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by

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DESINGULARIZATION OF LIE GROUPOIDS, THE EDGE PSEUDODIFFERENTIAL CALCULUS, AND FREDHOLM CONDITIONS FOR SINGULAR SPACES

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Abstract. Successive blow-ups of a singular space can be used to reduce its analysis to that on a non-compact manifold. In this paper, we are interested in constructing the integral kernel operators on the blown-up spaces and, especially, in studying when they are Fredholm using groupoid C^*-algebras. We thus introduce and study an analog of the blow-up of a space in the framework of Lie groupoids. This analog is a desingularization procedure for a groupoid $\mathcal{G}$ along an "$A(\mathcal{G})$-tame" submanifold L of the space of units M. An A-tame submanifold is one that has, by definition, a tubular neighborhood on which A becomes a pull-back Lie algebroid. The construction of the desingularization $[[\mathcal{G} : L]]$ is based on a canonical pull-back structure result for $\mathcal{G}$ in a neighbourhood of a tame $A(\mathcal{G})$-submanifold $L \subset M$. The space of units of the desingularization $[[\mathcal{G} : L]]$ is $[M : L]$, the blow up of M along L, however, the desingularization groupoid is not a blown-up space. We provide an explicit description of the structure of the desingularized groupoid and we identify its Lie algebroid. As an application, we obtain necessary and sufficient Fredholm conditions for the resulting pseudodifferential operators. More generally, we introduce and study the class of Fredholm groupoids. A *Fredholm groupoid* is a groupoid for which the Fredholm property is equivalent to the invertibility of the principal symbol and of its fiberwise boundary restrictions. We obtain a general characterization of Fredholm groupoids. In particular, using some results of Ionescu and Williams (Indiana Univ. Math. J. 2009), we show that amenable, second-countable, Hausdorff groupoids are Fredholm. Since there are no easy criteria to decide when a groupoid is amenable, we introduce the class of "stratified submersion groupoids." This class is defined by easily checked conditions and is invariant with respect to desingularization. Moreover, a stratified submersion groupoid is Fredholm if its isotropy groups are amenable. We then show how to use stratified submersion groupoids to obtain Fredholm conditions on many classes of manifolds that arise in practice, including asympototically Euclidean manifolds, asymptotically hyperbolic manifolds, manifolds with polycylindrical ends, and manifolds that are obtained by successive blow-ups.

Contents

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1. Introduction

We introduce a *desingularization procedure* for groupoids. This desingularization procedure is related to the blow-up of submanifolds and the associated edge operators [42, 35, 49, 64]. We then prove *Fredholm conditions* for the operators on groupoids obtained by iterating this desingularization procedure as well as on some other groupoids. We show that many groupoids and hence many pseudodifferential operators appearing in practice fit into our framework.

The paper is divided into three parts. In the first part is devoted mostly to the desingularization procedure of a groupoid with respect to a tame submanifold. The second part is devoted to Fredholm conditions and to the proof that the class of Fredholm groupoids is closed under desingularization. In the second part, we also introduce the class of stratified submersion groupoids, show that it is invariant under desingularization, and that it consists of Fredholm groupoids. In the last part, we show how to use our results in various situations that arise in practice.

1.1. **Desingularization.** In the first part of the paper, we introduce and study a desingularization procedure for a Lie groupoid $\mathcal{G}$ along an "$A(\mathcal{G})$-tame" submanifold L of the space of units M of $\mathcal{G}$, where $A(\mathcal{G})$ is the Lie algebroid of $\mathcal{G}$. The resulting groupoid is denoted $[[\mathcal{G} : L]]$ and has $[M : L]$ as space of units, where $[M : L]$ is the blow up of M along L. The construction of the desingularization groupoid $[[\mathcal{G} : L]]$ generalizes the blow-up of a manifold with respect to a submanifold and is useful in the analysis on singular spaces. We stress, however, that $[[\mathcal{G} : L]] \not\simeq [\mathcal{G} : L]$, the (usual) blow up of $\mathcal{G}$ with respect to the submanifold L.

The definitions of Lie groupoids and Lie algebroids, as well as many other definitions and results are reviewed in the first section, for the benefit of the reader, but also because we need slight extensions of classical results and defintions, from the framework of smooth manifolds (no corners), to the framework of manifolds with corners. For example, tame submanifolds are submanifolds that have tubular neighborhoods in a Lie algebroid sense. More precisely, given a Lie algebroid $A \to M$ and $L \subset M$, we say that L is *A-tame* if L has a tubular neighborhood in M on which A is a pull-back Lie algebroid. (To distinguish between the many pull-backs in this paper, the pull-back of a Lie algebroid will be called a *thick pull-back*, as in [3], whereas the pull-back of a Lie groupoid via a tame submersion will be called a *fibered pull-back* from now on.) The desingularization procedure is based on a fibered pull-back structure theorem for $\mathcal{G}$ on the tubular neighborhood of L in M, which is one of the main technical results of the first part of the paper (Theorem 3.3). We identify the Lie algebroid of the desingularization as the desingularization of the Lie algebroid of $\mathcal{G}$ [1], that is, desingularization and the Lie algebroid functor commute. In the last section, we apply this desingularization procedure to construct the groupoid associated to a polyhedral domain in dimension three and to other examples.

1.2. **Motivation.** Before discussing the second part of the paper, let us provide some motivation for this work. Our results, at least the ones in the first part of the paper, are motivated by an approach to analysis on singular spaces, which is to successively blow up the lowest dimensional singular strata. This procedure leads to the eventual removal of all singularities (one may keep a smooth boundary, though). This approach was used in [9] to obtain a well-posedness result for the Poisson problem in weighted Sobolev spaces on n-dimensional polyhedral domains using energy methods (the Lax-Milgram lemma). In three dimensions, this result was proved in [10]. One would like to use also other methods than the energy method to study singular spaces, such as the method of layer potentials, but then one has to study the resulting integral kernel operators. We thus build in this paper one of the necessary tools to use groupoids in order to desingularize the distribution kernels of resolvents of elliptic differential operators on suitable singular spaces. The next step is to combine the results in this paper with the construction of psedodifferential operators on groupoids [3, 7, 47, 54]. In the smooth case, this may allow to recover the pseudodifferential calculi of Grushin [24], Mazzeo [42], and Schulze [64].

After completing the iterated desingularization procedure (by iterating successively the lowest dimensional strata), we want to obtain Fredholm conditions for the operators associated to the resulting groupoids. We thus develop some general tools that turn out to be useful in other situtions as well. This is done in the second part of the paper, but before discussing the second part of the paper, let us give some more classical background for the problem of finding Fredholm conditions.

Let M be a smooth manifold and let P be an order m, pseudo-differential operator acting between Sobolev sections $P\colon H^s(M;E) \to H^{s-m}(M;F)$ of two smooth, hermitian vector bundles E, F on M. Recall that P is called *elliptic* if its principal symbol $\sigma_m(P)$ is invertible outside the zero section. When M is compact, a classical, well known (Fredholm) result [15, 65, 66] states that, for a pseudo-differential operator P of order m, the resulting operator $P\colon H^s(M;E) \to H^{s-m}(M;F)$ is Fredholm if, and only if, P is elliptic. This result has many applications to topology, gauge theory, index theory, geometry, and many other areas. In view of these applications, a natural question to ask is to what extent this result extends to (suitable) non-compact manifolds. The example of constant coefficient differential operators on $\mathbb{R}^n$ shows that that the classical Fredholm result above might be no longer true as stated if M is not compact.

A similar situation is encountered on singular spaces, where ellipticity again does not guarantee the Fredholm property. On certain classes of manifolds, however, it is possible to reformulate the classical result above as follows. Let M be a non-compact manifold with an appropriate condition on its ends. (The main goal of this paper is to obtain such conditions on the ends.) Then we can associate to M an index set I, certain Lie groups G_α, $\alpha \in I$, certain spaces M_α, $\alpha \in I$, and for each $\alpha \in I$, an action of G_α on M_α, with the following property:

Theorem 1.1. *If D is an order m pseudo-differential operator on M compatible with the geometry, then we can associate to D a family D_α, $\alpha \in I$, of G_α-invariant differential operators on M_α such that one has*

$$D : H^s(M) \to H^{s-m}(M) \text{ is Fredholm} \quad \Leftrightarrow \quad D \text{ is elliptic and } D_\alpha \text{ is invertible for all } \alpha \in I\,.$$

The same result hold for operators acting on sections of vector bundles.

We should stress that there is no issue in including vector bundles and actions of compact groups *on a suitable compactification of M* in the above results.

See for instance [18, 34, 35, 36, 40]. A similar type of result is used in spectral theory in relation to the N-body problem and its generalizations, in which case the operators D_α arise from so called *localization principles* [16, 22, 23, 57, 63]. The main goal of this papers is to study general conditions under which a similar result is valid. We refer to Fredholm conditions of the kind in Theorem 1.1 as *non-local Fredholm conditions.* (Slightly more general results than Theorem 1.1 are possible, see Theorems 5.14 and 5.17 and Corollary 5.16, but the extra generality does not see to be necessary in applications.)

It may be non-intuitive, but actually it was realized that the above mentioned results–such as Theorem 1.1–are related to the representation theory of certain C^*-algebras $\mathfrak{A}$. These C^*-algebras are such that they contain the operators of the form $a = (1+\Delta)^{(s-m)/2} D(1+\Delta)^{-s/2}$ (where Δ is the positive Laplacian) acting on $L^2(M)$. The reason for considering the operator a is that D is Fredholm if, and only if, a is Fredholm; moreover, a is bounded. In most the practical applications, these C^*-algebras can be chosen to be groupoid C^*-algebras. These ideas are used in the proof of Theorem 5.14, which contains Theorem 1.1 as a particular case.

1.3. **General Frehdolm conditions for groupoids.** In the second part of the paper, we introduce some analysis tools to study general Fredholm conditions for operators on groupoids. These operators are operators affiliated to a groupoid C^*-algebra of a locally compact groupoid $\mathcal{G}$ with a Haar system. *Unless stated otherwise, all our groupoids are locally compact with a Haar system and all our spaces are Hausdorff.*

The study of these operators is reduced to the study of operators in suitable algebras $\mathbf{\Psi}$ containing the reduced C^*-algebra $C^*_r(\mathcal{G})$ of $\mathcal{G}$ as an essential ideal. In particular, we obtain Fredholm conditions for operators on singular spaces obtained by the desingularization procedure introduced in the first part of the paper.

The Fredholm conditions established in this paper are formulated in terms of representations of groupoid C^*-algebras, and we found it particularly useful to describe the primitive ideal spectrum of such a C^*-algebra in terms of representations of their isotropy groups. Thus, along the way, we obtained some results that shed some fresh new light on regular representations of groupoids (see [33]), relating, in particular, our Fredholm conditions to the generalized Effros-Hahn conjecture on groupoids [29, 30]. In order to describe these results and other results of the second part of the paper, we need to introduce some notation.

Throughout the paper, $\mathcal{G}$ will denote a locally compact groupoid with a Haar system λ_x and with domain and range maps $d, r : \mathcal{G} \to M$. For our main results, we also need to assume that there exists an open, $\mathcal{G}$-invariant subset $U \subset M$ with $\mathcal{G}_U \simeq U \times U$ (the pair groupoid). The set U will necessarily be dense in M and hence uniquelly determined. We call $F := M \smallsetminus U$ the set of *boundary units* of $\mathcal{G}$. We shall therefore assume throughout this subsection that there exists such an open subset $U \subset M$. Then the reduced C^*-algebra of $\mathcal{G}$ contains the algebra of compact operators on $L^2(U)$ as an essential ideal. For every $x \in M$, let $\mathcal{G}_x := d^{-1}(x)$ and $\pi_x \colon C^*(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}_x))$ be the regular representation of $C^*(\mathcal{G})$ associated to the unit x, that is, $\pi_x(f)g := f * g$. In particular, we denote by $\pi_0 := \pi_{x_0}$ the corresponding (equivalence class of) representation(s) of $C^*(\mathcal{G})$ for any $x_0 \in U$ acting on $L^2(\mathcal{G}_{x_0}) \simeq L^2(U)$, via the bijection $r : \mathcal{G}_{x_0} \to U$. We interested in characterizing the groupoids that have the following property:

- For any $a \in C^*_r(\mathcal{G})$, we have that $1 + \pi_0(a)$ is Fredholm if, and only if, all operators $1 + \pi_x(a)$, $x \in F := M \smallsetminus U$, are invertible.

These groupoid will be called *Fredholm groupoids* (see Definition 5.2).

One of our main results (Theorem 5.3) gives a characterization of Fredholm groupoids in terms of invertibility sufficient families of representations. Namely, we show that a groupoid $\mathcal{G}$ as above if Fredholm if, and only if,

- The representation π_0 is injective on $C^*_r(\mathcal{G})$.
- The sequence

$$0 \to C^*_r(\mathcal{G}_U) \to C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}_F) \to 0$$

  is exact, where $F := M \setminus U$ is the set of boundary units of $\mathcal{G}$.
- The family of regular representations $\{\pi_x\}_{x\in F}$ is an invertibility sufficient family of representations of $C^*_r(\mathcal{G}_F)$.

(See Theorem 5.3 for the precise statement.) We notice that it is the first of the conditions above that implies that U is dense in M, and hence hence the $\mathcal{G}$-invariant open subset U of is uniquely determined.

In applications, the relevant operators are not in $C^*_r(\mathcal{G}) + \mathbb{C}1$, but rather in unital C^* algebras $\mathbf{\Psi}$ containing $C^*_r(\mathcal{G})$ as an essential ideal. We prove that for a Fredholm groupoid $\mathcal{G} \rightrightarrows M$ as considered above, for any $a \in \mathbf{\Psi}$, we have that $\pi_0(a)$ is Fredholm if, and only if, the image of a in $\mathbf{\Psi}/C^*_r(\mathcal{G})$ is invertible and all $\pi_x(a)$, $x \in F$, are invertible. Let $\mathcal{G}$ be a locally compact, second countable, Hausdorff groupoid $\mathcal{G} \rightrightarrows M$ with $\mathcal{G}_U \simeq U \times U$, as before. It follows from this result (Theorem 5.3) that if $\mathcal{G}$ satisfies the Effros-Hahn conjecture and all isotropy groups $\mathcal{G}^x_x$ of $\mathcal{G}$ are amenable, then $\mathcal{G}$ is a Fredholm groupoid. In particular, the results of Ionescu and Williams [29] imply that a locally compact, second countable, Hausdorff groupoid $\mathcal{G} \rightrightarrows M$ with $\mathcal{G}_U \simeq U \times U$ is Fredholm (Theorem 5.10).

1.4. **Examples and applications.** Neither of the conditions in Theorem 5.3 is very easy to check in practice. It is also not easy to check that a groupoid is amenable (in this paper, by "amenable" we shall mean "topologically amenable", see [5, 12, 29, 58]). To address this issue, we have introduced the notion of stratified submersion groupoid. We prove in Theorem 6.6) that every Hausdorff reduced stratified submersion groupoid with a Haar system is Fredholm.

The main appeal of stratified submersion groupoids is that many of the groupoids that appear in practice are stratified submersion groupoids. Also, the class of stratified submersiongroupoids is closed under desingularization, the desingularization being the analog in the category of groupoids of the blow-up construction in the category of manifolds with corners. The last three subsectons of the paper contains examples and applications of the results presented in the first two parts of the paper. A first example deals with the blow-up of a smooth submanifold of another smooth, compact manifold. The second example extends this construction to manifolds with boundary. Finally, the last example deals with the iterated blow-up of a singular stratified subset of dimension one.

1.5. **Organization of the paper.** The paper is organized as follows. The second section (first after the Introduction) is devoted mostly to background material. We thus review manifolds with corners and tame submersions and establish a canonical local form of a tame submersion that generalizes to manifolds with corners the classical result in the smooth case. We then recall the definitions of Lie groupoids and Lie algebroids and that of a Lie algebroid of a Lie groupoid, in the framework that we need, that is, that of manifolds with corners. Almost everything extends to the setting of manifolds with corners without any significant change. One must be careful, however, to use *tame fibrations*. One of the main results of this paper is the construction of the desingularization of a Lie groupoid $\mathcal{G}$ along an $A(\mathcal{G})$-tame submanifold. This is carried out in the third section and requires several other constructions, such as that of the adiabatic (deformation) groupoid and of the fibered pull-back Lie algebroid. We thus review in our framework all these examples as well as other, more basic ones that are needed in the construction of the desingularization groupoid. In particular, we introduce the so called GDS modification of a groupoid using results of Debord and Skandalis [19]. The third section contains most of our main results on desingularization groupoids. We first prove a local structure theorem for a Lie groupoid $\mathcal{G}$ with units M in a tubular neighbourhood $\pi : U \to L$ of an $A(\mathcal{G})$-tame submanifold $L \subset M$. More precisely, we prove that the reduction of $\mathcal{G}$ to U is isomorphic to $\pi^{\downarrow\downarrow}(\mathcal{G}_L^L)$, the fibered pull-back to U of the reduction of $\mathcal{G}$ to L. This allows us to define the desingularization first for this type of pull-back groupoids. In general, we first localize close to the tame submanifold and then use a glueing construction due to Gualtieri and Li [25]. We identify the Lie algebroid of the desingularization as the desingularization of its Lie algebroid (the desingularization of a Lie algebroid was introduced in [1]). We conclude the first part of the paper with a section devoted to examples, including an example related to the Grushin-Mazzeo-Schulze 'edge'-calculus. The second part of the paper is based in large part on an unpublished paper with Daniel and Ingrid Beltiţă [11]. In order to describe our results, let us assume that all groupoids are locally compact, Hausdorff, second countable and have a Haar system. We begin it with some preliminaries on exhausting families of representations from [52] and some general results on groupoid C^*-algebras 4. The following section, the sixth section contains some unpublished results from [11] on Fredholm groupoids. In particular, we provide a characterization of Fredholm groupoids and show that the groupoids that have amenable isotropy groups and satisfying the Effros-Hahn conjecture are Fredholm. In the last section, the seventth, we introduce almost continuous isotropy groupoids and show that they are Fredholm. We also show that if a Lie groupoid $\mathcal{G}$ is obtained from the pair groupoid by successive desingularization, then it is an almost continuous isotropy groupoid and hence it is Fredholm, thus tying the two parts of the paper.

This paper is addressed to both analysts and specialists in the theory of Lie groupoids. It attempts to deal with all the Lie groupoid results that we will need for some subsequent work on pseudodifferential operators. We also envision some applications to Poisson groupoids. We thus include all the necessary geometric background material to make the paper as easy as possible to read by analysts and other people interested in the subject.

1.6. **A note on notation and terminology.** We shall use manifolds with corners extensively. They are defined in Subsection 2.1. A manifold without corners will be called *smooth*. We take the point of view that all maps, submanifolds, and so on will be defined in the same way in the corner case as in the smooth case. Sometimes, we need maps and submanifolds with special properties, they will usually be termed "tame", for instance, a tame submersion of manifolds with corners will have the property that all its fibers are smooth manifolds. This property is not shared by general submersions, however. Also, we use only *real* vector bundles, to avoid confusion and simplify notation. The results extend without any difficulty to the complex case, when one wishes so. For simplicity, in this paper, we shall consider *Hausdorff* topological spaces, except where explicitly stated otherwise.

This manuscript is based on several other papers, including [51, 52, 11], and will not be published in the current form. In fact, to a large extent, this manuscript consits of surveys, extensions, and applications of the results in these papers.

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2. Lie groupoids and Lie algebroids

We now recall the needed definitions and properties of Lie groupoids and of Lie algebroids. Although our interest is mainly in Lie groupoids, we have found it convenient to consider also the general case of locally compact groupoids, so we shall discuss these in parallel.

Our manifolds with typically have corners, so we also recall some basic definitions and results on manifolds with corners as well. Even if one is interested only in smooth manifolds (meaning "no corners"), the blow-up procedure leads to manifolds with corners, as each blow-up increases the depth of a given manifold (i.e. the highest codimension of a corner) by one.

Few result in this section is new, so we do not include all the details. We refer to Mackenzie's books [38, 39] for more details and, in general, for a nice introduction to the subject of Lie groups and Lie groupoids, as well as to further references and historical comments. See also [12, 45, 58] for the more specialized issues relating to analytic applications.

2.1. **Manifolds with corners and notation.** In the following, by a *manifold*, we shall mean a *possibly non Hausdorff* $\mathcal{C}^\infty$-manifold, possibly with corners. By a *smooth manifold* we shall mean a possibly non Hausdorff $\mathcal{C}^\infty$-manifold without corners. All our manifolds will be assumed to be paracompact. Recall [31, 41, 44] that M is a *manifold with corners* of dimension n if it is locally diffeomorphic to an open subset of $[-1,1]^n$ with smooth changes of coordinates. A point $p \in M$ is called of *depth* k if it has a neighborhood V_p diffeomorphic to $[0,a)^k \times (-a,a)^{n-k}$, $a > 0$ by a diffeomorphism $\phi_p : V_p \to [0,a)^k \times (-a,a)^{n-k}$ mapping p to the origin: $\phi_p(p) = 0$. Such a neighborhood will be called *standard*. A function $f : M \to M_1$ between two manifolds with corners will be called *smooth* if it its components are smooth in all coordinate charts. A little bit of extra care is needed here in defining the derivatives. This is illustrated clearly in one dimension: if $f : [0,1] \to \mathbb{R}$, then $f'(0) := \lim_{h\to 0} h^{-1}(f(h) - f(0))$, whereas $f'(1) := \lim_{h\to 0} h^{-1}(f(1) - f(1-h))$. Since $\lim_{h\to 0} h^{-1}(f(x+h) - f(x)) = \lim_{h\to 0} h^{-1}(f(x) - f(x-h)) =: f'(x)$, for $0 < x < 1$, this defines unambigously $f'(x)$. A similar comment is in order in higher dimensions as well.

