BMO Functions on Compact Sets

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Abstract

Let P be an elliptic differential operator of order p with real analytic coefficients on an open set $X \subset \mathbf{R}^n$. Given a compact set $K \subset X$, we describe the closure in $BMO(K)$ of the space of solutions to $Pf = 0$ on neighborhoods of K.

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1 Introduction

Let P be an elliptic differential operator of order p on an open set X in $\mathbf{R}^n$; suppose that the coefficients of P are real analytic. Let K be a compact subset of X. We denote the space of C^∞ functions f which are solutions of the equation $Pf = 0$ in at least some neighborhood of K by $\mathrm{sol}(K)$.

If we are given a topological vector space $L(K)$ in which the C^∞ functions near K form a subspace, then $\mathrm{sol}(K)$ can be considered as a subspace of $L(K)$. The general *approximation problem* consists in describing the closure of the subspace of solutions.

Problem 1.1 *Describe the closure of* $\mathrm{sol}(K)$ *in* $L(K)$.

If $L(K) = C(K)$, the space of continuous functions on K under the supremum norm, then one speaks of "uniform approximation," and if $L(K) = L^q(K)$ for some $q < \infty$, of "approximation in the mean." The former is the more difficult of the two settings. The crucial reason for this is that the spaces $L^q(\mathbb{R}^n)$ $(1 < q < \infty)$ are locally invariant under Calderon-Zygmund operators, whereas $L^\infty(\mathbb{R}^n)$ is not.

This difference notwithstanding, it is possible to give a rather unified treatment of the approximation problem in these two settings. A theme emphasized in the paper of Gauthier and Tarkhanov [6] was the parallel between approximation in uniform norms and approximation in Sobolev norms.

It is well-known that the space of functions of bounded mean oscillation (*BMO*) is often an effective substitute for L^∞. As but one instance of this, we recall that *BMO* is locally invariant under such classical Calderon-Zygmund operators as the Hilbert and Riesz transforms. With this as our starting point, we examine in this paper in what sense approximation within *BMO* may be seen as intermediate between the approximation theories in uniform and Sobolev spaces.

On the other hand, *BMO* can be thought of as the limit as $s \to 0$ of the Lipschitz classes Λ^s. Thus it is to be expected that in the context of qualitative approximation the *BMO*-theorems should be obtained by replacing s by 0 in the Λ^s-theorems, for $0 < s < 1$. This is true for $P = \bar{\partial}$ or for $P = \Delta$ in dimension 2 (see Verdera [23]) but nothing else has been known. Our viewpoint sheds some new light on the position of the *BMO* approximation in the scale of Lipschitz approximations.

We mention yet another aspect of our interest in *BMO* approximation. One of the problems now intensively discussed in approximation theory is whether, given any compact set $K \subset \mathbb{C}^1$, each continuous function on K that is biholomorphic in the interior of K can be approximated uniformly on K by biholomorphic functions on neighborhoods of K (see Verdera [24] and references there). That this is the case for nowhere dense compacta is proved in Trent and Wang [22]. (For this reason, the above problem is referred to as *Trent and Wang's problem.*)

Note that the natural L^q version of this result fails for $2 \leq q$, owing to a clever example of Hedberg [9, p.77]. Gauthier and Tarkhanov [6] showed that Hedberg's construction also provides a counterexample to an analogous problem in $\mathbf{R}^n$ (for $n > 2$). On this basis, one might conjecture that the answer to the above uniform biholomorphic approximation problem is negative.

Mateu and Verdera [12], however, gave some suggestive evidence to the contrary. They considered the subspace *VMO* of *BMO* and showed, given any compact set $K \subset \mathbb{C}^1$, that each function in $VMO(K)$ that is harmonic in the interior of K can be approximated in the norm of $BMO(K)$ by functions harmonic near K. Hence, the formal reasoning above can be disputed because *BMO* is "between" L^q and C.

It is worth pointing out that in this research we are not able to catch any argument on behalf of a counterexample in Trent and Wang's problem. Moreover, we show that the result of Mateu and Verdera [12] is of purely "two-dimensional" character in the sense that to the *BMO* approximations in $\mathbb{R}^n$ for $n > 2$ there always is a counterexample. This demonstrates rather strikingly that one should expect the affirmative answer in Trent and Wang's problem.

For a deeper discussion of approximation by solutions of an elliptic equation, we refer the reader to the survey of Tarkhanov [19].

The important point to note here is a new type of *BMO* spaces. They are obtained by localizing the space $BMO(\mathbf{R}^n)$ (see, for instance, Stein [18]). $BMO(\mathbf{R}^n)$ is suitable for pure Fourier analysis while $BMO_{loc}(\mathbb{R}^n)$ is more suited to problems associated with partial differential equations. The main advantage of this space over the classical one is that pseudodifferential operators are bounded on it. In addition, the above spaces are well-defined on manifolds.

We now sketch the contents of this paper. The next section introduces the space *BMO*, its restriction to compact sets K, and its higher-order variants. The important point is that *BMO* is here given a topology which makes it a semi-local space; that is, multiplication by smooth functions becomes a continuous operation. Section 3 presents the corresponding *VMO* spaces (for "vanishing mean oscillation"); these play a role within *BMO* analogous to that played by the closure of $\mathcal{D}$ in L^∞. Section 4 discusses the local Hardy spaces of Goldberg [7] and adapts the classical results of C.Fefferman and Stein [5] and of Coifman and Weiss [4] on the dual and pre-dual of the Hardy space H^1 (*BMO* and a form of *VMO*, respectively) to the local setting considered here. The local continuity of pseudodifferential operators on *BMO* spaces is the key result of the following section; it is dual to Goldberg's result for local Hardy spaces. The final three sections treat the approximation problem in *BMO* spaces; we argue that the situation is much closer to Sobolev approximation than to uniform. In Section 6 we show that the approximation problem in higher order *BMO* spaces is easily answered in terms of *spectral synthesis* in these spaces. Section 7 establishes the relation between lower order approximations in *BMO* spaces and the dual problem of spectral synthesis in local Hardy spaces. Section 8 deals with the case of nowhere dense compact sets K; in particular, we show that if the order of P is equal to n (the dimension of $\mathbb{R}^n$), then any function in $VMO(K)$ can be approximated in the norm of $BMO(K)$ by solutions of the equation $Pf = 0$ near K. This extends a result of Mateu and Verdera [12] for the case when K has empty interior to the multi-dimensional setting.

2 The space $BMO^s(K)$

A locally integrable function f on $\mathbf{R}^n$ has *bounded mean oscillation* if the quantity

$$\|f\|_* = \sup_Q \frac{1}{|Q|}\int_Q |f - f_Q| \tag{2.1}$$

is finite. The supremum here runs over all cubes Q with sides 'parallel' to the coordinate axes, the symbol $|Q|$ denotes the Lebesgue measure of $|Q|$, and $f_Q = \frac{1}{|Q|}\int_Q f$ is the average of f over Q. The space of all such functions is denoted $BMO(\mathbf{R}^n)$ or simply BMO. Every bounded function is, of course, in BMO, but the converse is not true; the (even) logarithm $f(x) = \log|x|$ is the paradigmatic example of an unbounded BMO function.

Since the mean oscillation of every constant function vanishes, it is customary to identify any two functions in BMO that differ by a constant almost everywhere; the resulting quotient space becomes a Banach space under the norm (induced by) $\|\cdot\|_*$. This (otherwise extremely useful) topology has the disadvantage in the present context that it is not semilocal. In this paper, we choose instead to topologize the (full) space BMO under the norm

$$\nu(f) = \int_{Q_0} |f| + \|f\|_*, \tag{2.2}$$

where Q_0 is the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbf{R}^n$.

Lemma 2.1 *The norm (2.2) is equivalent to any norm of the form*

$$\nu'(f) = \int_{Q'} |f| + \|f\|_*,$$

for Q' an arbitrary cube in $\mathbb{R}^n$.

Proof. Indeed, there is a constant $c = c(Q_0, Q')$ such that $|f_{Q_0} - f_{Q'}| \le c\,\|f\|_*$ for all $f \in BMO$. (This follows, for instance, from inequality (2.7) below and the triangle inequality.) Since $\|\,|f|\,\|_* \le \|f\|_*$, then this implies that

$$\begin{aligned}\int_{Q'} |f| &= |Q'|\,(|f|_{Q'}) \\ &\le |Q'|\,(|f|_{Q_0} + c\|f\|_*) \\ &= \frac{|Q'|}{|Q_0|}\int_{Q_0} |f| + c\,|Q'|\,\|f\|_*.\end{aligned}$$

A similar inequality controls $\int_{Q_0}|f|$ by $\int_{Q'}|f|$.

□

BMO is actually a complete space with norm ν.

Lemma 2.2 *BMO is a Banach space under the norm ν.*

Proof. Let $\{f_i\}$ be a Cauchy sequence, i.e., suppose

$$\int_{Q_0}|f_i - f_j| + \|f_i - f_j\|_* < \varepsilon$$

for i, j sufficiently large (i.e., $i, j > N(\varepsilon)$). Then there is a function $f^{(0)} \in L^1(Q_0)$ such that $f_i \to f^{(0)}$ in $L^1(Q_0)$, and there is a subsequence $\{f_{i_k}\}$ such that $f_{i_k} \to f^{(0)}$ a.e. in Q_0.

