DEFORMATION QUANTIZATION WITH GENERATORS AND RELATIONS

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Abstract. In this paper we prove a conjecture of B. Shoikhet which claims that two quantization procedures arising from Fourier dual constructions actually coincide.

1. Introduction

There are two ways to quantize a polynomial Poisson structure π on the dual V^* of a finite dimensional vector space V, using Kontsevich's formality as a starting point.

The first (obvious) way is to consider the image $\mathcal{U}(\pi_\hbar)$ of $\pi_\hbar = \hbar\pi$ through Kontsevich's L_∞-quasi-isomorphism

$$\mathcal{U} : \mathrm{T}_{\mathrm{poly}}(V^*) \longrightarrow \mathrm{D}_{\mathrm{poly}}(V^*)\,,$$

and to take $\mathrm{m}_\star := \mathrm{m} + \mathcal{U}(\pi_\hbar)$ as a $\star$-product quantizing π, m being the standard product on $\mathrm{S}(V) = \mathcal{O}_{V^*}$.

The main idea, due to B. Shoikhet [8], behind the second (less obvious) way is to deform the relations of $\mathrm{S}(V)$ instead of the product m itself. Namely, one makes use of the graded version [3] of Kontsevich's formality theorem, applied to the Fourier dual space $V[1]$. We then have an L_∞-quasi-isomorphism

$$\mathcal{V} : \mathrm{T}_{\mathrm{poly}}(V^*) \cong \mathrm{T}_{\mathrm{poly}}(V[1]) \longrightarrow \mathrm{D}_{\mathrm{poly}}(V[1])\,,$$

and the image $\mathcal{V}(\widehat{\pi_\hbar})$ of $\widehat{\pi_\hbar}$, where $\widehat{\bullet}$ is the isomorphism $\mathrm{T}_{\mathrm{poly}}(V^*) \cong \mathrm{T}_{\mathrm{poly}}(V[1])$ of dg Lie algebras (graded Fourier transform), induces a deformation of the cobar differential. It then gives a deformation $\mathcal{I}_\star$ of the two-sided ideal $\mathcal{I}$ in $\mathrm{T}(V)$ of defining relations of $\mathrm{S}(V)$.

Reinterpreting the deformation of the cobar resolution of $\mathrm{S}(V)$ in the context of the formality with 2 branes [2], we are able to prove the following result, first conjectured by Shoikhet in [7, Conjecture 2.6]:

Theorem 1.1 (see Theorem 2.7). *The algebra* $A_\hbar := \big(\mathrm{S}(V)[\![\hbar]\!], \mathrm{m}_\star\big)$ *is isomorphic to the quotient of* $\mathrm{T}(V)[\![\hbar]\!]$ *by the two-sided ideal* $\mathcal{I}_\star$*; the isomorphism is an* $\hbar$*-deformation of the standard symmetrization map from* $\mathrm{S}(V)$ *to* $\mathrm{T}(V)$.

The paper is organized as follows. In Section 2 we start with a recollection on A_∞-algebras and bimodules. We then formulate the formality theorem with two branes of [2] in a form suitable for the application at hand. After this we describe the deformation of the cobar complex obtained from $\mathcal{V}(\widehat{\pi_\hbar})$ and prove Theorem 1.1. We conclude the paper with three examples, see Section 3: the cases of constant, linear, and quadratic Poisson structures.

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2. A deformation of the cobar construction of the exterior coalgebra

2.1. **A_∞-algebras and (bi)modules of finite type.** We first recall the basic notions of the theory of A_∞-algebras and modules, see [2, 5] to fix the conventions and settle some finiteness issues. Note that we allow non-flat A_∞-algebras in our definition. Let $\mathrm{T}(V) = \mathbb{C} \oplus V \oplus V^{\otimes 2} \oplus \cdots$ be the tensor coalgebra of a $\mathbb{Z}$-graded complex vector space V with coproduct $\Delta(v_1, \ldots, v_n) = \sum_{i=0}^n (v_1, \ldots, v_i) \otimes (v_{i+1}, \ldots, v_n)$ and counit $\eta(1) = 1$, $\eta(v_1, \ldots, v_n) = 0$ for $n \geq 1$. Here we write $(v_1, \ldots, v_n)$ as a more transparent notation for $v_1 \otimes \cdots \otimes v_n \in \mathrm{T}(V)$ and set $() = 1 \in \mathbb{C}$. Let $V[1]$ be the graded vector space with $V[1]^i = V^{i+1}$ and let the suspension $s\colon V \to V[1]$ be the map $a \mapsto a$ of degree -1. Then an A_∞-algebra over $\mathbb{C}$ is a $\mathbb{Z}$-graded vector space B together with a codifferential $\mathrm{d}_B\colon \mathrm{T}(B[1]) \to \mathrm{T}(B[1])$, namely a linear map of degree 1 which is a coderivation of the coalgebra and such that $\mathrm{d}_B \circ \mathrm{d}_B = 0$. A coderivation is uniquely given by its components $\mathrm{d}_B^k\colon B[1]^{\otimes k} \to B[1]$, $k \geq 0$ and any set of maps : $B[1]^{\otimes k} \to B[1]$ of degree 1 uniquely extends to a coderivation. This coderivation is a codifferential if and only if $\sum_{j+k+l=n} \mathrm{d}_B^n \circ (\mathrm{id}^{\otimes j} \otimes \mathrm{d}_B^k \otimes \mathrm{id}^{\otimes l}) = 0$ for all $n \geq 0$. The maps d_B^k are called *Taylor components* of the codifferential d_B. If $\mathrm{d}_B^0 = 0$, the A_∞-algebra is called *flat*. Instead of d_B^k it is convenient to describe A_∞-algebras through the product maps $\mathrm{m}_B^k = s^{-1} \circ \mathrm{d}_B^k \circ s^{\otimes k}$ of degree

