q-LEIBNIZ ALGEBRAS

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Abstract. An algebra $(A,\circ)$ is called Leibniz if $a\circ(b\circ c)=(a\circ b)\circ c-(a\circ c)\circ b$ for all $a,b,c\in A$. We study identities for the algebras $A^{(q)}=(A,\circ_q)$, where $a\circ_q b=a\circ b+q\,b\circ a$ is the q-commutator. We show that the class of q-Leibniz algebras is defined by one identity of degree 3 if $q^2\neq 1$, $q\neq -2$, by two identities of degree 3 if $q=-2$, and by the commutativity identity and one identity of degree 4 if $q=1$. In the case of $q=-1$ we construct two identities of degree 5 that form a base of identities of degree 5 for -1-Leibniz algebras. Any identity of degree <5 for -1-Leibniz algebras follows from the anti-commutativity identity.

1. Introduction

Denote by $A=(A,\circ)$ an algebra with vector space A over a field K of characteristic $\neq 2,3$ and multiplication $(a,b)\mapsto a\circ b$. Let $(a,b,c)=a\circ(b\circ c)-(a\circ b)\circ c$ be the associator and $a\circ_q b=a\circ b+q\,b\circ a$ be the q-commutator, where $q\in K$. Denote by $A^{(q)}=(A,\circ_q)$ the algebra with the q-commutator. Notice that $a\circ_{-1}b=a\circ b-b\circ a$ is a commutator (Lie bracket, usually denoted by $[a,b]$) and $a\circ_1 b=a\circ b+b\circ a$ is an anti-commutator (Jordan bracket, sometimes denoted by $\{a,b\}$).

Example. If A is an associative algebra, then $A^{(-1)}=(A,[\ ,\ ])$ is a Lie algebra,

$$[a,b]=-[b,a],$$

$$[[a,b],c]+[[b,c],a]+[[c,a],b]=0,$$

and $A^{(+1)}=(A,\{\ ,\ \})$ is a Jordan algebra,

$$\{a,b\}=\{b,a\},$$

$$\{\{a,a\},\{b,a\}\}=\{\{\{a,a\},b\},a\}.$$

Usually, q-commutators are studied in a frame of quantum groups. It seems that studying of q-identities has their own interest. We try to demonstrate it in a class of Leibniz algebras. We call an algebra A *Leibniz* (more exactly *right-Leibniz*) if for all $a,b,c\in A$

$$a\circ(b\circ c)=(a\circ b)\circ c-(a\circ c)\circ b.$$

Leibniz algebras were introduced in [2], [3]. In other words, Leibniz algebras are algebras with the identity $lei=0$, where

$$lei = lei(t_1,t_2,t_3) = t_1(t_2t_3) - (t_1t_2)t_3 + (t_1t_3)t_2.$$

Example. Let $(L,\star)$ be a Lie algebra under multiplication $\star$ and let M be an L-module under the right action $(M,L)\to M, (m,a)\mapsto ma$. Make M a trivial left L-module: $am=0, a\in L, m\in M$. Then the vector space $L\oplus M$ under multiplication

$$(a+m)\circ(b+n) = a\star b + mb$$

becomes right-Leibniz. Indeed,

$$(a+m)\circ((b+n)\circ(c+s)) = (a+m)\circ(b\star c+nc)$$
$$= a\star(b\star c) + m(b\star c) = (a\star b)\star c - (a\star c)\star b + (mb)c - (mc)b$$
$$= ((a+m)\circ(b+n))\circ(c+s) - ((a+m)\circ(c+s))\circ(b+n).$$

We call the so-obtained algebra $L+M$ (a semi-direct sum of Leibniz algebras) *standard* Leibniz.

Endow a standard Leibniz algebra $(L+M,\circ)$ with the commutator $[\ ,\ ]$. Then

$$[a+m,b+n] = (a+m)\circ(b+n)-(b+n)\circ(a+m) = (a\star b)+mb-(b\star a)-na$$
$$= 2[a,b]+mb-na,$$

where $[a,b]=a\star b - b\star a$. The algebra $(L+M,[\ ,\ ])$ (more exactly, $L+M$ under multiplication $[a,b]+(mb-na)/2$) is called *Omni-Lie* [4], [5].

Given non-associative polynomials $f_1,\dots,f_s$, we let $Var(f_1,\dots,f_s)$ denote the variety of algebras with identities $f_1=0,\dots,f_s=0$.

In this paper we construct identities for q-(right)-Leibniz algebras. In particular, we describe identities for Omni-Lie algebras.

We prove that the category of q-Leibniz algebras is equivalent to the category of Leibniz algebras if $q^2\neq 1, q\neq -2$. This means that, for $q\neq \pm1,-2$, every algebra with identity $lei^{(q)}=0$ can be obtained as $A^{(q)}$ from some Leibniz algebra A and, conversely, if B is an algebra with identity $lei^{(q)}=0$, then $B^{(-q)}$ is right-Leibniz. In the case of $q=-2$ we should add to the identity $lei^{(q)}=0$ the identity $lei_1^{(q)}=0$ in order to obtain equivalent categories.

Theorem 1.1. $(q\neq -1,1,-2)$ *The class of q-Leibniz algebras $\mathfrak{Lei}^{(q)}$ satisfies the identity $lei^{(q)}=0$, where*

$$lei^{(q)} = lei^{(q)}(t_1,t_2,t_3) =$$
$$(q^2-1)\,(t_1(t_2t_3)-t_2(t_1t_3)) + (q^2+q-1)\,(t_2t_1)t_3 + (t_2t_3)t_1 - t_1(t_3t_2) - q\,t_3(t_1t_2).$$

The varieties $\mathfrak{Lei}$, $\mathfrak{Lei}^{(q)}$ and $Var(lei^{(q)})$ are equivalent.

In particular, $Var(lei^{(q)}$ has no special identity for $\mathfrak{L}^{(q)}$ if $q \neq -2, q^2 \neq 1$. The identity $lei_1^{(q)} = 0$ is a consequence of the identity $lei^{(q)} = 0$ if $q \neq -2, q^2 \neq 1$.

Theorem 1.2. $(q^2 \neq 1)$ *Let $q = -2$. The class of q-Leibniz algebras $\mathfrak{Lei}^{(-2)}$ satisfies the identities $lei^{(-2)} = 0$ and $lei_1^{(-2)} = 0$, where $lei^{(q)}$ is given above and*

$$lei_1^{(q)} = lei_1^{(q)}(t_1, t_2, t_3) = -t_1(t_2t_3 + t_3t_2) + q(t_2t_3 + t_3t_2)t_1.$$

The varieties $\mathfrak{Lei}$, $\mathfrak{Lei}^{(-2)}$ and $Var(lei^{(-2)}, lei_1^{(-2)})$ are equivalent.

So the identity $lei_1^{(-2)} = 0$ is a special identity for $Var(lei^{(-2)})$ which does not follow from the identity $lei^{(-2)} = 0$, and there are no other special identities for $\mathfrak{Lei}^{(-2)}$.

Define non-commutative non-associative polynomials $leilie_1, leilie_2$ of degree five by

$$leilie_1(t_1, t_2, t_3, t_4, t_5) = 2\, ljac(ljac(t_1, t_2, t_3), t_4, t_5) - [ljac(t_1, t_2, t_3), [t_4, t_5]],$$

$$leilie_2(t_1, t_2, t_3, t_4, t_5)$$

$$= -\frac{1}{2} \sum_{\sigma \in Sym(2,3,4,5)} sign\,\sigma(-4((((t_{\sigma(2)}t_{\sigma(3)})t_{\sigma(4)})t_{\sigma(5)})t_1 + 2(((t_{\sigma(2)}t_{\sigma(3)})t_1)t_{\sigma(4)})t_{\sigma(5)}$$

$$+2(((t_{\sigma(2)}t_{\sigma(3)})t_{\sigma(4)})t_1)t_{\sigma(5)} + ((t_1t_{\sigma(2)}t_{\sigma(3)})(t_{\sigma(4)}t_{\sigma(5)}) + ((t_1t_{\sigma(2)})(t_{\sigma(4)}t_{\sigma(5)}))t_{\sigma(3)}).$$

For a non-commutative non-associative polynomial $f(t_1, \ldots, t_k)$, denote by $Alt(f)$ its skew-symmetrization

$$Alt\, f(t_1, \ldots, t_k) = \sum_{\sigma \in Sym_k} sign\,\sigma\, f(t_{\sigma(1)}, \ldots, t_{\sigma(k)}).$$

Let

$$leilie(t_1, t_2, t_3, t_4, t_5) = Alt(4(((t_1t_2)t_3)t_4)t_5 - ((t_1t_2)t_3)(t_4t_5)).$$

Theorem 1.3. $(q = -1)$ *Let A be a right-Leibniz algebra. Then $A^{(-1)}$ satisfies the identities $acom = 0$, $leilie_1 = 0$ and $leilie_2 = 0$. Any polylinear identity of $\mathfrak{Lei}^{(-1)}$ of degree no more than 4 follows from the anti-commutativity identity. Any polylinear identity of $\mathfrak{Lei}^{(-1)}$ of degree 5 follows from the identities $com = 0$, $leilie_1 = 0$ and $leilie_2 = 0$.*

Corollary 1.4. *Let A be a right-Leibniz algebra. Then $A^{(-1)}$ satisfies the identity $leilie = 0$.*

Corollary 1.5. *Every Omni-Lie algebra satisfies the polynomial identities* $acom = 0$, $leilie_1 = 0$, $leilie_2 = 0$ *and* $leilie = 0$. *The identities* $acom = 0$, $leilie_1 = 0$ *and* $leilie_2 = 0$ *form a base of identities in the space of polylinear identities of degree no more than 5 for the class of Omni-Lie algebras.*

Note that the polynomials $leilie_1$, $leilie_2$ and $leilie$ have 9 , 60 and 90 terms respectively.

Let

$$leijor(t_1, t_2, t_3, t_4) = (t_1t_2)(t_3t_4).$$

Theorem 1.6. $(q = 1)$ *Let* A *be a right-Leibniz algebra. Then* $A^{(1)}$ *satisfies the identities* $com = 0$ *and* $leijor = 0$. *Every polylinear identity which is true for any Leibniz-Jordan algebra follows from the identities* $com = 0$ *and* $leijor = 0$.

In other words, there are no special identities for the class of Leibniz-Jordan algebras.

2. Non-commutative non-associative polynomials

Let $K\langle t_1, t_2, \dots\rangle$ be the space of non-commutative non-associative polynomials in variables $t_1, t_2, \dots$ (free magma). For a polynomial $f = f(t_1, \dots, t_k) \in K\langle t_1, t_2, \dots\rangle$, we say that $f = 0$ is an *identity* on an algebra $(A, \circ)$ if $f(a_1, \dots, a_k) = 0$ for all $a_1, \dots, a_k \in A$.

Recall that there are exists $\frac{1}{k}\binom{2(k-1)}{k-1}$ types of bracketing for the string $t_1 \dots t_k$. For example, there are 5 types of bracketing for 4 elements:

$$((t_1t_2)t_3)t_4,\ (t_1t_2)(t_3t_4),\ t_1(t_2(t_3t_4)),\ t_1((t_2t_3)t_4),\ (t_1(t_2t_3))t_4.$$

Order the types of bracketing somehow. If σ is a type of bracketing, denote by $\sigma(t_{i_1}, \dots, t_{i_k})$ the string $t_{i_1} \dots t_{i_k}$ with bracketing type σ. For example, if $k = 4$ and σ is the bracketing type $(t_1(t_2t_3))t_4$ then $\sigma(t_1, t_2, t_1, t_3) = (t_1(t_2t_1))t_3$.

Let α be some bracketing type of $t_1, \dots, t_n$. We say that a monomial of the form $\alpha(t_{i_1}, \dots, t_{i_n})$ has *polydegree* $(r_1, \dots, r_k)$ if $\{i_1, \dots, i_n\} \subseteq \{1, \dots, k\}$ and $r_m = |\{s : i_s = m, s = 1, \dots, n\}|$ is the number of indexes i_s equal to m for any $m = 1, \dots, k$. Call $f = f(x_1, \dots, x_k)$ *homogeneous of degree* $(r_1, \dots, r_k)$ if f is a linear combination of monomials of polydegree $(r_1, \dots, r_k)$. Say that a homogeneous polynomial f has *degree* l if $r_1 + \dots + r_k = l$.

A homogeneous polynomial $f = f(t_1, \dots, t_k)$ of polydegree $(1, \dots, 1)$ is called *polylinear.* Notice that the degree of a polylinear polynomial $f \in K\langle t_1, \dots, t_k\rangle$ is equal to the number of variables, k . In other words a polynomial f is polylinear if f is a linear combination of monomials of the form $\alpha(t_{i_1},\cdots,t_{i_k})$, where $\binom{1\cdots k}{i_1\dots i_k}\in Sym_k$ is a permutation of the set $\{1,\dots,k\}$ and α is a bracketing.

Given polynomials $f_1,\dots,f_s,g\in K\langle t_1,\dots,t_k\rangle$, we say that the identity $g=0$ *follows from the identities* $f_1=0,\dots,f_s=0$, and write $\{f_1=0,\dots,f_s=0\}\Rightarrow g=0$, if $g=0$ is an identity for any algebra $A\in\mathcal{L}$ with the identities $f_1=0,\dots,f_s=0$.

Suppose that polynomials $f_1,\dots,f_s$ are homogeneous and have degrees $n_1\le\cdots\le n_s$. Suppose that $(A,\circ_q)\in\mathcal{L}^{(q)}$ has identities $f_1=0,\dots,f_s=0$. We say that these identities are $\mathcal{L}^{(q)}$*-minimal* if

- for any $r=1,\dots,s$ the identity $f_r=0$ does not follow from the identities $f_1=0,\dots,f_{r-1}=0,f_{r+1}=0,\dots,f_s=0$
- if $\{f_1=0,\dots,f_{r-1}=0,g=0,f_{r+1}=0,\dots,f_r=0\}\Rightarrow f_r=0$ then $\{f_1=0,\dots,f_{r-1}=0,f_r=0,f_{r+1}=0,\dots,f_s=0\}\Rightarrow g=0$.