Note that the L^1 convergence implies convergence of the mean values: $(f_i)_Q \to (f^{(0)})_Q$ for any cube $Q \subset Q_0$. Hence, for any such Q,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |(f_i - f^{(0)}) - (f_i - f^{(0)})_Q| \\
&\quad = \frac{1}{|Q|}\int_Q \liminf_{k\to\infty}\ |(f_i - f_{i_k}) - (f_i - f_{i_k})_Q| \\
&\quad \le \liminf_{k\to\infty}\ \frac{1}{|Q|}\int_Q |(f_i - f_{i_k}) + (f_i - f_{i_k})_Q| \qquad \text{(by Fatou's lemma)} \\
&\quad \le \liminf_{k\to\infty}\ \|f_i - f_{i_k}\|_* \\
&\quad < \varepsilon,
\end{aligned}$$

for i sufficiently large (i.e., $i > N(\varepsilon)$).

We thus have a suitable limiting function in Q_0. We can use Lemma 2.1 to get a limiting function defined elsewhere in $\mathbb{R}^n$. Indeed, for any cube $Q' \supset Q_0$, the Cauchy sequence $\{f_i\}$ in ν must also be Cauchy in $L^1(Q')$. Hence, there is a function $f' \in L^1(Q')$ with $f_i \to f'$ in $L^1(Q')$ and $f_{i_k} \to f'$ a.e. in Q', for some subsequence $\{f_{i_k}\}$. Repeating the argument in the first part shows that

$$\frac{1}{|Q|}\int_Q |(f_i - f') - (f_i - f')_Q| < \varepsilon$$

for any cube $Q \subset Q'$ and any i sufficiently large.

This procedure leads to a unique limiting function. Indeed, if $Q_0 \subset Q' \subset Q''$ and f' and f'' are two functions constructed as above, then $f_i \to f'$ in $L^1(Q')$ and $f_i \to f''$ in $L^1(Q'')$. Hence $f' = f''$ a.e. in Q', as desired.

Exhausting $\mathbb{R}^n$ by an expanding sequence of cubes thus leads to a limiting function $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\int_{Q_0}|f - f_i| + \|f - f_i\|_* < \varepsilon$$

for $i > N(\varepsilon)$, as desired. Thus, $\{f_i\}$ converges to f in ν. □

This topology has the advantage of making multiplication by smooth functions into a continuous operator, which is the content of the following result.

Lemma 2.3 *Let $\varphi \in \mathcal{D}$. Then the multiplication operator $f \mapsto \varphi f$ is a continuous mapping of $BMO \to BMO$.*

Proof. We wish to show that for each $\varphi \in \mathcal{D}$ there is a constant C_φ so that

$$\nu(\varphi f) \leq C_\varphi \, \nu(f) \qquad \text{for all } f \in BMO.$$

Since $\int_{Q_0} |\varphi f| \leq \|\varphi\|_\infty \int_{Q_0} |f|$, we need really only show that

$$\|\varphi f\|_* \leq C_\varphi \left(\int_{Q_0} |f| + \|f\|_* \right). \tag{2.3}$$

One further simplification arises from the triangle inequality and allows us to measure the mean oscillation of a function F on Q about any constant, not just F_Q. That is,

$$\frac{1}{2} \|F\|_* \leq \sup_Q \inf_{c \in \mathbf{R}} \frac{1}{|Q|} \int_Q |F - c| \leq \|F\|_*. \tag{2.4}$$

(Indeed, for any cube Q and any $c \in \mathbb{R}$,

$$\frac{1}{|Q|} \int_Q |F - F_Q| \leq \frac{1}{|Q|} \int_Q |F - c| + |c - F_Q| \leq \frac{2}{|Q|} \int_Q |F - c|.)$$

With $F = \varphi f$, it proves convenient to take $c = \varphi_Q f_Q$ rather then $(\varphi f)_Q$. For then

$$\begin{aligned} \frac{1}{|Q|} \int_Q |\varphi f - \varphi_Q f_Q| &\leq \frac{1}{|Q|} \int_Q (|f - f_Q| \, |\varphi|) + \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q| \, |f_Q| \\ &\leq \|\varphi\|_\infty \|f\|_* + \frac{1}{|Q|} \int_Q |\varphi - \varphi_Q| \, |f_Q|. \end{aligned}$$

It remains to dominate $\frac{1}{|Q|} \int_Q |\varphi - \varphi_Q| \, |f_Q|$ by the right-hand side of (2.3) uniformly over all cubes Q. The key observation here is that the mean values of any BMO function can only vary slowly as we change the scale. In particular,

$$|f_Q - f_{2Q}| \leq c_n \, \|f\|_* \tag{2.5}$$

for all cubes Q, where $2Q$ is any cube containing Q with twice the latter's side length. (This is simply just another application of the triangle inequality:

$$\begin{aligned} |f_Q - f_{2Q}| &= \left| \frac{1}{|Q|} \int_Q (f - f_{2Q}) \right| \\ &\leq \frac{1}{|Q|} \int_Q |f - f_{2Q}| \\ &\leq \frac{1}{|Q|} \int_{2Q} |f - f_{2Q}| \\ &\leq 2^n \, \|f\|_*, \end{aligned} \tag{2.6}$$

as desired.)

Iterating (2.5) yields $|f_Q - f_{2^jQ}| \le c_n\, j\, \|f\|_*$ and, in general,

$$|f_Q - f_{Q'}| \le c_n \left(1 + \log \frac{|Q'|}{|Q|}\right) \|f\|_* \tag{2.7}$$

whenever $Q \subset Q'$.

Now let Q'' be a fixed cube containing the support of φ and Q_0. Then, for any cube Q within Q'' of side length at least 1, we have from (2.7) that

$$|f_Q - f_{Q''}| \le C_\varphi \|f\|_*.$$

With Q_0 the unit cube, it follows that $|f_Q - f_{Q_0}| \le 2C_\varphi \|f\|_*$, so that $|f_Q| \le |f_{Q_0}| + 2C_\varphi \|f\|_* \le 2C_\varphi\, \nu(f)$ for all cubes Q within Q'' of side length at least 1. Hence, for any such cube Q,

$$\frac{1}{|Q|}\int_Q |\varphi - \varphi_Q|\, |f_Q| \le (2\|\varphi\|_\infty)\,(2C_\varphi\, \nu(f)), \tag{2.8}$$

as desired.

It remains only to bound the left-hand side of (2.8) for all smaller cubes which intersect the support of φ. For any such cube Q, let Q' be a cube of side length 1, with $Q \subset Q' \subset Q''$. (Here Q'' is as above.) Now,

$$\frac{1}{|Q|}\int_Q |\varphi - \varphi_Q| \le c_n \|\nabla\varphi\|_\infty |Q|^{\frac{1}{n}}, \tag{2.9}$$

and

$$\begin{aligned} |f_Q| &\le |f_{Q'}| + |f_Q - f_{Q'}| \\ &\le 2C_\varphi\, \nu(f) + c_n \left(1 + \log \frac{1}{|Q|}\right) \|f\|_* \end{aligned} \tag{2.10}$$

by (2.7). Since $|Q| \le 1$, then $|Q|^{\frac{1}{n}} \left(1 + \log \frac{1}{|Q|}\right) < c'_n < \infty$, and hence combining (2.9) and (2.10) yields

$$\frac{1}{|Q|}\int_Q |\varphi - \varphi_Q|\, |f_Q| \le C'_\varphi\, \nu(f),$$

as desired.

□

This observation is likely not new. The estimate (2.5) has been frequently used, beginning with the original paper of John and Nirenberg [11] introducing *BMO*. Estimate (2.7) appears in Christ [3]. Reimann and Rychener [15] prove a statement on the tensor product of a *BMO* function and a smooth function that is analogous to Lemma 2.3. There may well be an earlier source.

Remark 2.4 *The proof shows that φ need not be infinitely differentiable for the result to hold. For instance, if φ is merely in Λ^{λ}_{comp}, that is, φ has compact support and is Hölder continuous of index λ for some $0 < \lambda < 1$, then estimate (2.9) becomes*

$$\frac{1}{|Q|}\int_Q |\varphi - \varphi_Q| \le C''_{\varphi}\, |Q|^{\frac{\lambda}{n}}.$$

Since $x^{\lambda}\log\frac{1}{x} < c_{\lambda} < \infty$ for all $0 < x < 1$ and each $0 < \lambda < 1$, then the argument goes through as before.

This observation goes back at least as far as Stegenga [17], who gave a necessary and sufficient condition on a bounded function φ on the unit circle S in $\mathbb{R}^2$ in order that the multiplication by φ be a continuous operator in $BMO(S)$ endowed with a norm topology equivalent to ours.

As a consequence of the lemma, we can consider for any open set $X \subset \mathbb{R}^n$ the spaces

$$\begin{aligned} BMO_{loc}(X) &= \{f \in \mathcal{D}'(X) : \ \varphi f \in BMO \ \text{ for all } \varphi \in \mathcal{D}(X)\}, \\ BMO_{comp}(X) &= \{\varphi f : \ \varphi \in \mathcal{D}(X), \ f \in BMO\} \end{aligned}$$

with the standard topology on *local* and *compact* spaces (relative to the norm ν). The lemma then extends to this setting.

Corollary 2.5 *For every $\varphi \in \mathcal{D}(X)$, the multiplication operator $f \mapsto \varphi f$ is a continuous mapping of $BMO_{loc}(X) \to BMO_{comp}(X)$.*

We can then proceed to define higher-order BMO spaces by the usual construction. That is, for any $s \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, we set

$$\begin{aligned} BMO^s &= BMO^s(\mathbb{R}^n) \\ &= \{f \in \mathcal{D}' : \ D^{\alpha} f \in BMO \ \text{ for all } \alpha \ \text{ with } |\alpha| \le s\}. \end{aligned}$$

Imposing the topology of convergence in the norm $\nu(\cdot)$ in all derivatives up to order s turns BMO^s into a semilocal Banach space.

