"Applications of Calabi-Yau metric to the Weil-Petersson metric and the Teichmüller space of complex manifold with $c_1 \equiv 0$"

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Introduction. It is a well-known fact that if X is a compact complex simply connected Kähler manifold with $c_1(X) = 0$, then

$$X = \prod X_j \times \prod Y_i$$

where

a) for each j $\dim H^0(X_j, \Omega^2) = 1$ and if φ_j is a non-zero holomorphic two form on X_j and at each point $x \in X_j$, φ_j is a non-degenerate form, i.e. $\underbrace{\varphi_j \wedge \ldots \wedge \varphi_j}_{n} \neq 0$ is a holomorphic $2n = \dim_{\mathbb{C}} X$ form which has no zeroes and no poles.

b) for each i and $0 < p < \dim_{\mathbb{C}} Y_i = n$ $\dim H^0(Y_i, \Omega^p) = 0$ and $\dim H^0(Y_i, \Omega^n) = 1$ and it is spanned by a holomorphic n-form without zeroes and poles. Such manifolds will be called Calabi-Yau manifolds if $n \geq 3$.

This fact is due to Calabi and Bogolmolov. See [2]. An elegant proof based on Yau's solution and Calabi conjecture was given by M.L. Michelson in [12].

The aim of this article is to study the deformations of the Calabi-Yau manifold X and namely the so called Teichmüller space $\mathcal{T}(X)$ of X which is defined in the

§1. Preliminaries

Definition 1.1. Let X be a compact complex manifold with a holonomy group $SU(n)$ $(n = \dim_{\mathbb{C}} X)$ then X will be called a Calabi-Yau manifold.

Theorem 1.2. Let X be a Calabi-Yau manifold, then

a) X has a Ricci flat Kähler metric

b) $\dim_{\mathbb{C}} H^{\circ}(X,\Omega^p) = 0$ for $0 < p < n = \dim_{\mathbb{C}} X$ and $H^{\circ}(X,\Omega^n)$ is generated by a holomorphic form $w_X(n,0) \neq 0$ which has no zeroes and no poles on X.

Proof: a) Since the holonomy group is $SU(n)$ we get immediately that on X there is a Kähler, Ricci flat metric. For more details see [10].

b) For the proof of this fact see [2] and [12].

Q.E.D.

Cor. 1.2.1. Every Calabi-Yau manifold X is a projective algebraic manifold.

Proof: From theorem 1.2. $\Rightarrow$ X is a Kähler manifold. See [10] Kodaira proved that if X is a Kähler manifold and $H^2(X,\mathcal{O}_X) = 0$, then X is a projective manifold. Since X is Calabi-Yau manifold, then $H^{\circ}(X,\Omega^2) = H^2(X,\mathcal{O}_X) = 0$ and so 1.2.1. is proved. See [11].

Q.E.D.

Definition 1.3. Let $(g_{\alpha\overline{\beta}})$ be a Kähler metric on X such that

$$\text{Ricci } (g) = \partial\overline{\partial}\log \det (g_{\alpha\overline{\beta}}) \equiv 0$$

then $(g_{\alpha\overline{\beta}})$ will be called a Calabi-Yau metric.

Remark. Since on a Calabi-Yau manifold X there is a holomorphic n form $(n = \dim_{\mathbb{C}} X)$ which has no zeroes and no poles, it follows that $c_1(X)_{\mathbb{Z}} \equiv 0$. On each Kähler manifold X with $c_1(X)_{\mathbb{Z}} = 0$, it follows from the solution of the Calabi conjecture by Yau, that on X we can find a Calabi-Yau metric. From now on we will fix a Calabi-Yau metric $(g_{\alpha\overline{\beta}})$ on X [18].

Lemma 1.4.

$\omega_X(n,0)$ is a parrallel form with respect to the connection induced by the Calabi-Yau metric $(g_{\alpha\overline{\beta}})$, i.e.

$$\nabla\omega_X(n,0) = 0$$

where ∇ is the covariant differentiation, induced by the Calabi-Yau metric.

Proof: In [11] the following formula is proved:

$$(\square\varphi)_{\alpha_1,\dots,\alpha_p,\overline{\beta}_1,\dots,\overline{\beta}_q} = -g^{\alpha\overline{\beta}}\,\overline{\nabla}_\beta\nabla_\alpha\varphi_{\alpha_1,\dots,\alpha_p,\overline{\beta}_1,\dots,\overline{\beta}_q}$$

$$+\sum_{\mu=1}^{p}\sum_{\nu=1}^{q} R^{\kappa,\overline{\ell}}_{\alpha_\mu\;\;\overline{\beta}_\nu}\,\varphi_{\alpha_1,\dots,\underset{\mu}{\underset{\uparrow}{\kappa}},\dots\alpha_p,\overline{\beta}_1,\dots,\underset{\nu}{\underset{\uparrow}{\overline{\ell}}},\dots,\overline{\beta}_q}$$

$$- \sum R^{\overline{\ell}}{}_{\overline{\beta}_\nu} \varphi_{\alpha_1,\ldots,\alpha_p,\overline{\beta}_1,\ldots,\underset{\substack{\uparrow\\ \nu}}{\overline{\ell}_1},\ldots,\overline{\beta}_q}$$

Let us put in this formula $\varphi = \omega_X(n,0)$. Since on Kähler manifold every holomorphic form is harmonic one we get that

$$\square\varphi = 0 .$$

On the other hand since $(g_{\alpha\overline{\beta}})$ is a Ricci flat metric, so $R^{\overline{\ell}}{}_{\overline{\beta}_\nu} \equiv 0$ and since $\overline{\beta}_1 = \ldots = \overline{\beta}_q = 0$ we get that

$$\square\omega_X(n,0) = -g^{\alpha\overline{\beta}}\overline{\nabla}_\beta\nabla_\alpha\omega_X(n,0) = 0$$

$$<\square\omega_X(n,0),\omega_X(n,0)> = <-g^{\alpha\overline{\beta}}\overline{\nabla}_\beta\nabla_\alpha\omega_X(n,0),\omega_X(n,0)> = 0$$

On the other hand it is not difficult to prove that

$$<-g^{\alpha\overline{\beta}}\nabla_{\overline{\beta}}\nabla_\alpha\omega_X(n,0),\omega_X(n,0)> = \sum_{\alpha=1}^{n} ||\nabla_\alpha\omega_X(n,0)||^2 \quad . \text{(See [11])}$$

So since

$$<\square\omega_X(n,0),\omega_X(n,0)> = 0 \quad \Rightarrow$$

$$\sum_{\alpha=1}^{n} ||\nabla_\alpha\,\omega_X(n,0)||^2 = 0 \quad .$$

From the last equality we conclude that

$$\nabla\omega_X(n,0) = 0$$

i.e. $\omega_X(n,0)$ is a parallel tensor with respect to the Calabi-Yau metric.

Q.E.D.

1.5. Let $\varphi \in H^1(X,\Theta)$ and φ be a harmonic representative with respect to the Calabi-Yau metric $(g_{\alpha\overline{\beta}})$, so locally φ can be written in the following way:

$$(1.5.1) \qquad \varphi|_U = \Sigma\, \varphi^{\alpha}_{\overline{\beta}}\, d\overline{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}}$$

where

$$\overline{\partial}\varphi = \overline{\partial}^*\varphi = 0, \quad \overline{\partial}^* \text{ is the conjugate of } \overline{\partial}$$

with respect to $(g_{\alpha\overline{\beta}})$ and $\overline{\partial}^* = -*\nabla *, \nabla$ is the connection of $g_{\alpha\overline{\beta}}$ on Θ.

From (1.5.1.) it follows that

$$(1.5.2) \qquad \varphi \in \Gamma(X, \mathrm{Hom}((T^*)^{1,0}, \overline{(T^*)^{1,0}}) \quad .$$

<u>Definition 1.5.3.</u> By $\Lambda^K\varphi \in \Gamma(X, \mathrm{Hom}(\Lambda^K((T^*)^{1,0}), \overline{\Lambda^*((T^*)^{1,0})}))$ we will denote the endomorphism given by

$$(\Lambda^K\varphi)(\ell_{i_1} \wedge \ldots \wedge \ell_{i_K}) = \varphi(\ell_{i_1}) \wedge \ldots \wedge \varphi(\ell_{i_K}) \quad .$$

<u>§2. The main lemma.</u>

<u>Lemma.</u> Let $0 \neq [\varphi] \in H^1(X,\Theta)$ and $\varphi = \sum \varphi^{\alpha}_{\overline{\beta}}\, d\overline{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}}$ be the harmonic representative of $[\varphi]$ with respect to the Calabi-Yau metric that we fixed, then

$$(\Lambda^k\varphi) \lrcorner\ \omega_X(n,0)$$

is a harmonic form of type $(k,n-k)$ with respect to $(g_{\alpha\overline{\beta}})$.

Proof: The proof of this lemma is based on

Proposition 2.1. If $\varphi \in H^1(X,\Theta)$ is a harmonic representative of a Dolbault class, then $\Lambda^k\varphi$ is a harmonic representative in $H^k(X,\Lambda^k\Theta)$.

Proof: It is a well-known fact that

$$\Box_{\overline{\partial}}(\Lambda^k\varphi) = 0 \Leftrightarrow \overline{\partial}(\Lambda^k\varphi) = 0 \quad \text{and} \quad \overline{\partial}^*(\Lambda^k\varphi) = 0$$

where $\Box_{\overline{\partial}}$ is the Laplace-Beltrami on $\Omega^{k,0} \otimes \Lambda^k\Theta$ defined by the Calabi-Yau metric $(g_{\alpha\overline{\beta}})$, $\overline{\partial}^*$ is the adjoint operator of $\overline{\partial}$.

From the definition of $\Lambda^k\varphi$ it follows that locally $\Lambda^k\varphi$ can be written in the following manner:

$$\Lambda^k\varphi|_U = \sum_{\alpha_1<\ldots<\alpha_K} (\varphi^{\alpha_1}\wedge\ldots\wedge\varphi^{\alpha_k}) \otimes \left(\frac{\partial}{\partial z^{\alpha_1}}\wedge\ldots\wedge\frac{\partial}{\partial z^{\alpha_K}}\right)$$

where

$$\varphi^{\alpha_i}|_U = \sum_{\overline{\beta}} \varphi_{\overline{\beta}}^{\alpha_i} d\overline{z}^{\beta}$$

are the components of φ. Since $\bar\partial\varphi = 0 \Rightarrow \forall_\alpha \ \bar\partial\varphi^\alpha = 0$ we get that

$$\bar\partial(\Lambda^k\varphi) = 0.$$

Next we need to prove that

$$(2.1.1.) \qquad \bar\partial^*(\Lambda^k\varphi) = 0 \quad \text{if} \quad \bar\partial^*\varphi = 0 .$$

In order to prove (2.1.1.) we will use the following relation which is true in the case of a Kähler manifold:

$$-i\bar\partial^* = [\Lambda,\nabla] = \Lambda\nabla - \nabla\Lambda$$

where

$$\Lambda\varphi := \left(\sum g^{i\bar j} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial \bar z^j}\right) \lrcorner\, \varphi$$

and ∇ is the covariant derivative on Θ induced by the Calabi-Yau metric. For the proof of this relation see [Siu]. From the definition of Λ and since $\Lambda^k\varphi$ is a form of type $(0,k)$ with values in $\Lambda^k\Theta$ we get that

$$\Lambda(\Lambda^k\varphi) = 0$$

so

$$-i\bar\partial^* = [\Lambda,\nabla]\ (\Lambda^k\varphi) = \Lambda[\nabla(\Lambda^k\varphi)]$$

We need to prove that:

$$(2.1.2.) \qquad \Lambda[\nabla(\Lambda^k\varphi)] = 0$$

Proof of (2.1.2): From the fact that $(g_{\alpha\overline{\beta}})$ is a Kähler metric it follows that at each point x_0 we can choose the local coordinates $(z^1,\ldots,z^n)$ such that

$$(*) \qquad g_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} + O(|z|^2)\ , \text{ where } \delta_{\alpha\overline{\beta}} = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

and $\nabla = \partial$ at the point x_0. For the proof of this fact see [7]. From (*) it follows that

$$\nabla(\Lambda^k\varphi)(x_0) = \partial(\Lambda^k\varphi)(x_0)\ .$$

So if we prove that

$$(**) \qquad \Lambda[\nabla(\Lambda^k\varphi)](x_0) = \Lambda[\partial(\Lambda^k\varphi)]\ (x_0) = 0$$

then (2.1.2.) will be proved. Next we need to prove (**). First we need to compute

$$\partial(\Lambda^k\varphi)\ (x_o)$$

$$\partial(\Lambda^k\varphi)(x_o) = \sum_{\alpha_1<\ldots<\alpha_K} \partial(\varphi^{\alpha_1}\times \ldots \times \varphi^{\alpha_K}) \otimes \frac{\partial}{\partial z^{\alpha_1}}\wedge\ldots\wedge\frac{\partial}{\partial z^{\alpha_k}} =$$

$$= \sum_{\alpha_1<\ldots<\alpha_k} \left(\sum_i (-1)^i \varphi^{\alpha_1}\times\ldots\times \partial\varphi^{\alpha_i}\wedge\ldots\wedge\varphi^{\alpha_k}\right) \otimes \frac{\partial}{\partial z^{\alpha_1}}\wedge\ldots\wedge\frac{\partial}{\partial z^{\alpha_k}}$$

Remember that φ^{α_1} is a (0,1) form and it is a component of $\varphi \in H^1(X,\Theta)$. We need to prove

$$(***)\ \Lambda(\varphi^{\alpha_1}\wedge\ldots\wedge\partial\varphi^{\alpha_i}\wedge\ldots\wedge\varphi^{\alpha_k})\ (x_0) = (\varphi^{\alpha_1}\wedge\ldots\wedge(\Lambda\partial\varphi^{\alpha_i})\wedge\ldots\wedge\varphi^{\alpha_k})(x_0)$$

then clearly

$$\overline{\partial}^*(\Lambda^k\varphi) = 0.$$

So we need to show that (***) is true for each point x_0 .