The space $K\langle t_1,t_2,\dots\rangle$ has the natural multiplication $(f,g)\mapsto f\cdot g=fg$. Let us endow it with the multiplication $(f,g)\mapsto f\cdot_q g$ given by $f\cdot_q g=f\cdot g+q\,g\cdot f$. For example,

$$(t_1+3t_1t_2)\cdot((t_2t_3)t_1)=t_1((t_2t_3)t_1)+3(t_1t_2)((t_2t_3)t_1),$$

$$(t_1+3t_1t_2)\cdot_q((t_2t_3)t_1)$$
$$=t_1((t_2t_3)t_1)+3(t_1t_2)((t_2t_3)t_1)+q((t_2t_3)t_1)t_1+3q((t_2t_3)t_1)(t_1t_2).$$

Let

$$\tau_q:K\langle t_1,t_2,\dots\rangle\to K\langle t_1,t_2,\dots\rangle$$

be a linear map defined by

$$\tau_q(t_i)=t_i,$$

$$\tau_q(f\cdot g)=\tau_q(f)\cdot\tau_q(g)+q\,\tau(g)\cdot\tau_q(f),$$

for any $f,g\in K\langle t_1,t_2,\dots\rangle$. Then

$$\tau_q:(K\langle t_1,t_2,\dots\rangle,\cdot)\to(K\langle t_1,t_2,\dots\rangle,\cdot_q)$$

is the homomorphism

$$\tau_q(f\cdot g)=\tau_q(f)\cdot_q\tau_q(g).$$

Given a bracketing type σ, we set

$$\sigma_q=\tau_q\sigma.$$

In other words, $\sigma_q(t_1,\dots,t_k)$ is a polynomial obtained from $\sigma(t_1,\dots,t_k)$ by multiplication $\circ_q$. For example, if σ is the bracketing type $(t_1t_2)t_3$, then

$$\sigma_q(t_3,t_1,t_2)=(t_3t_1)t_2+q((t_1t_3)t_2+t_2(t_3t_1))+q^2t_2(t_1t_3).$$

Lemma 2.1. *For any braketing type* σ

$$\sigma_{-q}\sigma_q(t_{i_1},\dots,t_{i_k})=(1-q^2)^{k-1}\sigma_0(t_{i_1},\dots,t_{i_k}).$$

Proof. We use the induction on k. For $k=2$ the statement is true:

$$\sigma_q(t_{i_1},t_{i_2})=t_{i_1}t_{i_2}+q\,t_{i_2}t_{i_1},$$

and

$$\begin{gathered}\sigma_{-q}\sigma_q(t_{i_1},t_{i_2})\\ =t_{i_1}t_{i_2}-q\,t_{i_2}t_{i_1}+q\,t_{i_2}\cdot t_{i_1}-q^2t_{i_1}t_{i_2}\\ =(1-q^2)t_{i_1}t_{i_2}=(1-q^2)\sigma_0(t_{i_1},t_{i_2}).\end{gathered}$$

Suppose that our statement is true for $k-1$. Let

$$\sigma(t_{i_1},\dots,t_{i_k})=\sigma'(t_{i_1},\dots,t_{i_{k'}})\sigma''(t_{i_{k'+1}},\dots,t_{i_k})$$

for some $1\le k'\le k$ and for some bracketings σ', σ''. Then

$$\begin{gathered}\sigma_q(t_{i_1},\dots,t_{i_k})\\ \\ =\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ +q\,\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\end{gathered}$$

and

$$\begin{gathered}\sigma_{-q}\sigma_q(t_{i_1},\dots,t_{i_k})\\ \\ =\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ -q\,\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\\ +q\,\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\\ -q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ \\ =\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})\\ -q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k}).\end{gathered}$$

By the induction hypothesis

$$\begin{gathered}\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\\ =(1-q^2)^{k'-1}\sigma'_0(t_{i_1},\dots,t_{i_{k'}}),\end{gathered}$$

$$\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})$$

$$= (1-q^2)^{k-k'-1}\sigma''_0(t_{i_{k'+1}},\dots,t_{i_k}).$$

Therefore,

$$\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})$$
$$= (1-q^2)^{k-2}\sigma_0(t_{i_1},\dots,t_{i_k}),$$

$$-q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})$$
$$= -q^2(1-q^2)^{k-2}\sigma_0(t_{i_1},\dots,t_{i_k})$$

and

$$\sigma_{-q}\sigma_q(t_{i_1},\dots,t_{i_k})$$
$$= \sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})$$
$$-q^2\sigma'_{-q}\sigma'_q(t_{i_1},\dots,t_{i_{k'}})\sigma''_{-q}\sigma''_q(t_{i_{k'+1}},\dots,t_{i_k})$$
$$= (1-q^2)^{k-1}\sigma_0(t_{i_1},\dots,t_{i_k}).$$

¿From Lemma 2.1 we infer the following

Theorem 2.2. $(q^2\neq 1)$ *Let* $f_1,\dots,f_s$ *be homogeneous polynomials of degree* k*. Then the class of* q*-agebras* $Var(f_1,\dots,f_s)^{(q)}$ *forms a variety generated by the polynomial identities* $\sigma_{-q}f_1=0,\dots,\sigma_{-q}f_s=0$*. This variety is equivalent to* $Var(f_1,\dots,f_s)$ *and the equivalence can be given by* $A=(A,\star)\mapsto A^{(-q)}=(A,\star_{-q})$*.*

The equivalence of varieties means the following. There exist functors

$$F: Var(f_1,\dots,f_s)\to Var(\sigma_{-q}f_1,\dots,\sigma_{-q}f_s),\qquad (A,\circ)\to(A,\circ_q),$$
$$G: Var(\sigma_{-q}f_1,\dots,\sigma_{-q}f_s)\to Var(f_1,\dots,f_s),\qquad (A,\star)\to(A,\star'_q)$$

such that

$$GF(A,\circ)=(A,\circ),\qquad GF(A,\star)=(A,\star).$$

Here

$$a\star'_q b=\frac{1}{(1-q^2)^{k-1}}a\star_q b.$$

Recall that all polynomials $f_1,\dots,f_s$ are supposed homogeneous. Notice that, for any $(A,\circ),(B,\cdot)\in Var(f_1,\dots,f_s)$ and a morphism between them, i.e., a homomorphism $\psi:(A,\circ)\to(B,\cdot)$, there corresponds a morphism of algebras $\psi: F(A,\circ)\to F(B,\cdot)$ in the category $Var(\sigma_{-q}f_1,\dots,\sigma_{-q}f_s)$, i.e., a homomorphism $\psi:(A,\circ_q)\to(B,\cdot_q)$. Indeed,

$$\psi(a_1\circ_q a_2)$$
$$=\psi(a_1\circ a_2+q\,a_2\circ a_1)$$
$$=\psi(a_1\circ a_2)+q\,\psi(a_2\circ a_1)$$

$$= \psi(a_1)\cdot\psi(a_2) + q\,\psi(a_2)\cdot\psi(a_1)$$
$$= \psi(a_1)\cdot_q \psi(a_2).$$

If I is an ideal of $(A,\circ)$ then I is an ideal of $(A,\circ_q)$. Therefore, simplicity, nilpotency and solvability properties of algebras in the category $Var(f_1,\dots,f_s)$ remain the same for the corresponding algebras in $Var(\sigma_{-q}f_1,\dots,\sigma_{-q}f_s)$. If $(A,\circ)$ is free in the variety $Var(f_1,\dots,f_s)$, then $(A,\circ_q)$ is free in the variety $Var(\sigma_{-q}f_1,\dots,\sigma_{-q}f_s)$. We pay attention to the fact that the categories $Var(f_1,\dots,f_s)$ and $Var(\sigma_{-q}f_1,\dots,\sigma_{-q}f_s)$ are equivalent only in the case of $q^2\neq 1$.

Let $g_1,\dots,g_s,h$ be non-commutative non-associative polynomials. Suppose that, for a class $\mathfrak{L}$ of algebras, the corresponding class $\mathfrak{L}^{(q)}$ of q-algebras satisfies the identities $g_1=0,\dots,g_s=0$ and $h=0$. In this case we say that $h=0$ is a *special* identity or an s-identity for $Var(g_1,\dots,g_s)$.

We give another application of Lemma 2.1.

Theorem 2.3. *If $q\neq\pm1$, then the map*

$$\tau_q : (K\langle t_1,t_2,\dots\rangle,\cdot)\to(K\langle t_1,t_2,\dots\rangle,\cdot_q)$$

is an isomorphism.

Let $\mathfrak{L}$ be some class of algebras. For a polynomial $f\in K\langle t_1,t_2,\dots\rangle$, we say that $f=0$ is an identity for $\mathfrak{L}$ if every algebra $A\in\mathfrak{L}$ satisfies the identity $f=0$. Recall that each class of algebras satisfying polynomial identities forms a variety.

Let $\mathfrak{Lei}$ be the class of Leibniz algebras, i.e., the variety of algebras generated by the (right)-Leibniz identity $lei=0$. Denote by $\mathfrak{Lei}^{(q)}$ the class of q-Leibniz algebras, i.e., algebras of the form $A^{(q)}=(A,\circ_q)$, where $A\in\mathfrak{Lei}$.

Define non-commutative polynomials *com* (commutativity), *acom* (anticommutativity), *ljac* (left-Jacobian), *rjac* (right-Jacobian), *lalia* (left-anti-Lie-admissable), *ralia* (rigt-Anti-Lie-admissible), *lia* (Lie-admissibile), s^l_k (stanard left-skew-symmetric), s^r_k (standard right-skew-symmetric) and $s^{[r]}_k$ (s_k-Lie-admissible) by

$$com(t_1,t_2) = t_1t_2 - t_2t_1,$$
$$acom(t_1,t_2) = t_1t_2 + t_2t_1,$$

$$ljac(t_1,t_2,t_3) = (t_1t_2)t_3 + (t_2t_3)t_1 + (t_3t_1)t_2,$$
$$rjac(t_1,t_2,t_3) = t_1(t_2t_3) + t_2(t_3t_1) + t_3(t_1t_2),$$

$$lalia(t_1,t_2,t_3) = [t_1,t_2]t_3 + [t_2,t_3]t_1 + [t_3,t_1]t_2,$$

$$ralia(t_1,t_2,t_3) = t_1[t_2,t_3] + t_2[t_3,t_1] + t_3[t_1,t_2],$$
$$lia(t_1,t_2,t_3) = [[t_1,t_2],t_3] + [[t_2,t_3],t_1] + [[t_3,t_1],t_2],$$
$$alia^{(q)} = lalia + q\,ralia, \qquad q \in K.$$

Recall that for a non-commutative non-associative polynomial $f(t_1,\dots,t_k)$, we denote by $Alt(f)$ its skew-symmetrization

$$Alt\, f(t_1,\dots,t_k) = \sum_{\sigma\in Sym_k} sign\,\sigma\, f(t_{\sigma(1)},\dots,t_{\sigma(k)}).$$

Let

$$s^r_k(t_1,\dots,t_k) = Alt(t_1(t_2(\cdots(t_{k-1}t_k)))),$$
$$s^l_k(t_1,\dots,t_k) = Alt((\cdots(t_1t_2)\cdots t_{k-1})t_k,$$
$$s^{[r]}_k(t_1,\dots,t_k) = Alt([t_1,[t_2,\cdots,[t_{l-1},t_k]]]).$$

Notice that

$$com = s_2,$$
$$lalia = s^l_3, \quad ralia = s^r_3, \quad lia = s^l_3 - s^r_3 = lalia - ralia.$$

If polynomials are anti-commutative, i.e., satisfy the identity $acom = 0$, then

$$ljac = -rjac,$$
$$lia = 4\, ljac.$$

3. Right-center and Lie elements

Let $F = F(V)$ be a free right-Leibniz algebra generated by a space V. Let $(F^{lie},[\ ,\ ])$ be the subspace of F generated by V under the commutator $[\ ,\ ]$. We say that $a \in F$ is a *Lie-element* if $a \in (F^{lie},[\ ,\ ])$. Homomorphic images of Lie elements of any Leibniz algebras are called Lie elements as well.

Let $(A,\circ)$ be a right-Leibniz algebra. An element $z \in A$ is called *right-central* if

$$a \circ z = 0$$

for all $a \in A$. Let A^{rann} be the set of right-central elements of A. It was noticed in [3] that A^{rann} is an ideal with trivial left action, $a\circ z = 0, z \in A^{rann}, a \in A,$ such that

$$\{a,b\} = a\circ b + b\circ a \in A^{rann}$$

for all $a,b \in A$. We construct new right-central elements.

Observe that

$$s^l_{k+1}(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1}(-1)^i s^l_k(a_1,\dots,\hat{a}_i,\dots,a_{k+1})\circ a_i. \tag{1}$$

Lemma 3.1. *Let* $(A, \circ)$ *be a right-Leibniz algebra. Then* A^{rann} *is an ideal such that*

$$a \circ z = 0.$$

For any $q \in K$,

$$(a \circ_q b) \circ c = (a \circ c) \circ_q b + a \circ_q (b \circ c). \tag{2}$$

In particular,

$$\{a, b\} \circ c = \{a \circ c, b\} + \{a, b \circ c\}.$$

For any $k \geq 3$

$$s^l_k(a_1, \ldots, a_k), \quad s^r_k(a_1, \ldots, a_k) \in A^{rann}.$$

Moreover,

$$s^l_k(a_1, \ldots, a_k) = s^{[r]}_k(a_1, \ldots, a_k)$$

are Lie elements,

$$s^r_k(a_1, \ldots, a_k) = 0, k \geq 4,$$

and

$$s^r_3(a, b, c) = 2s^l_3(a, b, c).$$

In other words, any right-Leibniz algebra A *is* $-1/2$ *-Alia,*

$$alia^{(-1/2)}(a, b, c) = 0$$

for all $a, b, c \in A$.

Proof. We have

$$\begin{gathered}(a \circ_q b) \circ c \\ = (a \circ b + qb \circ a) \circ c \\ = a \circ (b \circ c) + (a \circ c) \circ b + qb \circ (a \circ c) + q(b \circ c) \circ a \\ = (a \circ c) \circ_q b + a \circ_q (b \circ c).\end{gathered}$$

So, (2) is established. Thus, in the case $q = 0$ we obtain the right-Leibniz identity

$$(a \circ b) \circ c = (a \circ c) \circ b + a \circ (b \circ c).$$

Let $k = 3$. Notice that

$$s^r_3(a, b, c) = ralia(a, b, c).$$

We have

$$\begin{gathered}ralia(a, b, c) \\ = a \circ [b, c] + b \circ [c, a] + c \circ [a, b] \\ = 2(a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b)) - a \circ \{b, c\} - b \circ \{c, a\} - c \circ \{a, b\} \\ = 2rjac(a, b, c).\end{gathered}$$

By the right-Leibniz identity

$$ljac(a, b, c) = [a, b] \circ c + [b, c] \circ a + [c, a] \circ b = lalia(a, b, c),$$

and

$$lia[a,b,c) = lalia(a,b,c) - ralia(a,b,c) = rjac(a,b,c) - 2\,rjac(a,b,c) = -rjac(a,b,c).$$

So, $s_3^r(a,b,c) = 2\,rjac(a,b,c) = -2\,lia(a,b,c)$ is a Lie-element.

By the right-Leibniz identity

$$u \circ rjac(a,b,c)$$

$$= ((u \circ a) \circ (b \circ c) - ((u \circ (b \circ c)) \circ a + ((u \circ b) \circ (c \circ a)$$

$$-((u \circ (c \circ a)) \circ b + ((u \circ c) \circ (a \circ b) - ((u \circ (a \circ b)) \circ c$$

$$= ((u \circ a) \circ b) \circ c - ((u \circ a) \circ c) \circ b - ((u \circ b) \circ c)) \circ a + ((u \circ c) \circ b) \circ a$$

$$+((u \circ b) \circ c) \circ a - ((u \circ b) \circ a) \circ c - ((u \circ c) \circ a) \circ b + ((u \circ a) \circ c) \circ b$$

$$+((u \circ c) \circ a) \circ b - ((u \circ c) \circ b) \circ a - ((u \circ a) \circ b) \circ c + ((u \circ b) \circ a) \circ c$$

$$= 0.$$

So, the element $s_3^l(a,b,c)$ is right-central.

Suppose that $s_k^l(a_1,\dots,a_k) = s^{[r]}(a_1,\dots,a_k)$ is a Lie element and is right-central. Prove that $s_{k+1}^l(a_1,\dots,a_{k+1})$ is also a Lie element which is right-central. Since $s_k^l(a_1,\dots,\hat{a}_i,\dots,a_{k+1}) \in A^{rann}$ for every $i = 1,\dots,k+1$ and since A^{rann} is an ideal, we have

$$s_{k+1}^l(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+k+1} s_k^l(a_1,\dots,\hat{a}_i,\dots,a_{k+1}) \circ a_i \in A^{rann}.$$

Further,

$$s_{k+1}^{[r]}(a_1,\dots,a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} [a_i, s_k^{[r]}(a_1,\dots,\hat{a}_i,\dots,a_{k+1})]$$

(by the induction hypothesis)

$$= \sum_{i=1}^{k+1} (-1)^{i+1} [a_i, s_k^l(a_1,\dots,\hat{a}_i,\dots,a_{k+1})]$$

(since $s_k^l(a_1,\dots,\hat{a}_i,\dots,a_{k+1}) \in A^{rann}$)

$$= \sum_{i=1}^{k+1} (-1)^i s_k^l(a_1,\dots,\hat{a}_i,\dots,a_{k+1}) \circ a_i = s_{k+1}^l(a_1,\dots,a_{k+1}).$$

4. q-commutators of Leibniz algebras in case $q^2\neq 1$.

