ON INVARIANTS OF MORSE KNOTS

JACOB MOSTOVOY AND THEODORE STANFORD

Abstract. We define and study Vassiliev invariants for (long) Morse knots. It is shown that there are Vassiliev invariants which can distinguish some topologically equivalent Morse knots. In particular, there is an invariant of order 3 for Morse knots with one maximum that distinguishes two different representations of the figure eight knot. We also present the results of computer calculations for some invariants of low order. It turns out that for Morse knots with two maxima there is a $\mathbf{Z}/2$-valued invariant of order 6 which is not a reduction of any integer-valued invariant.

Keywords: Morse knots, Vassiliev invariants.

Introduction

In the last few years knot theorists have studied so many species of knots (Legendrian, transverse, holonomic, harmonic, Lissajou, virtual knots etc) that perhaps "low-dimensional zoology" will develop as a subfield in its own right. As with bears, buffalos or bumble-bees, each species of knots has its own particular niche. In particular, our interest in Morse knots stems from the fact that Morse knots can be viewed as an intermediate class of objects between braids and knots and in this regard they appear at least in two important contexts in knot theory.

The first appearance of Morse knots (under the name of "plat representations") was in connection with the plat closure of braids in the work of J.Birman [2]. The main result of [2] is a Markov-type theorem for the plat closure. Similarly to the classical Markov theorem, the allowed moves on braids fall into two categories: moves preserving the number of strands and a stabilisation move. Morse knots then are equivalence classes of braids under the first class of moves; consequently, the theory of Morse knots can be regarded as an "unstable knot theory". It turned out that there exist topologically equivalent (i.e. isotopic) Morse knots with the same number of strands, or, equivalently, with the same number of maxima of the "height function", which belong to different classes with respect to the equivalence relation on Morse knots. Examples of such pairs of knots were given by Birman [2] and Montesinos [6].

Another context where Morse knots are relevant is Kontsevich's construction of a universal Vassiliev invariant for knots [5]. This invariant, known as the "Kontsevich integral", is defined in two steps. First one chooses an embedding of a given knot into $\mathbf{R}^3$ as a Morse knot and defines what is sometimes called the "preliminary" Kontsevich integral. The preliminary integral is invariant under *Morse* equivalence of embeddings and it has to be normalised in a suitable way to obtain a genuine knot invariant. It may seem that the only information specific to Morse knots that is lost while passing to the stabilised (i.e. normalised) version of the Kontsevich integral is the number of critical points of the "height function" on the knot. (In fact, on page 145 of [5] an erroneous claim is made that the complete set of invariants of a Morse knot is its topological type together with the number of maxima of the height function.) However, certain stabilisation is implicit in the definition of the algebra of chord diagrams for knots. We will see how to refine the relations on chord diagrams in order to make the Kontsevich integral sensitive to more subtle information specific to Morse knots.

The main purpose of this paper is to show that some "unstable" information about Morse knots can be captured by Vassiliev invariants. In the first section we consider the basic properties of Morse knots. In particular, it is shown that the monoid of Morse knots is "almost commutative" and Morse knots with one maximum are classified. In section 2 we define the Vassiliev invariants and give an example which shows that they can distinguish isotopic (but not Morse equivalent) knots. We also introduce "semi-stable" invariants; these, however, turn out to be "uninteresting" in the sense that almost all of them are invariants of knots up to isotopy. Section 3 contains rough estimates of the stability range for the Vassiliev invariants of Morse knots. Finally, in section 4 we present some results of computer calculations. Here we encounter a rather unexpected phenomenon: there is a $\mathbf{Z}/2$-valued Vassiliev invariant of Morse knots with two maxima which is not a mod 2 reduction of any integer-valued invariant.

We work with long knots, which are somewhat easier to deal with than compact knots in certain situations. We do not attempt to answer the question whether long and compact knots lead to the same theory.

We assume the reader to be familiar with the basics of Vassiliev knot invariants, suitable references are [1] and [3].

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1. Morse knots

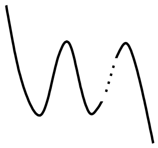
Throughout the paper we will work with oriented long knots. A long knot k is a smooth non-singular embedding $k : \mathbf{R} \to \mathbf{R}^3$ such that the tangent vector to k tends to $(0, 0, -1)$ when $|t|$ tends to infinity. Thus we adopt the convention that all knots "point downwards".

We say that k is a *Morse knot* if the height function, that is, the vertical component of the function $k(t)$ has only a finite number of critical points, all of which are non-degenerate. Two Morse knots are *Morse equivalent* if one can be deformed into the other through Morse knots. It is clear that Morse equivalent knots are isotopic.