A connected component F of the set of points of depth k will be called an *open face (of codimension k)* of M. The set of points of depth 0 of M is the *interior of* M, will be denoted M_0 and is also considered to be an open face of M when connected (which will usually be the case). Thus the *interior points of* M are the points that do not belong to any non-trivial face of M. The maximum depths of a point in M will be called the *rank* of M. Thus smooth manifolds will have rank zero. The closure in M of an open face F of M will be called a *closed* face of M. The closed faces of M may not be manifolds with corners on their own.

We define the tangent space to a manifold with corners TM as usual, that is, as follows: the vector space T_pM is the set of derivations $D_p : \mathcal{C}^\infty(M) \to \mathbb{R}$ satisfying $D_p(fg) = f(p)D_p(g) + D_p(f)g(p)$ and TM is the disjoint union of the vector spaces T_pM, with $p \in M$. It has a natural structure of a smooth vector bundle over M. Let v be a tangent vector to M (say $v \in T_pM$). We say that v is *inward pointing* if, by definition, there exists a smooth curve $\gamma : [0,1] \to M$ such that $\gamma'(0) = v$ (so $\gamma(0) = p$). The set of inward pointing vectors in $v \in T_x(M)$ will form a closed cone denoted $T_x^+(M)$. If, close to x, our manifold with corners is given by the conditions $\{f_i(y) \geq 0\}$ with df_i linearly independent at x, then the cone $T_x^+(M)$ is given by

$$T_x^+(M) \,=\, \{v \in T_xM,\, df_i(v) \geq 0\}. \tag{1}$$

Let M and M_1 be manifolds with corners and $f : M_1 \to M$ be a smooth map. Then f induces a vector bundle map $df : TM_1 \to TM$, as in the smooth case, satisfying also $df(T_z^+(M_1)) \subset T_{f(z)}^+M$. If the smooth map $f : M_1 \to M$ is injective, has injective differential df, and has locally closed range, then we say that $f(M_1)$ is a *submanifold* of M. We are thus imposing the least restrictions on smooth maps and submanifolds, unlike [31, 44], for example. For example, a smooth map f between manifolds with corners is a *submersion* if, by definition, the differential $df = f_*$ is surjective (as in the case of smooth manifolds). However, we will typically need a special class of submersions with additional, properties, the *tame submersions*. More precisely, we have the following definiton.

Definition 2.1. A *tame submersion* h between two manifolds with corners M_1 and M is a smooth map $h : M_1 \to M$ such that its differential dh is surjective everywhere and

$$(dh_x)^{-1}(T^+_{h(x)}M) \,=\, T^+_x M_1\,.$$

(That is, $dh(v)$ is an inward pointing vector if, and only if, v is an inward pointing vector.)

We do not require our tame submersions to be surjective (although, as we will see soon below, they are open, as in the smooth case). We shall need the following lemma.

Lemma 2.2. *Let $h : M_1 \to M$ be a tame submersion of manifolds with corners. Then x and $h(x)$ have the same depth.*

Proof. This is because the depth of x in M is the same as the depth of the origin 0 in $T^+_x M_1$, which, in turn, is the same as the depth of the origin 0 in $T^+_{h(x)}M$ since dh_x is surjective and $(dh_x)^{-1}(T^+_{h(x)}M) = T^+_x M_1$. □

The following lemma is undoubtly folklore, but we could not find a suitable reference.

Lemma 2.3. *Let $h : M_1 \to M$ be a tame submersion of manifolds with corners.*

(i) The rank of M_1 is $\leq$ the rank of M.
(ii) For $m_1 \in M_1$, there exists an open neighbourhood U of m_1 in M_1 such that $h(U)$ is open and the restriction of h to U is a fibration with basis $h(U)$.
(iii) Let $L \subset M$ be a submanifold, then $L_1 := h^{-1}(L)$ is a submanifold of M_1 of rank $\leq$ the rank of L.

Proof. We have already noticed that the depths of x and $h(x)$ are the same (Lemma 2.2), so the rank of M_1, which is the maximum of the depths of $x \in M_1$, is inferior or equal to the rank of M. This proves (i).

Let us now prove (ii). Let $m_1 \in M_1$ be of depth k. We can choose standard neighbourhoods W_1 of m_1 in M_1 and W of $h(m_1)$ in M such that $h(W_1) \subset W$. Since our problem is local, we may assume that $M_1 = W_1 = [0,a)^k \times (-a,a)^{n_1-k}$ and $M = W = [0,b)^k \times (-b,b)^{n-k}$, $a, b > 0$, with m_1 and $h(m_1)$ corresponding to the origins. Note that both M and M_1 will then be manifolds with corners of rank k; this is possible since h preserves the depth, by Lemma 2.2. We can then extend h to a map $h_0 : Y_1 := (-a,a)^{n_1} \to \mathbb{R}^n$ that is a (usual) submersion at $0 = m_1$. By decreasing a, if necessary, we may assume that h_0 is a (usual) submersion everywhere and hence that $h_0(Y_1)$ is open in $\mathbb{R}^n$. By standard differential geometry results, we can then choose an open neighbourhood V of $0 = h_0(m_1)$ in $\mathbb{R}^n$ and an open neighbourhood V_1 of $0 = m_1$ in $Y_1 := (-a,a)^{n_1}$ such that the restriction h_1 of h_0 to V_1 is a fibration $h_1 : V_1 \to V$ with fibers diffeomorphic to $(-1,1)^{n_1-n}$. By further decreasing V and V_1, we may assume that V is an open ball centered at 0.

Next, we notice that $M \cap V$ consists of the vectors in V that have the first k components ≥ 0. By construction, we therefore have that

$$h_1(M_1 \cap V_1) \,:=\, h_0(M_1 \cap V_1) \,\subset\, M \cap V \,=\, \Big([0,b)^k \times (-b,b)^{n-k}\Big) \cap V\,.$$

We will show that we have in fact more, namely, that we have

$$M_1 \cap V_1 \,=\, h_1^{-1}(M \cap V)\,, \tag{2}$$

which will prove (ii) for $U := M_1 \cap V_1$, since $h_1 : V_1 \to V$ is a fibration with fibers diffeomorphic to $(-1,1)^{n_1-n}$ and $h(U) = h_1(U) = M \cap V$ is open in M.

Indeed, in order to prove the relation in Equation (2) and thus to complete the proof of (ii), let us assume, by contradiction, that there exists $p = (p_i) \in V_1 \smallsetminus M_1$ such that $h_1(p) = h_0(p) \in M \cap V = \big([0,b)^k \times (-b,b)^{n-k}\big) \cap V$. Let us choose $q = (q_i)$ in $M_1 \cap V_1$ of depth zero. That is, we assume that q is an interior point of $M_1 \cap V_1$. Then the two points $h_1(p) = h_0(p)$ and $h_1(q) = h_0(q)$ both belong to M, more pricisely,

$$h_1(p), h_1(q) \in M \cap V = \big([0,b)^k \times (-b,b)^{n-k}\big) \cap V\,,$$

which is the first octant in a ball. Therefore $h_1(p)$ and $h_1(q)$ can be joined by a path $\gamma = (\gamma_i) : [0,1] \to M \cap V$, with $\gamma(1) = h_1(p)$. (All paths are continuous by definition.) Since h preserves the depth, $h_1(q) = h_0(q) = h(q)$ is moreover an interior point of $M \cap V$. Therefore we may assume that the path $\gamma(t)$ consists completely of interior points of M for $t < 1$.

We can lift the path γ to a path $\tilde\gamma : [0,1] \to V_1$ with $\tilde\gamma(0) = q$, $\tilde\gamma(1) = p$, $\gamma = h_1 \circ \tilde\gamma$, since

$$h_1 := h_0|_{V_1} : V_1 \to V$$

is a fibration. We have $\tilde\gamma_i(0) = q_i > 0$ for $i = 1, \ldots, k$, since $q = (q_i)$ is an interior point of $V_1 \cap M_1$. On the other hand, since $p \notin M_1$, there exists at least one i, $1 \le i \le k$, such that $\tilde\gamma_i(1) = p_i < 0$. Since $\tilde\gamma_i(0) = q_i > 0$, we obtain that the set

$$Z := \{\, t \in [0,1], \;\text{ there exists }\; 1 \le j \le k \;\text{ such that }\; \tilde\gamma_j(t) = 0 \,\}$$

is non-empty. Let $t_* = \inf Z$. Then $t_* > 0$ since $q = (q_i) = (\tilde\gamma_i(0))$ is of depth zero, meaning that $\tilde\gamma_j(0) > 0$ for $1 \le j \le k$. Moreover, $\tilde\gamma_i(s) > 0$ for all $0 \le s < t_*$, by the minimality of t_*. Hence $\tilde\gamma(s) \in M_1 \subset Y_1$ for $s < t_*$. (Recall that $h_0 : Y_1 := (-a,a)^{n_1} \to \mathbb{R}^n$ and that we are assuming $M_1 = [0,1)^k \times (-1,1)^{n-k}$.) We obtain that $\tilde\gamma(t_*) \in M_1 \cap V_1$, because M_1 is closed in Y_1. Therefore $t_* < 1$, because $p = \tilde\gamma(1) \notin M_1$. Since $\tilde\gamma_j(t_*) = 0$ for some j, we have that $\tilde\gamma(t_*)$ is a boundary point of M_1, and hence it has depth > 0. Hence the depth of $\gamma(t_*) = h_0(\tilde\gamma(t_*)) = h(\tilde\gamma(t_*))$ is also > 0 since h preserves the depth. But this is a contradition since $\gamma(t)$ was constructed to consist entirely of interior points for $t < 1$. This proves (ii).

The last part is a consequence of (ii), as follows. We use the same notation as in the proof of (ii). We may assume $h^{-1}(L)$ to be non-empty, because otherwise the statement is obviously true, and hence there is nothing to prove. Let us choose $m \in L$ and $m_1 \in M_1$ such that $h(m_1) = m$. (That is, we choose $m_1 \in h^{-1}(L)$ and let $m = h(m_1)$.) There exit neighborhoods U of m_1 and V of m such that the restriction of h to U induces a fibration $h_2 := h|_U : U \to V$, by (ii), which we have just proved. By decreasing U and V, we can assume that the fibers of h_2 are diffeomorphic to $(-1,1)^{n-n'}$. Let V_1 be a standard neighbourhood of $m = h(m_1)$ in L. Then $h_1^{-1}(V_1)$ is a standard neighbourhood of m_1 in $h^{-1}(L)$. This completes the proof of (iii) and, hence, also of the lemma. □

We shall use the above result in the following way:

Corollary 2.4. *Let $h : M_1 \to M$ be a tame submersion of manifolds with corners.*

(i) h is an open map.

(ii) The fibers $h^{-1}(m)$, $m \in M$, are smooth manifolds (that is, they have no corners).

(iii) *Let us denote by $\Delta \in M \times M$ be the diagonal and by $h \times h : M_1 \times M_1 \to M \times M$ the product map $h \times h(m, m') = (h(m), h(m'))$. Then $(h \times h)^{-1}(\Delta)$ is a submanifold of $M_1 \times M_1$ of the same rank as M_1.*

Proof. The first part follows from Lemma 2.3(ii). The second and third parts follows from Lemma 2.3(iii), by taking $L = \{m\}$ for (ii) and $L = \Delta$ for (iii). □

If $E \to X$ is a smooth vector bundle, we denote by $\Gamma(X; E)$ (respectively, by $\Gamma_c(X; E)$) the space of smooth (respectively, smooth, compactly supported) sections of E. Sometimes, when no confusion can arise, we simply write $\Gamma(E)$, or, respectively, $\Gamma_c(E)$ instead of $\Gamma(X; E)$, respectively $\Gamma_c(X; E)$. If M is a manifold with corners, we shall denote by

$$\mathcal{V}_b(M) := \{\, X \in \Gamma(M; TM),\ X \text{ tangent to all faces of } M \,\}$$

the set of vector fields on M that are tangent to all faces of M (see [44, Section 2.2]).

For further reference, let us recall a classical result of Serre and Swan [32], which we formulate in the form that we will use. Recall that in this paper all our topological spaces are Hausdorff, unless otherwise stated.

Theorem 2.5 (Serre-Swan, [32])**.** *Let M be a compact manifold with corners and $\mathcal{V}$ be a finitely generated, projective $\mathcal{C}^\infty(M)$-module. Then there exists a real vector bundle $E_\mathcal{V} \to M$, uniquely determined up to isomorphism, such that $\mathcal{V} \simeq \Gamma(M; E_\mathcal{V})$ as $\mathcal{C}^\infty(M)$-module. We can choose $E_\mathcal{V}$ to depend functorially on $\mathcal{V}$, in particular, any $\mathcal{C}^\infty(M)$-module morphism $f : \mathcal{V} \to \mathcal{W} \simeq \Gamma(M; E_\mathcal{W})$ induces a unique smooth vector bundle morphism $\tilde{f} : E_\mathcal{V} \to E_\mathcal{W}$ compatible with the isomorphisms $\mathcal{V} \simeq \Gamma(M; E_\mathcal{V})$ and $\mathcal{W} \simeq \Gamma(M; E_\mathcal{W})$.*

Proof. We include the standard proof in order to develop some intuition and in order to recall the standard constructions of the vector bundle $E_\mathcal{V}$. Let us denote, for $x \in M$, by $I_x := \{f \in \mathcal{C}^\infty(M),\ f(x) = 0\}$, the maximal ideal of functions vanishing at x. Then we let $E_x := \mathcal{V}/I_x\mathcal{V}$ and $E_\mathcal{V} := \cup_{x \in M} E_x$, where the union is disjoint. The Serre-Swan theorem is usually proved in the framework of *continuous* vector bundles and modules over the algebra of *real valued* continuous functions $\mathcal{C}(M)$. However, this gives right away the result in the smooth case, since a projective module $\mathcal{V}$ over $\mathcal{C}^\infty(M)$ gives rise to the projective module $\mathcal{C}(M) \otimes_{\mathcal{C}^\infty(M)} \mathcal{V}$ over $\mathcal{C}(M)$ and hence to a vector bundle $E \to M$ such that $\mathcal{V} \subset \mathcal{C}(M; E)$. The smooth structure on E is obtained by considering, in the neighbourhood of each point of M, a local basis of $\mathcal{V}$. The Serre-Swan Theorem and these considerations extend immediately to the non-compact case. Indeed, let M be a manifold with corners (hence paracompact, by our convention) and let $\mathcal{V}$ be a module over M that is *locally finitely generated, projective*, in the sense that the restriction $\mathcal{V}_K := \mathcal{C}^\infty(K) \otimes_{\mathcal{C}^\infty(M)} \mathcal{V}$ of $\mathcal{V}$ to K is a finitely generated, projective $\mathcal{C}^\infty(K)$-module for any compact submanifold $K \subset M$. We define $E_\mathcal{V} := \cup_{x \in M} E_x$ as before and we give it the topology and smooth structure that makes all the restriction maps $E_\mathcal{V} := \cup_{x \in M} E_x \to \cup_{x \in K} E_x$ smooth for any compact submanifold (possibly with corners) $K \subset M$. □

Remark 2.6. In particular, there exists a (unique up to isomorphism) vector bundle T^bM such that $\Gamma(T^bM) \simeq \mathcal{V}_b(M)$ as $\mathcal{C}^\infty(M)$-modules [43].

2.2. **Definition of Lie groupoids and Lie algebroids.** Recall that a *groupoid* $\mathcal{G}$ is a small category in which every morphism is invertible. We identify, by abuse of notation, $\mathcal{G}$ with its set of morphisms. Since we assumed that $\mathcal{G}$ is a small category, we have that $\mathcal{G}$ and the class of objects $M := \mathcal{G}^{(0)}$ of $\mathcal{G}$ are both sets. We shall write $\mathcal{G} \rightrightarrows M$ for a groupoid with units M. The domain and range of a morphism therefore give rise to maps $d, r : \mathcal{G} \to M$. We shall denote by $\mu(g, h) = gh$ the composition of two composable morphisms g and h, that is, the composition of two morphisms satisfying $d(g) = r(h)$. We denote by

$$\mathcal{G}^{(2)} := \{(g, h) \in \mathcal{G} \times \mathcal{G},\ d(g) = r(h)\} \tag{3}$$

the domain of the composition map μ. Since we have assumed that every morphism $g \in \mathcal{G}$ has an inverse, we obtain a well defined map $\iota : \mathcal{G} \to \mathcal{G}$, $\iota(g) = g^{-1}$, and an embedding $u : M \to \mathcal{G}$, which associates to each object its identity morphism. These maps are assumed to satisfy the usual axioms: associativity of the product, $gg^{-1} = r(g)$, $g^{-1}g = d(g)$, and $gd(g) = r(g)g$.

The objects of $\mathcal{G}$ will also be called *units* and the morphisms of $\mathcal{G}$ will also be called *arrows*. Recall then

Definition 2.7. A *locally compact groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ such that

(1) $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are locally compact spaces,
(2) the structural morphisms d, r, i, u, and μ are continuous,
(3) d is surjective and open.

We *do not assume* $\mathcal{G}$ to be Hausdorff in this definition, however, the space of units M is assumed to be Hausdorff.

Similarly, we define Lie groupoids roughly by replacing the continuity condition with a smoothness condition.

Definition 2.8. A *Lie groupoid* is a groupoid $\mathcal{G} \rightrightarrows M$ such that

(1) $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)}$ are manifolds with corners,
(2) the structural morphisms d, r, i, u are smooth,
(3) d is a tame submersion of manifolds with corners (and hence the set of composable arrows is a manifold), and
(4) μ is smooth.

Again, we *do not assume* $\mathcal{G}$ to be Hausdorff, however, the space of units M is assumed to be Hausdorff.

In particular, Lie groupoids are locally compact groupoids.

Since d and r are tame submersions, it follows from Corollary 2.4(ii) that the fibers $\mathcal{G}_x := d^{-1}(x)$, $x \in M$, are smooth manifolds (that is, they have no corners). Similarly, (iii) of the same corollary implies that the set of composable units in $\mathcal{G} \times \mathcal{G}$ is a manifold as well (but it may have corners).

Lie groupoids were introduced by Ehresmann, see [38, 39] for a comprehensive introduction to the subject as well as for more references.

A *subgroupoid* of a groupoid $\mathcal{G}$ is a subset $\mathcal{H}$ such that the structural morphisms of $\mathcal{G}$ induce a groupoid structure on $\mathcal{H}$. We shall need the notion of a *Lie subgroupoid* of a Lie groupoid, which is closely modelled on the definition in [39]. Recall that if M is a manifold with corners and $L \subset M$ is a subset, we say that L is a submanifold of M if it is a locally closed subset, it is a manifold with corners in its own with for topology induced from M, and the inclusion $L \to M$ is smooth and has injective differential.

Definition 2.9. Let $\mathcal{G}$ be a Lie groupoid with units M. A Lie groupoid $\mathcal{H}$ is a *Lie subgroupoid* of $\mathcal{G}$ with units $M_1 \subset M$ if M_1 is a submanifold of M and $\mathcal{H}$ is a submanifold of $\mathcal{G}$ and a subgroupoid with units M_1. We say that $\mathcal{H}$ is a *closed Lie subgroupoid* of $\mathcal{G}$ if $\mathcal{H}$ and M_1 are closed in $\mathcal{G}$, respectively, in M.

Every Lie group is a Lie groupoid. In fact, Lie groupoids provide a natural generalization of Lie groups and they enjoy many of the useful properties that Lie groups enjoy. In particular, they have an associated infinitesimal object, the *Lie algebroid* associated to a Lie groupoid. To define it, let us first recall the definition of a Lie algebroid. See Pradines' [56] for the original definition and Mackenzie's books for a nice introduction to their general theory.

Definition 2.10. A *Lie algebroid* $A \to M$ is a real vector bundle over a manifold with corners M together with a *Lie algebra* structure on $\Gamma(M; A)$ (with bracket $[\ ,\ ]$) and a vector bundle map $\varrho : A \to TM$, called *anchor*, such that the induced map $\varrho_* : \Gamma(M; A) \to \Gamma(M; TM)$ satisfies the following two conditions:

(i) $\varrho_*([X, Y]) = [\varrho_*(X), \varrho_*(Y)]$ and
(ii) $[X, fY] = f[X, Y] + (\varrho_*(X)f)Y$, for all $X, Y \in \Gamma(M; A)$ and $f \in \mathcal{C}^\infty(M)$.

We shall need the following simple lemma.

