Boundary Value Problems in Boutet de Monvel's Algebra for Manifolds with Conical Singularities II

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Abstract

We complete the work of Part I and present a pseudodifferential calculus for boundary value problems on a manifold D with finitely many conical singularities.

Outside the singular set, D is a smooth bounded manifold, and we use Boutet de Monvel's calculus in its standard form. Near a singularity, D is diffeomorphic to the cone $X \times [0, \infty)/X \times \{0\}$, where X is a smooth compact manifold with boundary. We then work on the cylinder $X \times \mathbf{R}_+$ with operators of Mellin type on $\mathbf{R}_+$ taking values in Boutet de Monvel's calculus on X.

First we construct the so-called *cone algebra without asymptotics*. It provides a framework in which all the relevant operations can be performed, although it is too coarse a tool to achieve a Fredholm theory. To this end we then develop the *cone algebra with asymptotics*, a calculus for boundary value problems based on meromorphic Mellin symbols. The associated operators act between Mellin Sobolev spaces with and without asymptotics.

A basic result is the construction of parametrices to elliptic operators in the algebra. The ellipticity is defined in terms of the symbols involved. It entails the Fredholm property of the associated operators and allows conclusions on regularity and asymptotics of solutions to elliptic equations.

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Key Words: Manifolds with conical singularities, boundary value problems, Boutet de Monvel's calculus, Mellin calculus

Contents

Introduction

In this paper we construct an algebra of pseudodifferential boundary value problems of Boutet de Monvel type on a manifold with conical singularities. We began this project with Part I, [40]. In order to keep the present article self-contained, we start in Section 1 with a review of the fundamental material, in particular, the concepts of the Mellin Sobolev spaces with and without asymptotics, meromorphic Mellin symbols, and the ideals of Green and smoothing Mellin operators.
We then construct two operator algebras with symbolic structure, namely the *cone algebra without asymptotics* in Section 2, and the *cone algebra with asymptotics* in Section 3.
The basic concept is the same in both case. A manifold with conical points and boundary, D, is a topological Hausdorff space, which, outside the finite set of the so-called singularities, is a smooth manifold with boundary. Close to a singularity, D is diffeomorphic to the cone $X \times [0, \infty)/X \times \{0\}$. The cross-section, X, is a compact manifold with boundary. We blow up the singularity and work on the cylinder $X \times \mathbf{R}_+$. Here we use Mellin operators with respect to $\mathbf{R}_+$ taking values in Boutet de Monvel's calculus on X.
The essential difference between the two calculi is that the cone algebra without asymptotics is based on smooth (parameter-dependent and operator-valued) Mellin symbols while in the cone algebra with asymptotics we employ meromorphic (operator-valued) Mellin symbols. In fact the cone algebra without asymptotics may be viewed as an analog of the pseudodifferential calculus with totally characteristic or Fuchs type symbols. The precise relation is given by the so-called Mellin quantization. Details can be found in Section 2.4. The operators in this calculus act on the Mellin Sobolev spaces $\mathcal{H}^{s,\gamma}$ introduced in Part I; the so-called weight γ here is fixed. The residual operators in this calculus are smoothing with respect to the regularity parameter s, but they do not improve the weight γ. In general, they are not compact. Therefore the cone algebra without asymptotics provides an excellent framework in which all relevant operations can be performed; it is, however, not precise enough to yield a Fredholm theory.
The cone algebra *with* asymptotics is a subalgebra of the cone algebra without asymptotics. The operators now have meromorphic Mellin symbols. In view of a decomposition theorem ([40, Theorem 4.1.8], here Theorem 1.7.6) the singularities can be confined to the regularizing part of the operator. The final algebra therefore consists of operators that are sums of

(i) a Mellin operator with a holomorphic (operator-valued) Mellin symbol, localized to a neighborhood of the singularities,
(ii) an operator in Boutet de Monvel's calculus for the smooth part of D, localized away from the singularities,
(iii) a Mellin operator with a regularizing meromorphic Mellin symbol, localized to a neighborhood of the singularities, and
(iv) a Green operator.

The operators in the cone algebra with asymptotics act on the Mellin Sobolev spaces with and without asymptotics. Here, the analyticity of the Mellin symbols plays a decisive role. The residual elements in this calculus are the Green operators. Apart from minor complications due to the existence of the 'type' in Boutet de Monvel's calculus, these operators are characterized by the fact that they, as well as their adjoints, map any weighted Mellin Sobolev space $\mathcal{H}^{s,\gamma}, s > -1/2$, to a Mellin Sobolev space $\mathcal{H}_P^{\infty,\delta}$ with infinite regularity and asymptotics described by P. These operators are compact, and this will enable us to obtain Fredholm results.

The construction of the cone algebra with asymptotics is carried out both for the case of classical symbols and for that of non-classical symbols. We define a notion of ellipticity. In general, it requires (i) the ellipticity of the interior symbol, (ii) Fuchs type ellipticity close to the singularities, and (iii) the invertibility of the principal conormal symbol. In the case of *classical* operators the situation is slightly simpler. We obtain three principal symbol levels: the interior principal pseudodifferential symbol, the principal boundary symbol, and the principal conormal symbol. The ellipticity condition asks that all three be invertible.

We eventually construct parametrices to elliptic elements: Given an elliptic operator A in the cone algebra with asymptotics, we find an operator B within the cone algebra with asymptotics, such that $AB - I$ and $BA - I$ both are Green operators. B therefore is a Fredholm inverse to A. Moreover, the fact that we know the structure of B rather precisely allows us to conclude for the regularity and the asymptotics of solutions u to an equation $Au = f$, given the regularity or the asymptotics of f.

It is the general intention of our approach to obtain 'pseudodifferential' algebras for singular spaces by an iterative process. Parallel to the geometric description of the singularities in terms of, say, repeatedly forming cones and wedges of increasing singularity orders, one would like to obtain higher pseudodifferential algebras by constructing cone or wedge algebras with correspondingly arranged symbolic structures, the (operator-valued) symbols taking values in the pseudodifferential algebras already treated. During the last few years, this has become realistic, and, by a sequence of papers and books of Schulze [41, 45, 44, 49], Dorschfeldt & Schulze [9], Egorov & Schulze [10], Schrohe [35, 38], more and more explicit. The symbols along the cone axis for example are modelled on a parameter-dependent version of the calculus for the base of the cone and adapted to the Mellin quantization, with a very precise control up to the conical singularities. The subsequent singularity, the wedge, locally is the Cartesian product of an infinite model cone and the edge. Following a recent point of view [49], the symbols therefore should be analogs of the boundary symbols in Boutet de Monvel's calculus, living on the cotangent bundle of the edge and acting along the model cone. Starting with a closed compact manifold as the cone base, the operator algebras on the corresponding manifolds with conical singularities regarded as a "surface" have been constructed in [44, 49]. In the present case the base of the cone is a manifold with boundary, hence the resulting singular manifold has a boundary. The context then is the analysis of boundary value problems, and for applications as well as for index problems Boutet de Monvel's algebra is the natural framework. One challenge in carrying out the above program is the complexity of the structures involved. The resulting theory on one hand contains Boutet de Monvel's calculus, namely by restricting to the cross-section; on the other hand, it includes the algebras for manifolds with conical singularities when the cross-section is a closed compact manifold, namely by restriction to the boundary. In order to facilitate the handling of the calculus we have introduced in Part I a new and very fast approach to Boutet de Monvel's calculus with parameters based on ideas from the edge pseudodifferential calculus. Parts of our paper also develop further the means on the cone algebra in the boundaryless case: For one thing, we treat the case of nonclassical symbols. This requires a more careful analysis of the smoothness of the operator-valued Mellin symbols up to the singularity, which in turn led us to a simplified parametrix construction. Also our approach to Mellin quantization is considerably easier than earlier presentations.

In Part I, we already commented on the relation of our calculus to the work of other authors. Let us recall that, for the early interest in phenomena near conical singularities, motivated by concrete models and applications in engineering, Kondrat'ev's paper [21] was a major breakthrough. It brought about a general understanding of the concept of ellipticity for differential boundary value problems, of the role of the Mellin transform, and of the weighted Mellin Sobolev spaces for the Fredholm property. The calculation of the asymptotics of solutions near conical points by means of the meromorphic inverses of the conormal symbols is often referred to today as Kondrat'ev's technique.
Another version of the analysis of operators near conical singularities in the boundaryless case has been developed by Plamenevskij [29], who emphasized specific Mellin transformation techniques with respect to the cone axis variable. This approach was extended by Derviz [6] to the case of boundary value problems. He constructed a Boutet de Monvel type calculus including a concept of ellipticity as well as parametrix constructions.
Our calculus, however, is more precise and gives more insight into the problem: By using Mellin symbols that are smooth up to zero we obtain an algebra with the ideal of Green operators as the residual elements. This in turn allows us to operate on Mellin Sobolev spaces with asymptotics and to obtain information on the asymptotics of solutions to elliptic equations.
The paper [26] by Melrose has established another approach to the analysis of pseudodifferential and Fourier integral operators with totally characteristic symbols. Applying these techniques, Melrose and Mendoza [28], in particular, established a Fredholm theory including ellipticity and parametrix construction for an algebra of pseudodifferential operators on manifolds with conical singularities, the cone bases being closed compact manifolds. In recent years, applications to Atiyah Patodi Singer type index theorems were given [27].
The results obtained here will play a role for the index theory on manifolds with conical singularities. Following the general concept, they will also serve as the necessary foundation for an iteration of the calculus, in particular, for manifolds with edges and boundaries, to be performed in a series of forthcoming papers.