The spaces $BMO^s_{loc}(X)$ and $BMO^s_{comp}(X)$ can be constructed analogously. If $X = \mathbb{R}^n$, we denote these simply by BMO^s_{loc} and BMO^s_{comp}.

Note that $(BMO^s)_{loc} = (BMO_{loc})^s$, etc., so that the notation is unambiguous. For this and other aspects of the general theory, see Tarkhanov [20, Ch.1].

We next define the spaces L^s for negative integers s, where L is one of the spaces BMO, $BMO_{loc}(X)$ or $BMO_{comp}(X)$. As there is no way to do this canonically, we opt for a method which allows us to remain in the framework of the foregoing approach. Namely, given any negative integer s, the space L^s is defined to consist of all distributions of the form $\sum_{|\alpha| \le -s} D^{\alpha} f_{\alpha}$, where $f_{\alpha} \in L$.

Proposition 2.6 *For every $s = -1, -2, \ldots$, it follows that*

$$\begin{aligned} (BMO^s)_{loc}(X) &= (BMO_{loc}(X))^s, \\ (BMO^s)_{comp}(X) &= (BMO_{comp}(X))^s. \end{aligned}$$

Proof. We only prove that $(BMO^s)_{loc}(X) \subset (BMO_{loc}(X))^s$. The other inclusions may be handled in the same way as in the proof of Proposition 1.1.17 in Tarkhanov [20].

Let $f \in (BMO^s)_{loc}(X)$. Fix a covering $\{U_i\}$ of X by relatively compact open subsets such that $U_i \subset\subset U_{i+1}$ and such that $X \setminus U_i$ has no compact connected components. For every i, choose a function $\varphi_i \in \mathcal{D}(X)$ which is equal to 1 in a neighborhood of $\overline{U_i}$.

By assumption, to each number i there correspond functions $f_\alpha^{(i)} \in BMO$ ($|\alpha| \le -s$) such that $\varphi_i f = \sum_{|\alpha|\le -s} D^\alpha f_\alpha^{(i)}$. The differences $f_\alpha^{(i+1)} - f_\alpha^{(i)}$ ($|\alpha| \le -s$) are therefore in BMO and satisfy $\sum_{|\alpha|\le -s} D^\alpha (f_\alpha^{(i+1)} - f_\alpha^{(i)}) = 0$ in a neighborhood of $\overline{U_i}$.

Since the differential operator $\{f_\alpha\} \mapsto \sum_{|\alpha|\le -s} D^\alpha f_\alpha$ has surjective symbol, it follows from Tarkhanov [20, Ch.5] that there are functions $a_\alpha^{(i)}$ ($|\alpha| \le -s$) in $BMO_{loc}(X)$ satisfying $\sum_{|\alpha|\le -s} D^\alpha a_\alpha^{(i)} = 0$ on X, such that

$$\nu\left(\varphi_i (f_\alpha^{(i+1)} - f_\alpha^{(i)} - a_\alpha^{(i)})\right) < \frac{1}{2^i}.$$

Therefore, the series

$$f_\alpha = f_\alpha^{(1)} + \sum_{i=1}^{\infty} (f_\alpha^{(i+1)} - f_\alpha^{(i)} - a_\alpha^{(i)})$$

converges in the topology of $BMO_{loc}(X)$. Moreover, as the differentiation operator is continuous in the space of distributions, we get

$$\begin{aligned}
\sum_{|\alpha|\le -s} D^\alpha f_\alpha &= \sum_{|\alpha|\le -s} D^\alpha f_\alpha^{(1)} + \lim_{I\to\infty} \sum_{i=1}^{I} \left(\sum_{|\alpha|\le -s} D^\alpha (f_\alpha^{(i+1)} - f_\alpha^{(i)} - a_\alpha^{(i)}) \right) \\
&= \lim_{I\to\infty} \sum_{|\alpha|\le -s} D^\alpha f_\alpha^{(I+1)} \\
&= \lim_{I\to\infty} \varphi_{I+1} f \\
&= f,
\end{aligned}$$

whence $f \in (BMO_{loc}(X))^s$, as desired.

□

Thus, it will cause no confusion if we write $BMO^s_{loc}(X)$ and $BMO^s_{comp}(X)$ for a negative integer s.

For negative integers s, the space L^s is topologized in the following way. The basis of neighborhoods of zero in L^s is declared to consist of the sets

$$\{ \sum_{|\alpha|\le -s} D^\alpha f_\alpha : \ f_\alpha \in U \ (|\alpha| \le -s)\},$$

where U varies over a basis of neighborhoods of zero in L.

The most important argument on behalf of this definition is that it immediately makes differentiation continuous.

Proposition 2.7 *For every multi-index α with $|\alpha| \le s$, the operator D^α acts from L to L^{-s} continuously.*

We close this section by defining the BMO spaces on arbitrary compact sets $K \subset \mathbf{R}^n$. For a general semilocal space L of distributions on $\mathbf{R}^n$ and an $s \in \mathbf{Z}_+$, we would like to define $L^s(K)$ as the quotient of L^s over the subspace of distributions which are "flat" on K in some sense. How can we capture the degree of flatness? The idea is to start with the space Σ of distributions vanishing on at least some neighborhood of K. Since this subspace might not be closed, we opt instead to use its closure in L^s and to define

$$L^s(K) = L^s/\overline{\Sigma}.$$

One advantage of this approach is that the three spaces $L^s(K)$, $L^s_{loc}(K)$ and $L^s_{comp}(K)$ are all topologically isomorphic. (See Tarkhanov [20].) For concreteness, we state this result for the BMO spaces considered here.

Lemma 2.8 *Given any compact $K \subset \mathbb{R}^n$ and $s \in \mathbf{Z}_+$, we have*

$$BMO^s(K) \overset{top.}{\cong} BMO^s_{loc}(K) \overset{top.}{\cong} BMO^s_{comp}(K).$$

3 The space $VMO^s(K)$

What happens when we demand that the mean oscillation of a function is arbitrarily small for all sufficiently small cubes? The first person to consider this question was Sarason [16], who defined the space of *functions of vanishing mean oscillation* by the condition

$$\begin{aligned} VMO &= VMO(\mathbb{R}^n) \\ &= \{f \in BMO : \limsup_{|Q|\to 0} \frac{1}{|Q|}\int_Q |f - f_Q| = 0\}. \end{aligned}$$

Sarason [16] actually showed that (in the conventional topology on BMO, under which functions which differ by a constant a.e. are identified) VMO is the closure of those BMO functions which are uniformly continuous on $\mathbb{R}^n$. We shall show that the same is true under the topology induced by the norm ν.

Let us look more closely at the regularization "a la Sarason." To this end, set

$$\|f\|_{*,\delta} = \sup_{Q:\, l(Q)\le\delta} \frac{1}{|Q|}\int_Q |f - f_Q|,$$

where $\delta > 0$ and $l(Q)$ is the side length of the cube Q.

Lemma 3.1 *Let χ be the characteristic function of the unit cube, and $\chi^\varepsilon(x) = \varepsilon^{-n}\chi(x/\varepsilon)$. Then*

$$|\chi^\varepsilon * f(x) - \chi^\varepsilon * f(y)| \le c_n \|f\|_{*,2\varepsilon}, \qquad \text{when } |x-y| < \varepsilon. \tag{3.1}$$

Proof. As above, we denote by Q_0 the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$ in $\mathbb{R}^n$. Given any x and y with $|x-y| < \varepsilon$, let Q be some cube of side length 2ε containing both $\varepsilon Q_0 + x$ and $\varepsilon Q_0 + y$. (Here εQ_0 is the cube $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]^n$.) Then

$$\begin{aligned} &|\chi^\varepsilon * f(x) - \chi^\varepsilon * f(y)| \\ &\quad = |f_{\varepsilon Q_0 + x} - f_{\varepsilon Q_0 + y}| \\ &\quad \le |f_{\varepsilon Q_0 + x} - f_Q| + |f_{\varepsilon Q_0 + y} - f_Q| \\ &\quad \le 2\, 2^n \|f\|_{*,2\varepsilon} \end{aligned}$$

by (2.6), as desired.

□

Lemma 3.1 shows that, for any $f \in VMO$, the convolution $\chi^\varepsilon * f$ is uniformly continuous on all of $\mathbb{R}^n$ (written $\chi^\varepsilon * f \in UC$). Note that rather than $\chi^\varepsilon * f$, Sarason [16] considers a piecewise constant function (i.e., a step function) which is equal on each step to the average of f there.

Lemma 3.2 *Under the assumptions of Lemma 3.1,*

$$\|\chi^\varepsilon * f - f\|_* \le c\, \|f\|_{*,2\varepsilon}. \tag{3.2}$$

Proof. If the side length $l(Q)$ of Q exceeds ε, then cover Q by nonoverlapping cubes $Q_1, \ldots, Q_N$ of side length ε such that

$$\left|\cup_{i=1}^N Q_i\right| = \cup_{i=1}^N |Q_i| \le C_n\, |Q|.$$

Then

$$\begin{aligned} \frac{1}{|Q|}\int_Q |\chi^\varepsilon * f - f| &\le \frac{1}{|Q|}\sum_{i=1}^N \int_{Q_i} |\chi^\varepsilon * f - f| \\ &\le \frac{1}{|Q|}\sum_{i=1}^N \left(\int_{Q_i} |\chi^\varepsilon * f - f_{Q_i}| + \int_{Q_i} |f_{Q_i} - f|\right) \\ &\le \frac{1}{|Q|}\sum_{i=1}^N (2c_n\, |Q_i|\, \|f\|_{*,2\varepsilon} + |Q_i|\, \|f\|_{*,\varepsilon}), \end{aligned}$$

by Lemma 3.1.