Denote by $\mathcal{M}_n$ the set of Morse equivalence classes of knots with n maxima (or, equivalently, n minima) of height function and set $\mathcal{M} = \bigcup \mathcal{M}_i$. Abusing the terminology, we will refer to elements of $\mathcal{M}$ as to "Morse knots"; this should not lead to confusion. $\mathcal{M}_0$ consists of a single element $[\mathbf{0}]$ which is the class of the embedding $t \to (0, 0, -t)$. The sets $\mathcal{M}_n$ are formed by Morse knots whose isotopy classes have bridge number less than or equal to $n + 1$.

On Morse knots there is an operation of connected sum: $k\#l$ is a knot formed by "putting k on top of l". If $k \in \mathcal{M}_n$ and $l \in \mathcal{M}_m$ the sum $k\#l$ belongs to $\mathcal{M}_{n+m}$, so Morse knots form a graded monoid under connected sum. The grading comes from the number of maxima and the identity is $[\mathbf{0}]$.

Denote by $[\mathbf{k}]$ the "k-hump" Morse knot as below:



Notice that $[\mathbf{m}]\#[\mathbf{k}] = [\mathbf{m}+\mathbf{k}]$.

The following theorem is a version of a statement in [2]:

Theorem 1. [7] *For any two isotopic Morse knots $k, l \in \mathcal{M}_n$ there exists a non-negative integer m such that $k\#[\mathbf{m}]$ and $l\#[\mathbf{m}]$ are Morse equivalent.*

Thus the theory of Morse knots can be thought of as an unstable version of the usual knot theory. Indeed, according to the theorem above the direct limit of the sequence

$$\mathcal{M}_n \overset{\#[\mathbf{1}]}{\to} \mathcal{M}_{n+1}$$

is precisely the set of isotopy classes of knots.

There are other "unstable knot theories" which stabilise to knots in a similar way. For example, closed braids can also be regarded as "unstable knots". Note, however, that the stabilisation move for closed braids does not commute with the unstable moves, that is, with conjugation. In this respect, the theory of Legendrian and transverse knots is a better example of an unstable knot theory; certain statements and proofs can be transferred from the subject of Legendrian and transverse knots to Morse knots without major changes. In particular, we will apply the technique developed by D.Fuchs and S.Tabachnikov in [4] to describe the semi-stable invariants of Morse knots (Theorem 7 below).

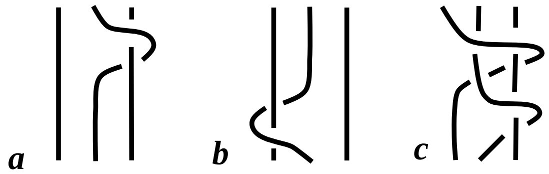
1.1. **Morse knots as closed braids.** The stabilisation phenomenon for Morse knots was discovered by J.Birman in [2] where she proved a Markov-type theorem for the plat closure. The counterpart of the plat closure for long knots (called "short-circuit" closure) is described in [7]. The short-circuit closure of a pure braid on an odd number of strands is obtained by stretching the top of the first strand up and the bottom of the last strand down to infinity and joining all strands in turn at the bottom and at the top. Every braid in the pure braid group $\mathcal{P}_{2n+1}$ closes to a knot in $\mathcal{M}_n$. For example, the closure of a trivial braid on $2n+1$ strands is the n-hump knot $[\mathbf{n}]$.

There is a Markov theorem for the short-circuit closure:

Theorem 2. [7] *For all $n \geqslant 0$ there exist finitely generated subgroups $H_n^T, H_n^B \subset \mathcal{P}_{2n+1}$ such that the fibres of the short-circuit map $\mathcal{P}_{2n+1} \to \mathcal{M}_n$ are the orbits of the simultaneous action on $\mathcal{P}_{2n+1}$ by H_n^T on the left (that is, on the top) and by H_n^B on the right.*

The stable case of the theorem, i.e. the corresponding statement for knots up to isotopy is obtained by letting n tend to infinity. For an explicit description of H_n^T and H_n^B for all $n \geqslant 1$ we refer to [7]. In the last section we list the generators for H_2^T and H_2^B; here we will discuss as an example the case $n = 1$.

1.2. **Morse knots with one maximum.** Recall that the pure braid group $\mathcal{P}_3$ is isomorphic to $F_2 \times \mathbf{Z}$ where F_2 is the free group on two generators. Let a and b be the generators of F_2 and c be the generator of $\mathbf{Z}$ shown below.



