If $p > 2q$ then $\phi^{p,q}(0)$ is finite and for $q > 1$, $u = 0$ is a branch point of the Gauss map. Since $\mathscr{C}_{p,q}$ is described in $(\omega,\mu\frac{d}{d\omega})$ – coordinates on T, where $\omega = 1/\zeta$, by the curve $\mu^q = (-1)^q \omega^{2q-p}$, observe that $p > 2q$ iff $\mathscr{C}_{p,q}$ intersects E_∞. So having only one end forces greater growth there.

In fact, since we also have $1 < p/q < 2$ iff $0 < 2 - p/q < 1$ and $0 < p/q < 1$ iff $1 < 2 - p/q < 2$, the growth behaviour at $u = 0$ is coupled in a simple way to that at $u = \infty$.

For $p > 2q + 1$, $u = 0$ is a branch point in the metric since $qu^{q-1}f'''(u^q) \sim u^{p-2q-1}$, however the surface is immersed for $p \leq 2q + 1$. The surfaces with $p = 2q+1$ may be of special interest; for $q = 1$ this is Enneper's surface.

(6.2) Let $\wp$ be the Weierstrass $\wp$–function associated with a lattice $\Lambda \subset \mathbb{C}$ and g_2, g_3 be the usual constants derived from the Eisenstein series for Λ. The curve $\eta^2 = 4\zeta^3 - g_2\zeta - g_3$ in $\mathbb{C}^2$ completes in T to a smooth elliptic curve $\mathscr{E}$, which lies in the linear system $|2E_0|$. $(\wp, \wp') : \mathbb{C}/\Lambda - \{0\} \longrightarrow \mathbb{C}^2$ extends to give a parameterization of $\mathscr{E}$

Since $\mathscr{E} \cdot E_0 = 4$ and $\mathscr{E} \cdot C = 2$, osculation of $\mathscr{E}$ gives a null meromorphic curve $\Omega : \mathscr{E} \longrightarrow \mathbb{C}^3 \cup \mathbb{P}_2$ of class 4 and with total Gaussian curvature -8π. The map $\pi|_{\mathscr{E}}$ has 4 branch points and since $\mathscr{E}$ is smooth they each give rise to a pole of Ω.

As a curve on $\mathscr{C}(Q_1)$, $\mathscr{E}$ is embedded in $\mathbb{P}_3$ and has degree 4. It follows from Hurwitz's theorem (see Ex. 4.6 [Ha]) that there are 16 points of hyperosculation on $\mathscr{E}$ (These are distinct because deg $\mathscr{E} = 4$ means that any point of $\mathscr{E}$ can count at most 4 times in the intersection with a hyperplane.) Each of the 4 branch points of $\pi|_{\mathscr{E}}$ is a point of hyperosculation: for, the osculating hyperplane at a branch point $b \in \mathscr{E}$ lies tangent to $\mathscr{C}(Q_1)$ along the fibre through b and hence it intersects $\mathscr{E}$ only at b and so it must intersect there with multiplicity 4. Consequently it follows from Remark 3.9 (i) that there are 12 zeros in the metric induced by Ω on $\mathcal{S}$–{poles of Ω}.

Recall that $\wp'(z) = 0$ for $z = \omega_1/2, \omega_2/2, \omega_3/2$ where $\{\omega_1, \omega_2\}$ is a basis for Λ and $\omega_3 = \omega_1 + \omega_2$. Differentiation of the equation $f \circ \wp = \wp'$ together with some elementary calculations yields the following global formulae for

$$\Omega : \mathbb{C}/\Lambda - \{0, \omega_1/2, \omega_2/2, \omega_3/2\} \longrightarrow \mathbb{C}^3:$$

$$\Omega_1 = \{2\wp^6 + 3(2+g_2)\wp^4 + 8g_3\wp^3 - 3g_2(1 + g_2/8)\wp^2$$

$$-3g_3(2 + g_2/2)\wp - (g_2^2/8 + g_3^2)\}/(\wp')^3$$

$$\Omega_2 = i\{-2\wp^6 + 3(2 - g_2)\wp^4 - 8g_3\wp^3 - 3g_2(1 - g_2/8)\wp^2$$

$$-3g_3(2 - g_2/2)\wp - (g_2^2/8 - g_3^2)\}/(\wp')^3$$

$$\Omega_3 = \{-12\wp^5 + 2g_2\wp^3 - 6g_3\wp^2 - 3/4g_2^2\wp - g_2g_3/2\}/(\wp')^3 .$$