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$2-k$. If $\mathrm{m}_B^k=0$ for all $k\neq 1,2$ then B with differential m_B^1 and product m_B^2 is a differential graded algebra. A *unital* A_∞-algebra is an A_∞-algebra B with an element $1\in B^0$ such that

$$\mathrm{m}_B^2(1,b)=\mathrm{m}_B^2(b,1)=b,\quad \forall b\in B,$$
$$\mathrm{m}_B^j(b_1,\dots,b_j)=0,\qquad \text{if } b_i=1 \text{ for some } 1\le i\le j \text{ and } j\neq 2.$$

The first condition translates to $\mathrm{d}_B^2(s1,b)=b=(-1)^{|b|-1}\mathrm{d}_B^2(b,s1)$, if $b\in B[1]$ has degree $|b|$. A *right module* over an A_∞-algebra B is a graded vector space M together with a degree one codifferential d_M on the cofree right $\mathrm{T}(B[1])$-comodule $M[1]\otimes\mathrm{T}(B[1])$ cogenerated by M. The Taylor components are $\mathrm{d}_M^j\colon M[1]\otimes B[1]^{\otimes j}\to M[1]$ and in the unital case we require that $\mathrm{d}_M^1(m,s1)=(-1)^{|m|-1}m$ and $\mathrm{d}_M^j(m,b_1,\dots,b_j)=0$ if some b_j is $s1$. Left modules are defined similarly. An A_∞-A-B-bimodule M over A_∞-algebras A, B is the datum of a codifferential on the $\mathrm{T}(A[1])$-$\mathrm{T}(B[1])$-bicomodule $\mathrm{T}(A[1])\otimes M[1]\otimes\mathrm{T}(B[1])$, given by its Taylor components $\mathrm{d}_M^{j,k}\colon A[1]^{\otimes j}\otimes M[1]\otimes B[1]^k\to M[1]$. The following is a simple but important observation.

Lemma 2.1. *If M is an A_∞-A-B-bimodule and A is a flat A_∞-algebra then M with Taylor components $\mathrm{d}_M^{0,k}$ is a right A_∞-module over B.*

Morphisms of A_∞-algebras (A_∞-(bi)modules) are (degree 0) morphisms of graded counital coalgebras (respectively, (bi)comodules) commuting with the codifferentials. Morphisms of tensor coalgebras and of free comodules are again uniquely determined by their Taylor components. For instance a morphism of right A_∞-modules $M\to N$ over B is uniquely determined by the components $f_j\colon M[1]\otimes B[1]^{\otimes j}\to N[1]$.

Definition 2.2. A morphism of free comodules over a tensor coalgebra, and in particular of A_∞-modules over an A_∞-algebra is of *finite type* if all but finitely many of its Taylor components vanish.

The identity morphism is of finite type and the composition of morphisms of finite type is again of finite type.

The unital algebra of endomorphisms of finite type of a right A_∞-module M over an A_∞-algebra B is the 0-th cohomology of a differential graded algebra $\underline{\mathrm{End}}_{-B}(M)=\oplus_{j\in\mathbb{Z}}\underline{\mathrm{End}}^j_{-B}(M)$. The component of degree j is the space of endomorphisms of degree j of finite type of the comodule $M[1]\otimes\mathrm{T}(B[1])$. The differential is the graded commutator $\delta f=[\mathrm{d}_M,f]=\mathrm{d}_M\circ f-(-1)^j f\circ\mathrm{d}_M$ for $f\in\underline{\mathrm{End}}^j_{-B}(M)$. If M is an A_∞-A-B-bimodule and A is flat, then $\underline{\mathrm{End}}_{-B}(M)$ is defined and the left A-module structure induces a *left action* L_A, which is a morphism of A_∞-algebras $A\to\underline{\mathrm{End}}_{-B}(M)$: its Taylor components are $\mathrm{L}_A^j(a)^k(m\otimes b)=\mathrm{d}_M^{j,k}(a\otimes m\otimes b)$, $a\in A[1]^{\otimes j}$, $m\in M[1]$, $b\in B[1]^{\otimes k}$.