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1 Review of Smoothing Mellin and Green Operators

In this section we will recall some of the concepts introduced in the first part of this paper [40]. When referring to a definition, lemma, or theorem in Part I, we shall write e.g. Definition I.3.1.3 to indicate Definition 3.1.3 of Part I.

1.1 Notation. Manifolds with Conical Singularities

An $n-$dimensional manifold with boundary is a topological (second countable) Hausdorff space M such that each point in M has a neighborhood which is diffeomorphic to either $\mathbf{R}^n$ or the closed half-space $\overline{\mathbf{R}}^n_+$. The former points are called the interior points of M, the latter the boundary points. We will use the standard notation $\operatorname{int} M$ and ∂M.

1.1.1 Definition. A *manifold with boundary and conical singularities D of dimension $n+1$* is a topological (second countable) Hausdorff space with a finite subset $\Sigma \subset D$ ('singularities') such that $D \backslash \Sigma$ is an $n+1$-dimensional manifold with boundary and, for every $v \in \Sigma$, there is an open neighborhood U of v, a compact manifold with boundary X of dimension n, and a system $\mathcal{F} \neq \emptyset$ of mappings with the following properties:

(1) For all $\phi \in \mathcal{F}$, $\phi : U \to X \times [0,1)/X \times \{0\}$ is a homeomorphism with $\phi(v) = X \times \{0\}/X \times \{0\}$.

(2) Given $\phi_1, \phi_2 \in \mathcal{F}$, the restriction $\phi_1 \phi_2^{-1} : X \times (0,1) \to X \times (0,1)$ extends to a *diffeomorphism* $X \times (-1,1) \to X \times (-1,1)$.

(3) The charts $\phi \in \mathcal{F}$ are compatible with the charts for the manifold for $D \backslash \Sigma$, i.e., the restriction $\phi : U \backslash \{v\} \to X \times (0,1)$ is a diffeomorphism.

We can and will assume that, for each singularity $v \in \Sigma$, the system $\mathcal{F}$ is maximal with respect to the properties (1), (2), and (3).

1.1.2 Definition and Remark. Let D be a manifold with boundary and conical singularities. By assumption, $D \backslash \Sigma$ is a manifold with boundary. Properties 1.1.1(1) and (2) imply that any neighborhood of a point $v \in \Sigma$ contains points of the topological boundary of $D \backslash \Sigma$, namely of $\partial X \times (0,1)$.
We may therefore define the interior and the boundary of D just as usual: $x \in D$ is an *interior point of D* if there is an open neighborhood of x which is homeomorphic to an open ball in $\mathbf{R}^{n+1}$, and $\operatorname{int} D$ is the collection of all interior points; $\partial D = D \backslash \operatorname{int} D$ is the boundary of D. We always have $\Sigma \subset \partial D$.

1.1.3 Definition. Let D be a manifold with boundary and conical singularities. Then the topological boundary ∂D of D is a (boundaryless) manifold with conical singularities in the sense of [47, 1.1.2, Definition 10].
By $I\!D$ (the "stretched object associated with D") denote the topological space constructed by replacing, for every singularity v, the neighborhood U in Definition 1.1.1 by $X \times [0,1)$ via glueing with any one of the diffeomorphisms ϕ. This procedure also defines a "stretched object" $I\!B$ associated with $B = \partial D$.

1.1.4 Notation and Assumptions. Throughout this article we will keep the following notation fixed.

- D is a manifold with conical singularities of dimension $n+1$ with singularity set Σ; $I\!D$ is the associated $(n+1)$–dimensional stretched object defined in 1.1.3.

- $B = \partial D$ is the boundary of D, cf. 1.1.2, it is of dimension n and a manifold with conical singularities (without boundary); $I\!B$ is the corresponding stretched boundary object defined in 1.1.3.

In a neighborhood of one of the singularities,

- X will denote the cross-section as in 1.1.1; by definition, X is a manifold with boundary of dimension n, in particular, X contains its boundary. For practical purposes, this is often inconvenient. We shall therefore agree to denote by X the open interior, and by $\overline{X}$ the manifold including the boundary. $X^\wedge = X \times \mathbf{R}_+$; $\overline{X}^\wedge = \overline{X} \times \mathbf{R}_+$.

- $Y = \partial X$ is the topological boundary of X; Y is a closed manifold of dimension $n-1$; $Y^\wedge = Y \times \mathbf{R}_+$.

We will assume that X is endowed with a Riemannian metric and embedded in a closed Riemannian manifold Ω and that $I\!D$ has a Riemannian metric which coincides with the canonical (cylindrical) metric on $X \times [0,1)$ near each singularity.

1.2 Symbols and Sobolev Spaces

As before, X will be the interior of a compact $n-$dimensional manifold with smooth boundary Y. We assume X to be embedded in a compact manifold Ω without boundary, e.g. the 'double' of X. In a collar neighborhood of the boundary we introduce normal coordinates. A point there can be written $x = (y,r)$ with $y \in Y, r \geq 0$. If U is an open subset of $\mathbf{R}^{n-1}$, then coordinates in $U \times \mathbf{R}$ will also be written in the form $x = (x', r)$ or likewise $x = (x', x_n)$, with $x' \in U$ and $r, x_n \in \mathbf{R}$.

1.2.1 Sobolev Spaces on $\mathbf{R}^n$ and $\mathbf{R}^n_+$. Let U be an open subset of $\mathbf{R}^{n-1}$. For a function or distribution u on $U \times \mathbf{R}$ let r^+u denote its restriction to $U \times \mathbf{R}_+$. We shall also use the operator r^+ to indicate the restriction of functions or distributions on Ω to X.
$H^s(\mathbf{R}^n), s \in \mathbf{R}$, is the usual Sobolev space over $\mathbf{R}^n$. We let $H^s(\mathbf{R}^n_+) = \mathrm{r}^+H^s(\mathbf{R}^n)$ and $H^s_0(\mathbf{R}^n_+) = \{u \in H^s(\mathbf{R}^n) : \text{supp } u \subseteq \overline{\mathbf{R}^n_+}\}$. Equivalently, $H^s_0(\mathbf{R}^n_+)$ is the closure of $C^\infty_0(\mathbf{R}^n_+)$ in the topology of $H^s(\mathbf{R}^n)$.
For functions or distributions in $H^s(\mathbf{R}^n_+), s > -1/2$, we let e^+ denote the operator of extension (by zero) to $\mathbf{R}^n$. For $-1/2 < s < 1/2$ this yields a bounded map

$$\mathrm{e}^+ : H^s(\mathbf{R}^n_+) \to H^s(\mathbf{R}^n).$$

The notation extends to the case of compact manifolds via a partition of unity. This defines the spaces $H^s(\Omega), H^s(X)$, and $H^s_0(X)$. We shall employ the notation $H^s(\Omega \times \mathbf{R}), H^s(X \times \mathbf{R}), H^s_0(X \times \mathbf{R})$, etc., understanding that we use $L^2(X \times \mathbf{R})$ with the measure $dxdt$.
$\mathcal{S}(\mathbf{R}^n)$ denotes the space of all rapidly decreasing functions on $\mathbf{R}^n$, and $\mathcal{S}(\mathbf{R}^n_+)$ is the space of all restrictions of functions in $\mathcal{S}(\mathbf{R}^n)$ to $\mathbf{R}^n_+$. We have the following relations

$$\begin{aligned}\mathcal{S}(\mathbf{R}_+) &= \text{proj} - \lim_{\sigma,\tau \in \mathbf{N}} H^{\sigma,\tau}(\mathbf{R}_+),\\ \mathcal{S}'(\mathbf{R}_+) &= \text{ind} - \lim_{\sigma,\tau \in \mathbf{N}} H_0^{-\sigma,-\tau}(\mathbf{R}_+),\end{aligned}$$

where $H^{\sigma,\tau}(\mathbf{R}_+), H^{\sigma,\tau}_0(\mathbf{R}_+)$ are the weighted Sobolev spaces defined by

$$\begin{aligned}H^{\sigma,\tau}_0(\mathbf{R}_+) &= \{\langle r\rangle^{-\tau} u : u \in H^\sigma_0(\mathbf{R}_{+,r})\},\\ H^{\sigma,\tau}(\mathbf{R}_+) &= \{\langle r\rangle^{-\tau} u : u \in H^\sigma(\mathbf{R}_{+,r})\}.\end{aligned}$$

1.2.2 Group Actions and Operator-Valued Symbols. Let E, F be Banach spaces with strongly continuous group actions $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ and $\{\tilde\kappa_\lambda : \lambda \in \mathbf{R}_+\}$ respectively. By definition this means that

(i) $\lambda \mapsto \kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(E))$, $\lambda \mapsto \tilde\kappa_\lambda \in C(\mathbf{R}_+, \mathcal{L}_\sigma(F))$ (strong continuity of κ and $\tilde\kappa$), and

(ii) $\kappa_\lambda\kappa_\mu = \kappa_{\lambda\mu}, \tilde\kappa_\lambda\tilde\kappa_\mu = \tilde\kappa_{\lambda\mu}$.

Here $\mathcal{L}_\sigma(\cdot)$ refers to the space $\mathcal{L}(\cdot)$ endowed with the strong topology.