Lemma 2.11. *Let $A \to M$ be a Lie algebroid and $f \in \mathcal{C}^\infty(M)$ be such that $\{f = 0\}$ has an empty interior. Then $f\Gamma(M; A) \subset \Gamma(M; A)$ is a finitely generated, projective module and a Lie subalgebra. Thus there exists a Lie algebroid, denoted fA, such that $\Gamma(fA) := \Gamma(M; fA) \simeq f\Gamma(A)$. Moreover, $f\Gamma(fA) = \Gamma(f^2A)$ is a Lie ideal in $\Gamma(fA)$.*

Proof. The proof of the Lemma relies on two simple calculations, which nevertheless will be useful in what follows. Let $X, Y \in \Gamma(A) := \Gamma(M; A)$. We have

$$[fX, fY] \,=\, fX(f)Y - fY(f)X + f^2[X, Y] \,\in\, \Gamma(fA)\,. \tag{4}$$

On the other hand,

$$[fX, f^2Y] = 2f^2X(f)Y - f^2Y(f)X + f^3[X, Y] \,\in\, \Gamma(f^2A)\,. \tag{5}$$

The proof is complete. □

Recall the following definition (see [39, 61]).

Definition 2.12. Let R be a commutative associative unital real algebra and let $\mathfrak{g}$ be a Lie algebra and an R-module such that $\mathfrak{g}$ acts by derivations on R and the Lie bracket satisfies

$$[X, rY] \,=\, r[X, Y] + X(r)Y\,, \qquad \text{for all } \ r \in R \ \text{ and } \ X, Y \in \mathfrak{g}\,.$$

Then we say that $\mathfrak{g}$ is an *R-Lie-Rinehart algebra.*

Let us recall now the definition of the Lie algebroid $A(\mathcal{G})$ associated to a Lie groupoid $\mathcal{G}$. Let $\mathcal{G}$ be a Lie groupoid with units M, then we let

$$A(\mathcal{G}) \,:=\, \ker(d_* : T\mathcal{G} \to TM)|_M\,,$$

that is, $A(\mathcal{G})$ is the restriction to the units of the kernel of the differential of the domain map d. The sections of $A(\mathcal{G})$ identify with the space of d-horrizontal, right invariant vector fields on $\mathcal{G}$ (that is, vector fields on $\mathcal{G}$ that are tangent to the submanifolds $\mathcal{G}_x := d^{-1}(x)$ and are invariant with respect to the natural action of

$\mathcal{G}$ by right translations). In particular, the space of sections of $A(\mathcal{G}) \to M$ has a natural Lie bracket that makes it into a Lie algebroid.

Definition 2.13. Let $\mathcal{G}$ be a Lie groupoid with units M, then the Lie algebroid $A(\mathcal{G})$ is called the *Lie algebroid associated to* $\mathcal{G}$.

2.3. **Examples of groupoids.** We continue with various examples of constructions of Lie groupoids and Lie algebroids that will be needed in what follows. Most of these constructions work in the category of locally compact groupoids, and often will be done in both settings (locally compact and Lie). We begin with the following three basic examples. Most of these examples are extensions to the smooth category of some examples from the locally compact category. We will not treat the locally compact examples separately, however.

Example 2.14. Any locally compact topological group is a locally compact groupoid with set of units reduced to one point: the identity element of G. If G is a Lie group, then we obtain a Lie groupoid with associated Lie algebroid $A(G) = Lie(G)$, the Lie algebra of G.

Example 2.15. Let M be a locally compact *Hausdorff* space and let $\mathcal{G}^{(1)} = \mathcal{G}^{(0)} = M$, so the groupoid of this example contains only units. We shall call a groupoid with these properties a *space*. If M is also a manifold with corners, we obtain a Lie groupoid with associated Lie algebroid $A(M) = M \times \{0\}$, the 0 vector bundle over M.

We thus see that the category of locally compact groupoids contains the subcategories of locally compact groups and of locally compact spaces. A similar comment applies to the category of Lie groupoids, which contains the subcategories of Lie groups and of manifolds (possibly with corners). The last basic example is that of a product.

Example 2.16. Let $\mathcal{G}_i \to M_i$, $i = 1, 2$, be two locally compact groupoids. Then $\mathcal{G}_1 \times \mathcal{G}_2$ is a locally compact groupoid with units $M_1 \times M_2$. If $\mathcal{G}_i$ are Lie groupoids, then so is $\mathcal{G}_1 \times \mathcal{G}_2$, and its associated Lie algebroid is $A(\mathcal{G}_1 \times \mathcal{G}_2) \simeq A(\mathcal{G}_1) \boxtimes A(\mathcal{G}_2)$, (see Proposition 4.3.10 in [39]).

We shall need the following more specific classes of Lie groupoids. The goal is to build more and more general examples that will lead us to a (slight extension of a) construction due to Debord and Skandalis [19]. We proceed by small steps, mainly due to the complicated nature of this construction, but also because particular or intermediate cases of this construction are needed on their own.

Example 2.17. Let G be a locally compact group with automorphism group $\mathrm{Aut}(G)$ and let $P \to M$ be a locally trivial, principal $\mathrm{Aut}(G)$-bundle with both P and M Hausdorff. Then the associated fiber bundle $\mathcal{G} := P \times_{\mathrm{Aut}(G)} G$ with fiber G is a locally compact groupoid with units M. A groupoid of this form will be called a *bundle of groups*. It satisfies $d = r$. If G is a Lie group and $P \to M$ is differentiable (with P and M manifolds, possibly with corners), then $\mathcal{G}$ is a Lie groupoid and $A(\mathcal{G}) \simeq P \times_{\mathrm{Aut}(G)} Lie(G)$.

Example 2.18. Let M be a Hausdorff, locally compact space. Then we define the *pair groupoid* of M as $\mathcal{G} := M \times M$, a groupoid with units M and with d the second projection, r the first projection, and $(m_1, m_2)(m_2, m_3) = (m_1, m_3)$. If M is a smooth manifold (so M has no corners), then the pair groupoid is a Lie groupoid and we have $A(M \times M) = TM$, with anchor map the identity map. A related example is that of the *path groupoid* of M, which will have the same Lie algebroid as the pair groupoid.

We extend the above example by defining *fibered pull-back groupoids* [26, 27].

Example 2.19. Let again M and L be Hausdorff, locally compact spaces and $f : M \to L$ be a continuous map. Let $\mathcal{H}$ be a locally compact groupoid with units L, the *pull-back groupoid* is then

$$f^{\downarrow\downarrow}(\mathcal{H}) := \{\, (m, g, m') \in M \times \mathcal{H} \times M, f(m) = r(g),\, d(g) = f(m') \,\},$$

with product $(m, g, m')(m', g', m'') = (m, gg', m'')$. We shall also sometimes write $M \times_f \mathcal{H} \times_f M = f^{\downarrow\downarrow}(\mathcal{H})$ for the pull-back groupoid. If f is a local fibration, then $\mathcal{H}$ is a locally compact groupoid with units M and a Haar system. If, moreover, M and L are manifolds, f is a tame submersion, and $\mathcal{H}$ is a Lie groupoid, then $\mathcal{G}$ is a Lie groupoid, called the *fibered pull-back* Lie groupoid. Indeed, to see that d is a tame submersion, if is enough to write that f is locally a product, see Lemma 2.3(ii). (See also Proposition 4.3.11 in [39] for more general conditions on f that ensure that the pull-back by f is a Lie groupoid.) It is a subgroupoid of the product $M \times M \times \mathcal{H}$ of the pair groupoid $M \times M$ and $\mathcal{H}$. Also by Proposition 4.3.11 in [39], we have

$$A\big(f^{\downarrow\downarrow}(\mathcal{H})\big) \simeq f^{\downarrow\downarrow}\big(A(\mathcal{H})\big) \tag{6}$$

(see Definition **??**). Thus the Lie algebroid of the fibered pull-back groupoid $f^{\downarrow\downarrow}\mathcal{H}$ is the pull-back Lie algebroid $f^{\downarrow\downarrow}\big(A(\mathcal{H})\big)$ and hence it contains as a Lie algebroid the space $\ker(df)$ of f-vertical tangent vector fields on M.

Remark 2.20. In Example 2.19, if $\mathcal{H}$ is a bundle of groups (see Example 2.17), then the orbits of the fibered pull-back groupoid $M \times_f \mathcal{H} \times_f M$ are the fibers $f^{-1}(x)$ with $x \in L$, hence this is not a bundle of groups unless f is injective. Nevertheless, the isotropy subgroupoid of $M \times_f \mathcal{H} \times_f M$ is a bundle of groups on every connected component of M.

We note that more general functions f (more general than submersions) can be used to define pull-back Lie groupoids [39].

Of all the groupoids considered so far, we shall need the last example in the topological case, since it is the prototypical example of a groupoid with the generalized Effros-Hahn property (see Section 5). The other constructions, especially the ones that will follow, will be needed in the smooth (Lie) setting. In fact, they do not make sense in the topological category. We continue with the adiabatic groupoid.

Recall that if $\mathcal{G}$ is a groupoid with units M and $A, B \subset M$, then we use the notation $\mathcal{G}_A^B := r^{-1}(B) \cap d^{-1}(A)$. In particular, $\mathcal{G}_A^A$ is a groupoid with units A. It is called the *reduction of* $\mathcal{G}$ *to* A. In general, it will not be a Lie groupoid even if $\mathcal{G}$ is. If A is invariant, then $\mathcal{G}_A^A = \mathcal{G}_A = \mathcal{G}^A$ and $\mathcal{G}_A$ is thus a groupoid, called the *restriction* of $\mathcal{G}$ to (the invariant subset) A.

We now recall the important example of the adiabatic groupoid [14, 19, 54]. It will be useful to recall the following glueing result from [53].

Example 2.21. Let $\mathcal{G}$ be a Lie groupoid with units M and Lie algebroid $A := A(\mathcal{G}) \to M$. The adiabatic groupoid $\mathcal{G}_{ad}$ [14, 19, 54] will have units $M \times [0, \infty)$. To define it, we proceed as in [54] to first define its Lie algebroid $A_{ad} = A(\mathcal{G}_{ad})$. As vector bundles over $M \times [0, \infty)$, we have

$$A_{ad} = A(\mathcal{G}_{ad}) \, = \, A \times [0, \infty) \to M \times [0, \infty)\,.$$

To define the Lie algebra structure on the space of sections of A, let $X(t)$ and $Y(t)$ be sections of A, regarded as smooth functions $[0, \infty) \to \Gamma(M; A(\mathcal{G}))$. Then

$$[X, Y](t) \, := \, t[X(t), Y(t)]\,. \tag{7}$$

Let us denote by $\pi : M \times [0, \infty) \to M$ the natural projection and by $\pi^*(A)$ the Lie algebroid defined by the vector bundle pull-back. Thus we see that $A_{ad} \simeq \pi^*(A)$ as *vector bundles* but **not** as Lie algebroids. Nevertheless, we do have a natural Lie groupoid morphism (not injective!)

$$A_{ad} \, \simeq \, t\pi^*(A) \, \to \pi^*(A)\,, \tag{8}$$

where the second Lie algebroid is defined by Lemma 2.11 and the isomorphism is by Equation (7). Next, as a set, we define the adiabatic groupoid $\mathcal{G}_{ad}$ associated to $\mathcal{G}$ as the *disjoint* union

$$\mathcal{G}_{ad} \, := \Big(A(\mathcal{G}) \times \{0\}\Big) \sqcup \Big(\mathcal{G} \times (0, \infty)\Big). \tag{9}$$

We endow $\mathcal{G}_1 := A(\mathcal{G}) \times \{0\}$ with the Lie groupoid structure of a bundle of Lie groups (with fiber $\mathbb{R}^N$, where N is the rank of $A(\mathcal{G})$) and zero anchor map. Then $A(\mathcal{G}_1)$ is isomorphic as a vector bundle to $A(\mathcal{G})$, but with zero Lie bracket and coincides with the restriction of $A_{ad} = A(\mathcal{G}_{ad})$ to $M \times \{0\}$. We endow $\mathcal{G}_2 := \mathcal{G} \times (0, \infty)$ with the product Lie groupoid structure, where $(0, \infty)$ is regarded as a space (that is $(0, \infty)$ has only units, and all orbits are reduced to a single point, as in Example 2.15). On $\mathcal{G} := \mathcal{G}_1 \cup \mathcal{G}_2$, we shall use the smooth structure defined by the deformation to the normal cone [28] associated to the inclusion $M \subset \mathcal{G}$. More precisely, we use the method in [53] as follows. Let us choose as in [54] connections ∇ on all the manifolds $\mathcal{G}_x$ such that the resulting family of connections is invariant with respect to right translations. This gives rise to a smooth map $\exp_\nabla : A = A(\mathcal{G}) \to \mathcal{G}$ that maps the zero section of $A(\mathcal{G})$ to the set of units of $\mathcal{G}$. There exists a neighbourhood U of the zero section of $A(\mathcal{G})$ on which $\exp_\nabla$ is a diffeomorphism onto its image. Let us define then $W \subset A \times [0, \infty) = A_{ad}$ to be the set of pairs $(X, t) \in A \times [0, \infty)$ such that $tX \in U$ and define $\Phi : W \to \mathcal{G}_{ad}$ by the formula

$$\Phi(X, t) \, := \begin{cases} (\exp_\nabla(tX), t) \in \mathcal{G} \times (0, \infty) & \text{if } t > 0 \text{ and } tX \in U \\ (X, 0) \in A(\mathcal{G}) \times \{0\} & \text{if } t = 0\,. \end{cases} \tag{10}$$

We define the smooth structure on $\mathcal{G}_{ad}$ such that $\mathcal{G} \times (0, \infty)$ and the image of Φ are open subsets. The fact that the resulting smooth structure makes $\mathcal{G}_{ad}$ a Lie groupoid follows from the differentiability with respect to parameters (including initial data) of solutions of ordinary differential equations. Note that by [53, 54], it is known that there exists a unique Lie groupoid structure on $\mathcal{G}_{ad}$ such that the associated Lie algebroid is A_{ad}. For the pair groupoid $\mathcal{G} = M \times M$ with M smooth, compact, this example is due to Connes [14] and was studied in connection with the index theorem for smooth, compact manifolds.

We shall need the following slight generalization of the above example, that will be called also an *adiabatic groupoid*. We shall do that by combining the construction of the pull-back with that of the adiabatic groupoid. We shall use the reduction of a groupoid G to a subset A, which, we recall, is denoted $G_A^A := r^{-1}(A) \cap d^{-1}(A)$.

Example 2.22. Let again M and L be manifolds with corners and $f : M \to L$ be a tame submersion of manifolds with corners. Let $\mathcal{H}$ be a Lie groupoid with units L and adiabatic groupoid $\mathcal{H}_{ad}$. Let $\mathcal{G} := f^{\downarrow\downarrow}(\mathcal{H}) = M \times_f \mathcal{H} \times_f M$ be the fibered pull-back groupoid. Then the *adiabatic groupoid of* $\mathcal{G}$ (with respect to f) has units $M \times [0, \infty)$ and is defined by

$$\mathcal{G}_{ad,f} \;:=\; f_1^{\downarrow\downarrow}(\mathcal{H}_{ad})\,,$$

where $f_1 := (f, id) : M \times [0, \infty) \to L \times [0, \infty)$.

Let us take a closer look at this example in the following particular case. Assume all manifolds are smooth and $\mathcal{H} = L \times L$. Then we obtain $\mathcal{G} = M \times M$ (so both $\mathcal{H}$ and $\mathcal{G}$ are pair groupoids in this particular case). However, unlike $\mathcal{G}_{ad}$, the groupoid $\mathcal{G}_{ad,f}$ will not be commutative at time 0, but will be the fibered pull-back of the Lie groupoid $A(\mathcal{H}) \to L$, regarded as a bundle of Lie groups, by the map $f : M \to L$. More precisely, let $X := \{0\} \times M$, which is an invariant subset of the set of units of $M \times [0, \infty)$. Then the restriction of $\mathcal{G}_{ad,f}$ to X satisfies

$$(\mathcal{G}_{ad,f})_X \;\simeq M \;\times_f A(\mathcal{H}) \times_f M \;=:\; f^{\downarrow\downarrow}\big(A(\mathcal{H})\big)\,. \tag{11}$$

In this particular case, the associated differential operators on $\mathcal{G}_{ad,f}$ model adiabatic limits, hence the name of these groupoids (this explains the choice of the name “adiabatic groupoid” in [54]).

We need an even more general class of groupoids, generalizing the above construction to obtain a (slight extension of a) construction by [19]. We use the acronym GDS as a shorthand for Grushin-Debord-Skandalis.

Example 2.23. We use the same setting and notation as in Example 2.22 above and let $\mathbb{R}_+^* = (0, \infty)$ act by dilations on the time variable $[0, \infty)$. This action induces a family of automorphisms of $\mathcal{H}_{ad}$, as in [19] if we let $s \in \mathbb{R}_+^* = (0, \infty)$ act by $s \cdot (g, t) = (g, s^{-1}t)$ on $(g, t) \in \mathcal{H} \times (0, \infty) \subset \mathcal{H}_{ad}$. Refering to Equation (10) that defines a parametrization of a neighbourhood of $A(\mathcal{H}) \times \{0\} \subset \mathcal{H}_{ad}$, we obtain

$$\begin{aligned} s \cdot \Phi(X, t) \;:=\; s \cdot (\exp_\nabla(tX), t) \;:=\; & (\exp_\nabla(tX), s^{-1}t) \\ =\; & (\exp_\nabla(s^{-1}tsX), s^{-1}t) \;=:\; \Phi(sX, s^{-1}t)\,. \end{aligned}$$

By setting $t = 0$ in this equation, we obtain by continuity that the action of s on $(X, 0)$ is $s(X, 0) = (sX, 0)$.

Let $f_1 := (f, id) : M \times [0, \infty) \to L \times [0, \infty)$. The action of $\mathbb{R}_+^*$ then commutes with f_1 and induces a family of automorphisms of $\mathcal{G}_{ad,f} := f_1^{\downarrow\downarrow}(\mathcal{H}_{ad})$. We then let

$$GDS(M, f, \mathcal{H}) \;:=\; \mathcal{G}_{ad,f} \rtimes \mathbb{R}_+^* := f_1^{\downarrow\downarrow}(\mathcal{H}_{ad}) \rtimes \mathbb{R}_+^* \;\simeq := f_1^{\downarrow\downarrow}(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*)\,,$$

the associated semi-direct product groupoid [39, 46]. The groupoid $\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*$ was introduced and studied in [19]. The space of units of the groupoid $GDS(M, f, \mathcal{H}) := \mathcal{G}_{ad,f} \rtimes \mathbb{R}_+^*$ is $M \times [0, \infty)$. To describe $GDS(M, f, \mathcal{H})$ as a set, we shall describe its reductions to $M \times \{0\}$ and to $M \times (0, \infty)$ (that is, we shall describe its reductions at time $t = 0$ and at time $t > 0$).

Let us endow $A(\mathcal{H})$ with the Lie groupoid structure of a (commutative) bundle of Lie groups with units $L \times \{0\}$. Then, at time $t = 0$, $GDS(M, f, \mathcal{H})$ is the semi-direct product $(M \times_f A(\mathcal{H}) \times_f M) \rtimes \mathbb{R}_+^*$, with $\mathbb{R}_+^*$ acting by dilations on the fibers of $A(\mathcal{H})$. That is

$$GDS(M, f, \mathcal{H})_{\{0\} \times M} \;\simeq\; (M \times_f A(\mathcal{H}) \times_f M) \rtimes \mathbb{R}_+^* \;=\; M \times_f (A(\mathcal{H}) \rtimes \mathbb{R}_+^*) \times_f M$$

groups on L. On the other hand, the complement, that is, the reduction of $GDS(M, f, \mathcal{H})$ to $M \times (0, \infty)$ is isomorphic to the product groupoid

$$(M \times_f \mathcal{H} \times_f M) \times (0, \infty)^2 ,$$

where the first factor in the product is the fibered pull-back of $\mathcal{H}$ to M and the second factor is the pair groupoid of $(0, \infty)$. We therefore have

$$GDS(M, f, \mathcal{H}) := M \times_f (\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*) \times_f M := f^{\downarrow\downarrow}(\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*) ,$$

with $\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*$ introduced and studied in [19].

3. Desingularization groupoids and their geometric properties

We now introduce a desingularization construction of a Lie groupoid that is related to some earlier results of Grushin [24] and Debord-Skandalis [19], Mazzeo [42], Schulze [64], Lauter-Nistor [36], and many others.

3.1. **A structure theorem near tame submanifolds.** We now recall the definition of a Lie subalgebroid, see for example [39, Definition 4.3.14].

Definition 3.1. f Let $A \to M$ be a Lie algebroid with anchor $\varrho : A \to TM$. A *Lie subalgebroid* of A is a subbundle $B \subset A|_L$, where L is a closed submanifold of M (possibly with corners) with the following properties:

(1) The anchor ϱ maps B to TL.
(2) If $X, Y \in \Gamma(M; A)$ are such that their restrictions to L are sections of B, then the restriction of $[X, Y]$ to L is also a section of B.
(3) If $X, Y \in \Gamma(M; A)$ satisfy $X = 0$ on L and the restriction $Y|_L$ of Y to L is a section of B, then $[X, Y] = 0$ on L.

We now introduce the concept of a *tame Lie subalgebroid* $B \subset A$ over $L \subset M$.

