THE COHOMOLOGY OF HOMOTOPY CATEGORIES AND THE GENERAL LINEAR GROUP

by

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MPI/89-5

This paper describes and exploits connections of the following topics:

(A) The cohomology of categories as introduced by A. Grothendieck [13].

(B) Secondary operations of homotopy theory like Toda brackets [29].

(C) The first k–invariant of a classifying space and Cooke's first obstruction for realizing homotopy actions [8].

(D) Elements in the cohomology of the general linear group $GL_N(\mathbb{Z})$ and Igusa's associativity class [16].

(E) The automorphism group of free nil (2) – groups.

We need S–normalized cohomology groups $H^n_S(\underline{\underline{C}},D)$ of a category $\underline{\underline{C}}$ where S is an ideal of morphisms in $\underline{\underline{C}}$, see (1.8). The following three fundamental properties of such cohomology groups are proved.

(1) The normalization theorems (1.9) and (1.10).

(2) The classification of normalized linear extensions by $H^2_S(\underline{\underline{C}},D)$, see (2.3).

(3) The classification of normalized linear track extensions by $H^3_S(\underline{\underline{C}},D)$, see (4.6).

Let $\underline{\underline{Top}}^*/\simeq$ be the homotopy category of pointed spaces. Using the classification (3) we associate with each small subcategory $\underline{\underline{C}} \subset \underline{\underline{Top}}^*/\simeq$ (consisting of co–H–groups) a cohomology class

$$(4) \qquad \langle\underline{\underline{C}}\rangle \in H^3(\underline{\underline{C}},D_\Sigma)$$

which we call the bracket of $\underline{\underline{C}}$. For a subcategory $\iota : \underline{\underline{K}} \subset \underline{\underline{C}}$ we have the restriction formula $\langle\underline{\underline{K}}\rangle = \iota^*\langle\underline{\underline{C}}\rangle$. Each triple Toda bracket $\langle f,g,h\rangle$ in $\underline{\underline{C}}$ can be deduced from $\langle\underline{\underline{C}}\rangle$, see (3.3), more generally the element $\langle\underline{\underline{C}}\rangle$ determines secondary homotopy operations in $\underline{\underline{C}}$. We consider the following examples.

(5) If $\underline{\underline{C}}$ contains all spheres S^n and all maps $S^n \longrightarrow S^m$ $(n,m \geq 2)$ then $\langle\underline{\underline{C}}\rangle$ is an element of infinite order, see (3.4).

(6) If all objects of $\underline{\underline{C}}$ are simply connected rational spaces then $\langle\underline{\underline{C}}\rangle = 0$ is trivial, see (3.4).

(7) If $\underline{C} = \underline{\mathrm{Aut}}(X)$ consists merely of the group of homotopy equivalences of a space X then $<\underline{\mathrm{Aut}}(X)>$ can be identified with the first k–invariant of the classifying space $B\mathcal{E}(X)$ where $\mathcal{E}(X)$ is the topological monoid of all homotopy equivalences of X, see (3.10). Whence the class $<\underline{\mathrm{Aut}}\ X>$ is Cooke's first obstruction.

(8) Now let $\underline{C} = \underline{S}(n)$ be the full homotopy category consisting of finite one point unions of n–spheres $\overset{N}{\vee} S^n = S^n \vee \dots \vee S^n$. Our main result shows that in this case the bracket

$$<\underline{S}(n)> \in H^3(\underline{S}(n), D_\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$$

is the generator, see (3.7); all triple Toda brackets in $\underline{S}(n)$, however, are trivial. For the proof we construct an explicit algebraic model of the 'track category for $\underline{S}(n)$' in terms of the category $\underline{\mathrm{nil}}$ of free nil(2)–group, see (5.2). This shows that one has the formula

$$<\underline{S}(n)> = \beta_j\{\underline{\mathrm{nil}}\}$$

where the class $\{\underline{\mathrm{nil}}\}$ is given by the classification (2) and where β_j is a Bockstein homomorphism, see (6.9), moreover $2\{\underline{\mathrm{nil}}\} = 0$.

(9) A free nil(2)–group $G_N = F/\Gamma_3 F$ is the quotient defined by a free group F of N generators and the lower central series $\Gamma_* F$; whence one has $G_N^{ab} = F/\Gamma_2 F = \mathbb{Z}^N$. We show that the projection of automorphism groups

$$p : \mathrm{Aut}(G_N) \twoheadrightarrow \mathrm{Aut}(\mathbb{Z}^N) = GL_N(\mathbb{Z})$$

(obtained by abelianization) has a splitting if and only if $N \leq 2$, see (7.2). The extension p represents an element $\{\mathrm{Aut}(G_N)\}$ in the second cohomology of $GL_N(\mathbb{Z})$ which is the restriction of the class $\{\underline{\mathrm{nil}}\}$ in (8) above.

(10) We deduce from $<\underline{S}(n)>$ in (8) a formula for the first k–invariant $k^{N,n}$ of the classifying space $B\mathcal{E}(\overset{N}{\vee} S^n)$ which plays a role in Waldhausen's algebraic K–theory. By restricting the formula in (8) and by (7) one gets

$$k^{N,n} = <\underline{\mathrm{Aut}}(\overset{N}{\vee} S^n)> = \beta_j\{\mathrm{Aut}\ G_N\},$$

see (7.5). This shows that $k^{2,n}$ is trivial. By the work of Igusa [16] we know that $k^{N,n}$ is non trivial for $N \geq 4$. It seems likely that also $k^{3,n}$ for $n \geq 3$ is trivial though $\{\mathrm{Aut}\ G_3\} \neq 0$.

(11) In the final example we show that Igusa's mysterious associativity class $\chi(1)$ [16] admits a new interpretation by the formula

$$\chi(1) = <\underline{\underline{\mathrm{End}}}\,(\overset{N}{\vee}\, S^n)> = \beta_j\{\underline{\underline{\mathrm{End}}}(G_N)\}, \quad n \geq 3.$$

Here $\underline{\underline{\mathrm{End}}}(X)$ denotes the category of endomorphisms of the object X and the second equation is again a restriction of the equation in (8). The proof uses a new and simple algebraic characterization of Igusa's associativity cocycle, see (7.9). On the other hand the topological interpretation of $\chi(1)$ as the bracket of the homotopy category $\underline{\underline{\mathrm{End}}}(\overset{N}{\vee} S^n)$ is more direct than Igusa's topological construction in B (12.1) [16].

The authors would like to thank Mamuka Jibladze and Teimuraz Pirashvili for explaining results of their fundamental work on the Cohomology of Algebraic Theories; the first named author remembers with pleasure discussions on the subject in Tblissi and Baku. Moreover the authors would like to acknowledge the support of the Max–Planck–Institut für Mathematik, Bonn.

§1 Normalized cohomology of a small category

In this section we introduce the S–normalized cohomology of a pair $(\underline{\underline{C}},\underline{\underline{K}})$ where $\underline{\underline{C}}$ is a small category and where $\underline{\underline{K}}$ is a subcategory of $\underline{\underline{C}}$. Moreover we consider the Toda–category $\underline{\underline{T}}$ and its normalized cohomology which, as we shall see, corresponds to the classical definition of Toda brackets in topology. We use the following notations: A boldface letter like $\underline{\underline{C}}$ denotes a category, $\mathrm{Ob}(\underline{\underline{C}})$ and $\mathrm{Mor}(\underline{\underline{C}})$ are the classes of objects and morphisms respectively. We identify an object A with its identity $1_A = 1 = A$ so that $Ob(\underline{C}) \subset Mor(\underline{C})$. The set of morphisms $A \longrightarrow B$ is $\underline{C}(A,B)$, and the group of automorphisms of A is $Aut_{\underline{C}}(A)$. The *category of factorizations* in $\underline{C}$, denoted by $F\underline{C}$, is given as follows: Objects are the morphisms $f,g,\ldots$ in $\underline{C}$ and morphisms $f \longrightarrow g$ are pairs (α,β) for which

$$\begin{array}{ccc} B & \xrightarrow{\ \alpha\ } & B' \\ f\uparrow & & \uparrow g \\ A & \xleftarrow[\ \beta\ ]{} & A' \end{array}$$

commutes in $\underline{C}$. Composition is defined by $(\alpha',\beta')(\alpha,\beta) = (\alpha'\alpha,\beta\beta')$ so that $(\alpha,\beta) = (\alpha,1)(1,\beta) = (1,\beta)(\alpha,1)$. A *natural system* (of abelian groups) on $\underline{C}$ is a functor

$$D : F\underline{C} \longrightarrow \underline{Ab} \tag{1.1}$$

from $F\underline{C}$ to the category of abelian groups. The functor D carries the object f to $D_f = D(f)$ and carries the morphism (α,β) to $D(\alpha,\beta) = \alpha_*\beta^* : D_f \longrightarrow D_{\alpha f\beta} = D_g$ where $D(\alpha,1) = \alpha_*$ and $D(1,\beta) = \beta^*$. We have obvious functors

$$F\underline{C} \xrightarrow{\ \pi\ } \underline{C}^{op} \times \underline{C} \xrightarrow{\ p\ } \underline{C} \tag{1.2}$$

which show that a *$\underline{C}$–module* $F : \underline{C} \longrightarrow \underline{Ab}$ and a *$\underline{C}$–bimodule* $G : \underline{C}^{op} \times \underline{C} \longrightarrow \underline{Ab}$ and a *$\underline{C}$–bimodule* $G : \underline{C}^{op} \times \underline{C} \longrightarrow \underline{Ab}$ yield in a canonical way natural systems $(p\pi)^*F$, π^*G as well denoted by F and G respectively. Let $M,M' : \underline{Ab} \longrightarrow \underline{Ab}$ be functors then we get as an example the $\underline{Ab}$–bimodule $Hom(M',M) : \underline{Ab}^{op} \times \underline{Ab} \longrightarrow \underline{Ab}$ which carries the object (A,B) to the group $Hom(M'A,MB)$; in case M' is the identical functor we write $Hom(-,M)$.

We now recall the definition of the cohomology of $\underline{\underline{C}}$ with coefficients in a natural system, see [4]. Let $\underline{\underline{C}}$ be a small category and let $N_n(\underline{\underline{C}})$ be the set of sequences $(\lambda_1,\ldots,\lambda_n)$ of n composable morphisms in $\underline{\underline{C}}$ (which are the n–simplices of the nerve of $\underline{\underline{C}}$). For $n = 0$ let $N_0(C) = Ob(\underline{\underline{C}})$ be the set of objects in $\underline{\underline{C}}$. The n–th cochain group $F^n = F^n(\underline{\underline{C}},D)$ is the abelian group of all functions

$$c : N_n(\underline{\underline{C}}) \longrightarrow \bigcup_{g\in Mor(\underline{\underline{C}})} D_g$$

with $c(\lambda_1,\ldots,\lambda_n) \in D_{\lambda_1\circ\ldots\circ\lambda_n}$. Addition in F^n is given by adding pointwise in the abelian groups D_g. The coboundary $\delta : F^{n-1} \longrightarrow F^n$ is defined by the formula

(1.3)
$$\begin{aligned}(\delta c)(\lambda_1,\ldots,\lambda_n) = {} & \lambda_{1*}c(\lambda_2,\ldots,\lambda_n) \\ & + \sum_{i=1}^{n-1} (-1)^i c(\lambda_1,\ldots,\lambda_i\lambda_{i+1},\ldots,\lambda_n) \\ & + (-1)^n \lambda_n^* c(\lambda_1,\ldots,\lambda_{n-1}) .\end{aligned}$$

For $n = 1$ we have $(\delta c)(\lambda) = \lambda_* c(A) - \lambda^* c(B)$ for $\lambda : A \longrightarrow B \in N_1(\underline{\underline{C}})$. One can check that $\delta c \in F^n$ for $c \in F^{n-1}$ and that $\delta\delta = 0$. Whence the cohomology groups

(1.4)
$$H^n(\underline{\underline{C}},D) = H^n(F^*(\underline{\underline{C}},D),\delta)$$

are defined, $n \geq 0$. These groups are discussed in [4]; in particular they coincide for $D = (p\pi)^*F$ and $D = \pi^*G$, see (1.2), with the cohomology groups introduced by A. Grothendieck [13] and B. Mitchell [23] respectively. We now introduce normalized and relative versions of these cohomology groups.

(1.5) Definition: Let $S \subset Mor(\underline{C})$ be a subclass of morphisms in $\underline{C}$. We say that S is an ideal in $\underline{C}$ if $(f,g,h) \in N_3(\underline{C})$ with $g \in S$ satisfies $f \circ g \in S$ and $g \circ h \in S$. A natural system D on $\underline{C}$ is S–trivial if $f_* = 0$ and $f^* = 0$ for all $f \in S$.

(1.6) Example: Assume $\underline{C}$ has a zero object $*$ (i.e. the object $*$ is an initial and a final object of $\underline{C}$). Then the class $O(\underline{C})$ of all zero morphisms $O: A \longrightarrow * \longrightarrow B$ in $\underline{C}$ is an ideal. A natural system D on $\underline{C}$ is $O(\underline{C})$–trivial iff $D_f = 0$ for all $f: A \longrightarrow *$, $f: * \longrightarrow B$, $A,B \in Ob(\underline{C})$. For a subclass $S \subset Mor(\underline{C})$ we define the subgroups

$$\not{F}^n(S),\ F^n(S) \subset F^n(\underline{C},D) = F^n\ . \tag{1.7}$$

Here $F^n(S)$ contains all $c \in F^n$ which satisfy $c(\lambda_1,\ldots,\lambda_n) = 0$ if $\lambda_i \in S$ for all $\iota \in \{1,\ldots,n\}$, $n \geq 1$; for $n = 0$ the group $F^0(S)$ contains all $c \in F^0$ with $c(A) = 0$ for $1_A \in Ob(\underline{C}) \cap S$. In this case we call c a cochain relative S. On the other hand let $\not{F}^n(S)$ be the set of all $c \in F^n$ which satisfy $c(\lambda_1,\ldots,\lambda_n) = 0$ if there exists $i \in \{1,\ldots,n\}$ with $\lambda_i \in S$. We say that the elements in $\not{F}^n(S)$ are S–normalized cochains; they are simply called normalized cochains if $S = Ob(\underline{C})$. We now consider such S for which δ in (1.3) induces maps

(1) $\qquad \delta : F^{n-1}(S) \longrightarrow F^n(S)$ and

(2) $\qquad \delta : \not{F}^{n-1}(S) \longrightarrow \not{F}^n(S)$

respectively. Here (1) is well defined if $S = Mor(\underline{K})$ is the class of morphisms of a subcategory $\underline{K} \subset \underline{C}$. Moreover (2) is well defined if one of the following conditions (3), (4) is satisfied.

(3) S is an ideal in $\underline{C}$ and D is S–trivial.

(4) $S = S' \cup Ob(\underline{C})$ where S' is an ideal in $\underline{C}$ and where D is S'–trivial.

Since the empty class $S' = \phi$ is an ideal and since any natural system D is ϕ–trivial we have by (4) the special case $S = Ob(\underline{C})$. We now are ready for the definition of the following cohomology groups.