Let $U \subseteq \mathbf{R}^k$ and $p \in C^\infty(U \times \mathbf{R}^n, \mathcal{L}(E,F)), \mu \in \mathbf{R}$. We shall write $p \in S^\mu(U, \mathbf{R}^n; E, F)$ provided that, for every $K \subset\subset U$ and all multi-indices α, β, there is a constant $C = C(K, \alpha, \beta)$ with

$$\|\tilde{\kappa}_{\langle\eta\rangle^{-1}}\{D_\eta^\alpha D_y^\beta p(y,\eta)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \leq C \langle\eta\rangle^{\mu-|\alpha|}, \quad y \in K, \eta \in \mathbf{R}^n, \tag{1}$$

cf. [44, 3.2.1, Definition 1]. The space $S^\mu(U, \mathbf{R}^n; E, F)$ is a Fréchet space topologized by the choice of the best constants C.
A symbol $p \in S^\mu(U, \mathbf{R}^n; E, F)$ is said to be *classical*, if it has an asymptotic expansion $p \sim \sum_{j=0}^\infty p_j$ with $p_j \in S^{\mu-j}(U, \mathbf{R}^n; E, F)$ satisfying the homogeneity relation

$$p_j(y, \lambda\eta) = \lambda^{\mu-j}\tilde{\kappa}_\lambda p_j(y,\eta)\kappa_{\lambda^{-1}}$$

for all $\lambda \geq 1, |\eta| \geq 1$.
For the usual or weighted Sobolev spaces on $\mathbf{R}_+$, we will always employ the group action

$$[\kappa_\lambda f](r) = \lambda^{1/2} f(\lambda r). \tag{2}$$

On $E = \mathbf{C}$ we use the trivial group action $\kappa_\lambda \equiv id$. For $E = F = \mathbf{C}$ we shall write $S^\mu(U, \mathbf{R}^n)$ instead of $S^\mu(U, \mathbf{R}^n; \mathbf{C}, \mathbf{C})$. The above definition then coincides with the standard symbol class notation.
If $F_1 \hookleftarrow F_2 \hookleftarrow \ldots$ is a sequence of Banach spaces with the same group action, and F is the Fréchet space given as the projective limit of the F_k, then let

$$S^\mu(U, \mathbf{R}^n; E, F) = \text{proj} - \lim_k S^\mu(U, \mathbf{R}^n; E, F_k). \tag{3}$$

Vice versa, if E is the inductive limit of the Banach spaces $E_1 \hookrightarrow E_2 \hookrightarrow \ldots$ with the same group action, then

$$S^\mu(U, \mathbf{R}^n; E, F) = \text{ind} - \lim_k S^\mu(U, \mathbf{R}^n; E_k, F). \tag{4}$$

Finally, a symbol p belongs to $S^\mu(U, \mathbf{R}^n; E, F)$, $E = \text{ind} - \lim E_k$, $F = \text{proj} - \lim F_l$, if the group actions coincide on the E_k and F_l, respectively, and $p \in S^\mu(U, \mathbf{R}^n; E_k, F_l)$ for all k and l. We give it the topology induced by all the topologies of the spaces $S^\mu(U, \mathbf{R}^n; E_k, F_l)$.
We will, in particular, deal with the spaces $S^\mu(U, \mathbf{R}^n; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$. For the inductive and projective limit constructions we shall then use the representation of $\mathcal{S}'(\mathbf{R}_+)$ and $\mathcal{S}(\mathbf{R}_+)$, respectively, as limits of weighted Sobolev spaces over $\mathbf{R}_+$, cf. 1.2.1.
In view of the nuclearity of $C^\infty(U)$ we have

$$S^\mu(U, \mathbf{R}^n; E, F) = C^\infty(U)\hat{\otimes}_\pi S^\mu(\mathbf{R}^0, \mathbf{R}^n; E, F), \tag{5}$$

the functions in the last space on the right hand side being independent of y.

1.2.3 Definition. Let $U = U_1 \times U_2 \subseteq \mathbf{R}^n \times \mathbf{R}^n$ be open and $p \in S^\mu(U, \mathbf{R}^n; E, F)$ an operator-valued symbol. Then the pseudodifferential operator $\operatorname{op} p$ is defined by

$$[\operatorname{op} p(f)](y) = (2\pi)^{-n} \int\int_{U_2} e^{i(y-y')\eta} p(y, y', \eta) f(y') dy' d\eta \tag{1}$$

for $f \in C_0^\infty(U_2, E), y \in U_1$. This reduces to

$$[\operatorname{op} p(f)](y) = (2\pi)^{-n/2} \int e^{iy\eta} p(y,\eta) \hat{f}(\eta) d\eta, \tag{2}$$

for 'simple' symbols, i.e. those that are independent of y'. Here, $\hat{f}(\eta) = (2\pi)^{-\frac{n}{2}} \int e^{-iy\eta} f(y)dy$ is the vector-valued Fourier transform of f.
We may also consider the case, where a part of the covariables serves as parameters: $p \in S^\mu(U, \mathbf{R}^n_\eta \times \mathbf{R}^l_\lambda; E, F)$ defines a parameter-dependent operator $\operatorname{op} p(\lambda)$ by

$$[\operatorname{op} p(\lambda) f](y) = (2\pi)^{-n} \int \int_{U_2} e^{i(y-y')\eta} p(y, y', \eta, \lambda) f(y') dy' d\eta \tag{3}$$

for $f \in C_0^\infty(U_2, E)$.

1.2.4 The Manifold Case. Let Ω be a smooth manifold, and E, F Banach spaces with strongly continuous group actions. Moreover, let $P : C_0^\infty(\Omega, E) \to C^\infty(\Omega, F)$ be a continuous operator. We shall say that $P \in \operatorname{op}S^\mu(\Omega, \mathbf{R}^n; E, F)$ if the following holds:

(i) For all C_0^∞ functions ϕ, ψ, supported in the same coordinate neighborhood, the operator $(\phi P \psi)_* : C_0^\infty(U, E) \to C^\infty(U, F)$ induced on $U \subseteq \mathbf{R}^n$ by $\phi P \psi$ and the coordinate maps has the form $(\phi P \psi)_* = \operatorname{op} p$ for some $p \in S^\mu(U, \mathbf{R}^n; E, F)$.

(ii) For all C_0^∞ functions ϕ, ψ, with disjoint supports, the operator $\phi P \psi$ is given as an integral operator with a kernel in $C^\infty(\Omega \times \Omega, \mathcal{L}(E, F))$ (more precisely a kernel section, see [7, Section 23.4]).

If P depends on a parameter $\lambda \in \mathbf{R}^l$, then (i) carries over, while in (ii) we ask that the integral kernel belongs to $\mathcal{S}(\mathbf{R}^l, C^\infty(\Omega \times \Omega, \mathcal{L}(E, F)))$.
Suppose we are given a locally finite covering of the manifold by relatively compact coordinate neighborhoods $\{\Omega_j\}$ with associated coordinate maps $\chi_j : \Omega_j \to U_j$. Then we can find $p_j \in S^\mu(U_j, \mathbf{R}^n; E, F)$ and an integral operator K_j with C^∞ kernel such that $P(f \circ \chi_j)(\chi_j^{-1}(x)) = \operatorname{op} p_j(f)(x) + K_j f(x)$ for all $f \in C_0^\infty(U_j, E)$. We shall call the tuple $\{p_j\}$ the symbol of P.
Let now $\Omega_j \cap \Omega_k \neq \emptyset$, and suppose that both ϕ and ψ are supported in the intersection. Denote by P_j and P_k the operators on $C_0^\infty(\Omega, E)$ induced by $(\phi \circ \chi_j^{-1}) \operatorname{op} p_j\, (\psi \circ \chi_j^{-1})$ and $(\phi \circ \chi_k^{-1}) \operatorname{op} p_k\, (\psi \circ \chi_k^{-1})$. Then $P_j - P_k$ is an integral operator with a kernel in $C^\infty(\Omega \times \Omega, \mathcal{L}(E, F))$. Vice versa, given a tuple $\{p_j\}$ with this property, we can define an operator $P : C_0^\infty(\Omega, E) \to C_0^\infty(\Omega, F)$ whose symbol is $\{p_j\}$. Hence the notion $S^\mu(\Omega, \mathbf{R}^n; E, F)$ makes sense.

1.3 Boutet de Monvel's Calculus

1.3.1 Definition. Let $\mu \in \mathbf{R}, d \in \mathbf{N}$ and $U \subseteq \mathbf{R}^{n-1}$ open. In the following definition the parameter-dependence will always refer to the parameter $\lambda \in \mathbf{R}^l$.
(a) A regularizing parameter-dependent singular Green operator (s.G.o.) of type 0 on $U \times \mathbf{R}_+$ is a family of integral operators

$$G_0(\lambda) : C_0^\infty(U, \mathcal{S}(\mathbf{R}_+)) \to C^\infty(U, C^\infty(\overline{\mathbf{R}}_+))$$

given by a kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times \overline{\mathbf{R}}_+ \times U \times \overline{\mathbf{R}}_+))$. Here we identify $C_0^\infty(U, \mathcal{S}(\mathbf{R}_+))$ and $C^\infty(U, C^\infty(\overline{\mathbf{R}}_+))$ with subsets of $C^\infty(U \times \overline{\mathbf{R}}_+)$. A regularizing s.G.o. G_0 of type d is a parameter-dependent operator of the form $G_0(\lambda) = \sum_{j=0}^d G_{0j}(\lambda)\partial_r^j$ with regularizing parameter-dependent s.G.o's G_{0j} of type zero and the derivative ∂_r on $\mathbf{R}_+$.
A parameter-dependent s.G.o. of order μ and type d on U is an operator

$$G(\lambda) : C_0^\infty(U, \mathcal{S}(\mathbf{R}_+)) \to C^\infty(U, \mathcal{S}(\mathbf{R}_+))$$