Definition 3.2. A *tame Lie subalgebroid* B of $A \to M$ over a submanifold $L \subset M$ is a Lie subalgebroid of A as in Definition 3.1 such that B is a subbundle $A|_L$ and there exists a a tubular neighborhood V of L in M with projection map $\pi : V \to L$ such that the restriction of A to V is isomorphic to the fibered pull-back Lie algebroid of B to V via π, that is,

$$A|_V \simeq \pi^{\downarrow\downarrow}(B) . \tag{12}$$

A submanifold L for which there exists a tame Lie subalgebroid $B \to L$ of $A \to M$ is called an A-*tame* submanifold of M.

Recall that if $\mathcal{G}$ is a groupoid with units M and $A \subset M$, then $\mathcal{G}_A^A := r^{-1}(A) \cap d^{-1}(A)$ is the *reduction of* $\mathcal{G}$ *to* A. Also, recall that a topological space is called *simply-connected* if it is path connected and its first homotopy group $\pi_1(X)$ is trivial. A groupoid $\mathcal{G}$ is called *d-simply connected* if the fibers $\mathcal{G}_x := d^{-1}(x)$ of the domain map are simply-connected. Here is one of our main technical results that provides a canonical form for a Lie groupoid in the neighborhood of a tame submanifold. All the isomorphisms of Lie groupoids are smooth morphisms.

Theorem 3.3. *Let $\mathcal{G}$ be a Lie groupoid with units M and let $L \subset M$ be an $A(\mathcal{G})$-tame submanifold of M. Let $\pi : U \to L \subset U$ a tubular neighborhood as in Definition 3.2. Assume that the fibers of $\pi : U \to L$ are simply-connected. Then the* *reduction groupoids* $\mathcal{G}_L^L$ *and* $\mathcal{G}_U^U$ *are Lie groupoids and there exists an isomorphism of Lie groupoids*

$$\mathcal{G}_U^U \simeq \pi^{\downarrow\downarrow}(\mathcal{G}_L^L) := U \times_\pi \mathcal{G}_L^L \times_\pi U$$

that is the identity on the set of units U*.*

See [51] for a proof.

3.2. **Definition of the desingularization.** We shall now use the structure theorem, Theorem 3.3 to provide a modification (or desingularization) of a Lie groupoid in the neighborhood of a tame submanifold of its set of units. We need, however, to first discuss the desingularization (or blow-ups) of tame submanifolds. We follow [1]. Let L be a submanifold of M and we assume that L has a tubular neighborhood U in M. We denote by N the normal bundle of L in M and by SN its unit sphere bundle (for some fixed metric). The desingularization procedure yields a new manifold by removing L from M and glueing back SN. More precisely, we can arrange so that the tubular neighborhood U in M identifies with the interior of S, so it is such that $U \smallsetminus L \simeq SN \times (0,1)$. Thus we glue $M \smallsetminus L$ and $SN \times [0,1)$ along the common open subset (diffeomorphic to) $SN \times (0,1)$. We denote the resulting manifold by $[M : L]$ and call it the *blow-up* of M with respect to L, as usual. By construction, there exists an associated natural smooth map $\kappa : [M : L] \to M$, the *blow-down map*, which is the identity on $M \setminus L$. For example,

$$[\mathbb{R}^{n+k} : \{0\} \times \mathbb{R}^k] \simeq S^{n-1} \times [0,\infty) \times \mathbb{R}^k\,, \tag{13}$$

with $r \in [0,\infty)$ representing the distance to the submanifold $L = \{0\} \times \mathbb{R}^k$ and S^p denoting the sphere of dimension p (the unit sphere in $\mathbb{R}^{p+1}$. Locally, all blow-ups that we consider are of this form. The definition of the blow-up in this paper is the one common in Analysis [1, 9, 24, 42, 49], however, it is *different* from the one in [8, 25, 55], who consider $PN = SN/Z_2$, the projectivization of SN, instead of SN.

We are ready now to introduce the desingularization of a Lie groupoid with respect to a tame submanifold in the particular case of a pull-back.

Definition 3.4. Let $\pi : E \to L$ be a orthogonal vector bundle. Let $U \subset E$ be the set of vector of lenght < 1 and $SNL := \partial U \subset E$. The various restrictions of π to subsets of E will be denoted also by π. Let $\mathcal{H}$ be a Lie groupoid with units L and $\mathcal{G} := U \times_\pi \mathcal{H} \times_\pi U$. Then the GDS modification $\mathcal{G}_{U,GDS}$ of $\mathcal{G} := U \times_\pi \mathcal{H} \times_\pi U$ is the fibered pull-back groupoid

$$GDS(SNL, \pi, \mathcal{H}) := (SNL \times_\pi \mathcal{H}_{ad} \times_\pi SNL) \rtimes \mathbb{R}_+^* \simeq SNL \times_\pi (\mathcal{H}_{ad} \rtimes \mathbb{R}_+^*) \times_\pi SNL$$

introduced in the example 2.23. It is a Lie groupoid with units $SNL \times [0,\infty)$. We extend in an obvious way the definition of the GDS modification to groupoids isomorphic to groupoids of the form $\mathcal{G} = U \times_\pi \mathcal{H} \times_\pi U$.

We shall need also the following construction very closely related to a construction in [25, Theorem 3.4]. For $i = 1, 2$, let $\mathcal{G}_i$ be a Lie groupoid with units M_i, each M_i being a manifold with corners. Let us assume that we are given open subsets $U_i \subset M_i$ such that the reductions $(\mathcal{G}_i)_{U_i}^{U_i}$, $i = 1, 2$, are isomorphic via an isomorphism $\phi : (\mathcal{G}_1)_{U_1}^{U_1} \to (\mathcal{G}_2)_{U_2}^{U_2}$. Let

$$\mathcal{H} := \mathcal{G}_1 \cup_\phi \mathcal{G}_2 := (\mathcal{G}_1 \sqcup \mathcal{G}_2)/\sim\,, \tag{14}$$

where $\sim$ identifies $(\mathcal{G}_1)_{U_1}^{U_1}$ with $(\mathcal{G}_2)_{U_2}^{U_2}$ via ϕ and $\sqcup$ disjoint the disjoint union. Thus we glue $\mathcal{G}_1$ and $\mathcal{G}_2$ along the open subsets $(\mathcal{G}_1)_{U_1}^{U_1} \subset \mathcal{G}_1$ and $(\mathcal{G}_2)_{U_2}^{U_2} \subset \mathcal{G}_2$. We shall denote by $U_1^c := M_1 \smallsetminus U_1$ the complement of U_1 in M_1 and by $M_1 \cap \mathcal{G}_1 U_1^c \mathcal{G}_1$ the orbit of U_1^c in M_1. We shall use a similar notation for $\mathcal{G}_2$. The presumed groupoid $\mathcal{H}$ will have as units $M := M_1 \cup_\phi M_2$.

Proposition 3.5. *Let us assume that $\phi(U_1 \cap \mathcal{G}_1 U_1^c \mathcal{G}_1)$ does not intersect $U_2 \cap \mathcal{G}_2 U_2^c \mathcal{G}_2$ and that the set $\mathcal{H}$ of Equation* (14) *is a Hausdorff manifold (possibly with corners). Then $M := M_1 \cup_\phi M_2$ is a Hausdorff manifold (again, possibly with corners) and $\mathcal{H}$ has a natural Lie groupoid structure with units M containing $(\mathcal{G}_i)_{U_i}^{U_i}$ as open subsets. We have $\mathcal{G}_i \simeq (\mathcal{H})_{M_i}^{M_i}$.*

Proof. This is basically a consequence of the definitions. To define the multiplication in $\mathcal{H}$ just note that the hypothesis ensures that, if $g_j \in \mathcal{G}_j$, for $j = 1, 2$, with $r(g_1) = d(g_2)$, then either $d(g_1) \in U_1$ or $r(g_2) \in \phi(U_1)$. The structural maps d and r of $\mathcal{H}$ are defined using the ones of $\mathcal{G}_i$ since they coincide on their common domain. □

One of the differences between our result and Theorem 3.4 in [25] is that we are not starting with a Lie algebroid that needs to be integrated, thus we do not have orbits that we could use. See however [25] for a discussion of the glueing procedure in the framework of manifolds (and many other useful results). We are ready now to define the desingularization of any Lie groupoid with respect to a tame submanifold. We begin by fixing some notation.

Notations 3.6. In what follows, $\mathcal{G}$ will denote a Lie groupoid with units M and $L \subset M$ be an $A(\mathcal{G})$-tame submanifold. (Thus L will have coners in general.) By $\pi : U \to L$ we shall denote a tubular neighborhood of L, as in Definition 3.2, so $L \subset U$. Let S be the boundary of U. By decreasing U, if necessary, we can assume that $U \to L$ is the set of vectors of length < 1 in a suitable metric. Using Theorem 3.3, we obtain that the reduction $\mathcal{G}_U^U$ is of the form $U \times_\pi \mathcal{H} \times_\pi U$, and hence its GDS-modification $GDS(SNL, \pi, \mathcal{H})$ is defined. Let $M_1 = S \times [0, 1)$, which is an open subset of the set $S \times [0, \infty)$ of units of $GDS(SNS, \pi, \mathcal{H})$. We shal denote by $\mathcal{G}_1$ the reduction of $GDS(SNS, \pi, \mathcal{H})$ to M_1 and by $U_1 := U \smallsetminus L = S \times (0, 1) \subset M_1$. Similarly, $\mathcal{G}_2$ will denote the reduction of the groupoid $\mathcal{G}$ to $M \smallsetminus L$.

Remark 3.7. Using the notation and assumptions of Definition 3.4 and the notation introduced in 3.6, we have that the smooth morphism $\Psi : \mathcal{G}_{U,GDS} \to E \times_\pi \mathcal{H} \times_\pi E$ restricts to a morphism $\mathcal{G}_1 \to U \times_\pi \mathcal{H} \times_\pi U$ that is a bijection outside $d^{-1}(L) \subset \mathcal{G}_1 \subset \mathcal{G}_{U,GDS}$.

Remark 3.8. Using again the notation introduced in 3.6, we then have that the reduction of $\mathcal{G}_1$ to U_1 is isomorphic to

$$(SNL \times_\pi \mathcal{H} \times_\pi SNL) \times (0,1)^2 \simeq U_1 \times_\pi \mathcal{H} \times_\pi U_1 \,, \tag{15}$$

where $(0, 1)^2$ is the pair groupoid. Since the reduction of $\mathcal{G}$ to U is isomorphic to $U \times_\pi \mathcal{H} \times_\pi U$, it follows that the reduction of $\mathcal{G}$ to U_1 is isomorphic to $U_1 \times_\pi \mathcal{H} \times_\pi U_1$. Hence the reduction of $\mathcal{G}_2$ to U_1 is also isomorphic to $(U_1 \times_\pi \mathcal{H} \times_\pi U_1)$. We are in position then to glue the groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ along their reductions to U_1, using Proposition 3.5 (for $U_2 = U_1$). We can now define the blow-up of a groupoid with respect to a tame submanifold.

Definition 3.9. Let $L \subset M$ be an $A(\mathcal{G})$-tame manifold. Using the notation just defined, the result of glueing the groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ along their isomorphic reductions to $U_1 = SNL \times (0,1)$ using Proposition 3.5 is denoted $[[\mathcal{G} : L]]$ and is called the *desingularization of* $\mathcal{G}$ *along* L.

One should not confuse $[[\mathcal{G} : L]]$ with $[\mathcal{G} : L]$, the blow-up of the manifold $\mathcal{G}$ with respect to the submanifold L. We denote by $SNL \subset NL$ the units normal sphere bundle of L. In particular, $[M : L] \smallsetminus (M \smallsetminus L) = SNL$.

Proposition 3.10. *The space of units of* $[[\mathcal{G} : L]]$ *is* $[M : L]$. *We endow* $A(\mathcal{H}) \to L$ *with the Lie groupoid structure of a bundle of Lie groups. Then* $SNL := [M : L] \smallsetminus (M \smallsetminus L)$ *is a closed, invariant subset for* $[[\mathcal{G} : L]]$ *and* $[[\mathcal{G} : L]]_{SNL} = [[\mathcal{G} : L]] \smallsetminus \mathcal{G}_2$ *is a closed subgroupoid isomorphic to the fibered pull-back of* $A(\mathcal{H}) \rtimes \mathbb{R}_+^*$ *to* SNL *via the natural projection* $\pi : SNL \to L$.

Proof. When we glue groupoids, we also glue their units, which gives that the set of units of $[[\mathcal{G} : L]]$ is indeed $[M : L]$. The rest follows from the construction of $[[\mathcal{G} : L]]$ and the discussion in Example 2.23. □

The class of Hausdorff groupoids is closed under desingularizations. Indeed, we have the following lemma.

Lemma 3.11. *Let* $U_i \subset M_i$ *be open subsets of some Hausdorff manifolds* M_i, $i = 0, 1$. *Let us assume that we are given a diffeomorphism* $\phi : U_0 \to U_1$ *and let* $M := M_0 \cup_\phi M_1$ *the union of the two manifolds after identifying* U_0 *and* U_1. *Let us assume that there exists a continuous function* $\psi : M \to [0, 1]$ *such that* $\psi(M \smallsetminus M_i) = \{i\}$. *Then* M *is Hausdorff.*

More precisely, $M_0 \cup_\phi M_1$ is defined as follows. Let us consider on the disjoint union $M_0 \sqcup M_1$ the equivalence relation $\sim_\phi$ generated by $x \sim_\phi \phi(x)$ if $x \in U_0$. Then $M_0 \cup_\phi M_1 := M_0 \sqcup M_1 / \sim_\phi$.

Proof. Let $x_i \in M$. If $x_i \in M_j$, then we can find open, disjoint neighbourhoods of x_i, since M_j is Hausdorff. Let us assume then that $x_0 \notin M_0$ and $x_1 \notin M_1$. Then $\{\psi < 1/2\}$ and $\{\psi > 1/2\}$ are two open sets that will separate the two points. □

This gives the following.

Theorem 3.12. *Let* $L \subset M$ *be a tame* $A(\mathcal{G})$*-submanifold, for some Lie groupoid* $\mathcal{G}$ *with units* M. *If* $\mathcal{G}$ *is Hausdorff, then* $[[\mathcal{G} : L]]$ *is also Hausdorff.*

Proof. We use the notation of 3.6. We can assume that $U_1 = \{r < 1\}$, where r is the distance to L. Then the result follows from Lemma 3.11 by taking $\psi := \min\{r, 1\}$. □

This completes the first part of the paper.

4. Preliminaries on reprentations and groupoid C^*-algebras

We now begin the second part of the paper, where applications further analytic properties of desingularization groupoids will be discussed. In the two subsections that comprise this first section of the second part of the paper, we introduce and study several basic concepts related first to representations of C^*-algebras and, then, to groupoid C^*-algebras. More precisely, in the first subsection we introduce and study *exhaustive* and *invertibility sufficient* families of representations. In the following subsections we introduce locally compact groupoids, their Haar systems, and their C^*-algebras. We refer to [20] for general results on C^*-algebras. The basic definitions needed in this paper were reviewed also in [52], on which part of this section is based.

4.1. **Exhausting families of representations of C^*-algebras.** In what follows, we shall make extensive use of C^*-algebras. We review here some needed results from [20, 52].

We begin with a review of some needed general C^*-algebra results. We recall [20] that a *C^*-algebra* is a complex algebra A together with a conjugate linear involution $*$ and a complete norm $\| \ \|$ such that $(ab)^* = b^*a^*$, $\|ab\| \le \|a\|\|b\|$, and $\|a^*a\| = \|a\|^2$, for all $a, b \in A$. (The fact that $*$ is an involution means that $a^{**} = a$ and that $a \to a^*$ is conjugate linear: $(\lambda a)^* = \overline{\lambda} a^*$ for $a \in A$ and $\lambda \in \mathbb{C}$.) In particular, a C^*-algebra is also a Banach algebra. Let $\mathcal{H}$ be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of linear, bounded operators on $\mathcal{H}$.

One of the main reasons why C^*-algebras are important in applications is that every norm-closed subalgebra $A \subset \mathcal{L}(\mathcal{H})$ that is also closed under taking Hilbert space adjoints is a C^*-algebra. In this paper, we are interested mainly in C^*-algebras obtaining by completing algebras of order zero pseudodifferential operators acting on L^2-spaces. Nevertheless, abstract C^*-algebras have many non-trivial properties that can then be used to study the concretely given algebra A. A *representation* of a C^*-algebra A on the Hilbert space $\mathcal{H}_\pi$ is a $*$-morphism $\pi : A \to \mathcal{L}(\mathcal{H}_\pi)$ to the algebra of bounded operators on $\mathcal{H}_\pi$. We shall use the fact that every morphism ϕ of C^*-algebras (and hence any representation of a C^*-algebra) has norm $\|\phi\| \le 1$. Consequently, every bijective morphism of C^*-algebras is an isometric isomorphism. A basic results states that every abstract C^*-algebra is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(\mathcal{H})$ (the Gelfand-Naimark theorem, see Theorem 2.6.1 of [20]).

Throughout this paper, we shall denote by A a generic C^*-algebra. Also, by $\phi : A \to \mathcal{L}(\mathcal{H}_\phi)$ we shall denote generic representations of A. A two-sided ideal $I \subset A$ is called *primitive* if it is the kernel of an irreducible representation of A. We shall denote by $\operatorname{Prim}(A)$ the set of primitive ideals of A.

Remark 4.1. For any two-sided ideal $J \subset A$, we have that its primitive ideal spectrum $\operatorname{Prim}(J)$ identifies with the set of all the primitive ideals I of A *not* containing the two-sided ideal $J \subset A$. The correspondence is $I \to I \cap J \in \operatorname{Prim}(J)$. It turns out then that the sets of the form $\operatorname{Prim}(J)$, where J ranges through the set of two-sided ideals $J \subset A$, define a topology on $\operatorname{Prim}(A)$, called the *Jacobson topology* on $\operatorname{Prim}(A)$. We have the following canonical identification

$$\operatorname{Prim}(A/J) \,=\, \operatorname{Prim}(A/J)) \smallsetminus \operatorname{Prim}(J) \tag{16}$$

The identification is $I \to I/J$. Thus the closed sets of $\operatorname{Prim}(A)$ are the sets of the form $\operatorname{Prim}(A/J)$, with J a closed, two-sided ideal of A.

Primitive ideals will play a crucial role in what follows.

Examples 4.2. Let us list now a two basic examples primitive ideal spectra.

(1) Let us denote by $\mathcal{K}$ the algebra of compact operators on a generic, infinite dimensional, separable Hilbert space $\mathcal{H}$. Then $\operatorname{Prim}(\mathcal{K})$ consists of a single ideal, the zero ideal.

(2) If $A = C(K)$, the algebra of continuous functions on a compact space K, then K and $\mathrm{Prim}(A)$ are canonically homeomorphic. The correspondence associates to $x \in K$ the maximal ideal $\mathfrak{m}_x$ continuous functions on K that vanish at x.

We continue with a more involved, but basic for us, example.

Example 4.3. Let M be a smooth compact manifold and let $\Psi^0(M)$ be the algebra of order-zero, classical pseudodifferential operators on M. Let $\mathfrak{A}(M)$ denote the norm closure of $\Psi^0(M)$. Let us denote by S^*M the unit cosphere bundle of M, which is diffeomorphic to the set of vectors of length one in the cotangent space T^*M of M. Then the principal symbol map $\sigma_0 : \mathfrak{A}(M) \to \mathcal{C}(S^*M)$ yields the following exact sequence

$$0 \to \mathcal{K} \to \mathfrak{A}(M) \to \mathcal{C}(S^*M) \to 0\,.$$

According to the previous two examples and Equation (16), we have that

$$\mathrm{Prim}(\mathfrak{A}) \;=\; \mathrm{Prim}(\mathcal{C}(S^*M)) \sqcup \mathrm{Prim}(\mathcal{K}) \;=\; S^*M \sqcup \{0\}\,, \tag{17}$$

a disjoint union. The topology on S^*M is the standard one and the global topology is such that the point of $\mathrm{Prim}(\mathfrak{A})$ defined by the minimal ideal 0 is dense in the whols of $\mathrm{Prim}(\mathfrak{A})$.

We shall need the following definitions [21, 52, 62].

Definition 4.4. Let $\mathcal{F}$ be a set of representations of a unital C^*-algebra A.

(i) We say that $\mathcal{F}$ is *exhausting* if $\mathrm{Prim}(A) = \bigcup_{\phi\in\mathcal{F}} \mathrm{supp}(\phi)$.
(ii) We say that $\mathcal{F}$ is *strictly norming* if, for any $a \in A$, there exists $\phi \in \mathcal{F}$ such that $\|\phi(a)\| = \|a\|$.
(iii) We say that $\mathcal{F}$ is *invertibility sufficient* if, for any $a \in A$, we have that a is invertible in A if, and only if, $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$.

The same definitions can be formulated for sets of morphisms or sets of primitive ideals.

If A is nonunital, we change this definition as follows. We denote by $A^+ = A \oplus \mathbb{C}$ and by $\chi_0\colon A^+ \to \mathbb{C}$ the morphism defined by $\chi_0 = 0$ on A and $\chi_0(1) = 1$. We then modify Definition 4.4 by replacing A with A^+ and $\mathcal{F}$ with $\mathcal{F} \cup \{\chi_0\}$.