Then H_1^T is generated by a and c, H_1^B is generated by b and c, so the quotient $H_1^T \backslash \mathcal{P}_3 / H_1^B = \mathcal{M}_1$ can be easily described:

Theorem 3. *Non-trivial knots in $\mathcal{M}_1$ are in one-to-one correspondence with all such reduced words on two letters a and b that start with a non-zero power of b and end with a non-zero power of a.*

By the "trivial knot" we mean here the hump $[\mathbf{1}]$.

Notice that the mirror image of a knot k in $\mathcal{M}_1$ with respect to a reflection in any vertical plane can be obtained by replacing a and b by a^{-1} and b^{-1} respectively and vice versa in the reduced word that represents k. This implies the following

Corollary 4. *Any non-trivial knot in $\mathcal{M}_1$ is distinct from its mirror image with respect to any vertical plane.*

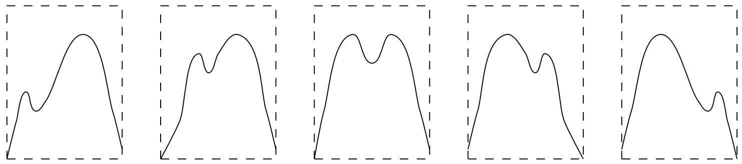
In particular, the figure eight knot (which is isotopic to its mirror image) has at least two distinct representatives in $\mathcal{M}_1$. In the next section we will see an example of two Morse knots which are both isotopic to the figure eight knot and can be distinguished by Vassiliev invariants.

1.3. **Are Morse knots commutative?** We do not know if the monoid of Morse knots is commutative. The proof of the commutativity of the connected sum operation for usual knots consists of making one of the knots very small and running it through the other summand. This does not work with Morse knots for the following reason: while running one knot through the other one we have turn the smaller knot upside down at the maxima and minima of the bigger knot. It is not clear if this can be done without creating new critical points. Still, we have the following

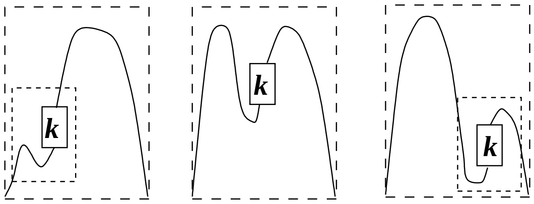
Theorem 5.

$$\begin{array}{lll} \text{(a)} & k\#[\mathbf{1}] = [\mathbf{1}]\#k & \text{for any } k \in \mathcal{M}; \\ \text{(b)} & k\#l\#[\mathbf{1}] = l\#k\#[\mathbf{1}] & \text{for any } k, l \in \mathcal{M}. \end{array}$$

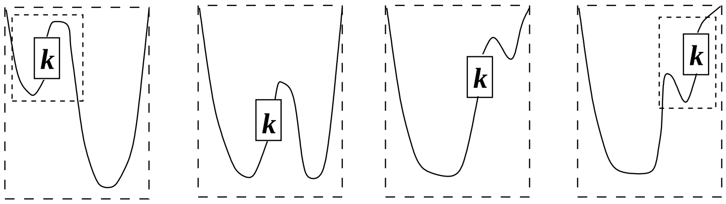
Part (a) of the above theorem was proved in [7]. Essentially, one has to show that a "hump" can be run through a critical point. The following picture illustrates how to do it:



To verify part (b) notice that the knot $k\#[\mathbf{1}]$ can be passed through a maximum as shown below.



Let k' be the Morse knot obtained by passing $k\#[\mathbf{1}]$ through a maximum. Passing k' through a minimum we again obtain $k\#[\mathbf{1}]$ as illustrated by the following picture:



Passing $k\#[\mathbf{1}]$ through maxima and minima of l we see that $l\#k\#[\mathbf{1}]$ equals to $k\#[\mathbf{1}]\#l$ which is the same as $k\#l\#[\mathbf{1}]$.

A possible method of showing that the connected sum is not commutative would be to prove the non-commutativity of the algebra of Morse chord diagrams defined in the next section. Our calculations up to degree 7 in $\mathcal{M}_2$ failed to find any such phenomenon.

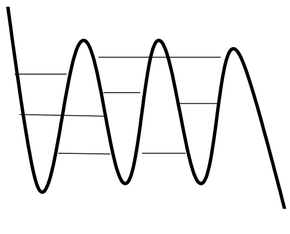
2. Vassiliev invariants

2.1. Vassiliev invariants of Morse knots. Let v be a function from $\mathcal{M}_n$ to some abelian group (which henceforth will be assumed to be the additive group of real numbers, unless stated otherwise). It can be extended inductively to a function on Morse knots with transversal double points by means of the Vassiliev skein relation:

$v($  $) = v($  $) - v($  $).$

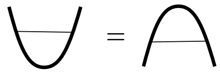
The function v is a Vassiliev invariant of type (or order) k if its extension vanishes on all knots with more than k double points. Notice that we define Vassiliev invariants separately for each $\mathcal{M}_n$.