So,

$$\begin{aligned} \frac{1}{|Q|}\int_Q |\chi^\varepsilon * f - f| &\le \frac{2c_n + 1}{|Q|}\sum_{i=1}^N |Q_i|\, \|f\|_{*,2\varepsilon} \\ &\le c'\, \|f\|_{*,2\varepsilon}. \end{aligned} \tag{3.3}$$

Hence, if $l(Q) \geq \varepsilon$, then

$$\frac{1}{|Q|}\int_Q |(\chi^\varepsilon * f - f) - (\chi^\varepsilon * f - f)_Q| \leq 2\,c'\,\|f\|_{*,2\varepsilon}.$$

On the other hand, if $l(Q) < \varepsilon$, then (with x_0 the center of Q)

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |(\chi^\varepsilon * f - f) - (\chi^\varepsilon * f\,(x_0) - f_Q)| \\
&\quad \leq \frac{1}{|Q|}\int_Q |\chi^\varepsilon * f - \chi^\varepsilon * f\,(x_0)| + \frac{1}{|Q|}\int_Q |f - f_Q| \\
&\quad \leq 2^{n+1}\,\|f\|_{*,2\varepsilon} + \|f\|_{*,\varepsilon} \\
&\quad \leq c''\,\|f\|_{*,2\varepsilon},
\end{aligned}$$

the second estimate being due to Lemma 3.1.

For such cubes Q,

$$\frac{1}{|Q|}\int_Q |(\chi^\varepsilon * f - f) - (\chi^\varepsilon * f - f)_Q| \leq 2c''\,\|f\|_{*,2\varepsilon}$$

by (2.4), which completes the proof.

□

It is not the case in general, however, that $\|\chi^\varepsilon * f - f\|_* \to 0$ as $\varepsilon \to 0$, as the following example shows.

Example 3.3 Consider the *BMO* function $f(x) = \log\frac{1}{|x|}$ on $\mathbf{R}^1$. An easy computation shows that

$$\chi^\varepsilon * f\,(x) = -\frac{x}{\varepsilon}\log\left|\frac{x+\frac{1}{2}\varepsilon}{x-\frac{1}{2}\varepsilon}\right| - \frac{1}{2}\log\left|x+\frac{1}{2}\varepsilon\right|\left|x-\frac{1}{2}\varepsilon\right| + 1.$$

To see that

$$\sup_Q \frac{1}{|Q|}\int |(\chi^\varepsilon * f - f) - (\chi^\varepsilon * f - f)_Q| \not\to 0,$$

it's enough to take $Q = [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. Indeed, for all $x \in Q$, we have

$$\log\frac{1}{\varepsilon} + 1 \leq \chi^\varepsilon * f\,(x) \leq \log\frac{2}{\varepsilon} + 1.$$

So, the same is true for the average:

$$\log\frac{1}{\varepsilon} + 1 \leq (\chi^\varepsilon * f)_Q \leq \log\frac{2}{\varepsilon} + 1.$$

Therefore, for all $x \in Q$,

$$|\chi^\varepsilon * f\,(x) - (\chi^\varepsilon * f)_Q| \leq \log 2.$$

Since $f_Q = \chi^\varepsilon * f(0) = \log\frac{2}{\varepsilon} + 1$, we get

$$\begin{aligned}\frac{1}{|Q|}\int_Q &|(\chi^\varepsilon * f - f) - (\chi^\varepsilon * f - f)_Q| \\ &\geq \frac{1}{|Q|}\int |f - f_Q| - \log 2 \\ &= \frac{2}{\varepsilon}\int_Q \left(\left|\log\frac{1}{x} - \left(\log\frac{2}{\varepsilon} + 1\right)\right| - \log 2\right) dx \\ &\geq \frac{2}{\varepsilon}\int_0^\delta \left(\log\frac{1}{x} - \log\frac{4e}{\varepsilon}\right) dx\end{aligned}$$

for $\delta = \delta(\varepsilon) > 0$ such that $\log\frac{1}{\delta} \geq \log\frac{4e}{\varepsilon}$, i.e., $\delta \leq \frac{\varepsilon}{4e}$. Thus,

$$\begin{aligned}\frac{1}{|Q|}\int &|(\chi^\varepsilon * f - f) - (\chi^\varepsilon * f - f)_Q| \\ &\geq \frac{2}{\varepsilon}\left(\delta \log\frac{e}{\delta} - \delta\log\frac{4e}{\varepsilon}\right) \\ &\geq \frac{1}{2e}\end{aligned}$$

for all $\varepsilon > 0$, which is the desired conclusion.

□

Lemma 3.2 gives a regularization of *VMO* functions by uniformly continuous *BMO* functions. We next obtain a regularization by smooth functions.

Let $\omega \in \mathcal{D}$, $\omega \geq 0$, and $\int \omega = 1$, and suppose that ω is supported in the unit cube.

Lemma 3.4 *For any BMO function f,*

$$\|\omega^\varepsilon * \chi^\varepsilon * f - f\|_* \leq c\,\|f\|_{*,2\varepsilon}. \tag{3.4}$$

Proof. Using Lemma 3.1, we obtain the pointwise estimate

$$\begin{aligned}\|\omega^\varepsilon &* \chi^\varepsilon * f - \chi^\varepsilon * f\|_\infty \\ &= \sup_x \left|\int \frac{1}{\varepsilon^n}\omega\left(\frac{x-y}{\varepsilon}\right)(\chi^\varepsilon * f(y) - \chi^\varepsilon * f(x))\,dy\right| \\ &\leq \sup_x \int \omega(z)\,|(\chi^\varepsilon * f(x+\varepsilon z) - \chi^\varepsilon * f(x))|\,dz \\ &\leq c\,\|f\|_{*,2\varepsilon}.\end{aligned} \tag{3.5}$$

Together with (3.2), this gives the desired result.

□

It easily follows from (3.1) that, for $f \in VMO$ and $\varepsilon > 0$, the smooth function $\omega^\varepsilon * \chi^\varepsilon * f$ is actually uniformly continuous on all of $\mathbb{R}^n$. We thus obtain the following result, analogous to that of Sarason [16]:

Corollary 3.5 *VMO is the closure of $C^\infty_{loc} \cap UC \cap BMO$ in the norm ν.*

Proof. If $f \in VMO$, then (3.4) implies that

$$\|\omega^\varepsilon * \chi^\varepsilon * f - f\|_* \to 0 \qquad \text{as } \varepsilon \to 0.$$

Since

$$\int_{Q_0} |\omega^\varepsilon * \chi^\varepsilon * f - f| \\ \leq \|\omega^\varepsilon * \chi^\varepsilon * f - \chi^\varepsilon * f\|_\infty + \int_{Q_0} |\chi^\varepsilon * f - f|,$$

then (3.3) and (3.5) combine to show that $\nu(\omega^\varepsilon * \chi^\varepsilon * f - f) \to 0$ as $\varepsilon \to 0$.

Conversely, since $UC \cap BMO \subset VMO$, any function f in the closure of $UC \cap BMO$ in the norm ν satisfies $\|f\|_{*,\delta} \to 0$ as $\delta \to 0$, which finishes the proof.

□

Let

$$\begin{aligned} VMO_{loc}(X) &= \{f \in \mathcal{D}'(X) : \varphi f \in VMO \text{ for all } \varphi \in \mathcal{D}(X)\}, \\ VMO_{comp}(X) &= \{\varphi f : \varphi \in \mathcal{D}(X),\ f \in VMO\}, \end{aligned}$$

with the usual topology on *local* and *compact* spaces derived from ν. It now follows that the smooth, compactly supported functions $\mathcal{D}(X)$ are dense in $VMO_{loc}(X)$.

Proposition 3.6 *The closure of $\mathcal{D}(X)$ in $BMO_{loc}(X)$ is $VMO_{loc}(X)$.*

Proof. Without loss of generality we can assume that $X = \mathbf{R}^n$.

Let $f \in VMO_{loc}$ and $\varphi \in \mathcal{D}$ be given. If $\varphi(x) = 0$ for all $|x| > R > 1$, choose $\tilde{\varphi} \in \mathcal{D}$, with $\tilde{\varphi} = 1$ for all $|x| \leq 2R$. So, for all $\varepsilon < 1$,

$$\begin{aligned} \nu(\varphi(\omega^\varepsilon * \chi^\varepsilon * (\tilde{\varphi} f) - f)) &= \nu(\varphi(\omega^\varepsilon * \chi^\varepsilon * (\tilde{\varphi} f) - (\tilde{\varphi} f))) \\ &\leq C_\varphi \nu(\omega^\varepsilon * \chi^\varepsilon * (\tilde{\varphi} f) - (\tilde{\varphi} f)), \end{aligned}$$

by Lemma 2.3. The last term vanishes as $\varepsilon \to 0$, by Corollary 3.5. The other inclusion is immediate.

□

For $s \in \mathbf{Z}$, we define the spaces VMO^s, $(VMO_{loc}(X))^s$ and $(VMO_{comp}(X))^s$ as in Section 2.

Proposition 3.7 *For every $s = -1, -2, \ldots$, it follows that*

$$\begin{aligned} (VMO^s)_{loc}(X) &= (VMO_{loc}(X))^s, \\ (VMO^s)_{comp}(X) &= (VMO_{comp}(X))^s. \end{aligned}$$

Proof. This is analogous to the proof of Proposition 2.6.

□

Thus, it will cause no confusion if we write $VMO^s_{loc}(X)$ and $VMO^s_{comp}(X)$ for a negative integer s.

