BOUNDED COHOMOLOGY CHARACTERIZES HYPERBOLIC GROUPS

IGOR MINEYEV

Abstract. For a finitely presentable group G, G is hyperbolic if and only if the map $H_b^2(G,V) \to H^2(G,V)$ is surjective for any bounded G-module. We prove the remaining "if" direction. The words "bounded G-module" here can be understood in several different ways. We also consider several ways to define a linear homological isoperimetric inequality.

1. Introduction.

The question of cohomological description of hyperbolicity was considered by S. M. Gersten who proved the following

Theorem 1 (Gersten [3]). *The finitely presented groups G is hyperbolic iff* $H^2_{(\infty)}(G,\ell_\infty)=0$.

Here $H^n_{(\infty)}(G,V)$ is the ℓ_∞-*cohomology* defined by bounded (not necessarily equivariant) cellular cochains in the universal cover of a $K(G,1)$ complex with finitely many cells in the dimensions up to n. This theorem was generalized by the author [9] to higher dimensions: if G is hyperbolic then $H^n_{(\infty)}(G,V)=0$ for any $n\geq 2$ and any normed vector space V (over $\mathbb{Q}$ or $\mathbb{R}$).

The *bounded cohomology* of a group is defined by bounded equivariant cochains in the homogeneous bar construction (see the precise definition in the next section). G. A. Noskov [11, p.1068], using results of B. E. Johnson [8], showed a characterization of *amenable* groups, which we restate in the following form.

Theorem 2 (Noskov [11]). *For a group G the following statements are equivalent.*
(a) *G is amenable.*
(b) *$H^1_b(G,V^*)=0$ for any bounded G-module V.*
(c) *$H^i_b(G,V^*)=0$ for any $i\geq 1$ and any bounded G-module V.*

The expression "bounded G-module V" here is what we call a "bounded Banach $\mathbb{R}G$-module", and V^* is the space conjugate to V.

The result presented in this paper is in a sense an analog of the above two theorems: we characterize hyperbolic groups by bounded cohomology. It was shown in [10] that if G is a hyperbolic group, then the map $H_b^i(G,V)\to H^i(G,V)$, induced by inclusion, is

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surjective for any bounded $\mathbb{Q}G$-module V and any $i \geq 2$ (see the definitions in the next section). The goal of the present paper is to show the converse of this result. We only need to require the group G to be finitely presentable for this converse to be true. If this is the case, then surjectivity of the above maps *only in dimension 2* is already sufficient for hyperbolicity. Namely, we prove

Theorem 3. *For a finitely presentable group G, the following statements are equivalent.*
(a) *G is hyperbolic.*
(b) *The map $H_b^2(G, V) \to H^2(G, V)$ is surjective for any bounded G-module V.*
(c) *The map $H_b^i(G, V) \to H^i(G, V)$ is surjective for any $i \geq 2$ and any bounded G-module V.*

Here by a "bounded G-module" we mean one of the six concepts $\mathcal{M}_1(G) - \mathcal{M}_6(G)$ described in the next section. (See Theorem 9 for a more precise statement.)

This looks quite interesting that the same property can be characterized by the ℓ_∞-cohomology and the bounded cohomology, the two theories which do not seem to have much in common. Also, the characterizations of hyperbolic and amenable groups seem strikingly similar. This similarity may be worth being investigated further.

There are two crucial points in our proof. Firstly, for finitely presentable groups, hyperbolicity is equivalent to the existence of a linear isoperimetric inequality *for real 1-cycles*. This equivalence was proved by S. M. Gersten. (It follows, for example, from Proposition 3.6, Theorem 5.1, and Theorem 5.7 (stated above) in [3].) This isoperimetric inequality for real cycles is a homological version of the usual combinatorial isoperimetric inequality *for loops*. We give a direct proof of the fact that the linear isoperimetric inequality for filling (usual) real 1-cycles with *summable* 2-chains implies hyperbolicity.

Secondly, one needs to pick appropriate coefficients V. We take V to be the space of all boundaries of summable 2-chains in a cell complex with a nice G-action. Similar coefficients (but $\mathbb{Z}$-modules rather than $\mathbb{R}$-modules), and the associated with them universal cocycle were used by S. M. Gersten [7, Chapter 12] to introduce *$\mathbb{Z}$-metabolic* (or simply *metabolic*) groups and to give several equivalent definitions. In particular, metabolic groups are hyperbolic, and it was shown in [9] that the corresponding properties *$\mathbb{Q}$-metabolic* and *$\mathbb{R}$-metabolic* are actually equivalent to hyperbolicity.

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2. Preliminaries.

First we need some definitions. Let $\mathbb{F}$ stand for either $\mathbb{Q}$ or $\mathbb{R}$, and $\mathbb{A}$ stand for $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

2.1. **Abelian group norms.** A <u>*normed abelian group*</u> A is an abelian group with an <u>*abelian group norm*</u> $|\cdot| : A \to \mathbb{R}_+$ satisfying

- $|a| = 0$ iff $a = 0$ and

- $|a + a''| \leq |a| + |a'|$.

for all $a, a' \in A$.

2.2. **Norms.** A *normed vector space* W over $\mathbb{F}$ is a vector space with a *norm* $|\cdot| : W \to \mathbb{R}_+$ satisfying

- $|w| = 0$ iff $w = 0$,
- $|w + w''| \leq |w| + |w'|$, and
- $|\alpha w| = |\alpha| \cdot |w|$

for all $w, w' \in W$ and $\alpha \in \mathbb{F}$.

Of course, each norm on W is an abelian group norm, but not conversely.

2.3. **ℓ_1-norm.** Let a free $\mathbb{A}$-module M have a preferred basis $\{m_i, i \in I\}$. The *ℓ_1-norm* $|\cdot|_1$ on M (with respect to this basis) is given by

$$\left|\sum_{i\in I} \alpha_i m_i\right|_1 := \sum_{i\in I} |\alpha_i|.$$

The ℓ_1-norm is an abelian group norm. Moreover, if M happens to be a vector space over $\mathbb{Q}$ or $\mathbb{R}$, then it is a norm.