Lemma 4.1. *For any Leibniz algebra A its q-algebra $A^{(q)}$ satisfies the identities $lei^{(q)}=0$ and $lei_1^{(q)}=0$.*

Proof. We have

$$lei^{(q)}(a,b,c)$$

$$= (q^2-1)\,a\circ_q(b\circ_q c) - a\circ_q(c\circ_q b) - (q^2-1)\,b\circ_q(a\circ_q c) - q\,c\circ_q(a\circ_q b)$$
$$+(q^2+q-1)\,(b\circ_q a)\circ_q c + (b\circ_q c)\circ_q a$$

$$= (q^2-1)(a\circ(b\circ c)+(1+q)a\circ(c\circ b)-b\circ(a\circ c)-qb\circ(c\circ a)+(q+q^2)c\circ(a\circ b)$$
$$+qc\circ(b\circ a)+q(a\circ b)\circ c-q(a\circ c)\circ b+(1-q)(b\circ a)\circ c$$
$$+(q-1)(b\circ c)\circ a-q^2(c\circ a)\circ b+q^2(c\circ b)\circ a)$$

$$= (q^2-1)(a\circ(b\circ c)+(1+q)a\circ(c\circ b)-b\circ(a\circ c)+qb\circ(a\circ c)$$
$$+q^2c\circ(a\circ b)+q(a\circ b)\circ c-q(a\circ c)\circ b+(1-q)(b\circ a)\circ c$$
$$+(q-1)(b\circ c)\circ a-q^2(c\circ a)\circ b+q^2(c\circ b)\circ a)$$

$$= (q^2-1)(qa\circ(c\circ b)+(q-1)b\circ(a\circ c)+q^2c\circ(a\circ b)+q((a\circ b)\circ c-(a\circ c)\circ b)$$
$$+(1-q)((b\circ a)\circ c-(b\circ c)\circ a)-q^2((c\circ a)\circ b-(c\circ b)\circ a))$$

$$= (q^2-1)(q(a\circ(c\circ b)+(a\circ b)\circ c-(a\circ c)\circ b)+(1-q)(-b\circ(a\circ c)+(b\circ a)\circ c-(b\circ c)\circ a)$$
$$-q^2(-c\circ(a\circ b)+(c\circ a)\circ b-(c\circ b)\circ a))$$

(by the right-Leibniz identity)

$$=0.$$

Similarly,

$$lei_1^{(q)}(a,b,c) = -a\circ_q(b\circ_q c) - a\circ_q(c\circ_q b) + q(b\circ_q c)\circ_q a + q(c\circ_q b)\circ_q a$$

$$= -a\circ(b\circ c)-qa\circ(c\circ b)-q(b\circ c)\circ a-q^2(c\circ b)\circ a$$
$$-a\circ(c\circ b)-qa\circ(b\circ c)-q(c\circ b)\circ a-q^2(b\circ c)\circ a$$
$$+q(b\circ c)\circ a+q^2(c\circ b)\circ a+q^2a\circ(b\circ c)+q^3a\circ(c\circ b)$$
$$+q(c\circ b)\circ a+q^2(b\circ c)\circ a+q^2a\circ(c\circ b)+q^3a\circ(b\circ c)$$

$$= -(a\circ b)\circ c+(a\circ c)\circ b-q(a\circ c)\circ b+q(a\circ b)\circ c$$

$$-q(b\circ c)\circ a - q^2(c\circ b)\circ a$$
$$-(a\circ c)\circ b + (a\circ b)\circ c - q(a\circ b)\circ c + q(a\circ c)\circ b$$
$$-q(c\circ b)\circ a - q^2(b\circ c)\circ a$$
$$+q(b\circ c)\circ a + q^2(c\circ b)\circ a$$
$$+q^2(a\circ b)\circ c - q^2(a\circ c)\circ b + q^3(a\circ c)\circ b - q^3(a\circ b)\circ c$$
$$+q(c\circ b)\circ a + q^2(b\circ c)\circ a$$
$$+q^2(a\circ c)\circ b - q^2(a\circ b)\circ c + q^3(a\circ b)\circ c - q^3(a\circ c)\circ b$$

$$= (-1+q+1-q+q^2-q^3-q^2+q^3)(a\circ b)\circ c$$
$$+(1-q-1+q-q^2+q^3+q^2-q^3)(a\circ c)\circ b$$
$$+(-q-q^2+q+q^2)(b\circ c)\circ a$$
$$+(-q^2-q+q^2+q)(c\circ b)\circ a$$

$$= 0.$$

Lemma 4.2. *If* $q\neq -2,$ *then*

$$Alt(lei^{(q)}) = -(q+2)(q-1)alia^{(\frac{-(2q+1)}{q+2})}.$$

If $q=-2,$ *then*

$$Alt(lei^{(-2)}) = 9\,ralia.$$

Proof. Consider the case $q\neq -2$. We have

$$lei^{(q)}(t_1,t_2,t_3) + lei^{(q)}(t_2,t_3,t_1) + lei^{(q)}(t_3,t_1,t_2)$$
$$-lei^{(q)}(t_2,t_1,t_3) - lei^{(q)}(t_3,t_2,t_1) - lei^{(q)}(t_1,t_3,t_2)$$
$$= (q-1)\{(2q+1)(t_1[t_2,t_3]+t_2[t_3,t_1]]+t_3[t_1,t_2]])$$
$$-(q+2)([t_1,t_2]t_3+[t_3,t_1]t_2+[t_2,t_3]t_1\}$$
$$= (2-q-q^2)ralia^{(\frac{-(2q+1)}{q+2})}.$$

The case $q=-2$ is considered in a similar manner.

Lemma 4.3. *Let* L *be a free Leibniz algebra with* 3 *generators,* $q\in K, q\neq 0,\pm 1.$ *Then any polylinear identity of* $L^{(q)}$ *of degree* 3 *follows from the identities* $lei^{(q)}=0$ *and* $lei_1^{(q)}=0.$ *If* $q\neq -2$ *then* $lei_1^{(q)}=0$ *is a consequence of the identity* $lei^{(q)}=0.$ *If* $q=-2,$ *then* $lei^{(q)}=0$ *and* $lei_1^{(q)}=0$ *are independent identities.*

Proof. Let $L = (L, \circ)$ be a free Leibniz algebra generated by three elements a, b, c. Write the q-commutator in $L^{(q)}$ by $uv = u\circ v + qv\circ u$.

The polylinear part of the free magma algebra (the algebra of non-commutative nonassociative polynomials) in degree 3 has dimension 12. It is generated by the following 12 monomials:

$$e_1 = e_1(t_1,t_2,t_3) = t_1(t_2t_3), \quad e_2 = e_2(t_1,t_2,t_3) = t_2(t_3t_1),$$
$$e_3 = e_3(t_1,t_2,t_3) = t_3(t_1t_2), \quad e_4 = e_4(t_1,t_2,t_3) = t_2(t_1t_3),$$
$$e_5 = e_5(t_1,t_2,t_3) = t_3(t_2t_1), \quad e_6 = e_6(t_1,t_2,t_3) = t_1(t_3t_2),$$
$$e_7 = e_7(t_1,t_2,t_3) = (t_1t_2)t_3, \quad e_8 = e_8(t_1,t_2,t_3) = (t_2t_3)t_1,$$
$$e_9 = e_9(t_1,t_2,t_3) = (t_3t_1)t_2, \quad e_{10} = e_{10}(t_1,t_2,t_3) = (t_2t_1)t_3,$$
$$e_{11} = e_{11}(t_1,t_2,t_3) = (t_3t_2)t_1, \quad e_{12} = e_{12}(t_1,t_2,t_3) = (t_1t_3)t_2.$$

Let $X = X(t_1,t_2,t_3) = \sum_{i=1}^{12} \lambda_i e_i(t_1,t_2,t_3)$ be a polynomial such that $X(a,b,c) = 0$ is an identity on $L^{(q)}$.

Substitute the generator elements $a, b, c \in L$ for the parameters t_1, t_2, t_3. Write e_i instead of $e_i(a,b,c)$. We have

$$e_1 = a\circ(b\circ c) + qa\circ(c\circ b) + q(b\circ c)\circ a + q^2(c\circ b)\circ a$$
$$= (a\circ b)\circ c - (a\circ c)\circ b + q(a\circ c)\circ b - q(a\circ b)\circ c$$
$$+q(b\circ c)\circ a + q^2(c\circ b)\circ a.$$

Similar calculations show that

$$e_2 = (b\circ c)\circ a - (b\circ a)\circ c + q(b\circ a)\circ c - q(b\circ c)\circ a$$
$$+q(c\circ a)\circ b + q^2(a\circ c)\circ b,$$

$$e_3 = (c\circ a)\circ b - (c\circ b)\circ a + q(c\circ b)\circ a - q(c\circ a)\circ b$$
$$+q(a\circ b)\circ c + q^2(b\circ a)\circ c,$$

$$e_4 = (b\circ a)\circ c - (b\circ c)\circ a + q(b\circ c)\circ a - q(b\circ a)\circ c$$
$$+q(a\circ c)\circ b + q^2(c\circ a)\circ b,$$

$$e_5 = (c\circ b)\circ a - (c\circ a)\circ b + q(c\circ a)\circ b - q(c\circ b)\circ a$$
$$+q(b\circ a)\circ c + q^2(a\circ b)\circ c,$$

$$e_6 = (a\circ c)\circ b - (a\circ b)\circ c + q(a\circ b)\circ c - q(a\circ c)\circ b$$
$$+q(c\circ b)\circ a + q^2(b\circ c)\circ a,$$

$$e_7 = (a\circ b)\circ c + q(b\circ a)\circ c + q(c\circ a)\circ b - q(c\circ b)\circ a$$
$$+q^2(c\circ b)\circ a - q^2(c\circ a)\circ b,$$

$$e_8 = (b\circ c)\circ a + q(c\circ b)\circ a + q(a\circ b)\circ c - q(a\circ c)\circ b \\ +q^2(a\circ c)\circ b - q^2(a\circ b)\circ c,$$

$$e_9 = (c\circ a)\circ b + q(a\circ c)\circ b + q(b\circ c)\circ a - q(b\circ a)\circ c \\ +q^2(b\circ a)\circ c - q^2(b\circ c)\circ a,$$

$$e_{10} = (b\circ a)\circ c + q(a\circ b)\circ c + q(c\circ b)\circ a - q(c\circ a)\circ b \\ +q^2(c\circ a)\circ b - q^2(c\circ b)\circ a,$$

$$e_{11} = (c\circ b)\circ a + q(b\circ c)\circ a + q(a\circ c)\circ b - q(a\circ b)\circ c \\ +q^2(a\circ b)\circ c - q^2(a\circ c)\circ b,$$

$$e_{12} = (a\circ c)\circ b + q(c\circ a)\circ b + q(b\circ a)\circ c - q(b\circ c)\circ a \\ +q^2(b\circ c)\circ a - q^2(b\circ a)\circ c.$$

So,

$$X =$$

$$\lambda_1(a\circ b)\circ c - \lambda_1(a\circ c)\circ b + q\lambda_1(a\circ c)\circ b - q\lambda_1(a\circ b)\circ c \\ +q\lambda_1(b\circ c)\circ a + q^2\lambda_1(c\circ b)\circ a) \\ +\lambda_2(b\circ c)\circ a - \lambda_2(b\circ a)\circ c + q\lambda_2(b\circ a)\circ c - q\lambda_2(b\circ c)\circ a \\ +q\lambda_2(c\circ a)\circ b + q^2\lambda_2(a\circ c)\circ b \\ +\lambda_3(c\circ a)\circ b - \lambda_3(c\circ b)\circ a + q\lambda_3(c\circ b)\circ a - q\lambda_3(c\circ a)\circ b \\ +q\lambda_3(a\circ b)\circ c + q^2\lambda_3(b\circ a)\circ c \\ \lambda_4(b\circ a)\circ c - \lambda_4(b\circ c)\circ a + q\lambda_4(b\circ c)\circ a - q\lambda_4(b\circ a)\circ c \\ +q\lambda_4(a\circ c)\circ b + q^2\lambda_4(c\circ a)\circ b \\ \lambda_5(c\circ b)\circ a - \lambda_5(c\circ a)\circ b + q\lambda_5(c\circ a)\circ b - q\lambda_5(c\circ b)\circ a \\ +q\lambda_5(b\circ a)\circ c + q^2\lambda_5(a\circ b)\circ c \\ \lambda_6(a\circ c)\circ b - \lambda_6(a\circ b)\circ c + q\lambda_6(a\circ b)\circ c - q\lambda_6(a\circ c)\circ b \\ +q\lambda_6(c\circ b)\circ a + q^2\lambda_6(b\circ c)\circ a \\ +\lambda_7(a\circ b)\circ c + q\lambda_7(b\circ a)\circ c + q\lambda_7(c\circ a)\circ b - q\lambda_7(c\circ b)\circ a \\ +q^2\lambda_7(c\circ b)\circ a - q^2\lambda_7(c\circ a)\circ b \\ +\lambda_8(b\circ c)\circ a + q\lambda_8(c\circ b)\circ a + q\lambda_8(a\circ b)\circ c - q\lambda_8(a\circ c)\circ b \\ +q^2\lambda_8(a\circ c)\circ b - q^2\lambda_8(a\circ b)\circ c \\ +\lambda_9(c\circ a)\circ b + q\lambda_9(a\circ c)\circ b + q\lambda_9(b\circ c)\circ a - q\lambda_9(b\circ a)\circ c \\ +q^2\lambda_9(b\circ a)\circ c - q^2\lambda_9(b\circ c)\circ a$$

$$+\lambda_{10}(b\circ a)\circ c+q\lambda_{10}(a\circ b)\circ c+q\lambda_{10}(c\circ b)\circ a-q\lambda_{10}(c\circ a)\circ b$$
$$+q^2\lambda_{10}(c\circ a)\circ b-q^2\lambda_{10}(c\circ b)\circ a$$
$$+\lambda_{11}(c\circ b)\circ a+q\lambda_{11}(b\circ c)\circ a+q\lambda_{11}(a\circ c)\circ b-q\lambda_{11}(a\circ b)\circ c$$
$$+q^2\lambda_{11}(a\circ b)\circ c-q^2\lambda_{11}(a\circ c)\circ b$$
$$+\lambda_{12}(a\circ c)\circ b+q\lambda_{12}(c\circ a)\circ b+q\lambda_{12}(b\circ a)\circ c-q\lambda_{12}(b\circ c)\circ a$$
$$+q^2\lambda_{12}(b\circ c)\circ a-q^2\lambda_{12}(b\circ a)\circ c$$

$$=(\lambda_1-q\lambda_1+q\lambda_3+q^2\lambda_5-\lambda_6+q\lambda_6+\lambda_7+q\lambda_8-q^2\lambda_8+q\lambda_{10}-q\lambda_{11}+q^2\lambda_{11})(a\circ b)\circ c$$
$$+(-\lambda_1+q\lambda_1+q^2\lambda_2+q\lambda_4+\lambda_6-q\lambda_6-q\lambda_8+q^2\lambda_8+q\lambda_9+q\lambda_{11}-q^2\lambda_{11}+\lambda_{12})(a\circ c)\circ b$$
$$+(-\lambda_2+q\lambda_2+q^2\lambda_3+\lambda_4-q\lambda_4+q\lambda_5+q\lambda_7-q\lambda_9+q^2\lambda_9+\lambda_{10}+q\lambda_{12}-q^2\lambda_{12})(b\circ a)\circ c$$
$$+(q\lambda_1+\lambda_2-q\lambda_2-\lambda_4+q\lambda_4+q^2\lambda_6+\lambda_8+q\lambda_9-q^2\lambda_9+q\lambda_{11}-q\lambda_{12}+q^2\lambda_{12})(b\circ c)\circ a$$
$$+(q\lambda_2+\lambda_3-q\lambda_3+q^2\lambda_4-\lambda_5+q\lambda_5+q\lambda_7-q^2\lambda_7+\lambda_9-q\lambda_{10}+q^2\lambda_{10}+q\lambda_{12})(c\circ a)\circ b$$
$$+(q^2\lambda_1-\lambda_3+q\lambda_3+\lambda_5-q\lambda_5+q\lambda_6-q\lambda_7+q^2\lambda_7+q\lambda_8+q\lambda_{10}-q^2\lambda_{10}+\lambda_{11})(c\circ b)\circ a$$
.