Just like in the usual knot theory one can introduce *Morse* chord diagrams. A Morse chord diagram with k chords and n maxima is a graph as below:



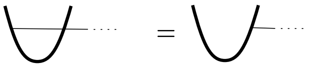
Here all k chords are horizontal and the Wilson loop is a graph of a function on $\mathbf{R}$ with n non-degenerate maxima. Let $\mathcal{A}_{k,n}$ be the $\mathbf{R}$-vector space generated by all chord diagrams with k chords and n maxima modulo several types of relations:

- homotopy which preserves the number and non-degeneracy of critical points and horizontality of chords;
- braid-type 4T-relations;
- framing independence:

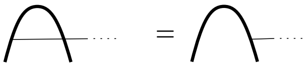


$= 0;$

- strand exchange:



and



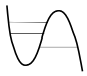
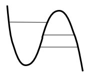
Vector spaces $\mathcal{A}_{0,n}$ are one-dimensional; they are generated by diagrams without chords and with n maxima.

The connected sum of Morse knots induces a structure of a bigraded algebra on $\mathcal{A}_{*,*} = \bigoplus_{k,n} \mathcal{A}_{k,n}$.

Theorem 6. *The Vassiliev invariants of Morse knots form a dual of the algebra $\mathcal{A}_{*,*}$. In particular, $\mathcal{A}_{k,n}$ is dual to the vector space of the invariants of type k modulo invariants of type $k-1$ for Morse knots with n maxima.*

The tool used to prove this theorem is the universal Vassiliev invariant for Morse knots known as the "Kontsevich integral". The only difference between the Kontsevich integral for Morse knots and the Kontsevich integral (without normailisation) for knots up to isotopy is that by the "class of a diagram" one should understand its class in the algebra $\mathcal{A}_{*,*}$ rather than $\mathcal{A}_*$. The proof is identical to that of the Kontsevich theorem for knots up to isotopy apart from that there is no need to check the invariance under an insertion of a hump. For details we refer to [1] or [3]. Note that all terms of the Kontsevich integral of a knot in $\mathcal{M}_n$ belong to $\mathcal{A}_{*,n}$. In particular, its first term is the chord diagram without chords and with n maxima.

The algebra $\mathcal{A}_{*,*}$ contains some unstable information. For instance, an easy calculation shows that the vector space $\mathcal{A}_{3,1}$ is generated by two diagrams

$d_1 =$  $d_2 =$ 

while there is only one Vassiliev invariant of knots up to isotopy in degree 3.

This implies, for example, that the two singular Morse knots k_1 and k_2 shown on Figure 1 are distinguished by Vassiliev invariants of Morse knots of type 3, as their underlying chord diagrams are precisely d_1 and d_2. On the other hand, their isotopy classes differ only by the reversal of orientation. It is well-known that all 2-bridge knots are invertible so k_1 and k_2 are in the same isotopy class. Thus Vassiliev invariants can distinguish isotopic but not Morse equivalent knots.

In fact, both singular knots k_1 and k_2 (up to isotopy) are a sum of a trefoil and a figure eight knot. It is easily checked that a trefoil and its reverse are Morse equivalent so, as we have seen in the previous section, the figure eight knot has at least two distinct representatives in $\mathcal{M}_1$.

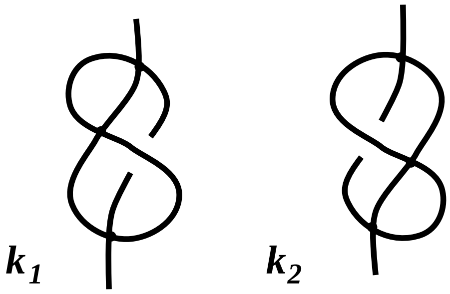


FIGURE 1. Singular knots that are isotopic but not Morse equivalent.