2.4. **ℓ_∞-norm.** Suppose that $(W, |\cdot|_1)$ and $(W', |\cdot|)$ are normed vector spaces, where W is equipped with the ℓ_1-norm $|\cdot|_1$ with respect to some preferred basis $\{w_i,\ i \in I\}$. For a linear map $\varphi : W \to W'$, the *ℓ_∞-norm* of φ, $|\varphi|_\infty$, is the operator norm of φ, i.e. $|\varphi|_\infty$ is the smallest number K (possibly infinity) such that $|\varphi(w)| \leq K|w|_1$ for each $w \in W$. One checks that

$$|\varphi|_\infty = \sup_{i\in I} \big|\varphi(w_i)\big|.$$

2.5. **Cell complexes.** By a *cell complex* we mean a *combinatorial cell complex*, i.e. the one in which the boundary of each cell σ is cellulated and the gluing map of σ restricts to homeomorphisms on the open cells of this cellulation. In particular, each 2-cell can be viewed as a polygon.

We always put the path metric on the 1-skeleta of cell complexes, which is induced by assigning length 1 to each edge. A cellular ball of radius r centered at a vertex x is the union of all closed cells whose vertices lie r-close to x.

If X is a cell complex and $C_i(X, \mathbb{A})$ is the space of cellular i-chains, we always give $C_i(X, \mathbb{A})$ the ℓ_1-norm with respect to the standard basis consisting of i-cells.

2.6. **Filling norm for compactly supported chains.** If G is a finitely presentable group, there exists a contractible cell complex X with a free cellular G-action such that the induced action on the 2-skeleton of X is cocompact. In particular, the boundary homomorphism $\partial_2 : C_2(X, \mathbb{A}) \to C_1(X, \mathbb{A})$ is bounded (with respect to the ℓ_1-norms in the domain and the target).

The *filling norm* $|\cdot|_{fc,\mathbb{A}}$ on the space $B_1(X,\mathbb{A})$ of 1-boundaries is given by

$$|b|_{fc,\mathbb{A}} := \inf\big\{|a|_1 \ \big|\ a \in C_2(X,\mathbb{A}),\ \partial a = b\big\}.$$

In other words, $|\cdot|_{fc,\mathbb{A}}$ is the abelian group norm induced by the map $\partial_2 : C_2(X,\mathbb{A}) \to C_1(X,\mathbb{A})$ on its image. One checks that $|\cdot|_{fc,\mathbb{Z}}$ is an abelian group norm, and $|\cdot|_{fc,\mathbb{Q}}$ and $|\cdot|_{fc,\mathbb{R}}$ are norms. (The condition "$|w| = 0$ iff $w = 0$" holds because $|\cdot|_{fc,\mathbb{A}}$ dominates the ℓ_1-norm on $B_1(X,\mathbb{A})$.)

2.7. **ℓ_1-completion.** Now we restrict to the real coefficients. For X as above, let $C_i^{(1)}(X,\mathbb{R})$ be the completion of $C_i(X,\mathbb{R})$ with respect to the ℓ_1-norm, $i = 1,2$, i.e. it is the set of all *(absolutely) summable* i-chains in X. The boundary homomorphism $\partial_i : C_i(X,\mathbb{R}) \to C_{i-1}(X,\mathbb{R})$, $i = 1,2$, is continuous, so it extends to a continuous linear map $\hat{\partial}_i : C_i^{(1)}(X,\mathbb{R}) \to C_{i-1}^{(1)}(X,\mathbb{R})$ which commutes with the G-action.

Denote $B_i^{(1)}(X,\mathbb{R}) := \operatorname{Im}\hat{\partial}_{i+1}$, $Z_i^{(1)}(X,\mathbb{R}) := \operatorname{Ker}\hat{\partial}_i$.

2.8. **Filling norm for summable chains.** Now we consider "the complete version" of the filling norm. The norm $|\cdot|_{f1,\mathbb{R}}$ on the space $B_1^{(1)}(X,\mathbb{R})$ is given by

$$|b|_{f1,\mathbb{R}} := \inf\big\{|a|_1 \ \big|\ a \in C_2^{(1)}(X,\mathbb{R}),\ \partial a = b\big\}.$$

Again, since $\hat{\partial}_2 : C_2^{(1)}(X,\mathbb{R}) \to C_1^{(1)}(X,\mathbb{R})$ is bounded, $|\cdot|_{f1,\mathbb{R}}$ dominates $|\cdot|_1$ on $B_1^{(1)}(X,\mathbb{R})$ and is therefore a norm indeed.

2.9. **Bounded cohomology.** An $\mathbb{F}G$-module is called bounded if it is normed as a vector space over $\mathbb{F}$ and G acts on it by linear operators of uniformly bounded norm. For a bounded $\mathbb{F}G$-module V, the *bounded cohomology* of G with coefficients in V, $H_b^*(G,V)$, is the homology of the cochain complex

$$0 \longrightarrow C_b^0(G,V) \xrightarrow{\delta_0} C_b^1(G,V) \xrightarrow{\delta_1} C_b^2(G,V) \xrightarrow{\delta_2} \ldots ,$$

where

$$C_b^i(G,V) := \{\alpha : G^{i+1} \to V \mid \alpha \text{ is a bounded } G\text{-map}\} \tag{1}$$

and the coboundary map δ_i is defined by

$$\delta_i\alpha\big([x_0,\ldots,x_{i+1}]\big) := \sum_{k=0}^{i+1}(-1)^k\alpha\big([x_0,\ldots,\widehat{x_k},\ldots,x_{i+1}]\big). \tag{2}$$

Here G^{i+1} is considered with the diagonal G-action by left multiplication, and "bounded" means "has bounded image with respect to the norm on V".