We thus we obtain the following system of equations

$$(1-q)\lambda_1+q\lambda_3+q^2\lambda_5+(q-1)\lambda_6+\lambda_7+(q-q^2)\lambda_8+q\lambda_{10}+(q^2-q)\lambda_{11}=0,$$
$$(q-1)\lambda_1+q^2\lambda_2+q\lambda_4+(1-q)\lambda_6+(q^2-q)\lambda_8+q\lambda_9+(q-q^2)\lambda_{11}+\lambda_{12}=0,$$
$$(q-1)\lambda_2+q^2\lambda_3+(1-q)\lambda_4+q\lambda_5+q\lambda_7+(q^2-q)\lambda_9+\lambda_{10}+(q-q^2)\lambda_{12}=0,$$
$$q\lambda_1+(1-q)\lambda_2+(q-1)\lambda_4+q^2\lambda_6+\lambda_8+(q-q^2)\lambda_9+q\lambda_{11}+(q^2-q)\lambda_{12}=0,$$
$$q\lambda_2+(1-q)\lambda_3+q^2\lambda_4+(q-1)\lambda_5+(q-q^2)\lambda_7+\lambda_9+(q^2-q)\lambda_{10}+q\lambda_{12}=0,$$
$$q^2\lambda_1+(q-1)\lambda_3+(1-q)\lambda_5+q\lambda_6+(q^2-q)\lambda_7+q\lambda_8+(q-q^2)\lambda_{10}+\lambda_{11}=0.$$

The transposed matrix of this system is

$$\begin{matrix}
1-q & q & 0 & 0 & q^2 & q-1\\
0 & 1-q & q & q-1 & 0 & q^2\\
q & 0 & 1-q & q^2 & q-1 & 0\\
0 & q-1 & q^2 & 1-q & 0 & q\\
q^2 & 0 & q-1 & q & 1-q & 0\\
q-1 & q^2 & 0 & 0 & q & 1-q\\
1 & 0 & q-q^2 & q & q^2-q & 0\\
q-q^2 & 1 & 0 & 0 & q & q^2-q\\
0 & q-q^2 & 1 & q^2-q & 0 & q\\
q & 0 & q^2-q & 1 & q-q^2 & 0\\
q^2-q & q & 0 & 0 & 1 & q-q^2\\
0 & q^2-q & q & q-q^2 & 0 & 1
\end{matrix}$$

The determinant of the 6×6-matrix composed of the first 6 rows is $(1-q)^5q^3(1+q)^3(q+2)$. So, this system has rank 6 if $q^2\neq 1, q\neq 0,-2$.

One can choose $\lambda_i, 7 \le i \le 12$ as free parameters. Now, we consider two cases.

Suppose that $q \neq -2$. In this case the system has the following solution

$$\lambda_1 = -\frac{-1+q+q^2}{(q+2)q}(\lambda_7+\lambda_8+\lambda_9+(1-q-q^2)\lambda_{10}+(1+q)\lambda_{11}-\lambda_{12}),$$

$$\lambda_2 = -\frac{1}{(q+2)q}(\lambda_7+(q^2+q-1)\lambda_8+\lambda_9-\lambda_{10}+(1-q-q^2)\lambda_{11}+(q+1)\lambda_{12}),$$

$$\lambda_3 = -\frac{1}{(q+2)q}(\lambda_7+\lambda_8+(q^2+q-1)\lambda_9+(q+1)\lambda_{10}-\lambda_{11}-(q^2+q-1)\lambda_{12}),$$

$$\lambda_4 = -\frac{1}{(q+2)q}((1-q-q^2)\lambda_7-\lambda_8+(q+1)\lambda_9+(q^2+q-1)\lambda_{10}+\lambda_{11}+\lambda_{12}),$$

$$\lambda_5 = -\frac{1}{(q+2)q}((1+q)\lambda_7-(q^2+q-1)\,\lambda_8-\lambda_9+\lambda_{10}+(q^2+q-1)\lambda_{11}+\lambda_{12}),$$

$$\lambda_6 = -\frac{1}{(q+2)q}(-\lambda_7+(q+1)\lambda_8+(1-q-q^2)\lambda_9+\lambda_{10}+\lambda_{11}+(q^2+q-1)\lambda_{12}).$$

Substitute these expressions for λ_i, $1 \le i \le 6$, in $X(t_1,t_2,t_3)$ and collect the coefficients of $\lambda_j, 7 \le j \le 12$. We obtain a presentation of the polynomial $X(t_1,t_2,t_3)$ as a linear combination of the following 6 polynomials

$$f_1 = (q-1)t_1(t_2t_3)-(q^3-q+1)t_1(t_3t_2)-(q-1)t_2(t_1t_3)-(q^2+q-1)t_2(t_3t_1) +(q^3-q)t_3(t_1t_2)+(q^3+q^2-q)(t_1t_3)t_2+q(t_2t_3)t_1$$

$$f_2 = (-1+q^2)t_1(t_2t_3)-t_1(t_3t_2)-(q^2-1)t_2(t_1t_3)-qt_3(t_1t_2)+(q^2+q-1)(t_2t_1)t_3+(t_2t_3)t_1$$

$$f_3 = (-q^3+q-1)t_1(t_2t_3)-t_1(t_3t_2)+(q^3-q+1)t_2(t_1t_3)-(q^2+q-1)t_2(t_3t_1) -qt_3(t_1t_2)+(q^3+q^2-q)(t_1t_2)t_3+(q^2+q)(t_2t_3)t_1$$

$$f_4 = -t_1(t_2t_3)-(1+q)t_1(t_3t_2)+t_2(t_1t_3)-(q^2+q-1)t_2(t_3t_1)-t_3(t_1t_2) +(q^2+q-1)t_3(t_2t_1)+(q^2+2q)(t_2t_3)t_1$$

$$f_5 = (1-q)t_1(t_2t_3) + (q^3-q+1)t_1(t_3t_2) - q^2t_2(t_1t_3) - (q^3-q)t_3(t_1t_2) - q(t_2t_3)t_1 + (q^3+q^2-q)(t_3t_1)t_2$$

$$f_6 = -t_1(t_2t_3) - t_1(t_3t_2) + q(t_2t_3)t_1 + q((t_3t_2)t_1).$$

We see that if $q^2 \neq 1, q \neq -2$, then

$$f_1 = \frac{1}{(q-1)(q+1)(q+2)}(-lei^{(q)}(t_1,t_2,t_3) - (-1+q+q^2)lei^{(q)}(t_2,t_1,t_3) + (-1+q+q^2)^2 lei^{(q)}(t_3,t_1,t_2) + (-1+q+q^2)lei^{(q)}(t_3,t_2,t_1)),$$

$$f_2 = lei^{(q)},$$

$$f_3 = \frac{1}{(q-1)(q+1)(q+2)}(-(1+q)lei^{(q)}(t_1,t_2,t_3) + (-1+q+q^2)^2 lei^{(q)}(t_2,t_1,t_3) - (-1+q+q^2)lei^{(q)}(t_3,t_1,t_2) + (1+q)(-1+q+q^2)lei^{(q)}(t_3,t_2,t_1)),$$

$$f_4 = \frac{1}{(q+1)(q-1)}(-lei^{(q)}(t_1,t_2,t_3) + (-1+q+q^2)lei^{(q)}(t_3,t_2,t_1)),$$

$$f_5 = \frac{1}{(q-1)(q+1)(q+2)}(lei^{(q)}(t_1,t_2,t_3) + (-1+q+q^2)^2\, lei^{(q)}(t_1,t_3,t_2) - (-1+q+q^2)lei^{(q)}(t_2,t_3,t_1) - (-1+q+q^2)lei^{(q)}(t_3,t_2,t_1)),$$

$$f_6 = \frac{1}{(q-1)(q+1)(q+2)}(-lei^{(q)}(t_1,t_2,t_3) - lei^{(q)}(t_1,t_3,t_2) + (-1+q+q^2)lei^{(q)}(t_2,t_3,t_1) + (-1+q+q^2)lei^{(q)}(t_3,t_2,t_1)).$$

Now, we consider the case $q = -2$. In this case, similar arguments show that X is a linear combination of the following polynomials

$$g_1 = t_3(t_2t_1) + 2/3(t_1t_2)t_3 + 4/3(t_1t_3)t_2 + 4/3(t_2t_1)t_3 - 4/3(t_2t_3)t_1 + 5/3(t_3t_1)t_2 - 5/3(t_3t_2)t_1,$$

$$g_2 = t_2(t_3t_1) + 4/3(t_1t_2)t_3 + 2/3(t_1t_3)t_2 + 5/3(t_2t_1)t_3 - 5/3(t_2t_3)t_1 + 4/3(t_3t_1)t_2 - 4/3(t_3t_2)t_1,$$

$$g_3 = t_1(t_2t_3) - 5/3(t_1t_2)t_3 + 5/3(t_1t_3)t_2 - 4/3(t_2t_1)t_3 + 4/3(t_2t_3)t_1$$

$$+4/3(t_3t_1)t_2+2/3(t_3t_2)t_1,$$

$$g_4=t_1(t_3t_2)+5/3(t_1t_2)t_3-5/3(t_1t_3)t_2+4/3(t_2t_1)t_3+2/3(t_2t_3)t_1$$
$$-4/3(t_3t_1)t_2+4/3(t_3t_2)t_1,$$

$$g_5=t_2(t_1t_3)-4/3(t_1t_2)t_3+4/3(t_1t_3)t_2-5/3(t_2t_1)t_3+5/3(t_2t_3)t_1$$
$$+2/3(t_3t_1)t_2+4/3(t_3t_2)t_1,$$

$$g_6=t_3(t_1t_2+4/3(t_1t_2)t_3-4/3(t_1t_3)t_2+2/3(t_2t_1)t_3+4/3(t_2t_3)t_1$$
$$-5/3(t_3t_1)t_2+5/3(t_3t_2)t_1.$$

We have

$$g_1-1/3(4lei^{(-2)}(t_1,t_2,t_3)+3lei^{(-2)}(t_1,t_3,t_2)+2lei^{(-2)}(t_2,t_1,t_3)$$
$$+4lei_1^{(-2)}(t_1,t_2,t_3)-lei_1^{(-2)}(t_2,t_1,t_3))=1/3\,ralia(t_1,t_2,t_3),$$

$$g_2-1/3(5lei^{(-2)}(t_1,t_2,t_3)+6lei^{(-2)}(t_1,t_3,t_2)+4lei^{(-2)}(t_2,t_1,t_3)$$
$$+5lei_1^{(-2)}(t_1,t_2,t_3)+lei_1^{(-2)}(t_2,t_1,t_3))=8/3\,ralia(t_1,t_2,t_3),$$

$$g_3-1/3(-4lei^{(-2)}(t_1,t_2,t_3)-6lei^{(-2)}(t_1,t_3,t_2)-5lei^{(-2)}(t_2,t_1,t_3)$$
$$-4lei_1^{(-2)}(t_1,t_2,t_3)-5lei_1^{(-2)}(t_2,t_1,t_3))=-10/3\,ralia(t_1,t_2,t_3),$$

$$g_4-1/3(4lei^{(-2)}(t_1,t_2,t_3)+6lei^{(-2)}(t_1,t_3,t_2)+5lei^{(-2)}(t_2,t_1,t_3)$$
$$+lei_1^{(-2)}(t_1,t_2,t_3)+5lei_1^{(-2)}(t_2,t_1,t_3))=10/3\,ralia(t_1,t_2,t_3),$$

$$g_5-1/3(-5lei^{(-2)}(t_1,t_2,t_3)-6lei^{(-2)}(t_1,t_3,t_2)-4lei^{(-2)}(t_2,t_1,t_3)$$
$$-5lei_1^{(-2)}(t_1,t_2,t_3)-4lei_1^{(-2)}(t_2,t_1,t_3))=-8/3\,ralia(t_1,t_2,t_3),$$

$$g_6-1/3(2lei^{(-2)}(t_1,t_2,t_3)+3lei^{(-2)}(t_1,t_3,t_2)+4lei^{(-2)}(t_2,t_1,t_3)$$
$$-lei_1^{(-2)}(t_1,t_2,t_3)+4lei_1^{(-2)}(t_2,t_1,t_3))=8/3\,ralia(t_1,t_2,t_3).$$

By Lemma 4.2 $ralia=0$ is a consequence of the identity $lei^{(q)}=0$. Therefore, all the identities $g_i=0$ are consequences of the identities $lei^{(q)}=0$ and $lei_1^{(q)}=0$.

We have proved that any identity of degree 3 of $L^{(q)}$ for $q=-2$ follows from the identities $lei^{(q)}=0$ and $lei^{(q)}_1=0$. Notice that the equation

$$lei^{(q)}_1(a,b,c)=\mu_1 lei^{(q)}(a,b,c)+\mu_2 lei^{(q)}(b,c,a)+\mu_3 lei^{(q)}(c,a,b)$$

$$+\mu_4 lei^{(q)}(b,a,c)+\mu_5 lei^{(q)}(c,b,a)+\mu_6 lei^{(q)}(a,c,b)$$

in $L^{(q)}$ with unknowns $\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6$ is not solvable. Therefore, this system of identities $lei^{(q)}=0, lei^{(q)}_1=0$ is $\mathfrak{Lei}^{(q)}$-minimal if $q=-2$.

Lemma 4.4. *Suppose that $q\neq 0,\pm 1$ and an algebra $(A,\star)$ satisfies the identities $lei^{(q)}=0$ and $lei^{(q)}_1=0$. Then the algebra $(A,\circ)$, where $a\circ b=(1-q^2)^{-1}(a\star b-q\,b\star a)$, is a (right)-Leibniz algebra, and the algebras $(A,\star)$ and $(A,\circ_q)$ are isomorphic.*

Proof. One checks that

$$lei(t_1,t_2,t_3)=$$

$$-2\,lei^{(q)}(t_1,t_2,t_3)-2/3\,lei^{(q)}(t_1,t_3,t_2)-lei^{(q)}(t_2,t_1,t_3)+2/3\,(lei^{(q)}(t_2,t_3,t_1)$$

$$-2\,lei^{(q)}_1(t_1,t_2,t_3)-lei^{(q)}_1(t_2,t_1,t_3)$$

for $q^2\neq 1, q=-2$, and

$$lei(t_1,t_2,t_3)=$$

$$\frac{1}{(q^2-1)(q+2)}(q(q+1)lei^{(q)}(t_1,t_2,t_3)-(-1+2q+q^2)lei^{(q)}(t_1,t_3,t_2)$$

$$-(q+1)\,lei^{(q)}(t_2,t_1,t_3)+(1-q+q^2+q^3)lei^{(q)}(t_2,t_3,t_1)$$

$$+(q+1)lei^{(q)}(t_3,t_1,t_2)-(q+q^2)lei^{(q)}(t_3,t_2,t_1))$$

for $q^2\neq 1, q\neq -2$.