4 Duality

It immediately follows from Proposition 3.6 that the subspace $\mathcal{D}(X)$ is dense in $VMO^s_{comp}(X)$ for any integer s. Thus, every continuous linear functional F on $VMO^s_{comp}(X)$, if restricted to $\mathcal{D}(X)$, is a distribution in X, and this correspondence is one-to-one. Moreover, the inclusion map $(VMO^s_{comp}(X))' \hookrightarrow \mathcal{D}'(X)$ is continuous. Our next objective is to describe this dual space of distributions on X.

In the classical setting in which BMO is topologized as a quotient space under the seminorm $\|\cdot\|_*$, a famous result is that BMO is the dual of the Hardy space $H = H^1(\mathbb{R}^n)$ (see Fefferman and Stein [5]). On the other hand, H is the dual of the space CMO, the closure in the norm $\|\cdot\|_*$ of the space C_{comp} of continuous functions with compact support on $\mathbb{R}^n$ (see Coifman and Weiss [4]).

The very property of the classical Hardy space H which enables the main theorems to hold ($f \in H \Rightarrow \int f = 0$) also causes H to fail to behave properly with respect to multiplication by functions in $\mathcal{S}$ (the space of rapidly decreasing functions). To see this note that if $\varphi \in \mathcal{S}$, then $f \mapsto \varphi f$ is not bounded on H (since $\int \varphi f \neq 0$). (In particular, $H_{loc} = \{0\}$.) This map is a pseudodifferential operator and also is a "patching" map of the kind necessary for working on a manifold.

David Goldberg [7] gave an account of a local version of Hardy space (denoted h). The main advantage of this space over the classical one is that $\mathcal{S} \subset h$ and that h is stable under multiplication by functions in $\mathcal{S}$. Thus, h is suitable for working with manifolds and pseudodifferential operators.

We begin with two definitions. First, the local Hardy space $h = h^1(\mathbb{R}^n)$ is defined by

$$h = \left\{ f \in L^1(\mathbb{R}^n) : \ \|f\|_h := \|f\|_{L^1(\mathbb{R}^n)} + \sum_{j=1}^{n} \|r_j f\|_{L^1(\mathbb{R}^n)} < \infty \right\},$$

where r_j is the "modified Riesz transform" given by $\widehat{r_j f}\,(\xi) = (1-\omega(\xi))\frac{\xi_j}{|\xi|}\hat{f}(\xi)$, for any fixed $\omega \in \mathcal{D}$ with $\omega(0) = 1$.

Second, the local BMO space bmo is defined by

$$bmo = \left\{ f \in L^1(\mathbb{R}^n) : \ \|f\|_{bmo} := \sup_{Q:|Q|\leq 1} \frac{1}{|Q|}\int_Q |f - f_Q| + \sup_{Q:|Q|\geq 1} \frac{1}{|Q|}\int_Q |f| < \infty \right\}.$$

Goldberg's results [7] include the following:

Theorem 4.1 ([7]) *The dual of h is bmo; that is, every F in bmo defines a linear functional $f \mapsto \int Ff$ on $\mathcal{S}$ (which is dense in h), and every linear functional is of this form.*

Theorems 4.1 and 5.2 (see below) imply that both h and bmo are semilocal spaces, so we can consider their *loc* and *comp* variants as well as the corresponding higher order spaces.

Lemma 4.2 *Given any open set $X \subset \mathbf{R}^n$, it follows that*

$$\begin{aligned} BMO_{loc}(X) &\overset{top.}{\cong} bmo_{loc}(X), \\ BMO_{comp}(X) &\overset{top.}{\cong} bmo_{comp}(X). \end{aligned}$$

Proof. We give the proof only for the second isomorphism; the first statement is an easy consequence of the second one.

It is a simple matter to see that $BMO_{comp}(X)$ and $bmo_{comp}(X)$ coincide as vector spaces. What is left is to show that the topologies on these spaces are also the same.

To this end, it suffices to prove that a set b is bounded in $BMO_{comp}(X)$ if and only if it is bounded in $bmo_{comp}(X)$. If however b is a bounded subset of $BMO_{comp}(X)$ or $bmo_{comp}(X)$, then there is a compact set $K \subset X$ such that $\operatorname{supp} f \subset K$ for all $f \in b$. Hence it follows that we only need to show that for any compact set $K \subset \mathbb{R}^n$ there are positive constants c_1, c_2 with the property that

$$c_1\ \nu(f) \leq \|f\|_{bmo} \leq c_2\ \nu(f) \qquad \text{for all } f \in BMO_K; \tag{4.1}$$

the subscript K of BMO indicates that the functions are supported in K.

We begin by proving the right estimate in (4.1). To do this, pick any cube Q' containing $K \cup Q_0$. It follows from (2.7) that

$$|f_{Q_0} - f_{Q'}| \leq c_n\,(1 + \log|Q'|)\,\|f\|_*,$$

where c_n is independent of $f \in BMO$.

Given any $f \in BMO_K$, we have

$$\|f\|_{bmo} \leq \|f\|_* + \sup_{Q:|Q|\geq 1} \frac{1}{|Q|}\int_Q |f|.$$

We restrict attention to those cubes which intersect K and which have side length at least 1. A geometric argument shows that, for any such cube Q, there exists a larger cube Q'' with the following properties: $Q \cup Q' \subset Q''$ and $|Q''| \leq C\,|Q|$, the constant C depending only on K and n. Then

$$\begin{aligned} &\frac{1}{|Q|}\int_Q |f| \\ &\quad \leq \|f\|_* + |f_Q - f_{Q''}| + |f_{Q''} - f_{Q'}| + |f_{Q'} - f_{Q_0}| + |f_{Q_0}| \\ &\quad \leq \nu(f) + c_n\,(1 + \log C)\,\|f\|_* + |f_{Q''} - f_{Q'}| + c_n\,(1 + \log|Q'|)\,\|f\|_* \\ &\quad \leq c\ \nu(f) + |f_{Q''} - f_{Q'}|, \end{aligned} \tag{4.2}$$

where c depends only on K and n. Since $\operatorname{supp} f \subset Q'$ and $Q' \subset Q''$, then

$$\begin{aligned} |f_{Q''} - f_{Q'}| &\leq |f_{Q'}| \\ &\leq |f_{Q'} - f_{Q_0}| + |f_{Q_0}| \\ &\leq c_n\ (1 + \log|Q'|)\,\nu(f). \end{aligned} \tag{4.3}$$

Combining (4.2) (4.3) yields the right estimate in (4.1).

The left estimate in (4.1) is much easier and holds uniformly over all functions $f \in BMO$. Indeed,

$$\begin{aligned} \nu(f) &\le \|f\|_{bmo} + \sup_{Q:|Q|\ge 1} \frac{1}{|Q|}\int_Q |f - f_Q| \\ &\le \|f\|_{bmo} + \sup_{Q:|Q|\ge 1} \frac{2}{|Q|}\int_Q |f| \\ &\le 3\,\|f\|_{bmo}, \end{aligned}$$

as desired. This completes the proof.

□

Combining this lemma with Theorem 4.1 we see that our BMO spaces are in fact dual spaces.

Proposition 4.3 *For every $s \in \mathbf{Z}_+$, it follows that*

$$\begin{aligned} h^s_{loc}(X)' &\overset{top.}{\cong} BMO^{-s}_{comp}(X), \\ h^s_{comp}(X)' &\overset{top.}{\cong} BMO^{-s}_{loc}(X). \end{aligned}$$

Proof. We give the proof only for the first isomorphism; the second isomorphism is proved analogously.

A simple argument from functional analysis (cf. Proposition 1.1.19 in Tarkhanov [20]) shows that the dual space to $h^s_{loc}(X)$ is topologically isomorphic to $(h_{loc}(X)')^{-s}$. The dual space to $h_{loc}(X)$ is nothing other than $BMO_{comp}(X)$, by Theorem 4.1 and Lemma 4.2.

□

Letting *cmo* be the closure of C_{comp} in *bmo*, we see at once, with the help of standard regularization, that $\mathcal{D}$ is dense in *cmo*. The following theorem is a local version of the result of Coifman and Weiss [4] mentioned above.

Theorem 4.4 *h is the dual of cmo. More precisely, each continuous linear functional on cmo has the form $F \mapsto \int fF$ for all $F \in C_{comp}$, where $f \in h$, and $\|f\|_h$ is equivalent to the linear functional norm.*

Proof. This is analogous to the proof of Coifman and Weiss [4, Theorem 4.1] that H^1 is the dual of CMO (the closure of C_{comp} in BMO in the seminorm topology). We merely substitute the atomic decomposition of h given in [7] for that given in [4, Theorem 4.1].

□

We can now proceed analogously to the proof of Proposition 4.3 in describing the dual spaces for our VMO spaces.

Lemma 4.5 *Given any open set $X \subset \mathbb{R}^n$, it follows that*

$$\begin{aligned} VMO_{loc}(X) &\overset{top.}{\cong} cmo_{loc}(X), \\ VMO_{comp}(X) &\overset{top.}{\cong} cmo_{comp}(X). \end{aligned}$$

Proof. This follows immediately from Lemma 4.2 and Proposition 3.6.

□

Proposition 4.6 *For every $s \in \mathbf{Z}_+$, it follows that*

$$\begin{aligned} VMO^s_{loc}(X)' &\overset{top.}{\cong} h^{-s}_{comp}(X), \\ VMO^s_{comp}(X)' &\overset{top.}{\cong} h^{-s}_{loc}(X). \end{aligned}$$

Proof. We give the proof only for the first isomorphism; the second isomorphism is proved analogously.

A simple argument from functional analysis (cf. Proposition 1.1.19 in Tarkhanov [20]) shows that the dual space to $VMO^s_{loc}(X)$ is topologically isomorphic to $(VMO_{loc}(X)')^{-s}$.

The only point remaining concerns the dual space to $VMO_{loc}(X)$, which is $h_{comp}(X)$ by Theorem 4.4 and Lemma 4.5.