Equivalently, $C_b^i(G,V) \subseteq Hom_{\mathbb{F}G}(C_i(G,\mathbb{F}),V)$ is the subspace of all $\mathbb{F}G$-morphisms $C_i(G,\mathbb{F}) \to V$, which are bounded as linear maps, where $C_i(G,\mathbb{F})$ is the space of all chains (= finite support functions) $G^{i+1} \to \mathbb{F}$ given the ℓ_1-norm with respect to the standard basis G^{i+1}. The coboundary homomorphism δ in $C_b^i(G,V)$ is the dual of the boundary morphism ∂ in $C_*(G,\mathbb{F})$. The spaces $C_b^i(G,V)$ are naturally given the ℓ_∞-norm dual to the ℓ_1-norm on $C_i(G,\mathbb{F})$.

The definition of bounded cohomology is usually given for $\mathbb{F}=\mathbb{R}$. We give it both for $\mathbb{Q}$ and $\mathbb{R}$. This might cause the following problem: if V is a normed *real* vector space, the bounded cohomology $H_b^*(G,V)$ is defined in two different ways, because V may be viewed as a normed rational vector space as well. But it turns out that these two definitions coincide in this case, because formulas (1) and (2) make no use of the field $\mathbb{F}$.

3. Summable 1-chains.

In this section we prove that certain summable 1-chains on a graph can be approximated in a good way by good compactly supported chains. These results (though not in full generality) will be needed in section 4 for isoperimetric inequalities. Essentially, we generalize the following theorem proved by D. J. Allcock and S. M. Gersten.

Theorem 4 (Theorem 3.3 in [1])**.** *If Γ is a graph and f is a summable real-valued 1-cycle on Γ, then there is a countable coherent family C of simple circuits in Γ and a function $g : C \to [0,\infty)$ such that $f=\sum_C g(c)c$.*

"Coherent" here essentially means that $|f|_1=\sum_C g(c)|c|_1$.

Let Γ be a graph. As a part of the structure, we assign an orientaton to each edge of Γ. Therefore "a 1-chain on Γ" is the same thing as "a function on the edges of Γ". The orientation on Γ determines *the initial vertex* ιe and *the terminal vertex* τe for each edge e. A directed path p in Γ is a sequence of edges $(e_1,\ldots,e_n)$ with $\tau e_i=\iota e_{i+1}$. The initial vertex ιp of the path p is ιe_1 and the terminal vertex τp of p is τe_n. A directed path $(e_1,\ldots,e_n)$ is *simple* if the vertices $\iota e_1,\ldots,\iota e_n$ are all distinct. If T is a set of vertices in Γ, we say that p is a T-path if it is a directed path in Γ such that $\iota p, \tau p\in T$ or $\iota p=\tau p$ (in other words, the homological boundary of p is supported in T).

Let f be a(n absolutely) summable 1-chain in Γ. For a vertex v in Γ, define

$$Div_v(f):=\sum_{\iota e=v} f(e)-\sum_{\tau e=v} f(e),$$

where e stands for edges in Γ (with the fixed orientation).

By $\Gamma(f)$ we denote the same graph Γ but with an orientation on the edges chosen so that $f(e)\geq 0$ for each edge e. Let $\Gamma_+(f)$ be the minimal subgraph of $\Gamma(f)$ containing all the edges e with $f(e)\neq 0$ (i.e. $f(e)>0$).

Lemma 5. *Let Γ be a graph and T be a set of vertices in Γ. If h is a summable real 1-chain such that supp$(\partial h)\subseteq T$ and $\Gamma_+(h)$ contains no nontrivial simple T-paths, then $h=0$.*

Proof. (cf Lemma 3.2 in [1].) Suppose to the contrary that $h\neq 0$, i.e. there is an edge e' in $\Gamma_+(h)$ (hence $h(e')>0$). Let Γ' be the (non-empty) union of all directed paths $(e_1,\ldots,e_n)$ in $\Gamma_+(h)$ with $e_1=e'$. Then e' is the only edge in Γ' incident on $\iota e'$ (otherwise there would be a non-trivial directed edge cycle in $\Gamma'\subseteq\Gamma_+(h)$.)

Case 1. We assume for the moment that Γ' does not contain vertices from T, except possibly for $\iota e'$. Let h' be the summable chain in Γ which coincides with h on Γ' and takes value 0 outside. If v is any vertex in Γ' different from $\iota e'$, then $v \notin T \supseteq supp(\partial f)$ and also, by the definition of Γ', Γ' contains *all* the edges e of $\Gamma_+(h)$ with $\iota e = v$, so

$$0 = Div_v(h) \leq Div_v(h').$$

Obviously, for the vertex $\iota e'$ the strong inequality holds:

$$0 < h(e') = h'(e') = Div_{\iota e'}(h'). \tag{3}$$

Since h' is (absolutely) summable, rearranging the terms we get

$$0 \leq \sum_{v \ vertex \ in \ \Gamma} Div_v(h') = \sum_{v \ vertex \ in \ \Gamma} \left(\sum_{\iota e = v} h'(e) - \sum_{\tau e = v} h'(e) \right) =$$
$$= \sum_{e \ edge \ in \ \Gamma} \big(h'(e) - h'(e) \big) = 0.$$

Thus $Div_v(h') = 0$ for *any* v, which contradicts (3).

Case 2. If Γ' contains s vertex v' from T and $v' \neq \iota e'$, then do the same construction in the other direction: let Γ'' be the union of all the paths $(e_1, ..., e_n)$ in $\Gamma_+(h)$ with $e_n = e'$. Now Γ'' does not contain a vertex from T, except possibly for $\tau e'$, because otherwise we could connect such a vertex to v' with a T-path in $\Gamma_+(h)$, which would contradict the assumptions of the lemma. So the same argument as in Case 1 works for Γ''. $\square$

Each T-path p in $\Gamma(f)$ gives rise to an integer 1-chain in Γ which we also denote p. As before, $\mathbb{A}$ denotes $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$.

Theorem 6. *Let Γ be a graph, T be a set of vertices in Γ and f be a summable 1-chain on Γ with coefficients in $\mathbb{A}$ and $supp(\partial f) \subseteq T$.*

(a) *There is a countable family $P = \{p_1, p_2, ...\}$ of simple T-paths in $\Gamma_+(f)$ and a sequence $\{\alpha_i\}$ in $[0, \infty) \cap \mathbb{A}$ such that $f = \sum_i \alpha_i p_i$ and $|f|_1 = \sum_i \alpha_i |p_i|_1$.*

(b) *If the above f happens to have finite support, then P can be chosen to be finite.*

This is a generalization of Theorem 4 and our proof of Theorem 6 follows the lines of the proof of Theorem 3.3 in [1] (cf also Lemma 3.6 in [4]), though we make the proof a bit more explicit without referring to the Zorn's lemma.