Proof of (***): Since X is a Kähler manifold we can find local coordinates in X_0 $(z^1,\dots z^n)$ such that

1) $\{dz^i\}$ is an orthonormal basis in $(T^{1,0}_{x_0})$.

2) $g_{\alpha\bar\beta}(z) = \delta_{\alpha\bar\beta} + O(|z|^2)$ at x_0 .

See [7].

The condition 2) means that at the point x_0 we have $D = \partial$.

Since φ^{α_i} is a $\bar\partial$-closed form, it follows from Dalbault's lemma that there exists a function F^{α_i} in a neighborhood of x_0 such that

$$\varphi^{\alpha_i} = \bar\partial F^{\alpha_i} = \sum \bar\partial_\beta F^{\alpha_i} d\bar z^\beta .$$

Let $F^{\alpha_i} = \operatorname{Re} F^{\alpha_i} + i \operatorname{Im} F^{\alpha}$, where $\operatorname{Re} F^{\alpha_i}$ and $\operatorname{Im} F^{\alpha_i}$ are real C^∞ functions. Clearly

$$\varphi^{\alpha_i} = \bar\partial \operatorname{Re} F^{\alpha_i} + i\bar\partial \operatorname{Im} F^{\alpha_i} .$$

From here we get that

$$(\varphi^{\alpha_1}\wedge\dots\wedge\partial\varphi^{\alpha_i}\wedge\dots\wedge\varphi^{\alpha_K})(x_0) = (\varphi^{\alpha_1}\wedge\dots\wedge(\partial\bar\partial \operatorname{Re} F^{\alpha_i})\wedge\dots\wedge\varphi^{\alpha_K})(x_0) +$$
$$+ [\varphi^{\alpha_1}\wedge\dots\wedge(i\partial\bar\partial \operatorname{Im} F^{\alpha_i})\wedge\dots\wedge\varphi^{\alpha_K}](x_0) .$$

Next we will prove that:

$$(****)\text{a)}\ \Lambda(\varphi^{\alpha_1}\wedge\ldots\wedge(\partial\bar\partial \operatorname{Re} F^{\alpha_i})\wedge\ldots\wedge\varphi^{\alpha_K})(x_0)=(\varphi^{\alpha_1}\wedge\ldots\wedge[\Lambda(\partial\bar\partial \operatorname{Re} F^{\alpha_i})]\wedge\ldots\wedge\varphi^{\alpha_K})(x_0)$$

$$\text{b)}\ \Lambda(\varphi^{\alpha_1}\wedge\ldots\wedge(i\partial\bar\partial \operatorname{Im} F^{\alpha_i})\wedge\ldots\wedge\varphi^{\alpha_K})(x_0)=(\varphi^{\alpha_1}\wedge\ldots\wedge[\Lambda(i\partial\bar\partial \operatorname{Im} F^{\alpha_i})]\wedge\ldots\wedge\varphi^{\alpha_K})(x_0)$$

<u>Proof of (****)</u>: Since $\operatorname{Re} F^{\alpha_i}$ is a real function, then the matrix $A = (\partial_\alpha\bar\partial_\beta \operatorname{Re} F^{\alpha_i})$ is such that $\bar A^t = A$. From the standart fact of linear algebra we can make an orthogonal change of the orthogonal basis in $(T^{1,0}_{x_0})^*$ such that $UA\bar U^t$ will be a diagonal matrix, where $U \in U(n)$. This means that we can find orthogonal basis $(dz^1,\ldots dz^u)$ of $(T^{1,0}_{x_0})^*$ such that

$$\partial\bar\partial \operatorname{Re} F^{\alpha_i} = \sum \varphi^{\alpha_i}_{\beta\bar\beta}\, dz^\beta\wedge d\bar z^\beta .$$

Since in this basis $i\sum g_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta = \sum dz^\beta\wedge d\bar z^\beta$, the definition of the operator Λ, and the fact that $\varphi^{\alpha_1},\ldots,\varphi^{\alpha_K}$ are forms of type (0,1) we get that

$$\Lambda(\varphi^{\alpha_1}\wedge\ldots\wedge(\sum_{\beta\bar\beta}\varphi^{\alpha_i}_{\beta\bar\beta} dz^\beta\wedge d\bar z^\beta)\wedge\ldots\wedge\varphi^{\alpha_K}) = (\sum\frac{\partial}{\partial z^\beta}\wedge\frac{\partial}{\partial\bar z^\beta}) \lrcorner\ (\varphi^{\alpha_1}\wedge\ldots\wedge$$

$$(\sum_{p\bar p}\varphi^{\alpha_i}_{p\bar p} dz^p\wedge d\bar z^p)\wedge\ldots\wedge\varphi^{\alpha_K}) := (\varphi^{\alpha_1}\wedge\ldots\wedge[\Lambda(\sum_{\beta\bar\beta}\varphi^{\alpha_i}_{\beta\bar\beta} dz^\beta\wedge d\bar z^\beta)]\wedge\ldots\wedge\varphi^{\alpha_K}) .$$

So (****)a) is proved. Repeating the same arguments we get (****)b). So (****) is proved.

Q.E.D.

From (****)a) and b) we get (***). Since for each i φ^{α_i} is a harmonic form we get that

$$\bar{\partial}^*(\Lambda^K \varphi) = 0 .$$

So $\Lambda^K \varphi$ is a harmonic form.

Q.E.D.

The end of the proof of the lemma.

The lemma will follow from the following formula:

$$(2.2)\qquad \bar\partial^*[(\Lambda^k\varphi)\lrcorner\,\omega_X(n,0)] = [\bar\partial^*(\Lambda^k\varphi)]\lrcorner\,\omega_X(n,0).$$

Proof of (2.2.)

From the well-known formula:

$$\nabla_\alpha(\Psi\lrcorner\,\omega_X(n,0)) = (\nabla_\alpha\Psi)\lrcorner\,\omega_X(n,0) \pm \Psi\lrcorner\,(\nabla_\alpha\omega_X(n,0))$$

and the fact that $\nabla_\alpha\omega_X(n,0) = 0$ we get that

$$(2.2.1.)\qquad \nabla_\alpha(\Psi\lrcorner\,\omega_X(n,0)) = (\nabla_\alpha\Psi)\lrcorner\,\omega_X(n,0)$$

(2.2.) follows from (2.2.1.) and the following formula for $\bar\partial^*$ that can be found in [15]

$$(\bar\partial^*\psi^\alpha)_{I_p,\bar j_1,\ldots,\bar j_{q-1}} = (-1)^{p+1}\sum g^{i\bar j}\,\nabla_i\psi^\alpha_{I_p,\bar j,\bar j_1,\ldots,\bar j_{q-1}}$$

Q.E.D.

From (2.2.1.) and the fact that $\bar\partial^*(\Lambda^k\varphi) = 0$ we get that

$$\bar\partial^*[(\Lambda^k\varphi)\lrcorner\,\omega_X(n,0)] = [(\bar\partial^*\Lambda^k\varphi)\lrcorner\,\omega_X(n,0)] = 0$$

Clearly

$$\bar\partial[(\Lambda^k\varphi)\lrcorner\,\omega_X(n,0)] = 0$$

since $\bar{\partial}(\Lambda^k\varphi) = 0$. So this proves that

$$(\Lambda^k\varphi) \lrcorner \omega_X(n,0)$$

is a harmonic form.

Q.E.D.

§3. Local deformation theory for Calabi-Yau manifolds.

Theorem. Let X be a Calabi-Yau manifold, then the Kuranishi space U of local deformations of X is a non-singular and has dimension equal to $\dim_{\mathbb{C}} H^1(X,\Theta)$.

Proof: The proof of this theorem is based on the following Proposition:

Proposition 3.1. Let $\varphi \in H^1(X,\Theta)$ be a harmonic representative with respect to the Calabi-Yau metric, then

$$[\varphi,\varphi] \equiv 0$$

where $[\varphi,\varphi]$ means the Lie bracket.

Proof: From the definition of the Lie bracket it follows

$$[\varphi,\varphi] = 0 \Leftrightarrow \sum_{\mu=1}^{n} \left(\varphi^{\mu}_{\bar{\alpha}} \frac{\partial\varphi^{\tau}_{\bar{\beta}}}{\partial z^{\mu}} - \varphi^{\mu}_{\bar{\beta}} \frac{\partial\varphi^{\tau}_{\bar{\alpha}}}{\partial z^{\mu}}\right) = 0$$

for each $\alpha \neq \beta$ and τ. The proof of Proposition 3.1 is based on the following observation:

$$\partial[(\Lambda^2\varphi) \lrcorner\, \omega_X(n,0)] = 0 \Rightarrow \sum_{\mu=1}^{u} (\varphi^{\mu}_{\alpha} \frac{\partial\varphi^{\tau}_{\beta}}{\partial z^{\mu}} - \varphi^{\mu}_{\beta} \frac{\partial\varphi^{\tau}_{\alpha}}{\partial z^{\mu}}) = 0$$

We will prove this observation:

From the definition of $\Lambda^2\varphi$ we get that

$$\Lambda^2\varphi = \sum_{\mu<\tau} (\varphi^{\mu} \wedge \varphi^{\tau}) \otimes \frac{\partial}{\partial z^{\mu}} \wedge \frac{\partial}{\partial z^{\tau}}$$

where

$$\varphi^{\mu} = \sum \varphi^{\mu}_{\bar{\nu}}\, d\bar{z}^{\nu}$$

is a component of φ .

Let us fix α,β,τ and compute the coefficient in front of $d\bar{z}^{\alpha} \wedge d\bar{z}^{\beta} \wedge dz^1 \wedge \ldots \wedge dz^{\mu_1} \wedge \ldots \wedge d\hat{z}^{\tau} \wedge \ldots \wedge dz^k$ in the form of type $(n-2,2)$ $(\Lambda^2\varphi) \lrcorner\, \omega_X(n,0)$. We suppose that

$$\omega_X(n,0)|_U = dz^1 \wedge \ldots \wedge dz^n$$

From the definition of $(\Lambda^2\varphi) \lrcorner\, \omega_X(n,0)$ we get that

$$(3.1.1.) \qquad [\sum_{\mu<\tau} (\varphi^{\mu} \wedge \varphi^{\tau}) \otimes \frac{\partial}{\partial z^{\tau}} \wedge \frac{\partial}{\partial z^{\mu}}] \lrcorner\, \omega_X(n,0) =$$

$$= \sum_{\mu<\tau} (-1)^{\mu+\tau} (\varphi^{\mu}_{\alpha}\, \varphi^{\tau}_{\beta} - \varphi^{\mu}_{\beta}\, \varphi^{\tau}_{\alpha})\, d\bar{z}^{\alpha} \wedge d\bar{z}^{\beta} \wedge dz^1 \wedge \ldots \wedge d\hat{z}^{\mu} \wedge \ldots$$

$$\wedge d\hat{z}^{\tau} \wedge \ldots \wedge dz^n \quad .$$

Now let us compute the coefficient of

$$\partial[(\Lambda^2\varphi) \lrcorner\, \omega_X(n,0)]$$

in front of $d\bar{z}^\alpha \wedge d\bar{z}^\beta \wedge dz^1 \wedge\ldots\wedge d\hat{z}^\tau \wedge\ldots\wedge dz^k$. From (3.1.1.) we get that this coefficient is

$$(-1)^\tau \sum_{\mu=1}^{n} \left(\varphi_\alpha^\mu \frac{\partial\varphi_{\bar\beta}^\tau}{\partial z^\mu} - \varphi_\alpha^\mu \frac{\partial\varphi_{\bar\beta}^\tau}{\partial z^\mu}\right) .$$

So this proves our observation.

From the fact that $(\Lambda^2\varphi) \lrcorner\, \omega_X(n,0)$ is a harmonic form we get that

$$\partial[(\Lambda^2\varphi) \lrcorner\, \omega_X(n,0)] = 0$$

and so

$$[\varphi,\varphi] = 0.$$

Let $\varphi_1,\ldots,\varphi_N$ be a basis of harmonic forms of $H^1(X,\Theta)$ with respect to a fixed Calabi-Yau metric. Let

$$\varphi(t) = \sum t_\nu\varphi_\nu \text{ , where } t = (t_1,\ldots,t_N) \in \mathbb{C}^N .$$

Clearly $\varphi(t)$ is such that

$$\varphi(0) = 0$$

$$\bar\partial\varphi(t) - \tfrac{1}{2}[\varphi(t),\varphi(t)] = 0 .$$

Obviously since φ_ν are harmonic form we have

$$\bar{\partial}\varphi(t) = \sum t_\nu \bar{\partial}\varphi_\nu = 0 \quad .$$

On the other hand §2 we know that

$$[\varphi(t),\varphi(t)] = 0 \quad .$$

Also we have

$$\left(\frac{\partial\varphi(t)}{\partial t_\nu}\right)_{t=0} = \varphi_\nu \in H^1(X,\Theta) \quad .$$

From Newlander-Nirenberg theorem it follows that $\varphi(t)$ determines a complex analytic family of Calabi-Yau manifolds for each $t \in \mathbb{C}^N$. Let us prove this for the completeness of the paper.