(6.3) There is a natural real structure $\tau : T \longrightarrow T$ given in coordinates by $\tau(\zeta, \eta) = (-1/\bar{\zeta}, \bar{\eta}/\bar{\zeta}^2)$ which, viewing T as the set of oriented lines in $\mathbb{R}^3$, simply reverses the orientation along lines [H1]. For $\sigma_z \in H^0 \cong \mathbb{C}^3$ (via (z_1, z_2, z_3)–coordinates, see Remark 2.3 (ii)), it is easy to see that $\tau \circ \sigma_z = \sigma_{\bar{z}}$.

Suppose that $A \subset T$ is τ–invariant, for example A could be the spectral curve of a magnetic monopole [H1]. If σ_z osculates A at α then $\sigma_{\bar{z}}$ osculates A at $\tau(\alpha)$. Consequently, $\Omega(\tau(\alpha)) = \overline{\Omega}(\alpha)$ and hence $\phi(\alpha) = \frac{1}{2}(\Omega(\alpha) + \Omega(\tau(\alpha)))$ and satisfies $\phi(\tau(\alpha)) = \phi(\alpha)$. Thus ϕ factors through A/τ.

Clearly, α is a pole of Ω iff $\tau(\alpha)$ is, and α is a point of hyperosculation iff $\tau(\alpha)$ is. So if an elliptic curve $\mathscr{E} \in |2E_0|$ is τ–invariant then $\phi : \mathscr{E}/\tau - \{2 \text{ points}\} \longrightarrow \mathbb{R}^3$ gives a complete branched minimal immersion of a Klein bottle into $\mathbb{R}^3$ with total Gaussian curvature -4π, 6 branch points in the metric and 2 ends.

Remarks (i) Monopole spectral curves are τ–invariant and those of charge 2 enjoy (at least) $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry [Hu]. This symmetry is reflected in the geometry of the associated minimal surface. For example $\eta^2 = 4\zeta(\zeta^2 - 1)$ is invariant under the action of the bundle automorphisms of T induced by differentiating elements in the following subgroup of PGL(2,$\mathbb{C}$):

$\{\zeta \longmapsto \zeta,\ \zeta \longmapsto -\zeta,\ \zeta \longmapsto \zeta^{-1},\ \zeta \longmapsto -\zeta^{-1}\}$. This subgroup corresponds to $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset SO(3,\mathbb{R})$ given by the rotations through π–degrees about the coordinate axes in $\mathbb{R}^3$. This might be exploited in the graphical construction of such surfaces.

(ii) There does not exist a complete non–orientable minimal immersion with total Gaussian curvature -4π, [M]. The branch points of ϕ above contribute to the Chern–Osserman inequality and remove the obstruction.

(iii) Any τ–invariant algebraic curve on T gives rise to a complete non–orientable branched minimal immersion in $\mathbb{R}^3$. A familiar example is Henneberg's surface, whose Gauss transform is given by the meromorphic section $\eta = 1/3(\zeta^{-1} + \zeta^3)$.

(iv) In order to have an explicit example of a complete branched minimal surface in $\mathbb{R}^3$ which is genuinely a punctured Klein bottle it remains to check that ϕ constructed from a τ–invariant elliptic curve does not factor through $\mathbb{RP}_2$.

(v) Note that $\mathscr{C}_{3,2}$ lies in $|2E_0|$. Also, a family of elliptic curves in $|2E_0|$ may degenerate into a pair of global sections; for example this phenomena is associated with monopole scattering [A–H]. Osculation of a reducible divisor in $|2E_0|$ gives a pair of points in $\mathbb{C}^3$: if the sections are τ–invariant then the pair lies in $\mathbb{R}^3$.

(vi) It is not hard to see that osculation of the spectral curve of a monopole of charge k induces a metric whose total Gaussian curvature is $-4\pi k$: we discuss this in more detail in [S3].

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