Lemma 2.3. *Let M be a right A_∞-module over a unital A_∞-algebra B. Then the subspace $\underline{\mathrm{End}}_{-B^+}(M)$ of endomorphisms f such that $f_j(m,b_1,\dots,b_j)=0$ whenever $b_i=s1$ for some i, is a differential graded subalgebra.*

We call this differential graded subalgebra the subalgebra of *normalized* endomorphisms.

Proof. It is clear from the formula for Taylor components of the composition that normalized endomorphisms form a graded subalgebra: $(f\circ g)^k=\sum_{i+j=k}f^j\circ(g^i\otimes\mathrm{id}_{B[1]}^{\otimes j})$. The formula for the Taylor components of the differential of an endomorphism f is

$$(\delta f)^k\ =\ \sum_{i+j=k}\Big(\mathrm{d}_M^j\circ(f^i\otimes\mathrm{id}_{B[1]}^{\otimes j})-(-1)^{|f|}f^i\circ(\mathrm{d}_M^j\otimes\mathrm{id}_{B[1]}^{\otimes i})$$
$$-(-1)^{|f|}f^{k-j+1}\circ(\mathrm{id}_{M[1]}\otimes\mathrm{id}_{B[1]}^{\otimes i}\otimes\mathrm{d}_B^j\otimes\mathrm{id}_{B[1]}^{\otimes(k-i-j)})\Big).$$

If f is normalized and $b_i=s1$ for some i, then only two terms contribute nontrivially to $(\delta f)^k(m,b_1,\dots,b_k)$, namely $f^{k-1}(m,b_1,\dots,\mathrm{d}_B^2(s1,b_{i+1}),\dots)$ (or $\mathrm{d}_M^1(f^{k-1}(m,b_1,\dots,b_{k-1}),s1)$ if $i=k$) and $f^{k-1}(m,b_1,\dots,\mathrm{d}_B^2(b_{i-1},s1),\dots)$ (or $f^{k-1}(\mathrm{d}_M^1(m,s1),b_2,\dots)$ if $i=1$). Due to the unital condition these two terms are equal up to sign, hence cancel together. □

The same definitions apply to A_∞-algebras and A_∞-bimodules over $\mathbb{C}[\![\hbar]\!]$ with completed tensor products and continuous homomorphisms for the $\hbar$-adic topology, so that for vector spaces V,W we have $V[\![\hbar]\!]\otimes_{\mathbb{C}[\![\hbar]\!]}W[\![\hbar]\!]=(V\otimes_{\mathbb{C}}W)[\![\hbar]\!]$ and $\mathrm{Hom}_{\mathbb{C}[\![\hbar]\!]}(V[\![\hbar]\!],W[\![\hbar]\!])=\mathrm{Hom}_{\mathbb{C}}(V,W)[\![\hbar]\!]$. A flat deformation of an A_∞-algebra B is an A_∞-algebra $B_\hbar$ over $\mathbb{C}[\![\hbar]\!]$ which, as a $\mathbb{C}[\![\hbar]\!]$-module, is isomorphic to $B[\![\hbar]\!]$ and such that $B_\hbar/\hbar B_\hbar\simeq B$. Similarly we have flat deformations of (bi)modules. A right A_∞-module $M_\hbar$ over $B_\hbar$ which is a flat deformation of M over B is given by Taylor coefficients $\mathrm{d}_{M_\hbar}^j\in\mathrm{Hom}_{\mathbb{C}}(M[1]\otimes B[1]^{\otimes j},M[1])[\![\hbar]\!]$. The differential graded algebra $\underline{\mathrm{End}}_{-B_\hbar}(M_\hbar)$ of endomorphism of finite type is then defined as the direct sum of the homogeneous components of $\mathrm{End}^{\mathrm{finite}}_{\mathrm{comod}-\mathrm{T}(B[1])}(M[1]\otimes\mathrm{T}(B[1]))[\![\hbar]\!]$ with differential $\delta_\hbar=[\mathrm{d}_{M_\hbar},\ ]$. Thus its degree j part is the $\mathbb{C}[\![\hbar]\!]$-module

$$\underline{\mathrm{End}}^j_{B_\hbar}(M_\hbar)=\big(\oplus_{k\ge0}\mathrm{Hom}^j(M[1]\otimes B[1]^{\otimes k},M[1])\big)[\![\hbar]\!],$$

where Hom^j is the space of homomorphisms of degree j between graded vector spaces over $\mathbb{C}$.