that can be written $G = \sum_{j=0}^d [\operatorname{op} g_j]\partial_r^j + G_0$, where each g_j is a (parameter-dependent and operator-valued) symbol g_j in $S^{\mu-j}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$ and G_0 is a regularizing parameter-dependent s.G.o. of type d.
(b) A regularizing parameter-dependent trace operator of type 0 on $U \times \mathbf{R}_+$ is an operator

$$T_0(\lambda) : C_0^\infty(U, \mathcal{S}(\mathbf{R}_+)) \to C^\infty(U)$$

with an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times U \times \overline{\mathbf{R}}_+))$. A regularizing trace operator T_0 of type d is a sum $T_0(\lambda) = \sum_{j=0}^d T_{0j}\partial_r^j$; each T_{0j} being regularizing of type 0.
A parameter-dependent trace operator T of order μ and type d on U is an operator that can be written $T = \sum_{j=0}^d [\operatorname{op} t_j]\partial_r^j + T_0$, with t_j in $S^{\mu-j}(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathcal{S}'(\mathbf{R}_+), \mathbf{C})$ and a regularizing parameter-dependent trace operator T_0 of type d.
(c) A regularizing parameter-dependent potential operator on U is an operator

$$K_0(\lambda) : C_0^\infty(U) \to C^\infty(U, C^\infty(\overline{\mathbf{R}}_+))$$

given by an integral kernel in $\mathcal{S}(\mathbf{R}^l, C^\infty(U \times \overline{\mathbf{R}}_+ \times U))$; a parameter-dependent potential operator K of order μ is a sum $K = \operatorname{op} k + K_0$ with a symbol k in $S^\mu(U, \mathbf{R}^{n-1} \times \mathbf{R}^l; \mathbf{C}, \mathcal{S}(\mathbf{R}_+))$ and a regularizing parameter-dependent potential operator K_0.
(d) All these spaces of operators carry Fréchet topologies in a natural way: We use the topology of non-direct sums of Fréchet spaces in connection with the natural topologies on the symbol spaces and on the spaces $\mathcal{S}(\mathbf{R}^l, \ldots)$ for the integral kernels.
(e) We call $g = \sum_{j=0}^d g_j\partial_r^j$, $t = \sum_{j=0}^d t_j\partial_r^j$, and k symbols for G, T, and K, respectively. We shall say that they are *classical*, if the g_j, t_j, and k are classical in the sense of 1.2.2.

1.3.2 Remark. Let E, F be Fréchet spaces and suppose both are continuously embedded in the same Hausdorff vector space. The exterior direct sum $E \oplus F$ is Fréchet and has the closed subspace $\Delta = \{(a, -a) : a \in E \cap F\}$. The non-direct sum of E and F then is the Fréchet space $E + F := E \oplus F/\Delta$.

1.3.3 Parameter-Dependent Operators and Symbols in Boutet de Monvel's Calculus. Let $U \subseteq \mathbf{R}^{n-1}$ be open. A *parameter-dependent operator of order* $\mu \in \mathbf{R}$ *and type* $d \in \mathbf{N}$ *in Boutet de Monvel's calculus* on $U \times \mathbf{R}_+$ is a family $\{A(\lambda) : \lambda \in \mathbf{R}^l\}$ of operators

$$A(\lambda) = \begin{bmatrix} P_+(\lambda) + G(\lambda) & K(\lambda) \\ T(\lambda) & S(\lambda) \end{bmatrix} : \begin{matrix} C_0^\infty(U \times \overline{\mathbf{R}}_+) \\ \oplus \\ C_0^\infty(U) \end{matrix} \to \begin{matrix} C^\infty(U \times \overline{\mathbf{R}}_+) \\ \oplus \\ C^\infty(U) \end{matrix}, \qquad (1)$$

where

$P(\cdot)$	$= \operatorname{op} p(\cdot)$ with $p \in S^{\mu}_{tr}(U \times \overline{\mathbf{R}}_+ \times U \times \overline{\mathbf{R}}_+, \mathbf{R}^n; \mathbf{R}^l)$, $P_+ = \mathrm{r}^+ P \mathrm{e}^+$,
$G(\cdot)$	is a parameter-dependent singular Green operator of order μ and type d,
$K(\cdot)$	is a parameter-dependent potential operator of order μ,
$T(\cdot)$	is a parameter-dependent trace operator of order μ and type d,
$S(\cdot)$	is a parameter-dependent pseudodifferential operator of order μ on U.

The subscript 'tr' indicates that the symbol p satisfies the transmission condition (see [30, Section 2.2.2.1] or 1.2.1.5) at the boundary $U \times \{0\}$. Note that the decomposition $P_+ + G$ is not unique; the regularizing pseudodifferential operators provide examples for operators that belong to both classes. We shall write $A \in \mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$. The topology on this space is that of a non-direct sum of Fréchet spaces induced by (1) and the topologies on the spaces of pseudodifferential, singular Green, trace, and potential operators.

A parameter-dependent *regularizing* operator A of type d in Boutet de Monvel's calculus on U is one that can be written in the form (1) with all entries being regularizing operators. We shall write $A \in \mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$, and give this space the obvious Fréchet topology. It is a consequence of 1.3.1 that the operators in (1) indeed have the desired mapping properties.

In general, all entries will be matrix-valued.

Given an operator $A \in \mathcal{B}^{\mu,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ there is the quintuple $a = \{p, g, k, t, s\}$ of the symbols for the operators P, G, K, T, and S, respectively, cf. 1.3.1(e). As pointed out before, there is a certain ambiguity in the choice of the symbols; we understand them as equivalence classes of tuples inducing the same operator modulo $\mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$. We usually refer to this symbol as the full symbol of A.

Moreover, we have a parameter-dependent operator-valued symbol $\sigma_\wedge(A)$, the so-called complete (parameter-dependent) boundary symbol of A, namely

$$\sigma_\wedge(A)(x', \xi', \lambda) = \begin{bmatrix} \operatorname{op}^+_{x_n} p(x, \xi, \lambda) + g(x', \xi', \lambda) & k(x', \xi', \lambda) \\ t(x', \xi', \lambda) & s(x', \xi', \lambda) \end{bmatrix} : \begin{matrix} C_0^\infty(\overline{\mathbf{R}}_+)^{n_1} \\ \oplus \\ \mathbf{C}^{m_1} \end{matrix} \to \begin{matrix} C^\infty(\overline{\mathbf{R}}_+)^{n_2} \\ \oplus \\ \mathbf{C}^{m_2} \end{matrix} \qquad (2)$$

with p, g, t, k, s as before, $x' \in U, \xi' \in \mathbf{R}^{n-1}, \lambda \in \mathbf{R}^l$. Again, we understand the symbol $\sigma_\wedge(A)$ as an equivalence class of tuples in the corresponding symbol classes with the property that

$$A - \operatorname{op}'\sigma_\wedge(A) \in \mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l).$$

Here, $a_1 \sim a_2$ iff $\operatorname{op}'a_1 - \operatorname{op}'a_2 \in \mathcal{B}^{-\infty,d}(U \times \mathbf{R}_+; \mathbf{R}^l)$ and op' denotes the pseudodifferential action with respect to the x'-variables.

1.3.4 Boutet de Monvel's Algebra on a Manifold. Symbol Levels. Let X be an n-dimensional C^∞ manifold with boundary Y, embedded in an n-dimensional manifold Ω without boundary, all not necessarily compact. Let V_1, V_2 be vector bundles over Ω and W_1, W_2 be vector bundles over Y.

Let $\{\Omega_j\}$ denote a locally finite open covering of Ω, and suppose that the coordinate charts map $X \cap \Omega_j$ to $U_j \times \mathbf{R}_+ \subset \mathbf{R}^n_+$ and $Y \cap \Omega_j$ to $U_j \times \{0\}$ for a suitable open set $U_j \subseteq \mathbf{R}^{n-1}$, unless $\Omega_j \cap Y = \emptyset$.

For a smooth function ϕ on Ω write M_ϕ for the multiplication operator with the diagonal matrix $\operatorname{diag}\{\phi, \phi|_Y\}$. We will say that $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$, if

$$A(\lambda) : \begin{matrix} C_0^\infty(\overline{X}, V_1) \\ \oplus \\ C_0^\infty(Y, W_1) \end{matrix} \to \begin{matrix} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{matrix}, \qquad (1)$$

is an operator with the following properties:

(i) For all C_0^∞ functions ϕ, ψ, supported in the same coordinate neighborhood Ω_j intersecting the boundary, the operator

$$(M_\phi A(\lambda) M_\psi)_* : \begin{array}{c} C_0^\infty(U_j \times \overline{\mathbf{R}}_+, V_1) \\ \oplus \\ C_0^\infty(U_j, W_1) \end{array} \to \begin{array}{c} C^\infty(U_j \times \overline{\mathbf{R}}_+, V_2) \\ \oplus \\ C^\infty(U_j, W_2) \end{array},$$

induced on $U_j \times \mathbf{R}_+$ by $M_\phi A(\lambda) M_\psi$ and the coordinate maps, is an operator in the class $\mathcal{B}^{\mu,d}(U_j \times \mathbf{R}_+; \mathbf{R}^l)$ of Boutet de Monvel's calculus on $\mathbf{R}^n_+$ in the sense of 1.3.3.

(ii) If ϕ, ψ are as before, but the coordinate chart does not intersect the boundary, then all entries in the matrix $(M_\phi A(\lambda) M_\psi)_*$ – except for the pseudodifferential part – are regularizing.