(1.8) Definition: Let $\underline{K}$ be a subcategory in $\underline{C}$ and let S be a class which satisfies (3) or (4) above. Then we obtain the cohomology groups

$$H^n_S(\underline{C},\underline{K};D) = H^n(F^*(\text{Mor }\underline{K}) \cap \check{F}^*(S),\delta)$$

which we call the S–normalized cohomology groups of the pair $(\underline{C},\underline{K})$.

These cohomology groups are natural in D and in the triple $(\underline{C},\underline{K},S)$, see (1.9) [4]. We omit S or $\underline{K}$ in the notation if $S = \phi$ or $\underline{K} = \phi$ respectively. In particular we get for $\underline{K} = \phi$, $S = \phi$ the cohomology (1.4). Cup products for the cohomology group (1.8) are defined in the same way as in (IV. 5.19) [1]. In the literature one can find various definition of cohomology groups of a category, a detailed description of the connections between such cohomology notions is contained in [4], in particular, one has

(1) $$H^n(\underline{C},D) = \text{Ext}^n_{F\underline{C}}(\mathbb{Z},D)$$

where the right hand side is defined in the functor category of functors $D : F\underline{C} \longrightarrow \underline{Ab}$, compare Grothendieck [13]. We also point out that Igusa's definition of the cohomology of a monoid §1 [16] in a special case of the cohomology (1.8). Further properties of cohomology groups in (1) are discussed in the work of Jibladze–Pirashvili on the cohomology of algebraic theories.

For the inclusion $\iota : \underline{K} \subset \underline{C}$ let ι^*D be the induced natural system on $\underline{K}$ and let $\iota^*S = S \cap \text{Mor}(\underline{K})$. Then we get as usual the long exact sequence

(2) $$\dots \xrightarrow{\iota^*} H^{n-1}_{\iota^* S}(\underline{\underline{K}}, \iota^* D) \xrightarrow{\partial} H^n_S(\underline{\underline{C}},\underline{\underline{K}},D) \xrightarrow{j} H^n_S(\underline{\underline{C}};D) \xrightarrow{\iota^*} H^n_{\iota^* S}(\underline{\underline{K}}, \iota^* D)$$

Here j is induced by the inclusion $(\underline{\underline{C}},\phi) \longrightarrow (\underline{\underline{C}},\underline{\underline{K}})$ which is the identity on $\underline{\underline{C}}$. We prove the following two <u>normalization theorems</u> for the cohomology groups (1.8).

(1.9) <u>Theorem</u>: Let S be an ideal in $\underline{\underline{C}}$ and let D be an S–trivial natural system on $\underline{\underline{C}}$. Then the inclusion $S \subset S \cup Ob(\underline{\underline{C}})$ induces an isomorphism $(n \geq 0)$

$$H^n_{S \cup Ob(\underline{\underline{C}})}(\underline{\underline{C}},\underline{\underline{K}},S) \xrightarrow{\cong} H^n_S(\underline{\underline{C}},\underline{\underline{K}};D).$$

The theorem describes the "normalization with respect to identities". We also have the following "normalization with respect to zero morphisms".

(1.10) <u>Theorem</u>: Let $*$ be a zero object in $\underline{\underline{C}}$ and let $O(\underline{\underline{C}})$ be the ideal of zero morphism. Let $\underline{\underline{K}}$ be a subcategory of $\underline{\underline{C}}$ which contains the zero morphisms $0 : A \longrightarrow A$ with $A \in Ob(\underline{\underline{K}})$. Moreover let S be an ideal in $\underline{\underline{C}}$ and let D be a natural system on $\underline{\underline{C}}$ which is $S \cup O(\underline{\underline{C}})$–trivial, see (1.6). Then the inclusion $S \subset S \cup O(\underline{\underline{C}})$ induces an isomorphism

$$H^n_{S \cup O(\underline{\underline{C}})}(\underline{\underline{C}},\underline{\underline{K}};D) \xrightarrow{\cong} H^n_S(\underline{\underline{C}},\underline{\underline{K}};D).$$

We shall prove these results in the Appendix B below. In the next section we give a simple illustration of these results for $n = 2$. Combining (1.9), (1.10) we get with the assumptions in (1.10) the "<u>strong normalization</u>" isomorphism

(1.11) $$H^n_{S \cup O(\underline{\underline{C}}) \cup Ob(\underline{\underline{C}})}(\underline{\underline{C}},\underline{\underline{K}};D) = H^n_S(\underline{\underline{C}},\underline{\underline{K}};D).$$

The following example is relevant for the classical Toda brackets discussed below in §3. The Toda–category $\underline{\underline{T}}$ is generated by the diagram $E \xleftarrow{f} F \xleftarrow{g} G \xleftarrow{h} H$. We define an ideal S in $\underline{\underline{T}}$ by the set $S = \{fg, gh, fgh\}$. Then we get for any natural system D on $\underline{\underline{T}}$ which is S–trivial the isomorphism

(1.12) Lemma: $$H^3_{S\cup Ob(\underline{\underline{T}})}(\underline{\underline{T}},D) = D_{fgh}/(f_*D_{gh} + h^*D_{fg})$$

where the right hand side denotes the quotient group. For $\mathcal{F}^n = \mathcal{F}^n(S \cup Ob(\underline{\underline{T}}))$ one readily checks: $\mathcal{F}^2 = D_{fg} \times D_{gh}$, $\mathcal{F}^3 = D_{fgh}$ and $\mathcal{F}^n = 0$ for $n \geq 4$. Moreover $\delta : \mathcal{F}^2 \longrightarrow \mathcal{F}^3$ carrries $(x,y) \in D_{fg} \times D_{gh}$ to $\delta(x,y) = f_*y - h^*x$. This proves (1.12).

§2 Normalized linear extensions of categories

We consider normalized linear extensions of categories and we show that the set of equivalence classes of such extensions is classified by the second normalized cohomology H^2_S defined in Section 1. Moreover we use the classification to give a simple proof of the normalization theorems (1.9), (1.10) in case $n = 2$.

Let $\underline{\underline{C}}$ be a category and let D be a natural system on $\underline{\underline{C}}$. We say that

(2.1) $$D+ \longrightarrow \underline{\underline{E}} \xrightarrow{p} \underline{\underline{C}}$$

is a linear extension of $\underline{\underline{C}}$ by D (see [4] or [1]) if (a), (b) and (c) holds.

(a) $\underline{\underline{E}}$ and $\underline{\underline{C}}$ have the same objects and p is a full functor which is the identity on objects.

(b) For each morphism $f : A \longrightarrow B$ in $\underline{\underline{C}}$ the abelian group D_f acts transitively and effectively on the subset $p^{-1}(f)$ of morphisms in $\underline{\underline{E}}$. We write $f_0 + \alpha$ for the action of $\alpha \in D_f$ on $f_0 \in p^{-1}(f)$.

(c) The action satisfies the linear distributative law $(f_0 + \alpha)(g_0 + \beta) = f_0 g_0 + f_* \beta + g^* \alpha$.

We now extend this notion slightly as follows.

(2.2) Definition. Let $\underline{\underline{K}}$ be a subcategory in $\underline{\underline{C}}$, let S be an ideal of morphisms in $\underline{\underline{C}}$ and let D be an S–trivial natural system on $\underline{\underline{C}}$. Then we say that the linear extensions $\underline{\underline{E}}$ in (2.1) together with a function $j : S \cup \mathrm{Mor}(\underline{\underline{K}}) \longrightarrow \mathrm{Mor}(\underline{\underline{E}})$ is an S–normalized linear extension of the pair $(\underline{\underline{C}},\underline{\underline{K}})$ by D if (1)...(3) hold.

(1) $pj(f) = (f)$ for $f \in S \cup \mathrm{Mor}\, \underline{\underline{K}}$,

(2) $j \,|\, \mathrm{Mor}\, \underline{\underline{K}}$, is a functor $\underline{\underline{K}} \longrightarrow \underline{\underline{E}}$,

(3) $j(f \circ g) = f_0 \circ j(g)$ and $j(gh) = j(g) \circ h_0$ for $g \in S$, $(f,g,h) \in N_3(\underline{\underline{C}})$, $f_0 \in p^{-1}(f)$, $h_0 \in p^{-1}(h)$.

Two such extensions $(\underline{\underline{E}},j)$, $(\underline{\underline{E}}',j')$ are equivalent if there is an isomorphism $\epsilon : \underline{\underline{E}} \longrightarrow \underline{\underline{E}}'$ of categories with $\epsilon(f_0 + \alpha) = \epsilon(f_0) + \alpha$, $p'\epsilon = p$, $\epsilon j = j'$. Moreover the extension is split if there is a functor $s : \underline{\underline{C}} \longrightarrow \underline{\underline{E}}$ with $ps = 1$, $s \,|\, S \cup \mathrm{Mor}\, \underline{\underline{K}} = j$.

We now assume that $\underline{\underline{C}}$ is a small category.

(2.3) Theorem: Let $M_S(\underline{\underline{C}},\underline{\underline{K}},D)$ be the set of equivalence classes of S–normalized linear extensions of $(\underline{\underline{C}},\underline{\underline{K}})$ by D. Then there are canonical bijections

$$H^2_S(\underline{\underline{C}},\underline{\underline{K}};D) \underset{\Psi}{\cong} M_S(\underline{\underline{C}},\underline{\underline{K}};D) \underset{\chi}{\cong} H^2_{S\cup \mathrm{Ob}(\underline{\underline{C}})}(\underline{\underline{C}},\underline{\underline{K}};D)$$

which carry the split extension to the trivial cohomology class. The bijection $\Psi\chi^{-1}$ coincides with the isomorphism in (1.9).

Let $(\underline{\underline{E}},j)$ be an extension as in (2.2). There exists a function $s : \mathrm{Mor}(\underline{\underline{C}}) \longrightarrow \mathrm{Mor}(\underline{\underline{E}})$ with $s \mid S \cup \mathrm{Mor}\,\underline{\underline{K}} = j$ and with $ps = 1$. For $(y,x) \in N_2(\underline{\underline{C}})$ the formula

(1) $$s(y \circ x) = s(y) \circ s(x) + \Delta_s(y,x)$$

determines an element

(2) $$\Delta_s \in F^2(\underline{\underline{K}}) \cap \breve{F}^2(S)\ ,$$

here we use (2.2)(2), (3). If s is a splitting in the sense of (2.2) then $\Delta_s = 0$. We now define the function Ψ in (2.3) by $\Psi\{\underline{\underline{E}}\} = \{\Delta_s\}$. We can modify the construction of $\Psi\{\underline{\underline{E}}\}$ as follows. We choose a function s with the additional property that s carries identities to identities. Then the cocycle Δ_s in (2) satisfies

(3) $$\Delta_s \in F^2(\underline{\underline{K}}) \cap \breve{F}^2(S \cup \mathrm{Ob}(\underline{\underline{C}}))$$

and we define χ in (2.3) by $\chi\{\underline{\underline{E}}\} = \{\Delta_s\}$. As in (2.3) [4] we see that χ and Ψ are bijections. This proves (2.3).

Now assume that $\underline{\underline{C}}$ has a zero object $*$ and that D is $O(\underline{\underline{C}})$–trivial. Then we define

(4) $$\chi_0 : M(\underline{\underline{C}},\underline{\underline{K}};D) \cong H^2_{O(\underline{\underline{C}})}(\underline{\underline{C}},\underline{\underline{K}},D)$$

by $\chi_0\{\underline{\underline{E}}\} = \{\Delta_s\}$. Here s is a function as in (1) which in addition carries zero morphisms in $\underline{\underline{C}}$ to zero morphisms in $\underline{\underline{E}}$. For this we observe that $*$ is actually as well the zero object of $\underline{\underline{E}}$ since $D_f = 0$ for $f : A \longrightarrow *$ and $f : * \longrightarrow A$, see (1.6).

§3 Toda brackets

The classical triple Toda bracket was one of Toda's main tools to construct elements in homotopy groups of spheres, [29]. We show that all triple Toda brackets in a homotopy category $\underline{\underline{C}}$ can be deduced from a unique cohomology class $<\underline{\underline{C}}> \in H^3(\underline{\underline{C}},D_\Sigma)$. We describe some examples of such cohomology classes; the construction of $<\underline{\underline{C}}>$, however, is postponed to the next section since it involves the definition of linear track extensions of categories.

Let $\underline{\underline{Top}}^*$ be the category of topological spaces with basepoint $*$ which is the zero object and let $\underline{\underline{Top}}^*/\simeq$ be the homotopy category. We consider the full subcategories $\underline{\underline{coH}}$ and $\underline{\underline{H}}$ of $\underline{\underline{Top}}^*/\simeq$ consisting of co–H–spaces and $\underline{\underline{H}}$–spaces respectively and we choose a small subcategory $\underline{\underline{C}}$,

(3.1) $$\underline{\underline{C}} \subset \underline{\underline{coH}} \text{ or } \underline{\underline{C}} \subset \underline{\underline{H}}\,.$$

Now assume that $*$ is the zero object also in $\underline{\underline{C}}$. For homotopy classes f,g,h of maps in $\underline{\underline{C}}$ with $E \xleftarrow{f} F \xleftarrow{g} G \xleftarrow{h} H$ and with $fg = 0$ and $gh = 0$ the classical <u>Toda</u> <u>bracket</u>

(3.2) $$<f,g,h> \in [\Sigma H,E]/(f_*[\Sigma H,F] + h^*[\Sigma G,E])$$

is defined, see [29]. Here

(1) $$D_\Sigma(H,E) = [\Sigma H,E]$$

is an abelian group which defines the bimodule D_Σ on $\underline{\underline{C}}$. For the Toda category $\underline{\underline{T}}$ any functor

(2) $$\varphi : (\underline{\underline{T}},S) \longrightarrow (\underline{\underline{C}},O(\underline{\underline{C}})),\ \text{see (1.12)},$$

which carries S to $O(\underline{\underline{C}})$ corresponds equivalently to a triple (f,g,h) as above. The functor φ induces a homomorphism $(f,g,h)^*$ by the commutative diagram

(3) $$\begin{array}{ccc} H^3_{O(\underline{\underline{C}})}(\underline{\underline{C}},D_\Sigma) & \xrightarrow{\varphi^*} & H^3_S(\underline{\underline{T}},\varphi^*D_\Sigma) \\ \cong\uparrow & & \downarrow\cong \\ H^3(\underline{\underline{C}},D_\Sigma) & \xrightarrow{(f,g,h)^*} & [\Sigma H,E]/(f_*[\Sigma H,D]+h^*(\Sigma G,F]) \end{array}$$

Here the isomorphism are given by the normalization (1.9), (1.10) and by (1.12) respectively. The next result is proved in (4.8).

(3.3) Theorem: There is a cohomology class $\langle \underline{C} \rangle \in H^3(\underline{C}, D_\Sigma)$ well defined by $\underline{C}$. If $*$ is the zero object in $\underline{C}$ we get all triple Toda brackets in $\underline{C}$ by the formula $\langle f,g,h \rangle = (f,g,h)^* \langle \underline{C} \rangle$ .We therefore call $\langle \underline{C} \rangle$ the bracket of $\underline{C}$. For a subcategory $i : \underline{K} \subset \underline{C}$ we have $i^* \langle \underline{C} \rangle = \langle \underline{K} \rangle$.

