On the usefulness of an index due to Leray for studying the intersections of Lagrangian and symplectic paths

Maurice de Gosson
Max-Planck-Institut für Mathematik
Pf. 7280, DE-53072 Bonn
degosson@mpim-bonn.mpg.de
and
Universität Wien
Fakultät für Mathematik, NuHAG
Nordbergstrasse 15, AT-1090 Wien
maurice.de.gosson@univie.ac.at

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Abstract

Using the ideas of Keller, Maslov introduced in the mid 1960's an index for Lagrangian loops, whose definition was clarified by Arnol'd. Leray extended Arnol'd' s results by defining an index depending on two paths of Lagrangian planes with transversal endpoints. We show that the combinatorial and topological properties of Leray's index suffice to recover all Lagrangian and symplectic intersection indices commonly used in symplectic geometry and its applications to Hamiltonian and quantum mechanics. As a by-product we obtain a new simple formula for the Hörmander index, and a definition of the Conley–Zehnder index for symplectic paths with arbitrary endpoints. Our definition leads to a formula for the Conley–Zehnder index of a product of two paths.

Entia non sunt multiplicanda praeter necessitatem (William of Ockham)

1 Introduction

In the Preface to his *Lagrangian Analysis and Quantum Mechanics* [22] Jean Leray adds a *Historical note* where he tells us that *[... In Moscow in 1967*

I.V. Arnold asked me my thoughts on Maslov's work. The present book is an answer to that question...]. One of the most original features of Leray's "answer" to Arnol'd's question –and perhaps one of the most forgotten parts of Leray's mathematical work– is the introduction of a function m associating an integer to each pair of Lagrangian paths with same origin and transversal end points. This function –which Leray calls "Maslov index"– is uniquely characterized by two properties. The first of these properties is of combinatorial nature: if $\Lambda, \Lambda', \Lambda$ are three such Lagrangian paths defined on $[0, 1]$ and then

$$m(\Lambda, \Lambda') - m(\Lambda, \Lambda'') + m(\Lambda', \Lambda'') = \mathrm{Inert}(\Lambda(1), \Lambda'(1), \Lambda(1))$$

(Inert is the index of inertia of a triple of Lagrangian planes), and the second is topological:

$$m(\Lambda, \Lambda') \text{ is locally constant on its domain.}$$

In [13] we have proposed an extension of Leray's index to the non-transversal case using the properties of the signature of a triple of Lagrangian planes, due to Wall [32] and Kashiwara [24]. Our construction was taken up by Cappell, Lee, and Miller in [2] who compared our index to other indices appearing in the literature (beware: the reference "M. de Gosson" is misspelled "E. Gossen" in this paper). We should add at this point that Dazord [7] had previously proposed an extension of Leray's index, using different methods; however (but neither Dazord nor I were aware of this at that time) Leray himself had constructed an extension of his index, in [23], using reduction techniques. We notice that Lion and Vergne sketched in [24] the construction of an index which is identical with ours (the aim of [13], which was preceded by two "Notes aux Comptes Rendus" [11, 12], was actually to advertise and make totally rigorous the constructions in [24]).

The aim of this paper is to propose an unifying approach to the theory of Lagrangian and symplectic intersection indices ("Maslov indices") bases on the properties of the Leray index; we will show that the combinatorial and topological properties of Leray's index allow a simple and elegant construction of all Maslov indices for Lagrangian and symplectic paths available in the literature. In addition our approach leads to a very simple formula expressing the so-called Hörmander index in terms of the signature of a triple of Lagrangian planes, and to a redefinition of the Conley–Zehnder index for symplectic path with arbitrary endpoints; this redefinition allows us to prove a general product formula.

We shortly discuss some related by other authors in the Conclusion to this article.

Notation 1 (General) *Let $X = \mathbb{R}^n$; the vector space $Z = X \times X^*$ is endowed with the canonical symplectic form defined by:*

$$\omega(z, z') = \langle p, x' \rangle - \langle p', x \rangle$$

if $z = (x, p)$, $z' = (x', p')$. The symplectic group of (Z, ω) will be denoted by $\mathrm{Sp}(n)$. The unitary group $U(n)$ is identified with a subgroup of $\mathrm{Sp}(n)$. We denote by $\mathrm{Lag}(n)$ the Lagrangian Grassmannian of (Z, ω). We will write $X = X \times 0$ and $X^* = 0 \times X^*$.

Notation 2 (Cohomological) *Let E be a set, $k \in \mathbb{Z}_+$, and $(G, +)$ an abelian group. A k-cochains on E with values in G is a function $f : E^{k+1} \longrightarrow G$. The coboundary ∂f of a k-cochain is the $(k+1)$-cochain defined by:*

$$\partial f(a_0, ..., a_{k+1}) = \sum_{j=0}^{k+1} (-1)^j f(a_0, ..., \widehat{a}_j, ..., a_{k+1}),$$

where the cap $\hat{}$ deletes the term it covers. We have $\partial^2 f = 0$. A k-cochain f is a coboundary if there exists a cochain g such that $f = \partial g$; a cochain f is a cocycle if $\partial f = 0$.

2 The Leray index

We denote by $C_{\ell_0}(\mathrm{Lag}(n))$ the set of all continuous paths $[0, 1] \longrightarrow \mathrm{Lag}(n)$ joining a given base point ℓ_0 to ℓ in $\mathrm{Lag}(n)$. Let $\overset{\ell_0}{\sim}$ be the equivalence relation on $C_{\ell_0}(\mathrm{Lag}(n))$ defined by $\Lambda \overset{\ell_0}{\sim} \Lambda'$ if and only if Λ and Λ' are homotopic with fixed endpoints. Let $\pi^{\mathrm{Lag}} : \mathrm{Lag}_\infty(n) \longrightarrow \mathrm{Lag}(n)$ be the universal covering of the Lagrangian Grassmannian; as a set $\mathrm{Lag}_\infty(n) = C_{\ell_0}(\mathrm{Lag}(n))/ \overset{\ell_0}{\sim}$; for $\ell_\infty \in \mathrm{Lag}_\infty(n)$ we write $\pi^{\mathrm{Lag}}(\ell_\infty) = \ell$, and we will say that ℓ_∞ covers ℓ.

2.1 Leray's index m

Using the intersection theory of Lefschetz chains, Leray constructs in [22], Ch.I, §2.5, a function

$$m : \mathcal{C}_{X^*}(\mathrm{Lag}(n)) \times \mathcal{C}_{X^*}(\mathrm{Lag}(n)) \longrightarrow \mathbb{Z}$$

defined for all pairs (Λ, Λ') with transversal endpoints; this function has the following homotopy property: $m(\Lambda, \Lambda') = m(\Lambda'', \Lambda''')$ if and only if $\Lambda \overset{\ell_0}{\sim} \Lambda''$ and $\Lambda' \overset{\ell_0}{\sim} \Lambda'''$. We can thus view m as a function

$$m : \{(\ell_\infty, \ell'_\infty) : \ell \cap \ell' = 0\} \longrightarrow \mathbb{Z}.$$

Leray's index is characterized by the two following properties:

$$m(\ell_\infty, \ell'_\infty) - m(\ell_\infty, \ell''_\infty) + m(\ell'_\infty, \ell''_\infty) = \mathrm{Inert}(\ell, \ell', \ell'') \tag{1}$$

((ℓ, ℓ', ℓ'') covering $(\ell_\infty, \ell'_\infty, \ell''_\infty)$), and

$$m \text{ is locally constant on its domain.} \tag{2}$$

The integer $\mathrm{Inert}(\ell_1, \ell_2, \ell_3)$ is the index of inertia of the Lagrangian triple (ℓ, ℓ', ℓ''); it is is defined in the following way (Leray [22], Ch.I, §2.5): the transversality condition $\ell \cap \ell' = \ell' \cap \ell'' = \ell'' \cap \ell = 0$ being equivalent to $Z = \ell \oplus \ell' = \ell' \oplus \ell'' = \ell'' \oplus \ell$ the relation $z + z' + z'' = 0$ ($z \in \ell$, $z' \in \ell'$, $z'' \in \ell''$) defines three quadratic forms $z \longmapsto \omega(z', z'')$, $z' \longmapsto \omega(z'', z)$, $z'' \longmapsto \omega(z, z')$

such that $\omega(z', z'') = \omega(z'', z) = \omega(z, z')$. These quadratic forms have the same index of inertia $\mathrm{Inert}(\ell, \ell', \ell'')$.