Proof of Theorem 6. Since f is summable, then $\Gamma_+(f)$ has only countably many edges. Let $P = \{p_1, p_2, ...\}$ be the (countable) set of all simple T-paths in $\Gamma_+(f)$. If the support of f is finite, then P is finite.

For two summable 1-chains f' and f'' in Γ we will write $f' \preceq f''$ if $f'(e) \leq f''(e)$ for each edge in $\Gamma(f)$ (with the orientation defined in $\Gamma(f)$).

Let α_1 be the (non-negative) real number which is maximal among those satisfying

$$\alpha_1 p_1 \preceq f.$$

Note that $\alpha_1 \in [0, \infty) \cap \mathbb{A}$ since α_1 coincides with the minimal value of f on the edges of p_1. Continue inductively: if $\alpha_1, ..., \alpha_{i-1}$ are constructed, let α_i be the maximal real

number satisfying
$$\alpha_1 p_1 + ... + \alpha_{i-1} p_{i-1} + \alpha_i p_i \preceq f.$$
Inductively we see that each α_i is in $[0, \infty) \cap \mathbb{A}$.

Denote $f_i := \alpha_1 p_1 + ... + \alpha_i p_i$. Since all the chains are non-negative,
$$|f_i|_1 = \alpha_1 |p_1|_1 + ... + \alpha_i |p_i|_1.$$
The sequence f_i is monotone and bounded by the summable chain f, so the chain
$$\bar{f} := \sum_i \alpha_i p_i$$
is well defined and satisfies
$$|\bar{f}|_1 := \sum_i \alpha_i |p_i|_1.$$
Also, $0 \preceq \bar{f} \preceq f$, therefore $0 \preceq f - \bar{f} \preceq f$.

We prove now that $\Gamma_+(f - \bar{f})$ does not contain nontrivial simple T-paths. If it contained a nontrivial simple T-path p, then $\alpha p \preceq f - \bar{f} \preceq f$ for some $\alpha > 0$. Therefore p would be a simple path in $\Gamma_+(f)$ as well, i.e. $p = p_i$ for some i, and
$$f_i + \alpha p_i \preceq \bar{f} + \alpha p_i \preceq f.$$
This contradicts the maximality in the definition of α_i.

Also $supp(\partial(f - \bar{f})) \subseteq supp(\partial f) \cup supp(\partial \bar{f}) \subseteq T$, so, by Lemma 5, $f - \bar{f} = 0$, i.e. $f = \bar{f} = \sum_i \alpha_i p_i$ and $|f|_1 = |\bar{f}|_1 = \sum_i \alpha_i |p_i|_1$. This finishes the proof of Theorem 6. $\square$

4. Linear isoperimetric inequalities.

One could mean many different things by "a linear homological isoperimetric inequality". In this section we attempt presenting a comprehensive list of possible interpretations and show that they are all equivalent to hyperbolicity. Essentially all this was shown by S. M. Gersten (see [1, 3, 5, 7]), we just collect the statements in one theorem and sketch the proofs.

Theorem 7. *Let G be a finitely presentable group G and X be a simply connected cellular 2-complex with a free cocompact G action. Then the following statements are equivalent.*

(0) *G is hyperbolic.*
(1) *There exists $K_1 \geq 0$ such that, for any $b \in B_1(X, \mathbb{Z})$, $|b|_{fc,\mathbb{Z}} \leq K_1 |b|_1$.*
(1') *There exists $K_1' \geq 0$ such that for any $b \in B_1(X, \mathbb{Z})$ there exists $a \in C_2(X, \mathbb{Z})$ with $\partial a = b$ and $|a|_1 \leq K_1' |b|_1$.*
(2) *There exists $K_2 \geq 0$ such that, for any $b \in B_1(X, \mathbb{Z})$, $|b|_{fc,\mathbb{Q}} \leq K_2 |b|_1$.*
(3) *There exists $K_3 \geq 0$ such that, for any $b \in B_1(X, \mathbb{Q})$, $|b|_{fc,\mathbb{Q}} \leq K_3 |b|_1$.*
(3') *There exists $K_3' \geq 0$ such that for any $b \in B_1(X, \mathbb{Q})$ there exists $a \in C_2(X, \mathbb{Q})$ with $\partial a = b$ and $|a|_1 \leq K_3' |b|_1$.*
(4) *There exists $K_4 \geq 0$ such that, for any $b \in B_1(X, \mathbb{R})$, $|b|_{fc,\mathbb{R}} \leq K_4 |b|_1$.*
(4') *There exists $K_4' \geq 0$ such that for any $b \in B_1(X, \mathbb{R})$ there exists $a \in C_2(X, \mathbb{R})$ with $\partial a = b$ and $|a|_1 \leq K_4' |b|_1$.*

(5) *There exists* $K_5 \geq 0$ *such that, for any* $b \in B_1(X,\mathbb{R}) \subseteq B_1^{(1)}(X,\mathbb{R})$, $|b|_{f1,\mathbb{R}} \leq K_5|b|_1$.
(6) *There exists* $K_6 \geq 0$ *such that, for any* $b \in B_1^{(1)}(X,\mathbb{R})$, $|b|_{f1,\mathbb{R}} \leq K_6|b|_1$.
(6') *There exists* $K_6' \geq 0$ *such that, for any* $b \in B_1^{(1)}(X,\mathbb{R})$, *there exists* $a \in C_2^{(1)}(X,\mathbb{R})$ *with* $\hat{\partial}_2 a = b$ *and* $|a|_1 \leq K_6'|b|_1$.
(7) *There exists* $K_7 \geq 0$ *such that, for any* $z \in Z_1^{(1)}(X,\mathbb{R})$, $|z|_{f1,\mathbb{R}} \leq K_7|z|_1$.
(7') *There exists* $K_7' \geq 0$ *such that, for any* $z \in Z_1^{(1)}(X,\mathbb{R})$, *there exists* $a \in C_2^{(1)}(X,\mathbb{R})$ *with* $\partial a = z$ *and* $|a|_1 \leq K_7'|z|_1$.