(3.4) Example Let $\underline{S} \subset \underline{Top}^*/\simeq$ be the full subcategory consisting of all spheres S^n, $n \geq 2$, and of the basepoint $*$. The morphism set $S^n \longrightarrow S^m$ is the homotopy group of spheres $\pi_n(S^m)$. For the bimodule D_Σ on $\underline{S}$ with $D_\Sigma(S^n, S^m) = \pi_{n+1}(S^m)$ we have by (3.3) the bracket

(1) $$\langle \underline{S} \rangle \in H^3(\underline{S}, D_\Sigma) .$$

All triple Toda brackets $\langle f,g,h \rangle$ in $\underline{S}$ can be deduced from $\langle \underline{S} \rangle$ by (3.3). Since there are triple Toda brackets $\langle f,g,h \rangle$ in $\underline{S}$ of arbitrary high order, see for example [12], we conclude that $\langle \underline{S} \rangle$ is actually an element of infinite order. It is an open problem to compute $\langle \underline{C} \rangle$ and $H^3(\underline{C}, D_\Sigma)$ even for simple subcategories of $\underline{S}$. It seems to be more appropriate to replace the category in (1) by the category $\underline{VS}$, the full subcategory of $\underline{Top}^*/\simeq$ consisting of finite one point unions $S^{n_1} \vee \dots \vee S^{n_r}$ of spheres, $n_i \geq 2$, $r \geq 0$. The element

(2) $$\langle \underline{VS} \rangle \in H^3(\underline{VS}, D_\Sigma)$$

contains all the information of "secondary operations" on homotopy groups of spheres. In [9] the second named author describes the category $\underline{VS}$ only in terms of "primary operations" on homotopy groups of spheres, see also [3] where the corresponding problem is solved for R–localized spheres, $R \subset \mathbb{Q}$ with $1/2, 1/3 \in R$.

Now let $R \subset \mathbb{Q}$ and let $p : \underline{\underline{VS}} \longrightarrow \underline{\underline{VS}}_R$ be the localization functor. The bimodule $D_\Sigma = D_\Sigma^R$ on $\underline{\underline{VS}}_R$ yields the bimodule $p^* D_\Sigma^R$ on $\underline{\underline{VS}}$ which can be identified with $D_\Sigma \otimes R$; in fact, we have for X,Y in $\underline{\underline{VS}}$ the natural isomorphism

$$(3) \qquad (p^* D_\Sigma^R)(X,Y) = [\Sigma X_R, Y_R] = [\Sigma X, Y] \otimes R = (D_\Sigma \otimes R)(X,Y).$$

Let $\otimes 1_R : D_\Sigma \longrightarrow D_\Sigma \otimes R$ be the natural transformation given by $x \longrightarrow x \otimes 1_R$, $1_R = 1 \in R$. Then we obtain the formula

$$(4) \qquad p^* <\underline{\underline{VS}}_R> = (\otimes 1_R)_* <\underline{\underline{VS}}>$$

in $H^3(\underline{\underline{VS}}, D_\Sigma \otimes R)$. We point out that $H^n(\underline{\underline{C}}, D \otimes R)$, in general, does not coincide with $H^n(\underline{\underline{C}}, D) \otimes R$ even for a field R, see below. (Equation (4) can be deduced from the fact that the localization functor is a model functor, see (II 4.4)(2) [1], and whence induces a functor between track categories, see §4, and (II §5a) [1].) Using the Quillen equivalence of rational homotopy theories, see [27], [1], we see that the bracket

$$(5) \qquad <\underline{\underline{VS}}_{\mathbb{Q}}> = 0 \in H^3(\underline{\underline{VS}}_{\mathbb{Q}}, D_\Sigma) \cong H^3(\underline{\underline{FL}}_0, D)$$

is trivial. Here the bimodule D is given by $D(L,L') = \mathrm{Hom}(sQL, L')$ with $QL = L/[L,L]$. We do not know whether the cohomology group in (5) is trivial as well, but we expect this to be true. Finally note that the natural map

$$(6) \qquad (\otimes 1_{\mathbb{Q}})_* : H^3(\underline{\underline{VS}}, D_\Sigma) \otimes \mathbb{Q} \longrightarrow H^3(\underline{\underline{VS}}, D_\Sigma \otimes \mathbb{Q})$$

is not an isomorphism. This follows since $<\underline{\underline{VS}}>$ is an element of infinite order in $H^3(\underline{\underline{VS}}, D_\Sigma)$ which is mapped by $(\otimes 1_{\mathbb{Q}})_*$ to the trivial element, see (4) and (5).

(3.5) Example: Let $\underline{\underline{K}} \subset \underline{\mathrm{Top}}^*/\simeq$ be the full subcategory consisting of all Eilenberg–Mac Lane spaces $K(G,n)$ where G is a finitely generated abelian group. The morphism set $K(G,n) \longrightarrow K(H,m)$ is the cohomology group $H^m(K(G,n),H)$ which can be computed by the work of Eilenberg–Mac Lane [10] and Cartan [7]. We have the bimodule D_Σ on $\underline{\underline{K}}$ with $D_\Sigma(K(G,n),K(H,m)) = H^{m-1}(K(G,n),H)$. Now (3.3) yields the bracket $<\underline{\underline{K}}> \in H^3(\underline{\underline{K}}, D_\Sigma)$. The subcategory $\underline{\underline{K}}_{\mathbb{Q}} \subset \underline{\underline{K}}$ of rational Eilenberg–Mac Lane spaces satisfies $<\underline{\underline{K}}_{\mathbb{Q}}> = 0$.

Next we consider an example which we compute completely. Let

(3.6) $$\underline{\underline{S}}(n) \subset \underline{\underline{Top}}^*/\simeq, \quad n \geq 2,$$

be the full subcategory consisting of finite one point unions of n–dimensional spheres $S^n \vee \dots \vee S^n$. The homology functor gives us the equivalence of categories

(1) $$H_n : \underline{\underline{S}}(n) \xrightarrow{\sim} \underline{\underline{M}}_{\mathbb{Z}}$$

where $\underline{\underline{M}}_{\mathbb{Z}}$ is the category of finitely generated free abelian groups. For objects X,Y in $\underline{\underline{S}}(n)$ with homology groups $H_nX = A$, $H_nY = B$ we get the natural isomorphism

(2) $$D_\Sigma(X,Y) = [\Sigma X,Y] = \mathrm{Hom}(A,\Gamma_n^1 B)$$

where

(3) $$\pi_{n+1}Y \cong \Gamma_n^1 B = \begin{cases} \Gamma B & , n = 2 \\ B \otimes \mathbb{Z}/2 , & n \geq 3 \end{cases} .$$

Here Γ is the quadratic functor of J.H.C. Whitehead [30]. Using the equivalence (1) and the natural isomorphism (2) we get the isomorphism of cohomology groups

(4) $$H^3(\underline{\underline{S}}(n),D_\Sigma) \cong H^3(\underline{\underline{M}}_{\mathbb{Z}},\mathrm{Hom}(\_,\Gamma_n^1))$$

which we use as an identification; compare (1.11) [4].

(3.7) Theorem: The cohomology group in (4) is a cyclic group of order 2 and the bracket $<\underline{\underline{S}}(n)>$ is the generator of this group.

Though $<\underline{\underline{S}}(n)>$ is non–trivial all triple Toda brackets in $\underline{\underline{S}}(n)$ vanish, that is:

(3.8) Proposition: For f,g,h in $\underline{\underline{S}}(n)$ with $fg = 0$ and $gh = 0$ we have $0 = <f,g,h> = (f,g,h)^*<\underline{\underline{S}}(n)>$, $n \geq 2$.

We prove (3.7) and (3.8) in (6.9) and (6.11) respectively, moreover we give explicit descriptions of cocycles representing $<\underline{S}(n)>$ in (6.9).

Finally we consider an important relation of the brackets $<\underline{C}>$ with the first Postnikov invariant of a classifying space. Let X be an H–group or a co–H–group and let $\mathcal{E}(X)$ be the topological monoid of all pointed maps $X \longrightarrow X$ which are homotopy equivalences. Let X^{*X} be the space of all pointed maps $X \longrightarrow X$ with the compact open topology. Then $\mathcal{E}(X)$ and X^{*X} have the same path components of the identity 1_X. Since X is an H–group or a co–H–group we know that X^{*X} is an H–group as well. The set of path components of $\mathcal{E}(X)$ is the group

(3.9) $$\mathrm{Aut}(X)^* = \pi_0(\mathcal{E}(X))$$

of pointed homotopy equivalences of X. Let $\underline{\mathrm{Aut}}(X)^*$ be the corresponding subcategory of $\underline{\mathrm{Top}}^*/\simeq$ with the single object X and with morphisms given by the elements in the group $\mathrm{Aut}(X)^*$. The $\underline{\mathrm{Aut}}(X)^*$–bimodule D_Σ yields the right $\mathrm{Aut}(X)^*$–module $[\Sigma X,X]$ with the action $x^\alpha = (\alpha^{-1})_*\alpha^*(x)$ for $x \in [\Sigma X,X]$ and $\alpha \in \mathrm{Aut}(X)^*$. The bracket is as well defined for the subcategory $\underline{C} = \underline{\mathrm{Aut}}(X)^*$ so that we have the element

(1) $$<\underline{\mathrm{Aut}}(X)^*> \in H^3(\underline{\mathrm{Aut}}(X)^*,D_\Sigma) \cong H^3(\mathrm{Aut}(X)^*,[\Sigma X,X]).$$

Here the isomorphism is given as in (2.5)(3) [4]; the right hand side of (1) is the usual cohomology of the group $\mathrm{Aut}(X)^*$ with coefficients in the right $\mathrm{Aut}(X)^*$–module $[\Sigma X,X]$ described above.

Now let $B\,\mathcal{E}(X)$ be the classifying space of the topological monoid $\mathcal{E}(X)$. We have the isomorphisms of groups

(2) $$\pi_1 = \pi_1 B\,\mathcal{E}(X) \cong \pi_0\,\mathcal{E}(X) = \mathrm{Aut}(X)^*$$

(3) $$\pi_2 = \pi_2 B\,\mathcal{E}(X) \cong \pi_1\,\mathcal{E}(X) = \pi_1(X^{*X},1_X) \underset{h_*}{\cong} \pi_1(X^{*X},0) = [\Sigma X,X]$$

The isomorphism h_* in (3) is given by the H–group structure $\oplus$ of X^{*X} which yields the homotopy equivalence $h : X^{*X} \longrightarrow X^{*X}$ with $h(f) = f \oplus 1_X$, $f \in X^{*X}$. One can check that the usual action of the fundamental group π_1 on the homotopy group π_2 coincides via the isomorphisms (2) and (3) with the action of $\mathrm{Aut}(X)^*$ on $[\Sigma X,X]$ described in (1) above. The first <u>k–invariant</u> k_2 of the space $B\,\mathfrak{F}(X)$ is an element in the group

(4) $$k_2 \in H^3(\pi_1,\pi_2) \cong H^3(\mathrm{Aut}(X)^*,[\Sigma X,X])$$

compare [1]. Here the isomorphism is induced by (2), (3). The next result is proved in (4.9).

(3.10) <u>Theorem</u>. Using the isomorphism in (1) and (4) above the bracket $<\underline{\underline{\mathrm{Aut}}}(X)^*>$ coincides with the first k–invariant k_2 of the classifying space $B\,\mathfrak{F}(X)$.

We point out that the first k–invariant in the theorem determines the first obstruction in Cooke's theory [8] for realizing homotopy G–actions on X by a topological G–action. A further example, relevant for algebraic K–theory, is considered in §7 below.

§4 <u>Normalized linear track extensions of categories</u>

Track categories are essentially the same as groupoid enriched categories; typical examples are given by the category $\underline{\mathrm{Top}}^*$ with 2–morphisms given by homotopies. Given a category $\underline{\underline{C}}$ and a natural system D on $\underline{\underline{C}}$ we introduce the notion of a linear track extension $\mathfrak{F}$ of $\underline{\underline{C}}$ to D in such a way that $\mathfrak{F}$ is a track category with homotopy category $\underline{\underline{C}}$ and with the number of tracks measured by D. Such linear track extension arise naturally in topology and lead to the bracket $<\underline{\underline{C}}>$ discussed in §3. For this we show that equivalence classes of linear track extensions are classified by the cohomology group $H^3(\underline{\underline{C}},D)$.

Recall that a groupoid is a small category whose morphisms are invertible. A track category denoted by $T\underline{\underline{K}}$ or by

(4.1) $$T \rightrightarrows \underline{\underline{K}}$$

is a category $\underline{\underline{K}}$ together with the following 'track structure' T. For all $A',A,B \in \mathrm{Ob}\,\underline{\underline{K}}$ groupoids $\underline{\underline{T}}(A,B)$ with $\mathrm{Ob}\,\underline{\underline{T}}(A,B) = \underline{\underline{K}}(A,B)$ and functors

(1) $$\underline{\underline{T}}(A,B) \times \underline{\underline{T}}(A',A) \xrightarrow{*} \underline{\underline{T}}(A',B)$$

are given. For $f,f^1 \in \underline{\underline{K}}(A,B)$ we call $T(f,f^1) = \underline{\underline{T}}(A,B)(f^1,f)$ the set of tracks (or homotopies) from f to f^1 and we write $H : f^1 \longrightarrow f$ or $H : f \simeq f^1$ for $H \in T(f,f^1)$. Composition in the groupoid $\underline{\underline{T}}(A,B)$ is written $+$ and is called addition of tracks. The functor (1), defined on the product groupoid $\underline{\underline{T}}(A,B) \times \underline{\underline{T}}(A',A)$, coincides on objects (f,g) with the composition in $\underline{\underline{K}}$, that is $*(f,g) = f \circ g$. Moreover $*$ carries the pair of tracks (H,G) with $H : f \simeq f^1$, $G : g \simeq g^1$ to the following track in $T(fg,f^1g^1)$,

(2) $$H*G = f_*G + (g^1)^*H = g^*H + (f^1)_*G.$$

Here we set $f_*G = o_f*G$, $g^*H = H*o_g$ where $o = o_f : f \simeq f$ denotes the trivial or zero track (which is the identity of f in the category $\underline{\underline{T}}(A,B)$). The negative of H is $-H$ with $H + (-H) = o_f$. The operation $*$ in (2) is associative and satisfies

(3) $$o_A*G = (1_A)_*G = G, \quad H*o_A = (1_A)^*H = H\,.$$

Up to the convention on track addition above a track category is the same as a "groupoid enriched category" or equivalently a category "based on the monoidal category of groupoids", compare [11]. We define a functor

(4) $$t : T\underline{\underline{K}} \longrightarrow T'\underline{\underline{K}}'$$

between track categories by a functor $t : \underline{\underline{K}} \longrightarrow \underline{\underline{K}}'$ and by functions $t = t_{f,g}: T(f,g) \longrightarrow T'(tf,tg)$ which are compatible with the structure above, that is $t(0) = 0$, $t(H + G) = tH + tG$, $t(f_*H) = (tf)_*(tH)$, $t(g^*H) = (tg)^*(tH)$. One readily checks that the relation $\simeq$ on morphisms of $\underline{\underline{K}}$, given by

(5) $$f \simeq g \Leftrightarrow T(f,g) \neq \phi,$$

is a natural equivalence relation which yields the quotient category $\underline{\underline{K}}/\simeq$. Clearly a functor as in (4) induces a functor $t : \underline{\underline{K}}/\simeq \longrightarrow \underline{\underline{K}}'/\simeq$ between homotopy categories.

