On the birational classification of linear representations

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§ 0.

The ground field assumed to be the field of complex number $\mathbb{C}$.

Let G be a linear algebraic group, $G : V_1,\ V_2$ be linear representations. These respresentations are called (linearly) isomorphic, iff there exist a linear $G-$equivariant isomorphism

$$V_1 \to V_2 .$$

The natural generalization of this classical definition is as follows:

Definition. *The representations $G : V_1,\ V_2$ are called birationally isomorphic, iff there exist a birational $G-$ equivariant isomorphism*

$$V_1 \to V_2 .$$

Hypothesis *. Let $\dim V_1 = \dim V_2$, the kernel H of the representation $G : V_1$ is equal to the kernel of the representation $G : V_2$, the representations $G/H : V_1,\ V_2$ are locally free. Then the representations $G : V_1,\ V_2$ are birationally isomorphic.

Recall that a regular action of a linear algebraic group G on an irreducible variety X is called locally free, iff the stabilizer in G of the point $x \in X$ in general position is trivial. Maybe this hypothesis or its particular cases were proposed before this preprint. But the author has not the corresponding information. The hypothesis * is related to the rationality problem (stable rationality problem) in invariant theory. Recall this problem:

Let $G : V$ be a linear representation. Is the field $\mathbb{C}(V)^G$ rational (stable rational)?

The results concerning the rationality problem can be devided by 3 types by means of its characters and methods.

1. Groups and their representations for which the problem of (stable) rationality has the negative answer were found and investigated.
2. The stable rationality was proved for representations of some groups.
3. The rationality was proved for some representations of some groups.

It appeared in all investigated cases that the answer to the question on (stable) rationality of the field $\mathbb{C}(V)^G$ depends on the group G, but not on the representation of this group. This fact approves the hypothesis * . Moreover, all proofs of the rationality of invariant fields have an interesting feature. Namely, the crucial step of all these proofs is the proof of the hypothesis * in the particular case. All these facts encourage me to propose the hypothesis * .

In § 1 of this preprint we recollect some folklore the 1960's. Namely, we formulate and prove Noname lemma in the equivariant form. It seems, this form of Noname lemma is the most natural. Then we formulate and prove the stable form of the hypothesis *.

In § 2 we prove the hypothesis * for group S_n, $n \leq 4$. In § 2 we discuss (G, H)-sections, and then (§§ 5, 6) we prove the hypothesis * for the group SL_2 and for the solvable connected group.

§ 1.

We consider bundles with irreducible base only. A rational morphism under a base X of a vector bundle $\pi : E \to X$ in a vector bundle $\pi_1 : E_1 \to X$ is a rational morphism $\varphi : E \to E_2$ such that there exist a nonempty open subset $X_0 \subset X$, such that

a. φ is defined in all points of $\pi^{-1}(X_0)$,
b. $\varphi(\pi^{-1}(X_0)) \subset \pi_1^{-1}(X_0)$,
c. $\varphi|\pi^{-1}(X_0) : \pi^{-1}(X_0) \to \pi_1^{-1}(X_0)$ is a morphism of vector bundles under base X_0 .

The rational morphism of affine, projective bundles is defined analogously. As usually, the projectivization of a vector bundle E is denoted by PE.

Let $\pi : E \to X$ be a locally trivial (in Zariski topology) vector bundle and a linear algebraic group G acts on E by means of automorphisms. Thus, the regular action $G : X$ defined canonically. Consider a trivial vector bundle $\mathbb{C}^n \times X \to X$, $n = rkE$ and a regular action $G : \mathbb{C}^n \times X$, $q(v, x) = (v, gx)$. The question arises in a natural way:

> Does there exists a birational $G-$equivariant isomorphism under X of the vector bundle $\pi : E \to X$ in the trivial vector bundle $\mathbb{C}^n \times X \to X$?

The evident necessary condition is the coinsideness of the kernel of action $G : E$ with the stabilizer in group G of point $x \in X$ in general position. Folklore the 1960's tells us that this condition is sufficient.