□

Given any compact set $K \subset \mathbb{R}^n$, we define the VMO spaces on K within the abstract framework of Section 2. Namely, for an $s \in \mathbf{Z}_+$, we set

$$VMO^s(K) = VMO^s/\overline{\Sigma},$$

where Σ is the subspace of VMO^s consisting of distributions vanishing near K.

As mentioned, we obtain the same quotient if we begin with one of the spaces $VMO^s_{loc}(X)$ and $VMO^s_{comp}(X)$, provided $K \subset X$.

As for the dual space for $VMO^s(K)$, we have the following result.

Proposition 4.7 *Let K be a compact subset of X, and let $s \in \mathbf{Z}_+$. Then*

$$VMO^s(K)' \overset{top.}{\cong} h^{-s}_K(X),$$

where $h^{-s}_K(X)$ is the subspace of $h^{-s}_{comp}(X)$ consisting of distributions supported in K.

Proof. Using the duality theory for normed spaces (see Bourbaki [2, IV.8]), we conclude that $VMO^s(K)'$ is topologically isomorphic to the annihilator of the subspace in VMO^s consisting of the functions which vanish in a neighborhood of K. Since K is closed, this annihilator is just the subspace of $(VMO^s)'$ consisting of the distributions supported on K. To finish the proof, it suffices to invoke Proposition 4.6.

□

5 A boundedness theorem

The key result of this section is that pseudodifferential operators are continuous on the local Hardy and BMO spaces.

Let X be an open set in $\mathbb{R}^n$, and m a nonnegative integer. We recall that a smooth function $p \in C^\infty_{loc}(X \times \mathbb{R}^n)$ is termed a *symbol* in the class $S^m(X)$ if, for all compact K in X and all multi-indices $\alpha, \beta \in \mathbf{Z}^n_+$, there is a constant $c_{K,\alpha,\beta}$ such that

$$\left|D^\alpha_x D^\beta_\xi p(x,\xi)\right| \le c_{K,\alpha,\beta}(1+|\xi|)^{m-|\alpha|-|\beta|}. \tag{5.1}$$

Such a symbol p induces an operator $\mathrm{op}(p)$ on $\mathcal{D}(X)$, defined via the Fourier transform:

$$(\mathrm{op}(p)\, f)(x) = \frac{1}{(2\pi)^n}\int e^{\sqrt{-1}\langle x,\xi\rangle} p(x,\xi)\hat{f}(\xi)\, d\xi \qquad (f \in \mathcal{D}(X)),$$

where $\hat{f}(\xi) = \int e^{-\sqrt{-1}\langle \xi, x\rangle} f(x)\, dx$.

A result of fundamental importance is that $L^2(X)$ is locally invariant under $\mathrm{op}(p)$, for $p \in S^0(X)$. In other words, $\mathrm{op}(p)$ maps $L^2_{comp}(X)$ continuously into $L^2_{loc}(X)$. For the local Hardy space h, an analogous result holds.

Theorem 5.1 *Let* $p \in S^0(X)$*. Then* $\mathrm{op}(p) : h_{comp}(X) \to h_{loc}(X)$ *is continuous.*

This theorem follows readily from the main result of Goldberg [7], which we now state.

Theorem 5.2 ([7]) *If* $q \in S^0(\mathbb{R}^n)$ *satisfies (5.1) with* $c_{K,\alpha,\beta}$ *independent of* K*, then* $\|\mathrm{op}(p)\, f\|_h \le C\, \|f\|_h$.

Proof of Theorem 5.1. The proof is simply a standard adaptation of Goldberg's result to the local setting.

Fix a bounded set B in $h_{comp}(X)$. By definition, $B \subset \varphi\tilde{B}$ for some bounded set $\tilde{B}$ in h and some $\varphi \in \mathcal{D}(X)$. To prove that the image of B under $\mathrm{op}(p)$ is bounded in $h_{loc}(X)$, it suffices to show that the set $(\chi\, \mathrm{op}(p)\, \varphi)\, \tilde{B}$ is bounded in h for each function $\chi \in \mathcal{D}(X)$.

To this end, we denote by q the symbol in $S^0(\mathbb{R}^n)$ such that

$$\mathrm{op}(q) = \chi\, \mathrm{op}(p)\, \varphi.$$

The existence and uniqueness of such a symbol on $\mathbb{R}^n$ is well-known (see, for instance, Taylor [21]). Moreover, q satisfies the estimate (5.1) with $c_{K,\alpha,\beta}$ independent of K.

According to Theorem 5.2, $\mathrm{op}(q)$ is a bounded operator in h. Therefore, the set $(\chi\, \mathrm{op}(p)\, \varphi)\, \tilde{B} = \mathrm{op}(q)\, \tilde{B}$ is bounded in h, as desired.

□

With BMO topologized as a semilocal space via the norm ν, as in Section 2, then an analogous result holds.

Corollary 5.3 *Let $p \in S^0(X)$. Let BMO be topologized as in Section 2. Then* $\mathrm{op}(p): BMO_{comp}(X) \to BMO_{loc}(X)$ *is continuous.*

Proof. Fix a bounded set B in $BMO_{comp}(X)$. By definition, there is a function $\varphi \in \mathcal{D}(X)$ such that $B = \varphi B$. Our goal is to prove, given any $\chi \in \mathcal{D}(X)$, that the set $\chi\,\mathrm{op}(p)\,B = \chi\,\mathrm{op}(p)\,\varphi\,B$ is bounded in BMO under the norm $\nu(\cdot)$.

To this end, we denote by q the symbol in $S^0(\mathbb{R}^n)$ such that $\mathrm{op}(q)$ is the transposed operator to $\chi\,\mathrm{op}(p)\,\varphi$; that such a symbol exists and is unique is a basic fact from pseudodifferential operators (see, for instance, Taylor [21]). Moreover, q satisfies the estimate (5.1) with $c_{K,\alpha,\beta}$ independent of K.

According to Theorem 5.2, $\mathrm{op}(q)$ is a bounded operator in the local Hardy space h. Combining this with Theorem 4.1, we conclude by duality that $\chi\,\mathrm{op}(p)\,\varphi$ is a bounded operator in bmo. By Lemma 4.2, B is a bounded set in bmo_{comp}. Therefore, B is a bounded set in bmo, hence the image of B under $\chi\,\mathrm{op}(p)\,\varphi\,b$ is bounded in bmo. Lemma 4.2 also shows that $bmo \hookrightarrow BMO_{loc}$. (The latter inclusion is strict; the identity function, $f(x) = x$, lies in BMO_{loc}, but not in bmo.) Hence $\chi\,\mathrm{op}(p)\,\varphi\,B$ is a bounded subset of BMO_{loc}, and so bounded in BMO. This proves the corollary.

□

Our next concern will be the behavior of pseudodifferential operators in the the higher-order Hardy and BMO spaces, which we constructed in Sections 2 and 4. To this end, it is convenient to use an abstract framework suggested by O'Farrell [13] and Tarkhanov [19].

Let L be a semilocal space of distributions on an open set $X \subset \mathbb{R}^n$, with continuous embeddings $\mathcal{D}(X) \hookrightarrow L \hookrightarrow \mathcal{D}'(X)$.

Definition 5.4 *The space L is locally invariant under the operator $T: \mathcal{E}'(X) \to \mathcal{D}'(X)$, if T maps L_{comp} continuously into L_{loc}.*

We also say that L is locally invariant under *op* $S^0(X)$ if it is under each operator in *op* $S^0(X)$.

In order to simplify the statements of our next results, we adopt the clever notation of O'Farrell [13] who suggested writing $\int L$ for L^1 and DL for L^{-1}.

Proposition 5.5 *If L is locally invariant under op $S^0(X)$, then so are $\int L$ and DL.*

Proof. Fix $p \in S^0(X)$. For $1 \le j \le n$, the commutator $[\mathrm{op}(p), D_j]$ is a pseudodifferential operator of order 0, and hence maps L_{comp} continuously into L_{loc}. This is enough.

□

Repeated application of Proposition 5.5 gives that if L is locally invariant under *op* $S^0(X)$, then so is L^s for any integer s.

In order to handle pseudodifferential operators of any order, we need the following "fundamental theorem of calculus." This result appears in various guises in the context of partial differential equations and function spaces. We state and prove a specific version.

Theorem 5.6 *If L is locally invariant under* $\operatorname{op} S^0(X)$, *then*

$$L_{comp} = \int DL_{comp} = D \int L_{comp}. \tag{5.2}$$

Proof. For general spaces L, we have $L \subset \int DL$ and $D \int L \subset L$.

To see that $\int DL_{comp} \subset L_{comp}$, fix $f \in \int DL_{comp}$. For $|\alpha| \leq 1$, we have $D^\alpha f = \sum_{|\beta|\leq 1} D^\beta f_\beta^{(\alpha)}$ for some $f_\beta^{(\alpha)}$ in L_{comp}. Note that (from $\alpha = 0$) this implies that f has compact support.

If $G(x)$ is the Newtonian potential in $\mathbb{R}^n$, then, as distributions,

$$f = G * \Delta f = -\sum_{\substack{1\leq j\leq n \\ |\beta|\leq 1}} G * D_j D^\beta f_\beta^{(e_j)}.$$

Here e_j denotes the multi-index with 1 in the j th place, and zeroes elsewhere.

Since $G * D^\alpha$ is in $\operatorname{op} S^0(X)$ provided $|\alpha| \leq 2$, we conclude by assumption that $f \in L_{loc}$, and hence $f \in L_{comp}$, as desired.