2.2. **Semi-stable invariants.** The definition of Vassiliev invariants for Morse knots given above is not the only possible one. Along with double points one can consider yet another type of singularity, namely a degenerate critical point where the 3rd derivative of the height function does not vanish. The relation for resolving degenerate critical points is analoguous to the Vassiliev skein relation:

$$v(\ \bullet\ ) = v(\ \ ) - v(\ \ ).$$

We will call the Vassiliev invariants defined in this manner *semi-stable*. An example of a semi-stable invariant is the number of maxima of a Morse knot, this is an invariant of type 1. It is clear that each semi-stable Vassiliev invariant is a Vassiliev invariant.The converse is not true:

Theorem 7. *The algebra of semi-stable Vassiliev invariants of Morse knots is generated by Vassiliev invariants of knots up to isotopy and the single invariant of type 1, which is the number of maxima.*

The proof is identical to that of Theorems 4.5 and 5.5 of [4] where similar statements about Legendrian and transverse knots are proved. The only modification needed is a substitution of a "hump" instead of a "zigzag" in Legendrian case or a "double loop" in the case of transverse knots.

3. STABILISATION OF VASSILIEV INVARIANTS.

Consider the algebra $\mathcal{A}_*$ of usual chord diagrams, i.e. of the chord diagrams for knots up to isotopy. Each Morse chord diagram naturally gives rise to a chord diagam of knots up to isotopy. This "forgetful" map sends the relations between Morse chord diagams to relations (or trivial identities) in the algebra $\mathcal{A}_*$ so there is a natural algebra homomorphism

$$\mathcal{A}_{*,*} \to \mathcal{A}_*$$

which preserves the grading by the number of chords. The unstable Vassiliev invariants of Morse knots form the kernel of this map and, as we have seen above, this kernel is non-trivial. It is clearly of importance to understand when the Vassiliev invariants for Morse knots coincide with those of knots up to isotopy. In this section we give a partial answer to this question.

Theorem 8. *The natural map $\mathcal{A}_{k,n} \to \mathcal{A}_k$ is an epimorphism for $k \leqslant 2n$ and an isomorphism for $k \leqslant n$.*

The proof occupies the rest of the section.

We will say that a chord diagram D of knots up to isotopy can be presented on $2n+1$ strands if it is the diagram of some singular Morse knot with n maxima. The corresponding Morse chord diagram will be called a *presentation* of D (on $2n+1$ strands).

The first assertion of Theorem 8 follows from

Lemma 9. *Any chord diagram with $2n$ chords can be presented on $2n+1$ strands.*

Proof. For $n = 1$ the statement is obvious. Suppose it is established for some $n = p$ and let D be a diagram on $2p + 2$ chords. Recall that we think of knots and diagrams as being "long" so the ends of chords of a chord diagram are naturally ordered. Consequently, the chords themselves can also be ordered according to the order of their larger ends. Let c_1 be the last (i.e. the largest) chord of D and c_2 be the last chord of the diagram $D\backslash\{c_1\}$ obtained from D by deleting c_1. The diagram $D' = D\backslash\{c_1 \cup c_2\}$ can be presented on $2p + 1$ chords, so it can also be presented on $2p + 3$ chords in such a way that no chords have ends on any of the last two strands. Now it is possible to add the chord c_2 to the presentation of D' in such a way that (a) it is horizontal; (b) one of its ends is on one of the first $2p+1$ strands and the second end is on the $(2p + 2)$nd strand. Similarly, one can add the chord c_1 to the obtained diagram in such a way that its last end is situated on the last strand; this gives a presentation of D on $2p + 3$ strands. □

To prove the second assertion of Theorem 8 it suffices to verify the following two lemmas:

Lemma 10. *If a diagram on k chords can be presented in two different ways on $2n+1$ strands, where $n \geqslant k$, these presentations are equivalent modulo the relations of strand exchange.*

Lemma 11. *Every 4T and framing independence relation in $\mathcal{A}_k$ come from relations in $\mathcal{A}_{k,n}$ for $n \geqslant k$;*

Proof of Lemma 10. Say that a diagram with n chords on $2n+1$ strands is *arranged* if it has exactly one end of a chord on each of the first $2n$ strands. We describe an algorithm of connecting any diagram with n chords on $2n + 1$ strands to an arranged one by relations of strand exchange; the lemma follows from the existence of such an algorithm.

Let $N(m)$ be the number of ends of chords on the mth strand of a given Morse diagram. Suppose the diagram is not arranged.

Find such m that $N(m) = 0$ and that $N(i) > 0$ for all $1 \leqslant i < m$. If $N(i) = 1$ for all $1 \leqslant i < m$ and m' is the smallest number such that $m' > m$ and $N(m') \neq 0$ we move one chord end from the m'th strand to $(m'-1)$st strand by a strand exchange relation. (We can do it as $N(m' - 1) = 0$.) Otherwise, move one chord end from $(m - 1)$st strand to the mth strand.