Later in the paper we will only need implication (5) $\Rightarrow$ (0), and this is the only implication we actually prove here. It also follows from Gersten's results and we present a direct proof which in turn is based on ideas of [1, 3, 12].

Sketch of proof of Theorem 7. Equivalences $(\mathbf{1}) \Leftrightarrow (\mathbf{1'})$, $(\mathbf{3}) \Leftrightarrow (\mathbf{3'})$, $(\mathbf{4}) \Leftrightarrow (\mathbf{4'})$, $(\mathbf{6}) \Leftrightarrow (\mathbf{6'})$, $(\mathbf{7}) \Leftrightarrow (\mathbf{7'})$ follow from the definition of the filling norm.

$(\mathbf{0}) \Rightarrow (\mathbf{1'})$ Hyperbolic groups are defined as the ones having a linear isoperimetric inequality for filling edge-loops with combinatorial disks. Each $b \in B_1(X,\mathbb{Z})$ is a sum of such loops (viewed as chains). Let a be the sum of fillings for these loops. (See [6] where the converse was also proved.)

$(\mathbf{1}) \Rightarrow (\mathbf{2})$ is obvious because $|\cdot|_{fc,\mathbb{Q}} \leq |\cdot|_{fc,\mathbb{Z}}$ on $B_1(X,\mathbb{Z})$. Similarly, $(\mathbf{4}) \Rightarrow (\mathbf{5})$.

$(\mathbf{2}) \Rightarrow (\mathbf{3})$ Let $\Gamma := X^{(1)}$ and $b \in B_1(X,\mathbb{Q})$. Since b is a summable cocycle, being the boundary of a summable 2-chain, then, by the above Theorem 4, $b = \sum_C g(c)c$ for some $g : C \to [0,\infty)$. The main point here is that the elements of C are *integer* 1-cycles in X. Also, since g is rational, g can be chosen to take rational values (see Theorem 6 for the proof.) Fill each $c \in C$ by an integer 2-chain a_c with a linear isoperimetric inequality, then $\sum_C g(c)a_c$ is a rational filling of b with a linear isoperimetric inequality.

$(\mathbf{3'}) \Rightarrow (\mathbf{4'})$ and $(\mathbf{5}) \Rightarrow (\mathbf{7})$ are analogous to (2) $\Rightarrow$ (3). (See also Proposition 3.6 in [1] for the proof of (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4).)

$(\mathbf{7'}) \Rightarrow (\mathbf{6'})$, $(\mathbf{6}) \Rightarrow (\mathbf{5})$ are obvious.

We devote the rest of this section to the proof of the remaining implication

$(\mathbf{5}) \Rightarrow (\mathbf{0})$ It immediately follows from the following proposition (cf Proposition 5.4 in [3].)

Proposition 8. *Let* G *and* X *be as in the hypotheses of Theorem 7 and suppose* G *is not hyperbolic. Then there exist* $C > 0$ *and a sequence of geodesic quadrilaterals* w *in* $X^{(1)}$ *with*

$$|w|_1 \to \infty \qquad \text{and} \qquad |w|_{f1,\mathbb{R}} \geq C(|w|_1)^2.$$

Here, abusing the notations, we identify the quadrilateral w and the corresponding integral 1-chain.

Proof. We modify the techniques of [2, 3] which in turn used a modification of the Ol'shanskii's method of layers [12].

Each 2-cell in X is a polygon whose sides are glued to edges in $X^{(1)}$. Let M be the maximal number of sides over all 2-cells in X. In particular, all combinatorial boundaries of 2-cells in X have diameter at most M, and also, for any 2-chain a in X,

$$|\partial a|_1 \leq M|a|_1. \tag{4}$$

If G is not hyperbolic, Ol'shanskii shows that there are arbitrarily *thick* geodesic quadrilaterals in $X^{(1)}$, and Gersten states more precisely that (see Proposition 5.2 in [3] in which we make the substitution $t = 8Mr$)
there exist

- a sequence of integers r tending to infinity,
- a geodesic quadrilateral $w = w(r)$ for each r,
- a (geodesic) side S in each w, and
- a subinterval $[x, y] \subseteq S$ with the midpoint z

such that

- the length of $[x, y]$ is $4Mr$,
- the distance from z to the sides of w other than S is $\geq 4Mr$, and
- the perimeter of w is $\leq 80Mr$.

The last property implies that

$$|w|_1 \leq 80Mr. \tag{5}$$

Since the interval $[x, y]$ does not intersect other sides of the quadrilateral,

$$|w|_1 \geq 4Mr \to \infty \quad as \quad r \to \infty.$$

For each positive integer $k \leq r$, let B_k be the simplicial ball of radius $4Mk$ centered at z. If a is any summable chain with $\partial a = w$, denote a_k the restriction of a to (the 2-cells of) $B_k \setminus B_{k-1}$ (defining by zero outside). Let s_k and t_k be the two points on $[x, y]$ which are at distance $4Mk - 2M$ from z (see Fig. 1), and

$$A_k := \{v \in [x, y] \mid 4M(k-1) \leq d(z, v) \leq 4Mk - 2M\}$$
$$A'_k := \{v \in [x, y] \mid 4Mk - 2M \leq d(z, v) \leq 4Mk\}.$$

∂a_k is a 1-cycle which takes value 1 on the edges of $[x, y]$ incident on s_k and t_k (since these edges are "deep inside" $B_k \setminus B_{k-1}$). Also, ∂a_k splits as the sum of a 1-chain q_k supported in

$$\{M - neighborhood\ of\ B_{k-1}\} \cup A_k,$$

and a 1-chain q'_k supported in

$$\{M - neighborhood\ of\ \overline{X \setminus B_k}\} \cup A'_k.$$

These two supports intersect only in $T := \{s_k, t_k\}$, and the above observation implies that

$$\partial q_k = t_k - s_k \qquad \text{and} \qquad \partial q'_k = s_k - t_k. \tag{6}$$