Consider $\varphi(t) = \sum \varphi_{\bar{\alpha}}^{\beta}(t)\, d\bar{z}^{\alpha} \otimes \frac{\partial}{\partial z^{\beta}}$ as a vector (0,1)-form defined on $X \times B_\varepsilon$, where $B_\varepsilon = \{t \in \mathbb{C}^N \mid |t| < \varepsilon\}$. Clearly $\varphi(t)$ satisfies the integrability condition

$$\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t),\varphi(t)] = 0.$$

$(\varphi = \sum t_\nu \varphi_\nu \quad \varphi$ is holomorphic in $t.)$

Thus φ determines a complex structure $\mathfrak{X}$ on $X \times B_\varepsilon$. The local complex coordinates of $\mathfrak{X}$ are solutions ζ of

$$(*) \qquad \bar{\partial}\zeta - \sum \varphi^{\beta}(t) \frac{\partial\zeta}{\partial z^{\beta}} = 0.$$

This equation is satisfied if and only if

$$(**) \qquad \frac{\partial}{\partial \bar{z}^{\alpha}}\zeta - \sum_{\beta} \varphi_{\bar{\alpha}}^{\beta} \frac{\partial \zeta}{\partial z^{\beta}} = 0 \qquad \alpha = 1,\dots,n$$

$$\frac{\partial\zeta}{\partial \bar{t}^{\nu}} = 0 \ , \quad 0 = 1,\dots,N \ .$$

That (*) has a solution follows from Newlander-Nirenberg theorem. Hence, on some coordinate chart

$$\mathcal{U}_j = U_j \times B_\varepsilon \subseteq \mathcal{X}$$

we have $n + N$ independent solutions

$$t_1, \ldots, t_N, \zeta_j^1(z,t), \ldots, \zeta_j^n(z,t)$$

of (*). So $\mathcal{X}$ is a complex manifold such that the projection:

$$\pi : \mathcal{X} \longrightarrow B_\varepsilon$$

is holomorphic of rank N and for each fixed t, $\pi^{-1}(t) = X_t$ is a complex manifold with a complex structure given by

$$A_t J_0 A_t^{-1} = J_t$$

where J_0 is the complex structure operator on X and

$$A_t = (\mathrm{id} + t\sum(\varphi_{\bar\beta}^{\alpha})) \oplus (\mathrm{id} + \overline{t\sum\varphi_{\bar\beta}^{\alpha}})$$

Q.E.D.

§ 4. Construction of the moduli space of Calabi-Yau manifolds

<u>Definition 4.1.</u> The Teichmüller space $\mathcal{T}(X)$ of a given Calabi-Yau manifold is defined in the following way:

$\mathcal{T}(X) = \{$all complex structures on X as C^∞ manifold $\} / \mathrm{Diff}_0(X)$

where $\mathrm{Diff}_0(X)$ = {all diffeomorphisms isotopic to the identity}.

Theorem 4. Let X be a Calabi-Yau manifold, then $\mathcal{T}(X)$ exists and $\mathcal{T}(X)$ is a complex manifold of dimension equal to $\dim_{\mathbb{C}} H^1(X,\Theta)$.

Proof.

Lemma 4.1. Let $L \in H^{1,1}(X,\mathbf{Z})$ and suppose that L is fixed and $L = [\mathrm{Im}\, g_{\alpha\overline{\beta}}]$, where $g_{\alpha\overline{\beta}}$ is a Kähler metric on X. Let

$$\mathrm{Aut}_0(X) = \{\varphi \in \mathrm{Aut}\,(X) \mid \varphi \text{ acts as identity on } H^n(X,\mathbf{Z}) \text{ and } \varphi^*(L) = L\}$$

where $\mathrm{Aut}\,(X)$ =: {the group of the biholomorphic automorphism of X} then

$$\mathrm{Aut}_0(X) \text{ is a finite group.}$$

Proof: Since $H^0(X,\Theta) \cong H^0(X,\Omega^{n-1})$ (this isomorphism is obtained by the following map $\varphi \in H^0(X,\Theta) \longrightarrow \varphi \lrcorner\, \omega_X(n,0)$) and on a Calabi-Yau manifold

$$H^0(X,\Omega^{n-1}) = 0$$

we get that $\mathrm{Aut}\,(X)$ is a discrete group and so $\mathrm{Aut}_0(X)$ is also a discrete group. Since if $\varphi \in \mathrm{Aut}_0(X) \Rightarrow \varphi^*(L) = L$ and from the solution of the Calabi-Conjecture by Yau in L there exists a unique Ricci-flat metric we get that if $\varphi \in \mathrm{Aut}_0(X)$, then

$$\varphi^*(g_{\alpha\overline{\beta}}) = g_{\alpha\overline{\beta}}$$

where $(g_{\alpha\overline{\beta}})$ is the Calabi-Yau metric corresponding to the class $L \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$. From $\varphi^*(g_{\alpha\overline{\beta}}) = g_{\alpha\overline{\beta}}$ it follows that φ is an isometry, which means that $\mathrm{Aut}_0(X)$ is a discrete subgroup of a compact group. From here we get that $\#|\mathrm{Aut}_0(X)| < \infty$.

Q.E.D.

<u>Lemma 4.2.</u> Let $\varphi \in \mathrm{Aut}_0(X)$ and let $\pi: \mathcal{X} \longrightarrow U$ be the Kuranishi family of X, then φ induces an action on U and this action is just the identity map.

<u>Proof:</u> If $\varphi \in \mathrm{Aut}_0(X)$, then φ induces the following action on U:

Let $s \in U$ and let $J_s \in \Gamma(X, \mathrm{Hom}(TX,TX))$ be the complex structure operator that defines the complex manifold X_s, then we define $\psi^*(s)$ to be the point of U that correspond to the complex structure operator $\psi^*(J_s)$. Here we look at ψ as an element of $\mathrm{Diff}(X)$. We know from §3 that all complex structures on X that correspond to $s \in U$ are given by

$$J_s = (\mathrm{id} + s\varphi_1)J_0(\mathrm{id} + s\varphi_1)^{-1}, \quad \varphi_1 = \varphi + \bar{\varphi}$$

where $[\varphi] \in H^1(X,\Theta)$ and φ is a harmonic representative of $[\varphi]$. If we prove that

(4.2.1.) $\quad \psi^*(\varphi) \equiv \varphi$

then lemma 4.2. will be proved

Proof of 4.2.1.: Since $\varphi \in \mathrm{Aut}_0(X) \Rightarrow$

1) $\varphi^*(g_{\alpha\overline{\beta}}) = (g_{\alpha\overline{\beta}})$, where $g_{\alpha\overline{\beta}}$ is the Calabi-Yau metric

2) from the fact that φ induces the identity on $H^n(X,\mathbb{C})$ $\Rightarrow \varphi^*(\omega_X(n,0)) = \omega_X(n,0)$ and $\varphi^*(\omega(n-1,1)) = \omega(n-1,1)$, where $\omega(n-1,1)$ is any harmonic form of type $(n-1,1)$.

Since

$$\varphi \lrcorner \omega_X(n,0) = \omega_X(n-1,1)$$

is a harmonic form by the lemma in § 2 and from the canonical isomorphism:

$$H^1(X,\Theta) \cong H^1(X,\Omega^{n-1})$$

given by:

$$\varphi \longrightarrow \varphi \lrcorner \omega_X(n,0)$$

we get that:

$$\psi^*(\varphi) = \varphi$$

and so lemma 4.2. is proved.

Q.E.D.

Remark. From local Torelli theorem it follows that 4.2 is true without the assumption that $\varphi^*(L) = L$.

4.3. Construction of the Teichmüller space of Polarized marked Calabi-Yau manifolds.

Definition 4.3.1. Let X be a Calabi-Yau manifold. We will call the tripple $(X;\gamma_1,\ldots,\gamma_{b_n};L)$ a marked polarized Calabi-Yau manifold if $\gamma_1,\ldots,\gamma_{b_n}$ is a basis in $H_n(X,\mathbb{Z})$ and $L = [\operatorname{Im} g_{\alpha\overline{\beta}}]$, (where $(g_{\alpha\overline{\beta}})$ is a Kähler metric) is a fixed class in $H^2(X,\mathbb{Z})$.

Clearly $(S;\gamma_1,\ldots,\gamma_{b_n};L)$ and $(X,\gamma_1,\ldots,\gamma_{b_n};L)$ are isomorphic if there exists a biholomorphic map

$$\varphi:X \longrightarrow S$$

such that

$$\varphi^*(\gamma_i) = \gamma_i \quad \text{and} \quad \varphi^*(L) = L.$$

We want to construct a universal family of marked polarized Calabi-Yau manifolds. In order to construct it we will need the period map. For this construction we will need some definitions.

Definition 4.3.2. Let $H_{\mathbb{Z}}$ be a free abelian group equipped with a) (if n is even)non-degenerate symmetric bilinear form $<,>:H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ b) (if n is odd)bilinear skew symmetric non-degenerate form

$$<,>:H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}.$$

Then we define the Hodge structure on $H_{\mathbb{Z}}$ in the following way: This is a filtration, the so called Hodge filtration

(4.3.2.1) $F^0 \subset F^1 \subset F^2 \subset \ldots \subset F^n = F = H_{\mathbb{Z}} \otimes \mathbb{C}$

which fulfills the following conditions:

a) The Hodge filtration is isotropic, which means that:

(4.3.2.2) $(F^q)^{\perp} = F^{n-q-1}$ ($\perp$ means orthogonal with with respect to $\langle , \rangle$ induced on $H_{\mathbb{Z}} \otimes \mathbb{C}$)

b) We have the Hodge decomposition

$$H_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{q=0}^{n} F^{n-q,q}$$

where

(4.3.2.3) $F^{n-q,q} = F^q \cap F^{\overline{n-q}} = F^q \cap (F^{\overline{q-1}})^{\perp}$

c) The following Riemann bilinear relations

$$\langle F^{n-q,q}, F^{\overline{n-p,p}} \rangle = 0 \quad (p \neq q)$$

(4.3.2.4) $(-i)^n(-1)^q \langle F^{n-q,q}, F^{\overline{n-q},q} \rangle > 0$.

<u>Remark</u> If X is an algebraic manifold, L is the polarization class, then for each k on the primitive cohomology $H^k(X,\mathbb{Z})_0$ we can define in a natural way a Hodge structure of weight k in the following way

1) the bilinear form on $H^k(X,\mathbb{Z})_0$ is defined as follows:

$$\langle u,v \rangle = \int L^{n-k} \wedge u \wedge n \quad (u = \dim_{\mathbb{C}} X)$$

2) the Hodge filtration is defined $n:H^k(X,\mathbb{C})_0$ as

$$F^p \overset{def}{=} H^{k,0} + H^{k-1,1} + \dots + H^{k-p,p} .$$

Now it is easy to check that in such a way we get a Hodge structue of weight k on $H^k(X,\mathbb{Z})_0$.

4.3.3. Classifying spaces for Hodge structures.

Let $H_{\mathbb{Z}}$ be a free abelian group equiped with non-singular bilinear form $<,> : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ such that if

a) n is odd, then $<,>$ is skew-symmetric

b) n is even, then $<,>$ is symmetric. Let

$$0 < h_0 \leq h_1 \leq \dots \leq h_{n-1} < h_n = \dim_{\mathbb{C}} H_Z \otimes_{\mathbb{Z}} \mathbb{C}$$

be an increasing sequence of integers which is self-dual in the sense that

$$h_{n-q-1} = h_n - h_q \quad \text{for} \quad 0 \leq q \leq n \quad .$$

Consider the set Gr of all filtrations

$$F^0 \subset \dots \subset F^n = H_{\mathbb{Z}} \otimes \mathbb{C}, \ \dim F^q = h_q$$

which satisfy

$$\begin{cases} (F^q)^{\perp} = F^{n-q-1} \\ <E^q, E^{n-q-1}> = 0 \\ (-i)^n <E^a, \overline{E}^a> \text{ is non-singula} \\ (i)^q (-i)^n <E^a, \overline{E}^a> > 0 \end{cases}$$

where $E^q = F^q/F^{q-1}$

Proposition 4.3.3.1 Gr is a homogeneous complex manifold

$$Gr = G/H$$

of a real, simple, non-compact Lie group G divided by a compact subgroup H. See [6].

Examples. a) When $n = 2m$ is even,

$$G = SO(a,b;\mathbb{R}) \quad (a = h^0 + h^2 + \ldots + h^{2m}, b = h^1 + h^3 + \ldots + h^{2m-1})$$

is the orthogonal group of the quadratic form $\sum_{i=1}^{a} (x_i)^2 - \sum_{j=1}^{b} (x_j)^2$, the compact isotropy group is

$$H = U(h^0) \times \ldots U(h^{m-1}) \times SO(h^m)$$

and the maximal compact subgroup of G is

$$K = SO(a;\mathbb{R}) \times SO(b;\mathbb{R})$$

b) when $n = 2m + 1$ is odd $G = Sp(2a;\mathbb{R})$ $(a = h^0 + \ldots + h^m)$ is the group leaving the skew-form $\sum_{j=1}^{a} (x_j \wedge x_{j+a})$ invariant, the compact isotropy group is $H = U(h^0) \times \ldots \times U(h^m)$ and the maximal compact group is $K = U(a)$.

4.3.3.2. According to [6] we may identify, the tangent bundle to Gr as

$$(4.3.3.2.1) \quad T_F(Gr) = \bigoplus_{q=0}^{m} \bigoplus_{p=1}^{m-q} \mathrm{Hom}_{<,>}(E^q, E^{p+q})$$

where

$$F \in Gr \quad \text{and} \quad m = [\frac{n-1}{2}] \ .$$

The identification (4.3.3.2.1) is G-invariant, and the positive definite metrics $(-1)^q\langle u,\bar{u}\rangle$ on E^q induce a G-invariant Hermitian metric ds^2_{Gr} on Gr. Group theoretically, ds^2_{Gr} is the metric induced by the Cartan-Killing form on the Lie algebra of G. This metric we will call standart.