Remark 4.5. Let A be a non-unital C^*-algebra and let $\mathcal{F}$ be a set of representations. Then $\mathcal{F}$ is exhausting if, and only if, $\mathrm{Prim}(A) = \bigcup_{\phi\in\mathcal{F}} \mathrm{supp}(\phi)$. This follows immediately by the equality

$$\mathrm{Prim}(A^+) = \mathrm{Prim}(A) \cup \{A\}$$

and the fact that $\ker \chi_0 = A$. Similarly, $\mathcal{F}$ is exhausting if $1 + a \in A^+$, $a \in A$, is invertible if, and only if, $1 + \phi(a)$ is invertible for any $\phi \in \mathcal{F}$.

The set of all irreducible representations of a unital C^*-algebra is invertibility sufficient (see [20, 21]). Therefore any exhausting family is strictly norming [52]. Strictly norming and exhausting families are interesting because of the following result [21, 52, 62].

Theorem 4.6. *Let $\mathcal{F}$ be a set of non-degenerate representations of a C^*-algebra A. Then $\mathcal{F}$ is strictly norming if, and only if, it is invertibility sufficient.*

Every exhausting set of representations is invertibility sufficient. If A is furthermore separable, then the converse is also true.

Remark 4.7. We use the above result for "concrete" C^*-algebras, where by a concrete C^*-algebra we mean a C^*-algebra of bounded operators acting on a Hilbert space. Let thus $A \subset \mathcal{L}(\mathcal{H})$ be a concrete C^*-algebra. We assume that $\mathcal{K} := \mathcal{K}(\mathcal{H})$, the ideal of compact operators on $\mathcal{H}$, is contained in A. Then we have the following consequence of Theorem 4.6: If $\mathcal{F} = \{(\phi, \mathcal{H}_\phi)\}$ is a strictly norming set of representations of $A/\mathcal{K}$, then $a \in A$ is Fredholm if, and only if, $\phi(a)$ is invertible in $\mathcal{L}(\mathcal{H}_\phi)$ for all $\phi \in \mathcal{F}$. The converse is also true, in the sense that a family $\mathcal{F}$ of representations of $A/\mathcal{K}$ with the above properties must be strictly norming. This converse, although not needed, it justifies our interest in strictly norming families.

Let A be a C^*-algebra and $I \subset A$ be a closed two-sided ideal. Recall that any nondegenerate representation $\pi\colon I \to \mathcal{L}(\mathcal{H})$ extends to a unique representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$. (See [20, Proposition 2.10.4]. This extension is an instance of the Rieffel induction [60] corresponding to I, regarded as an A–I bimodule. We have the following result [52].

Proposition 4.8. *Let $I \subset A$ be an ideal of a C^*-algebra. Let $\mathcal{F}_I$ be a set of nondegenerate representations of I and $\mathcal{F}_{A/I}$ be a set of representations of A/I. Let $\mathcal{F} := \mathcal{F}_I \cup \mathcal{F}_{A/I}$, regarded as a family of representations of A. If $\mathcal{F}_I$ and $\mathcal{F}_{A/I}$ are both exhausting, then $\mathcal{F}$ is also exhausting. The same result holds by replacing exhausting with strictly norming.*

The following corollary will be used later on.

Corollary 4.9. *Let $I \subset A$ be an ideal of a unital C^*-algebra A and let $\mathcal{F}_I$ be an invertibility sufficient set of nondegenerate representations of I. Let $a \in A$. Then a is invertible in A if, and only if, it is invertible in A/I and $\phi(a)$ is invertible for all $\phi \in \mathcal{F}_I$.*

Proof. Since $\mathcal{F}$ is an invertibility preserving set of representations of I, it consists of non-degenerate representations, which will hence extend uniquely to A. Let π be a faithful representation of A/I and put $\{\pi\} = \mathcal{F}_{A/I}$. The result then follows from Proposition 4.8 applied to families of representations $\mathcal{F}_I$ and $\mathcal{F}_{A/I}$. □

4.2. **Locally compact groupoids and their C^*-algebras.** We now recall the definition of a Haar system of a locally compact groupoid and we use this oportunity to fix some more notation to be used throughout the rest of the paper. We refer to [12, 29, 58] for more information on the topics discussed in this subsection.

The definition of a locally compact groupoid was recalled in Definition 2.7. If $\mathcal{G}$ is a locally compact groupoid, we shall denote by $\mathcal{C}_c(\mathcal{G})$ the space of continuous, complex valued, compactly supported functions on $\mathcal{G}$. We are not assuming that $\mathcal{G}$ is Hausdorff, which means that some extra care needs to be taken in defining $\mathcal{C}_c(\mathcal{G})$, see [33] and the references therein. Nevertheless, all of our applications are for Hausdorff groupoids, so the reader may safely ignore the non-Hausdorff case.

Definition 4.10. A *left Haar system* for a locally compact groupoid $\mathcal{G}$ is a family $\lambda = \{\lambda^x\}_{x\in M}$, where λ^x is a Borel regular measure on $\mathcal{G}$ with $\operatorname{supp}\lambda^x = r^{-1}(x) =: \mathcal{G}^x$ for every $x \in M = \mathcal{G}^{(0)}$, satisfying

(i) *The continuity condition* that
$$M \ni x \mapsto \lambda(\varphi) := \int \varphi \mathrm{d}\lambda^x \in \mathbb{C}$$
is continuous for $\varphi \in \mathcal{C}_c(\mathcal{G})$.

(ii) *The invariance condition* that

$$\int \varphi(gh) \mathrm{d}\lambda^{d(g)}(h) = \int \varphi(h) \mathrm{d}\lambda^{r(g)}(h)$$

for all $g \in \mathcal{G}$ and $\varphi \in \mathcal{C}_c(\mathcal{G})$.

Right Haar systems are defined in the same way.

Remark 4.11. One defines similarly a *right Haar system* λ_x and notices that $\lambda_x(g) := \lambda^x(g^{-1})$ establishes a bijection between left and right Haar systems. We shall therefore simply use the term *Haar system* from now on.

Lie groupoids always have Haar systems.

Remark 4.12. Let us assume that $\mathcal{G}$ is a Lie groupoid with Lie algebroid $A(\mathcal{G})$. Let $D := |\Lambda^n A(\mathcal{G})|$, where n is the dimension of the Lie algebroid of $\mathcal{G}$. The pull-back vector bundle $r^*(D)$ is the bundle of 1-densities along the fibers of d. A trivialization of D will hence give rise to a right invariant set of measures on $\mathcal{G}_x$ and hence to a right Haar system.

Let now $\mathcal{G}$ be a locally compact groupoid with a Haar system λ_x. We now recall the definition of the basic C^*-algebras associated to $\mathcal{G}$ and its Haar system, following [58]. See also [12, 67]. We first define the convolution product on the space $\mathcal{C}_c(\mathcal{G})$ by the formula

$$(\varphi_1 * \varphi_2)(x) := \int\limits_{r^{-1}(d(g))} \varphi_1(gh)\varphi_2(h^{-1}) \mathrm{d}\lambda^{r(g)} , \quad \text{for } g \in \mathcal{G} \text{ and } \varphi_1, \varphi_2 \in \mathcal{C}_c(\mathcal{G}).$$

This makes $\mathcal{C}_c(\mathcal{G})$ into an associative $*$-algebra with the involution defined by $\varphi^*(g) := \overline{\varphi(g^{-1})}$ for all $g \in \mathcal{G}$ and $\varphi \in \mathcal{C}_c(\mathcal{G})$. There also exists a natural algebra norm on $\mathcal{C}_c(\mathcal{G})$ defined by

$$\|f\|_I := \max\Big\{ \sup_{x\in M} \int |\varphi| \mathrm{d}\lambda^x, \sup_{x\in M} \int |\varphi^*| \mathrm{d}\lambda^x \Big\}.$$

Definition 4.13. The (full) C^*-algebra associtated to $\mathcal{G}$, denoted $C^*(\mathcal{G})$, is defined as the completion of $\mathcal{C}_c(\mathcal{G})$ with respect to the norm

$$\|\varphi\| := \sup_{\pi} \|\pi(\varphi)\| ,$$

where π ranges over all bounded $*$-representations of $\mathcal{C}_c(\mathcal{G})$. Let us define as usual for any $x \in M$ the *regular* representation π_x of $\mathcal{C}_c(\mathcal{G})$ (and hence also of $C^*(\mathcal{G})$) by the formula

$$\pi_x : \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_x, \lambda_x)) , \quad \pi_x(\varphi)\psi := \varphi * \psi .$$

We then define similarly the reduced C^*-algebra $C^*_r(\mathcal{G})$ as the completion of $\mathcal{C}_c(\mathcal{G})$ with respect to the norm.

$$\|\varphi\|_r := \sup_{x\in M} \|\pi_x(\varphi)\| .$$

Here are some quick comments. It follows from this defition, that there is a canonical surjective $*$-homomorphism $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$. If this canonical morphism is also injective, then the groupoid $\mathcal{G}$ is called *metrically amenable.* Also, for further use, we note that if $\mathcal{G}$ second countable, then $C^*(\mathcal{G})$ is a separable C^*-algebra.

Remark 4.14. For any $\mathcal{G}$-invariant, locally closed subset $A \subseteq M$, the *reduced* groupoid $\mathcal{G}_A = \mathcal{G}_A^A$ has a Haar system λ_A obtained by restricting the Haar system λ of $\mathcal{G}$ to $\mathcal{G}_A$. In particular, we can construct as above the corresponding C^*-algebra $C^*(\mathcal{G}_A)$ and the reduced C^*-algebra $C^*_r(\mathcal{G}_A)$. For any closed subset $A \subseteq M$, the subset $d^{-1}(A) \subseteq \mathcal{G}$ is also closed, so the restriction map $\mathcal{C}_c(\mathcal{G}) \to \mathcal{C}_c(d^{-1}(A))$ is well defined. If A is also $\mathcal{G}$-invariant then the restriction extends by continuity to both a $*$-homomorphism $\mathcal{R}_A \colon C^*(\mathcal{G}) \to C^*(\mathcal{G}_A)$ and a $*$-homomorphism $(\mathcal{R}_A)_r \colon C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}_A)$ that are related by the commutative diagram

$$\begin{array}{ccc} C^*(\mathcal{G}) & \xrightarrow{\mathcal{R}_A} & C^*(\mathcal{G}_A) \\ \downarrow & & \downarrow \\ C^*_r(\mathcal{G}) & \xrightarrow{(\mathcal{R}_A)_r} & C^*_r(\mathcal{G}_A) \end{array} \tag{18}$$

where the vertical arrows are the natural quotient homomorphisms.

We have the following well known, but important result [48, 59] that we record for further reference.

Proposition 4.15. *Let $\mathcal{G} \rightrightarrows M$ be a second countable, locally compact groupoid with a Haar system.*

(i) Let $U \subset M$ be an open $\mathcal{G}$-invariant subset, $F := M \smallsetminus U$. Then $C^*(\mathcal{G}_U)$ is a closed two-sided ideal of $C^*(\mathcal{G})$ that yields the short exact sequence

$$0 \to C^*(\mathcal{G}_U) \to C^*(\mathcal{G}) \xrightarrow{\mathcal{R}_F} C^*(\mathcal{G}_F) \to 0$$

where $F := M \setminus U$.

(ii) If $\mathcal{G}_F$ is metrically amenable, then one has the exact sequence

$$0 \to C^*_r(\mathcal{G}_U) \to C^*_r(\mathcal{G}) \xrightarrow{(\mathcal{R}_F)_r} C^*_r(\mathcal{G}_F) \to 0\,.$$

Proof. The first assertion is well known, see for instance [48, Lemma 2.1]. The second statement is in [59, Remark 4.1]. □

We shall need the following consequence

Corollary 4.16. *Let $\mathcal{G} \rightrightarrows M$ be a locally compact groupoid with a Haar system and $U \subset M$ be an open subset, $F := M \smallsetminus U$, as usual. Let us assume that the set of regular representations of $C^*_r(\mathcal{G}_U)$ is an invertibility sufficient set of representations of $C^*_r(\mathcal{G}_U)$ and that the set of regular representations of $C^*(\mathcal{G}_F)$ is an invertibility sufficient set of representations of $C^*(\mathcal{G}_F)$. Then the set of regular representations of $C^*_r(\mathcal{G})$ is an invertibility sufficient set of representations of $C^*_r(\mathcal{G})$. The same result holds if we replace "invertibility sufficient" with "exhaustive."*

Proof. First of all, we have that $\mathcal{G}_F$ is metrically amenable. The result then follows from Propositions 4.15 and 4.8 applied to $I := C^*_r(\mathcal{G}_U)$, $A := C^*_r(\mathcal{G})$, since $A/I \simeq C^*(\mathcal{G}_F) \simeq C^*_r(\mathcal{G}_F)$ and the union of the set of regular representations of $C^*_r(\mathcal{G}_U)$ and of $C^*_r(\mathcal{G}_F)$ is the set of regular representations of $C^*_r(\mathcal{G}_F)$. □

5. Fredholm groupoids and the generalized Effros-Hahn conjecture

We now study Fredholm conditions for operators in algebras $\mathbf{\Psi}$ containing a reduced groupoid C^*-algebra $C^*_r(\mathcal{G})$ as an essential ideal. The groupoids for which we obtain the kind of Fredholm conditions that we want (the kind that are typically used in practice) will be called "Fredholm groupoids." They are introduced and discussed next. Our main abstract results on Fredholm groupoids are in the second subsection. Some more applicable results on Fredholm groupoids will be given in the next section, based on the results in this section.

5.1. Fredholm groupoids and their characterization.

We now introduce Fredholm groupoids and give a first characterization of these groupoids. As usual, $\mathcal{G} \rightrightarrows M$ denotes a locally compact groupoid with a Haar system λ_x. We shall use the following notation throughout the rest of the paper.

Notations 5.1. Recall that $\pi_x \colon C^*(\mathcal{G}) \to \mathcal{L}(L^2(\mathcal{G}^x, \lambda_x))$ denotes be the regular representation on $\mathcal{G}_x$, given by left convolution, $\pi_x(\phi)\psi := \phi * \psi$, Definition (4.13). Let us assume that $U \subset M$ is an open, $\mathcal{G}$-invariant subset with $\mathcal{G}_U \simeq U \times U$ (the pair groupoid, see Example 2.18). For any $x_0 \in U$, the range map then defines a bijection $r : \mathcal{G}_{x_0} \to U$ and hence a measure μ on U corresponding to λ_{x_0}. This measure does not depend on the choice of $x_0 \in U$ and leads to isometries $L^2(\mathcal{G}_{x_0}, \lambda_{x_0}) \simeq L^2(U; \mu)$ that commute with the action of $\mathcal{G}$. In particular π_0 and π_{x_0} are canonically unitarily equivalent. We then denote by π_0 the corresponding (equivalence class of) representation(s) of $C^*(\mathcal{G})$ on $L^2(U; \mu)$. It is often called the *vector representation* of $C^*(\mathcal{G})$. We shall usually write $L^2(U) := L^2(U; \mu)$.

Definition 5.2. Let $\mathcal{G} \rightrightarrows M$ is a locally compact, second countable groupoid with a Haar system. Then $\mathcal{G}$ is called a *Fredholm groupoid* if:

(i) There is an open, $\mathcal{G}$-invariant subset $U \subset M$ such that $\mathcal{G}_U \simeq U \times U$.
(ii) For any $a \in C^*_r(\mathcal{G})$, we have that $1 + \pi_0(a)$ is Fredholm if, and only if, all $1 + \pi_x(a)$, $x \in F := M \smallsetminus U$, are invertible, where we have used the notation introduced in 5.1.

The set $F := M \smallsetminus U$ will be called the set of *boundary units of* $\mathcal{G}$.

For a Fredholm groupoid $\mathcal{G}$, we shall always denote by U the $\mathcal{G}$-invariant open subset of the units of $\mathcal{G}$ as in the definition of a Fredholm groupoid. The set U is uniquely determined by $\mathcal{G}$, since it is a dense orbit of $\mathcal{G}$ in M. We have the following abstract characterization of Fredholm groupoids.

Theorem 5.3. *Let $\mathcal{G} \rightrightarrows M$ be a locally compact groupoid with a Haar system. If $\mathcal{G}$ is Fredholm, then following three conditions are satisfied:*

(i) The vector representation $\pi_0 \colon C^*_r(\mathcal{G}) \to \mathcal{L}(L^2(U))$ is injective.
(ii) The canonical projection $C^*_r(\mathcal{G}) \to C^*_r(\mathcal{G}_F)$ induces an isomorphism

$$C^*_r(\mathcal{G})/C^*_r(\mathcal{G}_U) \simeq C^*_r(\mathcal{G}_F) , \quad F := M \smallsetminus U .$$

(iii) The set of regular representations π_x, $x \in F$, form an invertibility sufficient set of representations of $C^*_r(\mathcal{G}_F)$.

The following strong converse holds: if the three conditions (i-iii) are satisfied, then, for any unital C^*-algebra $\mathbf{\Psi}$ containing $C^*_r(\mathcal{G})$ as an essential ideal and for any $a \in \mathbf{\Psi}$, we have that $\pi_0(a)$ if Fredholm if, and only if, the image of a in $\mathbf{\Psi}/C^*_r(\mathcal{G})$ is invertible and all $\pi_x(a)$, $x \in M \smallsetminus U$, are invertible.

Proof. The representation π_0 defines an isomorphism $C^*_r(\mathcal{G}_U) \simeq \mathcal{K}$, the algebra of compact operators on $L^2(U)$. Let us assume first that $\mathcal{G}$ is a Fredholm groupoid and check the three conditions of the statement.

First, if π_0 is not injective, then let $0 \neq a^* = a \in \ker(\pi_0)$. The family of representations $(\pi_x)_{x\in M}$ is a faithful family of representations of $C_r^*(\mathcal{G})$, hence there is $x \in M$ such that $\pi_x(a) \neq 0$. Moreover, we have $x \notin U$, since for $y \in U$, the representation π_y is unitarily equivalent to π_0, and hence $\ker(\pi_0) = \ker(\pi_y)$. Since $\pi_x(a) = \pi_x(a)^* \neq 0$, there is $0 \neq \lambda \in \mathbb{R}$ in the spectrum of $\pi_x(a)$. We may assume $\lambda = 1$, by rescaling. Then $1 - \pi_x(a)$ is not invertible, but $1 = 1 - \pi_0(a)$ is Fredholm. This contradicts our assumption that $\mathcal{G}$ is a Fredholm groupoid, and hence π_0 is injective.

Let $p\colon C_r^*(\mathcal{G}) \to C_r^*(\mathcal{G}_F)$ be the natural projection. It is known that $C_r^*(\mathcal{G}_U) \subseteq \ker(p)$ (see the proof of [58, Prop.II.4.5 (a)]). To prove (ii), we need to show that we have equality $C_r^*(\mathcal{G}_U) = \ker(p)$. Let us again proceed by contradiction, that is, let us assume that $C_r^*(\mathcal{G}_U) \neq \ker(p)$. Then we can choose $a = a^* \in \ker(p) \smallsetminus C_r^*(\mathcal{G}_U)$. Since a is self-adjoint and non-zero, in the quotient $\ker(p)/C_r^*(\mathcal{G}_U)$, there is $0 \neq \lambda \in \mathbb{R}$ such that $\lambda - a$ is not invertible in $\ker(p)/C_r^*(\mathcal{G}_U)$. Again, by rescaling, we may assume $\lambda = 1$ and thus $\lambda - a$ is not invertible in $C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_U)$. By the isomorphism $C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_U) \simeq \pi_0(C_r^*(\mathcal{G}))/\mathcal{K}(L^2(U))$, it then follows that $1 - \pi_0(a)$ is not Fredholm. However $1 - \pi_x(a) = 1$ is invertible for all $x \notin U$. This is a contradiction.

Let $c \in C_r^*(\mathcal{G})^+$ be arbitrary. The first two parts show that there is an isomorphism

$$C_r^*(\mathcal{G}_F) \simeq C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_U) \simeq \pi_0(C_r^*(\mathcal{G}))/\mathcal{K}(L^2(U))$$

hence $\pi_0(c)$ is Fredholm if, and only if, $c + \mathcal{K} \in C_r^*(\mathcal{G})^+/C_r^*(\mathcal{G}_U) = C_r^*(\mathcal{G}_F)^+$ is invertible. Therefore we have that $c := 1 + a \in C_r^*(\mathcal{G}_F)^+$, $a \in C_r^*(\mathcal{G}_F)$, is invertible if, and only if, all $1 + \pi_y(a) = \pi_y(1 + a)$, $y \in F$, are invertible. This means precisely that the family $\mathcal{F} := \{\pi_y, y \in F\}$ is invertibility sufficient. This proves (iii) and hence the direct implication.

To prove the converse, let us assume that (i-iii) are satisfied and let $\mathbf{\Psi}$ be a C^*-algebra containing $C_r^*(\mathcal{G})$ as an essential ideal. Property (i) implies that π_0 is injective on $\mathbf{\Psi}$ since $C^*(\mathcal{G})$ is an essential ideal of $\mathbf{\Psi}$. By our assumptions on $\mathcal{G}$, we obtain that the algebra $\Psi/C_r^*(\mathcal{G}_U) \simeq \pi_0(\Psi)/\mathcal{K} =: B$ contains $B_0 := \pi_0(C_r^*(\mathcal{G}))/\mathcal{K}$ as an ideal and $B/B_0 \simeq \mathbf{\Psi}/C_r^*(\mathcal{G})$. Moreover, $B_0 \simeq C_r^*(\mathcal{G}_F)$ by (ii).