Finally, the following notation will be used: if $\phi\colon V_1[1]\otimes\cdots V_n[1]\to W[1]$ is a linear map and V_i, W are graded vector spaces or free $\mathbb{C}[\![\hbar]\!]$-modules, we set

$$\phi(v_1|\cdots|v_n)=s^{-1}\phi(sv_1\otimes\cdots\otimes sv_n),\qquad v_i\in V_i.$$

2.2. **Formality theorem for two branes and deformation of bimodules.** Let $A=\mathrm{S}(V)$ be the symmetric algebra of a finite dimensional vector space V, viewed as a graded algebra concentrated in degree 0. Let $B=\wedge(V^*)=\mathrm{S}(V^*[-1])$ be the exterior algebra of the dual space with $\wedge^i(V^*)$ of degree i. For any graded vector space W, the augmentation module over $\mathrm{S}(W)$ is the unique one-dimensional module on which W acts by 0. Let $A_\hbar=(A[\![\hbar]\!],\star)$ be the Kontsevich deformation quantization of A associated with a Poisson bivector field $\hbar\pi$. It is an associative algebra over $\mathbb{C}[\![\hbar]\!]$ with unit 1. The graded version of the formality theorem, applied to the same Poisson bracket, also defines a deformation quantization $B_\hbar$ of the graded commutative algebra B. However $B_\hbar$ is in general a unital A_∞-algebra with non-trivial Taylor components $\mathrm{d}^k_{B_\hbar}$ for all k including $k=0$. Still, the differential graded algebra $\underline{\mathrm{End}}_{-B_\hbar}(M_\hbar)$ is defined since $A_\hbar$ is an associative algebra and thus a flat A_∞-algebra. The following result is a consequence of the formality theorem for two branes (=submanifolds) in an affine space, in the special case where one brane is the whole space and the other a point, and is proved in [2]. It is a version of the Koszul duality between $A_\hbar$ and $B_\hbar$.

Proposition 2.4. *Let $A=\mathrm{S}(V)$, $B=\wedge(V^*)$ for some finite dimensional vector space V and let $A_\hbar$, $B_\hbar$ be their deformation quantizations corresponding to a polynomial Poisson bracket.*

- (i) *There exists a one-dimensional A_∞-A-B-bimodule K, which, as a left A-module and as a right B-module, is the augmentation module, and such that $\mathrm{L}_A\colon A\to\underline{\mathrm{End}}_{-B}(K)$ is an A_∞-quasiisomorphism.*
- (ii) *The bimodule K admits a flat deformation $K_\hbar$ as an A_∞-$A_\hbar$-$B_\hbar$-bimodule such that $\mathrm{L}_{A_\hbar}\colon A_\hbar\to\underline{\mathrm{End}}_{-B_\hbar}(K_\hbar)$ is an A_∞-quasiisomorphism.*
- (iii) *The bimodule $K_\hbar$ is in particular a right module over the unital A_∞-algebra $B_\hbar$. The first Taylor component $\mathrm{L}^1_{A_\hbar}$ sends $A_\hbar$ to the differential graded subalgebra $\underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar)$ of normalized endomorphisms.*

The proof of (i) and (ii) is contained in [2]. The claim (iii) follows from the explicit form of the Taylor components $\mathrm{d}^{1,j}_{K_\hbar}$, given in [2], appearing in the definition of L^1_A:

$$\mathrm{L}^1_{A_\hbar}(a)^j(1|b_1|\cdots|b_j)=\mathrm{d}^{1,j}_{K_\hbar}(a|1|b_1|\dots|b_j).$$

Namely $\mathrm{d}^{1,j}_{K_\hbar}$ is a power series in $\hbar$ whose term of degree m is a sum over certain directed graphs with $j+m+1$ vertices. Each graph contributes a multidifferential operator acting on $a,b_1,\dots,b_j$ times a weight, which is an integral of a differential form on a configuration space of m points in the upper half-plane and 1 point (associated with a) on the negative real axis and j ordered points on the positive real axis (associated with $b_1,\dots,b_j$) modulo dilations. By construction, if a b_i is scalar then the multidifferential operator vanishes unless the vertex of the graph associated with b_i is not an endpoint of an edge. But it is a general feature of the weights that the integral is zero if the dimension of the configuration space is positive and there is a vertex that is not the endpoint of an edge.

We turn to the description of the differential graded algebra $\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)$. Let $B^+=\oplus_{j\geq1}\wedge^j(V^*)=\wedge(V^*)/\mathbb{C}$. We have

$$\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)=(\oplus_{k\geq0}\mathrm{Hom}^j(K[1]\otimes B^+[1]^{\otimes k},K[1]))[\![\hbar]\!],$$

with product

$$(\phi\cdot\psi)(1|b_1|\cdots|b_n)=\sum_k\psi(1|b_1|\dots|b_k)\phi(1|b_{k+1}|\cdots|b_n).$$

It follows that the algebra $\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)$ is isomorphic to the tensor algebra $\mathrm{T}(B^+[1]^*)[\![\hbar]\!]$ generated by $\mathrm{Hom}(K[1]\otimes B^+[1],K[1])\simeq B^+[1]^*$. In particular it is concentrated in non-positive degrees.