<u>Definition 4.3.3.</u> Let $\pi:\mathfrak{X} \longrightarrow U$ be the Kuranishi family of a marked polarized Calabi-Yau manifold $(X;\gamma_1,\dots,\gamma_{b_n};L)$ then we can define the period map in the following way

$$p:U \longrightarrow Gr$$

$p(t) \overset{\text{def}}{=} \{$The Hodge polarized structure of weight n on $H^n(X,\mathbb{Z})_0$ induced from the complex structure on $X_t = \pi^{-1}(t)\}$ where $H^n(X,Z)_0$ are the primitive cohomologies. p is a holomorphic map [6].

<u>Remark:</u> From the theorem proved in §3 it follows that if $\mathfrak{X} \longrightarrow U$ is the Kuranishi family of the Calabi-Yau manifold X, then U is a non-singular complex manifold of dimension equal to $\dim_{\mathbb{C}} H^1(X,\Theta)$. So from here it follows that $\mathfrak{X}$ as a C^∞ manifold is just:

$$\mathfrak{X} \cong X \times U \ .$$

If we fix the basis in $H_n(X,\mathbb{Z})$ we get that we have fixed the basis in $H_n(X_t,\mathbb{Z})$ for all $t \in U$, where $X_t = \pi^{-1}(t)$. After shrinking U we may suppose that for each $t \in U$ we can find a Kähler metric $g_{\alpha\overline{\beta}}(t)$ such that

$$[\operatorname{Im} g_{\alpha\overline{\beta}}(t)] = L.$$

So from this remark it follows that the period map

$$p : U \longrightarrow G$$

is correctly defined.

The local Torelli theorem says that:

$$p : U \hookrightarrow G \quad .$$

For the proof of this fact see [ ]. Now using lemma 4.2. we can "patch" together all Kuranishi families, i.e. we define

$$\begin{matrix} \mathcal{X} \\ \downarrow \\ F(X) \end{matrix} \overset{\text{def}}{=} \coprod_{U_i} \left(\begin{matrix} \mathcal{X} \\ \downarrow \\ U_i \end{matrix} \right) / \sim$$

where $\sim$ means that we idenify $t \in U_i$ with $s \in U_j$ if X_t and X_s are isomorphic as marked polarized Calabi-Yau manifolds. Notice that $\mathcal{X} \longrightarrow F(X)$ is a universal family for all marked polarized Calabi-Yau manifolds since from the proof of the theorem in § 3 it follows that the Kuranishi family is complete in the sence of Kodaira-Spencer, i.e.

$$p : T_{0,U} \simeq H^1(X_0,\Theta)$$

and so from a theorem of Kodaira-Spencer and lemma 4.2. it follows that $\mathcal{X} \longrightarrow U$, i.e. the Kuranishi family is the local universal family of X_0. See [11].

From lemma 4.2. and the fact that if $\varphi \in \mathrm{Diff}_0(X)$, then φ induces the identity map on $H^n(X,\mathbb{Z})$ and $\varphi^*(L) = L$ we get that $F(X)$ is really the Teichmüller space of X.

Q.E.D.

§ 5. The Weil-Peterson metric on the Teichmüller space $T(X)$

5.1. Let $s_0 \in T(X)$ and let $X_{s_0} = \pi^{-1}(s_0)$, where

$$\pi: \mathcal{X} \longrightarrow T(X).$$

If $\varphi_1,\ldots,\varphi_N$ is a basis of harmonic forms of type (0,1) with values in Θ_{s_0}, i.e. a basis of $H^1(X_{s_0},\Theta)$ and let J_0 be the complex structure operator of X_{s_0}, then locally around s_0 the complex structure operator J_t of $X_t = \pi^{-1}(t)$ is given by:

$$5.1.1. \qquad J_t = (\mathrm{id} + \sum t^i\varphi_i)\, J_0(\mathrm{id} + \sum t^i\varphi_i)^{-1}$$

where $t = (t_n^1,\ldots,t^N) \in \mathbb{C}^N$. So we can view $(t_1^1,\ldots,t^N)$ as local coordinates of $T(X)$ around the point $s_0 \in T(X)$. From (5.1.1.) it follows that the Kodaira-Spencer map:

$$p: T_{s_0,T(X)} \longrightarrow H^1(X_{s_0},\Theta)$$

is given by:

$$p: \frac{\partial}{\partial t_i} = v_i \longrightarrow \left(\frac{\partial}{\partial t_i}(J_t)\right)_{t=0} = \varphi_i \quad .$$

So we can identify $T_{s_0},\mathcal{T}(X) \cong H^1(X,\Theta)$.

5.3. Let dV denote the volume form of the Calabi-Yau metric $g_{\alpha\overline{\beta}}$. We define the Weil-Petersson metric

$$\sum h_{i\overline{j}}\, dt^i \otimes \overline{dt^j}$$

on $\mathcal{T}(X)$ by:

$$h_{i\overline{j}}(t_0) := \int_{X_{t_0}} (\varphi_i)^p_a \, \overline{(\varphi_j)^q_b}\, g_{p\overline{q}}\, g^{b\overline{a}}\, dV$$

5.4. In the case of polarized symplectic holomorphic manifolds, i.e. compact Kähler manifold on which there exists a unique up to a constant holomorphic two form $\omega_X(2,0)$ which is a non-degenerate at each point $x \in X$, i.e. if

$$\omega_X(2,0)\,|_{U\ni x} = \sum \omega_{ij} dz^i \wedge dz^j$$

then

$$\det(\omega_{ij}) \in \Gamma(U, \mathcal{O}_U^*)$$

it is proved in [17] that

$$\mathcal{T}(X) = SO_0(2,b_2-3)/SO(2)\times SO(b_2-3)$$

where $b_2 = \dim H_2(X,\mathbb{R})$.

5.5. Theorem. Let X be a compact polarized symplectic holomorphic manifold, then the Weil-Petersson metric on $\mathcal{T}(X) \cong SO_0(2,b_2-3)/SO(2) \times SO(b_2-3)$ is just the Bergman metric. See also [13] & [14].

For the definition of the Bergman metric see [9].

Proof. First we need some facts about the space

$$\mathcal{T}(X) \cong SO_0(2,b_2-3)/SO(2) \times SO(b_2-3) \; .$$

Definition 5.5.1. The tripple $(X;\gamma_1,\ldots,\gamma_{b_2};L)$ will be called a marked polarized symplectic holomorphic manifold iff

a) $\gamma_1,\ldots,\gamma_{b_2}$ is a basis of $H_2(X,\mathbb{Z})$

b) $L \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ and L is the cohomology class of the imaginary part of a Kähler metric on X.

Definition 5.5.2. We can define a scalar product on $H^2(X,\mathbb{Z}) \otimes \mathbb{R}$ in the following way:

$$\langle\omega_1,\omega_2\rangle = \int_X \omega_1 \wedge \omega_2 \wedge L^{n-2}, \text{ where } \omega_1,\omega_2 \in H^2(X,\mathbb{R})$$

and $n = \frac{1}{2}\dim_{\mathbb{C}} X$.

In [17] the following proposition is proved.

Proposition 5.5.3. The scalar product $\langle,\rangle$ has signature $(3,b_2-3)$, where $b_2 = \dim_{\mathbb{R}} H^2(X,\mathbb{R})$.

The scalar product $<,>$ defines a non-singular quadric Q in $\mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C}$ in the following way:

(5.5.4.) $$Q \overset{\text{def}}{=} \{u \in \mathbb{P}(H^2(x,\mathbb{C})) \mid <u,u> = 0\} .$$

Let Ω be

(5.5.5.) $$\Omega = \{u \in \mathbb{P}(H^2(X,\mathbb{C})) \mid <u,\overline{u}> > 0\}$$

Ω is an open subset in Q. Let

(5.5.6.) $$\Omega(L) = \{u \in \Omega \mid <u,L> = 0\} \subset \mathbb{P}(H^2(X,\mathbb{C})) .$$

It is easy to prove the following proposition.

<u>Proposition (5.5.7.)</u> $\Omega(L) \cong SO_0(2,b_2 - 3)/SO(2) \times SO(b_2 - 3)$ [17]. From the description (5.5.6.) of $SO_0(2,b_2 - 3)/SO(2) \times SO(b_2 - 3)$ we get that if $t \in \Omega(L) = SO_0(2,b_2 - 3)/SO(2) \times SO(b_2 - 3)$ then t corresponds to a line ℓ_t in $H^2(X,\mathbb{C})$ and the tangent space at the point t, i.e. $T_{t,\Omega(L)}$ can be described in the following way:

(5.5.8.) $$T_{t,\Omega(L)} = \{u \in H^2(X,\mathbb{C}) \mid <u,\ell_t> = <u,L>=0\} \subset H^2(X,\mathbb{C}) .$$

From this description of the tangent space $T_{t,\Omega(L)}$ follows a very nice description of the Bergman metric

(5.5.9.) The Bergman metric on $T_{t,\Omega(L)}$ can be described in the following way: The Bermgna metric on $T_{t,\Omega(L)}$, viewed as a subspace in $H^2(X,\mathbb{C})$ is just defined in the following way

$$\langle u,v\rangle_{\text{Bergman}} = -\int_X u \wedge \overline{v} \wedge L^{n-2}$$

After the geometric desciption of the Bergman metric on $\Omega(L) \hookrightarrow \mathbb{P}(H^2(X,\mathbb{C}))$ we need to connect it with a geometry. Let $\pi: \mathfrak{X} \longrightarrow U$, be a family of non-singular marked polarized holomorphic symplectic manifolds, then we can define the period map

$$p: S \longrightarrow \mathbb{P}(H^2(X,\mathbb{C}))$$

in the following manner:

$$p(s) = (\ldots, \int_{\gamma_i} \omega_s(2,0), \ldots)$$

$s \in S$ and $\omega_s(2,0)$ is the unique holomorphic two form defined up to a constant.

In [17] it is proved that $p(S) \subset \Omega(L)$. From

(5.5.9.) it follows that

$$T_{s,\Omega(L)} = \{u \in H^{1,1}(X_s,\mathbb{C}) \mid \langle u,\ell\rangle = 0\} = H^{1,1}(X,\mathbb{C})_0$$

i.e. $T_{s,\Omega(L)} = H^{1,1}(X,\mathbb{C})_0 = \{$all primitive (1,1) classes.$\}$

Lemma 5.5.10. Let φ and ψ are two non-zero harmonic elements of $H^1(X_s,\Theta)$ with respect to the Calabi-Yau metric $g_{\alpha\bar{\beta}}$, such that $[\mathrm{Im}\, g_{\alpha\bar{\beta}}] = L$, then

$$\int_{X_s} \varphi^p_a \,\overline{(\psi)^a_b}\, g_{p\bar{q}}\, g^{b\bar{a}}\, dV =$$

$$= (+i)^{2n} \int_{X_s} (\varphi \lrcorner\, \omega_X(2,0)) \wedge (\psi \lrcorner\, \omega_X(2,0)) \wedge L^{n-2}$$

where we suppose that $\int_{X_s} (\omega_X(2,0))^n \wedge \overline{(\omega_X(2,0))}^n = 1$, $L = \mathrm{Im}\, g_{\alpha\bar{\beta}}$ and $\int_{X_s} dV = 1$.

Proof. Since $(\omega_{X_s}(2,0))^n = \omega_{X_s}(2n,0)$ is a parallel form we get that

$$(*) \qquad (\omega_{X_s}(2,0)^n) \wedge \overline{(\omega_{X_s}(2,0)^n)} = dV \quad .$$

So from (*) we get that if we choose in one point $x \in X$ the coordinate $(z^1,\ldots,z^n)$ in such a way that $dz^1,\ldots,dz^{2n}$ is an orthonormal basis, then we may suppose that

$$(**) \qquad \omega_{X_s}(2,0) = \sum_{i=1}^{n} dz^i \wedge dz^{i+n}.$$ For the proof see [17].

So from (*) and (**) it follows that

$$\varphi^{p}_{\bar a}\,\overline{(\psi)^{q}_{\bar b}}\, g_{p\bar q}\, g^{b\bar a} = -(\varphi \lrcorner\, \omega_{X_s}(2,0)) \wedge (\psi \lrcorner\, \omega_{X_s}(2,0)) \wedge L^{n-2}$$

by direct computation. So lemma 5.5.10 is proved.

Q.E.D.

Lemma 5.5.11. Let φ be a harmonic non zero class in $H^1(X,\Theta)$ with respect to the Calabi-Yau metric $g_{\alpha\bar\beta}$ then $\varphi \lrcorner\, \omega_X(2,0)$ is a harmonic form of type (1,1) with respect to $g_{\alpha\bar\beta}$.

Proof: The proof is exactly the same as the lemma in § 2.

Q.E.D.

From lemma 5.5.10 and 5.5.11 if follows that

$$(5.5.12) \qquad \int_{X_s} \varphi^{p}_{\bar a}\,\overline{(\psi)^{q}_{\bar b}}\, g_{p\bar q}\, g^{\overline{ba}} dV = -\langle \varphi \lrcorner\, \omega_s(2,0), \psi \lrcorner\, \omega_s(2,0)\rangle .$$

From (5.5.12) and (5.5.9) our theorem follows.

Q.E.D.

Theorem 5.6. The Weil-Petersson metric on the Teichmüller space of a Calabi-Yau manifold has negative holomorphic sectional curvature bounded away from zero.