(4.2) Example Let $\underline{\underline{A}}$ be a cofibration category with an initial object $*$. Then the full subcategory $\underline{\underline{K}} = \underline{\underline{A}}_{cf}$ consisting of cofibrant and fibrant objects in $\underline{\underline{A}}$ is a track category with tracks given by the homotopy set under (f,g), $T(f,g) = [I_*A,B]^{(f,g)}$, where $A \vee A \longrightarrow I_*A \longrightarrow A$ is a cylinder on A, compare (II S.6)[1]. The dual result holds for fibration categories.

(4.3) Definition: Let $\underline{\underline{C}}$ be a category and let D be a natural system on $\underline{\underline{C}}$. A linear track extension $\mathscr{E}$ of $\underline{\underline{C}}$ by D, denoted by

(1) $$D \xrightarrow{+} T \rightrightarrows \underline{\underline{K}} \xrightarrow[p]{} \underline{\underline{C}},$$

is defined by a track category as in (4.1), a functor p and an action of D on T as follows. The functor p is the identity on objects and is full, moreover p satisfies

(2) $$p(f) = p(g) \Leftrightarrow f \simeq g$$

so that p induces an isomorphism $\underline{\underline{K}}/\simeq \cong \underline{\underline{C}}$. The action of D on T is given by isomorphisms of groups

(3) $$\sigma = \sigma_f : D_{pf} \cong T(f,f), \quad f \in \operatorname{Mor} \underline{\underline{K}},$$

such that (4) and (5) hold:

(4) $$\sigma_f(\alpha) + H = H + \sigma_h(\alpha) \text{ for } H \in T(f,h),$$

(5) $$\begin{cases} g^*\sigma_g(\beta) = \sigma_{fg}(g^*\alpha), & \alpha \in D_{pf}, \\ f_*\sigma g(\beta) = \sigma_{fg}(f_*\beta), & \beta \in D_{pg}. \end{cases}$$

(4.4) Definition: Consider a linear track extension $\mathcal{E}$ as in (4.3). We choose functions

(1) $$\begin{cases} t : \operatorname{Mor} \underline{\underline{C}} \longrightarrow \operatorname{Mor} \underline{\underline{K}} \\ H : N_2\underline{\underline{C}} \longrightarrow \bigcup_{f,g \in \operatorname{Mor}(\underline{\underline{K}})} T(f,g) \end{cases}$$

with $pt = 1$ and $H(f,g) \in T(tf \circ tg, t(fg))$. We define the cochain

(2) $$c_{\mathcal{E}}(t,H) : N_3(\underline{\underline{C}}) \longrightarrow \bigcup_{f \in \operatorname{Mor} \underline{\underline{C}}} D(f)$$

by the element $c_{\mathcal{E}}(t,H)(f,g,h) \in D(fgh)$ which is obtained by the "operation of pasting" that is

(3) $$c_{\mathcal{E}}(t,H)(f,g) = \sigma^{-1}_{t(fgh)}(\Delta) \text{ with}$$

$$\Delta = -H(f,gh) - (tf)_*H(g,h) + (th)^*H(f,g) + H(fg,h).$$

By lemma (A.1) below we see that $c_{\mathcal{E}}(t,H)$ is a cocycle.

(4.5) Definition: Let S be an ideal in $\underline{\underline{C}}$, let D be S–trivial and let $\underline{\underline{C}}^1 \subset \underline{\underline{C}}$ be a subcategory. We say that $(\mathcal{E}, j, J)$ is an S–normalized linear track extension of $(\underline{\underline{C}}, \underline{\underline{C}}^1)$ by D if a function $j : S \cup \operatorname{Mor} \underline{\underline{C}}^1 \longrightarrow \operatorname{Mor} \underline{\underline{K}}$ and tracks $J(f_0,g) \in T(f_0 \circ j(g), j(fg))$ $J(g,h_0) \in T(j(g) \circ h_0, j(gh))$ are given for $g \in S$, $(f,g,h) \in N_3(\underline{\underline{C}})$, $f_0 \in p^{-1}(f)$, $h_0 \in p^{-1}(h)$ and for $(f,g) \in N_2(\underline{\underline{C}}^1)$ with $f_0 = j(f)$. Moreover the following two properties are satisfied:
(1) For $g \in S$, $(f,g,h) \in N_3(\underline{\underline{C}})$ and $F \in T(f_0,f_1)$ with $f_0,f_1 \in p^{-1}(f)$, $H \in T(h_0,h_1)$ with (h_0,h_1) with $h_0,h_1 \in p^{-1}(h)$ the equations $J(f_0,g) = j(g)^*F + J(f_1,g)$, $J(g,h_0) = j(g)_*H^* + J(g,f_1)$ hold.
(2) For any choice of t and H in (4.4) which extend j and J respectively the cocycle

$c_{\mathcal{T}}(t,H)$ is an S–normalized cocycle rel $\underline{\underline{C}}^1$. This in particular means that the restriction of (j,J) to $\underline{\underline{C}}^1$ is a pseudo functor in the sense of [11].

A map $t : (\mathcal{T},j,J) \longrightarrow (\mathcal{T}',j',J')$ between such track extensions is a functor t as in (4.1)(4) such that $pt = p$, $tj = j'$, $t\sigma_f = \sigma_{tf,tf}$ and $tJ = J'$. Whence S–normalized linear track extensions of $(\underline{\underline{C}},\underline{\underline{C}}^1)$ by D form a category which we denote by Track. Two objects are equivalent and we write $(\mathcal{T},j,J) \sim (\mathcal{T}',j',J')$ if there exist maps $(\mathcal{T},j,J) \longleftarrow (\mathcal{T}'',j'',J'') \longrightarrow (\mathcal{T}',j',J')$ in Track. The equivalence classes form the class of connected components of the category Track which we denote by

(3) $$\pi_0\mathrm{Track}_S(\underline{\underline{C}},\underline{\underline{C}}^1;D) = \mathrm{Ob}(\mathrm{Track})/\sim.$$

(4.6) Theorem: There is a canonical bijection

$$\Psi : \pi_0\mathrm{Track}_S(\underline{\underline{C}},\underline{\underline{C}}^1;D) \cong H^3_S(\underline{\underline{C}},\underline{\underline{C}}^1;D)$$

The bijection carries the equivalence class of $(\mathcal{T},j,J)$ to the cohomology class of the cocycle $c_{\mathcal{T}}(t,H)$ in (4.5)(2).

We define the trivial track extension in Track by $\underline{\underline{K}} = \underline{\underline{C}}$, $p = 1_{\underline{\underline{C}}}$, and $T(f,g) = D(f)$ for $f = g$ and $T(f,g) = \phi$ for $f \neq g$, moreover $\sigma_f = 1_{D(f)}$. The tracks J are given by the zero elements in $D(f)$, $f \in \mathrm{Mor}(\underline{\underline{C}})$. Clearly the bijection Ψ in (4.6) carries the trivial track extension to the zero cohomology class.

(4.7) Example: Let $\underline{\underline{C}} \subset \underline{\underline{coH}} \subset \underline{\underline{Top}}^*/\simeq$ be a category as in (3.1) and let $E\underline{\underline{C}}$ be the full subcategory of $\underline{\underline{Top}}^*$ consisting of objects in $\underline{\underline{C}}$. Then we obtain by (4.2) the linear track extension $\mathcal{T}(\underline{\underline{C}})$:

(1) $$D_\Sigma \xrightarrow{+} T \rightrightarrows E\underline{\underline{C}} \longrightarrow \underline{\underline{C}}\,.$$

Here we use for $f: A \longrightarrow B$ in $\underline{\underline{E}}\underline{\underline{C}}$ the isomorphism

(2) $$\sigma_f^{-1} : T(f,f) = [I_*A,B]^{f,f} = \pi_1(B^{*A},f) \cong [\Sigma A,B]$$

which is defined similarly as in (3.9)(3) by the co–H–space structure of A. One can check that σ_f satisfies all properties in (4.3). The <u>bracket</u> $<\underline{\underline{C}}>$ is now simply defined by the cohomology class

(3) $$<\underline{\underline{C}}> = \Psi\{\chi(\underline{\underline{C}})\}$$

in (4.6) above. The class $<\underline{\underline{C}}>$ is more generally defined for any subcategory $\underline{\underline{C}} \subset \underline{\underline{\text{co–H}}}(\underline{\underline{A}})$ where $\underline{\underline{A}}$ is a cofibration category and where $\underline{\underline{\text{co–H}}}(\underline{\underline{A}}) \subset \underline{\underline{A}}_{cf}/\simeq \text{rel}_*$ is the full subcategory consisting of co–H–groups in $\underline{\underline{A}}$, see [1]. In this case we get σ_f in (2) similarly as in (II. 10.18) [1]. Dually we obtain the class $<\underline{\underline{C}}>$ as well for any subcategory $\underline{\underline{C}} \subset \underline{\underline{H}}(\underline{\underline{B}})$ where $\underline{\underline{B}}$ is a fibration category and where $\underline{\underline{H}}(\underline{\underline{B}}) \subset \underline{\underline{B}}_{fc}/\simeq \text{rel}_*$ is the full subcategory consisting of H–groups in $\underline{\underline{B}}$. This shows that $<\underline{\underline{C}}>$ is as well defined for $\underline{\underline{C}} \subset \underline{\underline{H}}$ in (3.1) when we use the fibration category $\underline{\underline{\text{Top}}}^*$.

(4.8) <u>Proof of theorem</u> (3.3): Using the definition of $<\underline{\underline{C}}>$ above we see immediately by (4.4)(3) that $(f,g,h)^*<\underline{\underline{C}}>$ represents the triple Toda bracket $<f,g,h>$; compare Toda's definition of $<f,g,h>$ in [29].

(4.9) <u>Example</u>: We identify a monoid $(M,\cdot,1)$ with the category $\underline{\underline{M}}$ with a single object $*$ and with $M = \underline{\underline{M}}(*,*)$. Now let M be a <u>topological</u> <u>monoid</u> and let $\pi_0 M$ be the monoid of path components of M with the projection $p: M \longrightarrow \pi_0 M$ which carries $m \in M$ to the component $p(m) = M_m$ with $m \in M_m$. We assume that the Hurewicz homomorphism

(1) $$\sigma : \pi_1(M_m,m) \xrightarrow{\cong} H_1(M_m)$$

is an isomorphism for all $m \in M$. In this case the <u>fundamental groupoid</u> πM of M yields the linear track extension

$$\pi_1 M \xrightarrow{+} \pi M \rightrightarrows M \longrightarrow \pi_0 M . \tag{2}$$

Here $\pi_1 M$ is the natural system on $\pi_0 M$ which carries $M_m \in \pi_0 M$ to the homology $H_1(M_m)$; induced functions are $p(m)_* = H_1(\ell_m)$ and $p(n)^* = H_1(r_n)$ with $m \cdot n = \ell_m(n) = r_n(m)$. Let

$$\Psi\{\pi M\} \in H^3(\pi_0 M; \pi_1 M) \tag{3}$$

be the cohomology class associated to the track extension (2), see (4.4). In case $\pi_0 M$ is a group we have as well the first k–invariant

$$k_2 \in H^3(\pi_1 BM, \pi_2 BM) \cong H^3(\pi_0 M, \pi_1 M) \tag{4}$$

of the classifying space BM of M. Here the isomorphism is induced by $\pi_n BM \cong \pi_{n-1} M$, $n = 1,2$. More generally as in (3.10) we get

$$k_2 = \Psi\{\pi M\} . \tag{5}$$

For the proof it is enough to consider monoids $M = \Omega X$ where ΩX is the Moore loop space of a 2–dimensional CW–complex X^2 with trivial 0–skeleton $X^0 = *$. In this case k_2 is represented by a cocyle $c(\partial)$ which is determined by the crossed module $\partial : \pi_2(X^2,X^1) \longrightarrow \pi_1(X^1)$; see [22] or [15]. This cocycle corresponds exactly to the cocycle (4.4)(3) which represents $\Psi\{\pi M\}$. On the other hand (5) can be deduced from a result of Igusa, compare B. 1.1. [16].

§5 <u>An algebraic model for the track category of one point unions of n–spheres</u>.

Using free nil(2)–groups we define an algebraic track category which is equivalent to the topological track category of one point unions of n–spheres. This is the crucial step for the computation of the bracket $<\underline{\underline{S}}(n)>$ in (3.7). Recall that $\underline{\underline{S}}(n) \subset \underline{\underline{Top}}^*/\simeq$ denotes the full subcategory consisting of finite one point unions of n–spheres. By (4.7) we have the linear track extension $\mathscr{E}(\underline{\underline{S}}(n))$,

$$(5.1) \qquad D_\Sigma \xrightarrow{+} T \rightrightarrows E\underline{\underline{S}}(n) \longrightarrow \underline{\underline{S}}(n), \ n \geq 2,$$

which is defined topologically. We now describe an algebraic model of this track extension. For this we identify $\underline{\underline{S}}(n)$ with the category of finitely generated free abelian groups $\underline{\underline{M}}_{\mathbb{Z}}$, see (3.6)(1), and we identify the bimodule D_Σ in (5.1) via (3.6)(2) with the bimodule D_n. Therefore (5.1) corresponds to a linear track extension of $\underline{\underline{M}}_{\mathbb{Z}}$ by D_n.

(5.2) Theorem: The topological track extension $\mathscr{E}(\underline{\underline{S}}(n))$ in (5.1) is equivalent to the algebraic track extension $\mathscr{E}_n(\underline{\underline{M}}_{\mathbb{Z}})$ defined via free nil(2)–groups in (5.5) below.

Here we use the notion of equivalence in (4.3)(8) where we set $S = \phi$, $\underline{\underline{C}}^1 = \phi$.