Lemma 2.5. *The restriction $\delta_\hbar\colon B^+[1]^*\to\mathrm{T}(B^+[1]^*)[\![\hbar]\!]$ of the differential of $\underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar)\simeq\mathrm{T}(B^+[1]^*)[\![\hbar]\!]$ to the generators is dual to the A_∞-structure $\mathrm{d}_{B_\hbar}$ in the sense that*

$$(\delta_\hbar f)^k(z\otimes b)=(-1)^{|f|}f(z\otimes\mathrm{d}^k_{B_\hbar}(b)),\qquad z\in K[1],\quad b\in B[1]^{\otimes k},$$

for any $f\in\mathrm{Hom}(K[1]\otimes B^+[1],K[1])\simeq B^+[1]^*$

Proof. The A_∞-structure of $B_\hbar$ is given by Taylor components $\mathrm{d}^k_{B_\hbar}\colon B[1]^{\otimes k} \to B[1]$. By definition the differential on $\underline{\mathrm{End}}^j_{-B^+_\hbar}(K_\hbar)$ is the graded commutator $\delta_\hbar f = [\mathrm{d}_{K_\hbar}, f]$. In terms of Taylor components,

$$\begin{aligned}(\delta_\hbar f)^k(z\otimes b_1\otimes\cdots\otimes b_k) &= \mathrm{d}^{k-1}_{K_\hbar}(f(z\otimes b_1)\otimes b_2\otimes\cdots\otimes b_k)\\ &-(-1)^{|f|}f(\mathrm{d}^{k-1}_{K_\hbar}(z\otimes b_1\otimes\cdots\otimes b_{k-1})\otimes b_k)\\ &+(-1)^{|f|}f(z\otimes \mathrm{d}^k_{B_\hbar}(b_1\otimes\cdots\otimes b_k)).\end{aligned}$$

The first two terms vanish if $b_i\in B^+[1]$ for degree reasons. □

Thus $\mathrm{L}_{A_\hbar}$ induces an isomorphism from $A_\hbar$ to the cohomology in degree 0 of $\underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar)\simeq \mathrm{T}(B^+[1]^*)[\![\hbar]\!]$.

Remark 2.6. For $\hbar = 0$ this complex is Adam's cobar construction of the graded coalgebra B^*, which is a free resolution of $\mathrm{S}(V)$.

Theorem 2.7. *The composition*

$$\mathrm{L}^1_{A_\hbar}\colon A_\hbar \to \underline{\mathrm{End}}_{-B^+_\hbar}(K_\hbar) \xrightarrow{\simeq} \mathrm{T}(B^+[1]^*)[\![\hbar]\!],$$

induces on cohomology an algebra isomorphism

$$\mathrm{L}^1_{A_\hbar}\colon A_\hbar \to \mathrm{T}(V)/\mathrm{T}(V)\otimes\delta_\hbar((\wedge^2V^*)^*)\otimes\mathrm{T}(V),$$

where $\delta_\hbar\colon(\wedge^2V^*)^*\to\mathrm{T}(V)[\![\hbar]\!]$ *is dual to* $\oplus_{k\geq 0}\mathrm{d}^k_{B_\hbar}\colon (B^+[1]^0)^{\otimes k} = V^{\otimes k}\to B^+[1]^1 = \wedge^2V^*$.

Proof. The fact that the map is an isomorphism follows from the fact that it is so for $\hbar = 0$, by the classical Koszul duality. As the cohomology is concentrated in degree 0 it remains so for the deformed differential $\delta_\hbar$ over $\mathbb{C}[\![\hbar]\!]$.

As a graded vector space, $B^+[1]^* = V\oplus(\wedge^2V^*)^*\oplus\cdots$, with $(\wedge^iV^*)^*$ in degree $1-i$. Therefore the complex $\mathrm{T}(B^+[1]^*)[\![\hbar]\!]$ is concentrated in non-positive degrees and begins with

$$\cdots\to\left(\mathrm{T}(V)\otimes(\wedge^2V^*)^*\otimes\mathrm{T}(V)\right)[\![\hbar]\!]\to\mathrm{T}(V)[\![\hbar]\!]\to 0.$$

Thus to compute the degree 0 cohomology we only need the restriction of the Taylor components $\mathrm{d}^k_{B_\hbar}$ on $\mathrm{T}(V^*) = \mathrm{T}(B^+[1])^0$, whose image is in $B[1]^1 = \wedge^2V^*$. □