Proof: Let $Gr = G/H$, where G is a simple Lie group and H is a compact and G and H are like examples on page 20. On Gr we have the standart metric defined in 4.3.3.2.

5.6.1. Review of some results of Griffiths and W. Schmid.

We have an equivariant fibering:

$$\tilde{\omega}: G/H \longrightarrow G/K$$

where K is a maximal compact-subgroup in G. Clearly the fibre is K/H. Griffiths proved in [6] that the fibre through each point $F \in G/H$ Z_F is a compact complex submanifold. Let $F \in Gr = G/H$, then the ds^2_{Gr} (standard metric) defines an equivariant splitting of $T_F(Gr)$ namely:

$$T_F(Gr) = T_F^v + T_F^v$$

where

$$T_F^v = \{u \in T_F(Gr) \mid u \in T_F(Z_F)\}$$

$$T_F^h = (T_F^v)^{\perp} = \{u \in T_F(Gr) \mid \langle u, T_F^v\rangle = 0\} .$$

The following theorem is proved in [8] :

<u>Theorem</u>. The holomorphic sectional curvature in $Gr = G/H$ corresponding to directions in $T^h(Gr)$ are negative and bounded away from zero.

5.6.2. Let $\mathfrak{X} \to U$ be the Kuranishi family of the marked Calabi-Yau manifold with a fix class of polarization. From local Torelli theorem we know that

5.6.2. a) $p:U \hookrightarrow Gr$, i.e. the period map gives us an embedding locally.

From Griffiths transversality theorem we know that:

5.6.2. b) $p_*:T(U) \hookrightarrow T^h(Gr)$.

So from 5.6.1. a and 5.6.2.b. follows that if we prove that Weil-Peterson metric on $T_t(U)$ is the restriction of the standart metric ds^* on G under the map p_*, then our theorem will follow from the theorem of Griffiths and Schmid. That Weil-Petersson metric is the restriction of the standart metric ds^2 on Gr follows from the following proposition.

<u>Proposition 5.6.2.1.</u> We have the following equality on each Calabi-Yau manifold with a fixed Calabi-Yau metric $g_{\alpha\bar{\beta}}$:

$$\int_X \varphi^p_{\bar{a}} \, \overline{(\psi^q_{\bar{b}})} \, g_{p\bar{q}} \, g^{b\bar{a}} dV =$$

$$(i)^n \cdot \int_X (\varphi \lrcorner \, \omega_X(n,0)) \wedge \overline{(\psi \lrcorner \, \omega_X(n,0))}$$

where $\int_X \omega_X(n,0) \wedge \overline{\omega_X(n,0)} = \int_X dV = 1$ and

$\varphi,\psi \in H^1(X,\Theta)$ are harmonic representative with respect to the metric $g_{\alpha\overline{\beta}}$.

<u>Proof:</u> The proof is absolutely the same as the proof of 5.5.10.

Q.E.D.

From the main lemma in §2 we know that $\varphi \lrcorner\, \omega_X(n,0)$ and $\psi \lrcorner\, \omega_X(n,0)$ are harmonic forms of type $(n-1,1)$ on X. So from 5.6.2.1. we get that the Weil-Peterson metric is purelly topologically defined. So it is invariant under the action of the group G. This is so since the scalar product on $H^n(X,\mathbf{Z})_0$ is coming from the intersection of cycles. From here we get that the Weil-Peterson metric is just the restriction of the standart metric on G. So our theorem follows from Griffiths Schmid's theorem.

Q.E.D.

§6. The Torelli problem for Calabi-Yau manifolds.

<u>Theorem 6.1.</u> The period map $p:\mathcal{T}(X) \hookrightarrow Gr$ is an embedding, where X is a Calabi-Yau manifold, $\mathcal{T}(X)$ is the Teichmuller space of X and Gr is the Griffiths domain that parametrizes all Hodge structues of weight n on $H^n(X,\mathbf{Z})$, where $n = \dim_{\mathbb{C}} X$.

<u>Proof:</u> Let X_0 be a fixed Calabi-Yau manifold, $g_{\alpha\overline{\beta}}(0)$ be a Ricci flat Kähler metric on X_0 and

$$\varphi^1,\dots,\varphi^N \in H^1(X_0,\Theta_{X_0})$$

be a basis of harmonic forms with respect to $g_{\alpha\overline{\beta}}(0)$.

In §2 we define for each $t = (t_1,\dots,t_N) \in B \subset \mathbb{C}^N$ a new complex structure X_t on X_0 in the following way: Let $\{U_i\}$ be a covering of X_0 and let

$$\varphi^k|_U = \sum (\varphi^k)^{\alpha}_{\overline{\beta}}\, d\overline{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}} \quad \text{for each} \quad U \in \{U_i\}$$

then

$$(*) \qquad \Theta^{\alpha}_t = dz^{\alpha} + \sum_{\beta} \Big(\sum_k (t_k \varphi^k)^{\alpha}_{\overline{\beta}}\, \overline{dz}^{\beta}\Big)$$

will be a basis of $(T^{1,0}_t)^*$ for each point $z \in U$, in another words the new complex structure operator J_t is defined as follows:

$$(**) \quad J_t = A_t\, J_0\, A_t^{-1}$$

where

$$A_t = (\mathrm{id} + \sum t^k \varphi_k) \oplus (\mathrm{id} + \sum t^k \overline{\varphi}_k)$$

and J_0 is the complex structure operator defining X_0.

<u>Definition 6.1.1.</u> The Kuranishi family defined as above will be called the standart Kuranishi family.

Lemma 6.1.2. Let $\mathcal{X} \longrightarrow U$ be the standart Kuranishi family, then for each $t \in U \subset \mathbb{C}^N$ the holomorphic n-form $\omega_t(n,0)$ on $X_t = \pi^{-1}(t)$ is a harmonic form with respect to the Ricci-flat metric $g_{\alpha\overline{\beta}}(0)$ on $X_0 = \pi^{-1}(0)$.

Proof: Let $\{U_i\}$ be a covering of X_0 and let $(z^1,\ldots,z^k)$ be local coordinates in U_i such that

$$\omega_{X_0}(n,0)|_{U_i} = dz^1\wedge\ldots\wedge dz^n \quad .$$

Let $\varphi \in H^1(X_0,\Theta_{X_0})$ and $\varphi = \sum t_i\varphi^i$, where

$$\varphi|_U = \sum\varphi^{\alpha}_{\overline{\beta}}\, \overline{dz}^{\beta} \oplus \frac{\partial}{\partial z^{\alpha}} \quad .$$

Let

$$\Theta^{\alpha}_t = dz^{\alpha} + t\sum \varphi^{\alpha}_{\overline{\beta}}\, d\overline{z}^{\beta}$$

then

$$\Theta^1_t\wedge\ldots\wedge\Theta^n_t = dz^1\wedge\ldots\wedge dz^n + \sum(-1)^{\mu(k)}[\Lambda^k\varphi\lrcorner(dz^1\wedge\ldots\wedge dz^n)] \qquad (6.1.2.1.)$$

where $\Lambda^k\varphi$ is defined as in §2, $\mu(k)$ is an integer >0, which can be computed very easily. Since

$$\omega_{X_0}(n,0)|_U = dz^1\wedge\ldots\wedge dz^n|_U$$

we get

(6.1.2.2.) $\theta_t^1\wedge\ldots\wedge\theta_t^n = \omega_{X_0}(n,0)|_U + \sum(-1)^{\mu(k)}[(\Lambda^k\varphi)\lrcorner\omega_X(n,0)]|_U$.

From 6.1.2.2. we get that $\theta_t^1\wedge\ldots\wedge\theta_t^n$ is globally defined since $\omega_{X_0}(n,0)$ and $\Lambda^k\varphi$ are globally defined forms.
From the definition of $\{\theta_t^i\}$ it follows that $\theta_t^1\wedge\ldots\wedge\theta_t^n$ on X_t is a form of type $(n,0)$, i.e.

$$\theta_t^1\wedge\ldots\wedge\theta_k^n \in \Gamma(U,\Lambda^n((T_t^{1,0})^*)).$$

In § 2 we proved that $(\Lambda^k\varphi)\lrcorner\omega_{X_0}(n,0)$ is a harmonic form with respect to $g_{\alpha\overline{\beta}}(0)$ so

$$d(\theta_t^1\wedge\ldots\wedge\theta_t^n) = 0 .$$

From here we get that

$$\omega_t(n,0)|_U = \theta_t^1\wedge\ldots\wedge\theta_t^n.$$

So this proves that for each $t\in B$, $\omega_t(n,0)$ is a harmonic form in the standart Kuranishi family with respect to $g_{\alpha\overline{\beta}}(0)$.

Q.E.D.

6.2. Let L be the cohomology class of $\mathrm{Im}_{\alpha\overline{\beta}}(0)$. From Yau's solution of the Calabi conjecture it follows that for each $t\in B$ L defines a unique Ricci flat Kähler metric $g_{\alpha\overline{\beta}}(t)$ on $X_t = \pi^{-1}(t)$, where $\mathcal{X}\longrightarrow B$ is the standart Kuranishi family. Since

$$[\mathrm{Im}\ g_{\alpha\overline{\beta}}(t)] \equiv L$$

we have on X_0

6.2.1. $\operatorname{Im} g_{\alpha\overline{\beta}}(t) = \operatorname{Im} g_{\alpha\overline{\beta}}(0) + d\psi_t$

where ψ_t is a real one form on X_0.

<u>Lemma 6.2.2.</u> In the equality 6.2.1. $d\psi_t = 0$ for the standart Kuranishi family.

<u>Proof:</u> We will need several propositions.

<u>Proposition 6.2.2.1.</u> For each $t \in B$ we have on X_0

$$\omega_t(n,0) \wedge \overline{\omega_t(n,0)}|_U = \theta_t^1 \wedge \ldots \wedge \theta_t^n \wedge \overline{\theta_t^1} \wedge \ldots \wedge \overline{\theta_t^n} = \varphi(t) dz^1 \wedge \ldots \wedge dz^n \wedge$$

$$\wedge \overline{dz}^1 \wedge \ldots \wedge \overline{dz}^n = \varphi(t)[\omega_{X_0}(n,0) \wedge \omega_{X_0}(n,0)]|_U$$

for $n = 3$.

where $\varphi(t)>0$ is a function of $|t|^2$ which do not depend on z and $\overline{z}$. This equality is true on every open set $U \subset X_0$. Here again $\omega_t(n,0)$ is the $(n,0)$ holomorphic form on $X_t = \pi^{-1}(t)$, where $\pi: \mathcal{X} \longrightarrow B$ is the standart Kuranishi family.

<u>Proof.</u> From (6.1.2.2.) it follows that

$$(6.2.2.1.1.) \quad \theta_t^1 \wedge \ldots \wedge \theta_t^n \wedge \overline{\theta_t^1 \wedge} \quad \wedge \overline{\theta_t^n} = \overline{\omega_t(n,0)}|_U =$$

$$= \omega_0(n,0) \wedge \overline{\omega_0(n,0)} + \sum_{k=1}^{n} |t|^{2k} [(\Lambda^k \varphi) \lrcorner\, \omega_0(n,0)] \wedge \overline{[(\Lambda^k \varphi) \lrcorner\, \omega_0(n,0)]} .$$

$n = 3$.

In order to prove 6.2.2.1. it will be enough to prove the following observation:

(6.2.2.1.3.) Let $\omega_1(1,n-1)$ and $\omega_2(n-1,1)$ be two harmonic form of type $(1,n-1)$ and $(n-1,1)$ on a Calabi-Yau manifold with respect to the Kähler Ricci-flat metric $g_{\alpha\overline{\beta}}$, then $\omega_1(1,n-1) \wedge \omega_2(n-1,1)$ is a harmonic form of type (n,n), i.e.

$$\omega_1(1,n-1) \wedge \omega_2(n-1,1) = c\omega_X(n,0) \wedge \overline{\omega_X(n,0)}$$

where c is a constant.

<u>Proof:</u> Repeating the arguments in §2 we see that there is a natural isomorphism for each k:

$$\mu_k : \mathbb{H}^k(X_0, \Lambda^k \Theta_{X_0}) \xrightarrow{\sim} \mathbb{H}^k(X, \Omega^{n-k}_{X_0})$$

where $\mathbb{H}^k(X_0, \Lambda^k\Theta_{X_0})$ is the space of harmonic forms of type $(0,k)$ with coefficients in $\Lambda^k\Theta_{X_0}$, $\mathbb{H}^p(X, \Omega^{n-p}_{X_0})$ is the space of harmonic forms of type $(n-p,p)$. In both cases the forms are harmonic with respect to $g_{\alpha\overline{\beta}}(0)$. The map μ_k is defined as follows:

$$\mu_k(\varphi) = \varphi \lrcorner \omega_{X_0}(n,0) .$$

Let

$$\varphi_1 \overset{\text{def}}{=} \mu^{-1}_{n-1}(\omega_1(1,n-1)) \in \mathbb{H}^{n-1}(X_0, \Lambda^{n-1}\Theta_{X_0})$$

$$\varphi_2 =: \mu^{-1}_1(\omega_2(n-1,1)) \in \mathbb{H}^1(X_0, \Lambda^1\Theta_{X_0}) .$$

If

$$\varphi_1 = \sum \varphi_{1\overline{B}_{n-1}}^{A_{n-1}} d\overline{z}^{\overline{B}_{n-1}} \otimes \frac{\partial}{\partial z}^{A_{n-1}}$$

$$\varphi_2 = \sum (\varphi_2)_{\overline{\beta}}^{\alpha} d\overline{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}}$$

where $d\overline{z}^{B_k} = d\overline{z}^{\beta_1}\wedge\ldots\wedge d\overline{z}^{\beta_k}$ and $\frac{\partial}{\partial z}^{A_k} = \frac{\partial}{\partial z^{\alpha_1}}\wedge\ldots\wedge\frac{\partial}{\partial z^{\alpha_k}}$

$B_k = (\beta_1,\ldots,\beta_k)$; $A_k = (\alpha_1,\ldots,\alpha_k)$. We define

$$\varphi_1 \wedge \varphi_2 = \sum (\varphi_{1\overline{B}_{n-1}}^{A_{n-1}} \times (\varphi_2)_{\beta}^{\alpha}) \, d\overline{z}^{\overline{B}_{n-1}} \wedge d\overline{z}^{\beta} \otimes \frac{\partial}{\partial z^{A_{n-1}}} \frac{\partial}{\partial z^{\alpha}} .$$

<u>Sublemma.</u> $\varphi_1 \wedge \varphi_2$ is a harmonic form of type $(0,n)$ with coefficients in $\Lambda^n\Theta_{X_0}$ with respect to $g_{\alpha\overline{\beta}}(0)$.