A free nil(2)–group is the quotient $F/\Gamma_3 F$ where F is a free group and where $\Gamma_3 F$ is the subgroup generated by triple commutators in F. For a free abelian group A we choose a free nil(2)–group G_A with abelianization $G_A^{ab} = A$. We write the group structure of G_A additively. One has the well known exact sequence

$$(5.3) \qquad 0 \longrightarrow \Lambda^2 A \xrightarrow[i_A]{} G_A \xrightarrow[p_A]{} A \longrightarrow 0$$

where $\Lambda^2 A = A \wedge A$ is the exterior product and where $i_A(x \wedge y) = -x - y + x + y$ is the commutator. Let $\underline{\underline{nil}}$ be the full subcategory of the category of groups consisting of all G_A with $A \in Ob(\underline{\underline{M}}_{\mathbb{Z}})$. There is the canonical linear extension of categories

(5.4) $$\mathrm{Hom}(\_,\Lambda^2) \xrightarrow{+} \underline{\underline{\mathrm{nil}}} \xrightarrow{p} \underline{\underline{M}}_{\mathbb{Z}}$$

where p is the abelianization functor and where the action of $\alpha \in \mathrm{Hom}(A,\Lambda^2 B)$ on morphisms $f_0 : G_A \longrightarrow G_B$ in $\underline{\underline{\mathrm{nil}}}$ is given by $f_0 + \alpha = f_0 + i_B \alpha p_A$. Moreover, we obtain for $n \geq 2$ the following linear track extension $\mathcal{E}_n(\underline{\underline{M}}_{\mathbb{Z}})$:

(5.5) $$\mathrm{Hom}(\_,\Gamma_n^1) \xrightarrow{+} T_n \rightrightarrows \underline{\underline{\mathrm{nil}}} \xrightarrow{p} \underline{\underline{M}}_{\mathbb{Z}} .$$

Here the functor p is the same as in (5.4). For morphism $f,f^1 : G_A \longrightarrow G_B$ in $\underline{\underline{\mathrm{nil}}}$ with $p(f) = p(f^1)$ there is a unique homomorphism

(1) $$\Delta(f,f^1) : A \longrightarrow B \wedge B$$

with $f + \Delta(f,f^1) = f^1$. We now consider the commutative diagram

(2) $$\begin{array}{ccccccccc} & & & & & & A & & \\ & & & & & \overset{H}{\swarrow} & \downarrow{\scriptstyle \Delta(f,f^1)} & & \\ 0 \longrightarrow & \Gamma B & \xrightarrow{\tau} & B \otimes B & \xrightarrow{q} & & B \wedge B & \longrightarrow 0 \\ & \sigma\downarrow & O & \downarrow\bar{\sigma} & & & \| & & \\ 0 \longrightarrow & B \otimes \mathbb{Z}/2 & \xrightarrow{\bar{\tau}} & B \hat{\otimes} B & \xrightarrow{\bar{q}} & & B \wedge B & \longrightarrow 0 \end{array}$$

in which the rows are natural exact and in which the subdiagram O is a push out of abelian groups. For the definition of the homomorphism in (2) recall that the functor Γ is equipped with the universal quadratic map $\gamma : B \longrightarrow \Gamma B$ and that τ and σ are defined by $\tau\gamma(b) = b \otimes b$, $\sigma\gamma(b) = b \otimes 1$, $b \in B$; moreover $q(b \otimes b') = b \wedge b'$. We now define the track structure T_n in (5.5) by use of q and $\bar{q}$ in (2), namely

(3) $$\begin{cases} T_2(f,f^1) = \{H : qH = \Delta(f,f^1)\} \subset \mathrm{Hom}(A,B \otimes B) \text{ and} \\ T_n(f,f^1) = \{H : \bar{q}H = \Delta(f,f^1)\} \subset \mathrm{Hom}(A,B \hat{\otimes} B) \text{ for } n \geq 3. \end{cases}$$

Addition of tracks is defined by addition of homomorphisms and induced functions (4.1)(2) are given by composition of homomorphisms

(4)
$$\begin{cases} f_*G = \begin{cases} (\otimes^2 p(f))G & \text{for } n = 2, \\ (\hat{\otimes}^2 p(f))G & \text{for } n \geq 3, \end{cases} \\ g^* H = H\, p(g) \quad . \end{cases}$$

Next we define the isomorphism

(5)
$$\sigma = \sigma_f : \mathrm{Hom}(A, \Gamma_n^1 B) \cong T_n(f,f)$$

by $\sigma(\alpha) = \tau\alpha$ for $n = 2$ and $\sigma(\alpha) = \bar{\tau}\alpha$ for $n \geq 3$. This completes the definition of the linear track extension (5.5); one readily checks that all properties in (4.1) and (4.3) are actually satisfied.

For the proof of (5.2) it is enough to construct maps between linear track extensions

(5.6)
$$\mathcal{E}(\underline{\underline{S}}(n)) \xleftarrow{\ i\ } \mathcal{E}_n \xrightarrow{\ t\ } \mathcal{E}_n(\underline{\underline{M}}_{\mathbb{Z}}) \,.$$

Here $\mathcal{E}_n$ is the subextension of (5.1) given by the subcategory $\underline{\underline{E}}_n \subset E\underline{\underline{S}}(n)$ consisting of all maps in $E\underline{\underline{S}}(n)$ which are (n–1)–fold suspensions $\Sigma^{n-1}f$ with f in $E\underline{\underline{S}}(1)$. Moreover the map i in (5.6) is the inclusion; clearly this is a map in <u>Track</u> since each homotopy class in $\underline{\underline{S}}(n)$ can be represented by a map in E_n. The map t is more complicated. The functor

(1)
$$t : \underline{\underline{E}}_n \longrightarrow \underline{\underline{nil}}$$

carries the object $S^n \vee \dots \vee S^n = \Sigma^{n-1}X$ to the free nil(2)–group

(2)
$$G_X = \pi_1(X)/\Gamma_3\pi_1(X)$$

with $X = S^1 \vee \dots \vee S^1$. Moreover t carries the morphism $\Sigma^{n-1}f$ to the homomorphism induced by $\pi_1(f)$. In (2) we identify $G_X^{ab} = \pi_1(X)^{ab} = H_n(\Sigma^{n-1}X)$, $n \geq 1$. This shows that t induces the functor (3.6)(1) on homotopy categories. Next we define the map t in (5.6) on tracks, that is

$$(3) \qquad t = t_{f,g} : T(\Sigma^{n-1}f, \Sigma^{n-1}g) \xrightarrow{\approx} T_n(tf,tg),$$

as follows. We first consider the case $n = 2$. Let $f,g : X \longrightarrow Y$ be maps in $\underline{\mathrm{Top}}^*$ where X and Y are finite one point unions of 1–spheres. Then we get $t_{f,g}$ in (3) by the composition of the following bijections.

$$\begin{aligned}
& T(\Sigma f,\Sigma g) = [I_*\Sigma X,\Sigma Y]^{\Sigma f,\Sigma g} \\
(4) \quad & \cong [I_*X,\Omega\Sigma Y]^{if,ig} \\
(5) \quad & \cong [I_*X,J(Y)]^{jf,jg} \\
(6) \quad & \cong [\rho I_*X,\rho J(Y)]^{f_*,g_*} \\
(7) \quad & \cong [I_*\rho X,\rho^Y]^{hf_*,hg_*} \\
(8) \quad & \cong T_2(tf,tg) .
\end{aligned}$$

In (4) we use the adjunction and the map $i : Y \longrightarrow \Omega\Sigma Y$ adjoint to the identity of ΣY. For the infinite reduced product $J(Y)$ of James [17] we have the homotopy equivalence $J(Y) \simeq \Omega\Sigma Y$ which carries the inclusion $j : Y \subset J(Y)$ to i and which induces the bijection (5). Next we use the crossed chain complex $\rho(Z)$ of a CW–complex Z with $Z^0 = *$, this is the homotopy system of Z introduced by J.H.C. Whitehead [31], compare also VI, §1 [1]. Recall that $\rho = \rho(Z)$ is given by the homotopy groups $\rho_n = \pi_n(Z^n,Z^{n-1})$, $n \geq 2$, and $\rho_1 = \pi_1 Z^1$ and by the obvious boundary maps $\partial_n : \rho_n \longrightarrow \rho_{n-1}$. The cylinder $I_*\rho$ is defined in such way that $I_*\rho(Z) = \rho(I_*Z)$. Using this cylinder we define relative homotopy sets as in (II 2.3)a) [1] so that the set in (6) is defined with $f_* = \rho(if)$. The bijection (6) carries a track represented by a cellular map $H : I_*X \longrightarrow J(Y)$ to the class of $\rho(H)$ which extends (f_*,g_*). The crucial point for the existence of t in (3), however, is the construction of the natural map

$$(9) \qquad h : \rho(J(Y)) = \rho \longrightarrow \rho^Y$$

in the category of crossed chain complexes. We define ρ^Y by $\rho^Y_n = 0$ for $n \geq 3$ and by $\rho^Y_2 = H_1Y \otimes H_1Y$, $\rho^Y_1 = G_Y$, see (2). The boundary $\partial : \rho^Y_2 \to \rho^Y_1$ is the composition

$$\partial : H_1Y \otimes H_1Y \xrightarrow[q]{} H_1Y \wedge H_1Y \rightarrowtail_{i} G_Y$$

where i is the commutator map as in (5.3) and where q is the projection (5.5)(2). We define h in (9) by the commutative diagram

$$\begin{array}{ccccc} \rho_2 = \pi_2(J^2,J^1) & \xrightarrow[h_2]{} & H_2(J^2, J^1) \cong \rho^Y_2 \\ \downarrow \partial_2 & & \downarrow \partial \\ \rho_1 = \pi_1 J^1 = \pi_1 Y & \xrightarrow[h_1]{} & G_Y = \rho^Y_1 \end{array} \tag{10}$$

where h_1 is the projection and where h_2 is the Hurewicz map with $J = J(Y)$. The isomorphism in the top row of (10) is given by the multiplication in J. The map ∂ in (10) is a crossed module with the trivial action of ρ^Y_1 on $\rho_2{}^Y$ and the map ∂_2 is a free crossed module generated by the attaching maps of 2–cells in J. This shows that the diagram commutes, moreover, one can check:

(11) Lemma: The map h in (9) is a well defined map between crossed chain complexes which induces isomorphisms $h_* : \pi_n\rho \cong \pi_n\rho^Y$, for $n \leq 2$, where $\pi_n\rho = \ker \partial_n / \mathrm{im}\, \partial_{n+1}$.

The lemma implies that the induced map h_* in (7) is a bijection as well. Moreover the bijection (8) is a direct consequence of the definitions. Finally define t in (3) for $n \geq 3$ by the commutative diagram

$$\begin{array}{ccc} T(\Sigma f, \Sigma g) & \xrightarrow{t} & T_2(tf, tg) \\ \downarrow \Sigma^{n-2} & & \downarrow \mathrm{Hom}(1,\overline{\sigma}) \\ T(\Sigma^{n-1} f, \Sigma^{n-1} g) & \xrightarrow[t]{} & T_n(tf, tg) \end{array} \tag{12}$$

where Σ^{n-2} is the (n–2)–fold suspension and where $\overline{\sigma}$ is the homomorphism in (2). This completes the definition of t in (5.6); one can check that all properties of a map in Track are satisfied by t.

§6 Bockstein homomorphisms and cup products

In this section we study certain cohomology groups of the category $\underline{\underline{M}}_{\Pi}$ of finitely generated free abelian groups. Let $M, M', M'' : \underline{\underline{M}}_{\Pi} \longrightarrow \underline{\underline{Ab}}$ be functors, then we write for short

(6.1)
$$\begin{cases} H^n(M) = H^n(\underline{\underline{M}}_{\Pi}, \mathrm{Hom}(\_, M)), \\ H^n(M'', M') = H^n(\underline{\underline{M}}_{\Pi}, \mathrm{Hom}(M'', M')), \end{cases}$$

compare (1.2). A short exact sequence

(1)
$$\mathcal{M} : 0 \longrightarrow M' \xrightarrow{j} M \xrightarrow{q} M'' \longrightarrow 0$$

of functors as usual induces an exact sequence

(2)
$$\longrightarrow H^n(M') \longrightarrow H^n(M) \longrightarrow H^n(M'') \xrightarrow{\beta} H^{n+1}(M') \longrightarrow \ldots$$

where $\beta = \beta(\mathcal{M})$ is the Bockstein homomorphism. Now assume that $\mathcal{M}$ admits a pointwise splitting r (i.e. a family of homomorphisms $r_A : MA \longrightarrow M'A$ with $r_A j_A = 1$, $A \in \mathrm{Ob}\underline{\underline{M}}_{\Pi}$). Then we have the cohomology class

(3)
$$\eta = \eta(\mathcal{M}) = \{c_r\} \in H^1(M'', M')$$

which is the obstruction for the existence of a natural splitting r of $\mathcal{M}$. We define the cocycle c_r by $c_r(\lambda)q_A = -r_B M(\lambda) + M'(\lambda)r_A$ for $\lambda : A \longrightarrow B$ in $\underline{\underline{M}}_{\Pi}$. Using the composition pairing

(4)
$$\mathrm{Hom}(A, M''B) \otimes \mathrm{Hom}(M''B, M'C) \longrightarrow \mathrm{Hom}(A, M'C)$$

with A,B,C $\in Ob(\underline{\underline{M}}_{\mathbb{Z}})$ we get the cup product

(5) $$\cup : H^n(M'') \otimes H^m(M'',M') \longrightarrow H^{n+m}(M'),$$

compare (IV. 5.18)[1].

(6.2) Lemma: The Bockstein β and the class η above satisfy the equation $\beta(\xi) = -\xi \cup \eta$ for all $\xi \in H^n(M'')$, $n \in \mathbb{Z}$.

Proof: Let c'' be a cocycle in ξ. Then $\xi \cup \eta$ is represented by the cocycle $c'' \cup c_r$ with

(1) $$(c'' \cup c_r)(\lambda_1,\dots,\lambda_{n+1}) = c_r(\lambda_1) \circ c''(\lambda_2,\dots,\lambda_{n+1})$$

for $(\lambda_1,\dots,\lambda_{n+1}) \in N_{n+1}(\underline{\underline{M}}_{\mathbb{Z}})$. On the other hand a cocycle $j^{-1}\delta c \in \beta\{c''\}$ is obtained by any cochain c for which $qc(\lambda_1,\dots,\lambda_n) = c''(\lambda_1,\dots,\lambda_n)$ with q in (6.1). Since $q(\delta c) = 0$ the cocycle $j^{-1}\delta c$ is well defined. Now let a splitting $\bar{r} : M''A \longrightarrow MA$ be given by $\bar{r}q(x) = x - jr_A x$, $x \in MA$. Then we can choose c by $c(\lambda_1,\dots,\lambda_n) = \bar{r} \circ c''(\lambda_1,\dots,\lambda_n)$ and we check $\delta c = -c'' \cup c_r$ by definition of δ in (1.3). In fact we get

(2) $$\begin{aligned} j^{-1}(\delta c)(\lambda_1,\dots,\lambda_{n+1}) &= r(\lambda_1)_* \bar{r} c''(\lambda_2,\dots,\lambda_{n+1}), \\ &= -(c'' \cup c_r)(\lambda_1,\dots,\lambda_{n+1}) . \end{aligned}$$

We know that for the functor $\otimes^2$ with $\otimes^2 A = A \otimes A$ the cohomology

(6.3) $$H^n \otimes^2 = 0$$

is trivial for all n, compare theorem (iii) [18]. The following exact sequences of functors $\underline{\underline{M}}_{\mathbb{Z}} \longrightarrow \underline{\underline{Ab}}$ are of importance to us:

$$\begin{aligned}
\mathcal{M}_0 &: 0 \longrightarrow \otimes\mathbb{Z}/2 \xrightarrow{\bar\tau} \hat\otimes^2 \xrightarrow{\bar q} \Lambda^2 \longrightarrow 0\\
\mathcal{M}_1 &: 0 \longrightarrow \Gamma \xrightarrow{\tau} \otimes^2 \xrightarrow{q} \Lambda^2 \longrightarrow 0\\
\mathcal{M}_2 &: 0 \longrightarrow \Lambda^2 \xrightarrow{\tilde\tau} \otimes^2 \xrightarrow{\tilde q} SP^2 \longrightarrow 0\\
\mathcal{M}_3 &: 0 \longrightarrow SP^2 \xrightarrow{\bar\omega} \otimes^2 \xrightarrow{\bar\sigma} \hat\otimes^2 \longrightarrow 0\\
\mathcal{M}_4 &: 0 \longrightarrow SP^2 \xrightarrow{\omega} \Gamma \xrightarrow{\sigma} \otimes\mathbb{Z}/2 \longrightarrow 0\,.
\end{aligned}$$

The sequences $\mathcal{M}_0$ and $\mathcal{M}_1$ are defined in (5.4)(2), moreover we define the <u>symmetric product</u> $SP^2(B) = B \,\tilde\otimes\, B$ by the cokernel of $\tilde\tau$ with $\tilde\tau(a \wedge b) = a \otimes b - b \otimes a$. For $\tilde q(a \otimes b) = a \,\tilde\otimes\, b$ we set $\omega(a \,\tilde\otimes\, b) = \gamma(a + b) - \gamma(a) - \gamma(b)$ and $\bar\omega(a \,\tilde\otimes\, b) = \tau\omega(a \,\tilde\otimes\, b) = a \otimes b + b \otimes a$, compare also [28]. By (6.3) the Bockstein homomorphisms $\beta_i = \beta(\mathcal{M}_i)$ $(i = 1,2,3)$ are isomorphisms

(6.4) $$\beta_1\beta_2\beta_3 : H^n\hat\otimes^2 \cong H^{n+1}SP^2 \cong H^{n+2}\Lambda^2 \cong H^{n+3}\Gamma$$

for $n \in \mathbb{Z}$. The sequences $\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2$ are pointwise split so that the classes

(1) $$\eta_i = \eta(\mathcal{M}_i) \text{ with } \beta_i(\xi) = -\xi \cup \eta_i$$

are defined for $i = 0,1,2$. Here, however, β_0 needs not to be an isomorphism.