Noname lemma. *Let the stabilizer in G of the point $x \in X$ in general position be equal to the kernel of the action $G : E$. Then there exist a birational $G-$equivariant isomorphism under the base X of the vector bundle $\pi : E \to X$ in the trivial vector bundle $\mathbb{C}^n \times X \to X$.*

Proof: We have to construct rational $G-$equivariant sections $\eta_1, \cdots, \eta_n$ of vector bundle, such that $\eta_1, \cdots, \eta_n$ are defined and linearly independent on the nonempty open in X subset. Let Y be an irreducible subvariety in X, such that $\overline{G \cdot Y} = X$ and the intersection of Y and the $G-$orbit of general position is a finite number of points (for example, m points), Y_0 is the nonempty affine open in a Y subset, such that the vector bundle $\pi^{-1}(Y_0) \to Y_0$ is trivial, Gy_1^0 is a $G-$ orbit of $y_1^0 \in Y_0$, such that $Gy_1^0 \cap Y = \{y_1^0, \cdots, y_m^0\} \subset Y_0$. Let $y_j^0 = g_j^0 y_1^0$, $g_j^0 \in G$, $1 \leq j \leq m$. Choose regular sections $\overline{\eta}_1, \cdots, \overline{\eta}_n$ of the vector bundle $\pi^{-1}(Y_0) \to Y_0$, such that $\overline{\eta}_1(y_1^0), \cdots, \overline{\eta}_n(y_1^0)$ are linearly independent and $\overline{\eta}_i\left(y_j^0\right) = g_j^0(\overline{\eta}_i(y_1^0))$, $1 \leq i \leq n$, $1 \leq j \leq m$. Put

$$\eta_i(x) = \sum_{G \cdot x \cap Y = \{g_1 x, \cdots, g_m x\}} g_j^{-1}(\overline{\eta}_i(g_j x))$$

Corollary (Stable form of the hypothesis *). *Let G be a linear algebraic group, $G : V_1, V_2$ be linear representations, the kernel H of the representation $G : V_1$ coincides with the kernel of representation $G : V_2$, the representations $G/H : V_1, V_2$ are locally free, $G : U$*

be a linear representation, H *is the kernel of the representation* $G : U$*, the representation* $G/H : U$ *is locally free. Then there exist a birational* $G-$ *equivariant isomorphism*

$$V_1 \times U_1 \to V_2 \times U_2 \ .$$

Here $G : U_1, U_2$ *is an arbitrary linear representation such that* $\dim U_1 + \dim V_1 = \dim U_2 + \dim V_2$, $\dim U_1 \geq \dim U$, $\dim U_2 \geq \dim U$, *actions* $H : U_1, U_2$ *are trivial.*

Proof:

$$V_1 \times U_1 \to V_1 \times \mathbb{C}^{\dim U_1} \to V_1 \times \mathbb{C}^{\dim U_1 - \dim U} \times U \to$$
$$\mathbb{C}^{\dim V_1 + \dim U_1 - \dim U} \times U \to V_2 \times \mathbb{C}^{\dim U_2 - \dim U} \times U \to$$
$$V_2 \times \mathbb{C}^{\dim U_2} \to V_2 \times U_2 \ .$$

All birational $G-$ equivariant isomorphism from this diagram exists according to Noname lemma.

Remark. Let a linear algebraic group G act on a locally trivial (in Zariski topology) projective bundle $\overline{\pi} : \overline{E} \to \overline{X}$. So, the regular action $G : \overline{X}$ is defined. Consider a trivial projective bundle $\overline{X} \times \mathbf{P}^n \to \overline{X}$, $n = \dim \overline{E} - \dim \overline{X}$ and a regular action $G : \overline{X} \times \mathbf{P}^n$, $g \cdot (\overline{x}, \overline{v}) = (g\overline{x}, \overline{v})$.

Is the projective bundle $\overline{\pi} : \overline{E} \to \overline{X}$ birationally $G-$ equivariantly isomorphic to the trivial projective bundle $\overline{X} \times \mathbf{P}^n \to \overline{X}$?