Let now $a \in \mathbf{\Psi}$ be arbitrary. By Atkinsons' Theorem, we know that $\pi_0(a)$ is Fredholm if, and only if, its image in B is invertible. But by (iii) and by Corollary 4.9, $c \in B$ is invertible if, and only if, the image of c in $\mathbf{\Psi}/C_r^*(\mathcal{G})$ and all $\pi_x(c)$ are invertible for all $x \in F$. □

We have the following useful variation of the above result.

Proposition 5.4. *Let $\mathcal{G} \rightrightarrows M$ be a Fredholm groupoid and let $\mathcal{F}$ be an invertibility preserving family of representations of $C^*(\mathcal{G}_F)$, where F is the set of boundary units of $\mathcal{G}$. Let $\mathbf{\Psi}$ be a C^*-algebra containing $C_r^*(\mathcal{G})$ as an essential ideal. Then, for any $a \in \mathbf{\Psi}$, we have that $\pi_0(a)$ if Fredholm if, and only if, the image of a in $\mathbf{\Psi}/C_r^*(\mathcal{G})$ is invertible and all $\phi(a)$, $\phi \in \mathcal{F}$, are invertible.*

Proof. The proof is essentially as that of the corresponding statement in Theorem 5.3. Indeed, by (i) of Theorem 5.3, we have that π_0 is injective on $C^*(\mathcal{G})$. Since $C^*(\mathcal{G})$ is an essential ideal in $\mathbf{\Psi}$, the extension of π_0 to $\mathbf{\Psi}$ is injective as well. By our assumptions on $\mathcal{G}$, we obtain that the algebra $\Psi/C_r^*(\mathcal{G}_U) \simeq \pi_0(\Psi)/\mathcal{K} =: B$ contains $B_0 := \pi_0(C_r^*(\mathcal{G}))/\mathcal{K}$ as an ideal and $B/B_0 \simeq \mathbf{\Psi}/C_r^*(\mathcal{G})$. Moreover, $B_0 \simeq C_r^*(\mathcal{G}_F)$ by (ii) of Theorem 5.3.

Let now $a \in \mathbf{\Psi}$ be arbitrary. By Atkinsons' Theorem, we know that $\pi_0(a)$ is Fredholm if, and only if, its image in B is invertible. But by Corollary 4.9, $c \in B$ is invertible if, and only if, the image of c in $\mathbf{\Psi}/C^*_r(\mathcal{G})$ and $\phi(c)$ is invertible each $\phi \in \mathcal{F}$. □

5.2. **The Effros-Hahn conjecture and Fredholm groupoids.** We now want to obtain some easier to use conditions for a groupoid to be Fredholm. It will be convenient to make some connections with the Effros-Hahn conjecture. Recall that a locally compact groupoid $\mathcal{H}$ with a Haar system *has the generalized Effros-Hahn property* if every primitive ideal of $C^*(\mathcal{H})$ is induced from an isotropy subgroup $\mathcal{H}_y^y$ of $\mathcal{H}$ [29, 59].

From now on and throughout the rest of the paper, we shall assume that $\mathcal{G}$ is a Hausdorff, second countable, locally compact groupoid with a fixed Haar system.

Definition 5.5. Let $\mathcal{H} \rightrightarrows M$ be a locally compact groupoid with a Haar system. If $\mathcal{H}$ has the generalized Effros-Hahn property and all the isotropy groups $\mathcal{H}_y^y$, $y \in M$ are amenable, we say that $\mathcal{H}$ is *EH-amenable.*

Example 5.6. Let $\mathcal{H} \rightrightarrows B$ be a locally trivial bundle of groups (so $d = r$) with typical fiber isomorphic to the locally compact group G, see Example 2.17. Also, let $f : M \to B$ be a continuous map that is a local fibration (that is, f is open and each point $m \in M$ has a neighborhood V_m such that the map $f : V_m \to f(V_m)$ is a locally trivial fibration). Then $f^{\downarrow\downarrow}\mathcal{H}$ is a locally compact groupoid with a Haar system and it has the generalized Effros-Hahn property. It will be EH-amenable if, and only if, the group G is amenable.

We shall need the folowing two result from [52]. Recall that a groupoid is *metrically amenable* if the canonical surjection $C^*(\mathcal{G}) \to C^*_r(\mathcal{G})$ is surjective.

Proposition 5.7. *Let $\mathcal{H} \rightrightarrows F$ be a locally compact groupoid with a Haar system. We assume that $\mathcal{H}$ is EH-amenable. Then the family of regular representations $\mathcal{R} := \{\pi_y, y \in F\}$ of $C^*(\mathcal{H})$ is exhausting. In particular, $\mathcal{H}$ is metrically amenable.*

Proof. Let I be any primitive ideal of $C^*(\mathcal{H})$. Then I is induced from the isotropy group $\mathcal{H}_y^y$, $y \in M$, by the assumption that $\mathcal{H}$ has the generalized Effros-Hahn property. Since $\mathcal{H}_y^y$ is amenable, every irreducible representation of $\mathcal{H}_y^y$ is weakly contained in the regular representation ρ_y of $\mathcal{H}_y^y$. But $\mathrm{Ind}_y^{\mathcal{H}}(\rho_y)$ is the regular representation π_y of $C^*(\mathcal{H})$ on $L^2(\mathcal{H}_y)$. Since induction preserves the weak containment of representations (see Proposition 6.26 of [60]), we obtain that I contains $\ker(\pi_y)$. This proves that the family $\mathcal{R} := \{\pi_y, y \in M\}$ is exhausting. Therefore $\mathcal{R}$ is also faithful, and hence $C^*(\mathcal{H}) \simeq C^*_r(\mathcal{H})$. The family $\mathcal{R}$ is invertibility sufficient since it is exhausting (Theorem 4.6). □

The class of EH-amenable groupoids is closed under extensions.

Proposition 5.8. *Let $\mathcal{H} \rightrightarrows M$ be a locally compact groupoid with a Haar system. Let $U \subset M$ be an open $\mathcal{H}$-invariant subset and $F := M \smallsetminus U$. We have that $\mathcal{H}$ is EH-amenable if, and only if, $\mathcal{H}_F$ and $\mathcal{H}_U$ are EH-amenable.*

Proof. It is clear that the isotropy groups $\mathcal{H}_x^x$ of $\mathcal{H}$ are given by the isotropy groups of the restrictions $\mathcal{H}_F$ and $\mathcal{H}_U$. This gives that all the isotropy groups of $\mathcal{H}$ are amebable if, and only if, the same property is shared by all the isotropy groups of the restrictions $\mathcal{H}_F$ and $\mathcal{H}_U$.

Let A be a C^*-algebra and $J \subset A$ be a two-sided ideal, then we have that $\mathrm{Prim}(A)$ is the disjoint union of $\mathrm{Prim}(J)$ and $\mathrm{Prim}(A/J)$ [20]. This correspondence sends a primitive ideal I of A to $I \cap J$, if $I \cap J \neq J$, and otherwise it sends I to I/J, which is an ideal of A/J. We shall use this correspondence as follows. Let I be primitive ideal of $C^*(\mathcal{H})$. Since $C^*(\mathcal{H}_U)$ is an ideal of $C^*(\mathcal{H})$ and $C^*(\mathcal{H})/C^*(\mathcal{H}_U) \simeq C^*(\mathcal{H}_F)$ by Renault's result [58, 59] (recalled in Proposition 4.15), we have that I corresponds uniquely to either a primitive ideal of $C^*(\mathcal{H}_U)$ or to a primitive ideal of $C^*(\mathcal{H}_F)$.

We next notice the following. If $I \supset C^*(\mathcal{H}_U)$ (so I comes from an ideal of to $C^*(\mathcal{H})/C^*(\mathcal{H}_U) \simeq C^*(\mathcal{H}_F)$), then I is induced from an isotropy group of $\mathcal{H}$ if, and only if, $I/C^*(\mathcal{H}_U)$ is induced from an isotropy group of $\mathcal{H}_F$. This follows directly from the definition of induced representations [60]. On the other hand, if I does not contain $C^*(\mathcal{H}_U)$, then the induced representation of I for a an isotropy group $\mathcal{H}_y^y$ is non zero if, and only if, $y \in U$, in which case, the inducing bimodule from $\mathcal{H}_y^y$ to $\mathcal{H}_U$ or $\mathcal{H}$ is the same. The induced representations from $\mathcal{H}_y^y$ to $\mathcal{H}_U$ and $\mathcal{H}$ will correspond to each other in the canonical way of extending non-degenerate representations of an ideal to the whole algebra (see the remark preceeding Propositioin 4.8), by Rieffel's Induction in Stages Theorem 5.9 of [60]. □

Here is an important for us technical result.

Proposition 5.9. *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff, second countable, locally compact groupoid with a Haar system such that there exists an open, dense, $\mathcal{G}$-invariant subset $U \subset M$ such that $\mathcal{G}_U \simeq U \times U$. Let us assume that $\mathcal{G}$ is EH-amenable. Then $\mathcal{G}$ is Fredholm.*

Proof. To check that $\mathcal{G}$ is a Fredholm groupoid, we shall check that the conditions (i–iii) of Theorem 5.3 are satisfied. Condition (i) follows directly from the fact that $\mathcal{G}$ is Hausdorff, by Corollary 2.4 of Khoshkam and Skandalis [33].

Since $\mathcal{G}$ is EH-amenable, the groupoid $\mathcal{H} := \mathcal{G}_F$ is also EH-amenable, By Proposition 5.8, and hence it is also metrically amenable by Proposition 5.7. We therefore obtain that condition (ii) is satisfied by Proposition 4.15. Finally, condition (iii) follows also from Proposition 5.7 applied again to $\mathcal{H} = \mathcal{G}_F$ and the fact that exhausting sets of representations are also invertibility sufficient (Theorem 4.6). □

Theorem 5.10. *Let $\mathcal{G} \rightrightarrows M$ be an amenable, Hausdorff, second countable, locally compact groupoid with a Haar system such that there exists an open, dense, $\mathcal{G}$-invariant subset $U \subset M$ such that $\mathcal{G}_U \simeq U \times U$. Then $\mathcal{G}$ is Fredholm.*

Proof. Since $\mathcal{G} \rightrightarrows M$ is an amenable, Hausdorff, second countable, locally compact groupoid with a Haar system, we have that $\mathcal{G} \rightrightarrows M$ satisfies the Effros-Hahn conjecture by the main result in [29], that is, it has the generalized Effros-Hahn property. Since $\mathcal{G}$ is amenable, all its isotropy groups $\mathcal{G}_x^x$ are amenable [5]. The result then follows from Proposition 5.9. □

We shall need the following lemma.

Lemma 5.11. *Let $\mathcal{G} \rightrightarrows M$ be a locally compact groupoid with a Haar system λ_x. Let us assume that there is given an open subset $W \subset M$ such that the set $\mathcal{G}_x^U := r^{-1}(U) \cap d^{-1}(x)$ has a complement of measure zero in $\mathcal{G}_x$. Then the inclusion $\mathcal{G}_U^U \to \mathcal{G}$ induces an isomorphism $C_r^*(\mathcal{G}_U^U) \to C_r^*(\mathcal{G})$.*

Proof. We have by definition that every orbit of $\mathcal{G}$ on M intersects U. Thus in the definition of the reduced norm, it suffices to take the representations π_x with

$x \in U$. The result then follows from the fact that $L^1(\mathcal{G}_U^U)$ is dense in $L^1(\mathcal{G})$ in the L^1-norm associated to π_x, for each $x \in U$. $\square$

We conclude with the following result.

Theorem 5.12. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Let us assume that $\mathcal{G}$ is Fredholm and second countable. Let $L \subset M$ is an $A(\mathcal{G})$-tame submanifold. Then the desingularization groupoid $[[\mathcal{G} : L]]$ is also Fredholm.*

Proof. We shall verify again the three conditions in Theorem 5.3. Let us fix and review some notation first.

Let $U \subset M$ be the dense open orbit that defines a Fredholm groupoid. That is, $\mathcal{G}_U = U \times U$. Also, let us denote by $W := M \smallsetminus L$, by $F := M \smallsetminus U$, by $\mathcal{M} := [[\mathcal{G} : L]]$, the desingularization groupoid, by $SNL := [M : L] \smallsetminus W$, the unit sphere bundle of the normal bundle of L in M, and by $\mathcal{H}$ the restriction of $\mathcal{M}$ to SNL.

We have that U is dense in $\mathcal{M}$, which is Hausdorff, and hence the vector representation of $C_r^*(\mathcal{M})$ is injective. This checks the first condition in Theorem 5.3.

By Example 5.6 and by Proposition 3.10, we have that $\mathcal{H}$ is EH-amenable, and hence also metrically amenable. Let then

$$A_1 := C_r^*(\mathcal{G}_{U\cap W}^{U\cap W}) = C_r^*(\mathcal{M}_{U\cap W}),\ A_2 := C_r^*(\mathcal{G}_W^W) = C_r^*(\mathcal{M}_W),\ \text{and } A_3 := C_r^*(\mathcal{M}).$$

Let $U^c := [M : L] \smallsetminus U$. We then have the following big commutative diagram, where all the maps are induced by natural morphisms.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_2/A_1 & \longrightarrow & A_3/A_1 & \longrightarrow & A_3/A_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_r^*(\mathcal{M}_{F\cap W}) & \longrightarrow & C_r^*(\mathcal{M}_{U^c}) & \longrightarrow & C_r^*(\mathcal{H}) & \longrightarrow & 0 \end{array} \tag{19}$$

The top line in this diagram is exact by elementary linear algebra. The bottom line of this diagram is exact since $\mathcal{H}$ is metrically amenable.

We have that the inclusion $\mathcal{G}_W^W \to \mathcal{G}$ induces an isomorphism $A_2 := C_r^*(\mathcal{G}_W^W) \to C_r^*(\mathcal{G})$ by Lemma 5.11, since $r^{-1}(L) \cap \mathcal{G}_x$ has measure zero in $\mathcal{G}_x$ for all $x \in M$. Similarly, the inclusion $\mathcal{G}_{U\cap W}^{U\cap W} \to \mathcal{G}_U$ also induces an isomorphism $A_1 := C_r^*(\mathcal{G}_{U\cap W}^{U\cap W}) \to C_r^*(\mathcal{G}_U) \simeq \mathcal{K}$, Hence $A_3/A_2 \simeq C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_U)$. Since we have assumed that $\mathcal{G}$ is Fredholm, Theorem 5.3(iii) then gives

$$A_2/A_1 \simeq C_r^*(\mathcal{G})/C_r^*(\mathcal{G}_U) \simeq C_r^*(\mathcal{G}_F) \simeq C_r^*(\mathcal{M}_{F\cap W}),$$

where the last isomorphism is again by 5.11. Hence the left-most down arrow in the diagram 19 is an isomorphism. Since $\mathcal{H}$ is metrically amenable, the right-most arrow in the same diagram is also an isomorphism, by 4.15(ii). A simple diagram chase then shows that the middle vertical arrow in the same diagram is also an isomorphism. This means exactly that condition (ii) in Theorem 5.3 is satisfied.

Finally, the sets of regular representations of $\mathcal{M}_{F\cap U}$ and of $\mathcal{H}$ form invertibility sufficient sets of representations of each of these algebras, and hence the result follows from Corollary 4.16 applied to the groupoid $\tilde{\mathcal{G}}_{U^c}$ and the invariant, open subset of units $W \smallsetminus U$. $\square$

5.3. **Pseudodifferential operators.** Let us see how to use these results in the case of pseudodifferential operators on groupoids. Let $\mathcal{G}$ be a Lie groupoid and consider the algebra $\Psi^*(\mathcal{G})$ whose definition we now briefly recall [4, 47, 54]. Then, for $m \in \mathbb{R} \cup \{\pm\infty\}$, $\Psi^m(\mathcal{G})$ consists of smooth families $(P_x)_{x\in M}$ of classical pseudodifferential operators $P_x \in \Psi^m(\mathcal{G}_x)$ of order m, that are right invariant with respect to the action of $\mathcal{G}$ and have compactly supported distribution kernels. In particular, $\Psi^{-\infty}(\mathcal{G})$ is nothing but the convolution algebra of smooth, compactly supported function on $\mathcal{G}$, that is, $\Psi^{-\infty}(\mathcal{G}) \simeq \mathcal{C}_c^\infty(\mathcal{G})$. We denote by $\overline{\Psi}(\mathcal{G})$ the C^*-algebra obtained as the closure of $\Psi^0(\mathcal{G})$ with respect to all contractive $*$-representations of $\Psi^{-\infty}(\mathcal{G})$, as in [36]. Let us denote, as usual, by S^*A the set of unit vectors in the Lie algebroid $A^*(\mathcal{G})$ associated with $\mathcal{G}$ with respect to some fixed metric on $A^*(\mathcal{G})$. Then $\overline{\Psi}(\mathcal{G})$ fits into the following exact sequence

$$0 \to C^*(\mathcal{G}) \to \overline{\Psi}(\mathcal{G}) \xrightarrow{\sigma_0} \mathcal{C}_0(S^*A) \to 0\,. \tag{20}$$

(See for instance [35] and the references therein.) Typically, Fredholm conditions are obtained for Fredholm Lie groupoids, by applying our results to the algebra $\mathbf{\Psi} = \overline{\Psi}(\mathcal{G})$. Let us see how this is done.

Let us fix in this subsection a Lie groupoid $\mathcal{G} \rightrightarrows M$. For the purpose of the next result, let us assume that its space of units M has an open, dense, $\mathcal{G}$-invariant subset $U \subset M$ such that the restriction $\mathcal{G}_U$ is isomorphic to the product groupoid $U \times U$. Let us also assume that the space of units M is compact. This is all we need to construct the Sobolev spaces. Indeed, there is an essentially unique class of metrics on $A(\mathcal{G})$, which, by restriction, gives rise to a class of metrics on U. All these metrics are Lipschitz equivalent and complete [2]. In fact, the Sobolev spaces of all these metrics will coincide. They are given as the domains of the powers of $1 + \Delta$, where Δ is the (geometer's, i.e. positive) Laplacian. We shall denote by $H^s(U) = H^s(M)$ these Sobolev spaces. See also [50] for a review.

We have the following result from [36]. The Sobolev spaces $H^s(M)$ are discussed in detail in [2]. Recall that we denote by $\pi_0 : C^*(\mathcal{G}) \to \mathcal{L}(L^2(M))$ the vector representation. It is unitarily equivalent to the regular representations π_x, $x \in U$.

Proposition 5.13. *Let $\mathcal{G}$ be as right above. Then $\overline{\Psi}(\mathcal{G})$ contains $C^*(\mathcal{G})$ as an essential ideal. Let $P \in \Psi^m(\mathcal{G})$ and $s \in \mathbb{R}$. Then P gives rise to a bounded map $P : H^s(M) \to H^{s-m}(M)$. Moreover, $a := (1 + \Delta)^{(s-m)/2} P (1 + \Delta)^{-s/2} \in \overline{\Psi}(\mathcal{G})$.*

Let us assume that π_0 is injective. We have that $P : H^s(M) \to H^{s-m}(M)$ is Fredholm if, and only if, a is Fredholm on $L^2(M)$. Similarly, $P : H^s(M) \to H^{s-m}(M)$ is invertible if, and only if, a is invertible.

All these results extend right away to operators acting between sections of vector bundles on M. The needed assumptions on $\mathcal{G} \rightrightarrows M$ are satisfied by any Lie Fredholm groupoid with M compact.

Proof. As we stated above, this is a direct consequence of the results in [36], except maybe the fact that $\overline{\Psi}(\mathcal{G})$ contains $C^*(\mathcal{G})$ as an essential ideal, which is a general fact–true for any Lie groupoid. This general fact is true because it is true for any non-compact manifold, in particular, for each of the manifolds $\mathcal{G}_x$. The fact that $P : H^s(M) \to H^{s-m}(M)$ is bounded is discussed in great detail (including its extension to L^p-type Sobolev spaces) in [2]. We have, by the definitions of Sobolev spaces and of Fredholm operators, that P is Fredholm if, and only if, a is Fredholm. The same applies to the statement about the invertibility.

To extend these results to operators acting between sections of vector bundles, we just need to introduce these bundles in the notation.

If $\mathcal{G}$ is a Fredholm Lie groupoid, then there exists an open set U as in the assumptions and π_0 is injective. ∎

This proposition then gives right away the following result. Recall from the discussion in the beginning of this section that an operator $P \in \Psi^m(\mathcal{G})$ consists of a right invariant family $P = (P_x)$, $x \in M$, the units of $\mathcal{G}$, with P_x acting on $\mathcal{G}_x$.