(ii) If the supports of the functions $\phi, \psi \in C_0^\infty(\Omega)$ are disjoint, then $(M_\phi A(\lambda) M_\psi)_*$ is an integral operator whose kernel density is C^∞ and a rapidly decreasing function of λ in all semi-norms defining the Fréchet topology of the smooth densities.

We topologize $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ as the corresponding non-direct sum of Fréchet spaces. $\mathcal{G}^{\mu,d}(X; \mathbf{R}^l)$ is the subspace of all elements in $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ where the pseudodifferential part can be taken to be zero. Note that $\mathcal{G}^{-\infty,d}(X; \mathbf{R}^l) = \mathcal{B}^{-\infty,d}(X; \mathbf{R}^l)$.
For each coordinate patch Ω_j intersecting the boundary, $A(\lambda)$ induces an operator

$$A_j(\lambda) = \begin{bmatrix} P_{j+}(\lambda) + G_j(\lambda) & K_j(\lambda) \\ T_j(\lambda) & S_j(\lambda) \end{bmatrix}$$

on $U_j \times \mathbf{R}_+$, cf. 1.3.3(1). We find a quintuple $a_j(\lambda) = \{p_j(\lambda), g_j(\lambda), k_j(\lambda), t_j(\lambda), s_j(\lambda)\}$ of symbols for $P_j(\lambda), G_j(\lambda), K_j(\lambda), T_j(\lambda), S_j(\lambda)$ in the sense of 1.3.3.
Given an interior chart Ω_j, $A(\lambda)$ induces a pseudodifferential operator $P_j(\lambda)$ with a symbol $p_j(\lambda)$ in the sense of equivalence classes modulo $S^{-\infty}$, cf. 1.2.4.
We call the system

$$\sigma_\psi(A) = \{p_j(\lambda)\}$$

a *complete (parameter-dependent) interior symbol for A*. For those j where Ω_j intersects the boundary, the system $\{\sigma_\wedge(A)_j\}$ given by

$$\sigma_\wedge(A)_j(x', \xi', \lambda) = \begin{pmatrix} \mathrm{op}^+_{x_n} p_j(x, \xi, \lambda) + g_j(x', \xi', \lambda) & k_j(x', \xi', \lambda) \\ t_j(x', \xi', \lambda) & s_j(x', \xi', \lambda) \end{pmatrix}$$

$$: \begin{array}{c} C_0^\infty(\overline{\mathbf{R}}_+)^{n_1} \\ \oplus \\ \mathbf{C}^{m_1} \end{array} \longrightarrow \begin{array}{c} C^\infty(\overline{\mathbf{R}}_+)^{n_2} \\ \oplus \\ \mathbf{C}^{m_2} \end{array}$$

is the *complete (parameter-dependent) boundary symbol for $A(\lambda)$*.

We shall call A classical, if all entries in the quintuples $a_j = \{p_j, g_j, k_j, t_j, s_j\}$ are classical elements in the respective symbol classes, i.e., p_j and s_j are classical pseudodifferential symbols, while g_j, k_j, t_j are classical operator-valued symbols, cf. 1.3.1(e). Write $A \in \mathcal{B}^{\mu,d}_{cl}(X; \mathbf{R}^l)$. The operator A then has:

- A principal pseudodifferential symbol, $\sigma^\mu_\psi(A) = \sigma^\mu_\psi(A)(x,\xi,\lambda)$, well-defined as a function on $(T^*X \times \mathbf{R}^l)\backslash 0$, where 0 denotes the zero-section in the sense that $(\xi,\lambda) = 0$, with values in vector bundle morphisms between V_1 and V_2; $\sigma^\mu_\psi(A)$ is the component of homogeneity μ with respect to (ξ,λ) of the complete interior symbol $\sigma_\psi(A)$.

- A principal boundary symbol, operator-valued, $\sigma^\mu_\wedge(A) = \sigma^\mu_\wedge(A)(x',\xi',\lambda)$, defined on $(T^*Y \times \mathbf{R}^l)\backslash 0$. The construction is as follows. In the complete boundary symbol $\sigma_\wedge(A)$ replace $\mathrm{op}^+_{x_n} p(x,\xi',D_n,\lambda)$ by $\mathrm{op}^+_{x_n} p(x',0,\xi',D_n,\lambda)$. We then obtain a classical operator-valued symbol, namely

$$\begin{pmatrix} \mathrm{op}^+_{x_n} p_j(x',0,\xi,\lambda) + g_j(x',\xi',\lambda) & k_j(x',\xi',\lambda) \\ t_j(x',\xi',\lambda) & s_j(x',\xi',\lambda) \end{pmatrix}.$$

  We define $\sigma^\mu_\wedge(A)$ as its component of homogeneity μ with respect to (ξ',λ). We then have a bundle morphism

$$\sigma^\mu_\wedge(A): \begin{matrix} \mathcal{S}(\mathbf{R}_+)\otimes\pi^*V_1 \\ \oplus \\ \pi^*W_1 \end{matrix} \to \begin{matrix} \mathcal{S}(\mathbf{R}_+)\otimes\pi^*V_2 \\ \oplus \\ \pi^*W_2 \end{matrix};$$

  here $\pi : (T^*Y\times\mathbf{R}^l)\backslash 0 \to Y$ is the canonical projection. This formulation differs from that in [30, Section 3.1.1.1] in that we have replaced H^+ by its Fourier preimage $\mathcal{S}(\mathbf{R}_+)$; it is, however, obviously equivalent.

1.3.5 Definition. We will say that $A \in \mathcal{B}^{\mu,d}(X;\mathbf{R}^l), d \le \mu_+ = \max\{\mu, 0\}$ is *parameter-elliptic* if there is an operator $B \in \mathcal{B}^{-\mu,d}(X;\mathbf{R}^l), d \le (-\mu)_+$, such that

- for each interior coordinate chart, the local components p_j, q_j of the complete symbols of A and B, respectively, satisfy the relations

$$p_jq_j - 1, q_jp_j - 1 \in S^{-1}(U_j \times U_j, \mathbf{R}^n;\mathbf{R}^l), \text{and} \tag{1}$$

- for each coordinate chart intersecting the boundary, the local complete boundary symbols a_j, b_j satisfy the following relations: For all functions $\varphi_j, \psi_j \in C_0^\infty(U_j)$ with $\varphi_j\psi_j = \varphi_j$ we have

$$\varphi_j a_j b_j \psi_j - \varphi_j I = c_{1j}, \tag{2}$$
$$\varphi_j b_j a_j \psi_j - \varphi_j I = c_{2j}, \tag{3}$$

  with suitable parameter-dependent symbols c_{1j}, c_{2j} of order -1 and types $d_1 = (-\mu)_+, d_2 = \mu_+$. Here I is the identity.

For classical operators, these two conditions are equivalent to the invertibility of both the principal pseudodifferential symbol and the principal boundary symbol.

1.3.6 Theorem. *Let $A \in \mathcal{B}^{\mu,d}(X;\mathbf{R}^l)$ be parameter-elliptic, $d \le \mu_+$. Then there is an operator $B \in \mathcal{B}^{-\mu,d'}(X;\mathbf{R}^l), d' = (-\mu)_+$ such that*

$$R_1 = AB - I \in \mathcal{B}^{-\infty,d_1}(X;\mathbf{R}^l) \text{ and } R_2 = BA - I \in \mathcal{B}^{-\infty,d_2}(X;\mathbf{R}^l),$$

where $d_1 = (-\mu)_+, d_2 = \mu_+$. *In particular, in the notation of* 1.3.4*:*

$$A(\lambda) : \begin{matrix} H^s(X, V_1) \\ \oplus \\ H^s(Y, W_1) \end{matrix} \to \begin{matrix} H^{s-\mu}(X, V_2) \\ \oplus \\ H^{s-\mu}(Y, W_2) \end{matrix}$$

is a Fredholm operator for $s, s-\mu > -1/2$.

1.3.7 Wedge Sobolev Spaces. Let E and κ_λ be as in 1.2.2, $q \in \mathbf{N}, s \in \mathbf{R}$. The *wedge Sobolev space* $\mathcal{W}^s(\mathbf{R}^q, E)$ is the completion of $\mathcal{S}(\mathbf{R}^q, E) = \mathcal{S}(\mathbf{R}^q)\hat{\otimes}_\pi E$ in the norm

$$\|u\|_{\mathcal{W}^s(\mathbf{R}^q,E)} = \left(\int \langle\eta\rangle^{2s} \|\kappa_{\langle\eta\rangle^{-1}} \mathcal{F}_{y\to\eta} u(\eta)\|_E^2 d\eta\right)^{1/2},$$

cf. [44, Section 3.1]. $\mathcal{W}^s(\mathbf{R}^q, E)$ is a subset of $\mathcal{S}'(\mathbf{R}^q, E) := \mathcal{L}(\mathcal{S}(\mathbf{R}^q), E)$. For $\kappa_\lambda \equiv id$ we obtain the usual Sobolev spaces of E-valued distributions.
Suppose $\{E_k\}$ is a sequence of Banach spaces, $E_{k+1} \hookrightarrow E_k$, $E = \text{proj} - \lim E_k$, and the group action coincides on all spaces. Then

$$\mathcal{W}^s(\mathbf{R}^q, E) = \text{proj} - \lim \mathcal{W}^s(\mathbf{R}^q, E_k).$$

Vice versa, if $E_k \hookrightarrow E_{k+1}$, $E = \text{ind} - \lim E_k$, and the group action is the same for all spaces, then

$$\mathcal{W}^s(\mathbf{R}^q, E) = \text{ind} - \lim \mathcal{W}^s(\mathbf{R}^q, E_k).$$

We shall write $u \in \mathcal{W}^s_{comp}(\mathbf{R}^q, E)$, if there is a function $\phi \in C_0^\infty(\mathbf{R}^q)$ such that $u = \phi u$. Similarly, for $u \in \mathcal{S}'(\mathbf{R}^q, E)$, write $u \in \mathcal{W}^s_{loc}(\mathbf{R}^q, E)$, if for arbitrary $\phi \in C_0^\infty(\mathbf{R}^q)$, $\phi u \in \mathcal{W}^s(\mathbf{R}^q, E)$.