(6.5) <u>Lemma</u>: $H^0(\hat\otimes^2) = \mathbb{Z}/2$ and $H^i(\hat\otimes^2) = 0$ for $i = 1,2,3$.

The Lemma and (6.4) imply

(1) $$H^n(\Gamma) = \begin{cases} \mathbb{Z}/2 & \text{for } n = 3 \\ 0 & \text{for } i < 3 \text{ and } i = 4,5,6\,. \end{cases}$$

Moreover we see that the Bockstein homomorphisms

(2) $\beta_0 : H^n(\Lambda^2) \cong H^{n+1}(\otimes \mathbb{Z}/2),\ n = 1,2,$

(3) $\beta_4 : H^n(\otimes \mathbb{Z}/2) \cong H^{n+1}(SP_2),\ n \le 5,$

are isomorphisms. By (3) one gets the isomorphism

(4) $\bar{\tau}_* : H^n(\otimes \mathbb{Z}/2) \cong H^n(\hat{\otimes}^2),\ n \le 5,$

since $\beta_3 \bar{\tau}_* = \beta_4$.

Proof of (6.5): The group $H^0\hat{\otimes}^2$ is the group of natural homomorphisms $A \longrightarrow \hat{\otimes}^2 A$ which can be computed by setting $A = \mathbb{Z}$. This shows that the generator of $H^0\hat{\otimes}^2 = \mathbb{Z}/2$ is the natural map $\epsilon : A \twoheadrightarrow A \otimes \mathbb{Z}/2 \rightarrowtail \hat{\otimes}^2 A$ given by $\mathcal{M}_0$ in (5.3)(2). Next we know $H^1\hat{\otimes}^2 = 0$ since $H^2SP^2 = 0$ by [18], see (6.4). Moreover, Hartl [14] shows $H^2\hat{\otimes}^2 = 0$ by use of his theory of 'quadratic rings'. Next we consider the exact sequence (6.1)(2) for $\mathcal{M}_0$ in (5.3)(2). This gives us the exact sequence

(1) $$0 \longrightarrow H^2\Lambda^2 \xrightarrow{\beta_0} H^3 \otimes \mathbb{Z}/2 \longrightarrow H^3\hat{\otimes}^2 \longrightarrow 0$$

where $H^2\Lambda^2 = \mathbb{Z}/2$ and $H^3\Lambda^2 = 0$ by (6.4). Thus β_0 is an isomorphism and $H^3\hat{\otimes}^2 = 0$ since we have

(2) $$H^3 \otimes \mathbb{Z}/2 = \mathbb{Z}/2.$$

For this we use the exact sequence in theorem (ii) [18] with $T(A) = A \otimes \mathbb{Z}/2$, namely

$$0 \longrightarrow \mathrm{Shukla}^3(\mathbb{Z},\mathbb{Z}/2) \longrightarrow H^3\otimes\mathbb{Z}/2 \longrightarrow \mathrm{Shukla}^0(\mathbb{Z},\mathbb{Z}/2) \longrightarrow \mathrm{Shukla}^4(\mathbb{Z},\mathbb{Z}/2)$$

where $\mathrm{Shukla}^n(\mathbb{Z},\mathbb{Z}/2) = \mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/2)$ by [26].

Remark: The result $H^2\hat{\otimes}^2 = 0$ used in the proof above is as well a consequence of Igusa's result $k^{N,n} \neq 0$ for $N \ge 4$, compare (7.6) below. In fact, $H^2\hat{\otimes}^2 \neq 0$ would imply that β_0 in (1) above is trivial and this would imply that $\langle \underline{S}(n) \rangle$ is trivial for $n \ge 3$.

Next we observe that via (2.3) the linear extension $\underline{\underline{\text{nil}}}$ in (5.4) yields an element $\{\text{nil}\} \in H^2\Lambda^2 \cong \mathbb{Z}/2$. This element is the generator since we show

(6.6) Lemma: $$\{\underline{\underline{\text{nil}}}\} = \beta_2\beta_3(\epsilon) = \beta_3(\epsilon) \cup \eta_2 \neq 0$$

Here $\epsilon \in H^0\, \hat{\otimes}^2 \cong \mathbb{Z}/2$ denotes the generator. It is easy to check that $\underline{\underline{\text{nil}}}$ admits no splitting on maps T,Δ where $\Delta : \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is the diagonal and where $T : \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ is the interchange map, this implies $\{\underline{\underline{\text{nil}}}\} \neq 0$. Whence we get (6.6) by (6.4) and (6.5).

(6.7) Lemma: The linear track extension $\mathscr{E}_2(\underline{\underline{M}}_{\mathbb{Z}})$ in (5.5) satisfies $\Psi\{\mathscr{E}_2(\underline{\underline{M}}_{\mathbb{Z}})\} = -\beta_1\{\underline{\underline{\text{nil}}}\} = \{\underline{\underline{\text{nil}}}\} \cup \eta_1$.

Proof: The second equation is a special case of (6.2) associated to the sequence $\mathscr{M}_1$. Now choose a cocycle $c'' = \Delta_s \in \underline{\underline{\text{nil}}}$ as in (2.3)(1). We define for the same s the cocycle $c(s,H) \in \Psi\{\mathscr{E}_2(\underline{\underline{M}}_{\mathbb{Z}})\}$, see (4.6)(3), by choosing H with $qH(f,g) = \Delta_s(f,g)$. Now H is a 2–cochain in $\text{Hom}(\_,\otimes^2)$ and we observe that for δ in (1.8) $(\delta H)(f,g,h) = -\Delta$ where Δ is defined in (4.4). By definition of σ in (5.5)(5) we obtain now the first equation in (6.7).

(6.8) Lemma: $$\eta_1 \neq 0 \text{ and } 2\eta_1 = 0.$$

Proof: Let $\alpha \in F^0(\underline{\underline{M}}_{\mathbb{Z}},\text{Hom}(\Lambda^2,\Gamma))$ be given by $\alpha_A : A \wedge A \longrightarrow \Gamma A$ with $\alpha_A(a \wedge b) = 2r_A(a \otimes b) - [a,b]$ where $[a,b] = \gamma(a + b) - \gamma(a) - \gamma(b)$. Then we have $2c_r(f) = \delta(\alpha)$ for $\{c_r\} = \eta_1$. Now let $A = \mathbb{Z} \oplus \mathbb{Z}$, if $\eta_1 = 0$ there is r_A with $r_A(T \otimes T) = \Gamma(T)r_A$; but this is not possible.

We are ready for the proof of theorem (3.7). For this we give the following descriptions of the cohomology class $<\underline{\underline{S}}(n)>$.

(6.9) Theorem: Using the identification (3.6)(4) we obtain the equations

$$<\underline{\underline{S}}(n)> = \beta_j\beta_2\beta_3(\epsilon) = \beta_j\{\underline{\underline{nil}}\} = \{\underline{\underline{nil}}\} \cup \eta_j = (\beta_3\epsilon) \cup \eta_2 \cup \eta_j$$

where $j = 1$ for $n = 2$ and $j = 0$ for $n \geq 3$.

All β_i $(i = 0,1,2,3)$ in (6.8) are isomorphisms by (6.4) and (6.5). This implies (3.7). The theorem shows that the cup products

$$\eta_2 \cup \eta_1 \in H^2(SP^2,\Gamma) \quad \text{and} \quad \eta_2 \cup \eta_2 \in H^2(SP^2,\otimes\mathbb{Z}/2) \tag{6.10}$$

are non–trivial elements of order 2. Moreover $\sigma : \Gamma \longrightarrow \otimes\mathbb{Z}/2$ yields the equations $\sigma_*\eta_1 = \eta_0$ and $\sigma_*\beta_1 = \beta_0$ by (5.5)(2).

Proof of (6.9): Using (5.2) and (6.7) we get $<\underline{\underline{S}}(2)> = \beta_1\{\underline{\underline{nil}}\}$. This yields the equations for $n = 2$ by (6.6) and (6.2). Moreover the construction of $\mathfrak{X}_n(\underline{\underline{M}}_{\mathbb{Z}})$ and the commutative diagram (5.6)(12) show that $<\underline{\underline{S}}(n)> = \sigma_*<\underline{\underline{S}}(2)>$ for $n \geq 3$. This completes the proof of (6.9).

(6.11) Proof of (3.8): For φ in (3.2)(2) we construct a splitting of $\varphi^*\underline{\underline{nil}} \longrightarrow \underline{\underline{T}}$. This implies (3.8) by (3.3) and (6.9). We can choose basis elements $\{b_i\}$ and $\{c_i\}$ such that $fb_i = 0$ for $i \leq i_0$ and $\mathrm{im}(g) \subset \mathrm{span}\,\{b_i : i > i_0\}$ and such that $g(c_j) = 0$ for $j \leq j_0$ and $\mathrm{im}(h) \subset \mathrm{span}\{c_j : j > j_0\}$. We can find sf,sg,sh with the same properties, this yields the splitting s.

§7 The automorphism group of a free nil(2)–group and elements in the cohomology of $GL_N(\mathbb{Z})$

Let G_N be the free nil(2)–group with N generators, that is $G_N = G_A$ for $A = \mathbb{Z}^N$, see (5.3). We have the extension of groups

$$(7.1) \qquad \mathrm{Hom}(\mathbb{Z}^N, \Lambda^2\mathbb{Z}^N) \rightarrowtail \mathrm{Aut}(G_N) \twoheadrightarrow GL_N(\mathbb{Z})$$

which is obtained by restricting the linear extension (5.4) to the subcategory $GL_N(\mathbb{Z})$ of $\underline{\underline{M}}_{\mathbb{Z}}$, see (2.5) [4]. The extension (7.1) represents the cohomology class

$$(1) \qquad \{\mathrm{Aut}\ G_N\} \in H^2(GL_N(\mathbb{Z}), \mathrm{Hom}(\mathbb{Z}^N, \Lambda^2\mathbb{Z}^N))$$

which is trivial if and only if the extension (7.1) is split. As in (2.5) [4] we see that

$$(2) \qquad \{\mathrm{Aut}\ G_N\} = i_N^*\{\underline{\underline{\mathrm{nil}}}\}$$

where $i_N : GL_N(\mathbb{Z}) \subset \underline{\underline{M}}_{\mathbb{Z}}$ is the inclusion. This implies that $\{\mathrm{Aut}\ G_N\}$ is an element of order ≤ 2. Moreover, we prove in (7.11) below the

(7.2) Theorem: For $N \geq 3$ the element $\{\mathrm{Aut}\ G_N\}$ is a non–trivial element of order 2. For $N = 2$, however, one has $\{\mathrm{Aut}\ G_2\} = 0$.

Now let

$$(7.3) \qquad \epsilon_N \in H^0(GL_N(\mathbb{Z}), \mathrm{Hom}(\mathbb{Z}^N, \hat{\otimes}^2\mathbb{Z}^N))$$

be given by the canonical homomorphism $\mathbb{Z}^N \twoheadrightarrow \mathbb{Z}^N \otimes \mathbb{Z}/2 \rightarrowtail \hat{\otimes}^2\mathbb{Z}^N$. Then we obtain by (6.6) and (7.1)(2) the formula

(7.4) <u>Theorem</u>: $\{\mathrm{Aut}\ G_N\} = \beta_2\beta_3(\epsilon_N) = \beta_3(\epsilon_N) \cup \eta_2^N$

Here β_i is the Bockstein homomorphism associated to the exact sequence of $GL_N(\mathbb{Z})$–modules $\mathrm{Hom}(\mathbb{Z}^N, \mathcal{M}_i(\mathbb{Z}^N))$, compare (6.4). Moreover $\eta_i^N = i_N^* \eta_i$ is the restriction of η_i to $GL_N(\mathbb{Z})$, see (6.4)(1).

Let $\overset{N}{\vee} S^n = S^n \vee \dots \vee S^n$ be an n–fold one point union of n–spheres. The classifying space $B\,\mathcal{E}(\overset{N}{\vee} S^N)$ plays an important role in the construction of Waldhausen's algebraic K–theory $A(*)$ of a point $*$. The first k–invariant of $B\,\mathcal{E}(\overset{N}{\vee} S^n)$ can be obtained via (3.10) by restricting the class $<S(n)>$ to the group of homotopy equivalences $\mathrm{Aut}(\overset{N}{\vee} S^n)^* = GL_N(\mathbb{Z})$. Therefore (7.2) and (6.9) yield the following result.

(7.5) <u>Theorem</u>:The first k–invariant $k^{N,n}$ of the classifying space $B\,\mathcal{E}(\overset{N}{\vee} S^n)$,

$$k^{N,n} \in H^3(GL_N(\mathbb{Z}),\mathrm{Hom}(\mathbb{Z}^N,\Gamma_n^1\mathbb{Z}^N)),$$

satisfies the equations $k^{N,n} = \beta_j\beta_2\beta_3(\epsilon_N) = \beta_j\{\mathrm{Aut}\ G_N\} = \{\mathrm{Aut}\ G_N\} \cup \eta_j^N$ where $j = 1$ for $n = 2$ and $j = 0$ for $n \geq 3$. Moreover $\mathrm{Hom}(1,\sigma)_* k^{N,2} = k^{N,n}$ for $n \geq 3$ and $2 \cdot k^{N,n} = 0$. For $N = 2$ the element $k^{2,n}$, $n \geq 2$, is trivial.