This theorem gives a presentation of the algebra $A_\hbar$ by generators and relations. Let $x_1,\ldots,x_d\in V$ be a system of linear coordinates on V^* dual to a basis $e_1,\ldots,e_d$. Let for $I=\{i_1<\cdots<i_k\}\subset\{1,\ldots,d\}$, $x_I\in(\wedge^kV^*)^*$ be dual to the basis $e_{i_1}\wedge\cdots\wedge e_{i_k}$. Then $A_\hbar$ is isomorphic to the algebra generated by $x_1,\ldots,x_d$ subject to the relations $\delta_\hbar(x_{ij}) = 0$. Up to order 1 in $\hbar$ the relations are obtained from the cobar differential and the graph of Figure 1.

$$\delta_\hbar(x_{ij}) = x_i\otimes x_j - x_j\otimes x_i - \hbar\mathrm{Sym}(\pi_{ij}) + O(\hbar^2).$$

Here Sym is the symmetrization map $\mathrm{S}(V)\to\mathrm{T}(V)$.

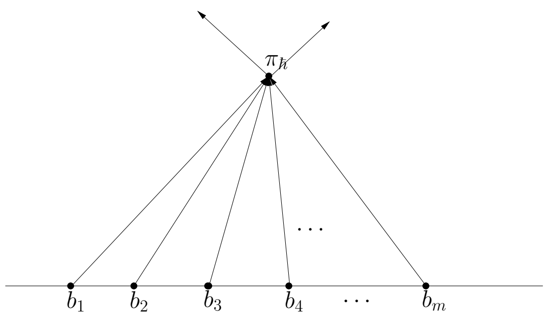

Figure 1 - The only admissible graph contributing to $\mathrm{d}^m_{B_\hbar}$ at order 1 in $\hbar$

The lowest order of the isomorphism induced by L^1_A on generators $x_i\in V$ of $A_\hbar = \mathrm{S}(V)[\![\hbar]\!]$ was computed in [2]:

$$\mathrm{L}^1_A(x_i) = x_i + O(\hbar).$$

The higher order terms $O(\hbar)$ are in general non-trivial (for example in the case of the dual of a Lie algebra, see below).

By comparing our construction with the arguments in [7], we see that the differential $\mathrm{d}_\hbar$ corresponds to the image of $\mathcal{V}(\widehat{\pi_\hbar})$, where the notations are as in the introduction, by the quasi-isomorphism Φ_1 in [7, Subsection 1.4]. Hence, Theorem 2.7 provides a proof of [7, Conjecture 2.6] with the amendment that the isomorphism $A_\hbar\to\mathrm{T}(V)/\mathcal{I}_\star$ is not just given by the symmetrization map but has non-trivial corrections.

3. Examples

We now want to examine more closely certain special cases of interest. We assume here that the reader has some familiarity with the graphical techniques of [2,3,6]. To obtain the relations $\delta_\hbar(x_{ij})$ we need $\mathrm{d}^m_{B_\hbar}(b_1|\cdots|b_m) \in \wedge^2 V^*[\![\hbar]\!]$, for $b_i \in V^* \subset B^+$. The contribution at order n in $\hbar$ to this is given by a sum over the set $\mathcal{G}_{n,m}$ of admissible graphs with n vertices of the first type and m of the second type.

3.1. **The Moyal–Weyl product on V.** Let $\pi_\hbar = \hbar\pi$ be a constant Poisson bivector on V^*, which is uniquely characterized by a complex, skew-symmetric matrix $d \times d$-matrix π_{ij}.

In this case, Kontsevich's deformed algebra $A_\hbar$ has an explicit description: the associative product on $A_\hbar$ is the Moyal–Weyl product

$$(f_1 \star f_2) = \mu \circ \exp \frac{1}{2}\pi_\hbar,$$

where $\pi_\hbar$ is viewed here as a bidifferential operator, the exponential has to be understood as a power series of bidifferential operators, and μ denotes the ($\mathbb{C}[\![\hbar]\!]$-linear) product on polynomial functions on V^*. On the other hand, it is possible to compute explicitly the complete A_∞-structure on $B_\hbar$.

Lemma 3.1. *For a constant Poisson bivector $\pi_\hbar$ on V^*, the A_∞-structure on $B_\hbar$ has only two non-trivial Taylor components, namely*

$$\mathrm{d}^0_{B_\hbar}(1) = \hbar\pi, \quad \mathrm{d}^2_{B_\hbar}(b_1|b_2) = (-1)^{|b_1|} b_1 \wedge b_2, \quad b_i \in B_\hbar, \quad i = 1,2. \tag{1}$$

Proof. We consider $\mathrm{d}^m_{B_\hbar}$ first in the case $m = 0$. Admissible graphs contributing to $\mathrm{d}^0_{B_\hbar}$ belong to $\mathcal{G}_{n,0}$, for $n \geq 1$. For $n \geq 2$, all graphs give contributions involving a derivative of π_{ij} and thus vanish. There remains the only graph in $\mathcal{G}_{1,0}$, whence the first identity in (1).