1.4 Sobolev Spaces Based on the Mellin Transform

1.4.1 Mellin Transforms. For $\beta \in \mathbf{R}$, Γ_β denotes the vertical line $\{z \in \mathbf{C} : \text{Re}\, z = \beta\}$. We recall that the classical Mellin transform Mu of a complex-valued $C_0^\infty(\mathbf{R}_+)$-function u is given by

$$(Mu)(z) = \int_0^\infty t^{z-1} u(t)\, dt. \tag{1}$$

M extends to an isomorphism $M : L^2(\mathbf{R}_+) \to L^2(\Gamma_{1/2})$. Of course, (1) also makes sense for functions with values in a Fréchet space E. The fact that, for $u \in C_0^\infty(\mathbf{R}_+)$, one has $Mu|_{\Gamma_{1/2-\gamma}}(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma)$ motivates the following definition of the *weighted Mellin transform* M_γ:

$$M_\gamma u(z) = M_{t\to z}(t^{-\gamma}u)(z+\gamma), \quad u \in C_0^\infty(\mathbf{R}_+, E).$$

The inverse of M_γ is given by

$$[M_\gamma^{-1}h](z) = \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^{-z} h(z) dz.$$

1.4.2 Parameter-Dependent Order Reductions on Ω. Let Ω be a closed compact manifold. For $\mu \in \mathbf{R}$ there is a parameter-elliptic pseudodifferential operator $\Lambda^\mu \in \text{op}S^\mu(\Omega, \mathbf{R}^n; \mathbf{R})$, depending on the parameter $\tau \in \mathbf{R}$ such that

$$\Lambda^\mu(\tau) : H^s(\Omega, V) \to H^{s-\mu}(\Omega, V)$$

is an isomorphism for all τ. *Parameter-ellipticity* here simply means that there is a symbol $q \in S^{-\mu}(\Omega, \mathbf{R}^n; \mathbf{R})$ such that $\lambda^\mu q - 1$ and $q\lambda^\mu - 1$ both are elements of $S^{-1}(\Omega, \mathbf{R}^n; \mathbf{R})$. In order to construct such an operator one can e.g. start with symbols of the form $\langle \xi, (\tau, C) \rangle^\mu \in S^\mu(\mathbf{R}^n, \mathbf{R}^n_\xi; \mathbf{R}_\tau)$ with a large constant $C > 0$ and patch them together to an operator on the manifold Ω with the help of a partition of unity and cut-off functions. Alternatively, one can choose a Hermitean connection on V and consider the operator $(C + \tau^2 - \Delta)^{\frac{\mu}{2}}$, where Δ denotes the connection Laplacian and C is a large positive constant.

1.4.3 Totally Characteristic Sobolev Spaces. (a) Let $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ be a family of parameter-elliptic pseudodifferential operators as in 1.4.2. For $s, \gamma \in \mathbf{R}$, the space $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ is the closure of $C_0^\infty(\Omega^\wedge)$ in the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} = \left\{ \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \|\Lambda^s(\operatorname{Im} z) M u(z)\|^2_{L^2(\Omega)} \, |dz| \right\}^{1/2}. \tag{1}$$

Recall that n is the dimension of X and Ω and that $\Gamma_\beta = \{z \in \mathbf{C} : \operatorname{Re} z = \beta\}$.
(b) We let $\mathcal{H}^{s,\gamma}(X^\wedge) = \{f|_{X^\wedge} : f \in \mathcal{H}^{s,\gamma}(\Omega^\wedge)\}$. The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ carries the quotient norm:

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \inf\{\|f\|_{\mathcal{H}^{s,\gamma}(\Omega^\wedge)} : f \in \mathcal{H}^{s,\gamma}(\Omega^\wedge), f|_{X^\wedge} = u\}.$$

(c) $\mathcal{H}_0^{s,\gamma}(X^\wedge)$ is the space of all distributions in $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ with support in $\overline{X}^\wedge = \overline{X} \times \mathbf{R}_+$. Since, by definition, $C_0^\infty(\Omega^\wedge)$ is dense in $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$, the space $\mathcal{H}_0^{s,\gamma}(X^\wedge)$ is the closure of $C_0^\infty(X^\wedge)$ in the topology of $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$.
(d) For $s = l \in \mathbf{N}$ we obtain the alternative description

$$u \in \mathcal{H}^{l,\gamma}(\Omega^\wedge) \quad \text{iff} \quad t^{n/2-\gamma}(t\partial_t)^k D u(x,t) \in L^2(\Omega^\wedge)$$

for all $k \le l$ and all differential operators D of order $\le l - k$ on Ω, cf. [44, Section 2.1.1, Proposition 2].
(e) The space $\mathcal{H}^{s,\gamma}(X^\wedge)$ is independent of the particular choice of the order-reducing family.
(f) $\mathcal{H}^{s,\gamma}(X^\wedge) \subseteq H^s_{loc}(X^\wedge)$; $\mathcal{H}^{s,\gamma}(X^\wedge) = t^\gamma \mathcal{H}^{s,0}(X^\wedge)$; $\mathcal{H}^{0,0}(X^\wedge) = t^{-n/2} L^2(X^\wedge)$.
(g) $\mathcal{H}^{0,0}(X^\wedge)$ has a natural inner product

$$(u, v)_{\mathcal{H}^{0,0}(X^\wedge)} = \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}}} (Mu(z), Mv(z))_{L^2(X)} \, dz.$$

(h) If ϕ is the restriction to $X^\wedge$ of a function in $C_0^\infty(\Omega \times \mathbf{R})$, then the operator M_ϕ of multiplication by ϕ,

$$M_\phi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge),$$

is bounded for all $s, \gamma \in \mathbf{R}$, and the mapping $\phi \mapsto M_\phi$ is continuous in the corresponding topology.
(i) Suppose that $\{\Omega_j : j = 1, \ldots, J\}$ is an open covering of Ω, and $\{\phi_j\}$ is a subordinate partition of unity. Let $\{R^\mu : \mu \in \mathbf{R}\}$ be an order-reducing family on $\mathbf{R}^n$. We can define the space $\mathcal{H}^{s,\gamma}(\mathbf{R}^n \times \mathbf{R}_+)$ as before and denote by $\|\cdot\|_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n \times \mathbf{R}_+)}$ the corresponding norm. Then

$$|||u|||_{s,\gamma} = \left(\sum_{j=1}^{J} \|(\phi_j u)_*\|^2_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n \times \mathbf{R}_+)} \right)^{1/2} \tag{2}$$

furnishes an equivalent norm on $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$. Here, $(\phi_j u)_*$ is the distribution induced on $\mathbf{R}^n \times \mathbf{R}_+$ via the coordinate functions.
(j) Near each singularity v, $I\!D$ is diffeomorphic to $X_v^\wedge$, with suitable X_v as in 1.1.1. We define $\mathcal{H}^{s,\gamma}(I\!D)$ as the space of all distributions belonging to $\mathcal{H}^{s,\gamma}(X_v^\wedge)$ near a singularity v and belonging to $H^s(I\!D)$ in the interior; for the precise construction use a cut-off function ω_v near each singularity v.
Notice that (e) is a simple consequence of the fact that if $\{\Lambda^\mu : \mu \in \mathbf{R}\}$ and $\{\tilde{\Lambda}^\mu : \mu \in \mathbf{R}\}$ are two order-reducing families, then for each μ, the operator $\Lambda^\mu \tilde{\Lambda}^{-\mu}$ is parameter-elliptic of order zero. (h) is immediate from (d) and interpolation in connection with 1.4.7, below.
We define the spaces $\mathcal{H}^{\infty,\gamma}(\Omega)$, $\mathcal{H}^{\infty,\gamma}(X)$ and $\mathcal{H}_0^{\infty,\gamma}(X)$ as the intersections of the corresponding spaces taken over all $s \in \mathbf{R}$.