To show that $L_{comp} \subset D \int L_{comp}$, pick $f \in L_{comp}$. Let $\varphi \in D(X)$ be identically 1 in a neighborhood of supp f. Writing

$$f = \Delta G * f = \Delta(\varphi G * f) + \Delta((1-\varphi)G * f)$$

and invoking our assumption on L, we have $\Delta(\varphi G * f) \in D \int L_{comp}$, as above, and $\Delta((1-\varphi)G * f) \in \mathcal{D}(X)$. It follows that $f \in D \int L_{comp}$.

This completes the proof.

□

Let us mention two important consequences of the theorem.

Corollary 5.7 *Let $p \in S^m(X)$, for $m \in \mathbb{Z}$. Let h^s be topologized as in Section 4, for $s \in \mathbb{Z}$. Then* $\operatorname{op}(p) : h^s_{comp}(X) \to h^{s-m}_{loc}(X)$ *is continuous.*

Proof. Given any nonnegative integer m_2, we are able to write $\operatorname{op}(p)$ in the form

$$\operatorname{op}(p) = p_2(x, D) \circ \operatorname{op}(p_1) \quad \textit{modulo smoothing operators},$$

$p_2(x, D)$ being a differential operator of order m_2 and p_1 being in $S^{m-m_2}(X)$. This reduces the proof to the case of $m < 0$, because for differential operators the result follows from Proposition 2.7.

Suppose now that $m < 0$. Given any multi-index α with $|\alpha| \leq -m$, the composition

$$\operatorname{op}(p) \circ D^\alpha$$

is a pseudodifferential operator of order 0. Therefore, $\operatorname{op}(p) \circ D^\alpha$ maps $h^{s-m}_{comp}(X)$ continuously into $h^{s-m}_{loc}(X)$, as follows from Proposition 5.5 and Theorem 5.1. Theorem 5.6 now shows that $\operatorname{op}(p)$ maps $h^s_{comp}(X)$ continuously into $h^{s-m}_{loc}(X)$, as desired.

□

Corollary 5.8 *Let $p \in op\,S^m(X)$, for $m \in \mathbf{Z}$. Let BMO^s be topologized as in Section 2, for $s \in \mathbf{Z}$. Then* $\mathrm{op}(p) : BMO^s_{comp}(X) \to BMO^{s-m}_{loc}(X)$ *is continuous.*

Proof. We apply the argument of the proof of Corollary 5.7 again, with Theorem 5.1 replaced by Corollary 5.3.

□

In addition, since $\mathcal{D}(X)$ is dense in $VMO^s_{loc}(X)$, then the boundedness result extends to VMO spaces, as well.

Corollary 5.9 *Let $p \in op\,S^m(X)$, for $m \in \mathbf{Z}$. Let VMO^s be topologized as in Section 3, for $s \in \mathbf{Z}$. Then* $\mathrm{op}(p) : VMO^s_{comp}(X) \to VMO^{s-m}_{loc}(X)$ *is continuous.*

Proof. Proposition 3.6 shows that $VMO^s_{comp}(X)$ is the closure of $\mathcal{D}(X)$ in $BMO^s_{comp}(X)$, and likewise for $VMO^s_{loc}(X)$. To finish the proof, use Corollary 5.8 and the fact that $C^\infty_{loc}(X)$ is locally invariant under pseudodifferential operators.

□

6 Higher order approximation

Let P be an elliptic differential operator of order p and $\mathrm{sol}(K)$ the space of smooth solutions to $Pf = 0$ near the compact set K, as described in Section 1.

What are the necessary conditions in order that a function $f \in BMO^s(K)$ be approximable with arbitrary degree of accuracy in the topology of this space by elements of $\mathrm{sol}(K)$? First of all, since $BMO^s(K)$ is continuously embedded in $\mathcal{D}'(\mathring{K})$, the condition $f \in \mathrm{sol}(\mathring{K})$ is necessary. Moreover, if $s \geq p$, then Pf must vanish along with its derivatives up to order $s - p$ on K. However, since $\mathcal{E}(K)$ is not dense in $BMO^s(K)$, there is an additional necessary condition, namely, that $f \in VMO^s(K)$.

The problem of approximation in $BMO^s(K)$ by solutions of the equation $Pf = 0$ is, in the first instance, the problem of describing those compact sets $K \subset X$ for which the conditions mentioned above are also sufficient. A fundamental step in the study of this problem is an adequate description of the annihilator of the subspace $\mathrm{sol}(K)$ in $VMO^s(K)$. For this we need the space $VMO^s_{loc}(X)$ for integral negative s, as it was defined in Section 3. The fact that the space $VMO_{loc}(X)$ is locally invariant under pseudodifferential operators of order 0 (see Corollary 5.9) is crucial here.

Lemma 6.1 *The transpose mapping $P' : h^{p-s}_{comp}(X) \to h^{-s}_{comp}(X)$ defines a (topological) isomorphism of $h^{p-s}_K(X)$, the subspace in $h^{p-s}_{comp}(X)$ consisting of distributions with supports in K, onto $sol(K)^\perp$, the annihilator of the subspace* $\mathrm{sol}(K)$ *in $VMO^s(K)$.*

Proof. If $v \in h^{p-s}_{comp}(X)$ is supported on K, then $P'v \in h^{-s}_{comp}(X)$ is supported on K, too. According to Proposition 4.7, $P'v$ can be considered as a continuous linear functional on $VMO^s(K)$. Moreover, we have

$$\langle P'v, f\rangle = \langle v, Pf\rangle = 0$$

for all $f \in \mathrm{sol}(K)$, and so $P'v \in \mathrm{sol}(K)^\perp$. Therefore, P' actually maps the subspace $h^{p-s}_K(X)$ continuously into $\mathrm{sol}(K)^\perp$. This mapping is certainly injective, because P is analytically hypoelliptic. It remains to prove that this mapping is surjective, and that the inverse mapping is continuous.

Let Φ be a fundamental solution of P. This is a pseudodifferential operator of order $-p$ on X, and its kernel $\Phi(x,y) \in \mathcal{D}'(X \times X)$ is a real analytic function away from the diagonal of $X \times X$.

According to Proposition 4.7, any continuous linear functional g on $VMO^s(K)$ can be identified with a distribution in $h^{-s}_K(X)$. Pick, then, any $g \in h^{-s}_K(X)$ that vanishes on $\mathrm{sol}(K)$. We see at once that $g = P'v$, where $v = \Phi'(g)$.

By Corollary 5.7, the operator

$$\Phi' : \; h^{-s}_{comp}(X) \to h^{p-s}_{loc}(X)$$

is continuous. Therefore, $v \in h^{p-s}_{loc}(X)$.

Moreover, since $P(x,D)\Phi(x,y) = \delta(x-y)$, the function $\Phi(\cdot,y)$ belongs to $\mathrm{sol}(K)$ for any fixed $y \in X \setminus K$. It follows that $v(y) = \langle g, \Phi(\cdot,y)\rangle = 0$ for $y \notin K$, and so $\mathrm{supp}\, v \subset K$. This is the desired conclusion.

□

With the help of Lemma 6.1 and the Hahn-Banach Theorem, it is now simple to describe the closure of the subspace $\mathrm{sol}(K)$ in $BMO^s(K)$ for $s \geq p$. For any compact $K \subset X$, it turns out that the necessary local conditions on $f \in BMO^s(K)$ at the beginning of this section also suffice for this function to be in the closure of $\mathrm{sol}(K)$ in $BMO^s(K)$.

Theorem 6.2 *Let $s \geq p$. A function $f \in BMO^s(K)$ is in the closure of* $\mathrm{sol}(K)$ *if and only if $f \in VMO^s(K)$ and $Pf = 0$ in $VMO^{s-p}(K)$.*

Proof. *Necessity.* Assume that $f \in BMO^s_{loc}(X)$ is approximable in the norm of $BMO^s(K)$ by elements of $\mathrm{sol}(K)$.

A simple argument shows that there is a Cauchy sequence $\{F_\nu\}$ in $BMO^s_{loc}(X)$ with the following properties:

- each F_ν is in $C^\infty_{loc}(X)$ (even in $\mathcal{D}(X)$) and satisfies $PF_\nu = 0$ near K;
- there exist suitable functions $\varphi_\nu \in BMO^s_{loc}(X)$ vanishing near K such that $f - F_\nu - \varphi_\nu \to 0$ as $\nu \to \infty$.

Denote by F the limit of $\{F_\nu\}$ in $BMO^s_{loc}(X)$. Proposition 3.6 implies that F is actually in $VMO^s_{loc}(X)$.

Our next claim is that f and F determine the same element in $BMO^s(K)$. Indeed, the difference

$$f - F = (f - F_\nu - \varphi_\nu) - (F - F_\nu) + \varphi_\nu$$

can be approximated in $BMO^s_{loc}(X)$ by the functions φ_ν vanishing near K. It follows that $f \in VMO^s(K)$.

Finally, since

$$Pf = P(f - F_\nu - \varphi_\nu) + P(F_\nu + \varphi_\nu),$$

then Pf can be approximated in $BMO^{s-p}_{loc}(X)$ by the functions $\{P(F_\nu + \varphi_\nu)\}$ vanishing near K. Hence the image of Pf in $BMO^{s-p}(K)$ is zero, which completes the proof of the necessity.

Sufficiency. Choose a continuous linear functional g on $VMO^s(K)$ equal to zero on sol(K). By Lemma 6.1, $g = P'v$ for some distribution $v \in h^{p-s}_K(X)$. We have $s \geq p$, and so, in view of Proposition 4.7, we may consider v as a continuous linear functional on $VMO^{s-p}(K)$. Thus, if $f \in VMO^s(K)$ and $Pf = 0$ in $VMO^{s-p}(K)$, then by the *transposition rule*,

$$\langle g, f \rangle = \langle v, Pf \rangle = 0.$$

The Hahn-Banach Theorem finishes the proof.