The function m (which Leray calls "Maslov index") is very simple to describe explicitly in when $n = 1$. We can identify $\Lambda_\infty(1)$ with the set of all pairs $\ell(\theta) = (e^{i\theta}, \theta)$, $\theta \in \mathbb{R}$; we have $\pi^{\mathrm{Lag}}(\ell(\theta)) = \ell = e^{i\theta}$, and

$$m(\ell(\theta), \ell(\theta')) = \left[\frac{\theta - \theta'}{2\pi}\right] \tag{3}$$

$[\cdot]$ being the integer part function. In the case $n > 1$ it can be explicitly computed using a formula due to Souriau [30]. Let $W(n, \mathbb{C})$ be the submanifold of $U(n, \mathbb{C})$ consisting of symmetric matrices:

$$W(n, \mathbb{C}) = \left\{u \in U(n, \mathbb{C}) : u = u^T\right\}.$$

($u^T = \overline{u}^*$ the transpose of u). The mapping

$$\mathrm{Lag}(n) \ni \ell = uX^* \longmapsto uu^T \in W(n, \mathbb{C})$$

is a homeomorphism identifying $\mathrm{Lag}(n)$ with $W(n, \mathbb{C})$ and $\mathrm{Lag}_\infty(n)$ with

$$W_\infty(n, \mathbb{C}) = \left\{(w, \theta) : w \in W(n, \mathbb{C}), \det w = e^{i\theta}\right\}.$$

Souriau's formula says that

$$m(\ell_\infty, \ell'_\infty) = \frac{1}{2\pi}\left[\theta - \theta' + i\,\mathrm{Tr}\,\mathrm{Log}(-w(w'^{-1}))\right] + \frac{n}{2}; \tag{4}$$

it is easily verified that this formula coincides with (3) when $n = 1$. The logarithm in (4) is well defined because $\ell \cap \ell' = 0$ if and only if $+1$ *is not an eigenvalue of* $w(w^{-'1})$ (Leray [22], Ch.I, §2.2). The function m possesses in addition following property: let γ and γ' be two elements of $\pi_1[\mathrm{Lag}(n)]$. We have

$$m(\gamma\ell_\infty, \gamma'\ell'_\infty) = m(\ell_\infty, \ell'_\infty) + m(\gamma) - m(\gamma') \tag{5}$$

where $m(\gamma)$ is the *Maslov index* of $\gamma \in \pi_1[\mathrm{Lag}(n)] \cong (\mathbb{Z}, +)$; it is defined as follows: the composition of the natural isomorphism $\pi_1[\mathrm{Lag}(n)] \cong \pi_1[W(n, \mathbb{C})]$ and of the morphism

$$\pi_1[W(n, \mathbb{C})] \ni [\gamma] \longmapsto \frac{1}{2\pi i}\oint_\gamma \frac{d(\det w)}{\det w} \in \mathbb{Z} \tag{6}$$

is an isomorphism

$$m : \pi_1[\mathrm{Lag}(n)] \ni [\gamma] \overset{\cong}{\longmapsto} m(\gamma) \in (\mathbb{Z}, +). \tag{7}$$

2.2 The index μ and the Wall–Kashiwara signature

We now define an index μ by the formula

$$\mu(\ell_\infty, \ell'_\infty) = 2m(\ell_\infty, \ell'_\infty) - n; \tag{8}$$

when $n = 1$ we have, in view of (3),

$$\mu(\ell(\theta), \ell(\theta')) = 2\left[\frac{\theta - \theta'}{2\pi}\right]_{\text{ant}}$$

where $[k]_{\text{ant}} = \frac{1}{2}([k] - [-k])$ is the antisymmetric part of the integer function.

Formula (1) becomes

$$\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'') \tag{9}$$

where

$$\tau(\ell, \ell', \ell'') = 2\,\text{Inert}(\ell, \ell', \ell'') - n. \tag{10}$$

One easily proves (de Gosson [13]) that $\tau(\ell, \ell', \ell'') = \tau^+ - \tau^-$ where τ^+ (resp. τ^-) is the number of > 0 (resp. < 0) eigenvalues of the quadratic form

$$Q(z, z', z'') = \omega(z, z') - \omega(z, z'') + \omega(z', z''); \tag{11}$$

this identifies $\tau(\ell, \ell', \ell'')$ with the Wall–Kashiwara index [2, 24, 32]. (Also see Py's [27] interesting discussion of Wall's contributions; also see the paper [1] of Barge and Ghys where related notions such as Euler's cocycle are studied in detail.)

Remark 3 *The signature τ is sometime called "Maslov index" in the literature. This is however somewhat misleading: the Maslov index is defined on loops (or paths) of Lagrangian planes, while τ depends on (triples of) points in the Lagrangian Grassmannian.*

The Wall–Kashiwara index τ is a totally antisymmetric 2-cocycle, that is $\varepsilon^*\tau = (-1)^{sgn(\varepsilon)}\tau$ (ε any permutation of (ℓ, ℓ', ℓ'')) and $\partial\tau = 0$; it has the following properties:

Let us mention the following properties of the signature:

- *τ is a symplectic invariant:*

$$\tau(s\ell, s\ell, s\ell'') = \tau(\ell, \ell', \ell'') \tag{12}$$

  for all $s \in \text{Sp}(n)$;

- *Let M be a symmetric automorphism of Z and $\ell_M = \{(x, Mx) : x \in X\}$. We have $\ell_M \in \text{Lag}(n)$ and*

$$\tau(X^*, \ell_M, X) = \text{sign}\, M \tag{13}$$

  where $\text{sign}\, M$ is the difference between the number of > 0 and < 0 eigenvalues of M;

- *Let τ' and τ'' the signatures on* $\mathrm{Lag}(n')$ *and* $\mathrm{Lag}(n'')$,. *Then* $\tau = \tau' \oplus \tau''$ *is the signature on* $\mathrm{Lag}(n)$, $n = n' + n''$, *and*

$$\tau(\ell_1' \oplus \ell_1'', \ell_2' \oplus \ell_2'', \ell_3' \oplus \ell_3'') = \tau'(\ell_1', \ell_2', \ell_3') + \tau''(\ell_1'', \ell_2'', \ell_3''). \tag{14}$$

Now, let us come to the crucial point: $\tau(\ell, \ell', \ell'')$ is defined for all triples (ℓ, ℓ', ℓ''); we may thus *define* $\mu(\ell_\infty, \ell_\infty')$ *for an arbitrary pair* $(\ell_\infty, \ell_\infty')$ by choosing $\ell_\infty'' \in \mathrm{Lag}_\infty(n)$ such that $\ell'' \cap \ell = \ell'' \cap \ell' = 0$ and setting

$$\mu(\ell_\infty, \ell_\infty') = \mu(\ell_\infty, \ell_\infty'') - \mu(\ell_\infty', \ell_\infty'') + \tau(\ell, \ell', \ell''). \tag{15}$$

In fact, using the cocycle property $\partial\tau = 0$ one shows (de Gosson [13]) that the right-hand side of (15) does not depend on the choice of ℓ_∞'', justifying the notation $\mu(\ell_\infty, \ell_\infty')$ in the left-hand side. We will call μ the *Leray index on* $\mathrm{Lag}_\infty(n)$.

Theorem 4 *(i) The Leray index is the only function*

$$\mu : \mathrm{Lag}_\infty(n) \times \mathrm{Lag}_\infty(n) \longrightarrow \mathbb{R}$$

having the two following properties:

$$\mu(\ell_\infty, \ell_\infty') - \mu(\ell_\infty, \ell_\infty'') + \mu(\ell_\infty', \ell_\infty'') = \tau(\ell, \ell', \ell'') \tag{16a}$$

$$\mu \textit{ is is locally constant on } \{(\ell_\infty, \ell_\infty') : \ell \cap \ell' = 0\}; \tag{16b}$$

(ii) In addition μ is locally constant on the sets

$$\mathrm{Lag}_\infty^2(n; k) = \{(\ell_\infty, \ell_\infty') : \dim(\ell \cap \ell') = k\}$$

for $1 \leq k \leq n$. (iii) We have

$$\mu(\gamma\ell_\infty, \gamma'\ell_\infty') = \mu(\ell_\infty, \ell_\infty') + 2(m(\gamma) - m(\gamma')) \tag{17}$$

for all $\gamma, \gamma' \in \pi_1 [\mathrm{Lag}(n)]$.