Lemma 1.1. *Let* $G : U$ *be a linear representation,* $X \subset U$ *be a conic* $G-$ *invariant irreducible subvariety in* U, $\overline{X}$ *be a projectivization of* X, $\pi : E \to \overline{X}$ *be a* $G-$ *linearized vector bundle,* $\overline{\pi} : \overline{E} \to \overline{X}$ *be a projectivization of* E*. Suppose that a stabilizer in* G *of point* $x \in X$ *in general position coicides with the kernel of the action* $G : E$*. Then the projective bundle* $\overline{\pi} : \overline{E} \to \overline{X}$ *is birationally* $G-$ *equivariantly isomorphic to the trivial projective bundle* $\overline{X} \times \mathbf{P}^n \to \overline{X}$, $n = rk\ E - 1$.

§ 2.

Lemma 2.1. *The hypothesis * is true for the group* S_n, $n \leq 4$.

Proof: Let $S_n : \chi$ be a nontrivial onedimensional representation of the groups S_n, $\sigma \mapsto Sgn\ \sigma$.

a. $n = 1$. There is nothing to prove.

b. $n = 2$. If $H \neq \{e\}$ then we have the case a). Let $H = \{e\}$,

$$V_1 = \chi \times \cdots,\ V_2 = \chi \times \cdots \ .$$

Here we apply Noname lemma.

c. $n = 3$. If $H \neq \{e\}$ then we have the case a) or b). Let $H = \{e\}$,

$$V_1 = U_2 \times \cdots,\ V_2 = U_2 \times \cdots \ ,$$

there $S_3 : U_2$ is an irreducible two-dimensional representation. Here we apply Noname lemma.

d. $n = 4$. If $H \neq \{e\}$ then we have the case a) or b) or c). Let $H = \{e\}$. We apply Noname lemma if the representations V_1 and V_2 contain the same irreducible three-dimensional representation as a direct product. In other cases we have

$$V_1 = U_3 \times \cdots,\ V_2 = \chi \otimes U_3 \times \cdots \ ,$$

there $S_4 : U_3$ is an irreducible three-dimensional representation. It is sufficient to prove the hypothesis * for $V_1 = U_3,\ V_2 = \chi \otimes U_3$. We have a diagram

$$\begin{array}{ccccc} U_3 & \leftarrow & \mathbb{C} \times PU_3 & \rightarrow & \chi \otimes U_3 \\ \downarrow & & \downarrow & & \downarrow \\ PU_3 & \overset{id}{\leftarrow} & PU_3 & \overset{id}{\rightarrow} & P(\chi \otimes U_3) \simeq PU_3 \end{array}$$

The vertical arrows in this diagram are projections of vector bundles on the base, the horizontal arrows in the upper row exist according to Noname lemma.

§ 3.

In this paragraph we discuss the concept of (G,H)-section in connection with hypothesis * .

Let X be an irreducible algebraic variety, G be a linear algebraic group, $G : X$ be a regular action. An irreducible subvariety $Y \subset X$ is called (G,H)-section of X iff a) $\overline{G \cdot Y} = X$, b) if $g \in G,\ y \in Y$, then $gy \in Y$ iff $g \in H$.

Let $G : V$ be a linear representation of a reductive algebraic group G. Suppose that the stabilizer in G of point $v_0 \in V$ in general position is not equal to the kernel of the representation $G : V$. Then we can construct (G,H)-section $Y \subset V$, such that Y is an open subset in the linear subspace $\overline{Y}$. Namely, put

$$H = N(G_{v_0}),\ \overline{Y} = V^{G_{v_0}},\ Y = \{y \in \overline{Y} | G_y = G_{v_0}\} .$$

Y is in fact $(G,H)-$ section, according to Richardsons theorem.

Let X be an irreducible algebraic variety, G be a linear algebraic group, $G : X$ be a regular action, $Y \subset X$ be (G,H)-section of X. We have a birational $G-$ equivariant isomorphism

$$G \times^H Y \to X ,$$
$$(g,y) \mapsto gy .$$

Definition. *Let a linear algebraic group G act regularily on irreducible algebraic varieties $Y_1,\ Y_2$. The actions $G : Y_1, Y_2$ are called birationally isomorphic iff there exist a birational $G-$ equivariant isomorphism*

$$Y_1 \to Y_2 .$$

Lemma 3.1. *Let $X_1,\ X_2$ be irreducible varieties, $G : X_1, X_2$ be regular actions, $Y_1,\ Y_2$ be (G,H)-sections of $X_1,\ X_2$. Suppose that actions $H : Y_1, Y_2$ are birationally isomorphic. Then actions $G : X_1, X_2$ are birationally isomorphic.*