1.4.4 Remark. On $\mathbf{R}^n$ we may choose a particularly simple order reduction, namely $\Lambda^\mu(\tau) = \operatorname{op}\langle \xi, \tau\rangle^\mu$. Using the transformation $\Phi_{n,\gamma}$ defined by

$$\Phi_{n,\gamma} v(r) = \exp(r(\frac{n+1}{2} - \gamma))v(e^r) = \left(t^{\frac{n+1}{2}-\gamma} v(t)\right)|_{t=e^r}$$

one can check that

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbf{R}^n\times\mathbf{R}_+)} = \|\Phi_{n,\gamma} u\|_{H^s(\mathbf{R}^n\times\mathbf{R})};$$

in other words,

$$\mathcal{H}^{s,\gamma}(\mathbf{R}^n \times \mathbf{R}_+) = \{t^{-\frac{n+1}{2}+\gamma} u(x, \ln t) : u \in H^s(\mathbf{R}^n \times \mathbf{R})\}, \tag{1}$$

cf. [44, 2.1.6(4)]. For $X = \mathbf{R}_+^n$ in $\Omega = \mathbf{R}^n$ we obtain

$$\mathcal{H}^{s,\gamma}(\mathbf{R}_+^n \times \mathbf{R}_+) = \{t^{-\frac{n+1}{2}+\gamma} u(x, \ln t) : u \in H^s(\mathbf{R}_+^n \times \mathbf{R})\}, \tag{2}$$

$$\mathcal{H}_0^{s,\gamma}(\mathbf{R}_+^n \times \mathbf{R}_+) = \{t^{-\frac{n+1}{2}+\gamma} u(x, \ln t) : u \in H_0^s(\mathbf{R}_+^n \times \mathbf{R})\}. \tag{3}$$

We have the following relation between the Fourier and the weighted Mellin transform:

$$[M_\gamma f(x,\cdot)](1/2 - \gamma + i\tau) = [\mathcal{F}u(x,\cdot)](\tau).$$

Here, $f(x,t) = t^{-1/2+\gamma} u(x, \ln t)$. As the notation indicates, both transforms act with respect to the last variable only. Therefore

$$M_\gamma \mathcal{H}^{s,\gamma+n/2}(\mathbf{R}_+^n \times \mathbf{R}_+) \text{ is isomorphic to } \mathcal{F}_{t\to\tau} H^s(\mathbf{R}_+^n \times \mathbf{R})$$

if we identify the lines $\Gamma_{1/2-\gamma}$ and $\mathbf{R}$.
The well-known fact that for $-1/2 < s < 1/2$ we have $H_0^s(\mathbf{R}_+^n) = H^s(\mathbf{R}_+^n)$ together with 1.4.4 then implies that

$$\mathcal{H}^{s,\gamma}(X^\wedge) = \mathcal{H}_0^{s,\gamma}(X^\wedge), \qquad -1/2 < s < 1/2. \tag{4}$$

The following lemma is new; it has not been given in Part I. We include it here since it gives more insight into the structure of $\mathcal{H}^{\infty,\gamma+n/2}(X^\wedge)$.

1.4.5 Lemma.

$$\mathcal{H}^{\infty,\gamma+n/2}(X^\wedge) = C^\infty(\overline{X})\hat{\otimes}_\pi \mathcal{H}^{\infty,\gamma}(\mathbf{R}_+).$$

Proof. In view of the nuclearity of $C^\infty(\overline{X})$ we have $C^\infty(\overline{X}, \mathcal{H}^{\infty,\gamma}(\mathbf{R}_+)) = C^\infty(\overline{X})\hat{\otimes}_\pi \mathcal{H}^{\infty,\gamma}(\mathbf{R}_+)$. So we only have to show the identity

$$\mathcal{H}^{\infty,\gamma+n/2}(X^\wedge) = C^\infty(\overline{X}, \mathcal{H}^{\infty,\gamma}(\mathbf{R}_+)) .$$

To this effect note that $\mathcal{H}^{\infty,\gamma+n/2}(X^\wedge)$ consists of smooth functions, so for $u = u(x,t) \in \mathcal{H}^{\infty,\gamma+n/2}(X^\wedge)$ we can consider the function $x \mapsto u(x,\cdot)$.
The norm on $\mathcal{H}^{\infty,\gamma+n/2}(X^\wedge)$ is given by the family of semi-norms

$$p_{k\alpha}(u)^2 = \|t^{-\gamma}(t\partial_t)^k\partial_x^\alpha u(x,t)\|^2_{L^2(X^\wedge)} = \int \|t^{-\gamma}(t\partial_t)^k\partial_x^\alpha u(x,\cdot)\|^2_{L^2(\mathbf{R}_+)}dx,$$

with suitable differential operators ∂_x^α on X, cf. 1.4.3. On the other hand, the last expression is of the form

$$\int q_k(\partial_x^\alpha u(x,\cdot))^2 dx, \tag{1}$$

where q_k is the semi-norm on $\mathcal{H}^{\infty,\gamma}(\mathbf{R}_+)$ defined by

$$q_k^2(f) = \int_0^\infty |t^{-\gamma}(t\partial_t)^k f(t)|^2 dt.$$

By Sobolev's lemma, (1) is a system of semi-norms that is defining for the topology of $C^\infty(\overline{X}, \mathcal{H}^{\infty,\gamma}(\mathbf{R}_+))$. This shows the assertion. $\lhd$

The following lemma also is new. We shall need it in Section 3.1.

1.4.6 Lemma. *Let $f \in \mathcal{H}^{1,\gamma}(\mathbf{R}_+)$ for some $\gamma > 1/2$. Then*

$$\lim_{t\to 0} f(t) = 0.$$

Proof. Without loss of generality we may assume that $f(t) = 0$ for $t \geq 1$. By definition,

$$t^{-\gamma}(t\partial_t)^j f \in L^2(\mathbf{R}_+), \qquad j = 0,1.$$

In particular, $f \in C(\mathbf{R}_+)$ by Sobolev's lemma, and it makes sense to consider $f(t)$ for $t > 0$. Moreover,

$$\begin{aligned} |f(t)| &\leq \int_t^1 |\partial_s f(s)|\, ds \ \ (1-t) \\ &\leq \left(\int_t^1 \left|s^{-\gamma}(s\partial_s f)(r)\right|^2 ds\right)^{1/2} \left(\int_t^1 s^{-2+2\gamma} ds\right)^{1/2} (1-t) \\ &\leq C_\gamma \|f\|_{\mathcal{H}^{1,\gamma}} \left(1 - t^{-1+2\gamma}\right)^{1/2}, \end{aligned}$$

so f is bounded whenever $\gamma > 1/2$. We have $\gamma - \varepsilon > 1/2$ for suitably small ε. The fact that $\mathcal{H}^{1,\gamma}(\mathbf{R}_+) = t^\varepsilon\mathcal{H}^{1,\gamma-\varepsilon}(\mathbf{R}_+)$ shows that $|f(t)| \leq C_{\gamma-\varepsilon}\|f\|_{\mathcal{H}^{1,\gamma-\varepsilon}} t^\varepsilon \to 0$ as $t \to 0^+$. $\lhd$

1.4.7 Proposition. *The inner product in* 1.4.3(g) *extends from $C_0^\infty(\overline{X}^\wedge) \times C_0^\infty(\overline{X}^\wedge)$ to a non-degenerate sesquilinear form*

$$\mathcal{H}^{s,\gamma}(X^\wedge) \times \mathcal{H}_0^{-s,-\gamma}(X^\wedge) \to \mathbf{C}$$

for all $s \in \mathbf{R}$. This admits the identification $\mathcal{H}_0^{-s,-\gamma}(X^\wedge) \cong (\mathcal{H}^{s,\gamma}(X^\wedge))'$. Moreover,

$$\|f\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \sup\left\{|(f,v)_{\mathcal{H}^{0,0}(X^\wedge)}| : \|v\|_{\mathcal{H}_0^{-s,-\gamma}(X^\wedge)} = 1\right\}$$

furnishes another equivalent norm on $\mathcal{H}^{s,\gamma}(X^\wedge)$.

1.4.8 Theorem. *Let $s > 1/2, \gamma \in \mathbf{R}, u \in \mathcal{H}^{s,\gamma}(\Omega^\wedge)$. Then the restriction $\gamma_0 u = u|_{Y^\wedge}$ of u to $Y^\wedge$ is well-defined and belongs to $\mathcal{H}^{s-1/2,\gamma-1/2}(Y^\wedge)$; the mapping*

$$\gamma_0 : \mathcal{H}^{s,\gamma}(\Omega^\wedge) \to \mathcal{H}^{s-1/2,\gamma-1/2}(Y^\wedge)$$

is continuous. Clearly, the same assertion holds if we replace $\Omega^\wedge$ by $X^\wedge$.
By r denote the normal coordinate in a neighborhood of Y. Then the operators $\gamma_j : u \mapsto \partial_r^j u|_{Y^\wedge}$ define continuous mappings

$$\gamma_j : \mathcal{H}^{s,\gamma}(\Omega^\wedge) \to \mathcal{H}^{s-j-1/2,\gamma-1/2}(Y^\wedge) \quad \text{and}$$

$$\gamma_j : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s-j-1/2,\gamma-1/2}(Y^\wedge).$$

1.4.9 Lemma. *Choose a smooth function ϕ equal to 1 in a neighborhood of Y and supported in the neighborhood of Y, where the normal derivative is defined. Then the operator $f \mapsto \partial_r(\phi f)$, defined for $f \in C^\infty(\Omega^\wedge)$ has a bounded extension to an operator*

$$\mathcal{H}^{s,\gamma}(\Omega^\wedge) \to \mathcal{H}^{s-1,\gamma}(\Omega^\wedge).$$

1.4.10 Theorem. *The spaces $\mathcal{H}^{s,\gamma}(\Omega^\wedge)$ are invariant under changes of coordinates if we restrict ourselves to the subspaces of functions with support in a compact set $\Omega \times \{t : 0 \le t \le R\}$, and if we ask that the diffeomorphism, say Φ,* respects the set $\{t = 0\}$, *i.e. Φ is the restriction of a diffeomorphism of $\Omega \times \overline{\mathbf{R}}_+$ (in particular, we will then have $\Phi(x,0) \in \Omega \times \{0\}$).*
More precisely: Let Φ be a diffeomorphism on $\Omega \times \mathbf{R}_+$, respecting $\{t = 0\}$. Then the space

$$\{u \in \mathcal{H}^{s,\gamma}(\Omega^\wedge) : u = 0 \text{ on } \{t > R\} \text{ for suitable } R\}$$

is invariant under the change of coordinates induced by Φ.