Proof. The statement (i) was proven in de Gosson [13]. (The uniqueness statement is obvious: if δ is the difference between two functions satisfying conditions (16) then $\delta(\ell_\infty, \ell_\infty') = \delta(\ell_\infty, \ell_\infty'') - \delta(\ell_\infty', \ell_\infty'')$ for all ℓ_∞'' hence δ is locally constant on $\mathrm{Lag}_\infty(n) \times \mathrm{Lag}_\infty(n)$; since $\mathrm{Lag}_\infty(n)$ is connected δ is in fact constant; taking $\ell_\infty = \ell_\infty'$ that constant is 0.) (ii) The kernel of the quadratic form Q is isomorphic to $(\ell \cap \ell') \times (\ell' \cap \ell'') \times (\ell'' \cap \ell)$ ([24], Proposition 1.9.3) hence τ is locally constant on each set $\mathrm{Lag}_\infty^2(n; k) \times \mathrm{Lag}_\infty^2(n; k') \times \mathrm{Lag}_\infty^2(n; k'')$. Let now $(\ell_\infty, \ell_\infty', \ell_\infty'')$ move continuously in such a way that $\dim(\ell \cap \ell') = k$ and $\ell \cap \ell' = \ell'' \cap \ell = 0$. Then $\mu(\ell_\infty, \ell_\infty'')$ and $\mu(\ell_\infty', \ell_\infty'')$ remain constant in view of property (2) of m and $\tau(\ell, \ell', \ell'')$ also remains constant. The claim follows in view of (15). (iii) Formula (17) immediately follows from (5), the definition of μ, and the fact that $\pi^{\mathrm{Lag}}(\gamma\ell_\infty) = \ell$. ∎

Let $\mathrm{Sp}_\infty(n)$ be the universal covering group of $\mathrm{Sp}(n)$. As a set, $\mathrm{Sp}_\infty(n)$ consists of the homotopy classes s_∞ of paths in $\mathrm{Sp}(n)$ joining the identity I to

s. The projection $\pi^{\mathrm{Sp}} : \mathrm{Sp}_\infty(n) \longrightarrow \mathrm{Sp}(n)$ associates to s_∞ its endpoint s. Let $\mathrm{St}_{X^*}(n)$ be the isotropy subgroup of X^* in $\mathrm{Sp}(n)$. The fibration

$$\mathrm{Sp}(n)/\,\mathrm{St}_{X^*}(n) = \mathrm{Lag}(n) \tag{18}$$

defines an isomorphism

$$\mathbb{Z} \cong \pi_1\left[\mathrm{Sp}(n)\right] \longrightarrow \pi_1\left[\mathrm{Lag}(n)\right] \cong \mathbb{Z}$$

which is multiplication by 2 on $\mathbb{Z}$. It follows (Leray [22], Theorem 3,3^o, p.36) that the action of $\mathrm{Sp}(n)$ on $\mathrm{Lag}(n)$ can be lifted to a transitive action of the universal covering $\mathrm{Sp}_\infty(n)$ on the Maslov bundle $\mathrm{Lag}_\infty(n)$ such that

$$(\alpha s_\infty)\ell_\infty = \beta^2(s_\infty \ell_\infty) = s_\infty(\beta^2 \ell_\infty) \tag{19}$$

for all $(s_\infty, \ell_\infty) \in \mathrm{Sp}_\infty(n) \times \mathrm{Lag}_\infty(n)$; α (resp. β) is the generator of $\pi_1\left[\mathrm{Sp}(n)\right]$ (resp. $\pi_1\left[\mathrm{Lag}(n)\right]$) whose image in $\mathbb{Z}$ is $+1$; note that the Maslov index of β is $m(\beta) = 1$.

The Leray index has the following properties of symplectic invariance: for all s_∞, ℓ_∞, ℓ'_∞ we have

$$\mu(s_\infty \ell_\infty, s_\infty \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty). \tag{20}$$

Set in fact, for fixed s_∞, $\mu'(\ell_\infty, \ell'_\infty) = \mu(s_\infty \ell_\infty, s_\infty \ell'_\infty)$. The index satisfies μ' satisfies condition (16a) because of the symplectic invariance (12) of the signature, it also satisfies condition (16b) because $s\ell \cap s\ell' = 0$ is equivalent to $\ell \cap \ell' = 0$, hence $\mu' = \mu$.

Let us finally mention the following dimensional additivity property of the Leray index: Let μ' and μ'' be the indices on $\mathrm{Lag}_\infty(n')$ and $\mathrm{Lag}_\infty(n'')$. Identifying $\mathrm{Lag}_\infty(n') \oplus \mathrm{Lag}_\infty(n'')$ with a submanifold of $\mathrm{Lag}_\infty(n)$, $n = n' + n''$, we have $\mu = \mu' \oplus \mu''$, that is:

$$\mu(\ell'_{1,\infty} \oplus \ell''_{1,\infty,}, \ell'_{2,\infty} \oplus \ell''_{2,\infty,}) = \mu'(\ell'_{1,\infty}, \ell'_{2,\infty,}) + \mu''(\ell''_{1,\infty}, \ell''_{2,\infty}). \tag{21}$$

This property readily follows from the dimensional additivity property (14) of the signature τ and definitions (15) and (8) of μ (that Leray's original index m is additive immediately follows from Souriau's formula (4), identifying $W(n', \mathbb{C}) \oplus W(n'', \mathbb{C})$ with a submanifold of $W(n, \mathbb{C})$ in the obvious way).

3 Maslov indices for Lagrangian paths

3.1 Axiomatic definition

For $0 \leq k \leq n$ the set

$$\mathrm{Lag}_\ell(n;k) = \{\ell' \in \mathrm{Lag}(n) : \dim(\ell \cap \ell') = k\}$$

is the *stratum of* $\mathrm{Lag}(n)$ of *order* k *with respect to* ℓ. The $\mathrm{Lag}_\ell(n;k)$ are connected submanifolds of $\mathrm{Lag}(n)$, of codimension $k(k+1)/2$ (see for instance Trèves [31]).

Let $[a,b]$ be an arbitrary compact interval and $\mathcal{C}(\mathrm{Lag}(n))$ the set of all continuous mappings $\Lambda : [a,b] \longrightarrow \mathrm{Lag}(n)$. We will write Λ_{ab} when we want to emphasize that Λ is defined on $[a,b]$, and set $\Lambda(a) = \ell_a$, $\Lambda(b) = \ell_b$

A "Maslov index" on $\mathrm{Lag}(n)$ is a mapping

$$\mathrm{Mas} : \mathcal{C}(\mathrm{Lag}(n)) \times \mathrm{Lag}(n) \ni (\Lambda, \ell) \longmapsto \mathrm{Mas}(\Lambda; \ell) \in \tfrac{1}{2}\mathbb{Z}$$

having the following four properties:

($\mathbf{L}_1$) Homotopy invariance: *If the paths* Λ and Λ' *in* $\mathrm{Lag}(n)$ *have same endpoints, then* $\mathrm{Mas}(\Lambda;\ell) = \mathrm{Mas}(\Lambda';\ell)$ *if and only if* Λ *and* Λ' *are homotopic with fixed endpoints;*

($\mathbf{L}_2$) Additivity: *If* Λ_{ab} *and* Λ'_{bc} *are two consecutive paths, the concatenation* $\Lambda''_{ac} = \Lambda_{ab} * \Lambda'_{bc}$ *satisfies*

$$\mathrm{Mas}(\Lambda''_{ac}, \ell) = \mathrm{Mas}(\Lambda_{ab}, \ell) + \mathrm{Mas}(\Lambda'_{bc}, \ell)$$

for all $\ell \in \mathrm{Lag}(n)$*;*

($\mathbf{L}_3$) Zero in strata: *If* $\Lambda(t) \in \mathrm{Lag}_\ell(n;k)$ *for all* t*, then* $Mas(\Lambda, \ell) = 0$;

($\mathbf{L}_4$) Restriction to loops: *If* Λ_{aa} *is a loop in* $\mathrm{Lag}(n)$*, then* $Mas(\Lambda_{aa};\ell) = m(\Lambda_{aa})$ *(the Maslov index (7) of* Λ_{aa}*).*

The two following properties are an immediate consequence of the axioms above:

($\mathbf{L}_5$) Antisymmetry: $\mathrm{Mas}(\Lambda^o_{ba}, \ell) = -\mathrm{Mas}(\Lambda_{ab}, \ell)$ *where* Λ^o_{ba} is the inverse path of Λ_{ab}: $\Lambda^o_{ba}(t) = \Lambda_{ab}(a+b-t)$ for $t \in [a,b]$;

[Follows from (L_2) and (L_4) noting that the Maslov index of a contractible loop is 0];