Theorem 5.14. *Let $\mathcal{G} \rightrightarrows M$ be a Fredholm Lie groupoid with M compact and let $U \subset M$ be the open subset such that $\mathcal{G}_U = U \times U$. Let $P \in \Psi^m(\mathcal{G})$. Then*

$$P : H^s(M) \to H^{s-m}(M) \text{ is Fredholm} \quad \Leftrightarrow \quad P \text{ is elliptic and}$$
$$P_x : H^s(\mathcal{G}_x) \to H^{s-m}(\mathcal{G}_x) \text{ is invertible for all } x \in M \smallsetminus U\,.$$

This result extends immediately to operators acting between sections of smooth vector bundles on M.

Proof. Let us use the notation of Proposition 5.13. Since $\mathcal{G}$ is Fredholm, Theorem 5.3, applied to $\mathbf{\Psi} := \overline{\Psi}(\mathcal{G})$, gives that $a \in \overline{\Psi}(\mathcal{G})$ is Fredholm if, and only if, its image in $\overline{\Psi}(\mathcal{G})C^*(\mathcal{G})$ is invertible and all the operators $\pi_x(a)$ are invertible. We then notice that $\pi_x(a) = (1+\Delta_x)^{(s-m)/2} P_x (1+\Delta_x)^{-s/2}$ since the extension of π_x to operators affiliated to $\overline{\Psi}(\mathcal{G})$ is given by $\pi_x(P) = P_x$ since this is true for $P \in \Psi^0(\mathcal{G})$ and $\pi_x(\Delta) = \Delta_x$, the Laplacian on $\mathcal{G}_x$ by [36]. □

Remark 5.15. We notice that the operator P_x of Theorem 5.14 is invariant for the (free) action of $\mathcal{G}_x^x$ on $\mathcal{G}_x$. Often in applications, the resulting bundle $\mathcal{G}_x \to \mathcal{G}_x/\mathcal{G}_x^x =: Z_x$ is trivial, which gives then right away Theorem 1.1. This is the case, for example, for stratified submersion groupoids.

Often a slight generalization of this theorem is useful.

Corollary 5.16. *Let $\mathcal{G}$ be as in Theorem 5.14 and let $I \subset M \smallsetminus U$ be a subset such that the family $\{\pi_x, x \in I\}$ is an invertibility sufficient family of representations of $C^*(\mathcal{G}_F)$, $F := M \smallsetminus U$. Let $P \in \Psi^m(\mathcal{G})$. Then*

$$P : H^s(M) \to H^{s-m}(M) \text{ is Fredholm} \quad \Leftrightarrow \quad P \text{ is elliptic and}$$
$$P_x : H^s(\mathcal{G}_x) \to H^{s-m}(\mathcal{G}_x) \text{ is invertible for all } x \in I\,.$$

Again, this result extends immediately to operators acting between sections of smooth vector bundles on M and to the action of P on the isotypical components for an action of a compact group.

Proof. This follows from Proposition 5.4 (and hence it parallels that of Theorem 5.3 that was used for Theorem 5.14. □

One could go beyond the class of Fredholm groupoids.

Theorem 5.17. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with M compact and let $U \subset M$ be an open $\mathcal{G}$-invariant subset such that $\mathcal{G}_U = U \times U$. Let $F := M \smallsetminus U$ and let us assume that π_0 is injective, that $\mathcal{G}_F$ is metrically amenable, and that $\mathcal{F}$ is a set of invertibility sufficient representations of $C^*(\mathcal{G}_F) = C_r^*(\mathcal{G}_F)$. Let $P \in \Psi^m(\mathcal{G})$. Then*

$$P : H^s(M) \to H^{s-m}(M) \text{ is Fredholm} \quad \Leftrightarrow \quad P \text{ is elliptic and} \tag{21}$$
$$\phi(P) \text{ is invertible for all } \phi \in \mathcal{F}\,.$$

This result extends immediately to operators acting between sections of smooth vector bundles on M and to the action of P on the isotypical components for an action of a compact group.

6. Stratified submersion Lie groupoids and examples

The characterization of Fredholm groupoids in Theorems 5.3 and 5.10 are not completely satisfactory in applications, since their conditions are not so easy to verify. For instance, it is well known to be difficult to check that a groupoid is amenable. Even Theorem 5.12 is not so useful as one might think, because one needs a large suply of Fredholm groupoids to start with and to which to apply the desingularization procedure. The easiest to use in applications seems to be Proposition 5.9. We thus provide in this section conditions for a groupoid to satisfy the conditions of Proposition 5.9 that are easy to use in practice and then apply them in concrete examples.

6.1. **Stratified submersion groupoids.** In this subsection, we introduce the stratified submersion groupoids and establish some of their basic properties. Recall from the previous section the pull-back constructions (or functors) $f^{\downarrow\downarrow}$: see Example 2.19 and Definition 2.19. If $\mathcal{H}$ is a groupoid, then the union of the connected components of the units in $\mathcal{H}_x$ defines a d-connected subgroupoid $\mathcal{H}^{conn}$ that is an open subgroupoid of $\mathcal{H}$ [39].

Definition 6.1. Let $\mathcal{G}$ be a locally compact groupoid with units M. We say that $\mathcal{G}$ is a *stratified submersion groupoid* if the following conditions are satisfied.

(i) We are given an increasing filtration of M with open, $\mathcal{G}$-invariant subsets:
$$\emptyset =: U_{-1} \subset U_0 \subset \ldots \subset U_{i-1} \subset U_i \subset \ldots \subset U_N := M\,.$$
(ii) For each connected component $S \subset U_i \smallsetminus U_{i-1}$ (called a *stratum*), there exist a bundle of groups $G_S \to B_S$ and a local fibration $f_S : S \to B_S$ such that
$$(\mathcal{G}_S^S)^{conn} \simeq (f_S^{\downarrow\downarrow}(G_S))^{conn} := (S \times_{f_S} G_S \times_{f_S} S)^{conn}\,.$$

We say that $\mathcal{G}$ is *reduced* if each S is $\mathcal{G}$-invariant and $\mathcal{G}_S \simeq f_S^{\downarrow\downarrow}(G_S)$. The least value of N with these properties is called the *depth* of $\mathcal{G}$.

For the rest of the paper, we shall keep fixed the notation and terminology of this definition (Definition 6.1). In particular, the connected components S of the sets $U_i \smallsetminus U_{i-1}$ are called *strata*.

The class of stratified submersion Lie groupoids is preserved by reduction to open subsets (which explains in part why we had to worry about connected components and allow for non d-connected stratified submersion Lie groupoids).

Proposition 6.2. *Let $\mathcal{G} \rightrightarrows M$ be a stratified submersion and $V \subset M$ be an open subset. Then the reduction $\mathcal{G}_V^V$ is also a stratified submersion groupoid with filtration $U_j' := V \cap U_j$ and maps $f_S' := f_S|_{S\cap V}$. The group bundles are $(G_S)|_{f_S(S\cap V)}$, where S ranges through the strata of $\mathcal{G}$. If $\mathcal{G}$ is a reduced stratified submersion groupoid, then $\mathcal{G}_V^V$ is also a reduced stratified submersion groupoid.*

Proof. We have that $f_S : S \cap V \to B_S$ is still a local fibration. Moreover, the connected component $(\mathcal{G}_V^V)^{conn}$ satisfies $(\mathcal{G}_V^V)^{conn} \subset (\mathcal{G}^{conn})_V^V$. The assertions hence follow directly from the definition. □

Proposition 6.3. *Let $\mathcal{G} \rightrightarrows M$ be a groupoid. Let $V \subset M$ be an open, invariant subset and $F := M \smallsetminus V$ be such that the reductions $\mathcal{G}_V$ and $\mathcal{G}_F$ are stratified submersion groupoids. Then $\mathcal{G}$ is a stratified submersion groupoid. More precisely, if U_i', $0 \le i \le N$ is the filtration corresponding to $\mathcal{G}_V$ (so $V = U_N'$) and $U_{N+1}', \ldots, U_M'$ is* *the filtration corresponding to $\mathcal{G}_F$, then the filtration corresponding to $\mathcal{G}$ is $U_i = U'_i$, if $i \leq N$, and $U_i = U'_N \cup U'_i$, if $i > N$. Consequently, the set of strata corresponding to $\mathcal{G}$ is the disjoint union of the sets of strata corresponding to $\mathcal{G}_U$ and $\mathcal{G}_F$, with the submersions and group bundles are the same for the corresponding strata. We have that $\mathcal{G}$ is reduced if, and only if, $\mathcal{G}_U$ and $\mathcal{G}_F$ are reduced.*

Proof. This follows directly from the definition. □

Corollary 6.4. *Let $\mathcal{G} \rightrightarrows M$ be a stratified submersion Lie groupoid and let $L \subset M$ be an $A(\mathcal{G})$-tame submanifold. Then $[[\mathcal{G} : L]]$ is also a stratified submersion Lie groupoid. If $\mathcal{G}$ is reduced, then $[[\mathcal{G} : L]]$ is also reduced.*

Proof. This follows from Propositions 3.10 and 6.3. □

The main reason we are interested stratified submersion groupoids is that they satisfy the generalized Effros-Hahn conjecture. For simplicity, we shall assume for the rest of the paper that $\mathcal{G}$ is reduced.

Proposition 6.5. *Let $\mathcal{G}$ be a reduced stratified submersion groupoid with a Haar system. Then $\mathcal{G}$ has the generalized EH-property.*

Proof. Each of the restriction groupoids $\mathcal{G}_{U_i \smallsetminus U_{i-1}}$ has the generalized EH-property by Example 5.6. The result then follows from Propositions 5.8 and 6.3 by induction on the depth N of $\mathcal{G}$. □

This gives the following result.

Theorem 6.6. *Let $\mathcal{G} \rightrightarrows M$ be a Hausdorff reduced stratified submersion groupoid with a Haar system. Let us assume that all the isotropy groups $\mathcal{G}_y^y$, $y \in M$, are amenable, then $\mathcal{G}$ is Fredholm.*

Proof. This follows from the Propositions 5.9 and 6.5. □

We conclude with our last (and hopefully most useful in applications) Theorem.

Theorem 6.7. *Let us assume that $\mathcal{G}$ is obtained from a pair groupoid $M \times M$ (with M smooth) by a sequence of desingularizations with respect to tame manifolds. Then $\mathcal{G}$ is a Hausdorff reduced stratified submersion Lie groupoid and all its isotropy groups are solvable. Consequently, $\mathcal{G}$ is Fredholm.*

Proof. This is obtained by induction as follows. First of all, we notice that the pair groupoid $M \times M$ is is a Hausdorff reduced stratified submersion Lie groupoid and all its isotropy groups are amenable (they are trivial). Corollary 6.4 gives, by induction, that $\mathcal{G}$ is a reduced stratified submersion Lie groupoid. The fact that $\mathcal{G}$ is Hausdorff follows by induction on the depth of $\mathcal{G}$ from Theorem 3.12. The fact that all the isotropy groups are amenable follows from the fact that they are solvable by Proposition 3.10. Since solvable groups are amenable, the fact that $\mathcal{G}$ is Fredholm follows from Theorem 6.6. □

Remark 6.8. Theorem 6.7 proves more properties of groupoids obtained by a sequence of desingularization. To prove just the resulting groupoid is Fredholm, it would be enough to proceed by induction using Theorem 5.12.

The rest of this section is devoted to examples. The first two examples are standard and involve blowing up submanifolds of a, respectively, smooth, compact manifold and of a compact manifold with boundary.

Notations 6.9. Before proceeding to the examples, let us however introduce a framework that will be used in all examples. For the rest of this section, M_k and L_k will denote manifolds with corners of depths k. Thus M_0 and L_0 will have, in fact, no corners or boundary (hence they will be "smooth".)

6.2. **Example: The blow-up of a smooth manifolds.** We now treat the desingularization of a groupoid with a smooth set of units over a smooth manifold. Thus neither the large manifold nor its submanifold have corners, following the convention in 6.9.

Remark 6.10. Let M_0 be a smooth, compact, connected manifold (so no corners). Recall the path groupoid of M_0, consisting of homotopy classes of end-point preserving paths $[0,1] \to M_0$. It is a d-simply-connected Lie groupoid integrating TM_0 (that is, its Lie algebroid is isomorphic to TM_0), so it is the maximal d-connected Lie groupoid with this property. On the other hand, the minimal groupoid integrating TM_0 is $\mathcal{G}_0 = M_0 \times M_0$. In general, a d-connected groupoid $\mathcal{G}_0$ integrating TM_0 will be a quotient of $\mathcal{P}(M_0)$, explicitly described in [25] (see also [46]), and thus it corresponds to a normal subgroup K of $\pi_1(M_0)$. For analysis questions, it is typically more natural to choose for $\mathcal{G}_0$ the minimal integrating groupoid $M_0 \times M_0$, whereas for questions related to topology and index theory, it may be convenient to choose another integrating groupoid. We notice that in analysis one has to use sometimes groupoids that are not d-connected [13].

We shall fix in what follows a smooth, compact, connected manifold M_0 (so M_0 has no corners) and a d-connected Lie groupoid $\mathcal{G}_0$ integrating the Lie algebroid $TM_0 \to M_0$.

The first example is related to some earlier results of Grushin [24], Debord-Skandalis [19], Lauter-Nistor [36], Mazzeo [42], Schulze [64], and others, and can be used to define the so-called "edge calculus".

Example 6.11. Let $L_0 \subset M_0$ be an embedded smooth submanifold. Let N be the normal bundle of L_0 in M_0 and denote by $S \subset N$ the set of unit vectors in N, that is, S is the unit sphere bundle of the normal bundle of L in M_0. We denote by $\pi : S \to L_0$ the natural projection. Then recall that the blow-up $M_1 := [M_0 : L_0]$ of M_0 with respect to L_0 is the disjoint union

$$M_1 := [M_0 : L_0] := (M_0 \smallsetminus L_0) \sqcup S\,,$$

with the topology of a manifold with boundary S. We have that L_0 is automatically $A(\mathcal{G}_0) = TM_0$ tame, so we can define $\mathcal{G}_1 := [\mathcal{G}_0 : L_0]$ (Definition 3.9), which is a Lie groupoid with base $M_1 := [M_0 : L_0]$. The filtration of M_1 has two sets, with $U_1 = M_1$ and $U_0 := M_0 \smallsetminus L_0 \subset M_1$, both of which are open and invariant for $\mathcal{G}_1$ (but U_0 is not invariant for $\mathcal{G}_0$, in general). We thus have two strata: U_0 and S (assuming that these sets are connected, otherwise we take their connected componets).

Let us spell out the structure of $\mathcal{G}_1$ in order to better understand the desingularization construction.

Remark 6.12. By the definition of the desingularization groupoid $\mathcal{G}_1 := [\mathcal{G}_0 : L_0]$, the reduction $(\mathcal{G}_1)_{U_0}$ coincides with the reduction $(\mathcal{G}_0)_{U_0}^{U_0}$. In particular, if $\mathcal{G}_0 = M_0 \times M_0$, then $(\mathcal{G}_1)_{U_0} = (\mathcal{G}_0)_{U_0}^{U_0} = U_0 \times U_0$, the pair groupoid. On the other hand, the restriction of $\mathcal{G}_1$ to $S := M_1 \smallsetminus U_0$ is a fibered pull-back groupoid defined as follows. We consider first $TL_0 \to L_0$, regarded as a bundle of (commutative)

Lie groups. We let $\mathbb{R}_+^*$ act on the fibers of $TL_0 \to L_0$ by dilation and define the structural bundle of Lie groups $G_S \to B_S$ (see Definition 6.1) by

$$G_S := TL_0 \rtimes \mathbb{R}_+^* \to L_0 := B_S ,$$

that is, the group bundle over L obtained by taking the semi-direct product of TL_0, by the action of $\mathbb{R}_+^*$ by dilations. In the notation of Definition 6.1, we thus have $B_S = L_0$ and $f_S : S \to B_S$ is nothing but the natural projection π introduced in the beginning of this example. Then

$$(\mathcal{G}_1)_S := \pi^{\downarrow\downarrow}(G) .$$

In particular, $(\mathcal{G}_1)_S$ does not depend on the choice of integrating groupoid $\mathcal{G}_0$. It is interesting to note that the stabilizers of $\mathcal{G}_1$ are as follows:

$$(\mathcal{G}_1)_x^x = \begin{cases} \pi_1(M_0)/K & \text{if } x \in U_0 \\ T_{\pi(x)}L_0 \rtimes \mathbb{R}_+^* & \text{if } x \in S := M_1 \smallsetminus U_0 , \end{cases}$$

with K as in Remark 6.10, that is, with $K = \{1\}$ corresponding to $\mathcal{G}_0 = \mathcal{P}(M_0)$, the maximal integrating groupoid of TM_0, and $K = \pi_1(M_0)$ corresponding to $\mathcal{G}_0 = M_0 \times M_0$, the minimal integrating groupoid of TM_0.

6.3. **Manifolds with boundary.** We now extend the previous example to manifolds with boundary.

Let M_1 be a compact manifold with smooth boundary. We denote by $F := \partial M_1$ its boundary and by $G := M_1 \smallsetminus F$ its interior. On M_1 we consider the Lie algebra of vector fields $\mathcal{V}_b$ *tangent* to ∂M_1. It is the algebra of sections of T^bM_1, the "b-tangent bundle"[43] of M_1, which is hence a Lie algebroid. We will not attempt to classify all the integrating groupoids of T^bM_1, because most of them will not be Hausdorff in general (this happens when $\pi_1(F) \to \pi_1(M_1)$ is not injective). We just content ourselves in the next remark to notice that the path construction groupoid extends to this case as well.

Remark 6.13. Denote by $\mathcal{P}(M_1)$ the path groupoid of M_1, as usual, and let $\mathcal{H}$ be any quotient of $\mathcal{P}(M_1)$ corresponding to a normal subgroup K of $\pi_1(M_1)$. Then $\mathcal{P}(M_1)$ is a groupoid, but is not a Lie groupoid in general, according to our conventions. Nevertheless, we can still define the reductions $\mathcal{H}_F^F$ and $\mathcal{H}_G^G$ to the boundary and the interior of M_1, respectively, and this time these reductions will be Lie groupoids. The groupoid $\mathcal{G}_1$ we are interested in will then be the disjoint union

$$\mathcal{G}_1 := (\mathcal{H}_F^F \times \mathbb{R}_+^*) \sqcup \mathcal{H}_G^G . \tag{22}$$

We call it the *covering b-calculus groupoid*. If $K = \pi_1(M_1)$, then we refer to $\mathcal{G}_1$ as simply the *b-calculus groupoid*. Let $r : M_1 \to [0, \infty)$ be the distance function to the boundary F. The topology on $\mathcal{G}_1$ is as follows. Let $\gamma_n \in \mathcal{H}_G^G$. We recall that each γ_n is a homotopy class of paths in M_1 modulo K. We have that $\gamma_n \to (\gamma, t) \in \mathcal{H}_F^F \times \mathbb{R}_+^*$ in $\mathcal{G}_1$ if, and only if, $\gamma_n \to \gamma$ in $\mathcal{H}$ and $r(\gamma_n(1))/r(\gamma_n(0)) \to t$ in $\mathbb{R}_+^*$. Thus, if $K = \pi_1(M_1)$, we recover the construction in [47, 54] that yields the b-calculus (thus we call this groupoid the *b-calculus groupoid* and denote it $M_1 \times^b M_1$). If $K = \{1\}$, on the other hand, our groupoid can be used to recover the pseudodifferential b-calculus on coverings defined by Leichtnam and Piazza [37].

We notice that the groupoid $\mathcal{G}_1$ will be d-connected if, and only if, the image of $\pi_1(F) \to \pi_1(M_1)/K$ is onto. We recall that if $K = \pi_1(M_1)$, then $\mathcal{G}_1$ is a stratified submersion Lie groupoid with the following structure.

Remark 6.14. The filtration of M_1 has two sets:

$$U_0 := M_1 \smallsetminus \partial M_1 \subset U_1 := M_1 .$$

which are open and $\mathcal{G}_2$ invariant (but not $\mathcal{G}_1$ invariant). The structure is such that there are two strata S, namely U_0 and $M_1 \smallsetminus U_0$ (assuming that they are connected, otherwise we consider their connected components). The base manifolds B_S are a point in each case, with $G_S = \{1\}$ for $S = U_0$ and $G_S = \mathbb{R}$ for $S = M_1 \smallsetminus U_0$. The groupoid structure is such that

$$(\mathcal{G}_1)_{U_0} = U_0^2 , \quad \text{and} \quad (\mathcal{G}_1)_{M_1 \smallsetminus U_0} = (M_1 \smallsetminus U_0)^2 \times \mathbb{R}_+^* ,$$

We are ready to extend Example 6.11 to manifolds with boundary.

Example 6.15. We consider now a manifold M_1 with smooth boundary ∂M_1. Let $\mathcal{G}_1$ be a groupoid integrating $T^b M_1$ as in Remark 6.13 (that is, $A(\mathcal{G}_1) = T^b M_1$). Let $L_1 \subset M_1$ be an embedded smooth submanifold assumed to be such that its boundary is $\partial L_1 = L_1 \cap \partial M_1$ and such that L_1 intersects ∂M_1 *transversely.* Then L_1 is $A(\mathcal{G}_1)$-tame, so we can define $\mathcal{G}_2 := [[\mathcal{G}_1 : L_1]]$, which is a Lie groupoid with base $M_2 := [M_1 : L_1]$. The filtration of M_1 has three sets:

$$U_0 := M_1 \smallsetminus (L_1 \cup \partial M_1) \subset U_1 := M_1 \smallsetminus L_1 \subset U_2 := M_2 ,$$

which are open and $\mathcal{G}_2$ invariant (but not $\mathcal{G}_1$ invariant).