Proof: This lemma is a corollary of the diagram of birational $G-$equivariant isomorphisms:

$$\begin{array}{cccccc} G \times^H Y_1 & & \rightarrow & & G \times^H Y_2 \\ & (g, y_1) & \mapsto & (g, \varphi(y_1)) & \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ X_1 & gy_1 & & g\varphi(y_1) & X_2 \end{array}$$

where $\varphi : Y_1 \to Y_2$ is the birational $H-$equivariant isomorphism.

The converse to lemma 2.1 statement is not true. For example, consider the representation

$$SL_3 : S^2\mathbb{C}^3 \times V = X \ ,$$

where $SL_3 : V$ is an arbitrary locally free representation. We indicate $(SL_3, SO_3)-$ sections Y_1, Y_2 of X , such that

1. action $SO_3 : Y_1$ has not a section,
2. action $SO_3 : Y_2$ has a section.

Put

$$Y_1 = \mathbb{C}^*\left(e_1^2 + e_2^2 + e_3^2\right) \times V \ .$$

The action $SO_3 : Y_1$ is birationally isomorphic to a linear representation $SO_3 : \mathbb{C} \times V$. But the representation $SO_3 : \mathbb{C} \times V$ has not a section (the group SO_3 is not special). Let Y be a section of X (the group SL_3 is special, so such Y exist). Put

$$Y_2 = SO_3 \cdot Y \ .$$

§ 4.

Lemma 4.1. *Hypothesis * is true for*

a. *solvable connected group,*
b. *locally free representations of the group* SL_2 *.*

Proof: In cases a) and b) the representation $G : V_i$ has the section Y_i . The section Y_i is birationally isomorphic to a rational factor V_i/G . But V_i/G is a rational variety ([1] for solvable connected group, [3] for SL_2). We have a diagram of birational $G-$ equivariant isomorphisms

$$V_1 \to G \times Y_1 \to G \times \mathbb{C}^n \leftarrow G \times Y_2 \leftarrow V_2 \ .$$

Remark. The hypothesis * is true for "almost all" linear representations of the group SL_3 . On the other hand, if the hypothesis * is true for representations

$$SL_3 : S^4\mathbb{C}^{3*}, \ SL_3 : \bigoplus_1^5 \mathbb{C}^3 \ ,$$

then the field $\mathbb{C}\left(S^4\mathbb{C}^{3*}\right)^{SL_3}$ is rational. But the rationality of this field is a classical problem. At the present moment for hypothesis * for group SL_3 new effective methods are needed.

§ 5.

In this paragraph we shall complete the proof of the hypothesis * for the group SL_2 .

Lemma 5.1. *The hypothesis * is true for the group* PSL_2 *.*

Proof: We shall prove that an arbitrary linear representation of the group PSL_2 birationally isomorphic to representation

$$PSL_2 : V(2) \times V(2) \times V(0) \times \cdots \times V(0) = V_0 \ .$$

As usually, we denote by $V(d)$ the space of forms of degree d in variable $z_1,\ z_2$. The group SL_2 acts on $V(d)$ canonically. If d is even, then the center H of the group SL_2 acts trivially on $V(d)$, so the linear representation $PSL_2 = SL_2/H : V(d)$ is defined.

Thanks to Noname lemma, it is sufficient to consider the next four cases:

1. $V_1 = V(2) \times V(4)$,
2. $V_2 = V(4) \times V(4)$,
3. $V_3 = V(d),\ d \geq 8$,
4. $V(6)$.

Let

$$T = \left\{ \overline{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}} | t \in \mathbb{C}^* \right\} \subset PSL_2 ,$$

$N(T)$ be a normalizer of the torus T in PSL_2 . The subvariety

$$\mathbb{C}^* z_1 z_2 \subset V(2)$$

is $(PSL_2, N(T))-$ section. We have $(PSL_2, N(T))-$ sections

$$Y_0 = \mathbb{C}^* z_1 z_2 \times V(2) \times V(0) \times \cdots \times V(0) \subset V_0 ,$$
$$Y_1 = \mathbb{C}^* z_1 z_2 \times V(4) \subset V_1 .$$

If $\dim V = \dim V_1$ then these $(PSL_2, N(T))-$ sections are birationally $N(T)-$ equivariantly isomorphic (Noname lemma). Thus, the representations $SL_2 : V_0, V_1$ are birationally isomorphic (Lemma 3.1).