Therefore, for any algebra $(A,\star)$ with identities $lei^{(q)}=0$ and $lei^{(q)}_1=0$ the algebra $(A,\circ)$, where $a\circ b=(1-q^2)^{-1}(a\star b-q\,b\star a)$, satisfies the identity $lei=0$. It is evident that

$$a\circ_q b$$

$$=(1-q^2)^{-1}(a\circ b+q\,b\circ a)=(1-q^2)^{-1}(a\star b-q\,b\star a+q\,b\star a-q^2\,a\star b)$$

$$=a\star b$$

Proof of Theorems 1.1 and 1.2. By Lemmas 4.1, 4.3, 4.4 our theorems are true.

5. Leibniz-Lie algebras

In this section we study identities for Leibniz-Lie algebras, i.e., algebras $(A, [\ ,\ ])$ under -1-commutator for Leibniz algebras $(A, \circ)$. Recall that

$$\begin{aligned} &leilie_1(t_1,t_2,t_3,t_4,t_5)\\ &= -ljac(t_1,t_2,t_3)(t_4t_5) + 2((ljac(t_1,t_2,t_3))t_4)t_5 - 2((ljac(t_1,t_2,t_3)t_5)t_4\\ &= -rjac(t_1,t_2,t_3)(t_4t_5) + 2((rjac(t_1,t_2,t_3))t_4)t_5 - 2((rjac(t_1,t_2,t_3)t_5)t_4\\ &= ((t_1t_2)t_3)(t_4t_5) - ((t_1t_3)t_2)(t_4t_5) + ((t_2t_3)t_1)(t_4t_5)\\ &-2(((t_1t_2)t_3)t_4)t_5 + 2(((t_1t_2)t_3)t_5)t_4 + 2(((t_1t_3)t_2)t_4)t_5\\ &-2(((t_1t_3)t_2)t_5)t_4 - 2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4), \end{aligned}$$

$$\begin{aligned} &leilie_2(t_1,t_2,t_3,t_4,t_5)\\ &= -((t_1t_2)t_3)(t_4t_5) + ((t_1t_2)t_4)(t_3t_5) - ((t_1t_2)t_5)(t_3t_4) - ((t_1t_2)(t_3t_4))t_5\\ &+((t_1t_2)(t_3t_5))t_4 - ((t_1t_2)(t_4t_5))t_3 + ((t_1t_3)t_2)(t_4t_5) - ((t_1t_3)t_4)(t_2t_5)\\ &+((t_1t_3)t_5)(t_2t_4) + ((t_1t_3)(t_2t_4))t_5 - ((t_1t_3)(t_2t_5))t_4 + ((t_1t_3)(t_4t_5))t_2\\ &-((t_1t_4)t_2)(t_3t_5) + ((t_1t_4)t_3)(t_2t_5) - ((t_1t_4)t_5)(t_2t_3) - ((t_1t_4)(t_2t_3))t_5\\ &+((t_1t_4)(t_2t_5))t_3 - ((t_1t_4)(t_3t_5))t_2 + ((t_1t_5)t_2)(t_3t_4) - ((t_1t_5)t_3)(t_2t_4)\\ &+((t_1t_5)t_4)(t_2t_3) + ((t_1t_5)(t_2t_3))t_4 - ((t_1t_5)(t_2t_4))t_3 + ((t_1t_5)(t_3t_4))t_2\\ &-2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4 - 2(((t_2t_3)t_4)t_1)t_5 + 4(((t_2t_3)t_4)t_5)t_1\\ &+2(((t_2t_3)t_5)t_1)t_4 - 4(((t_2t_3)t_5)t_4)t_1 + 2(((t_2t_4)t_1)t_3)t_5 - 2(((t_2t_4)t_1)t_5)t_3\\ &+2(((t_2t_4)t_3)t_1)t_5 - 4(((t_2t_4)t_3)t_5)t_1 - 2(((t_2t_4)t_5)t_1)t_3 + 4(((t_2t_4)t_5)t_3)t_1\\ &-2(((t_2t_5)t_1)t_3)t_4 + 2(((t_2t_5)t_1)t_4)t_3 - 2(((t_2t_5)t_3)t_1)t_4 + 4(((t_2t_5)t_3)t_4)t_1\\ &+2(((t_2t_5)t_4)t_1)t_3 - 4(((t_2t_5)t_4)t_3)t_1 - 2(((t_3t_4)t_1)t_2)t_5 + 2(((t_3t_4)t_1)t_5)t_2\\ &-2(((t_3t_4)t_2)t_1)t_5 + 4(((t_3t_4)t_2)t_5)t_1 + 2(((t_3t_4)t_5)t_1)t_2 - 4(((t_3t_4)t_5)t_2)t_1\\ &+2(((t_3t_5)t_1)t_2)t_4 - 2(((t_3t_5)t_1)t_4)t_2 + 2(((t_3t_5)t_2)t_1)t_4 - 4(((t_3t_5)t_2)t_4)t_1\\ &-2(((t_3t_5)t_4)t_1)t_2 + 4(((t_3t_5)t_4)t_2)t_1 - 2(((t_4t_5)t_1)t_2)t_3 + 2(((t_4t_5)t_1)t_3)t_2\\ &-2(((t_4t_5)t_2)t_1)t_3 + 4(((t_4t_5)t_2)t_3)t_1 + 2(((t_4t_5)t_3)t_1)t_2 - 4(((t_4t_5)t_3)t_2)t_1. \end{aligned}$$

We see that $leilie_1(t_1,t_2,t_3,t_4,t_5)$ has type $(3,2)$, i.e., it is skew-symmetric in t_1,t_2,t_3 and in t_4,t_5, and $leilie_2(t_1,t_2,t_3,t_4,t_5)$ has type $(1,4)$, is skew-symmetric in t_2,t_3,t_4,t_5.

Let

$$lei_3^{(-1)}(t_1,t_2,t_3,t_4,t_5) =$$

$$\begin{aligned} &((t_1t_2)t_3)(t_4t_5) + ((t_1t_2)t_5)(t_3t_4) + ((t_1t_2)(t_3t_4))t_5 - 2((t_1t_2)(t_3t_5))t_4\\ &+((t_1t_2)(t_4t_5))t_3 - ((t_1t_3)t_5)(t_2t_4) + ((t_1t_3)(t_2t_4))t_5 - ((t_1t_4)t_3)(t_2t_5) \end{aligned}$$

$$+((t_1t_4)t_5)(t_2t_3)+((t_1t_4)(t_2t_3))t_5-((t_1t_4)(t_2t_5))t_3+2((t_1t_4)(t_3t_5))t_2$$
$$+((t_1t_5)t_3)(t_2t_4)-((t_1t_5)(t_2t_4))t_3+((t_2t_3)t_5)(t_1t_4)+((t_2t_4)t_3)(t_1t_5)$$
$$-((t_2t_4)t_5)(t_1t_3)-2((t_2t_4)(t_3t_5))t_1-((t_2t_5)t_3)(t_1t_4)+((t_3t_5)t_4)(t_1t_2)$$
$$+2(((t_1t_2)t_3)t_5)t_4-6(((t_1t_2)t_4)t_3)t_5+6(((t_1t_2)t_4)t_5)t_3-2(((t_1t_2)t_5)t_3)t_4$$
$$-2(((t_1t_3)t_2)t_4)t_5+2(((t_1t_3)t_4)t_2)t_5+6(((t_1t_3)t_5)t_2)t_4-6(((t_1t_3)t_5)t_4)t_2$$
$$+6(((t_1t_4)t_2)t_3)t_5-6(((t_1t_4)t_2)t_5)t_3-2(((t_1t_4)t_3)t_5)t_2+2(((t_1t_4)t_5)t_3)t_2$$
$$+2(((t_1t_5)t_2)t_4)t_3-6(((t_1t_5)t_3)t_2)t_4+6(((t_1t_5)t_3)t_4)t_2-2(((t_1t_5)t_4)t_2)t_3$$
$$+2(((t_2t_3)t_1)t_4)t_5-2(((t_2t_3)t_4)t_1)t_5-6(((t_2t_3)t_5)t_1)t_4+6(((t_2t_3)t_5)t_4)t_1$$
$$-6(((t_2t_4)t_1)t_3)t_5+6(((t_2t_4)t_1)t_5)t_3+2(((t_2t_4)t_3)t_5)t_1-2(((t_2t_4)t_5)t_3)t_1$$
$$-2(((t_2t_5)t_1)t_4)t_3+6(((t_2t_5)t_3)t_1)t_4-6(((t_2t_5)t_3)t_4)t_1+2(((t_2t_5)t_4)t_1)t_3$$
$$+2(((t_3t_4)t_1)t_2)t_5-2(((t_3t_4)t_2)t_1)t_5-4(((t_3t_4)t_5)t_1)t_2+4(((t_3t_4)t_5)t_2)t_1$$
$$+4(((t_3t_5)t_1)t_2)t_4-4(((t_3t_5)t_1)t_4)t_2-4(((t_3t_5)t_2)t_1)t_4+4(((t_3t_5)t_2)t_4)t_1$$
$$+2(((t_3t_5)t_4)t_1)t_2-2(((t_3t_5)t_4)t_2)t_1+2(((t_4t_5)t_1)t_2)t_3-2(((t_4t_5)t_2)t_1)t_3$$
$$-4(((t_4t_5)t_3)t_1)t_2+4(((t_4t_5)t_3)t_2)t_1.$$

Lemma 5.1. *The identity $lei_3^{(-1)}=0$ is a consequence of the identities $leilie_1=0, leilie_2=0$ and the anti-commutativity identity.*

Proof. Let

$$R=$$
$$lei_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)+leilie_1(t_1,t_2,t_3,t_4,t_5)-2leilie_1(t_1,t_2,t_4,t_3,t_5)$$
$$+leilie_1(t_1,t_2,t_5,t_3,t_4)+leilie_1(t_1,t_3,t_4,t_2,t_5)+leilie_1(t_1,t_4,t_5,t_2,t_3)$$
$$-leilie_1(t_2,t_3,t_4,t_1,t_5)-leilie_1(t_2,t_4,t_5,t_1,t_3)+leilie_1(t_3,t_4,t_5,t_1,t_2)$$
$$+leilie_2(t_1,t_2,t_3,t_4,t_5)-leilie_2(t_2,t_1,t_3,t_4,t_5)-leilie_2(t_4,t_1,t_2,t_3,t_5).$$

Direct calculations show that

$$R=$$
$$+2((t_1t_3)(t_2t_4))t_5+((t_2t_4)(t_1t_3))t_5-((t_4t_2)(t_1t_3))t_5$$
$$+((t_2t_1)(t_3t_4))t_5+((t_4t_3)(t_1t_2))t_5$$
$$-((t_2t_3)(t_1t_4))t_5+((t_4t_1)(t_2t_3))t_5$$
$$-((t_1t_2)(t_3t_5))t_4-((t_2t_1)(t_3t_5))t_4$$
$$-((t_2t_5)(t_1t_3))t_4-((t_1t_3)(t_2t_5))t_4$$
$$+((t_1t_5)(t_2t_3))t_4+((t_2t_3)(t_1t_5))t_4$$
$$-2((t_1t_5)(t_2t_4))t_3-((t_2t_4)(t_1t_5))t_3+((t_4t_2)(t_1t_5))t_3$$
$$+((t_2t_5)(t_1t_4))t_3-((t_4t_1)(t_2t_5))t_3$$
$$+((t_2t_1)(t_4t_5))t_3-((t_4t_5)(t_1t_2))t_3$$
$$+((t_1t_3)(t_4t_5))t_2+((t_4t_5)(t_1t_3))t_2$$
$$+((t_1t_4)(t_3t_5))t_2+(((t_4t_1)(t_3t_5))t_2$$

$$
\begin{aligned}
&+((t_1t_5)(t_3t_4))t_2-((t_4t_3)(t_1t_5))t_2\\
&-((t_2t_3)(t_4t_5))t_1-((t_4t_5)(t_2t_3))t_1\\
&-((t_2t_4)(t_3t_5))t_1-((t_4t_2)(t_3t_5))t_1\\
&+((t_4t_3)(t_2t_5))t_1-((t_2t_5)(t_3t_4))t_1\\
&+((t_1t_2)t_3)(t_4t_5)+((t_2t_1)t_3)(t_4t_5)\\
&+((t_1t_4)t_2)(t_3t_5)+((t_4t_1)t_2)(t_3t_5)\\
&-((t_2t_4)t_1)(t_3t_5)-((t_4t_2)t_1)(t_3t_5))\\
&-((t_1t_2)t_4)(t_3t_5)-((t_2t_1)t_4)(t_3t_5)\\
&+((t_1t_2)t_5)(t_3t_4)+((t_2t_1)t_5)(t_3t_4)\\
&-((t_1t_4)t_3)(t_2t_5)-((t_4t_1)t_3)(t_2t_5)\\
&+((t_3t_4)t_1)(t_2t_5)+((t_4t_3)t_1)(t_2t_5)\\
&+((t_1t_4)t_5)(t_2t_3)+((t_4t_1)t_5)(t_2t_3)\\
&+((t_2t_4)t_3)(t_1t_5)+((t_4t_2)t_3)(t_1t_5)\\
&-(((t_3t_4)t_2)(t_1t_5)-((t_4t_3)t_2)(t_1t_5)\\
&-((t_2t_4)t_5)(t_1t_3)-((t_4t_2)t_5)(t_1t_3)\\
&+((t_3t_4)t_5)(t_1t_2)+((t_4t_3)t_5)(t_1t_2)\\
&=0,
\end{aligned}
$$

if $t_1t_2-t_2t_1=0$ is identity.

Lemma 5.2. *Let $(A,\circ)$ be a Leibniz algebra. Then the Leibniz-Lie algebra $(A,[\ ,\ ])$ satisfies the identities $lei_1^{(-1)}=0$, $leilie_2=0$.*

Proof. By Lemma 3.1 $leilie_1=0$ is an identity on $A^{(-1)}$.