($\mathbf{L}_6$) Stratum homotopy: *if there exits a continuous mapping* $h : [0,1] \times [0,1] \longrightarrow \mathrm{Lag}(n)$ *such that* $h(t,0) = \Lambda(t)$*,* $h(t,1) = \Lambda'(t)$ *for* $0 \leq t \leq 1$ *and two integers* k_0, k_1 $(0 \leq k_0, k_1 \leq n)$ *such that* $h(0,s) \in \mathrm{Lag}_\ell(n;k_0)$ *and* $h(1,s) \in \mathrm{Lag}_\ell(n;k_1)$ *for* $0 \leq s \leq 1$*, then* $\mathrm{Mas}(\Lambda;\ell) = \mathrm{Mas}(\Lambda';\ell)$;

[Define paths Γ_0 and Γ_1 joining $\Lambda'(0)$ to $\Lambda(0)$ and $\Lambda(1)$ to $\Lambda'(1)$, respectively, by $\Gamma_0(s) = h(0, 1-s)$ and $\Gamma_1(s) = h(1,s)$ $(0 \leq s \leq 1)$. Then $\Lambda * \Gamma_1 * \Lambda'^o * \Gamma_0$ is homotopic to a point, and hence, in view of (L_2) and (L_4):

$$\mathrm{Mas}(\Lambda, \ell) + \mathrm{Mas}(\Gamma_1, \ell) + \mathrm{Mas}(\Lambda'^o, \ell) + \mathrm{Mas}(\Gamma_0, \ell) = 0.$$

But, in view of (L_3) we have $\mathrm{Mas}(\Gamma_1, \ell) = \mathrm{Mas}(\Gamma_0, \ell) = 0$, hence $\mathrm{Mas}(\Lambda, \ell) + \mathrm{Mas}(\Lambda'^o, \ell) = 0$ so that $\mathrm{Mas}(\Lambda, \ell) = \mathrm{Mas}(\Lambda', \ell)$ using (L_3)].

3.2 Existence and uniqueness up to a coboundary

Let us state and prove the main result of this section:

Theorem 5 *(i) For $\Lambda_{ab} \in \mathcal{C}(\mathrm{Lag}(n))$ set $\Lambda(a) = \ell_a$ and $\Lambda(b) = \ell_b$. Let $\ell_{a,\infty} \in \mathrm{Lag}_\infty(n)$ be the homotopy class of an arbitrary path $\Lambda_{(a)}$ joining the base point ℓ_0 of $\mathrm{Lag}_\infty(n)$ to ℓ_a and $\ell_{b,\infty} \in \mathrm{Lag}_\infty(n)$ be the homotopy class of the concatenation $\Lambda_{(a)} * \Lambda_{ab}$ (thus $\pi^{\mathrm{Lag}}(\ell_{a,\infty}) = \ell_a$ and $\pi^{\mathrm{Lag}}(\ell_{b,\infty}) = \ell_b$). Let $\ell_\infty \in \mathrm{Lag}_\infty(n)$, $\pi^{\mathrm{Lag}}(\ell_\infty) = \ell$. The formula*

$$Mas_L(\Lambda_{ab};\ell) = \tfrac{1}{2}(\mu(\ell_{b,\infty},\ell_\infty) - \mu(\ell_{a,\infty},\ell_\infty)) \tag{22}$$

defines a Maslov index with respect to ℓ. (ii) Mas_L has the following property: let $\Lambda' \in \mathcal{C}(\mathrm{Lag}(n'))$ and $\Lambda'' \in \mathcal{C}(\mathrm{Lag}(n''))$ and identify $\Lambda' \oplus \Lambda''$ with an element of $\mathcal{C}(\mathrm{Lag}(n))$ with $n = n' + n''$. Then

$$Mas_L(\Lambda' \oplus \Lambda''; \ell' \oplus \ell'') = Mas'_L(\Lambda';\ell') + Mas''_L(\Lambda'';\ell''). \tag{23}$$

(iii) Let Mas be an arbitrary Maslov index on $\mathrm{Lag}(n)$; there exists a mapping $f : \{0,1,...,n\} \longrightarrow \frac{1}{2}\mathbb{Z}$ (only depending on Mas) such that

$$Mas(\Lambda;\ell) = Mas_L(\Lambda;\ell) + f(\dim(\ell_b \cap \ell)) - f(\dim(\ell_a \cap \ell)). \tag{24}$$

Proof. (i) We first note that the left-hand side of (22) does not depend on the choice of $\ell_{a,\infty}$ and ℓ_∞: if $\ell'_{a,\infty}$ and ℓ'_∞ correspond to other choices of paths, then there exist integers γ and γ' in $\pi_1[\mathrm{Lag}(n)]$ such that $\ell'_{a,\infty} = \gamma\ell_{a,\infty}$ and $\ell'_\infty = \gamma'\ell_\infty$. Of course we also have $\ell'_{b,\infty} = \gamma'\ell_{b,\infty}$ hence, using property (5) of the Leray index,

$$\mu(\ell_{b,\infty},\ell_\infty) - \mu(\ell_{a,\infty},\ell_\infty) = \mu(\ell'_{b,\infty},\ell'_\infty) - \mu(\ell'_{a,\infty},\ell'_\infty).$$

Let us now show that $\mathrm{Mas_L}$ satisfies the axioms (L_1)–(L_4) defining a Maslov index. If Λ_{ab} and Λ'_{ab} are homotopic with fixed end points then $\ell'_{b,\infty} = \ell_{b,\infty}$ where $\ell'_{b,\infty}$ is defined as $\ell_{b,\infty}$, replacing Λ_{ab} by Λ'_{ab}, hence

$$\mathrm{Mas_L}(\Lambda_{ab};\ell) = \mathrm{Mas_L}(\Lambda'_{ab};\ell).$$

Suppose conversely that two paths Λ_{ab} and Λ'_{ab} have same endpoints, and that $\mathrm{Mas_L}(\Lambda_{ab};\ell) = \mathrm{Mas_L}(\Lambda'_{ab};\ell)$. The concatenations $\Lambda_{(a)} * \Lambda_{ab}$ and $\Lambda_{(a)} * \Lambda'_{ab}$ have the same endpoints and we can therefore find $\gamma \in \pi_1[\mathrm{Lag}(n)]$ such that $\ell_{b,\infty} = \gamma\ell'_{b,\infty}$ where $\ell_{b,\infty}$ and $\ell'_{b,\infty}$ are the homotopy classes of $\Lambda_{(a)} * \Lambda_{ab}$ and $\Lambda_{(a)} * \Lambda'_{ab}$. In view of formula (17) in Proposition 4 we have $\mu(\ell'_{b,\infty},\ell_\infty) = \mu(\ell_{b,\infty},\ell_\infty) + 2m(\gamma)$; since $\mathrm{Mas_L}(\Lambda_{ab};\ell) = \mathrm{Mas_L}(\Lambda'_{ab};\ell)$ we must thus have $m(\gamma) = 0$ hence γ is homotopic to a point; it follows that $\ell_{b,\infty} = \ell'_{b,\infty}$ so that $\Lambda_{(a)} * \Lambda_{ab}$ and $\Lambda_{(a)} * \Lambda'_{ab}$ are homotopic, and Λ_{ab} and Λ'_{ab} are therefore also homotopic. We have proven that (L_1) holds. That property (L_2) is satisfied by $\mathrm{Mas_L}$ is obvious. Assume now that $\Lambda(t) \cap \ell = 0$ for $a \leq t \leq b$. Then $\mu(\ell_{b,\infty},\ell_\infty) = \mu(\ell_{a,\infty},\ell_\infty)$ in view of the topological property (16b) of μ, hence property (L_3). That (L_4) is satisfied by $\mathrm{Mas}_{\mathrm{L}}$ immediately follows from formula (17). (ii) Formula (23) immediately follows from formula (22) using the additivity property (21) of the Leray index μ. (iii) In view of (L_1) and (L_4) the difference $\mathrm{Mas}(\Lambda;\ell)-\mathrm{Mas}_{\mathrm{L}}(\Lambda;\ell)$ only depends on the triple (ℓ,ℓ_a,ℓ_b) let us denote it by $\delta_\ell(\ell_a,\ell_b)$. We claim that δ_ℓ is an antisymmetric cocycle: $\delta_\ell(\ell_a,\ell_b)=-\delta_\ell(\ell_b,\ell_a)$ and $\partial\delta_\ell=0$. The antisymmetry is clear by (L_5). To prove that $\partial\delta_\ell=0$, let Λ_{ab}, Λ_{bc}, and Λ_{ca} be three paths joining ℓ_a to ℓ_b, ℓ_b to ℓ_c, and ℓ_c to ℓ_a, respectively. In view of (L_1) and (L_4) we have

$$\begin{aligned}\mathrm{Mas}(\Lambda_{ab};\ell)-\mathrm{Mas}(\Lambda_{ac};\ell)+\mathrm{Mas}(\Lambda_{bc};\ell)&=m(\gamma)\\ \mathrm{Mas}_{\mathrm{L}}(\Lambda_{ab};\ell)-\mathrm{Mas}_{\mathrm{L}}(\Lambda_{ac};\ell)+\mathrm{Mas}_{\mathrm{L}}(\Lambda_{bc};\ell)&=m(\gamma)\end{aligned}$$

where γ is the loop $\Lambda_{ab}*\Lambda_{bc}*\Lambda_{ca}$ and $m(\gamma)$ its Maslov index. This proves that $\partial\delta_\ell=0$. It follows that

$$\mathrm{Mas}(\Lambda;\ell)-\mathrm{Mas}_{\mathrm{L}}(\Lambda;\ell)=\delta_\ell(\ell_a,\ell)-\delta_\ell(\ell_b,\ell).$$