The representations $PSL_2 : V_1, V_2, V_3$ have $(PSL_2, S_4)-$ sections Y_i such that the regular action $S_4 : Y_i$ is birationally isomorphic to the linear representation of S_4 (the cases V_1, V_2 are evident, the cases $V(d),\ d \neq 10$ see in [ ], in the case $V(10)$ such Y exist all the same (F.A. Bogomolov, unpublished)). Now we apply lemma 2.1 and lemma 3.1.

Let us consider the representation $PSL_2 : V(6)$. Put

$$\psi_2 : V(6) \times V(4) \to V(4) ,$$
$$X = \left\{ (\overline{f}, \overline{g}) | \psi_2(f, g) = 0 \right\} \subset PV(6) \times PV(4) .$$

The restriction of the canonical projection $PV(6) \times PV(4) \to PV(6)$ on X defines birational PSL_2- equivariant isomorphism of $PV(6)$ and X.

Remark 5.2. The space $V(6)$ is isomorphic to the linear subspace in $\Lambda^2 V(4)^*$. Then ψ_2 is a contraction. Since $\dim V(4) = 5$ is odd, then the kernel $\operatorname{Ker} \psi_2(f, \cdot)$ is nontrivial for $f \in V(6)$. The kernel $\operatorname{Ker} \psi_2(f, \cdot)$ is one-dimensional for all $f \in V(6)_0$, where $V(6)_0$ is an open in $V(6)$ and nonempty subset.

Consider bilinear PSL_2- equivariant mapping

$$\psi_3 : PV(6) \times PV(4) \to PV(2)$$

and put

$$\psi : X \to PV(2) \times PV(4) ,$$
$$(\overline{f}, \overline{g}) \mapsto \left(\overline{\psi(f, g)}, \overline{g} \right)$$

Let us prove that ψ is a birational PSL_2- isomorphism. Let P_2 be a restriction on X of the canonical projection $PV(6) \times PV(4) \to PV(4)$. Then the fiber $p_2^{-1}(\overline{g})$ is a projective plane for a point $\overline{g} \in PV(4)$ in general position. The rational morphism ψ map linearily isomorphically the fiber $p_2^{-1}(\overline{g})$ on the fiber $PV(2) \times \overline{g}$.

Remark 5.3. The birational PSL_2- equivariant isomorphism

$$PV(6) \to PV(2) \times PV(4)$$

gives us an another proof of the rationality of the moduli variety of curves of genus 2 (see [3]).

Let us apply Noname lemma to linear bundles

$$V(6) \to PV(6),\ PV(2) \times V(4) \to PV(2) \times PV(4) .$$

Then we obtain a birational PSL_2- equivariant isomorphism

$$V(6) \approx PV(2) \times V(4) .$$

The subvarieties

$$\overline{z_1 z_2} \times V(4) \subset PV(2) \times V(4),\ \mathbb{C}^* z_1 z_2 \times V(2) \times V(0) \subset V(2) \times V(2) \times V(0)$$

are ($PSL_2,\ N(T)$)-sections. The last remark is that regular actions

$$N(T) : \mathbb{C}^* z_1 z_2 \times V(2) \times V(0),\ \mathbb{C} z_1 z_2 \times V(2) \times V(0),\ V(4),\ \overline{z_1 z_2} \times V(4)$$

are birationally isomorphic.

[1] Vinberg E.B. "Rationality of the field of invariants of a triangular group", Vestnik Mosc. Univ. 1982, N2, p. 23–24.

[2] Katsylo P.I. "The Rationality of moduli spaces of hyperelliptic curves", Izvestiya Akad. Nauk SSSR, v. 48, 1984, N4, p. 705–710.

[3] Katsylo P.I. "Rationality of fields of invariants of reducible representations of the group SL_2 ", Vestnik. Mosc. Univ. 1984, N5, p. 77–79.