(7.6) <u>Remark</u>: K. Igusa [16] showed that $k^{N,n} \neq 0$ for $n \geq 3$, $N \geq 4$. The formula in (7.5), however, gives a new characterization of this element. Moreover, P. Kahn [19] observed by use of an example of Carlsson [6] that there is a representation

$$\rho_p : \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow GL_N(\mathbb{Z}),\ N \geq 8,$$

with $\rho_p^*(k^{N,n}) \neq 0$ for $n \geq 3$. The question whether $k^{3,n}$ is trivial or not remains open.

Using this remark we obtain the following corollaries of (7.5).

(7.7) <u>Corollary</u>: The representation ρ_p in (7.6) satisfies $\rho_p^*\{\mathrm{Aut}\ G_N\} \neq 0$.

It would be of interest to check this corollary directly.

(7.8) <u>Corollary</u>: The elements

$\eta_2^N \cup \eta_1^N \in H^2(GL_N\mathbb{Z}, \mathrm{Hom}(SP^2\mathbb{Z}^N, \Gamma\mathbb{Z}^N))$ and $\eta_2^N \cup \eta_0^N \in H^2(GL_N\mathbb{Z}, \mathrm{Hom}(SP^2\mathbb{Z}^N, \mathbb{Z}^N \otimes \mathbb{Z}/2))$

are non–trivial of order 2 for $N \geq 4$.

These elements correspond to certain non–trivial group extensions of $GL_N\mathbb{Z}$; in a similar way the elements (6.10) yield non–trivial linear extensions of the category $\underline{\underline{M}}_{\mathbb{Z}}$ by (2.3). Finally we get the following connection of $\langle \underline{\underline{S}}(n) \rangle$ with Igusa's <u>associativity class</u> $\chi(1)$ in [16]. Let

$$M_N(\mathbb{Z}) = \underline{\underline{\mathrm{End}}}(\overset{N}{\vee} S^n) \overset{i}{\hookrightarrow} \underline{\underline{S}}(n)$$

be the full subcategory of $\underline{\underline{S}}(n)$ consisting of the single object $\overset{N}{\vee} S^n$; using (3.6)(1) this is the monoid $M_N(\mathbb{Z})$ of integral $N \times N$ – matrices.

(7.9) <u>Theorem</u>: $$\langle \underline{\underline{\mathrm{End}}}(\overset{N}{\vee} S^n) \rangle = \chi(1),\ n \geq 3.$$

The equivariant version of this result for $\chi(M)$ where M is a monoid, will appear elsewhere. Using the restriction of the formulas in (6.9) we as well get

$$(7.10)\qquad \chi(1) = \beta_0\beta_2\beta_3(j^*\epsilon) = \beta_0\{\underline{\underline{\text{End}}}\, G_N\}$$
$$= \{\underline{\underline{\text{End}}}\, G_N\} \cup j^*\eta_0 = \beta_3(\epsilon) \cup j^*\eta_2 \cup j^*\eta_0$$

This, in fact, is a simpler algebraic characterization of $\chi(1)$ than the one in [16] where this class is defined by a somewhat mysterious cocycle of high complexity, see also (3) in the following proof.

Proof of (7.9): For a homomorphism $g : A \longrightarrow B$ in $\underline{\underline{M}}_{\mathbb{Z}}$ we obtain a homomorphism $s(g) : G_A \longrightarrow G_B$ as follows. We choose a basis $Z_B \subset B$ for each $B \in \text{Ob}\, \underline{\underline{M}}_{\mathbb{Z}}$ and we choose an ordering $<$ on Z_B. For $x' \in Z_A$, $x \in Z_B$ let $g(x',x) \in \mathbb{Z}$ be given by the formula $g(x') = \Sigma g(x',x)x$. We define $s(g)$ on generators $Z_A \subset G_A$ by the ordered sum

$$(1)\qquad s(g)(x') = \sum^{<} g(x',x)x$$

in G_B. This as well yields the cocycle $c'' = \Delta_s$ in (2.3)(1). For the exact sequence $\mathcal{M}_0$ we define a pointwise splitting $\bar{r} = \Lambda^2 B \longrightarrow \hat{\otimes}^2 B$ by $\bar{r}(x \wedge y) = x \hat{\otimes} y$ for $x,y \in Z_B$, $x < y$. Thus we obtain a cocycle $j^{-1}(\delta c)$ by

$$(2)\qquad j^{-1}(\delta c)(f,g,h) = rf_*\bar{r}\Delta_s(g,h),$$

compare (6.2)(2). This cocycle represents the element $\beta_0\{\text{nil}\}$. When we restrict the cocycle $j^{-1}(\delta c)$ to matrices $f,g,h \in \underline{\text{End}}(\mathbb{Z}^N) = M_N(\mathbb{Z})$ we get the equation of cocycles

$$(3)\qquad j^{-1}(\delta c)(f,g,h) = I(f,g,h).$$

Here I is Igusa's associativity cocycle which represents $\chi(1)$, (compare A(6.3) [16] where we set $M = 1$, $f = I$). Igusa's definition, however, is so complicated that it takes some effort to get an explicit formula for I by α in (4.2) [16]; in (4.2) [16] there is a missing summation index k in the first sum. A tedious but inevitable calculation shows that the equation (3) of cocycles is actually satisfied. An explicit formula for $c'' = \Delta_s$ is contained in (V. 7.17) [2]. Clearly (2) is the most elegant description of Igusa's cocycle since it involves no complicated summation formulas.

(7.11) Proof of (7.2):

We first show that (7.1) is split for $N = 2$. Let F_2 be the free group on two generators x,y, let $<i(c)> \subset Aut(F_2)$ be the subgroup generated by the inner automorphism $i(c)$ associated with the commutator $c = -x - y + x + y$, and let $K \subset Aut(F_2)$ be the subgroup consisting of all automorphisms φ such that $\varphi(c) = \pm c$. Then there is a short exact sequence

$$<i(c)> \rightarrowtail K \twoheadrightarrow GL_2(\mathbb{Z})$$

which is easily derived from the fact that the kernel of the canonical projection $Aut(F_2) \xrightarrow{p} GL_2(\mathbb{Z})$ is exactly the subgroup of inner automorphisms (see [24]). In particular, we see that the restriction of p is still surjective because $GL_2(\mathbb{Z})$ is generated by $p\alpha$, $p\beta$ and $p\tau$, where $\alpha,\beta,\tau \in K$ are the automorphisms sending x to $x, y + x, y$ and y to $x + y, y, x$, respectively. Now observe that conjugation by c is the identity in $Aut(G_2)$. Hence $K \subset Aut(F_2)$ induces a splitting for (7.1)

$$K/<i(c)> \cong GL_2(\mathbb{Z}) \longrightarrow Aut(G_2)\,.$$

The proof of (7.2) will be completed by showing that (7.1) is not split for $N = 3$. Assume that there is a splitting homomorphism $s : GL_3(\mathbb{Z}) \longrightarrow Aut(G_3)$. We shall use the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; B^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and the identities

(1) $$A^2 = C^3 = (AC)^3 = 1,\; AB = BA,\; AB^T = B^TA$$

to derive a contradiction. Note that there are unique functions $\sigma_i : GL_3(\mathbb{Z}) \longrightarrow \Lambda^2\mathbb{Z}^3$ such that

$$s(U)(e_i) = u_{i1}e_1 + u_{2i}e_2 + u_{3i}e_3 + \sigma_i(u) \in G_3$$

for $i = 1,2,3$ and $U = (u_{ij}) \in GL_3(\mathbb{Z})$, where e_1,e_2,e_3 are the generators of G_3 and $\Lambda^2\mathbb{Z}$ is identified with the commutator subgroup of G_3 via $i_{\mathbb{Z}^3}$, i.e. $e_i \wedge e_j = -e_i - e_j + e_i + e_j$, cf. (5.3). Let $\alpha_i, \beta_i, \beta_i^T, \gamma_i \in \Lambda^2\mathbb{Z}^3$ be the elements $\sigma_i(A)$, $\sigma_i(B)$, $\sigma_i(B^T)$, $\sigma_i(C)$, respectively. Then $A^2 = 1$ yields

(2) $$\alpha_3 = A_*\alpha_3$$

since we have $s(A)^2(e_3) = s(A)(-e_3 + \alpha_3) = e_3 - \alpha_3 + A_*\alpha_3$, where A_* denotes the induced endomorphism $A_* = \Lambda^2 A$ of $\Lambda^2\mathbb{Z}^3$. Similarly, applying the other identities in (1) to e_2,e_1,e_3 and e_1, respectively, yields

(3) $$\gamma_3 + C_*\gamma_1 + C_*^2\gamma_2 = 0$$

(4) $$\alpha_1 + A_*\gamma_2 - A_*C_*\alpha_2 - A_*C_*A_*\gamma_3 + A_*C_*A_*C_*\alpha_3 + A_*C_*A_*C_*A_*\gamma_1 = 0$$

(5) $$e_2 \wedge e_3 + \alpha_2 + \alpha_3 + A_*\beta_3 = -\beta_3 + B_*\alpha_3$$

(6) $$\alpha_1 + A_*\beta_1^T = \beta_1^T + B_*^T\alpha_1 .$$

Now, with respect to the basis $e_2 \wedge e_3$, $e_3 \wedge e_1$, $e_1 \wedge e_2$ of $\Lambda^2\mathbb{Z}^3$ the induced endomorphisms are represented by the matrices $A_* = A$, $B_* = (B^T)^{-1}$, $B_*^T = B^{-1}$, $C_* = C$. Write $\alpha_{ij}, \beta_{ij}, \ldots$ for the jth component of $\alpha_i, \beta_i, \ldots$ Then the 2nd, 1st, 2nd component of (2), (3) and (6), respectively, yield

$$\alpha_{32} = -\alpha_{32} = 0,\ \ \gamma_{31} + \gamma_{12} + \gamma_{33} = 0,\ \alpha_{13} = 2\beta_{12}^T,$$

whence the 3rd component of (4) yields

$$\alpha_{21} = -2\beta_{12}^T - 2\gamma_{31} .$$

On the other hand the 1st component of (5) yields

$$\alpha_{21} = -2\beta_{31} - 1$$

which is a contradiction.

Appendix A: Proof of the classification theorem for linear track extensions

The proof of theorem (4.6) is divided into three parts. In (A.1) and (A.3) we show that Ψ is well–defined and injective, respectively. The main part of the proof will be (A.2), where an S–normalized linear track extension $(\mathscr{E}^{\pi}_c, j_0, J_0)$ is constructed, for a given cocycle $c \in F^3(\mathrm{Mor}\ \underline{\underline{C}}^1) \cap \bar{F}^3(S)$, such that $\Psi\{(\mathscr{E}^{\pi}_c, j_0, J_0)\} = \{c\}$. The construction is based on well–known coherence properties in bicategories (cf. [5], [21]). The crucial observation is the fact that the cocycle condition corresponds exactly to the coherence conditions in a bicategory, see (A.2)(5) below. As T. Pirashivili points out in [25], M. Jibladze also used bicategories in order to represent cohomology classes in $H^3(\underline{\underline{C}}, D)$.

(A.1) Lemma: Let $(\mathscr{E}, j, J)$ be an S–normalized linear track extension of $(\underline{\underline{C}}, \underline{\underline{C}}^1)$ by D as in (4.5) and let t,H be functions as in (4.4) which extend j and J, respectively. Then:

a) $c_{\mathscr{E}}(t,H)$ is an S–normalized cocycle rel $\underline{\underline{C}}^1$.

b) $c_{\mathscr{E}}(t,H) + \delta c = c_{\mathscr{E}}(t,H-c)$, for all $c \in F^2(\mathrm{Mor}\ \underline{\underline{C}}^1) \cap \bar{F}^2(S)$, where $H - c$ denotes the extension of J given by $(H-c)(f,g) = H(f,g) - \sigma_{t(fg)}c(f,g)$.

c) $\{c_{\mathscr{E}}(t,H)\} \in H^3_S(\underline{\underline{C}}, \underline{\underline{C}}^1; D)$ does not depend on the choice of t and H.

d) $\{c_{\mathscr{E}}(t,H)\}$ depends only on the connected component of $(\mathscr{E}, j, J)$ in Track.

Proof: Using (4.5)(2) the proof of a), b) and d) is straightforward. To prove c), let t,H and t',H' be extensions of j and J, respectively. Assume first that H and H' satisfy

$$H'(f,g) = -(t'f)_*G(g) - (tg)^*G(f) + H(f,g) + G(fg),$$

with $G(f) = \sigma_{jf}(0)$ if $f \in \mathrm{Mor}(\underline{\underline{C}}^1) \cup S$, and $G(f) \in T(tf, t'f)$ arbitrary otherwise. Then we have $c_{\mathscr{E}}(t,H) = c_{\mathscr{E}}(t',H')$, whence the general case follows from b).

(A.2) Construction of $(\mathscr{E}^{\pi}_c, j_0, J_0)$: First we construct a certain bicategory $\underline{\underline{B}}(c)$ associated with c. Then $\mathscr{E}^{\pi}_c$ will be of the following form

(1) $$D \xrightarrow{+} T_c^{\pi} \rightrightarrows FC(\underline{\underline{C}}) \xrightarrow{q_0} \underline{\underline{C}}\,.$$

Here $FC(\underline{\underline{C}})$ denotes the free category generated by the morphisms of $\underline{\underline{C}}$, see (7) below, and T_c^{π} is obtained by pulling back the tracks, i.e. the 2–morphisms, from $\underline{\underline{B}}(c)$ via a suitable function π, see (8) below. We point out that δ_c^{π} will actually depend on π.

The set of morphisms of $\underline{\underline{B}}(c)$ is the disjoint union $B(\underline{\underline{C}})$ with

(2) $$B(\underline{\underline{C}}) = \coprod_{n\geq 1} B_n(\underline{\underline{C}}) \xrightarrow{q} Mor(\underline{\underline{C}})\,,$$

where $B_n(\underline{\underline{C}})$ denotes the following set of n–fold brackets, which is defined recursively,

$$B_1(\underline{\underline{C}}) = Mor\,(\underline{\underline{C}}) \coprod Ob(\underline{\underline{C}}) \quad \text{(disjoint union)}$$

$$B_{n+1}(\underline{\underline{C}}) = \{(u,v) \in \coprod_{i=1}^{n} B_i(\underline{\underline{C}}) \times B_{n+1-i}(\underline{\underline{C}}) : \partial_0 u = \partial_1 v\}\,,$$

with $\partial_0 A = \partial_1 A = A$, $\partial_0 f = A$, $\partial_1 f = B$, for $A \in Ob(\underline{\underline{C}})$ and $f \in Mor(\underline{\underline{C}})$, $f : A \longrightarrow B$, respectively, and $\partial_0(u,v) = \partial_0 v, \partial_1(u,v) = \partial_1 u$. $B(\underline{\underline{C}})$ is endowed with the canonical function q satisfying $q(A) = 1_A, q(f) = f$, and $q(u,v) = q(u)q(v)$. The subset $Ob(\underline{\underline{C}}) \subset B_1(\underline{\underline{C}})$ contains the identities (up to coherent isomorphisms) of $\underline{\underline{B}}(c)$. Next, for $A,B \in Ob(\underline{\underline{C}})$, let $\underline{\underline{B}}(A,B)$ be the groupoid with objects

$$Ob\,\underline{\underline{B}}(A,B) = \{u \in B(\underline{\underline{C}}) : \partial_0 u = A, \partial_1 u = B\}$$

and morphisms, which we call tracks,

(3) $$\underline{\underline{B}}(A,B)(v,u) = \begin{cases} (u,v) \times D_{qu} & \text{if } qu = qv \\ \phi & \text{otherwise} \end{cases}$$

subject to the composition $(u,v,\alpha) + (v,w,\beta) = (u,w,\alpha + \beta)$, cf. (4.1)(1). Moreover, for $A,B,C \in Ob(\underline{\underline{C}})$, let

(4) $$* = *_{A,B,C} : \underline{\underline{B}}(B,C) \times \underline{\underline{B}}(A,B) \rightarrow \underline{\underline{B}}(A,C)$$

be the functor given by $u^*v = uv = (u,v)$ and $(u,u',\alpha)*(v,v',\beta) = (uu',vv',\alpha + \beta)$. The composition in (3) and (4) will serve as the vertical resp. horizontal composition in $\underline{\underline{B}}(c)$. As usual, u_* and v^* will denote the induced functor $u_*v = uv$, $u_*(v,v',\alpha) = (uv,uv',\alpha)$, and $v^*u = uv$, $v^*(u,u',\alpha) = (uv,u'v,\alpha)$, respectively.