In view of Axiom (L_3) the function $\ell_a\longmapsto\delta_\ell(\ell_a,\ell)$ is locally constant on each stratum; formula (24) follows. ∎

The following result describes the effect of a change of Lagrangian plane ℓ in the Maslov index; it will be useful for the study of the Hörmander index in §3.3.2:

Proposition 6 *For all ℓ, ℓ' in* $\mathrm{Lag}(n)$ *we have*

$$\mathrm{Mas}(\Lambda_{ab};\ell)-\mathrm{Mas}(\Lambda_{ab};\ell')=\tau(\ell_b,\ell,\ell')-\tau(\ell_a,\ell,\ell'). \tag{25}$$

Proof. In view of formulas (22) and (24) in Theorem 5 we have

$$\begin{aligned}\mathrm{Mas}(\Lambda_{ab};\ell)-\mathrm{Mas}(\Lambda_{ab};\ell')=\ &\mu(\ell_{b,\infty},\ell_\infty)-\mu(\ell_{b,\infty},\ell'_\infty)\\ &-(\mu(\ell_{a,\infty},\ell_\infty)-\mu(\ell_{a,\infty},\ell'_\infty));\end{aligned}$$

in view of property (16a) of μ we have

$$\begin{aligned}\mu(\ell_{b,\infty},\ell_\infty)-\mu(\ell_{b,\infty},\ell'_\infty)&=-\mu(\ell_\infty,\ell'_\infty)+\tau(\ell_b,\ell,\ell')\\ \mu(\ell_{a,\infty},\ell_\infty)-\mu(\ell_{a,\infty},\ell'_\infty)&=-\mu(\ell_\infty,\ell'_\infty)+\tau(\ell_a,\ell,\ell')\end{aligned}$$

hence (25). ∎

Corollary 7 *Let Λ_{ab}, Λ_{bc}, and Λ_{ca} be paths in* $\mathrm{Lag}(n)$ *joining ℓ_a to ℓ_b, ℓ_b to ℓ_c, and ℓ_c to ℓ_a, respectively. The following "triangle equality":*

$$\mathrm{Mas}(\Lambda_{ab};\ell_c)+\mathrm{Mas}(\Lambda_{bc};\ell_a)+\mathrm{Mas}(\Lambda_{ca};\ell_b)=\tau(\ell_a,\ell_b,\ell_c) \tag{26}$$

holds for every Maslov index Mas *on* $\mathrm{Lag}(n)$*.*

Proof. This is an immediate consequence of (24) and property (9) of μ. ∎

Remark 8 *Formula (26) can be used to define a signature in infinitely dimensional symplectic spaces, as soon as a Maslov index (with adequate properties) is known.*

3.3 The Robbin–Salamon and Hörmander indices

3.3.1 The Robbin–Salamon index

In [28] Robbin and Salamon have constructed, using differentiability properties, a mapping $\mathrm{Mas}_{\mathrm{RS}} : \mathcal{C}(\mathrm{Lag}(n)) \times \ell \longrightarrow \frac{1}{2}\mathbb{Z}$ which they call "Maslov index". In addition to (L_1)–(L_4) that index satisfies the following property:

(L_7) Spectral flow formula: *Let the path $\Lambda_M : [a,b] \longrightarrow \mathrm{Lag}(n)$ be defined by $\Lambda_M(t) = \{(x, M(t)x) : x \in X\}$ where $M(t)$ is a symmetric linear automorphism of Z depending continuously on $t \in [a,b]$. Then*

$$\mathrm{Mas}_{\mathrm{RS}}(\Lambda_M, X) = \tfrac{1}{2}(\operatorname{sign} M(b) - \operatorname{sign} M(a)). \tag{27}$$

This condition identifies $\mathrm{Mas}_{\mathrm{RS}}$ with $\mathrm{Mas}_{\mathrm{L}}$:

Proposition 9 Mas_L *is the only Maslov index on* $\mathrm{Lag}(n)$ *satisfying* (L_7)*; hence* $\mathrm{Mas}_{RS} = \mathrm{Mas}_L$.

Proof. (See de Gosson [14] for an alternative proof). In view of formula (24) in Theorem 5 there exists f such that

$$\mathrm{Mas}_{\mathrm{RS}}(\Lambda_M; \ell) = \mathrm{Mas}_{\mathrm{L}}(\Lambda_M; \ell) + f(\dim(\ell_b \cap \ell)) - f(\dim(\ell_a \cap \ell)).$$

Set $\Lambda_M(a) = \ell_a$, $\Lambda_M(b) = \ell_b$. Since $\Lambda_M(t) \cap X^* = 0$ for $a \leq t \leq b$) we have $\mu(\ell_{b,\infty}, X^*_\infty)) = \mu(\ell_{a,\infty}, X^*_\infty)$ in view of property (16b) of μ, and hence $\mathrm{Mas}(\Lambda_M; X^*) = 0$ in view of (L_6) for every Maslov index Mas. Choosing in particular Mas= $\mathrm{Mas}_{\mathrm{L}}$ we have, in view of property (16a) of μ,

$$\begin{aligned}
\mu(\ell_{a,\infty}, X_\infty) &= \mu(\ell_{a,\infty}, X^*_\infty) - \mu(X_\infty, X^*_\infty) + \tau(\ell_a, X, X^*) \\
\mu(\ell_{b,\infty}, X_\infty) &= \mu(\ell_{b,\infty}, X^*_\infty) - \mu(X_\infty, X^*_\infty) + \tau(\ell_b, X, X^*)
\end{aligned}$$

hence, by subtraction,

$$\begin{aligned}
\mathrm{Mas}_{\mathrm{L}}(\Lambda_M, X) &= \tfrac{1}{2}(\tau(\ell_b, X, X^*) - \tau(\ell_a, X, X^*)) \\
&= \tfrac{1}{2}(\operatorname{sign} M(b) - \operatorname{sign} M(a))
\end{aligned}$$

where the second equality follows from the antisymmetry of τ and formula (13); $\mathrm{Mas}_{\mathrm{L}}$ thus satisfies (L_7), as claimed. Assume that Mas is another Maslov index satisfying satisfying (L_7). Then $\Delta = \mathrm{Mas} - \mathrm{Mas}_{\mathrm{L}}$ satisfies

$$\Delta(\Lambda_M, X) = f(\dim(\ell_b \cap X)) - f(\dim(\ell_a \cap X)) = 0 \tag{28}$$

for some function $f : \{0, 1, ..., n\} \longrightarrow \frac{1}{2}\mathbb{Z}$ only depending on Mas. Since $\dim(\Lambda_M(t) \cap X) = n - \operatorname{rank} M(t)$ can take any prescribed value in $\{0, 1, ..., n\}$ by choosing adequately $M(t)$ it follows that $\dim(\ell_a \cap X)$ and $\dim(\ell_b \cap X)$ can take arbitrary values in $\{0, 1, ..., n\}$ hence we must have $f = 0$. ■

3.3.2 The Hörmander index

In his study of pseudo-differential operators, Hörmander introduces in [21] a mapping

$$\mathrm{Hor} : \mathrm{Lag}(n)^4 \ni (\ell_1, \ell_2, \ell_3, \ell_4) \longrightarrow \mathrm{Hor}(\ell_1, \ell_2, \ell_3, \ell_4) \in \tfrac{1}{2}\mathbb{Z}$$

(this index is also discussed in Duistermaat [8]). Robbin and Salamon [28] show that the Hörmander index is related to their index $\mathrm{Mas}_{\mathrm{RS}}$ by the formula

$$\mathrm{Hor}(\ell_1, \ell_2, \ell_3, \ell_4) = \mathrm{Mas}_{\mathrm{RS}}(\Lambda_{34}, \ell_2) - \mathrm{Mas}_{\mathrm{RS}}(\Lambda_{34}, \ell_1) \tag{29}$$

where Λ_{34} is an arbitrary path in $\mathrm{Lag}(n)$ joining ℓ_3 to ℓ_4. In particular Hor is a symplectic invariant:

$$\mathrm{Hor}(s\ell_1, s\ell_2, s\ell_3, s\ell_4) = \mathrm{Hor}(\ell_1, \ell_2, \ell_3, \ell_4)$$

for every $s \in \mathrm{Sp}(n)$.