Let us check that $leilie_2=0$. Present

$$h=h(a,b,c,d,e)$$

$$\stackrel{def}{=}-4[[[[b,c],d],e],a]+2[[[[b,c],a],d],e]+2[[[[b,c],d],a],e]$$
$$+[[[a,b],c],[d,e]]+[[[a,b],[d,e]],c]$$

in the form h_1+h_2, where

$$h_1=h_1(a,b,c,d,e)=-4[[[[b,c],d],e],a]+2[[[[b,c],a],d],e]+2[[[[b,c],d],a],e],$$
$$h_2=h_2(a,b,c,d,e)=[[[a,b],c],[d,e]]+[[[a,b],[d,e]],c].$$

Notice that

$$
\begin{aligned}
&-2[[[[[u,b],c],a]+[[[[[u,a],d],e]+[[[[[u,d],a],e]\\
&=-8(((a\circ b)\circ u)\circ c)+8(((a\circ c)\circ b)\circ u)-8(((a\circ c)\circ u)\circ b)\\
&+2(((a\circ d)\circ u)\circ e)+8(((a\circ u)\circ b)\circ c)-3(((a\circ u)\circ d)\circ e)\\
&+2(((b\circ u)\circ c)\circ a)-4(((c\circ b)\circ u)\circ a)+4(((c\circ u)\circ b)\circ a)\\
&+2(((d\circ a)\circ u)\circ e)-3(((d\circ u)\circ a)\circ e)-4(((e\circ a)\circ d)\circ u)
\end{aligned}
$$

$$+8(((e\circ a)\circ u)\circ d)-4(((e\circ d)\circ a)\circ u)+8(((e\circ d)\circ u)\circ a)$$
$$-4(((e\circ u)\circ a)\circ d)-4(((e\circ u)\circ d)\circ a)+(((u\circ a)\circ d)\circ e)$$
$$-2(((u\circ b)\circ c)\circ a)+(((u\circ d)\circ a)\circ e).$$

Therefore,

$$h_1=-4[[[[b,c],d],e],a]+2[[[[b,c],a],d],e]+2[[[[b,c],d],a],e]$$
$$=20((((a\circ b)\circ c)\circ d)\circ e)-20((((a\circ c)\circ b)\circ d)\circ e)-24((((a\circ d)\circ b)\circ c)\circ e)$$
$$+24((((a\circ d)\circ c)\circ b)\circ e)-32((((a\circ e)\circ b)\circ c)\circ d)+32((((a\circ e)\circ c)\circ b)\circ d)$$
$$+32((((a\circ e)\circ d)\circ b)\circ c)-32((((a\circ e)\circ d)\circ c)\circ b)+2((((b\circ c)\circ a)\circ d)\circ e)$$
$$+2((((b\circ c)\circ d)\circ a)\circ e)-4((((b\circ c)\circ d)\circ e)\circ a)-2((((c\circ b)\circ a)\circ d)\circ e)$$
$$-2((((c\circ b)\circ d)\circ a)\circ e)+4((((c\circ b)\circ d)\circ e)\circ a)+8((((d\circ a)\circ b)\circ c)\circ e)$$
$$-8((((d\circ a)\circ c)\circ b)\circ e)-12((((d\circ b)\circ c)\circ a)\circ e)+8((((d\circ b)\circ c)\circ e)\circ a)$$
$$+12((((d\circ c)\circ b)\circ a)\circ e)-8((((d\circ c)\circ b)\circ e)\circ a)+32((((e\circ a)\circ b)\circ c)\circ d)$$
$$-32((((e\circ a)\circ c)\circ b)\circ d)-16((((e\circ a)\circ d)\circ b)\circ c)+16((((e\circ a)\circ d)\circ c)\circ b)$$
$$-16((((e\circ b)\circ c)\circ a)\circ d)+16((((e\circ c)\circ b)\circ a)\circ d)-16((((e\circ d)\circ a)\circ b)\circ c)$$
$$+16((((e\circ d)\circ a)\circ c)\circ b)+16((((e\circ d)\circ b)\circ c)\circ a)-16((((e\circ d)\circ c)\circ b)\circ a).$$

Further,

$$[[u,b],c]+[[u,c],b]$$
$$=2((b\circ c)\circ u)-3((b\circ u)\circ c)+2((c\circ b)\circ u)-3((c\circ u)\circ b)+((u\circ b)\circ c)+((u\circ c)\circ b).$$

Thus,

$$h_2=[[[a,b],c],[d,e]]+[[[a,b],[d,e]],c]$$
$$=2((((a\circ b)\circ c)\circ d)\circ e)-2((((a\circ b)\circ c)\circ e)\circ d)+2((((a\circ b)\circ d)\circ e)\circ c)$$
$$-2((((a\circ b)\circ e)\circ d)\circ c)-2((((b\circ a)\circ c)\circ d)\circ e)+2((((b\circ a)\circ c)\circ e)\circ d)$$
$$-2((((b\circ a)\circ d)\circ e)\circ c)+2((((b\circ a)\circ e)\circ d)\circ c)-12((((c\circ a)\circ b)\circ d)\circ e)$$
$$+12((((c\circ a)\circ b)\circ e)\circ d)+12((((c\circ b)\circ a)\circ d)\circ e)-12((((c\circ b)\circ a)\circ e)\circ d)$$
$$+8((((c\circ d)\circ e)\circ a)\circ b)-8((((c\circ d)\circ e)\circ b)\circ a)-8((((c\circ e)\circ d)\circ a)\circ b)$$
$$+8((((c\circ e)\circ d)\circ b)\circ a)-6((((d\circ e)\circ a)\circ b)\circ c)+6((((d\circ e)\circ b)\circ a)\circ c)$$
$$+4((((d\circ e)\circ c)\circ a)\circ b)-4((((d\circ e)\circ c)\circ b)\circ a)+6((((e\circ d)\circ a)\circ b)\circ c)$$
$$-6((((e\circ d)\circ b)\circ a)\circ c)-4((((e\circ d)\circ c)\circ a)\circ b)+4((((e\circ d)\circ c)\circ b)\circ a).$$

Hence, $h=h_1+h_2$ can be presented in the form

$$h=r_1+r_2+r_3+r_4+r_5,$$

where $r_i=r_i(a,b,c,d,e), i=1,2,3,4,5,$ and

$$r_1=22((((a\circ b)\circ c)\circ d)\circ e)-2((((a\circ b)\circ c)\circ e)\circ d)+2((((a\circ b)\circ d)\circ e)\circ c)$$
$$-2((((a\circ b)\circ e)\circ d)\circ c)-20((((a\circ c)\circ b)\circ d)\circ e)-24((((a\circ d)\circ b)\circ c)\circ e)$$
$$+24((((a\circ d)\circ c)\circ b)\circ e)-32((((a\circ e)\circ b)\circ c)\circ d)+32((((a\circ e)\circ c)\circ b)\circ d)$$
$$+32((((a\circ e)\circ d)\circ b)\circ c)-32((((a\circ e)\circ d)\circ c)\circ b),$$

$$r_2 = -2((((b\circ a)\circ c)\circ d)\circ e)+2((((b\circ a)\circ c)\circ e)\circ d)-2((((b\circ a)\circ d)\circ e)\circ c)$$
$$+2((((b\circ a)\circ e)\circ d)\circ c)-12((((c\circ a)\circ b)\circ d)\circ e)+12((((c\circ a)\circ b)\circ e)\circ d)$$
$$+8((((d\circ a)\circ b)\circ c)\circ e)-8((((d\circ a)\circ c)\circ b)\circ e)+32((((e\circ a)\circ b)\circ c)\circ d)$$
$$-32((((e\circ a)\circ c)\circ b)\circ d)-16((((e\circ a)\circ d)\circ b)\circ c)+16((((e\circ a)\circ d)\circ c)\circ b),$$

$$r_3 = 2((((b\circ c)\circ a)\circ d)\circ e)+10((((c\circ b)\circ a)\circ d)\circ e)-12((((c\circ b)\circ a)\circ e)\circ d)$$
$$-6((((d\circ e)\circ a)\circ b)\circ c)-10((((e\circ d)\circ a)\circ b)\circ c)+16((((e\circ d)\circ a)\circ c)\circ b),$$

$$r_4 = 2((((b\circ c)\circ d)\circ a)\circ e)-2((((c\circ b)\circ d)\circ a)\circ e)+8((((c\circ d)\circ e)\circ a)\circ b)$$
$$-8((((c\circ e)\circ d)\circ a)\circ b)-12((((d\circ b)\circ c)\circ a)\circ e)+12((((d\circ c)\circ b)\circ a)\circ e)$$
$$+6((((d\circ e)\circ b)\circ a)\circ c)+4((((d\circ e)\circ c)\circ a)\circ b)-16((((e\circ b)\circ c)\circ a)\circ d)$$
$$+16((((e\circ c)\circ b)\circ a)\circ d)-6((((e\circ d)\circ b)\circ a)\circ c)-4((((e\circ d)\circ c)\circ a)\circ b),$$

$$r_5 = -4((((b\circ c)\circ d)\circ e)\circ a)+4((((c\circ b)\circ d)\circ e)\circ a)-8((((c\circ d)\circ e)\circ b)\circ a)$$
$$+8((((c\circ e)\circ d)\circ b)\circ a)+8((((d\circ b)\circ c)\circ e)\circ a)-8((((d\circ c)\circ b)\circ e)\circ a)$$
$$-4((((d\circ e)\circ c)\circ b)\circ a)+16((((e\circ d)\circ b)\circ c)\circ a)-12((((e\circ d)\circ c)\circ b)\circ a).$$

Show that $Alt(r_1)=0$. Let us collect the coefficients of $((((a\circ b)\circ c)\circ d)\circ e)$ in r_1. We see that this yields

$$22\,sign\begin{pmatrix} a & b & c & d & e\\ a & b & c & d & e\end{pmatrix} - 2\,sign\begin{pmatrix} a & b & c & d & e\\ a & b & c & e & d\end{pmatrix}$$
$$+2\,sign\begin{pmatrix} a & b & c & d & e\\ a & b & d & e & c\end{pmatrix} - 2\,sign\begin{pmatrix} a & b & c & d & e\\ a & b & e & d & c\end{pmatrix}$$
$$-20\,sign\begin{pmatrix} a & b & c & d & e\\ a & c & b & d & e\end{pmatrix} - 24\,sign\begin{pmatrix} a & b & c & d & e\\ a & d & b & c & e\end{pmatrix}$$
$$+24\,sign\begin{pmatrix} a & b & c & d & e\\ a & d & c & b & e\end{pmatrix} - 32\,sign\begin{pmatrix} a & b & c & d & e\\ a & e & b & c & d\end{pmatrix}$$
$$+32\,sign\begin{pmatrix} a & b & c & d & e\\ a & e & c & b & d\end{pmatrix} + 32\,sign\begin{pmatrix} a & b & c & d & e\\ a & e & d & b & c\end{pmatrix}$$
$$-32\,sign\begin{pmatrix} a & b & c & d & e\\ a & e & d & c & b\end{pmatrix}$$
$$= 22+2+2+2+20-24-24+32+32-32-32=0.$$

So, by skew-symmetry, $r_1=0$. Analogious calculations show that

$$Alt(r_i)=0$$

for all $i = 1, 2, 3, 4, 5$. Thus,

$$leilie_2 = Alt(r_1) + \cdots + Alt(r_5) = 0.$$

Lemma 5.3. *Any identity of degree 4 for $\mathfrak{Lei}^{(-1)}$ follows from the identity $acom = 0$.*

Proof. Let

$$X_4(t_1,t_2,t_3,t_4) =$$

$$\lambda_1(t_1t_2)(t_3t_4)+\lambda_2(t_1t_3)(t_2t_4)+\lambda_3(t_2t_3)(t_1t_4)+\lambda_4((t_1t_2)t_3)t_4+\lambda_{10}((t_1t_2)t_4)t_3$$
$$+\lambda_5((t_1t_3)t_2)t_4+\lambda_{11}((t_1t_3)t_4)t_2+\lambda_6((t_1t_4)t_2)t_3+\lambda_{12}((t_1t_4)t_3)t_2+\lambda_7((t_2t_3)t_1)t_4$$
$$+\lambda_{13}((t_2t_3)t_4)t_1+\lambda_8((t_2t_4)t_1)t_3+\lambda_{14}((t_2t_4)t_3)t_1+\lambda_9((t_3t_4)t_1)t_2+\lambda_{15}((t_3t_4)t_2)t_1$$

be a generic skew-symmetric polynomial of degree 4. For t_1, t_2, t_3, t_4, we substitute the elements a, b, c, d of the free Leibniz algebra, and calculate $X_4(a, b, c, d)$ under the commutator $[u, v] = u \circ v - v \circ u$. We obtain

$$X_4(a,b,c,d) =$$
$$(2\lambda_1 + \lambda_4 - 2\lambda_7 - 4\lambda_{13} + 4\lambda_{15})(((a \circ b) \circ c) \circ d)$$
$$+(-2\lambda_1 - 2\lambda_8 + \lambda_{10} - 4\lambda_{14} - 4\lambda_{15})(((a \circ b) \circ d) \circ c)$$
$$+(2\lambda_2 + \lambda_5 + 2\lambda_7 + 4\lambda_{13} + 4\lambda_{14})(((a \circ c) \circ b) \circ d)$$
$$+(-2\lambda_2 - 2\lambda_9 + \lambda_{11} - 4\lambda_{14} - 4\lambda_{15})(((a \circ c) \circ d) \circ b)$$
$$+(-2\lambda_3 + \lambda_6 + 2\lambda_8 + 4\lambda_{13} + 4\lambda_{14})(((a \circ d) \circ b) \circ c)$$
$$+(2\lambda_3 + 2\lambda_9 + \lambda_{12} - 4\lambda_{13} + 4\lambda_{15})(((a \circ d) \circ c) \circ b)$$
$$+(-2\lambda_1 - \lambda_4 - 2\lambda_5 + 4\lambda_9 - 4\lambda_{11})(((b \circ a) \circ c) \circ d)$$
$$+(2\lambda_1 - 2\lambda_6 - 4\lambda_9 - \lambda_{10} - 4\lambda_{12})(((b \circ a) \circ d) \circ c)$$
$$+(2\lambda_3 + 2\lambda_5 + \lambda_7 + 4\lambda_{11} + 4\lambda_{12})(((b \circ c) \circ a) \circ d)$$
$$+(-2\lambda_3 - 4\lambda_9 - 4\lambda_{12} + \lambda_{13} - 2\lambda_{15})(((b \circ c) \circ d) \circ a)$$
$$+(-2\lambda_2 + 2\lambda_6 + \lambda_8 + 4\lambda_{11} + 4\lambda_{12})(((b \circ d) \circ a) \circ c)$$
$$+(2\lambda_2 + 4\lambda_9 - 4\lambda_{11} + \lambda_{14} + 2\lambda_{15})(((b \circ d) \circ c) \circ a)$$
$$+(-2\lambda_2 - 2\lambda_4 - \lambda_5 + 4\lambda_8 - 4\lambda_{10})(((c \circ a) \circ b) \circ d)$$
$$+(2\lambda_2 - 4\lambda_6 - 4\lambda_8 - \lambda_{11} - 2\lambda_{12})(((c \circ a) \circ d) \circ b)$$
$$+(-2\lambda_3 + 2\lambda_4 + 4\lambda_6 - \lambda_7 + 4\lambda_{10})(((c \circ b) \circ a) \circ d)$$
$$+(2\lambda_3 - 4\lambda_6 - 4\lambda_8 - \lambda_{13} - 2\lambda_{14})(((c \circ b) \circ d) \circ a)$$
$$+(-2\lambda_1 + 4\lambda_6 + \lambda_9 + 4\lambda_{10} + 2\lambda_{12})(((c \circ d) \circ a) \circ b)$$
$$+(2\lambda_1 + 4\lambda_8 - 4\lambda_{10} + 2\lambda_{14} + \lambda_{15})(((c \circ d) \circ b) \circ a)$$
$$+(2\lambda_3 - 4\lambda_4 - \lambda_6 + 4\lambda_7 - 2\lambda_{10})(((d \circ a) \circ b) \circ c)$$
$$+(-2\lambda_3 - 4\lambda_5 - 4\lambda_7 - 2\lambda_{11} - \lambda_{12})(((d \circ a) \circ c) \circ b)$$
$$+(2\lambda_2 + 4\lambda_4 + 4\lambda_5 - \lambda_8 + 2\lambda_{10})(((d \circ b) \circ a) \circ c)$$
$$+(-2\lambda_2 - 4\lambda_5 - 4\lambda_7 - 2\lambda_{13} - \lambda_{14})(((d \circ b) \circ c) \circ a)$$

$$+(2\lambda_1+4\lambda_4+4\lambda_5-\lambda_9+2\lambda_{11})(((d\circ c)\circ a)\circ b)$$
$$+(-2\lambda_1-4\lambda_4+4\lambda_7+2\lambda_{13}-\lambda_{15})(((d\circ c)\circ b)\circ a).$$

Since all 24 left-bracketed elements like $(((a\circ b)\circ c)\circ d$ are basis elements, the condition $X_4(a,b,c,d)=0$ gives us the system of 24 linear equations in 15 unknowns $\lambda_i, i=1,\dots,15$. We see that the rank of this system is 15 and our system has the trivial solution only: $\lambda_i=0$ for all $i=1,2,\dots,15$. In other words, any polylinear identity of degree 4 for $\mathfrak{Lei}^{(-1)}$ follows from the identity $acom=0$.