Let us now describe the structure of the Lie groupoid $\mathcal{G}_2 := [[\mathcal{G}_1 : L_1]]$ we have just defined in the case when $\mathcal{G}_1$ is the groupoid defining the b-calculus (that is, $K = \pi_1(M_1)$).

Remark 6.16. We use the notation and assumptions of Example 6.15. Recall that U_0 and U_1 are $\mathcal{G}_2$-invariant. The restriction $(\mathcal{G}_2)_{U_1}$ coincides with the reduction $(\mathcal{G}_1)_{U_1}^{U_1}$ by the definition of the desingularization groupoid (Definition 3.9). Hence the restriction $(\mathcal{G}_2)_{U_0} = (\mathcal{G}_2)_{U_0}^{U_0}$ coincides with the reduction $(\mathcal{G}_1)_{U_0}^{U_0}$. Therefore, using the structure of the groupoid $\mathcal{G}_1$ hence, assuming that U_0 and $U_1 \smallsetminus U_0$ are connected, we have

$$(\mathcal{G}_2)_{U_0} := (\mathcal{G}_1)_{U_0}^{U_0} = U_0^2 , \quad \text{and} \quad (\mathcal{G}_2)_{U_1 \smallsetminus U_0} := (\mathcal{G}_1)_{U_1 \smallsetminus U_0}^{U_1 \smallsetminus U_0} = (U_1 \smallsetminus U_0)^2 \times \mathbb{R}_+^* ,$$

where U_0^2 and $(U_1 \smallsetminus U_0)^2$ are pair groupoids. In the general case, if $U_1 \smallsetminus U_0$ is not connected, we write $U_1 \smallsetminus U_0 = \sqcup V_j$ as the disjoint union decomposition into connected subsets, then we have $(\mathcal{G}_2)_{U_1 \smallsetminus U_0} := \sqcup_j V_j^2 \times \mathbb{R}_+^*$. Let us denote as in the boundaryless case by S the unit sphere bundle of the normal bundle to L_1 in M_1. (This is going to be the last stratum $U_2 \smallsetminus U_1$ in M_2, assuming, of course, that it is connected.) Then the restriction of $\mathcal{G}_1$ to $S := M_2 \smallsetminus U_1$ is the following fibered pull-back groupoid. Let $\pi : S \to L$ be the natural projection, as before. Let again $G_S := T^b L \rtimes \mathbb{R}_+^* \to L$ be the group bundle over L obtained by taking the direct product of the b-tangent bundle to L with the action of $\mathbb{R}_+^*$ by dilation on the fibers of $TL \to L$. Then

$$\mathcal{G}_S := \pi^{\downarrow\downarrow}(G_S) .$$

Let us now comment on the vector fields resulting from our construction.

Remark 6.17. We continue to use the notation and assumptions of Example 6.15. Let $r : M_1 \to [0, \infty)$ denote the distance to the boundary, as before. Let $\rho : M_1 \to [0, \infty)$ denote the distance to L_1. Then r and ρ lift to smooth functions on $M_2 := [M_1 : L_1]$ that define a corner of codimension two of M_2. Near a point in this corner, we can choose a coordinate system with r and and ρ as coordinate functions and $x = (x', x'')$ denoting the remaining coordinates, with x'' coordinates in the fibers of $\pi : S \to L_1$ and x' coordinates in the boundary of L_1. Then the sections of $A(\mathcal{G}_2)$ are a Lie algebra of vector fields on M_2 that near a point in the codimension two corner $\{r = 0, \rho = 0\}$ are generated by the vector fields

$$\rho\partial_\rho\,,\ \rho r\partial_r\,,\ \rho\partial_{x'_i}\,,\ \partial_{x''_j}$$

by multiplication with smooth functions on M_2 (i.e. these vector fields form a local basis). Similarly, near a point in the open face $\{r > 0, \rho = 0\}$, we choose a coordinate system with ρ as one of the coordinate functions and $x = (x', x'')$ denoting the remaining coordinates, with x'' coordinates in the fibers of $\pi : S \to L_1$ and x' coordinates in L_1 (so r is now incorporated among the x'). Then a local basis for our vector fields is given by

$$\rho\partial_\rho\,,\ \rho\partial_{x'_i}\,,\ \partial_{x''_j}$$

On the other hand, close to a point of the open face $\{r = 0, \rho > 0\}$, we choose coordinates (r, x), with x coordinates in the boundary (we could choose one of the x coordinates to be ρ, for example), and then the corresponding vector fields have as local basis

$$r\partial_r\,,\ \partial_{x'_i}$$

that is, away from its boundary, we obtain all vector fields tangent to the open face $\{r = 0, \rho > 0\}$.

6.4. **Desingularization of a one-dimensional stratified subset.** We now deal with a more complicated example by combining the examples discussed in the previous two subsections. We now introduce the groupoid that is obtained from the desingularization of a stratified subset of dimension one.

Example 6.18. Let M_0 be a smooth, compact manifold (so no corners). Let $L_0 := \{P_1, P_2, \dots, P_k\} \subset M_0$ and let us assume that we are given a subset $\mathcal{S} \subset M_0$ such that

$$\mathcal{S} \,=\, L_0 \cup \cup_{j=1}^{l} \gamma_j\,, \tag{23}$$

where each γ_j is the image of a smooth map $c_j : [0, 1] \to M_0$, with the following properties:

(i) $c'_j(t) \neq 0$,
(ii) $c_j(0), c_j(1) \in L_0 := \{P_1, P_2, \dots, P_k\}$,
(iii) $c_j((0, 1))$ are disjoint and do not intersect L_0 and
(iv) the vectors $c'_j(0), c'_j(1) \in TM_0$, $j = 1, \dots, l$, are all distinct.

We now introduce the desingularization of M_0 (or, rather, of its pair groupoid) with respect to $\mathcal{S}$.

Let $\mathcal{G}_0$ be an arbitrary groupoid integrating TM_0. The set $L_0 := \{P_1, P_2, \dots, P_k\}$ defines then a TM_0-tame submanifold of M_0. We can first define $\mathcal{G}_1$ to be the desingularization of $\mathcal{G}_0$ with respect to L_0 as in Example 6.11:

$$\mathcal{G}_1 \,:=\, [[\mathcal{G}_0 : L_0]]\,, \tag{24}$$

which is a groupoid with units $M_1 := [M_0 : L_0]$. We denote by $A_1 = A(\mathcal{G}_1)$ its Lie algebroid. The continuous maps c_j then lift to continuous maps

$$\tilde{c}_j : [0, 1] \to [M_0 : L_0]\,.$$

The assumption that the vectors $c_j'(0)$ and $c_j'(1)$, $j = 1, \dots, l$, are all distinct then gives that the sets $\tilde\gamma_j := \tilde c_j([0,1])$ are all disjoint. Since $c_j'(t) \neq 0$, we obtain that all the curves parametrized by the $\tilde\gamma_j$ are A_1-tame submanifolds (their end points lie on the boundary of M_1 and they hit the boundary nontangentially). Let L_1 be the disjoint union of the embedded curves $\tilde\gamma_j$. Then we can perform a further desingularization along L_1, as in Example 6.15, thus obtaining

$$\mathcal{G}_2 := [[\mathcal{G}_1 : L_1]] ,$$

which is a boundary fibration Lie groupoid.

Definition 6.19. Assume $\mathcal{G}_0 = M_0 \times M_0$, the pair groupoid. Then the Lie groupoid $\mathcal{G}_2$ introduced in Example 6.18 is the *desingularization* groupoid of M_0 with respect to $\mathcal{S}$.

The structure of the *desingularization* groupoid of M_0 with respect to $\mathcal{S}$ is given by Remark 6.16. More precisely, we obtain the following:

Remark 6.20. Let U_1, U_2, and U_3 be as in Remark 6.16. Then $U_1 = M_0 \smallsetminus \mathcal{S}$,

$$U_1 \smallsetminus U_0 = \left(\cup_{j=1}^k \{P_j\} \times S^2\right) \smallsetminus \{\gamma_j'(0), \gamma_j'(1)\} ,$$

that is, we take the union of all the unit spheres S^2 around one of the points P_j and we remove the directions of the derivative vectors of the curves γ_j at their end points (which must be among the P_j points). Let us denote, for each $j = 1, \dots, k$, by $\{P_j\} \times S^2 \smallsetminus \{\gamma_j'(0), \gamma_j'(1)\}$. Then

$$(\mathcal{G}_2)_{U_0} = (M \smallsetminus \mathcal{S})^2 , \quad \text{and} \quad (\mathcal{G}_2)_{U_1 \smallsetminus U_0} = \sqcup V_j^2 \times \mathbb{R}_+^* .$$

To complete our description, let $H = \mathbb{R} \rtimes \mathbb{R}_+^*$ be the semidirect product of $\mathbb{R}_+^*$ acting on $\mathbb{R}$ by dilations (this is, of course, nothing but the "$ax+b$-group"). Then we have $U_2 \smallsetminus U_1 = \sqcup \tilde\gamma_j$, the union of the curves parametrized by γ_j, and

$$(\mathcal{G}_2)_{U_2 \smallsetminus U_1} \simeq \sqcup \tilde\gamma_j \times (S^1)^2 \times H ,$$

where $\sqcup \tilde\gamma_j$ is (the groupoid defined by) a space, $(S^1)^2$ is the pair groupoid, and H is the group just defined. The copy $(S^1)^2$ comes from the fibered pull-back and the other factors come from $T^b(U_2 \smallsetminus U_1) \rtimes \mathbb{R}_+^*$.

We denote by $[M_0 : \mathcal{S}] := [[M_0 : L_0] : L_1]$ and by $[[M_0 : \mathcal{S}]] := \mathcal{G}_2$.

A similar construction can be used to define the groupoid that is the desingularization of a polyhedral domain and its associated pseudodifferential operators.

References

[1] Bernd Ammann, Catarina Carvalho, and Victor Nistor. Regularity for eigenfunctions of Schrödinger operators. *Lett. Math. Phys.*, 101(1):49–84, 2012.

[2] Bernd Ammann, Alexandru D. Ionescu, and Victor Nistor. Sobolev spaces on Lie manifolds and regularity for polyhedral domains. *Doc. Math.*, 11:161–206 (electronic), 2006.

[3] Bernd Ammann, Robert Lauter, and Victor Nistor. On the geometry of Riemannian manifolds with a Lie structure at infinity. *Int. J. Math. Math. Sci.*, (1-4):161–193, 2004.

[4] Bernd Ammann, Robert Lauter, and Victor Nistor. Pseudodifferential operators on manifolds with a Lie structure at infinity. *Ann. of Math. (2)*, 165(3):717–747, 2007.

[5] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique]*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.

[6] Iakovos Androulidakis and Georges Skandalis. The holonomy groupoid of a singular foliation. *J. Reine Angew. Math.*, 626:1–37, 2009.

[7] Iakovos Androulidakis and Georges Skandalis. Pseudodifferential calculus on a singular foliation. *J. Noncommut. Geom.*, 5(1):125–152, 2011.
[8] Gregory Arone and Marja Kankaanrinta. On the functoriality of the blow-up construction. *Bull. Belg. Math. Soc. Simon Stevin*, 17(5):821–832, 2010.
[9] Constantin Bacuta, Anna L. Mazzucato, Victor Nistor, and Ludmil Zikatanov. Interface and mixed boundary value problems on n-dimensional polyhedral domains. *Doc. Math.*, 15:687–745, 2010.
[10] Constantin Bacuta, Victor Nistor, and Ludmil T. Zikatanov. Improving the rate of convergence of high-order finite elements on polyhedra. I. A priori estimates. *Numer. Funct. Anal. Optim.*, 26(6):613–639, 2005.
[11] Daniel Beltita, Ingrid Beltita, and Victor Nistor. Fredholm conditions for differential operators on singular spaces: an operator algebra approach.
[12] Mădălina Roxana Buneci. Groupoid C^*-algebras. *Surv. Math. Appl.*, 1:71–98 (electronic), 2006.
[13] Catarina Carvalho and Yu Qiao. Layer potentials C^*-algebras of domains with conical points. *Cent. Eur. J. Math.*, 11(1):27–54, 2013.
[14] A. Connes. *Noncommutative geometry.* Academic Press, San Diego, 1994.
[15] H. O. Cordes and E. A. Herman. Gel′fand theory of pseudo differential operators. *Amer. J. Math.*, 90:681–717, 1968.
[16] Mondher Damak and Vladimir Georgescu. Self-adjoint operators affiliated to C^*-algebras. *Rev. Math. Phys.*, 16(2):257–280, 2004.
[17] Claire Debord. Holonomy groupoids of singular foliations. *J. Differential Geom.*, 58(3):467–500, 2001.
[18] Claire Debord, Jean-Marie Lescure, and Frédéric Rochon. Pseudodifferential operators on manifolds with fibred corners. preprint arXiv:1112.4575, to appear in Annales de l'Institut Fourier.
[19] Claire Debord and Georges Skandalis. Adiabatic groupoid, crossed product by $\mathbb{R}_+^*$ and pseudodifferential calculus. *Adv. Math.*, 257:66–91, 2014.
[20] Jacques Dixmier. *Les C^*-algèbres et leurs représentations.* Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Paris, 1996. Reprint of the second (1969) edition.
[21] Ruy Exel. Invertibility in groupoid C^*-algebras. In *Operator theory, operator algebras and applications*, volume 242 of *Oper. Theory Adv. Appl.*, pages 173–183. Birkhäuser/Springer, Basel, 2014.
[22] V. Georgescu and A. Iftimovici. Crossed products of C^*-algebras and spectral analysis of quantum Hamiltonians. *Comm. Math. Phys.*, 228(3):519–560, 2002.
[23] Vladimir Georgescu and Victor Nistor. On the essential spectrum of N-body hamiltonians with asymptotically homogeneous interactions. submitted.
[24] V. V. Grušin. A certain class of elliptic pseudodifferential operators that are degenerate on a submanifold. *Mat. Sb. (N.S.)*, 84 (126):163–195, 1971.
[25] Marco Gualtieri and Songhao Li. Symplectic groupoids of log symplectic manifolds. *Int. Math. Res. Not. IMRN*, (11):3022–3074, 2014.
[26] Philip J. Higgins and Kirill Mackenzie. Algebraic constructions in the category of Lie algebroids. *J. Algebra*, 129(1):194–230, 1990.
[27] Philip J. Higgins and Kirill C. H. Mackenzie. Fibrations and quotients of differentiable groupoids. *J. London Math. Soc. (2)*, 42(1):101–110, 1990.
[28] Michel Hilsum and Georges Skandalis. Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes). *Ann. Sci. École Norm. Sup. (4)*, 20(3):325–390, 1987.
[29] Marius Ionescu and Dana Williams. The generalized Effros-Hahn conjecture for groupoids. *Indiana Univ. Math. J.*, 58(6):2489–2508, 2009.
[30] Marius Ionescu and Dana P. Williams. Irreducible representations of groupoid C^*-algebras. *Proc. Amer. Math. Soc.*, 137(4):1323–1332, 2009.
[31] Dominic Joyce. On manifolds with corners. In *Advances in geometric analysis*, volume 21 of *Adv. Lect. Math. (ALM)*, pages 225–258. Int. Press, Somerville, MA, 2012.
[32] Max Karoubi. *K-theory*, volume 226 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin-New York, 1978.

[33] Mahmood Khoshkam and Georges Skandalis. Regular representation of groupoid C^*-algebras and applications to inverse semigroups. *J. Reine Angew. Math.*, 546:47–72, 2002.
[34] R. Lauter and S. Moroianu. Fredholm theory for degenerate pseudodifferential operators on manifolds with fibered boundaries. *Comm. Partial Differential Equations*, 26:233–283, 2001.
[35] Robert Lauter, Bertrand Monthubert, and Victor Nistor. Pseudodifferential analysis on continuous family groupoids. *Doc. Math.*, 5:625–655 (electronic), 2000.
[36] Robert Lauter and Victor Nistor. Analysis of geometric operators on open manifolds: a groupoid approach. In *Quantization of singular symplectic quotients*, volume 198 of *Progr. Math.*, pages 181–229. Birkhäuser, Basel, 2001.
[37] Eric Leichtnam and Paolo Piazza. A higher Atiyah-Patodi-Singer index theorem for the signature operator on Galois coverings. *Ann. Global Anal. Geom.*, 18(2):171–189, 2000.
[38] K. Mackenzie. *Lie groupoids and Lie algebroids in differential geometry*, volume 124 of *London Math. Soc. Lect. Note Series*. Cambridge University Press, Cambridge, 1987.
[39] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
[40] Marius Măntoiu. C^*-algebras, dynamical systems at infinity and the essential spectrum of generalized Schrödinger operators. *J. Reine Angew. Math.*, 550:211–229, 2002.
[41] J. Margalef-Roig and E. Outerelo Domínguez. Topology of manifolds with corners. In *Handbook of global analysis*, pages 983–1033, 1216. Elsevier Sci. B. V., Amsterdam, 2008.
[42] R. Mazzeo. Elliptic theory of differential edge operators. I. *Commun. Partial Differ. Equations*, 16(10):1615–1664, 1991.
[43] R. Melrose. *The Atiyah-Patodi-Singer index theorem.* Research Notes in Mathematics (Boston, Mass.). 4. Wellesley, MA: A. K. Peters, Ltd.. xiv, 377 p. , 1993.
[44] Richard B. Melrose. Differential analysis on manifolds with corners. manuscript, 1996 (http://www-math.mit.edu/ rbm/book.html).
[45] I. Moerdijk and J. Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003.
[46] Ieke Moerdijk and Janez Mrčun. On integrability of infinitesimal actions. *Amer. J. Math.*, 124(3):567–593, 2002.
[47] Bertrand Monthubert. Pseudodifferential calculus on manifolds with corners and groupoids. *Proc. Amer. Math. Soc.*, 127(10):2871–2881, 1999.
[48] Paul S. Muhly, Jean N. Renault, and Dana P. Williams. Continuous-trace groupoid C^*-algebras. III. *Trans. Amer. Math. Soc.*, 348(9):3621–3641, 1996.
[49] Sergey A. Nazarov and Boris A. Plamenevsky. *Elliptic problems in domains with piecewise smooth boundaries*, volume 13 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter & Co., Berlin, 1994.
[50] V. Nistor. Analysis on singular spaces: Lie manifolds and operator algebras. Max Planck preprint 2015, submitted.
[51] V. Nistor. Desingularization of lie groupoids and pseudodifferential operators on singular spaces.
[52] V. Nistor and N. Prudhon. Exhausting families of representations and spectra of pseudodifferential operators. Preprint 2014, http://front.math.ucdavis.edu/1411.7921, submitted.
[53] Victor Nistor. Groupoids and the integration of Lie algebroids. *J. Math. Soc. Japan*, 52(4):847–868, 2000.
[54] Victor Nistor, Alan Weinstein, and Ping Xu. Pseudodifferential operators on differential groupoids. *Pacific J. Math.*, 189(1):117–152, 1999.
[55] A. Polishchuk. Algebraic geometry of Poisson brackets. *J. Math. Sci. (New York)*, 84(5):1413–1444, 1997. Algebraic geometry, 7.
[56] Jean Pradines. Théorie de Lie pour les groupoïdes différentiables. Calcul différenetiel dans la catégorie des groupoïdes infinitésimaux. *C. R. Acad. Sci. Paris Sér. A-B*, 264:A245–A248, 1967.
[57] Vladimir Rabinovich, Steffen Roch, and Bernd Silbermann. *Limit operators and their applications in operator theory*, volume 150 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2004.
[58] Jean Renault. *A groupoid approach to C^*-algebras*, volume 793 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[59] Jean Renault. The ideal structure of groupoid crossed product C^*-algebras. *J. Operator Theory*, 25(1):3–36, 1991. With an appendix by Georges Skandalis.
[60] Marc A. Rieffel. Induced representations of C^*-algebras. *Advances in Math.*, 13:176–257, 1974.
[61] George S. Rinehart. Differential forms on general commutative algebras. *Trans. Amer. Math. Soc.*, 108:195–222, 1963.
[62] Steffen Roch. Algebras of approximation sequences: structure of fractal algebras. In *Singular integral operators, factorization and applications*, volume 142 of *Oper. Theory Adv. Appl.*, pages 287–310. Birkhäuser, Basel, 2003.
[63] Steffen Roch, Pedro A. Santos, and Bernd Silbermann. *Non-commutative Gelfand theories.* Universitext. Springer-Verlag London, Ltd., London, 2011. A tool-kit for operator theorists and numerical analysts.
[64] B.-W. Schulze. *Pseudo-differential operators on manifolds with singularities*, volume 24 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1991.
[65] R. T. Seeley. Singular integrals on compact manifolds. *Amer. J. Math.*, 81:658–690, 1959.
[66] R. T. Seeley. The index of elliptic systems of singular integral operators. *J. Math. Anal. Appl.*, 7:289–309, 1963.
[67] Stéphane Vassout. Unbounded pseudodifferential calculus on Lie groupoids. *J. Funct. Anal.*, 236(1):161–200, 2006.

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