Proposition 10 *The Hörmander index* Hor *is given by*

$$\mathrm{Hor}(\ell_1, \ell_2, \ell_3, \ell_4) = \tfrac{1}{2}(\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_1, \ell_2, \ell_4)); \tag{30}$$

in particular it does not depend on the choice of the path Λ_{34}.

Proof. In view of formula (25) we can rewrite (29) as

$$\mathrm{Hor}(\ell_1, \ell_2, \ell_3, \ell_4) = \tfrac{1}{2}(\tau(\ell_4, \ell_2, \ell_1) - \tau(\ell_3, \ell_2, \ell_1))$$

which is (30) in view of the antisymmetry of the signature τ. ■

Remark 11 *Formula (30) generalizes formula (3) of Theorem 3.5 in Robbin and Salamon [28] to the non-transversal case: it makes sense for all* ℓ_j, $j \in \{1, 2, 3, 4\}$.

4 Symplectic Paths

The intersection theory for symplectic paths is very similar to that developed above for Lagrangian paths.

4.1 The index μ_ℓ

We denote by $\mathcal{C}_I(\mathrm{Sp}(n))$ the set of all continuous paths $[0,1] \longrightarrow \mathrm{Sp}(n)$ starting from the identity I in $\mathrm{Sp}(n)$. We will write $\Sigma \sim \Sigma'$ when $\Sigma, \Sigma' \in \mathcal{C}_I(\mathrm{Sp}(n))$ are homotopic with fixed endpoint. Denoting by $\pi^{\mathrm{Sp}} : \mathrm{Sp}_\infty(n) \longrightarrow \mathrm{Sp}(n)$ the universal covering of $\mathrm{Sp}(n)$ we have the identification $\mathrm{Sp}_\infty(n) = \mathcal{C}_I(\mathrm{Sp}(n))/\sim$. If $s = \pi^{\mathrm{Sp}}(s_\infty)$, $s_\infty \in \mathrm{Sp}_\infty(n)$, we will say that s_∞ covers s.

Let $(\Sigma, \ell) \in \mathcal{C}_I(\mathrm{Sp}(n)) \times \mathrm{Lag}(n)$ we define

$$\mu_\ell(\Sigma, \ell) = \mu(\Sigma\Lambda, \Lambda) \tag{31}$$

where Λ is an arbitrary element of $\mathcal{C}_{\ell_0}(\mathrm{Lag}(n))$ joining the base point ℓ_0 to ℓ. Equivalently, μ_ℓ can be viewed as the mapping $\mathrm{Sp}_\infty(n) \longrightarrow \mathbb{Z}$ defined, for $(s_\infty, \ell) \in \mathrm{Sp}_\infty(n) \times \mathrm{Lag}(n)$, by

$$\mu_\ell(s_\infty) = \mu(s_\infty \ell_\infty, \ell_\infty) \tag{32}$$

where ℓ_∞ covers ℓ. The notation μ_ℓ is motivated by following observations: assume that $\ell'_\infty \in (\pi^{\mathrm{Lag}})^{-1}(\ell)$, then there exists $k \in \mathbb{Z}$ such that $\ell'_\infty = \beta^k \ell_\infty$ and hence, taking (19) and formula (17) in Proposition 4(iii) into account, $\mu(s_\infty \ell'_\infty, \ell'_\infty) = \mu(s_\infty \ell_\infty, \ell_\infty)$. We will call μ_ℓ the *Leray index on* $\mathrm{Sp}_\infty(n)$ *relatively to* ℓ. Setting $\tau_\ell(s,s') = \tau(\ell, s\ell, ss'\ell)$ the index μ_ℓ is the *only* mapping $\mathrm{Sp}_\infty(n) \longrightarrow \mathbb{Z}$ satisfying the two following properties:

$$\mu_\ell(s_\infty s'_\infty) = \mu_\ell(s_\infty) + \mu_\ell(s'_\infty) + \tau_\ell(s,s') \tag{33a}$$

$$\mu_\ell \textit{ is is locally constant on } \{s_\infty : s\ell \cap \ell = 0\} \tag{33b}$$

(these properties immediately follow from the properties (16a), (16b) of μ; for the uniqueness see de Gosson [13]).

Assume that s and s' are such that

$$sX^* \cap X^* = s'X^* \cap X^* = 0 \tag{34}$$

and identify s and s' with their matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $\begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix}$ in the canonical symplectic basis of $(X \oplus X^*, \sigma)$ this condition is equivalent to $\det B \neq 0$ and $\det B' \neq 0$. We have shown in [15] (also see de Gosson [17], p. 216) that

$$\tau_{X^*}(s,s') = \mathrm{sign}(B^{-1}A + D'(B')^{-1});$$

note that $B^{-1}A$ and $D'(B')^{-1}$ are symmetric because s and s' are symmetric. Performing explicitly the matrix multiplication ss' one sees that $B^{-1}A + D'(B')^{-1} = B^{-1}B''(B')^{-1}$ hence the formula above can be written

$$\tau_{X^*}(s,s') = \mathrm{sign}(B^{-1}B''(B')^{-1}). \tag{35}$$

Remark 12 *In [28] Robbin and Salamon introduce a quadratic form they denote by $Q(s,s')$, and call it "composition form". In [15] we proved, using formula (13) that if condition (34) holds then $Q(s,s') = \tau_{X^*}(s,s')$; notice that $\tau_{X^*}(s,s')$ is however defined for arbitrary s, s' in* $\mathrm{Sp}(n)$.

4.2 Symplectic intersection indices

For $\ell \in \mathrm{Lag}(n)$ and $0 \leq k \leq n$ we set

$$\mathrm{Sp}_\ell(n;k) = \{s \in \mathrm{Sp}(n) : \dim(s\ell \cap \ell) = k\}$$

($\mathrm{Sp}_{X^*}(n;k)$ is the preimage of $\mathrm{Lag}_\ell(n;k)$ under the fibration $\mathrm{Sp}(n)/\mathrm{St}_{X^*}(n) = \mathrm{Lag}(n)$). $\mathrm{Sp}_\ell(n;k)$ is a submanifold of $\mathrm{Sp}(n)$ with codimension $k(k+1)/2$.

Let us denote by $\mathcal{C}(\mathrm{Sp}(n))$ the set of all continuous mappings $\Sigma : [a, b] \longrightarrow \mathrm{Sp}(n)$. By definition, the *symplectic Maslov index* on $\mathrm{Sp}(n)$ associated to a Maslov index Mas is the mapping

$$\mathrm{Symp} : \mathcal{C}(\mathrm{Sp}(n)) \times \mathrm{Lag}(n) \longmapsto \tfrac{1}{2}\mathbb{Z}$$

defined by

$$\mathrm{Symp}(\Sigma; \ell) = \mathrm{Mas}(\Sigma\ell; \ell)$$

where $\Sigma\ell$ is the path in $\mathrm{Lag}(n)$ defined by $\Sigma\ell(t) = \Sigma(t)\ell$. The properties of the index Symp immediately follow from the properties (L_1)–(L_6) of Mas:

$(\mathbf{S}_1)$ **Homotopy invariance:** *If the paths* Σ and Σ' *have the same endpoints, then* $\mathrm{Symp_L}(\Sigma; \ell) = \mathrm{Symp_L}(\Sigma'; \ell)$ *if and only if* Σ *and* Σ' *are homotopic with fixed endpoints;*

$(\mathbf{S}_2)$ **Additivity:** *If* Σ and Σ' *are two consecutive paths, then for all* $\ell \in \mathrm{Lag}(n)$*:*

$$\mathrm{Symp}(\Sigma * \Sigma', \ell) = \mathrm{Symp}(\Sigma, \ell) + \mathrm{Symp}(\Sigma', \ell)$$

$(\mathbf{S}_3)$ **Zero in strata:** *If* $\Sigma(t) \in \mathrm{Sp}_\ell(n; k)$ *for all* t*, then* $\mathrm{Symp_L}(\Sigma, \ell) = 0$;

$(\mathbf{S}_4)$ **Restriction to loops:** *If* $\Sigma \in \mathcal{C}(\mathrm{Sp}(n))$ *is a loop, then* $\mathrm{Symp}(\Sigma; \ell) = m(\Sigma)$ *(the Maslov index of* Σ*).*

$(\mathbf{S}_5)$ **Antisymmetry:** $\mathrm{Symp}(\Sigma^o, \ell) = -\mathrm{Symp}(\Sigma, \ell)$ *where* $\Sigma^o(t) = \Sigma(a+b-t)$ *if* Σ *is defined on* $[a, b]$

$(\mathbf{S}_6)$ **Stratum homotopy:** *if there exits a continuous mapping* $h : [0, 1] \times [0, 1] \longrightarrow \mathrm{Sp}(n)$ *such that* $h(t, 0) = \Sigma(t)$, $h(t, 1) = \Sigma'(t)$ *for* $0 \leq t \leq 1$ *and two integers* k_0, k_1 $(0 \leq k_0, k_1 \leq n)$ *such that* $h(0, s) \in \mathrm{Sp}_\ell(n; k_0)$ *and* $h(1, s) \in \mathrm{Sp}_\ell(n; k_1)$ *for* $0 \leq s \leq 1$*, then* $\mathrm{Symp}(\Sigma; \ell) = \mathrm{Symp}(\Sigma'; \ell)$.