By Theorem 6, each q_k splits as a (finite) linear combination of T-paths

$$q_k = \sum_i \alpha_i p_i$$

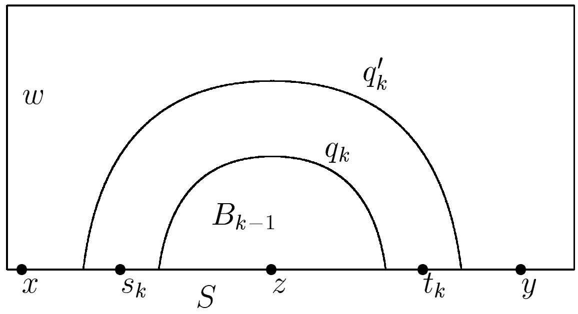

FIGURE 1. The filling a of b.

such that

$$\alpha_i \geq 0 \qquad \text{and} \qquad |q_k|_1 = \sum_i \alpha_i |p_i|_1.$$

Also, (6) says that $\sum_i \alpha_i \geq 1$. Those T-paths p_i which have non-trivial boundary must be of length at least $d(s_k, t_k) \geq 8Mk - 4M$, so

$$|q_k|_1 = \sum_i \alpha_i |p_i|_1 \geq \sum_i \alpha_i (8Mk - 4M) \geq 8Mk - 4M.$$

The same holds for q'_k, hence

$$|\partial a_k|_1 = |q_k|_1 + |q'_k|_1 \geq 2(8Mk - 4M) = 16Mk - 4M.$$

By (4),

$$|a_k|_1 \geq \frac{1}{M} |\partial a_k|_1 \geq 16k - 4,$$

and since the chains a_k have disjoint supports,

$$|a|_1 \geq \sum_{k=1}^{r} |a_k|_1 \geq \sum_{k=1}^{r} (16k - 4) = 16 \frac{r(r+1)}{2} - 4r \geq 8r^2.$$

Then, by (5),

$$|a|_1 \geq 8 \left(\frac{|w|_1}{80M} \right)^2 = \frac{1}{800M^2} \cdot (|w|_1)^2.$$

Since this is true for any summable filling a of w, then

$$|w|_{f1,\mathbb{R}} \geq \frac{1}{800M^2} \cdot (|w|_1)^2,$$

so we put $C := \frac{1}{800M^2}$. Proposition 8 and Theorem 7 are proved. $\square$

$\square$

5. The characterization.

In this section we characterize hyperbolic groups by bounded cohomology.

Let G be a finitely presentable group and $\mathcal{M}(G)$ be one of the following six classes of modules.

- $\mathcal{M}_6(G)$, the class of *bounded $\mathbb{Q}G$-modules*. A $\mathbb{Q}G$-module is called *bounded* if it is normed as a vector space over $\mathbb{Q}$ and G acts on it by linear operators of uniformly bounded norms.
- $\mathcal{M}_5(G)$, the class of *isometric $\mathbb{Q}G$-modules*. A $\mathbb{Q}G$-module is called *isometric* if it is bounded and, moreover, the G-action preserves its norm.
- $\mathcal{M}_4(G)$, analogously, the class of *bounded $\mathbb{R}G$-modules*.
- $\mathcal{M}_3(G)$, analogously, the class of *isometric $\mathbb{R}G$-modules*.
- $\mathcal{M}_2(G)$, analogously, the class of *bounded Banach $\mathbb{R}G$-modules*, i.e. bounded $\mathbb{R}G$-modules which are Banach spaces with respect to their norms.
- $\mathcal{M}_1(G)$, analogously, the class of *isometric Banach $\mathbb{R}G$-modules*.

Obviously, $\mathcal{M}_1(G) \subseteq \mathcal{M}(G) \subseteq \mathcal{M}_6(G)$ for each such $\mathcal{M}(G)$.

Theorem 3 promised in the introduction is stated more precisely as

Theorem 9. *Let G be a finitely presentable group, and $\mathcal{M}(G)$ be one of the classes $\mathcal{M}_1(G) - \mathcal{M}_6(G)$ described above. Then the following statements are equivalent.*

(a) *G is hyperbolic.*
(b) *The map $H^2_b(G,V) \to H^2(G,V)$ is surjective for any $V \in \mathcal{M}(G)$.*
(c) *The map $H^i_b(G,V) \to H^i(G,V)$ is surjective for any $i \geq 2$ and any $V \in \mathcal{M}(G)$.*

Proof. $(a) \Rightarrow (c)$ was shown in [10] for the largest class $\mathcal{M}_6(G)$. Property (c) for $\mathcal{M}_6(G)$ implies the same property for each of the classes $\mathcal{M}_1(G) - \mathcal{M}_6(G)$. The implication $(c) \Rightarrow (b)$ is obvious. It only remains to prove $(b) \Rightarrow (a)$ for the class $\mathcal{M}_1(G)$.

Since G is finitely presentable, there exists a contractible cell complex X with a free cellular G-action such that the induced action on the 2-skeleton of X is cocompact. We take V to be $B^{(1)}_1(X,\mathbb{R})$ with the filling norm $|\cdot|_{f1,\mathbb{R}}$ which we denote for simplicity by $|\cdot|_f$. The vector space V is Banach as the quotient of the Banach space $C^{(1)}_2(X,\mathbb{R})$ by the (closed) kernel of $\hat{\partial}_2 : C^{(1)}_2(X,\mathbb{R}) \to C^{(1)}_1(X,\mathbb{R})$. In particular, $V \in \mathcal{M}_1(G)$.