The above manipulations can be repeated until we get an arranged chord diagram. It is a straightforward check that the algorithm always terminates on such a diagram. □

Proof of Lemma 11. A 4T relation can be thought of as a resolution of a "singular" chord diagram whose two chords have a common end. So in order to show that every 4T relation for diagrams of knots up to isotopy comes from a 4T relation for Morse diagrams one can just check that every singular chord diagram on n chords can be presented on $2n+1$ strands. This can be easily seen using Lemma 10. The statement about the framing independence relations is similarly straightforward. $\square$

Remark. The estimates given by Theorem 8 are not sharp. Note, however, that nowhere in the proof we have used the framing independence relations. Also, the statement of the Lemma 11 can be easily improved. Namely, the condition $k \leqslant n$ can be replaced by $k \leqslant 2n$; the proof is very similar to that of Lemma 9.

4. Computations in $\mathcal{M}_2$.

We performed some computer calculations of invariants of $\mathcal{M}_2$, with the hope of establishing the noncommutativity of some Morse knots. Although we failed in this objective, we did discover that there is a $\mathbf{Z}/2$-valued sixth-order invariant of $\mathcal{M}_2$ which is not the mod 2 reduction of an integer-valued invariant. Moreover, this invariant can distinguish some elements of $\mathcal{M}_2$ from their reverses. We describe now how those calculations were carried out.

The basic idea behind our calculations is that Vassiliev invariants of Morse knots pull back under the short-circuit map to those Vassiliev invariants of pure braids which are preserved by the action of the subgroups H_n^T and H_n^B, see Theorem 2. Thus the problem of computing invariants in $\mathcal{M}_n$ is reduced to considering the action of H_n^T and H_n^B on pure braid invariants. We refer the reader to [8] for a description of the Vassiliev theory for pure braids.

Rather than considering the Vassiliev filtration on knot invariants we will work with the dual filtration on $\mathbf{Z}$-linear combinations of knots. Recall that a knot with k double points can be identified with a formal alternating sum of 2^k nonsingular knots obtained by resolving all k double points with the help of the Vassiliev skein relation. Consider the abelian group Q generated by all elements of $\mathcal{M}_2$, subject to the relations that set all knots with 7 double points equal to zero. A sixth-order Vassiliev invariant of Morse knots in $\mathcal{M}_2$ taking values in an abelian group G is a homomorphism from Q to G. We computed Q to be $\mathbf{Z}^{19} \times X$, where X is a finite group whose order is divisible by 2 but not by $3, 5, 7, 11, 13, 17, 19, 89$, or 131. The group X is most likely to be $\mathbf{Z}/2$, but since we did all our calculations modulo the primes listed, we do not know this for sure. In any event, there is a homomorphism $Q \to \mathbf{Z}/2$ which does not factor through a homomorphism $Q \to \mathbf{Z}$.

The elements of $\mathcal{M}_2$ pull back via the short-circuit map to equivalence classes in P_5, the equivalence being given by left multiplication by H_2^T and right multiplication by H_2^B. Denote by $p_{i,j}$ the standard generators of P_5. The generators of H_2^T and H_2^B may be chosen as below (see [7]):

H_2^T	$p_{2,3}$	$p_{4,5}$	$p_{1,2}p_{1,3}$	$p_{1,4}p_{1,5}$	$p_{2,4}p_{2,5}$	$p_{3,4}p_{3,5}$	$p_{2,4}p_{3,4}$	$p_{2,5}p_{3,5}$
H_2^B	$p_{1,2}$	$p_{3,4}$	$p_{1,3}p_{2,3}$	$p_{1,4}p_{2,4}$	$p_{1,5}p_{2,5}$	$p_{1,3}p_{1,4}$	$p_{2,3}p_{2,4}$	$p_{3,5}p_{4,5}$

The linear extension of the short-circuit map sends $\mathbf{Z}P_5$ to $\mathbf{Z}$-linear combinations of elements of $\mathcal{M}_2$ and knots with 7 singularities are the image of $I^7 \subset \mathbf{Z}P_5$ under this extension. (Here I denotes the augmentation ideal of $\mathbf{Z}P_5$.) We thus take Q to be the quotient of $\mathbf{Z}P_5/I^7$ by the action of H_2^T on the left and H_2^B on the right.

Let $q_{i,j} = p_{i,j} - 1$. We call a product of m elements from the set $\{q_{1,2}, q_{1,3}, \dots q_{4,5}\}$ *a monomial of degree* m. Let S be the set of monomials of degree less than 7. The set S generates $\mathbf{Z}P_5/I^7$ (as a $\mathbf{Z}$-module), subject to relations inherited from those among the $p_{i,j}$ in P_5. The simplest type of relation among the $p_{i,j}$ is a *commutation* *relation.* For example, $p_{1,2}p_{3,4} = p_{3,4}p_{1,2}$. This translates to $q_{1,2}q_{3,4} = q_{3,4}q_{1,2}$. This relation must then be multiplied on the left and on the right by all pairs of monomials, the sum of whose degrees is less than 5, to obtain part of the total list of relations in a presentation of $\mathbf{Z}P_5/I^7$. There are 10 commutation relations in P_5, and each generates a similar sublist of the relations of $\mathbf{Z}P_5/I^7$.