Suppose that Mas is the Maslov index Mas_L defined by formula (22) in Theorem 5; let us denote the corresponding symplectic Maslov index by $\mathrm{Symp_L}$. We have

$$\mathrm{Symp_L}(\Sigma; \ell) = \tfrac{1}{2}(\mu_\ell(s_{b,\infty}) - \mu_\ell(s_{a,\infty})) \tag{36}$$

where $s_{a,\infty}$ and $s_{b,\infty}$ are defined as follows (cf. Theorem 5(i)): let $s_a = \Sigma(a)$, $s_b = \Sigma(b)$. Then $s_{a,\infty}$ is the homotopy class of an arbitrary path Σ_{0a} in $\mathrm{Sp}(n)$ joining the base point of $\mathrm{Sp}_\infty(n)$ to s_a, and $s_{b,\infty}$ is that of the concatenation $\Sigma_{0a} * \Sigma$.

The properties (S_1)–(S_6) listed above do not characterize uniquely Symp. However:

Proposition 13 *Define* $\Sigma_{ab} \in \mathcal{C}(\mathrm{Sp}(n))$ *by* $\Sigma_{ab}(t)(x, p) = (x, M(t)x)$ *where* $M(t)$ *is a symmetric endomorphism of* $\mathbb{R}^n$*. Then*

$$\mathrm{Symp_L}(\Sigma; X) = \tfrac{1}{2}(\mathrm{sign}\, M(a) - \mathrm{sign}\, M(b)) \tag{37}$$

and $\mathrm{Symp_L}$ *is the only symplectic Maslov index having this property.*

Proof. Formula (37) is just a restatement of property (27) of $\mathrm{Mas_L} = \mathrm{Mas_{RS}}$. ■

5 The Conley–Zehnder index

The Conley–Zehnder is an index of symplectic paths generalizing the usual Morse index for closed geodesics on Riemannian manifolds. It arises from trivializing a symplectic vector bundle over a periodic orbit of a Hamiltonian vector field on a symplectic manifold (or the Reeb vector field on a contact manifold). The Conley–Zehnder was originally designed to compute the spectral flow of the Cauchy–Riemann-type operators arising in Floer homology (Salamon and Zehnder [29]). It plays a crucial role in the study of periodic orbits in Hamiltonian systems (Long [25], Long and Zhu [26]) and in their applications to semiclassical mechanics via "Gutzwiller's formula" and its variants (see de Gosson [18] and the references therein).

5.1 Definition, uniqueness, and existence

The subsets of $\mathrm{Sp}(n)$ defined by

$$\begin{aligned}
\mathrm{Sp}^{+}(n) &= \{s \in \mathrm{Sp}(n) : \det(s - I) > 0\} \\
\mathrm{Sp}^{-}(n) &= \{s \in \mathrm{Sp}(n) : \det(s - I) < 0\} \\
\mathrm{Sp}^{0}(n) &= \{s \in \mathrm{Sp}(n) : \det(s - I) = 0\}
\end{aligned}$$

partition $\mathrm{Sp}(n)$; moreover $\mathrm{Sp}^{+}(n)$ and $\mathrm{Sp}^{-}(n)$ are connected (see Conley–Zehnder [6]). Let

$$\mathrm{Sp}^{*}(n) = \mathrm{Sp}^{+}(n) \cup \mathrm{Sp}^{-}(n) = \mathrm{Sp}(n) \setminus \mathrm{Sp}^{0}(n).$$

The Conley–Zehnder index is the unique mapping i_{CZ} associating to every path $\Sigma : [0, b] \longrightarrow \mathrm{Sp}(n)$ such that $\Sigma(0) = I$ and $\Sigma(b) \in \mathrm{Sp}^{*}(n)$ an integer, and having the three following properties:

($\mathbf{CZ}_1$) Antisymmetry: *We have* $i_{\mathrm{CZ}}(\Sigma^{-1}) = -i_{\mathrm{CZ}}(\Sigma)$ (where $\Sigma^{-1}(t) = (\Sigma(t))^{-1}$ *for* $t \in [0, b]$);

($\mathbf{CZ}_2$) Homotopy invariance: $i_{\mathrm{CZ}}(\Sigma)$ *does not change when* Σ *is continuously deformed in such a way that its endpoint stays in* $\mathrm{Sp}^{+}(n)$ (or $\mathrm{Sp}^{-}(n)$);

($\mathbf{CZ}_3$) Action of $\pi_1[\mathrm{Sp}(n)]$: *We have* $i_{\mathrm{CZ}}(\alpha * \Sigma) = i_{\mathrm{CZ}}(\Sigma) + 2$.

Before we show the existence and uniqueness of the Conley–Zehnder index, let us remark that the homotopy invariance property (CZ_2) implies, in particular, that $i_{\mathrm{CZ}}(\Sigma) = i_{\mathrm{CZ}}(\Sigma')$ if the symplectic paths Σ and Σ' are homotopic with fixed endpoints. The integer $i_{\mathrm{CZ}}(\Sigma)$ thus only depends on the homotopy class $s_\infty \in \mathrm{Sp}_\infty(n)$ of Σ. We can thus view the Conley–Zehnder index as a mapping $i_{\mathrm{CZ}} : \mathrm{Sp}^*_\infty(n) \longrightarrow \mathbb{Z}$ where $\mathrm{Sp}^*_\infty(n) = \pi^{-1}(\mathrm{Sp}^*(n))$. We will therefore write indifferently $i_{\mathrm{CZ}}(\Sigma)$ or $i_{\mathrm{CZ}}(s_\infty)$.

Let us equip the vector space $Z \oplus Z$ with the symplectic form $\omega^\ominus = \omega \oplus (-\omega)$. We denote by $\mathrm{Sp}^\ominus(2n)$ and $\mathrm{Lag}^\ominus(2n)$ the corresponding symplectic group and Lagrangian Grassmannian, and by $\mu^\ominus$ (resp. $\mathrm{Mas}^\oplus_{\mathrm{L}}$) the Leray (resp. Maslov) index on $\mathrm{Lag}^\ominus_\infty(2n)$; the corresponding Leray index on $\mathrm{Sp}^\ominus_\infty(2n)$ relative to $\Delta \in \mathrm{Lag}^\ominus(2n)$ (*cf.* the notation (32)) is $\mu^\ominus_\Delta$.

Proposition 14 *The Conley–Zehnder index exists, is unique, and is given by the formula*

$$i_{CZ}(\Sigma) = \mathrm{Mas}^\oplus_L(\Sigma^\ominus \Delta; \Delta) \tag{38}$$

where $\Sigma^\ominus = I \oplus \Sigma$ *and* $\Delta = \{(z,z) : z \in \mathbb{R}^{2n}\}$*; equivalently*

$$i_{CZ}(\Sigma) = \tfrac{1}{2}\mu^\ominus((I \oplus s_1)_\infty \Delta_\infty, \Delta_\infty) \tag{39}$$

where $(I \oplus s)_\infty$ *is the homotopy class in* $\mathrm{Sp}^\ominus(2n)$ *of the path* $I \oplus \Sigma \in \mathcal{C}(\mathrm{Sp}^\ominus(2n))$*, that is*

$$i_{CZ}(\Sigma) = \tfrac{1}{2}\mu^\ominus_\Delta((I \oplus s_1)_\infty). \tag{40}$$