Let Y be the geometric realization of the homogeneous bar-construction for G, i.e. Y is the simplicial complex whose k-simplices are labeled by ordered $(k+1)$-tuples $[x_0, ..., x_k]$ of elements of the group G, and each simplex labeled $[x_0, ..., \widehat{x_i}, ..., x_k]$ is identified with the i-th face of $[x_0, ..., x_k]$. The action of G on Y is diagonal:

$$g \cdot [x_0, ..., x_k] := [g \cdot x_0, ..., g \cdot x_k].$$

Let $\mathcal{C}^X_*$ and $\mathcal{C}^Y_*$ be the augmented chain complexes

$$... \longrightarrow C_2(X,\mathbb{R}) \xrightarrow{\partial_2} C_1(X,\mathbb{R}) \xrightarrow{\partial_1} C_0(X,\mathbb{R}) \xrightarrow{\epsilon} \mathbb{R} \longrightarrow 0$$

and

$$... \longrightarrow C_2(Y,\mathbb{R}) \xrightarrow{\partial_2} C_1(Y,\mathbb{R}) \xrightarrow{\partial_1} C_0(Y,\mathbb{R}) \xrightarrow{\epsilon} \mathbb{R} \longrightarrow 0\ ,$$

respectively. Both X and Y are contractible, hence $\mathcal{C}_*^X$ and $\mathcal{C}_*^Y$ are acyclic. Both $\mathcal{C}_*^X$ and $\mathcal{C}_*^Y$ have free $\mathbb{R}G$-modules in each non-negative dimension. Also $\mathcal{C}_i^X$ is finitely generated as an $\mathbb{R}G$-module for $i = 0, 1, 2$.

By standard homological algebra, there exist homotopy equivalences $\varphi_* : \mathcal{C}_*^Y \to \mathcal{C}_*^X$ and $\psi_* : \mathcal{C}_*^X \to \mathcal{C}_*^Y$ which are identities in dimension -1. It is important that φ_* and ψ_* are chain maps *in the category of* $\mathbb{R}G$*-modules*, in particular, for each i, φ_i and ψ_i are linear maps commuting with the G-action.

For the dual cochain complexes

$$\mathcal{C}_X^i := C^*(X, V) = Hom_{\mathbb{R}G}(\mathcal{C}_i^Y, V) \quad \text{and} \quad \mathcal{C}_Y^i := C^*(Y, V) = Hom_{\mathbb{R}G}(\mathcal{C}_i^Y, V)$$

and the dual maps

$$\varphi^* : \mathcal{C}_X^* \to \mathcal{C}_Y^* \quad \text{and} \quad \psi^* : \mathcal{C}_Y^* \to \mathcal{C}_X^*,$$

the cochain map $\psi^* \circ \varphi^*$ is homotopic to the identity map, hence $\psi^* \circ \varphi^*$ induces the identity map on cohomology $H^*(G, V)$ in the positive dimensions.

The <u>*universal cocycle*</u> in $\mathcal{C}_X^2$ is the 2-cochain $u : \mathcal{C}_2^X \to V$ which coincides with the composition

$$C_2(X, \mathbb{R}) \xrightarrow{\partial_2} B_1(X, \mathbb{R}) \hookrightarrow B_1^{(1)}(X, \mathbb{R}).$$

One checks that u is indeed a cocycle. By the above observations,

$$u = (\psi^2 \circ \varphi^2)(u) + \delta v \tag{7}$$

for some 1-cochain $v : \mathcal{C}_1^X \to V$.

Since $\varphi^2(u)$ is a cocycle in $\mathcal{C}_Y^2$ and the map $H_b^2(G, V) \to H^2(G, V)$ is surjective by the assumption,

$$\varphi^2(u) = u' + \delta v', \tag{8}$$

for a 1-cochain $v' \in \mathcal{C}_Y^1$ and a *bounded* 2-cocycle $u' \in \mathcal{C}_Y^2$, i.e.

$$|u'|_\infty < \infty. \tag{9}$$

The above information is demonstrated by the diagrams

$$\begin{array}{ccc} a, b \in \mathcal{C}_*^X & = & C_*(X, \mathbb{R}) \\ {\scriptstyle \varphi_*}\big\uparrow\big\downarrow{\scriptstyle \psi_*} & & \\ \mathcal{C}_*^Y & = & C_*(Y, \mathbb{R}) \end{array} \quad \text{and} \quad \begin{array}{ccc} u, v \in \mathcal{C}_X^* & = & C^*(X, V) \\ {\scriptstyle \varphi^*}\big\downarrow\big\uparrow{\scriptstyle \psi^*} & & \\ u', v' \in \mathcal{C}_Y^* & = & C^*(Y, V). \end{array}$$

We also have the standard pairings $\langle \cdot, \cdot \rangle : \mathcal{C}_X^i \oplus \mathcal{C}_i^X \to V$ and $\langle \cdot, \cdot \rangle : \mathcal{C}_Y^i \oplus \mathcal{C}_i^Y \to V$.

Fix any vertex y in Y (i.e. $y \in G$). For each 1-chain $b = \sum_{y_0, y_1 \in G} \beta_{[y_0, y_1]}[y_0, y_1]$ in Y, <u>*the cone*</u> over b with vertex y is the 2-chain

$$[y, b] := \sum_{y_0, y_1 \in G} \beta_{[y_0, y_1]}[y, y_0, y_1].$$

If b happens to be a cycle, then $\partial[y, b] = b$. Obviously,

$$\Big|[y, b]\Big|_1 = |b|_1.$$

In other words, $[y, b]$ is a filling of b in Y, and the above formula says that these fillings satisfy a linear isoperimetric inequality in Y. We want the same property in X to show hyperbolicity, so we will use the maps φ^* and ψ^* to "transfer" the linear isoperimetric inequality from Y to X using the fact that the universal cocycle is cohomologous to a bounded cocycle.

Note also that if α is a cocycle in $\mathcal{C}^i_Y$ and $c \in \mathcal{C}_i^Y$, then $c - [y, \partial c]$ is a cycle, hence a boundary, so $\big\langle \alpha, c - [y, \partial c] \big\rangle = 0$ and

$$\big\langle \alpha, c \big\rangle = \big\langle \alpha, [y, \partial c] \big\rangle. \tag{10}$$

Now with the above setup we can finish the proof of Theorem 9. Pick any 1-boundary $b \in B_1(X, \mathbb{R})$ and any 2-chain a with $\partial a = b$. We will show that $|b|_f \le K|b|_1$ for some uniform constant K, therefore G is hyperbolic by implication (5) $\Rightarrow$ (0) in Theorem 7.