<u>Proof:</u> Clearly since $\overline{\partial}\varphi_1 = \overline{\partial}\varphi_2 \Rightarrow \overline{\partial}(\varphi_1 \wedge \varphi_2) = 0$. So we need to prove that

$$\overline{\partial}^*(\varphi_1 \wedge \varphi_2) = 0$$

where $\overline{\partial}^*$ is the conjugate operator of $\overline{\partial}$ with respect to $g_{\alpha\overline{\beta}}(0)$. For $\overline{\partial}^*$ we have the following formula:

$$i\overline{\partial}^* = [D^1,\Lambda] \quad \text{(See [15])}.$$

For the notation see §2. Since $\varphi_1 \wedge \varphi_2$ is a form of type $(0,n)$, then

$\Lambda(\varphi_1 \wedge \varphi_2) = 0$

So we need to prove that

$$(*) \qquad \Lambda[D(\varphi_1 \wedge \varphi_2)] = 0 \ .$$

D is the connection on Θ induced by the metric $g_{\alpha\bar{\beta}}(0)$. Like in § 2 we can choose the coordinates in a point $x_0 \in X_0$ $z^1,\ldots,z^u$ such that: a) $dz^1,\ldots,dz^4$ is an orthonormal basis in $T^*_{x_0,X_0}$ b) $D = \partial$ at the point x_0 b) can be done since $g_{\alpha\bar{\beta}}(0)$ is a Ricci-flat Kähler metric. So after these remarks and the following

(<u>Proposition</u>. Let X be a Calabi-Yau manifold and $(g_{\alpha\bar{\beta}})$ is a Ricci-flat Kähler metric on X , let $\varphi \in H^{n-1}(X,\Lambda^{n-1}\Theta)$ and φ be a harmonic form of type $(0,n-1)$ with coeff. in $\Lambda^{n-1}\Theta$, then there exists $\psi \in H^1(X,\Theta)$ such that 1) $\Lambda^{n-1}\psi = \varphi$ and 2) ψ is a harmonic form in $H^1(X,\Theta)$. Here $n = \dim_{\mathbb{C}} X$.)

(*) is reduced to the following

$$(**) \quad \Lambda[\partial\varphi_1 \wedge \Lambda^{n-1}\psi] = 0 \quad \text{and} \quad \Lambda[\varphi_1 \wedge [(\partial(\Lambda^{n-1}\psi))] = 0$$

Λ in (**) is the contraction with $i\zeta g^{\bar{\beta}\alpha} \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial \bar{z}^\beta}$ where $\varphi_1 \in H^1(X,\Theta)$, $\psi \in H^1(X,\Theta)$ and both are harmonic forms. Now (**) follows from the arguments, repeated word by word in § 2 on p.7, 8 and 9.

<u>Proof of the proposition</u>. <u>Step 1</u>. Let $\psi \in H^1(X,\Theta)$ and ψ

be a harmonic form, then $\Lambda^n\psi \neq 0$, $n = \dim_{\mathbb{C}} X$.

<u>Proof of Step 1</u>. Suppose that $\Lambda^K\psi \neq 0$ but $\Lambda^{K+1}\psi = 0$, $K < n$. From the results of [1] and since $\Lambda^K \psi$ is a harmonic form, then $\Lambda^K\psi$ is a non-zero form on an open and everywhere dense subset of X . Let χ be a point in this open set and let U be a small neighborhood of χ . Let $z^1,\ldots,z^n$ be local coordinates in U such that:

a) $dz^1\wedge\ldots\wedge dz^n|_V = w_X(n,0)|_U$

b) $\psi(dz^j) = 0$ $j = K+1,\ldots,n$ at the point $\chi \in V$, here we look at ψ as a linear map, i.e. $\psi : (T^{1,0})^* \to (T^{0,1})^*$

c) $\psi(dz^1)\wedge\ldots\wedge\psi(dz^K) \neq 0$ in U . This can be done since from $\Lambda^K\psi \neq 0$, $\Lambda^{K+1}\psi = 0 \Rightarrow \mathrm{rk}\psi = K$.

We know from § 3 that for small t $t\psi$ defines an one parameter family of complex structures on X . Let X_1 is a new complex structure defined by ψ , i.e. $t = 1$. This can be done since we can rescale ψ , i.e. consider $\psi = 1/N\ \psi$, for N big enough. The basis of $T^{1,0}(X_1)$ at the point χ is given by:

$$\theta^\alpha = dz^\alpha + \sum_{\beta=1}^{N} \psi\,{}^{\alpha}_{\beta} d\bar{z}^\beta \qquad \alpha = 1,\ldots,K$$

$$dz^\mu \qquad \mu = K+1,\ldots,u\ .$$

Let $w_1 = \theta^1\wedge\ldots\wedge\theta^K$. w_1 is a well defined form on U of type $(K,0)$ with respect to X_1 .

<u>Fact 1</u>. w_1 is a holomorphic form on U with respect to X_1 .

<u>Proof</u>. Since w_1 is a form of type $(K,0)$ with respect to X_1 it is enough to prove that $\bar{\partial}_1 w_1 = 0$ at each point $y \in U$. ($\bar{\partial}_1$ is $\bar{\partial}$ for X_1). First we will check that $\bar{\partial}_1 w_1 = 0$ at χ . We have proved that

$$w_1 \wedge dz^{K+1} \wedge \ldots \wedge dz^n \Big|_{\chi} = w_{X_1}(n,0)\Big|_{\chi} \quad \text{(See 6.1.2.)}$$

($w_{X_1}(n,0)$ is the holomorphic form on X_1 .)

Since $\bar{\partial}_1 w_{X_1}(n,0) \equiv 0$ we get that at the point χ

$$\bar{\partial}_1 w_1 \wedge dz^{K+1} \wedge \ldots \wedge dz^n \equiv 0 \ .$$

So from here it follows that at the point $\chi \in X$

$$\bar{\partial}_1 w_1 = 0 \ .$$

Now let y be any point in U . Then we fix $dz^1, \ldots, dz^K$ and choose $dy^{K+1}, \ldots dy^n$ such that $dz^1, \ldots, dz^K, dy^{K+1}, \ldots, dy^n$ fulfill a), b) & c) on p.42. Repeating the same arguments we get that

$$\bar{\partial}_1 w_1 = 0 \quad \text{at each point } y \in U \ .$$

Q.E.D.

Remark 1. w_1 defines a K-fimensional subspace in $T^*(X) \otimes \mathbb{C}$ E_1 which has the following property:

E_1 = {the maximum dimensional subspace in $T^*(X) \otimes \mathbb{C}$ | $E_1 \subset T^{1,0}(X_1)$ and $E_1 \cap T^{1,0}(X)=0$}. ($T^{1,0}(X)$ is the old (1,0) space.)

Clearly E_1 is defined at those points $\chi \in X$ where $\Lambda^K \psi \neq 0$ but $\Lambda^{K+1}\psi \equiv 0$ and this is an open and everywhere dense subset. Let us denote it by W . So we get a C^∞ family of K-dimensional subspaces in $T(X)^* \otimes \mathbb{C}$. Fact 1 shows that this family of K-dimensional subspaces is complex-analytical family with respect to X_1 .

Remark 2. Let us denote this complex analytic family of K-dimensional subspaces in $T^*(X) \otimes \mathbb{C}$ on W by K . Let $\zeta^1,\dots,\zeta^K$ are orthonormal vectors in U with respect to the Ricci-flat metric $g_{\alpha\bar\beta}(1)$ on X_1 and $\zeta^1,\dots,\zeta^K$ span $K|_U$.

Since w_1 is a holomorphic form of rank K we can find a function f on U such that

$$w_1 = f\ \zeta^1 \wedge \dots \wedge \zeta^K .$$

Let $\zeta^1,\dots,\zeta^K, \zeta^{K+1} \dots, \zeta^n$ be an orthonormal basis of $T^{1,0}(X_1)$ in U . We know that since $(g_{\alpha\bar\beta})$ is a Ricci-flat metric on X_1

$$(*) \qquad w_{X_1}(n,0)|_U = \zeta^1 \wedge \dots \wedge \zeta^K \wedge \zeta^{K+1} \wedge \dots \wedge \zeta^n .$$

Let F be a holomorphic function on $X_1 \cap U$ and $F \neq \mathrm{Const}. F \in \Gamma(U, \mathcal{O}^*_{X_1})$. Now we can find a function g such that

$$Fw_{X_1}(n,0)\Big|_U = (f\zeta^1 \wedge \ldots \wedge \zeta^K) \wedge (g\zeta^{K+1} \wedge \ldots \wedge \zeta^n) \ .$$

From $\bar{\partial}_1(Fw_{X_1}(n,0)|_U) = 0 \Rightarrow \bar{\partial}_1(fg) = 0$ since $\bar{\partial}_1(\zeta^1 \wedge \ldots \wedge \zeta^n) = 0$. So we get that $\bar{\partial}_1 f = 0$ and $\bar{\partial}_1 g = 0$. Since $\bar{\partial} w_1 = \bar{\partial}_1(f\zeta^1 \wedge .. \wedge \zeta^K) = 0$ and $\bar{\partial}_1 f = 0 \Rightarrow \bar{\partial}(\zeta^1 \wedge \ldots \wedge \zeta^K) = 0$. We have proved

Fact 2. $\zeta^1 \wedge \ldots \wedge \zeta^K$ is a holomorphic K form on $U \cap W \subset X_1$.
(**) From (*) and fact 2 $\Rightarrow \zeta^{K+1} \wedge \ldots \wedge \zeta^n$ is a holomorphic $(n-K)$ form on $U \cap X_1$.

From Fact 2 we will prove:

Fact 3. On X_1 there exist globally defined holomorphic K-form $w(K,0)$.

Proof. Let $\{U_i\}$ is a covering of W . On each U_i we can define a form $w_i = \zeta_i^1 \wedge \ldots \wedge \zeta_i^K$ where $\zeta_i^1, \ldots, \zeta_i^K$ are orthonormal vectors that spanned $K|_{U_i}$. For the definition of K see Remark 1 and 2.

Since for each $i, w_i = \zeta_i^1 \wedge \ldots \wedge \zeta_i^K$ is a holomorphic form on $U_i \cap U_j$ we have $w_i = f_{ij} w_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$. On the other hand we have $w_i \wedge \bar{w}_i = w_j \wedge \bar{w}_j$ on $U_i \cap U_j$ so from here we get that $|f_{ij}|^2 = 1$ and so from the maximum principle we obtain that $f_{ij} = \mathrm{const}$. So from here we get that on an open and everywhere dense subset $\mathcal{W} \subset X_1$ we have a holomorphic K-form $w_1(K,0)$. Since $\wedge^K \varphi$ is defined on the whole X_1 and form the definition of K we get that $w_1(K,0)$ is defined as C^∞-form on the whole X_1. So from here we get that $w_1(K,0)$ is a holomorphic form defined everywhere on X_1. From Bochner principle and the structure theorem we get that 1) $w_1(K,0)$ is a parallel form on X_1 2) $X_1 = Z \times Y$ as a complex manifold, where on Y there exists a holomorphic $(n-K)$ form $w_2(n-K,0)$ such that $w_1(K,0) \wedge w_2(n-K,0) = w_{X_1}(n,0)$. From the definition of K and so from the way we define $w_1(K,0)$ we obtain that $w_2(n-K,0)$ is a holomorphic form on X. So from Bochner principle we get that the holonomy group of X is $SU(n-K) \times SU(K)$ $(K>0)$. So we get a contradiction with the fact that X is a Calabi-Yau manifold, i.e. it has a holonomy group $SU(n)$. So $\Lambda^n \psi \neq 0$. From here and the fact that $\dim_{\mathbb{C}} H^1(X,\Theta) = \dim_{\mathbb{C}} H^{n-1}(X,\Lambda^{n-1}\Theta)$ (Serre's duality) we get that the map $\psi \to \Lambda^{n-1}\psi$ is a one to one map.

Q.E.D.

Remark. In the computation that follows we will use the fact that $\dim_{\mathbb{C}} X = 3$ in the following moment: We need to compute just $w_{X_0}(3,0) \wedge \overline{w_{X_0}(3,0)}$, $[\varphi \lrcorner w_{X_0}(3,0)] \wedge [\overline{\varphi \lrcorner w_{X_0}(3,0)}]$ and $[(\Lambda^2 \varphi \lrcorner w_{X_0}(3,0) \wedge (\overline{\Lambda^2 \varphi \lrcorner w_{X_0}(3,0)})$. In the computation of the second term of this sequence we will use the Proposition just proved, i.e. if $\psi \in H^{n-1}(X,\Lambda^{n-1}\Theta)$, then $\psi = \Lambda^{n-1}\varphi$.