Proof. The equivalence between the definitions (38), (39), (40) is obvious. Let us first prove the uniqueness statement. Let δ_{CZ} be the difference between two such indices. In view of (CZ_3) we have $\delta_{\mathrm{CZ}}(\alpha^k * \Sigma) = \delta_{\mathrm{CZ}}(\Sigma)$ for all $k \in \mathbb{Z}$ hence $\delta_{\mathrm{CZ}}(\Sigma)$ only depends on the endpoint s of Σ; δ_{CZ} is thus a function $\delta_{\mathrm{CZ}} : \mathrm{Sp}^*(n) \longrightarrow \mathbb{Z}$. Property ($\mathrm{CZ}_2$) then implies that δ_{CZ} is constant on both $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$. Since $\det(s^{-1} - I) = \det(s - I)$ the automorphisms s and s^{-1} always belong to the same set $\mathrm{Sp}^+(n)$ or $\mathrm{Sp}^-(n)$ if $\det(s - I) \neq 0$, property (CZ_1) implies that f must be zero on $\mathrm{Sp}^*(n)$. Let us prove that formula (38) indeed defines a Conley–Zehnder index. That (CZ_1) is satisfied follows at once from the equality $(s^\ominus_\infty)^{-1} = (I \oplus s^{-1})_\infty$ and the antisymmetry of $\mu^\ominus_\Delta$. To check property (CZ_2) it suffices to observe that to the generator α of $\pi_1[\mathrm{Sp}(n)]$ corresponds the generator $I_\infty \oplus \alpha$ of $\pi_1[\mathrm{Sp}^\ominus(2n)]$ (I_∞ the constant path through $I \in \mathrm{Sp}(n)$), and then to apply formula 17 in Proposition 4. Let us finally prove that (CZ_3) holds as well. Assume that s and s' belong to, say, $\mathrm{Sp}^+(n)$. Let Σ be a path joining I to s in $\mathrm{Sp}^+(n)$, and Σ' a path joining s to s' in $\mathrm{Sp}^+(n)$. Let $\Sigma'_{t'}$ be the restriction of Σ' to an interval $[0, t']$, $t' \leq t$ and consider the concatenation $\Sigma * \Sigma'_{t'}$. We have $\det(\Sigma(t) - I) > 0$ for all $t \in [0, t']$ hence $\Sigma(t)\Delta \cap \Delta \neq 0$ as t varies from 0 to 1. It follows from the fact that $\mu^\ominus_\Delta$ is locally constant on $\{s^\ominus_\infty : s^\ominus \Delta \cap \Delta = 0\}$ that the function $t \longmapsto \mu^\ominus_\Delta(s^\ominus_\infty(t))$ is constant, and hence

$$\mu^\ominus_\Delta(s^\ominus_\infty) = \mu^\ominus_\Delta(s^\ominus_\infty(0)) = \mu^\ominus_\Delta(s^\ominus_\infty(1)) = \mu^\ominus_\Delta(s'^\ominus_\infty)$$

which was to be proven. ∎

Formula (38) not only defines $i_{\mathrm{CZ}}(\Sigma)$ when the endpoint of Σ lies in $\mathrm{Sp}^0(n)$; it actually makes sense for arbitrary paths $\Sigma \in \mathcal{C}_I(\mathrm{Sp}(n))$.

5.2 A product formula

If $s \in \mathrm{Sp}^*(n)$ then $s - I$ is invertible and we may define

$$M_s = \frac{1}{2} J(s+I)(s-I)^{-1}. \tag{41}$$

One verifies without difficulty that M_s is a symmetric matrix; in [16] we called M_s the symplectic Cayley transform of s. (It plays an important role in determining the (covariant) Weyl symbol of metaplectic operators, as we showed in [16]). The notion has been generalized in Giambò and Girolimetti, elaborating on our joint work with Piccione [19].

We will write

$$\tau_\Delta^\ominus(s^\ominus, s'^\ominus) = \tau^\ominus(\Delta, s^\ominus\Delta, s^\ominus s'^\ominus\Delta).$$

Theorem 15 *Let $\Sigma, \Sigma' \in \mathcal{C}_I(\mathrm{Sp}(n))$ and define $\Sigma\Sigma' \in \mathcal{C}_I(\mathrm{Sp}(n))$ by $\Sigma\Sigma'(t) = \Sigma(t)\Sigma'(t)$ for $t \in [0,1]$. (i) We have*

$$i_{CZ}(\Sigma\Sigma') = i_{CZ}(\Sigma) + i_{CZ}(\Sigma') + \tfrac{1}{2}\tau_\Delta^\ominus(s^\ominus, s'^\ominus) \tag{42}$$

where $s = \Sigma(1)$ and $s' = \Sigma(1)$. (ii) If s and s' are in $\mathrm{Sp}^*(n)$ then

$$i_{CZ}(\Sigma\Sigma') = i_{CZ}(\Sigma) + i_{CZ}(\Sigma') + \tfrac{1}{2}\operatorname{sign}(M_s + M_{s'}) \tag{43}$$

where $M_s = \frac{1}{2}J(s+I)(s-I)^{-1}$.

Proof. (i) Formula (42) immediately follows from (39) applying property (33a) to $\mu_\Delta^\ominus$. Assume that the endpoints s, s' are in $\mathrm{Sp}^*(n)$. (ii) To prove (43) we need to show that when $\det(s-I) \neq 0$ and $\det(s'-I) \neq 0$ then

$$\tau^\ominus(\Delta, s^\ominus\Delta, s^\ominus s'^\ominus\Delta) = \operatorname{sign}(M_s + M_{s'}).$$

This can be done in the following way (we only sketch the argument since we gave a detailed proof ([18])). In view of (39) and the product property (33a) we have

$$\nu(s_\infty s'_\infty) = \nu(s_\infty) + \nu(s'_\infty) + \tfrac{1}{2}\tau_\Delta^\ominus(s^\ominus, s'^\ominus)$$

where $s^\ominus = I \oplus s$, $s'^\ominus = I \oplus s'$. One then uses the properties of the Wall–Kashiwara signature to calculate explicitly $\tau_\Delta^\ominus(s^\ominus, s'^\ominus)$; one finds (after some rather long calculations) that it is the signature of the quadratic form $Q(z) = \left\langle (M_s + M_{s'})^{-1}z, z\right\rangle$ on $\mathbb{R}^{2n}$ that is $\operatorname{sign}(M_s + M_{s'})$. ■

5.3 Relation with Morse's index of concavity

Assume that the endpoint s of $\Sigma \in \mathcal{C}_I(\mathrm{Sp}(n))$ satisfies the condition (34), that is $sX^* \cap X^* = 0$ and identify again s with $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The quadratic form W on $X \times X$ defined by

$$W(x, x') = \tfrac{1}{2}DB^{-1}x^2 - \left\langle B^{-1}x, x'\right\rangle + \tfrac{1}{2}B^{-1}Ax'^2$$

is called the *generating function* of s; in fact it is easy to check that the relation $(x,p) = s(x',p')$ is equivalent to $p = \partial_x W(x,x')$ and $p' = -\partial_{x'} W(x,x')$.

In [18] we proved the following result:

Theorem 16 *Assume that the endpoint* $s = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *of* $\Sigma \in \mathcal{C}_I(\mathrm{Sp}(n))$ *is such that* $\det B \neq 0$. *Then*

$$i_{CZ}(\Sigma) = \tfrac{1}{2}(\mu_{X^*}(\Sigma) + \operatorname{sign} W_{xx}) \tag{44}$$

where

$$W_{xx} = DB^{-1} - B^{-1} - (B^T)^{-1} + B^{-1}A \tag{45}$$

is the Hessian matrix of the quadratic form $x \longmapsto W(x,x)$.

The proof of formula (44) is rather lengthy, and makes repeated use of the properties of the signature cocycle τ so we do not duplicate it here.

The index of inertia $\operatorname{Inert} W_{xx}$ of the quadratic form $x \longmapsto W(x,x)$ is called *index of concavity*; it appears in Morse theory.

6 Concluding remarks

In addition to their simplicity, the constructions of the various intersection indices I have exposed have a conceptual appeal, in the sense that they do not make use of any supplementary hypothesis on the paths that are considered. In particular, there is no need to use any property of differentiability: the approach using Leray's index μ is purely combinatorial and topological. It is precisely the combinatorial property (16a) which makes it easy to use in all forms of practical calculations.

Clerc and Ørsted [3], Clerc [4], Clerc and Koufany [5] have extended the Leray index (and the associated Kashiwara signature) to Shilov boundary of Hermitian symmetric spaces of tube type. These constructions are highly nontrivial, and deserve to be studied further. For instance, is there an analogue of a Conley–Zehnder index in their context?

Professor Chaofeng Zhu (Nankai) has suggested that the methods used in this paper can be extended to the case of infinitely dimensional symplectic Hilbert spaces. We will come back to this possibility in future work; for progresses in the infinite-dimensional case see the paper [9] by Furutani.

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