By equality (7),

$$b = \partial a = \big\langle u, a \big\rangle = \big\langle (\psi^2 \circ \varphi^2)(u) + \delta v, a \big\rangle = \big\langle (\psi^2 \circ \varphi^2)(u), a \big\rangle + \big\langle v, b \big\rangle.$$

Since $\varphi^2(u)$ is a cocycle, and using (10) and (8),

$$\begin{aligned}
&\big\langle (\psi^2 \circ \varphi^2)(u), a \big\rangle = \big\langle \varphi^2(u), \psi_2(a) \big\rangle = \big\langle \varphi^2(u), [y, \partial(\psi_2(a))] \big\rangle = \\
&= \big\langle \varphi^2(u), [y, \psi_1(b)] \big\rangle = \big\langle u' + \delta v', [y, \psi_1(b)] \big\rangle = \big\langle u', [y, \psi_1(b)] \big\rangle + \big\langle v', \partial[y, \psi_1(b)] \big\rangle = \\
&= \big\langle u', [y, \psi_1(b)] \big\rangle + \big\langle v', \psi_1(b) \big\rangle = \big\langle u', [y, \psi_1(b)] \big\rangle + \big\langle \psi^1(v'), b \big\rangle.
\end{aligned}$$

So, combining the two formulas above,

$$\begin{aligned}
b = \quad & \big\langle u', [y, \psi_1(b)] \big\rangle + \big\langle \psi^1(v') + v, b \big\rangle, \\
|b|_f \le \quad & \Big|\big\langle u', [y, \psi_1(b)] \big\rangle\Big|_f + \Big|\big\langle \psi^1(v') + v, b \big\rangle\Big|_f \le \\
& \le |u'|_\infty \cdot \Big|[y, \psi_1(b)]\Big|_1 + \Big|\psi^1(v') + v\Big|_\infty \cdot |b|_1 = \\
& = |u'|_\infty \cdot \big|\psi_1(b)\big|_1 + \Big|\psi^1(v') + v\Big|_\infty \cdot |b|_1 \le \\
& \le \Big(|u'|_\infty \cdot |\psi_1|_\infty + \Big|\psi^1(v') + v\Big|_\infty\Big) \cdot |b|_1.
\end{aligned}$$

This will give the desired linear isoperimetric inequality once we prove that all the norms in the parentheses are finite.

The cochain u' is bounded by definition (by a constant depending only on the choice of G and X, see (9)). The maps $\psi_1 : \mathcal{C}_1^X \to \mathcal{C}_1^Y$ and $\psi^1(v') + v : \mathcal{C}_1^X \to V$ are both linear maps commuting with the G-action. Their boundedness (by constants depending only on G and X) is immediate from the following simple observation which deserves being stated as a separate lemma.

Lemma 10. *Let $(W, |\cdot|_1)$ and $(W', |\cdot|)$ be two normed vector spaces over $\mathbb{R}$, where $|\cdot|_1$ is the ℓ_1-norm on W with respect to some basis. Suppose a group G acts*

- *on W permuting the basis so that there are only finitely many orbits of basis elements, and also*

- *on W' preserving the norm $|\cdot|$.*

If $f : W \to W'$ is a linear map commuting with the G-action, then f is bounded, i.e. $|f|_\infty < \infty$.

Proof. If w_1 and w_2 are two basis elements of W in the same G-orbit, i.e. $w_1 = g \cdot w_2$, then

$$|f(w_1)| = |f(g \cdot w_2)| = |g \cdot f(w_2)| = |f(w_2)|.$$

Since there are only finitely many G-orbits of basis elements in W, $|f(\cdot)|$ takes only finitely many values on the basis elements, hence $|f|_\infty < \infty$. This proves Lemma 10 and Theorem 9. □

□

References

[1] D. J. Allcock and S. M. Gersten, *A homological characterization of hyperbolic groups*, Invent. Math., 135 (1999), pp. 723–742.

[2] J. Fletcher, 1998. Ph.D. thesis, University of Utah.

[3] S. M. Gersten, *A cohomological characterization of hyperbolic groups*. Preprint available at http://www.math.utah.edu/~gersten.

[4] ——, *Distortion and ℓ_1-homology*. To appear in the Proceedings of the Pusan Conference, Groups-Korea 1998, editor Ann-Chi Kim, preprint available at http://www.math.utah.edu/~gersten/.

[5] ——, *A note on cohomological vanishing and the linear isoperimetric inequality*. Preprint available at http://www.math.utah.edu/~gersten.

[6] ——, *Subgroups of word hyperbolic groups in dimension 2*, Jour. London Math. Soc., 54 (1996), pp. 261–283.

[7] ——, *Cohomological lower bounds for isoperimetric functions on groups*, Topology, 37 (1998), pp. 1031–1072.

[8] B. E. Johnson, *Cohomology in Banach algebras*, no. 127 in Mem. Amer. Math. Soc., American Mathematical Society, Providence, R.I., 1972.

[9] I. Mineyev, *Higher dimensional isoperimetric inequalities in hyperbolic groups*. To appear in Mathematische Zeitschrift, preprint available at http://www.math.utah.edu/~mineyev/math/.

[10] ——, *Straightening and bounded cohomology of hyperbolic groups*. Max-Planck-Institut preprint, also available at http://www.math.utah.edu/~mineyev/math/.

[11] G. A. Noskov, *Bounded cohomology of discrete groups with coefficients.*, Leningrad Math. J, (1991), pp. 1067–1084.

[12] A. Y. Ol'shanskii, *Hyperbolicity of groups with subquadratic isoperimetric inequality*, Internat. J. Algebra Comput., 1 (1991), pp. 281–289.

Max-Planck-Institut für Mathematik,
Vivatsgasse 7, D-53111 Bonn, Germany
e-mail: mineyev@mpim-bonn.mpg.de, mineyev@mathstat.usouthal.edu

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