Lemma 5.4. *Any identity of a free Leibniz algebra of degree 5 follows from the identities* $leilie_1=0$, $leilie_2=0$, $lei_3^{(-1)}=0$.

Proof. Let $f=f(t_1,\dots,t_5)$ be a non-commutative non-associative polynomial such that $f=0$ is an identity for any right-Leibniz algebra. Notice that there exist 105 anti-commutative non-associative polynomials. Present f as a linear combination of these 105 elements.

Insert in f the elements of the free Leibniz algebra generated by 5 elements u_1,u_2,u_3,u_4,u_5 and calculate the polynomial f under the commutator $[u,v]=u\circ v-v\circ u$, where $(u,v)\mapsto u\circ v$ is the multiplication in a free (right)-Leibniz algebra. Expand this expression in terms of the multiplication $\circ$ by using the Leibniz rule

$$u\circ(v\circ w)=(u\circ v)\circ w-(u\circ w)\circ v.$$

We obtain the element that is a linear combination of 120 elements of the form $(((u_{\sigma(1)}\circ u_{\sigma(2)})\circ u_{\sigma(3)})\circ u_{\sigma(4)})\circ u_{\sigma(5)}$, where $\sigma\in Sym_5$. The identity condition $f=0$ on $L^{(-1)}$ gives us 120 linear equations in 105 unknowns λ_i. Solve this system of equations. We do this using the computer system Matematica. We find out that the system has 14 free parameters. It shows that f can be presented as a linear combination of the following 14 polynomials

$$f_1=leilie_1,$$

$$f_2(t_1,t_2,t_3,t_4,t_5)=$$
$$((t_1t_2)t_4)(t_3t_5)-((t_1t_4)t_2)(t_3t_5)+((t_2t_4)t_1)(t_3t_5)$$
$$-2(((t_1t_2)t_4)t_3)t_5+2(((t_1t_2)t_4)t_5)t_3+2(((t_1t_4)t_2)t_3)t_5$$
$$-2(((t_1t_4)t_2)t_5)t_3-2(((t_2t_4)t_1)t_3)t_5+2(((t_2t_4)t_1)t_5)t_3,$$

$$f_3(t_1,t_2,t_3,t_4,t_5)=$$
$$((t_1t_2)t_5)(t_3t_4)-((t_1t_5)t_2)(t_3t_4)+((t_2t_5)t_1)(t_3t_4)$$
$$-2(((t_1t_2)t_5)t_3)t_4+2(((t_1t_2)t_5)t_4)t_3+2(((t_1t_5)t_2)t_3)t_4$$

$$-2(((t_1t_5)t_2)t_4)t_3 - 2(((t_2t_5)t_1)t_3)t_4 + 2(((t_2t_5)t_1)t_4)t_3,$$

$$f_4(t_1,t_2,t_3,t_4,t_5) =$$
$$((t_1t_3)t_4)(t_2t_5) - ((t_1t_4)t_3)(t_2t_5) + ((t_3t_4)t_1)(t_2t_5)$$
$$-2(((t_1t_3)t_4)t_2)t_5 + 2(((t_1t_3)t_4)t_5)t_2 + 2(((t_1t_4)t_3)t_2)t_5$$
$$-2(((t_1t_4)t_3)t_5)t_2 - 2(((t_3t_4)t_1)t_2)t_5 + 2(((t_3t_4)t_1)t_5)t_2,$$

$$f_5(t_1,t_2,t_3,t_4,t_5) =$$
$$((t_2t_3)t_4)(t_1t_5) - ((t_2t_4)t_3)(t_1t_5) + ((t_3t_4)t_2)(t_1t_5)$$
$$-2(((t_2t_3)t_4)t_1)t_5 + 2(((t_2t_3)t_4)t_5)t_1 + 2(((t_2t_4)t_3)t_1)t_5$$
$$-2(((t_2t_4)t_3)t_5)t_1 - 2(((t_3t_4)t_2)t_1)t_5 + 2(((t_3t_4)t_2)t_5)t_1,$$

$$f_6(t_1,t_2,t_3,t_4,t_5) =$$
$$((t_1t_3)t_5)(t_2t_4) - ((t_1t_5)t_3)(t_2t_4) + ((t_3t_5)t_1)(t_2t_4)$$
$$-2(((t_1t_3)t_5)t_2)t_4 + 2(((t_1t_3)t_5)t_4)t_2 + 2(((t_1t_5)t_3)t_2)t_4$$
$$-2(((t_1t_5)t_3)t_4)t_2 - 2(((t_3t_5)t_1)t_2)t_4 + 2(((t_3t_5)t_1)t_4)t_2,$$

$$f_7(t_1,t_2,t_3,t_4,t_5) =$$
$$((t_2t_3)t_5)(t_1t_4) - ((t_2t_5)t_3)(t_1t_4) + ((t_3t_5)t_2)(t_1t_4)$$
$$-2(((t_2t_3)t_5)t_1)t_4 + 2(((t_2t_3)t_5)t_4)t_1 + 2(((t_2t_5)t_3)t_1)t_4$$
$$-2(((t_2t_5)t_3)t_4)t_1 - 2(((t_3t_5)t_2)t_1)t_4 + 2(((t_3t_5)t_2)t_4)t_1,$$

$$f_8 = leilie_2,$$

$$f_9(t_1,t_2,t_3,t_4,t_5) =$$
$$-((t_1t_2)t_3)(t_4t_5) + ((t_1t_2)t_4)(t_3t_5) - ((t_1t_2)t_5)(t_3t_4) - ((t_1t_2)(t_3t_4))t_5$$
$$+((t_1t_2)(t_3t_5))t_4 - ((t_1t_2)(t_4t_5))t_3 + ((t_1t_3)t_2)(t_4t_5) - ((t_1t_3)t_4)(t_2t_5)$$
$$+((t_1t_3)t_5)(t_2t_4) + ((t_1t_3)(t_2t_4))t_5 - ((t_1t_3)(t_2t_5))t_4 + ((t_1t_3)(t_4t_5))t_2$$
$$-((t_1t_4)t_2)(t_3t_5) + ((t_1t_4)t_3)(t_2t_5) - ((t_1t_4)(t_2t_3))t_5 + ((t_1t_4)(t_2t_5))t_3$$
$$-((t_1t_4)(t_3t_5))t_2 + ((t_1t_5)t_2)(t_3t_4) - ((t_1t_5)t_3)(t_2t_4) + ((t_1t_5)(t_2t_3))t_4$$
$$-((t_1t_5)(t_2t_4))t_3 + ((t_1t_5)(t_3t_4))t_2 + ((t_4t_5)t_1)(t_2t_3) - 2(((t_1t_4)t_5)t_2)t_3$$
$$+2(((t_1t_4)t_5)t_3)t_2 + 2(((t_1t_5)t_4)t_2)t_3 - 2(((t_1t_5)t_4)t_3)t_2$$
$$-2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4 - 2(((t_2t_3)t_4)t_1)t_5 + 4(((t_2t_3)t_4)t_5)t_1$$
$$+2(((t_2t_3)t_5)t_1)t_4 - 4(((t_2t_3)t_5)t_4)t_1 + 2(((t_2t_4)t_1)t_3)t_5 - 2(((t_2t_4)t_1)t_5)t_3$$
$$+2(((t_2t_4)t_3)t_1)t_5 - 4(((t_2t_4)t_3)t_5)t_1 - 2(((t_2t_4)t_5)t_1)t_3 + 4(((t_2t_4)t_5)t_3)t_1$$
$$-2(((t_2t_5)t_1)t_3)t_4 + 2(((t_2t_5)t_1)t_4)t_3 - 2(((t_2t_5)t_3)t_1)t_4 + 4(((t_2t_5)t_3)t_4)t_1$$

$$+2(((t_2t_5)t_4)t_1)t_3-4(((t_2t_5)t_4)t_3)t_1-2(((t_3t_4)t_1)t_2)t_5+2(((t_3t_4)t_1)t_5)t_2$$
$$-2(((t_3t_4)t_2)t_1)t_5+4(((t_3t_4)t_2)t_5)t_1+2(((t_3t_4)t_5)t_1)t_2-4(((t_3t_4)t_5)t_2)t_1$$
$$+2(((t_3t_5)t_1)t_2)t_4-2(((t_3t_5)t_1)t_4)t_2+2(((t_3t_5)t_2)t_1)t_4-4(((t_3t_5)t_2)t_4)t_1$$
$$-2(((t_3t_5)t_4)t_1)t_2+4(((t_3t_5)t_4)t_2)t_1-4(((t_4t_5)t_1)t_2)t_3+4(((t_4t_5)t_1)t_3)t_2$$
$$-2(((t_4t_5)t_2)t_1)t_3+4(((t_4t_5)t_2)t_3)t_1+2(((t_4t_5)t_3)t_1)t_2-4(((t_4t_5)t_3)t_2)t_1,$$

$$f_{10}(t_1,t_2,t_3,t_4,t_5)=$$
$$((t_1t_2)(t_3t_4))t_5-((t_1t_2)(t_3t_5))t_4+((t_1t_2)(t_4t_5))t_3+((t_1t_3)t_2)(t_4t_5)$$
$$+((t_1t_3)(t_2t_4))t_5-((t_1t_3)(t_2t_5))t_4-((t_1t_4)t_2)(t_3t_5)-((t_1t_4)(t_2t_3))t_5$$
$$+((t_1t_4)(t_2t_5))t_3+((t_1t_5)t_2)(t_3t_4)+((t_1t_5)(t_2t_3))t_4-((t_1t_5)(t_2t_4))t_3$$
$$-((t_2t_3)t_4)(t_1t_5)+((t_2t_3)t_5)(t_1t_4)+((t_2t_3)(t_4t_5))t_1+((t_2t_4)t_3)(t_1t_5)$$
$$-((t_2t_4)t_5)(t_1t_3)-((t_2t_4)(t_3t_5))t_1-((t_2t_5)t_3)(t_1t_4)+((t_2t_5)t_4)(t_1t_3)$$
$$+((t_2t_5)(t_3t_4))t_1+2(((t_1t_2)t_3)t_4)t_5-2(((t_1t_2)t_3)t_5)t_4$$
$$-2(((t_1t_2)t_4)t_3)t_5+2(((t_1t_2)t_4)t_5)t_3+2(((t_1t_2)t_5)t_3)t_4-2(((t_1t_2)t_5)t_4)t_3$$
$$-4(((t_1t_3)t_2)t_4)t_5+4(((t_1t_3)t_2)t_5)t_4-2(((t_1t_3)t_4)t_2)t_5+4(((t_1t_3)t_4)t_5)t_2$$
$$+2(((t_1t_3)t_5)t_2)t_4-4(((t_1t_3)t_5)t_4)t_2+4(((t_1t_4)t_2)t_3)t_5-4(((t_1t_4)t_2)t_5)t_3$$
$$+2(((t_1t_4)t_3)t_2)t_5-4(((t_1t_4)t_3)t_5)t_2-2(((t_1t_4)t_5)t_2)t_3+4(((t_1t_4)t_5)t_3)t_2$$
$$-4(((t_1t_5)t_2)t_3)t_4+4(((t_1t_5)t_2)t_4)t_3-2(((t_1t_5)t_3)t_2)t_4+4(((t_1t_5)t_3)t_4)t_2$$
$$+2(((t_1t_5)t_4)t_2)t_3-4(((t_1t_5)t_4)t_3)t_2+2(((t_2t_3)t_1)t_4)t_5-2(((t_2t_3)t_1)t_5)t_4$$
$$-2(((t_2t_4)t_1)t_3)t_5+2(((t_2t_4)t_1)t_5)t_3+2(((t_2t_5)t_1)t_3)t_4-2(((t_2t_5)t_1)t_4)t_3$$
$$-2(((t_3t_4)t_1)t_2)t_5+4(((t_3t_4)t_1)t_5)t_2-2(((t_3t_4)t_2)t_1)t_5+2(((t_3t_4)t_2)t_5)t_1$$
$$-4(((t_3t_4)t_5)t_1)t_2+2(((t_3t_4)t_5)t_2)t_1+2(((t_3t_5)t_1)t_2)t_4-4(((t_3t_5)t_1)t_4)t_2$$
$$+2(((t_3t_5)t_2)t_1)t_4-2(((t_3t_5)t_2)t_4)t_1+4(((t_3t_5)t_4)t_1)t_2-2(((t_3t_5)t_4)t_2)t_1$$
$$-2(((t_4t_5)t_1)t_2)t_3+4(((t_4t_5)t_1)t_3)t_2-2(((t_4t_5)t_2)t_1)t_3+2(((t_4t_5)t_2)t_3)t_1$$
$$-4(((t_4t_5)t_3)t_1)t_2+2(((t_4t_5)t_3)t_2)t_1,$$

$$f_{11}(t_1,t_2,t_3,t_4,t_5)=$$
$$((t_1t_2)(t_3t_4))t_5-((t_1t_2)(t_3t_5))t_4+((t_1t_2)(t_4t_5))t_3+((t_1t_3)t_2)(t_4t_5)$$
$$+((t_1t_3)(t_2t_4))t_5-((t_1t_3)(t_2t_5))t_4-((t_1t_4)t_2)(t_3t_5)-((t_1t_4)(t_2t_3))t_5$$
$$+((t_1t_4)(t_2t_5))t_3+((t_1t_5)t_2)(t_3t_4)+((t_1t_5)(t_2t_3))t_4-((t_1t_5)(t_2t_4))t_3$$
$$-((t_2t_3)t_4)(t_1t_5)+((t_2t_3)t_5)(t_1t_4)+((t_2t_3)(t_4t_5))t_1+((t_2t_4)t_3)(t_1t_5)$$
$$-((t_2t_4)(t_3t_5))t_1-((t_2t_5)t_3)(t_1t_4)+((t_2t_5)(t_3t_4))t_1+((t_4t_5)t_2)(t_1t_3)$$
$$+2(((t_1t_2)t_3)t_4)t_5-2(((t_1t_2)t_3)t_5)t_4-2(((t_1t_2)t_4)t_3)t_5+2(((t_1t_2)t_4)t_5)t_3$$
$$+2(((t_1t_2)t_5)t_3)t_4-2(((t_1t_2)t_5)t_4)t_3-4(((t_1t_3)t_2)t_4)t_5+4(((t_1t_3)t_2)t_5)t_4$$
$$-2(((t_1t_3)t_4)t_2)t_5+4(((t_1t_3)t_4)t_5)t_2+2(((t_1t_3)t_5)t_2)t_4-4(((t_1t_3)t_5)t_4)t_2$$
$$+4(((t_1t_4)t_2)t_3)t_5-4(((t_1t_4)t_2)t_5)t_3+2(((t_1t_4)t_3)t_2)t_5-4(((t_1t_4)t_3)t_5)t_2$$