□

Note that whether or not the condition "$Pf = 0$ in $VMO^{s-p}(K)$" is equivalent to "$D^\alpha(Pf)|_K = 0$ for all $|\alpha| \leq s - p$" (the trace of $D^\alpha(Pf)$ on K being understood in some reasonable sense) is unknown. This question relates to "spectral synthesis" in the VMO spaces (cf. Hedberg and Wolff [10]).

7 Lower order approximation

On the other hand, if $0 < s < p$, then the necessary conditions at the beginning of Section 6 on the function $f \in BMO^s(K)$ are, in general, not sufficient.

Example 7.1 Gauthier and Tarkhanov [6, Example 4.1] constructed, for any $r = 1, 2, \ldots$ and $r - n < \delta \leq r - 1$, a compact set $K \subset X$ and a function v in $W^{r, n/(r-\delta)}(\mathbb{R}^n)$ with support on K such that v does not belong to the closure of $\mathcal{D}(\overset{\circ}{K})$ in $W^{\delta+1, n/(n-\delta)}(\mathbb{R}^n)$. The compact set K was constructed as a slight modification of Hedberg's example [9, p.77]. Supposing $n > 2$, $p \geq n$ and $s \leq p - n$, we apply this construction with $r = p - s$ and $\delta = r - n + 1$. Fix any $q > n$. By Theorem 4.2 of [6], there are measures m_α ($|\alpha| \leq \delta$) supported on the boundary of K such that the potential

$$f = \Phi\left(\sum_{|\alpha| \leq \delta} D^\alpha m_\alpha\right)$$

does not belong to the closure of sol(K) in $W^{s,q}(K)$. Note that this potential is in $W^{p-\delta-1,q}_{loc}(X)$ and satisfies $Pf = 0$ in the interior of K. Now we observe, by the Sobolev Embedding Theorem, that

$$W^{p-\delta-1,q}_{loc}(X) \hookrightarrow C^{p-\delta-2}_{loc}(X) \hookrightarrow VMO^{p-\delta-2}_{loc}(X).$$

Since $p - \delta - 2 \geq s$, it follows that the potential f belongs to $VMO^s(K) \cap sol(\overset{\circ}{K})$. However this potential cannot be approximated in $BMO^s(K)$ by solutions of sol(K) because otherwise it would be approximated also in $W^{s,q}(K)$ by solutions of sol(K). □

A result of Mateu and Verdera [12] shows that the bounds on n are sharp in Example 7.1.

Thus, it remains an open problem to describe those compact sets $K \subset X$ for which the above mentioned (necessary) local conditions on $f \in BMO^s(K)$ are also sufficient. The next result says that this problem is equivalent to a certain problem on the density of functions with compact support in $\overset{\circ}{K}$ in the space $h^{p-s}_K(X)$. The importance of this result is that it gives a criterion which is independent of the differential operator P.

Theorem 7.2 *For $0 \leq s < p$, the following conditions on the compact set K are equivalent:*

(1) sol(K) *is dense in* $VMO^s(K) \cap$ sol$(\overset{\circ}{K})$ *in the* $BMO^s(K)$*-norm;*

(2) $\mathcal{D}(\overset{\circ}{K})$ *is dense in* $h^{p-s}_K(X)$ *in the weak-* topology of the space* $VMO^{s-p}_{loc}(X)'$.

Proof.

(1) ⇒ (2). In view of Lemma 6.1 it is sufficient to show that if condition (1) holds, then $P'\mathcal{D}(\overset{\circ}{K})$ is dense in sol$(K)^\perp$ in the weak-* topology of the space dual to $VMO^s_{loc}(X)$. As above, sol$(K)^\perp$ denotes the annihilator of the subspace sol(K) in $VMO^s(K)$.

By the Hahn-Banach Theorem, $P'\mathcal{D}(\overset{\circ}{K})$ is dense in sol$(K)^\perp$ in the weak-* topology of $VMO^s_{loc}(X)'$, provided that each linear functional that is continuous in the topology on $VMO^s_{loc}(X)'$ and that vanishes on $P'\mathcal{D}(\overset{\circ}{K})$ also vanishes on sol$(K)^\perp$.

Let f be such a functional. Then $f \in VMO^s_{loc}(X)$, and $Pf = 0$ weakly on $\overset{\circ}{K}$. By condition (1), there is a sequence $\{f_\nu\}$ in sol(K) that tends to f in the $BMO^s(K)$-norm. Thus, if $g \in$ sol$(K)^\perp$, then

$$\langle g, f\rangle = \lim_{\nu\to\infty} \langle g, f_\nu\rangle = 0,$$

as desired.

(2) ⇒ (1). By the Hahn-Banach Theorem, it is sufficient to show that if condition (2) holds, then each continuous linear functional on $VMO^s(K)$, vanishing on sol(K), also vanishes on $VMO^s(K) \cap$ sol$(\overset{\circ}{K})$.

Let g be such a functional, i.e., $g \in \mathrm{sol}(K)^{\perp}$. By Lemma 6.1, there is a function $v \in h_K^{p-s}(X)$ such that $g = P'v$. Thus, if $f \in VMO^s_{loc}(X) \cap \mathrm{sol}(\mathring{K})$, then $\langle g, f\rangle = \langle v, Pf\rangle$. Now we can invoke condition (2) and for every $\varepsilon > 0$ find a function $v_\varepsilon \in \mathcal{D}(\mathring{K})$ such that

$$|\langle v - v_\varepsilon, Pf\rangle| < \varepsilon.$$

But $\langle v_\varepsilon, Pf\rangle = 0$, from which we conclude that $\langle g, f\rangle = 0$.

□

The results of Adams [1] show that functions in $h^{p-s}_{loc}(X)$ can be well defined except for sets of zero d-dimensional Hausdorff measure, where $d = n - p + s$. When so "strictly defined," they consequently enjoy some continuity properties measured by the d-dimensional Hausdorff content $\Lambda_d^{(\infty)}$.

8 Approximation on nowhere dense compact sets

We begin with an example.

Example 8.1 Polking [14, Theorem 4] constructed, for any real number $1 < r < \infty$, a nowhere dense compact set $K \subset X$ of positive Lebesgue measure and a nonzero bounded function in $W^{r,n/r}_{loc}(X)$ which is supported by K. The compact set K was constructed as a modification of the standard Sierpinski curve or "Swiss Cheese" in $\mathbb{R}^2$. (The term "Swiss Cheese" is traditionally applied to any compact set K obtained by removing from the closed unit disc an infinite sequence $\{B_\nu\}$ of disjoint open discs such that $\cup_\nu B_\nu$ is dense in the unit disc.) Supposing $s > p - n$, we apply this construction with $r = p - s$ and obtain a compact set K in X of positive Lebesgue measure for which $W_K^{p-s,n/(p-s)}(X) \neq \{0\}$. Fix a $1 < q < \infty$ large enough so that $s \geq (p - n) + n/q$. By Theorem 3.2 of [6], there exists a function $F \in C^\infty_{loc}(X)$ such that the potential

$$f = \Phi(\chi_K F)$$

does not belong to the closure of $\mathrm{sol}(K)$ in $W^{s,q}(K)$. Note that this potential is in $C^{p-1}_{loc}(X)$, and so in $VMO^{p-1}_{loc}(X)$. All the more, f does not belong to the closure of $\mathrm{sol}(K)$ in $BMO^s(K)$, because the topology of the latter space is stronger than the topology of $W^{s,q}(K)$.

□

For compact sets K of zero measure, such an example is impossible. The theorem of Hartogs and Rosenthal [8] can be also placed in the context of approximation in BMO spaces. A heuristic explanation of this fact is that if K has measure zero, then the complement of K is "massive" in the sense of any reasonable capacity.

Theorem 8.2 *Let K be a compact set of zero measure in X. Then, for any $0 \leq s < p$, the subspace* $\mathrm{sol}(K)$ *is dense in $VMO^s(K)$ in the $BMO^s(K)$-norm.*

Proof. The proof of the fact is essentially the same as that of Theorem 6.3.1 in Tarkhanov [20].

□

Let us now turn to some consequences of Theorem 7.2.

Theorem 8.3 *Let K be a compact subset of X with empty interior. Let $0 \leq s < p$. Then the subspace* sol(K) *is dense in $VMO^s(K)$ in the $BMO^s(K)$-norm if and only if $h_K^{p-s}(X) = \{0\}$.*

Proof. This immediately follows from Theorem 7.2, because for nowhere dense compact sets K the condition (2) of Theorem 7.2 becomes

$$h_K^{p-s}(X) = \{0\}.$$

□

The next result is a straightforward consequence of Theorem 8.3.

Corollary 8.4 *Let K be a compact subset of X with empty interior. Then, for any $0 \leq s \leq p - n$, the subspace* sol(K) *is dense in $VMO^s(K)$ in the $BMO^s(K)$-norm.*

Proof. If $p - s \geq n$, then

$$h_{loc}^{p-s}(X) \hookrightarrow W_{loc}^{n,1}(X) \hookrightarrow C_{loc}(X)$$

by the Sobolev Embedding Theorem. Hence the functions in $h_{loc}^{p-s}(X)$ are continuous provided that $p - s \geq n$.

Thus, for $0 \leq s \leq p - n$, we have $h_K^{p-s}(X) = \{0\}$. Indeed, if $g \in h_{loc}^{p-s}(X)$ is supported in K, then $g \equiv 0$, because it is continuous and $X \setminus K$ is dense in X.

Theorem 8.3 then completes the proof.

□

As follows from Example 8.1, the bounds on s here are sharp.

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