By the same reasons, $\mathrm{d}^m_{B_\hbar}$ is trivial, if $m \geq 1$ and $m \neq 2$: in the case $m = 1$, we have to consider contributions coming from admissible graphs in $\mathcal{G}_{n,1}$, with $n \geq 1$, which vanish for the same reasons as in the case $m = 0$.

For $m \geq 3$, contributions coming from admissible graphs in $\mathcal{G}_{n,m}$, $n \geq 1$, are trivial by a dimensional argument.

Finally, once again, the only possibly non-trivial contribution comes from the unique admissible graph in $\mathcal{G}_{0,2}$ which gives the product. □

As a consequence, the differential $\delta_\hbar$ be explicitly computed, namely

$$\delta_\hbar(x_{ij}) = x_i \otimes x_j - x_j \otimes x_i - \hbar\pi_{ij}.$$

This provides the description of the Moyal–Weyl algebra as the algebra generated by x_i with relations $[x_i, x_j] = \hbar\pi_{ij}$.

We finally observe that the quasi-isomorphism $\mathrm{L}^1_{A_\hbar}$ coincides, by a direct computation, with the usual symmetrization morphism.

3.2. **The universal enveloping algebra of a finite-dimensional Lie algebra $\mathfrak{g}$.** We now consider a finite-dimensional complex Lie algebra $V = \mathfrak{g}$: its dual space $\mathfrak{g}^*$ with Kirillov–Kostant-Souriau Poisson structure. With respect to a basis $\{x_i\}$ of $\mathfrak{g}$, we have

$$\pi = f^k_{ij} x_k \partial_i \wedge \partial_j,$$

where f^k_{ij} denote the structure constant of $\mathfrak{g}$ for the chosen basis.

It has been proved in [6, Subsubsection 8.3.1] that Kontsevich's deformed algebra $A_\hbar$ is isomorphic to the universal enveloping algebra $\mathrm{U}_\hbar(\mathfrak{g})$ of $\mathfrak{g}[\![\hbar]\!]$ for the $\hbar$-shifted Lie bracket $\hbar[\ ,\ ]$.

On the other hand, we may, once again, compute explicitly the A_∞-structure on $B_\hbar$.

Lemma 3.2. *The A_∞-algebra $B_\hbar$ determined by $\pi_\hbar$, where π is the Kirillov–Kostant–Souriau Poisson structure on $\mathfrak{g}^*$, has only two non-trivial Taylor components, namely*

$$\mathrm{d}^1_{B_\hbar}(b_1) = \mathrm{d}_{\mathrm{CE}}(b_1), \quad \mathrm{d}^2_{B_\hbar}(b_1|b_2) = (-1)^{|b_1|} b_1 \wedge b_2, \quad b_i \in B_\hbar, \quad i = 1,2, \tag{2}$$

where d_{CE} denotes the Chevalley–Eilenberg differential of $\mathfrak{g}$, endowed with the rescaled Poisson bracket $\hbar[\bullet,\bullet]$.

Proof. By dimensional arguments and because of the linearity of $\pi_\hbar$, there are only two admissible graphs in $\mathcal{G}_{1,0}$ and $\mathcal{G}_{2,0}$, which may contribute non-trivially to the curvature of $B_\hbar$, namely,

Figure 2 - The only admissible graphs in $\mathcal{G}_{1,0}$ and $\mathcal{G}_{2,0}$ respectively in the curvature of $B_\hbar$

The operator $\mathcal{O}_\Gamma^B$ for the graph in $\mathcal{G}_{1,0}$ vanishes, when setting $x = 0$. On the other hand, $\mathcal{O}_\Gamma^B$ vanishes in virtue of [6, Lemma 7.3.1.1].

We now consider the case $m \geq 1$. We consider an admissible graph Γ in $\mathcal{G}_{n,m}$ and the corresponding operator $\mathcal{O}_\Gamma^B$: the degree of the operator-valued form ω_Γ^B equals the number of derivations acting on the different entries associated to vertices either of the first or second type. Thus, the operator $\mathcal{O}_\Gamma^B$ has a polynomial part (since all structures are involved are polynomial on $\mathfrak{g}^*$): since the polynomial part of any of its arguments in $B_\hbar$ has degree 0, the polynomial degree of $\mathcal{O}_\Gamma^B$ must be also 0. A direct computation shows that this condition is satisfied if and only if $n + m = 2$, because $\pi_\hbar$ is linear.