Since $\varphi_1 \wedge \varphi_2$ is a harmonic form of type $(0,n)$ with coefficients in $\Lambda^n \Theta_{X_0}$ we get that

$$(\varphi_1 \wedge \varphi_2) \rfloor \omega_{X_0}(n,0)$$

is a harmonic form of type $(0,n)$, where $n = \dim_{\mathbb{C}} X$, so

$$(***) \qquad (\varphi_1 \wedge \varphi_2) \rfloor \omega_{X_0}(n,0) = a\, \overline{\omega_{X_0}(n,0)}, \text{ a constant}$$

from (***) and the following easy formulas:

$$\omega_1(1,n-1) \wedge \omega_2(n-1,1) = \pm[(\varphi_1 \wedge \varphi_2) \rfloor \omega_{X_0}(n,0)] \wedge \omega_{X_0}(n,0)$$

we get that

$$\omega_1(1,n-1) \wedge \omega_2(n-1,1) = c\, \omega_{X_0}(n,0) \wedge \overline{\omega_{X_0}(n,0)} = c \text{ vol } (g_{\alpha\overline{\beta}}(0)$$

where c is a constant.

So $\omega_t(3,0) \wedge \omega_t(0,3) = \varphi(t)\, \omega_0(3,0) \wedge \omega_0(0,3)$.

Q.E.D.

<u>The end of the proof of lemma 6.2.2.</u>

Let $\pi: \mathcal{X} \longrightarrow B$ be the standart Kuranishi family. Let $\{\mathcal{U}_i\}$ be a covering of $\mathcal{X}$, where $\mathcal{U}_i = U_i \times B$ and $\{U_i\}$ be a covering of X_0. Let for each $t \in B$

$$\int_{X_t} \omega_t(n,0) \wedge \overline{\omega_t(n,0)} = 1$$

where $X_t = \pi^{-1}(t)$.

Let $(\zeta_i^1(z,\bar{z};t),\ldots,\zeta_i^n(z,\bar{z};t),t^1,\ldots,t^N)$ be local coordinates in U_i such that for each t we have:

$$\omega_t(n,0)|_{U_i \cap X_t} = d\zeta_i^1 \wedge \ldots \wedge d\zeta_i^n.$$

According to Kodaira-Spencer theory of deformation on $U_i \cap X_0$ there exists a real vector field which induces a diffeomorphism $\psi_i(t):U_i \longrightarrow U_i$ such that

$$d((\psi_i(t))^*(z_i)) = d\zeta_i^\alpha(z,\bar{z};t).$$

Let for each $t \in B$ $g_{\alpha\bar{\beta}}(t)$ be the Calabi-Yau metric on X_t such that

$$[\operatorname{Im} g_{\alpha\bar{\beta}}(t)] = L$$

We have for each t

$$\operatorname{Im} g_{\alpha\bar{\beta}}(t)|_{U_i \cap X_t} = \frac{1}{2} \sum \omega_{\alpha\bar{\beta}}\, d\zeta_i^\alpha \wedge \overline{d\zeta_i^\beta}$$

and

$$(6.2.2.1.4.) \quad \frac{i}{2}\sum \omega_{\alpha\bar{\beta}}\, d\zeta_i^\alpha \wedge \overline{d\zeta}^\beta = \frac{i}{2}\sum \psi_i(t)^*(g_{\alpha\bar{\beta}}(0)|_{U_i})\, d(\psi_i^*(t)z^\beta) + $$
$$+\, i\partial\bar{\partial}\, f(\zeta_i,\bar{\zeta}_i;t)$$

where $f(\zeta_i,\overline{\zeta_i};t)$ is a function on U_i. Let $z_0 \in U_i \cap X_0$.

Since $\psi_i(t)^*(g_{\alpha\overline{\beta}}(0)|_{U_i})$ is a positive definite Kähler metric on $U_i \cap X_0$ we choose the coordinates $(z^1,\ldots,z^n)$ in such a way that at the point z_0 for a fix t

$$(\psi_i(t))^*(g_{\alpha\overline{\beta}}(0)(z_0)) = \delta_{\alpha\overline{\beta}}, \text{ where } \delta_{\alpha\overline{\beta}} = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha \neq \beta \end{cases}$$

and

$$\partial_\alpha \overline{\partial}_\beta \, f(\zeta_i,\overline{\zeta}_i;t) = \delta_{\alpha\overline{\beta}} \, f_{\alpha\overline{\beta}} \quad .$$

So from here we get that:

$$\det\left(\frac{i}{2} \cdot \sum \omega_{\alpha\overline{\beta}} \, d\zeta_i^\alpha \wedge d\overline{\zeta}_i^\beta\right) = \omega_t(n,0) \wedge \overline{\omega_t(n,0)} = \omega_0(n,0) \wedge \overline{\omega_0(n,0)} \ .$$

(This follows from prop. 6.2.2.1.) So at the point z_0 we have

$$\det\left(\frac{i}{2}\sum \omega_{\alpha\overline{\beta}} \, d\zeta_i^\alpha \wedge d\overline{\zeta}^\beta\right) = \det\left(\delta_{\alpha\overline{\beta}}(1 + f_{\alpha\overline{\beta}}) \, d\zeta^\alpha \wedge d\zeta^\beta\right) =$$

(6.2.2.1.5)

$$= \prod_{\alpha=1}^{n} (1+f_{\alpha\overline{\alpha}}) d\zeta^1 \wedge \ldots \wedge d\zeta^n \wedge d\overline{\zeta}^1 \wedge \ldots \wedge d\overline{\zeta}^n =$$

$$= \prod_{\alpha=1}^{n} (1+f_{\alpha\overline{\alpha}}) d(\psi_i(t)^* z^1) \wedge \ldots \wedge \overline{d(\psi_i^*(t) z^n)} =$$

$$= \prod_{\alpha=1}^{n} (1+f_{\alpha\overline{\beta}}) \overline{dz}^1 \wedge \ldots \wedge \overline{dz}^n \wedge \overline{dz}^1 \wedge \ldots \wedge \overline{dz}^n \ .$$

We have proved that $\det\left(\frac{i}{2}\sum \omega_{\alpha\overline{\beta}} d\zeta^\alpha \wedge \overline{d\zeta}^\beta\right)$ is constant along the trajectories of the vector field that defines the deformation of the standart Kuranishi family $\pi:\mathcal{X} \longrightarrow U$ locally on U_i. (This is reformulation of 6.2.2.1). So from here we get that for each t we must have

$$(6.2.2.1.5.) \qquad \prod_{\alpha=1}^{n} (1 + f_{\alpha\overline{\alpha}}) = 1 \ .$$

Since $f_{\alpha\overline{\alpha}}$ depends C^∞ on t and $f_{\alpha\overline{\alpha}} = 0$ when $t = 0$ we get that $f_{\alpha\overline{\alpha}} = tf^1_{\alpha\overline{\alpha}}$. So we must have that

$$\prod_{\alpha=1}^{n} (1+tf^1_{\alpha\overline{\alpha}}) = 1 \ .$$

So from here we get that $f^1_{\alpha\overline{\alpha}} \equiv 0$. So we have proved that for each t locally the Ricci-flat metric $g_{\alpha\overline{\alpha}}(t)$ on X_t that corresponds to L_1 is given locally on U_i by:

$$\frac{i}{2}\sum \omega_{\alpha\overline{\beta}}\, d\zeta_i^\alpha \wedge \overline{d\zeta_i^\beta} = \frac{i}{2}\sum (\psi_i(t)^* g_{\alpha\overline{\beta}}) \wedge d(\psi_i(t)^* z^\alpha) \wedge \overline{d(\psi_i^*(t) z^\beta)}$$

From this formula we get that on $U_i \subset X_0$ we have

$$(6.2.2.1.6.) \quad \frac{i}{2}\sum \omega_{\alpha\overline{\beta}}\, d\zeta_i^\alpha \wedge d\overline{\zeta}_i^\beta = \frac{i}{2} \sum g_{\alpha\overline{\beta}}(z,\overline{z})\, d\zeta^\alpha(z,\overline{z};t) \wedge d\overline{\zeta}^\beta(z,\overline{z}) .$$

Siu and Nannicini proved that $L_v(i\sum \omega_{\alpha\overline{\beta}}\, d\zeta \wedge d\overline{\zeta}\,)\,|_t = 0$, where v is a vector field that defines the trivialization (holomorphic) one, on U_i defined by the deformation coming from a harmonic $\varphi \in H^1(X_0, \Theta_{X_0})$. L_v means the Lie derivative. The result of Siu and Nannacini means that (6.2.2.1.6) can be expressed in terms of the local coordinates $(z^1,\ldots,z^n)$ in the following way:

(6.2.2.1.7.)

$$\frac{i}{2}\sum g_{\alpha\bar{\beta}}(z,\bar{z})\,d\zeta^{\alpha}\wedge d\bar{\zeta}^{\beta} = \frac{i}{2}\left(\sum g_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta} + t^2\sum \psi_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}^{U}\right) \quad .$$

Again we fix a point $z_0 \in U_i$ and we can choose the coordinates $(z^1,\ldots z^n)$ such that

(6.2.2.1.8.) $$\frac{1}{2}\left(\sum g_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta} + t^2\sum \psi_{\mu\bar{\nu}}dz^{\mu}\wedge d\bar{z}^{\nu}\right) =$$

$$= \frac{i}{2}\left(\sum_{\alpha=1}^{3} dz^{\alpha}\wedge d\bar{z}^{\alpha} + t^2\sum_{\alpha=1}^{3} \psi_{\alpha\bar{\alpha}}dz^{\alpha}\wedge d\bar{z}^{\alpha}\right) \quad .$$

This can be done since the deformation defined the harmonic form $\varphi \in H^1(X,\Theta)$ is defined to first order deformation. This means that $\frac{d^K}{dt^K}(d_{\zeta}\alpha) \equiv 0$ for $K \geq 2$. Since in (6.2.2.1.7) $\frac{d}{dt}(g_{\alpha\bar{\beta}}(z,\bar{z})) = 0$ we get from $\frac{d^K}{dt^K}(d_{\zeta}\alpha) = 0 (K \geq 2)$ that $\frac{d}{dt}(\psi_{\mu\nu}) = 0$.

Since the volume form of $\frac{i}{2}\sum g_{\alpha\bar{\beta}}d\zeta^{\alpha}\wedge d\bar{\zeta}^{\beta}$ is const., i.e. $w_0(3,0) \wedge \overline{w_0(3,0)}$ we get that

$$\prod_{\alpha=1}^{3}(1+t^2\psi_{\alpha\bar{\alpha}}) = 1 \quad .$$

So form here we conclude that $\psi_{\alpha\bar{\alpha}} = 0$. So our lemma is proved.

Q.E.D.

6.3. Geodesics on $\mathcal{T}(X)$ with respect to the Weil-Petersson metric.

<u>Remark 6.3.1.</u> Suppose that $\varphi \in \mathbb{H}^1(X_0, \Theta_{X_0})$, then for each $t \in \mathbb{C}$ we define $A_t: T(X_0) \otimes \mathbb{C} \longrightarrow T(X_0) \otimes \mathbb{C}$ as

$$A_t(dz\ ) = dz^\alpha + t\sum \varphi^{\alpha}_{\overline{\beta}}\, d\overline{z}^\beta$$

$$A_t(\overline{dz}^\mu) = \overline{dz}^\mu + t\sum \overline{\varphi^{\alpha}_{\overline{\beta}}}\, dz^\beta \quad .$$

From 6.2.2.1. it follows that for each $t \in \mathbb{C}$ 1 $n = 3$

$$A_t(dz^1)\wedge\ldots\wedge(A_t dz^n) \wedge (A_t\overline{dz}^1)\wedge\ldots\wedge(A_t\overline{dz}^\mu) = dz^1\wedge\ldots\wedge dz^n \wedge$$

$$\wedge\ \overline{dz}^1\wedge\ldots\wedge\overline{dz}^n$$

where c is a constant. So from here it follows that for each $t \in \mathbb{C}$ the operator $J_t = A_t J_0 A_t^{-1}$ defines a new complex structure on X_0, which is integrable. The deformations defined by $J_t = A_t J_0 A_t^{-1}$ for a fixed $\varphi \in \mathbb{H}^1(X_0, \Theta_{X_0})$ we will call the standart line in $\mathcal{T}(X)$ and will denote it by $\mathcal{X}_t \xrightarrow{\pi} S(t)$.

Since for each $t \in S(t)$ $\omega_t(n,0)$ is a harmonic form with respect to $g_{\alpha\overline{\beta}}(0)$ it follows that the period map $p: S(t) \hookrightarrow Gr$ is an embedding. Clearly we have that $S(t) \hookrightarrow \mathcal{T}(X)$.

<u>Lemma 6.3.2.</u> The standart line is a complex geodesics with respect to the Weil-Petersson metric on $\mathcal{T}(X)$.