$$-2(((t_1t_4)t_5)t_2)t_3+4(((t_1t_4)t_5)t_3)t_2-4(((t_1t_5)t_2)t_3)t_4+4(((t_1t_5)t_2)t_4)t_3$$
$$-2(((t_1t_5)t_3)t_2)t_4+4(((t_1t_5)t_3)t_4)t_2+2(((t_1t_5)t_4)t_2)t_3-4(((t_1t_5)t_4)t_3)t_2$$
$$+2(((t_2t_3)t_1)t_4)t_5-2(((t_2t_3)t_1)t_5)t_4-2(((t_2t_4)t_1)t_3)t_5+2(((t_2t_4)t_1)t_5)t_3$$
$$-2(((t_2t_4)t_5)t_1)t_3+2(((t_2t_4)t_5)t_3)t_1+2(((t_2t_5)t_1)t_3)t_4-2(((t_2t_5)t_1)t_4)t_3$$
$$+2(((t_2t_5)t_4)t_1)t_3-2(((t_2t_5)t_4)t_3)t_1-2(((t_3t_4)t_1)t_2)t_5+4(((t_3t_4)t_1)t_5)t_2$$
$$-2(((t_3t_4)t_2)t_1)t_5+2(((t_3t_4)t_2)t_5)t_1-4(((t_3t_4)t_5)t_1)t_2+2(((t_3t_4)t_5)t_2)t_1$$
$$+2(((t_3t_5)t_1)t_2)t_4-4(((t_3t_5)t_1)t_4)t_2+2(((t_3t_5)t_2)t_1)t_4-2(((t_3t_5)t_2)t_4)t_1$$
$$+4(((t_3t_5)t_4)t_1)t_2-2(((t_3t_5)t_4)t_2)t_1-2(((t_4t_5)t_1)t_2)t_3+4(((t_4t_5)t_1)t_3)t_2$$
$$-4(((t_4t_5)t_2)t_1)t_3+4(((t_4t_5)t_2)t_3)t_1-4(((t_4t_5)t_3)t_1)t_2+2(((t_4t_5)t_3)t_2)t_1,$$

$$f_{12}(t_1,t_2,t_3,t_4,t_5)=$$
$$((t_1t_2)t_5)(t_3t_4)-((t_1t_2)(t_3t_5))t_4+((t_1t_2)(t_4t_5))t_3-((t_1t_3)t_5)(t_2t_4)$$
$$+((t_1t_3)(t_2t_5))t_4-((t_1t_3)(t_4t_5))t_2+((t_1t_4)t_5)(t_2t_3)-((t_1t_4)(t_2t_5))t_3$$
$$+((t_1t_4)(t_3t_5))t_2+((t_1t_5)(t_2t_3))t_4-((t_1t_5)(t_2t_4))t_3+((t_1t_5)(t_3t_4))t_2$$
$$+((t_2t_3)t_5)(t_1t_4)+((t_2t_3)(t_4t_5))t_1-((t_2t_4)t_5)(t_1t_3)-((t_2t_4)(t_3t_5))t_1$$
$$-((t_2t_5)(t_3t_4))t_1+((t_3t_4)t_5)(t_1t_2)+4(((t_1t_2)t_3)t_4)t_5-2(((t_1t_2)t_3)t_5)t_4$$
$$-4(((t_1t_2)t_4)t_3)t_5+2(((t_1t_2)t_4)t_5)t_3-4(((t_1t_2)t_5)t_3)t_4+4(((t_1t_2)t_5)t_4)t_3$$
$$-4(((t_1t_3)t_2)t_4)t_5+2(((t_1t_3)t_2)t_5)t_4+4(((t_1t_3)t_4)t_2)t_5-2(((t_1t_3)t_4)t_5)t_2$$
$$+4(((t_1t_3)t_5)t_2)t_4-4(((t_1t_3)t_5)t_4)t_2+4(((t_1t_4)t_2)t_3)t_5-2(((t_1t_4)t_2)t_5)t_3$$
$$-4(((t_1t_4)t_3)t_2)t_5+2(((t_1t_4)t_3)t_5)t_2-4(((t_1t_4)t_5)t_2)t_3+4(((t_1t_4)t_5)t_3)t_2$$
$$+2(((t_1t_5)t_2)t_3)t_4-2(((t_1t_5)t_2)t_4)t_3-2(((t_1t_5)t_3)t_2)t_4+2(((t_1t_5)t_3)t_4)t_2$$
$$+2(((t_1t_5)t_4)t_2)t_3-2(((t_1t_5)t_4)t_3)t_2+4(((t_2t_3)t_1)t_4)t_5-2(((t_2t_3)t_1)t_5)t_4$$
$$-4(((t_2t_3)t_4)t_1)t_5+2(((t_2t_3)t_4)t_5)t_1-4(((t_2t_3)t_5)t_1)t_4+4(((t_2t_3)t_5)t_4)t_1$$
$$-4(((t_2t_4)t_1)t_3)t_5+2(((t_2t_4)t_1)t_5)t_3+4(((t_2t_4)t_3)t_1)t_5-2(((t_2t_4)t_3)t_5)t_1$$
$$+4(((t_2t_4)t_5)t_1)t_3-4(((t_2t_4)t_5)t_3)t_1-2(((t_2t_5)t_1)t_3)t_4+2(((t_2t_5)t_1)t_4)t_3$$
$$+2(((t_2t_5)t_3)t_1)t_4-2(((t_2t_5)t_3)t_4)t_1-2(((t_2t_5)t_4)t_1)t_3+2(((t_2t_5)t_4)t_3)t_1$$
$$+4(((t_3t_4)t_1)t_2)t_5-2(((t_3t_4)t_1)t_5)t_2-4(((t_3t_4)t_2)t_1)t_5+2(((t_3t_4)t_2)t_5)t_1$$
$$-4(((t_3t_4)t_5)t_1)t_2+4(((t_3t_4)t_5)t_2)t_1+2(((t_3t_5)t_1)t_2)t_4-2(((t_3t_5)t_1)t_4)t_2$$
$$-2(((t_3t_5)t_2)t_1)t_4+2(((t_3t_5)t_2)t_4)t_1+2(((t_3t_5)t_4)t_1)t_2-2(((t_3t_5)t_4)t_2)t_1$$
$$-2(((t_4t_5)t_1)t_2)t_3+2(((t_4t_5)t_1)t_3)t_2+2(((t_4t_5)t_2)t_1)t_3-2(((t_4t_5)t_2)t_3)t_1$$
$$-2(((t_4t_5)t_3)t_1)t_2+2(((t_4t_5)t_3)t_2)t_1,$$

$$f_{13}=lei_3^{(-1)},$$

$$f_{14}(t_1,t_2,t_3,t_4,t_5)=$$
$$((t_1t_2)t_3)(t_4t_5)+((t_1t_2)(t_3t_4))t_5-((t_1t_2)(t_3t_5))t_4+((t_1t_3)(t_2t_4))t_5$$

$$-((t_1t_3)(t_2t_5))t_4 + ((t_1t_3)(t_4t_5))t_2 - ((t_1t_4)t_3)(t_2t_5) + ((t_1t_4)(t_2t_3))t_5$$
$$+((t_1t_4)(t_3t_5))t_2 + ((t_1t_5)t_3)(t_2t_4) - ((t_1t_5)(t_2t_3))t_4 - ((t_1t_5)(t_3t_4))t_2$$
$$-((t_2t_3)(t_4t_5))t_1 + ((t_2t_4)t_3)(t_1t_5) - ((t_2t_4)(t_3t_5))t_1 - ((t_2t_5)t_3)(t_1t_4)$$
$$+((t_2t_5)(t_3t_4))t_1 + ((t_4t_5)t_3)(t_1t_2) - 4(((t_1t_2)t_3)t_4)t_5 + 4(((t_1t_2)t_3)t_5)t_4$$
$$-2(((t_1t_2)t_4)t_3)t_5 + 4(((t_1t_2)t_4)t_5)t_3 + 2(((t_1t_2)t_5)t_3)t_4 - 4(((t_1t_2)t_5)t_4)t_3$$
$$+2(((t_1t_3)t_2)t_4)t_5 - 2(((t_1t_3)t_2)t_5)t_4 - 2(((t_1t_3)t_4)t_2)t_5 + 2(((t_1t_3)t_4)t_5)t_2$$
$$+2(((t_1t_3)t_5)t_2)t_4 - 2(((t_1t_3)t_5)t_4)t_2 + 2(((t_1t_4)t_2)t_3)t_5 - 4(((t_1t_4)t_2)t_5)t_3$$
$$+4(((t_1t_4)t_3)t_2)t_5 - 4(((t_1t_4)t_3)t_5)t_2 + 4(((t_1t_4)t_5)t_2)t_3 - 2(((t_1t_4)t_5)t_3)t_2$$
$$-2(((t_1t_5)t_2)t_3)t_4 + 4(((t_1t_5)t_2)t_4)t_3 - 4(((t_1t_5)t_3)t_2)t_4 + 4(((t_1t_5)t_3)t_4)t_2$$
$$-4(((t_1t_5)t_4)t_2)t_3 + 2(((t_1t_5)t_4)t_3)t_2 - 2(((t_2t_3)t_1)t_4)t_5 + 2(((t_2t_3)t_1)t_5)t_4$$
$$+2(((t_2t_3)t_4)t_1)t_5 - 2(((t_2t_3)t_4)t_5)t_1 - 2(((t_2t_3)t_5)t_1)t_4 + 2(((t_2t_3)t_5)t_4)t_1$$
$$-2(((t_2t_4)t_1)t_3)t_5 + 4(((t_2t_4)t_1)t_5)t_3 - 4(((t_2t_4)t_3)t_1)t_5 + 4(((t_2t_4)t_3)t_5)t_1$$
$$-4(((t_2t_4)t_5)t_1)t_3 + 2(((t_2t_4)t_5)t_3)t_1 + 2(((t_2t_5)t_1)t_3)t_4 - 4(((t_2t_5)t_1)t_4)t_3$$
$$+4(((t_2t_5)t_3)t_1)t_4 - 4(((t_2t_5)t_3)t_4)t_1 + 4(((t_2t_5)t_4)t_1)t_3 - 2(((t_2t_5)t_4)t_3)t_1$$
$$-2(((t_3t_4)t_1)t_2)t_5 + 2(((t_3t_4)t_1)t_5)t_2 + 2(((t_3t_4)t_2)t_1)t_5 - 2(((t_3t_4)t_2)t_5)t_1$$
$$-2(((t_3t_4)t_5)t_1)t_2 + 2(((t_3t_4)t_5)t_2)t_1 + 2(((t_3t_5)t_1)t_2)t_4 - 2(((t_3t_5)t_1)t_4)t_2$$
$$-2(((t_3t_5)t_2)t_1)t_4 + 2(((t_3t_5)t_2)t_4)t_1 + 2(((t_3t_5)t_4)t_1)t_2 - 2(((t_3t_5)t_4)t_2)t_1$$
$$+4(((t_4t_5)t_1)t_2)t_3 - 2(((t_4t_5)t_1)t_3)t_2 - 4(((t_4t_5)t_2)t_1)t_3 + 2(((t_4t_5)t_2)t_3)t_1$$
$$-4(((t_4t_5)t_3)t_1)t2 + 4(((t_4t_5)t_3)t_2)t_1.$$

We see that

$$f_2 = leilie_1(t_1, t_2, t_4, t_3, t_5),$$

$$f_3 = leilie_1(t_1, t_2, t_5, t_3, t_4),$$

$$f_4 = leilie_1(t_1, t_3, t_4, t_2, t_5),$$

$$f_5 = leilie_1(t_2, t_3, t_4, t_1, t_5),$$

$$f_6 = leilie_1(t_1, t_3, t_5, t_2, t_4),$$

$$f_7 = leilie_1(t_2, t_3, t_5, t_1, t_4),$$

$$f_9 = leilie_1(t_1, t_4, t_5, t_2, t_3) + leilie_2(t_1, t_2, t_3, t_4, t_5),$$

$$f_{10} = (leilie_1(t_3, t_4, t_5, t_1, t_2) + 2\, leilie_2(t_1, t_2, t_3, t_4, t_5) + lei_3^{(-1)}(t_1, t_2, t_3, t_4, t_5)$$
$$-lei_3^{(-1)}(t_1, t_2, t_3, t_5, t_4) - lei_3^{(-1)}(t_1, t_2, t_4, t_3, t_5))/2,$$

$$f_{11} = (2\,leilie_1(t_2,t_4,t_5,t_1,t_3)+leilie_1(t_3,t_4,t_5,t_1,t_2)+2\,leilie_2(t_1,t_2,t_3,t_4,t_5)$$
$$+lei_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)-lei_3^{(-1)}(t_1,t_2,t_3,t_5,t_4)-lei_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2,$$

$$f_{12} = (leilie_1(t_3,t_4,t_5,t_1,t_2) + lei_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)$$
$$+lei_3^{(-1)}(t_1,t_2,t_3,t_5,t_4) - lei_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2,$$

$$f_{14} = (leilie_1(t_3,t_4,t_5,t_1,t_2) + lei_3^{(-1)}(t_1,t_2,t_3,t_4,t_5)$$
$$-lei_3^{(-1)}(t_1,t_2,t_3,t_5,t_4) + lei_3^{(-1)}(t_1,t_2,t_4,t_3,t_5))/2.$$

So, by Lemma 5.1 the 9-term polynomial $leilie_1$ and the 60-term polynomial $leilie_2$ form a base of polylinear identities of degree 5.

Proof of Theorem 1.4. Follows from Lemmas 5.1, 5.2, 5.3 and 5.4.

6. Leibniz-Jordan algebras

Proof of Theorem 1.6. It is easy to check that $leijor = 0$ is identity for any algebra of a form $A^{(1)}$, where A is a Leibniz algebra.

Let A be an associative algebra and M a right-module over A. Then $A^{(-1)}$ is a Lie algebra and M can be made into an antisymmetric $A^{(-1)}$-module. Let $L = A + M$ be the standard Lebniz algebra corresponding to these Lie and antisymmetric module structures. If we denote by $\star$ the multiplication in the Leibniz algebra L, then

$$(a+m)\star(b+n) = [a,b] + mb,$$

and

$$\{a+m, b+n\} = [a,b] + mb + [b,a] + na = na + mb.$$

In particular,

$$\{a,m\} = ma, \qquad \{a,b\} = 0, \qquad \{m,n\} = 0 \tag{3}$$

for all $a,b \in A, m,n \in M$. Recall that

$$\{t_1,t_2\} = t_1t_2 + t_2t_1$$

is the Jordan commutator.

Suppose that $f = 0$ is a minimal identity on the Leibniz-Jodan algebra $(L, \{\ ,\ \})$ which does not follow from the identity $leijor = 0$. We can assume that f is polylinear and $f = f(t_1,\dots,t_k)$ is a linear combination of left-bracketed monomials of the form $((t_{i_1}t_{i_2})\cdots)t_{i_k}$. So,

$$f(t_1,\dots,t_k) = \sum_{\sigma\in Sym_k} \lambda_\sigma((t_{\sigma(1)}t_{\sigma(2)})\cdots)t_{\sigma(k)}$$

for some $\lambda_\sigma \in K$. Write the condition $f(a_1, \dots , a_{k-1}, m) = 0$ by using the multiplication rules (3) for Leibniz-Jordan algebras. We have

$$f(a_1, \dots , a_{k-1}, m) = \sum_{\sigma \in Sym_{k-1}} \lambda_\sigma((ma_{\sigma(1)}) \cdots)a_{\sigma(k-1)} \tag{4}$$

for any $a_1, \dots , a_{k-1} \in A, m \in M$.

Take $A = Mat_n$ to be the matrix algebra and $M = K^n$ the n-dimensional natural module. Then conditions (4) imply that

$$\sum_{\sigma \in Sym_{k-1}} \lambda_\sigma((a_{\sigma(1)}a_{\sigma(2)}) \cdots)a_{\sigma(k-1)} = 0$$

is an identity on Mat_n. By the Amitsur-Levitsky theorem [1], matrix algebras have no identity of degree k if $k < 2n + 1$. So, $f = 0$ is not an identity for Leibniz-Jordan algebras of the form $Mat_n + K^n$ if $n > (k - 1)/2$. In other words, any s-identity for Leibniz-Jordan algebras follows from the identities $leijor = 0, com = 0$.

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