Obviously, the previous identity is never satisfied, if $m \geq 3$, which implies immediately that the only non-trivial Taylor components appear, when $m = 1$ and $m = 2$. When $m = 1$, the previous equality forces $n = 1$: there is only one admissible graph Γ in $\mathcal{G}_{1,1}$, whose corresponding operator is non-trivial, namely,

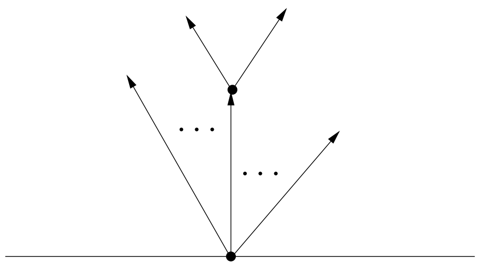

Figure 3 - The only admissible graph in $\mathcal{G}_{1,1}$ contributing to $\mathrm{d}^1_{B_\hbar}$

The weight is readily computed, and the identification with the Chevalley–Eilenberg differential is then obvious.

Finally, when $m = 2$, the result is clear by previous computations. □

Thus $\delta_\hbar$ is given by

$$\delta_\hbar(x_{ij}) = x_i \otimes x_j - x_j \otimes x_i - \hbar \sum_k f_{ij}^k x_k.$$

Hence we reproduce the result that $A_\hbar$ is isomorphic to $\mathrm{U}_\hbar(\mathfrak{g})$. The isomorphism $L^1_{A_\hbar}$ may be also evaluated explicitly: it is the composition of the symmetrization map with the "strange" automorphism, which appears in Duflo's Theorem.

This can be proved by evaluating the general admissible graph Γ, contributing to the deformed derived left action $\mathrm{L}^1_{A_\hbar}(a)$, for a general, homogeneous element of $\mathrm{S}(V)$:

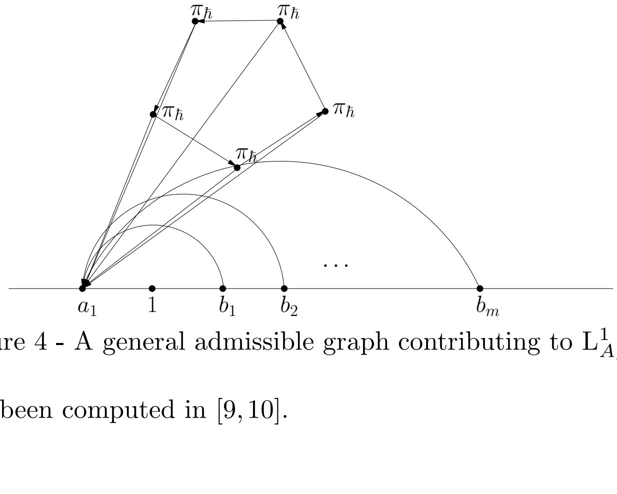

Figure 4 - A general admissible graph contributing to $\mathrm{L}^1_{A_\hbar}(a)$

The weight of such graphs has been computed in [9, 10].

3.3. **Quadratic algebras.** Here we briefly discuss the case where V^* is endowed with a quadratic Poisson bivector field π: this case has been already considered in detail in [2, Section 8], see also [8], where the property of the deformation associated $\pi_\hbar$ of preserving Koszulness has been proved.

The main feature of the quadratic case is the degree 0 homogeneity of the Poisson bivector field, which reflects itself in the homogeneity of all structure maps. In particular the Kontsevich star-product on a basis of linear functions has the form

$$x_i \star x_j = x_i x_j + \sum_{k,l} S_{ij}^{kl}(\hbar) x_k x_l,$$

for some $S_{ij}^{kl} \in \hbar\mathbb{C}[\![\hbar]\!]$. Our results implies that this algebra is isomorphic to the quotient of the tensor algebra in generators x_i by relations

$$x_i \otimes x_j - x_j \otimes x_i = \sum_{k,l} R_{ij}^{kl}(\hbar) x_k \otimes x_l,$$

for some $R_{ij}^{kl}(\hbar) \in \hbar\mathbb{C}[\![\hbar]\!]$. The isomorphism sends x_i to

$$L_{A_\hbar}(x_i) = x_i + \sum_j L_i^j(\hbar) x_j,$$

for some $L_i^j(\hbar) \in \hbar\mathbb{C}[\![\hbar]\!]$.

3.4. **A final remark.** We point out that, in [1], the authors construct a flat $\hbar$-deformation between a so-called non-homogeneous quadratic algebra and the associated quadratic algebra: the characterization of the non-homogeneous quadratic algebra at hand is in terms of two linear maps α, β, from R onto V and $\mathbb{C}$ respectively, which satisfy certain cohomological conditions. In the case at hand, it is not difficult to prove that the conditions on α and β imply that their sum defines an affine Poisson bivector on V^*: hence, instead of considering α and β separately, as in [1], we treat them together. Both deformations are equivalent, in view of the uniqueness of flat deformations yielding the PBW property, see [1].

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