Another type of pure braid relation is a *three-strand relation*, for example

$$p_{1,2}p_{1,3}p_{2,3} = p_{1,3}p_{2,3}p_{1,2}.$$

There are 20 such relations in P_5. This particular one translates to

$$(q_{1,2}+1)(q_{1,3}+1)(q_{2,3}+1) = (q_{1,3}+1)(q_{2,3}+1)(q_{1,2}+1),$$

or

$$q_{1,2}q_{1,3}q_{2,3} + q_{1,2}q_{1,3} + q_{1,2}q_{2,3} = q_{1,3}q_{2,3}q_{1,2} + q_{1,3}q_{1,2} + q_{2,3}q_{1,2}.$$

The lowest-order terms of such relations are the familiar 4T braid relations.

The last type of pure braid relation is a *four-strand relation.* There are five of these in P_5, for example $p_{1,3}p_{3,4}^{-1}p_{2,4}p_{3,4} = p_{3,4}^{-1}p_{2,4}p_{3,4}p_{1,3}$. Here things are complicated by the presence of inverses of the $p_{i,j}$. We write $p_{3,4}^{-1} = \sum_{i=0}^{6}(-1)^i q_{3,4}^i$ (since we are working modulo I^7). Our four-strand relation then expands to

$$\begin{gathered}
0 = q_{1,3}q_{2,4} - q_{2,4}q_{1,3} \\
+q_{1,3}q_{2,4}q_{3,4} - q_{1,3}q_{3,4}q_{2,4} - q_{2,4}q_{3,4}q_{1,3} + q_{3,4}q_{2,4}q_{1,3} \\
-q_{1,3}q_{3,4}q_{2,4}q_{3,4} + q_{1,3}q_{3,4}q_{3,4}q_{2,4} + q_{3,4}q_{2,4}q_{3,4}q_{1,3} - q_{3,4}q_{3,4}q_{2,4}q_{1,3} \\
+q_{1,3}q_{3,4}q_{3,4}q_{2,4}q_{3,4} - q_{1,3}q_{3,4}q_{3,4}q_{3,4}q_{2,4} - q_{3,4}q_{3,4}q_{2,4}q_{3,4}q_{1,3} + q_{3,4}q_{3,4}q_{3,4}q_{2,4}q_{1,3} \\
-q_{1,3}q_{3,4}q_{3,4}q_{3,4}q_{2,4}q_{3,4} + q_{1,3}q_{3,4}q_{3,4}q_{3,4}q_{3,4}q_{2,4} \\
+q_{3,4}q_{3,4}q_{3,4}q_{2,4}q_{3,4}q_{1,3} - q_{3,4}q_{3,4}q_{3,4}q_{3,4}q_{2,4}q_{1,3}
\end{gathered}$$

The lowest-order terms of such relations are the same as commutation relations. As with the commutation relations, each three-strand and four-strand relation must be multiplied on the left and right by pairs of monomials, ignoring of course those terms of degree greater than 6.

Then we must consider the relations generated by the action of H_2^T and of H_2^B. Multiplying a braid on the left by $p_{2,3}$, for example, does not change the Morse knot represented. This translates to any element of S which begins with $q_{2,3}$ being equal to 0. Similarly, any element of S which begins with $q_{4,5}$ or ends with $q_{1,2}$ or $q_{3,4}$ is also 0. We may call such relations *topological framing independence relations.* Multplying on the left by $p_{1,2}p_{1,3}$ doesn't change the Morse knot represented, and this implies all relations in Q of the form $q_{1,2}x + q_{1,3}x + q_{1,2}q_{1,3}x = 0$, where x is a monomial of degree less than 5. There are five more such relations from H_2^T and six from H_2^B. We may call such relations *topological strand exchange relations.*

We have now described all the relations we need for a presentation of Q. However, the set S has $1,111,111$ elements, and rather than attempt to solve over this many unknowns, we consider the set $T \subset S$ of all monomials which are generated by the set $\{q_{1,5}, q_{2,5}, q_{3,5}, q_{4,5}\}$, and moreover which begin with either $q_{1,5}$ $q_{3,5}$ and end with either $q_{2,5}$ or $q_{4,5}$. There are $1365 = 1+0+4+16+64+256+1024$ (breaking it down by degree) elements of T. We describe briefly an algorithm for writing any element of S as a linear ($\mathbf{Z}$-linear, as always in this section) combination of elements of T.