<u>Proof:</u> The proof of 6.3.2. is based on the following remark. For each t the imaginary part of the Calabi-Yau metric $g_{\alpha\overline{\beta}}(t)$ defines a symplectic structure, i.e. a skew symmetric scalar product $(\ ,\ )_t$ on X_t, where $X_t = \pi^{-1}(t)$ and $\pi: \mathcal{X} \longrightarrow U$ is the standart Kuranishi family. We have prove that for each $t \in U$ Im $(g_{\alpha\overline{\beta}}(t))$ defines one and the same symplectic structure. So the Riemannian structure for each t defined by $(g_{\alpha\overline{\beta}}(t))$ is given by

$$(6.3.2.1.) \qquad \langle u,v\rangle_t = (J_t u,v)_0 = (A_t J_0 A^{-1}_t u,v)_0 = = (J_0 A_t^{-1} u, A_t^{-1} v) = \langle A_t^{-1} u, A_t^{-1} v\rangle_0$$

at each point $x \in X_0$.
Using 6.3.2.1. we will prove

<u>Proposition 6.3.2.2.</u> WE have the following equality:

$$\langle \dot{S}(t), \dot{S}(t)\rangle_{w.p.} = \|\dot{S}(t)\|^2_{w.p.} = \text{wust for each } t \in \mathbb{C},$$

where $\dot{S}(t)$ is the tangent vector to the curve $S(t)$ at the point $t \in S(t)$.

Proof: The proof is based on the following two remarks

Remark 1. $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0} = \sum_{q=1}^{n} (-1)^{\alpha(q)} q s_0^{q-1} ([\Lambda^q\varphi] \lrcorner \omega_0(n,0))$

is a harmonic form of type $(n-1,1)$ with respect to the complex structure J_{s_0} and the Ricci-flat metric $(g_{\alpha\overline{\beta}}(s_0))$.

Proof: From the formula:

$$\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0} = \sum_{q=1}^{n} (-1)^{\alpha(q)} q s_0^{q-1} ([\Lambda^q\varphi \lrcorner \omega_0(n,0))$$

the fact that for each q the form $[\Lambda^q\varphi] \lrcorner \omega_0(n,0)$ is a harmonic form with respect to $g_{\alpha\overline{\beta}}(0)$ we get that the form $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}$ is a closed form, i.e.

$$(d\,\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0} = 0.$$

Let us reparametrize our curve $S(t)$ in such a way that we get a curve $S(s)$ such that the family: $\pi\ \mathcal{X} \longrightarrow S(s)$ is the same as $\mathcal{X} \longrightarrow S(t)$ and in the reparametrized family we have: $\pi^{-1}(0) = X_{s_0}$. This can be done very easily. Let $\{U_i\}$ be a covering of X_0 and $\{\mathcal{U}_i\} = \{U_i \times \mathbb{C}\}$ be a covering of $\mathcal{X} \xrightarrow{\pi} S(s)$. Let $(\zeta^1(z,s),\ldots,\zeta^n(k,s),s)$ be local coordinates in $\mathcal{U}_i$ such that

$$\omega_S(n,0)\,|_{U_i} = d\zeta^1(z,s)\wedge\ldots\wedge d\zeta^n(z,s).$$

From the fact that we have:

$$d\zeta^{\alpha}(z,s) = d\zeta^{\alpha}(z,s_0) + s \sum \psi^{\alpha}_{\beta} \, \overline{d\zeta}^{\beta}(z,s_0) + 0(s^2)$$

we get that

$$\frac{d\omega\,(n,0))}{ds}\Big|_{s=0} = \frac{d\omega_t(n,0)}{dt} =$$

$$= \frac{d}{ds} (d\zeta^1(z,s) \wedge \ldots \wedge d\zeta^n(z,s))\Big|_{=0} =$$

$$= \sum_{\alpha,\beta} \psi^{\alpha}_{\beta} \, d\zeta^{\beta}(z, {}_0) \wedge d\zeta^1(z,s_0) \wedge \ldots \wedge d\zeta^{\alpha-1}(z,s_0) \wedge d\zeta^{\alpha+1}$$

$$d\zeta^{\alpha+2}(z,s_0) \wedge \ldots \wedge d\zeta^n(z,s_0)$$

and this is clearly a form of type $(n-1,1)$ with respect to the complex structure J_{s_0}. Since for each $t \in \mathbb{C}$ $\omega_t(n,0)$ is a primitive form and the symplectic structure does not change we get that $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}$ is a primitve form. So $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}$ is a closed primitive form of type $(n-1,1)$ on X_{s_0}. From the following formula:

Let η be a primitive form of type (a,b), then

$$*\eta = \frac{i^{a-b}}{(n-a-b)!} (-1)^{\frac{(a+b)(a+b+1)}{2}} L^{n-a-b} \eta$$

we get that $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}$ is a harmonic form with respect to $g_{\alpha\overline{\beta}}(s_0)$ since

$$*\left(\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}\right) = i^{n-2}(-1)^{\frac{n(n+1)}{2}}\left(\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}\right)$$

and so $*\left(\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}\right)$ is a closed form. From here we get that $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}$ is a harmonic form with respect to $g_{\alpha\overline{\beta}}(s_0)$.

Q.E.D.

<u>Remark 2.</u> $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0} = \pm\sum \varphi^{\alpha}_{\overline{\beta}}\, \overline{\theta}^{\beta} \wedge \theta^1 \wedge \ldots \wedge \hat{\theta}^{\alpha} \wedge \ldots \wedge \theta^n$,

where $\varphi \in H^1(X_0, \Theta_{X_0})$ and $\varphi|_U = \sum \varphi^{\alpha}_{\overline{\beta}}\, \overline{dz}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}}$.

<u>Proof:</u> Locally $\omega_t(n,0)$ can be written as:

$$(*) \quad \omega_t(n,0) = dz^1\wedge\ldots\wedge dz^n + \sum_{q=1}^{n} (-1)^{\alpha(q)} t^q ([\Lambda^q\varphi] \lrcorner\, \omega_0(n,0))|_U$$

we know that $dz^{\mu} = A^{-1}_{s_0}\, \theta^{\mu}_{s_0}$. From the definition of the operator A_t, i.e.

$$A_t(dz^{\alpha}) = dz^{\alpha} + t\sum\varphi^{\alpha}_{\overline{\beta}}\, \overline{dz}^{\beta}$$

we get that $A_t = id - t\varphi + \ldots$. So from (*) we get

$$(**) \quad \omega_t(n,0) = (A^{-1}_{s_0}\theta^1_{s_0})\wedge\ldots\wedge(A^{-1}_{s_0}\theta^n_{s_0}) + \sum_{q=1}^{n} (-1)^{\alpha(q)} t^q$$

$$([\Lambda^q\varphi] \lrcorner\, (A^{-1}_{s_0}\, \theta^1_{s_0} \wedge\ldots\wedge A^{-1}_{s_0}\, \theta^n_{s_0}))$$

In (**) we need to compute the (n-1,1) part with respect to $\Theta^1_{s_0},\dots,\Theta^n_{s_0},\overline{\Theta^1_{s_0}},\dots,\overline{\Theta^n_{s_0}}$ taking into acount that

$$A^{-1}_{s_0} = \mathrm{id} - s_0\varphi + 0(s_0^2)\tilde{\varphi}\ , \text{ where } \varphi : T^{1,0} \longrightarrow \overline{T^{1,0}_{s_0}}$$

and $\tilde{\varphi} : T^{1,0}_{s_0} \longrightarrow T^{0,1}_{s_0}$.

So from here we get after direct calculation that the (n-1,1) part of $\omega_t(n,0)$ with respect to $\Theta^1_{s_0},\dots,\Theta^n_{s_0},\overline{\Theta^1_{s_0}},\dots,\overline{\Theta^n_{s_0}}$ will be:

$$\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0} = \pm\sum h^{\alpha}_{\overline{\beta}}\ \Theta^{\overline{\beta}}_{s_0} \wedge \Theta^1_{s_0} \wedge\dots\wedge \Theta^{\alpha-1}_{s_0} \wedge \Theta^{\alpha+1}_{s_0}\wedge\dots\wedge\Theta^n_{s_0} =$$

$$= \pm\sum h^{\alpha}_{\overline{\beta}}(A^{-1}_{s_0}\overline{dz^{\beta}}) \wedge (A^{-1}_{s_0}\, dz^1)\wedge\dots\wedge (A^{-1}_{s_0}dz^{\alpha-1})\wedge(A^{-1}_{s_0}dz^{\alpha-1}) \wedge$$

$$\wedge\dots\wedge (A^{-1}_{s_0}dz^n)$$

Q.E.D.

<u>The end of the proof of 6.3.2.2.</u>

From remark 2 it follows that

$$(6.3.2.2.1.) \qquad \frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0} = \Big(\sum h^{\alpha}_{\overline{\beta}}\ \overline{\Theta}^{\beta}_{s_0} \otimes \frac{\partial}{\partial\Theta^{\alpha}_{s_0}}\Big) \lrcorner$$

$$(\Theta^1_{s_0}\wedge\dots\wedge\Theta^n_{s_0}) \ .$$

Since we have a natural map

$$\mu_1 : \mathbb{H}^1(X_{s_0}, \Theta_{X_{0}}) \xrightarrow{\sim} \mathbb{H}^1(X_{s_0}, \Omega^{n-1}_{X_{s_0}})$$

where $\mu_1(\varphi) = \varphi \lrcorner\, \omega_{s_0}(n,0)$. Moreover μ_1 is an isometry with respect to the metric induced on $\mathbb{H}^1(X_{s_0}, \Theta_{X_{s_0}})$ and $\mathbb{H}^1(X_{s_0}, \Omega^{n-1}_{X_{s_0}})$ by $(g_{\alpha\bar{\beta}}(s_0))$. Moreover we know from remark 2 that $\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}$ is a harmonic form with respect to $g_{\alpha\bar{\beta}}(s_0)$. So from (6.3.2.2.1.) we get that

$$\mu_1^{-1}\left(\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}\right) = \sum \varphi^{\alpha}_{\bar{\beta}}\, \Theta^{\beta} \otimes \frac{\partial}{\partial \Theta_{\alpha}} =$$

$$= \tilde{A}^{-1}_{s_0}\left(\sum \varphi^{\alpha}_{\bar{\beta}}\, d\bar{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}}\right)$$

where $\tilde{A}^{-1}_{s_0}$ is a linear map from $T^* \otimes T \longrightarrow T^* \otimes T$ induced by A_{s_0}. From the following formula (6.3.2.1.)

$$\langle u,v\rangle_{s_0} = \langle A^{-1}_{s_0} u, A^{-1}_{s_0} v\rangle_0$$

we get that

$$\|\dot{S}(s_0)\|^2 = \left\|\mu_1^{-1}\left(\frac{d\omega_t(n,0)}{dt}\Big|_{t=s_0}\right)\right\|^2_{W.P.} = \|A^{-1}_{s_0}\varphi\|^2 = \|\varphi\|^2 = \|\dot{S}(0)\|^2$$

So 6.3.2.2. is proved.

Q.E.D.

From 6.3.2.2. we get that

$$\partial\bar{\partial}\| \dot{S}(t) \|^2 = 0 .$$

On the other hand we have:

$$\partial\bar{\partial}\| \dot{S}(t) \|^2 = -\bar{\partial}\partial<\dot{S}(t),\dot{S}(t)> = -\bar{\partial}<D'\dot{S}(t),\dot{S}(t)> =$$

$$= + <D'\dot{S}(t),D'(\dot{S}(t)> - <\bar{\partial}D'\dot{S}(t),\dot{S}(t)> = \| D'\dot{S}(t) \|^2 - <R\ (\dot{S}(t)),\dot{S}(t)>$$

where R is the curvature operator, which is negative. So we get that

$$\partial\bar{\partial}\|\dot{S}(t)\|^2 = 0 = \| D'\dot{S}(t) \|^2 - <R\ \ \dot{S}(t),\dot{S}(t)> \geq 0.$$

So it follows that $\| D'\dot{S}(t) \|^2 = 0 \Rightarrow D'\dot{S}(t) = 0$. So 6.3.2. is proved since $\bar{\partial}\ \dot{S}(t) = 0$ since $S(t)$ is a holomorphic curve in $\mathcal{T}(X)$.

Q.E.D.

Now we need to prove that we have an embedding

$$p:\mathcal{T}(X) \hookrightarrow Gr.$$

The proof that the period map $p:\mathcal{T}(X) \longrightarrow Gr$ is an embedding is based on the following three remarks.

Remark 1. p is a local isomorphism this is just local Torelli theorem for manifolds with $c_1 \equiv 0$.

Remark 2. Since the Weil-Petersson metric has negative sectional holomorphic curvature, then any two points of $T(X)$ p,q can be joint by a geodesics. For the proof of this fact see [9]. So let this geodesics be $S(t)$, i.e. the standart line.

Remark 3. Let $p \in S(t)$, then p corresponds to a complex Kähler Calabi-Yau manifold X_p with a fixed Calabi-Yau metric $g_{\alpha\overline{\beta}}(g)$. We know that each Calabi-Yau manifold X_t has a holomorphic form $\omega_t(n,0)$ where:

a) $$\omega_t(n,0) = \omega_p(n,0) + \sum_{q=1}^{n} (-1)^{\alpha(q)} [[\Lambda^q \varphi] \lrcorner \ \omega_p(n,0)]t^q$$

b) $\omega_t(n,0)$ is a harmonique form with respect to the Yau metric $g_{\alpha\overline{\beta}}(p)$.

So from a)b) it follows that each form $\omega_t(n,0)$ is different from $\omega_p(n,0)$ in $\mathbb{P}(H^n(X,\mathbb{C}))$.

So from Remarks 1,2 and 3 follows that p is an embedding. Since if p,q corresponds to two holomorphic form $\omega_p(n,0)$ and $\omega_q(n,0)$ such that

$$[\omega_p(n,0)] \equiv [\omega_q(n,0)] \quad \text{in} \quad H^n(X,\mathbb{C})$$

where $p,q \in T(X)$. Now we can joint p,q with a geodesics $S(t)$. From remark 3 it follows that

$$p \equiv q \ .$$

So from here we get the weak version of Global-Torelli theorem.

Q.E.D.

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