Next, c comes into the play. Note that c yields natural transformations, for all $A,B,C,D \in Ob(\underline{\underline{C}})$,

$$\lambda = \lambda_{A,B} : B_* \longrightarrow 1 : \underline{\underline{B}}(A,B) \longrightarrow \underline{\underline{B}}(A,B)$$
$$\rho = \rho_{A,B} : A^* \longrightarrow 1 : \underline{\underline{B}}(A,B) \longrightarrow \underline{\underline{B}}(A,B)$$
$$\alpha = \alpha_{A,B,C,D} : *(1 \times *) \longrightarrow *(* \times 1) : \underline{\underline{B}}(C,D) \times \underline{\underline{B}}(B,C) \times \underline{\underline{B}}(A,B) \longrightarrow \underline{\underline{B}}(A,D)$$

given by

$$\lambda(u) = (Bu,u,c(1_B,1_B,qu))$$
$$\rho(u) = (uA,u,-c(qu,1_A,1_A))$$
$$\alpha(u,v,w) = (u(vw),(uv)w,c(qu,qv,qw)).$$

Here A^*,B_* are the functors induced by $A,B \in Ob(\underline{\underline{C}}) \subset B_1(\underline{\underline{C}})$, and 1 denotes the identity functor on $\underline{\underline{B}}(A,B)$ and $\underline{\underline{B}}(C,D)$, respectively. The following coherence conditions for left and right identities and associativity are satisfied, for composable $u_1,A,u_2,u_3,u_4 \in B(\underline{\underline{C}})$, with $A \in Ob(\underline{\underline{C}}) \subset B_1(\underline{\underline{C}})$,

(i) $\lambda(A) = \rho(A)$

(5) (ii) $(u_1)_*\lambda(u_2) = \alpha(u_1,A,u_2) + u_2^*\rho(u_1)$

(iii) $\alpha(u_1,u_2,u_3u_4) + \alpha(u_1u_2,u_3,u_4)$
$= (u_1)_*\alpha(u_2,u_3,u_4) + \alpha(u_1,u_2u_3,u_4) + u_4^*\alpha(u_1,u_2,u_3)$,

because c is a cocycle. Whence we know that the categories $\underline{\underline{B}}(A,B)$ together with the compositions (3), (4) and λ,ρ,α actually from a bicategory which we denote $\underline{\underline{B}}(c)$, cf. [5]. Coherence in $\underline{\underline{B}}(c)$ yields a unique track

(6) $$h(u,u') \in \underline{\underline{B}}(A,B)(u',u),$$

for all $u,u' \in B(\underline{\underline{C}})$ satisfying $q_1u = q_1u' \in FC(\underline{\underline{C}})(A,B)$. The track $h(u,u')$ is obtained from the zero tracks $(u,u,0)$ and $\pm\lambda,\rho,\alpha$ by means of the pasting operations in $\underline{\underline{B}}(c)$, cf. the section on coherence in [21], VII which generalizes to bicategories. Moreover q_1 is part of the factorization $q = q_0q_1$

(7) $$B(\underline{\underline{C}}) \xrightarrow{q_1} \text{Mor } FC(\underline{\underline{C}}) \xrightarrow{q_0} \text{Mor } (\underline{\underline{C}})$$

given by $q_1(A) = A \in Ob(\underline{\underline{C}}) = N_0(\underline{\underline{C}})$, $q_1(f) = f \in Mor(\underline{\underline{C}}) = N_1(\underline{\underline{C}})$, and $q(uv) = q(u)q(v)$. Recall that $FC(\underline{\underline{C}})$ in (1) is the category with objects $Ob\ FC(\underline{\underline{C}}) = Ob(\underline{\underline{C}})$ and morphisms

$$\text{Mor } FC(\underline{\underline{C}}) = \coprod_{n\geq 0} N_n(\underline{\underline{C}}),$$

cf. (1.3), with $(\lambda_1,\dots,\lambda_n) \in FC(\underline{\underline{C}})(A,B)$ for $B \xleftarrow{\lambda_n} \dots \xleftarrow{\lambda_1} A$, subject to the obvious composition given by juxtaposition. q_0 in (7) then agrees with the canonical functor q_0 in (1). We shall now use (6) to pull back the track structure from $\underline{\underline{B}}(c)$ via a function

(8) $$\pi : \text{Mor } FC(\underline{\underline{C}}) \longrightarrow B(\underline{\underline{C}})$$

satisfying $q_1\pi = 1$. For $\lambda,\mu \in FC(\underline{\underline{C}})(A,B)$ set

$$T_c^{\pi}(\lambda,\mu) = \underline{\underline{B}}(A,B)(\pi\mu,\pi\lambda)$$

and let the track addition of T_c^{π} be the induced one, i.e. the same as in (3). Then, using (6), we define $\lambda_* : T_c^{\pi}(\mu,\mu') \longrightarrow T_c^{\pi}(\lambda\mu,\lambda\mu')$ and $\lambda^* : T_c^{\pi}(\mu,\mu') \longrightarrow T_c^{\pi}(\mu\lambda,\mu'\lambda)$ as follows,

$$\lambda_*(\pi\mu,\pi\mu',\alpha) = (\pi(\lambda\mu),\pi(\lambda\mu'), -\vartheta^{\pi}(\lambda,\mu) + \alpha + \vartheta^{\pi}(\lambda,\mu'))$$
$$\lambda^*((\pi\mu,\pi\mu',\alpha) = (\pi(\mu\lambda),\pi(\mu'\lambda), -\vartheta^{\pi}(\mu,\lambda) + \alpha + \vartheta^{\pi}(\mu',\lambda)),$$

where $\vartheta^{\pi}(\lambda,\mu) \in D_{q_0(\lambda\mu)}$ is given by the canonical track, i.e.

$$(\pi(\lambda)\pi(\mu),\pi(\lambda\mu),\vartheta^{\pi}(\lambda,\mu)) = h(\pi(\lambda)\pi(\mu),\pi(\lambda\mu)) .$$

It is now easily checked that $\mathcal{E}_c^\pi$ in (1) is a linear track extension with D acting via the canonical isomorphisms

$$D_{q_0(\lambda)} \cong T_c^\pi(\lambda,\lambda) = (\pi\lambda,\pi\lambda) \times D_{q_0(\lambda)} \cdot$$

Moreover, for $\mathcal{E} = \mathcal{E}_c^\pi$, we see that $c_{\mathcal{E}}(t_0,H_0)(f,g,h)$ is determined by the canonical track from $f(gh)$ to $(fg)h$ in $\underline{B}(c)$, i.e.

(9) $$c_{\mathcal{E}}(t_0,H_0) = c.$$

Here $c_{\mathcal{E}}(t_0,H_0)$ is defined as in (4.4)(3), with respect to the functions t_0 and H_0 given by $t_0(f) = f \in \mathrm{Mor}\,(\underline{C}) = N_1(\underline{C})$, and

$$H_0(f,g) = (\pi(t_0f \circ t_0g),\pi t_0(fg),0) \in T_c^\pi(t_0f \circ t_0f(g)).$$

We use the restriction of t_0 and H_0 to turn (1) into an S–normalized linear track extension $(\mathcal{E}_c^\pi,j_0,J_0)$ with $\Psi\{(\mathcal{E}_c^\pi,j_0,J_0)\} = \{c\}$; note that (4.5) (1), (2) are satisfied.

(A.3) <u>Injectivity of Ψ:</u> It suffices to find a map of S–normalized linear track extensions

$$\bar{t} : (\mathcal{E}_c^\pi,j_0,J_0) \longrightarrow (\mathcal{E},j,J),$$

cf. (4.5), for all such track extensions $(\mathcal{E},j,J)$ and all cocycles $c \in \Psi\{(\mathcal{E},j,J)\}$. By (A.1)b) any such cocycle is of the form

$$c = c_{\mathcal{E}}(t,H)$$

for some extension t and H of j and J, respectively. Let $\mathcal{E}$ be of the form (4.3)(1). Then there exists a unique functor

$$\bar{t} : FC(\underline{C}) \longrightarrow \underline{K}$$

such that $\bar{t}t_0 = t$. Moreover, for $\lambda,\mu \in FC(\underline{C})(A,B)$, let

$$\bar{t} = \bar{t}_{\lambda,\mu} : T_c^\pi(\lambda,\mu) \longrightarrow T(\bar{t}\lambda,\bar{t}\mu)$$

be the function given by

$$\bar{t}(\pi\lambda,\pi\mu,\alpha) = H(\pi\lambda) - H(\pi\mu) + (\pi\mu,\pi\mu,\alpha)$$

where $H(u) \in T(\bar{t}q_1u,tqu)$ is defined recursively as follows,

$$\begin{aligned} &H(A) = -t(1_A)_*^{-1}H(1_A,1_A) = -(t(1_A)^*)^{-1}H(1_A,1_A) \\ &H(f) = \sigma_{tf}(0) \\ &H(uv) = (\bar{t}q_1u)_*H(v) + (tqv)^*H(u) + H(qu,qv). \end{aligned} \tag{10}$$

Note that both $t(1_A)_*$ and $t(1_A)^*$ are bijections and that the second equation in (10) holds since $c(1_A,1_A,1_A) = 0$. Using the definition of λ,ρ,α and the coherence in $\underline{B}(c)$ one can check that $\bar{t}$ is a map of S–normalized linear track extensions.

Appendix B: Proof of the normalization theorems

The proof of (1.9) is based on the following lemma which is easily checked:

(B.1) Lemma: $F^*(\text{Mor }\underline{K}) \cap \bar{F}^*(S)$, provided with the following coface and codegeneracy operators d^i and s^i,

$$d^i c(\lambda_1,\dots,\lambda_n) = \begin{cases} (\lambda_1)_* c(\lambda_2,\dots,\lambda_n) & i = 0 \\ c(\lambda_1,\dots,\lambda_i\lambda_{i+1},\dots,\lambda_n) & 0 < i < n \\ \lambda_n^* c(\lambda_1,\dots,\lambda_{n-1}) & i = n \end{cases}$$

$$s^i c(\lambda_1,\dots,\lambda_n) = c(\lambda_1,\dots,\lambda_i,1,\lambda_{i+1},\dots,\lambda_n),$$

where $d_c^0(\lambda) = \lambda_* c(A)$, $d^1 c(\lambda) = \lambda^* c(B)$ for $n = 1$, $\lambda : A \longrightarrow B$, and $s^0 c(A) = c(1_A)$ for $n = 0$, $A \in \text{Ob}(\underline{C})$, is a cosimplicial abelian group.

Now (1.9) is a special case of the well–known normalization theorem for simplicial objects in an abelian category (cf. [20], VIII, Thm. 6.1) since we have $\delta = \Sigma(-1)^i d^i$, i.e. $(F^*(\mathrm{Mor}\,\underline{\underline{K}}) \cap \not{F}^*(S), \delta)$ is the associated cochain complex.

The proof of (1.10) is similar. Instead of codegeneracies, however, we shall use some additional operators t^i to show that the inclusion

$$F^*(\mathrm{Mor}\,\underline{\underline{K}}) \cap \not{F}^*(S \cup O(\underline{\underline{C}}) \xrightarrow{\sim} F^* = F^*(\mathrm{Mor}\,\underline{\underline{K}}) \cap \not{F}^*(S)$$

is a chain equivalence. For $i = 0,\dots,n$, let $t^i : F^{n+1} \longrightarrow F^n$ be defined as follows:

$$t^i c(\lambda_1,\dots,\lambda_n) = \begin{cases} c(\lambda_1,\dots,\lambda_i,0,\lambda_{i+1},\dots,\lambda_n) & \text{if } \lambda_1 \circ \dots \circ \lambda_n = 0 \\ 0 & \text{otherwise} \end{cases}$$

(in particular $t^0 c = 0$ for $c \in F^1$). Recall that 0 denotes zero morphisms as well as zero elements in abelian groups. In the definition of t^i we made use of the assumption that the zero morphism $0 : A \longrightarrow A$ is in $\underline{\underline{K}}$ if $A \in \mathrm{Ob}(\underline{\underline{K}})$. Note that the following identies are satisfied:

(1) $\quad t^j t^i = t^i t^{j+1}$ if $i \leq j$

(2) $$t^j d^i = \begin{cases} d^i t^{j-1} & i < j \\ d^{i-1} t^j & i > j+1 \end{cases}$$

(3) $\quad t^j t^j d^j = t^j d^j t^{j-1}$

(4) $\quad t^j t^j d^{j+1} = t^j,$

where d^i denotes the coface operator as above, and that

$$F^n_k = F^n \cap \bigcap_{i=0}^{\min(k-1,n-1)} \ker t^i = \{c \in F^n : c(\lambda_1,\dots,\lambda_n) = 0 \text{ if } \lambda_i = 0 \text{ for some } i \leq k\}$$

defines a decreasing sequence of subcomplexes of (F^*,δ) such that $F_0^n = F^n$ and $F_k^n = F^n(\text{Mor}\ \underline{K}) \cap F^*(S \cup O(\underline{C}))$ for $k \geq n$, whence it suffices to show that the inclusion $F_{k+1}^* \hookrightarrow F_k^*$ is a chain equivalence for $k \geq 0$. Let $h^k : F_k^* \longrightarrow F_k^*$ be the chain transformation $h^k = 1 - \tilde{t}^k\delta - \delta\tilde{t}^k$, where $\tilde{t}^k c = (-1)^{k+1} t^k c$ if $k \leq n-1$, and $\tilde{t}^k c = 0$ otherwise, for $c \in F^n$. Then the fact that F_k^* is a subcomplex and (1) – (4) combine to give

(5) $$h^k c = c \quad \text{for } c \in F_{k+1}^*$$

(6) $$h^k(F_k^*) \subset F_{k+1}^*,$$

which completes the proof since h^k is chain homotopic to the identity.

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