Given a monomial x, we first replace it with a linear combination of monomials which do not contain $q_{1,2}$. If $q_{1,2}$ occurs at the end of x, then $x = 0$, and we are done. If $q_{1,2}$ occurs in the middle of x, then there exists a commutation relation or a three-strand relation which moves $q_{1,2}$ closer to the end of x at the expense of adding several more linear terms, each of which is either of higher degree than x or contains one fewer occurences of $q_{1,2}$. For example, $q_{1,2}$ moves past $q_{3,4}$ with a commutation relation and past $q_{2,3}$ with a three-strand relation. We may then proceed inductively to eliminate all occurences of $q_{1,2}$.

The next step is to eliminate $q_{1,3}$ and $q_{2,3}$. Both of these may be moved toward the end of a monomial x by a process similar to that just described. Four-strand relations may now be necessary, for example to move $q_{1,3}$ past $q_{2,4}$. One has to be careful not to reintroduce $q_{1,2}$, or at least to raise the degree of the monomial if it is introduced, but this is always possible.

If $q_{1,3}$ occurs at the end of x, then a topological strand exchange will change it to $q_{1,4}$ (a similar exchange will turn $q_{2,3}$ to $q_{2,4}$), introducing an extra term whose degree is one greater than that of x. Similarly, $q_{i,4}$ may be moved toward the beginning of x, and then exchanged for $q_{i,5}$. Now all the $q_{i,j}$ occuring in the linear expansion of x so far have $j = 5$. Any monomial which begins with $q_{4,5}$ is 0, and any monomial which begins with $q_{2,5}$ may be exchanged for one which begins with $q_{3,5}$. Topological strand exchanges may also be employed to ensure that x ends in $q_{2,5}$ or $q_{4,5}$.

We took all the relations—commutation, three-strand, four-strand, and topological strand exchange and framing independence—and applied the above algorithm to write each one as a linear combination of elements of T. This was the most time-consuming part of the calculation, taking several days using compiled C code on a 300 MHz processor. It was not unusual for a single monomial in S to require several hundred million iterations of the algorithm to resolve it into a linear combination of monomials in T. What we did was overkill, of course, since many of the relations that we processed were accounted for by being used in the algorithm. In fact, most of the relations came out to be trivial in the end, though there still remained several hundred thousand nontrivial relations in the 1365 variables.

Exact rational solution of these equations did not seem feasible, so we solved these equations modulo various primes. For every prime we tried except 2, the dimension of the solution space was 19. For 2, the dimension was 20. Thus we obtain $Q = \mathbf{Z}^{19} \times X$, as noted above. The reason we know that $\mathbf{Z}^{19}$ is really $\mathbf{Z}^{19}$, and not 19 factors of the form $\mathbf{Z}/P$ for various large numbers P divisible by many small primes, is that $19 = 1 + 0 + 1 + 1 + 3 + 4 + 9$, and these numbers are the dimensions of the spaces of rational Vassiliev invariants (of isotopy). Moreover, we verified in our calculations that the map $\mathcal{A}_{6,2} \to \mathcal{A}_6$ described in Section 3 is an epimorphism.

To investigate what happens when a knot is reversed, we generated all the relations of the form "x equals its reverse", where $x \in S$. (The reverse of Morse knot is obtained by rotating the braid presenting it 180° in the plane, and this rotation induces an obvious involution on the set S.) We processed these relations with the same algorithm to obtain relations among the elements of T. When we added these relations to all the previous ones, the dimension of the solution space was 19 for all primes, including 2, thus indicating that the mod 2 invariant which we had found sometimes detects the difference between a knot and its reverse.

We note that there are various modifications of the algorithm described. It is possible, for example, to avoid using the framing independence relation at all. One must add monomials that begin with $q_{4,5}$ to the set T, and instead of pushing $q_{1,2}$

to the end of a monomial x, as described above, one pushes it to the beginning and exchanges it for $-q_{1,3}-q_{1,2}q_{1,3}$. We attempted to run this version of the algorithm, but (in degree six) it was taking many times as long as the version described above, and so we gave up.

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Instituto de Matemáticas (Unidad Cuernavaca), Universidad Nacional Autónoma de México, A.P. 273-3 Cuernavaca, Morelos, MEXICO
E-mail address: `jacob@matcuer.unam.mx`

Mathematics Department, United States Naval Academy, 572C Holloway Road, Annapolis, MD 21402, USA
E-mail address: `stanford@nadn.navy.mil`