We say that a diffeomorphism Φ of $X^\wedge$ is *boundary-preserving* if there are open neighborhoods U_1, U_2 of $X^\wedge$ in $\Omega^\wedge$, and Φ extends to a diffeomorphism $\Phi : U_1 \to U_2$ respecting $\{t = 0\}$. This immediately leads to the following corollary.

1.4.11 Corollary. Also the subspace of $\mathcal{H}^{s,\gamma}(X^\wedge)$ consisting of the distributions that vanish for large t is invariant under changes of coordinates induced by boundary-preserving diffeomorphisms.

1.4.12 Definition. Let $\mathcal{F}$ be a subspace of $\mathcal{D}'(X^\wedge)$ or $\mathcal{D}'(\Omega^\wedge)$ with a stronger topology. Suppose that ϕ is a smooth function on $\overline{\mathbf{R}}_+$ and that multiplication by ϕ is continuous on $\mathcal{F}$. Then $[\phi]\mathcal{F}$ denotes the closure of the space $\{\phi u : u \in \mathcal{F}\}$ in $\mathcal{F}$.

1.4.13 Theorem. *Let $\omega \in C_0^\infty(\overline{\mathbf{R}}_+), \omega \equiv 1$ near zero. Then for $s \ge s', \gamma \ge \gamma'$*

$$[\omega]\mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow [\omega]\mathcal{H}^{s',\gamma'}(X^\wedge)$$

is continuous. For $s > s', \gamma > \gamma'$ the embedding

$$[\omega]\mathcal{H}^{s,\gamma}(X^\wedge) \hookrightarrow [\omega]\mathcal{H}^{s',\gamma'}(X^\wedge)$$

is compact.

1.4.14 Definition. For $s, \gamma \in \mathbf{R}, \omega$ a cut-off function on $\mathbf{R}_+$, let

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \{u \in \mathcal{D}'(X^\wedge) : \omega u \in \mathcal{H}^{s,\gamma}(X^\wedge), (1-\omega)\, u \in H^s_{cone}(X^\wedge)\}. \tag{1}$$

Here, $H^s_{cone}(X^\wedge)$ is as in 4.2.1. The definition is independent of the choice of ω by 1.4.3(f). In the notation of 1.4.12,

$$\mathcal{K}^{s,\gamma}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]\, H^s_{cone}(X^\wedge); \tag{2}$$

similarly,

$$\mathcal{K}^{s,\gamma}_0(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}_0(X^\wedge) + [1-\omega]\, H^s_{0,cone}(X^\wedge), \tag{3}$$

cf. 1.4.3(c). We shall give $\mathcal{K}^{s,\gamma}(X^\wedge)$ the Banach topology induced by (2):

$$\|u\|_{\mathcal{K}^{s,\gamma}(X^\wedge)} = \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} + \|(1-\omega)u\|_{H^s_{cone}(X^\wedge)}.$$

1.4.15 Remark. Note that, in contrast to Definition I.3.1.18, we have slightly changed the notation, replacing $[1-\omega]H^s(X^\wedge)$ and $[1-\omega]H^s_0(X^\wedge)$ by $[1-\omega]H^s_{cone}(X^\wedge)$ and $[1-\omega]H^s_{0,cone}(X^\wedge)$ respectively. The results of Part I hold with both conventions.

1.4.16 Definition. Let Θ be the interval $(\theta, 0], \theta < 0$, and let $s, \gamma \in \mathbf{R}$. $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ is defined as the intersection $\bigcap_{\epsilon>0} \mathcal{K}^{s,\gamma-\theta-\epsilon}(X^\wedge)$. We endow this space with the projective limit topology.
For $\Theta = (-\infty, 0]$ define $\mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ as the intersection of all the above spaces for $\theta < 0$.

1.4.17 Remark. (a) Let $u \in \mathcal{K}^{s,\gamma}(X^\wedge), s > 1/2$. By Theorem 1.4.8 the restriction $u|_Y$ belongs to $\mathcal{K}^{s-1/2,\gamma-1/2}(Y^\wedge)$.
(b) From Remark 1.4.4 we obtain natural dualities

$$\mathcal{K}^{s,\gamma}(X^\wedge)' \cong \mathcal{K}^{-s,-\gamma}_0(X^\wedge) \quad \text{and} \quad \mathcal{K}^{s,\gamma}_0(X^\wedge)' \cong \mathcal{K}^{-s,-\gamma}(X^\wedge)$$

for all $s, \gamma \in \mathbf{R}$.
(c) Let ϕ be as in 1.4.3(h). Then the multiplication operator

$$M_\phi : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge) \quad \text{and} \quad M_\phi : \mathcal{K}^{s,\gamma}_\Theta(X^\wedge) \to \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$$

is continuous.
(d) Of course, all these distributions may take values in finite-dimensional vector bundles with a Hermitean structure which are restrictions of smooth Hermitean bundles on $\Omega \times \mathbf{R}$.

1.5 Spaces with Asymptotics

1.5.1 Definition. (a) A *weight datum* $\mathbf{g} = (\gamma, \Theta)$ consists of a number $\gamma \in \mathbf{R}$ and an interval $\Theta = (\theta, 0]$ with $-\infty \le \theta < 0$.
(b) The *collection of asymptotic types* $As(X, \mathbf{g})$ for a weight datum $\mathbf{g} = (\gamma, (\theta, 0])$ with $\theta > -\infty$ ("finite weight interval") is the set of all finite vectors

$$P = \{(p_j, m_j, L_j) : j = 0, \ldots, N(P) \in \mathbf{N}\}$$

consisting of

(i) $p_j \in \mathbf{C}$ with $\frac{n+1}{2} - \gamma + \theta < \operatorname{Re} p_j < \frac{n+1}{2} - \gamma$, where $n = \dim X$,
(ii) $m_j \in \mathbf{N}$, and
(iii) L_j a finite-dimensional subspace of $C^\infty(\overline{X})$.
The elements P of $As(X, \mathbf{g})$ are called *asymptotic types.*
If $\mathbf{g}$ is a weight datum with $\theta = -\infty$, ("infinite weight interval") then $As(X, \mathbf{g})$ is the family of all vectors $P = \{(p_j, m_j, L_j) : j = 0, \ldots, N(P) \le \infty\}$ with the additional assumption that

(iv) $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$, whenever P is infinite.

By $\pi_{\mathbf{C}} P$ denote the set $\{p_j : j = 0, \ldots, N(P)\}$.
Correspondingly, $As(Y, \mathbf{g})$ is the set of all $P = \{(p_j, m_j, L_j) : j \in \mathbf{N}\}$ with $\frac{n}{2} - \gamma + \theta < \operatorname{Re} p_j < \frac{n}{2} + \gamma, m_j \in \mathbf{N}$, and L_j a finite-dimensional subspace of $C^\infty(Y)$. As before we assume that $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ whenever P is infinite. Finally we let for $\mathbf{g} = (\gamma, \Theta)$

$$As(X, Y, \mathbf{g}) = \{P = (P_1, P_2) : P_1 \in As(X, \mathbf{g}), P_2 \in As(Y, (\gamma - 1/2, \Theta))\}.$$

(c) The space $\mathcal{K}_P^{s,\gamma}(X^\wedge)$, for $P = \{(p_j, m_j, L_j) : j = 0, \ldots, N\} \in As(X, \mathbf{g})$ with finite weight interval consists of all $u = u(x,t) \in \mathcal{K}^{s,\gamma}(X^\wedge)$ such that for suitable $c_{jk} \in L_j, 0 \le j \le N, 0 \le k \le m_j$, and all cut-off functions near zero, ω,

$$u - \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x) t^{-p_j} \ln^k t \; \omega(t) \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge);$$

cf. 1.4.16 for the definition of $\mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$. In the case of an infinite weight interval first let $\mathbf{g}_k = (\gamma, (-k, 0])$, $k = 1, 2, \ldots$, and define $P_k \in As(X, \mathbf{g})$ by

$$P_k = \{(p_j, m_j, L_j) \in P : \frac{n+1}{2} - \gamma - k < \operatorname{Re} p_j \le \frac{n+1}{2} - \gamma\}.$$

Then let

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \bigcap_k \mathcal{K}_{P_k}^{s,\gamma}(X^\wedge). \tag{1}$$

$\mathcal{K}_P^{\infty,\gamma}(X^\wedge)$ is the intersection of all $\mathcal{K}_P^{s,\gamma}(X^\wedge), s \in \mathbf{R}$. It is a Fréchet space.
(d) Near each singularity v, $I\!D$ is diffeomorphic to $X_v \times \overline{\mathbf{R}}_+$, with suitable X_v as in 1.1.1. We define $\mathcal{H}_P^{s,\gamma}(I\!D)$ as the space of all distributions belonging to $\mathcal{H}_P^{s,\gamma}(X_v^\wedge)$ near a singularity v and belonging to $H^s(I\!D)$ in the interior; for the precise construction use a cut-off function ω_v near each singularity v.

1.5.2 Remark. The representation of a function in the form

$$u(x,t) = \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x)\, t^{-p_j} \ln^k t\, \omega(t) + f(x,t) \tag{1}$$

with $f \in \mathcal{K}_\Theta^{s,\gamma}(X^\wedge)$ as in 1.5.1(c) depends on the particular choice of coordinates. Under a change of coordinates, the function $\sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk}(x)\, t^{-p_j} \ln^k t\, \omega(t)$ transforms to a function $\sum_{j=0}^{N'} \sum_{k=0}^{m_j'} c_{jk}'(x) t^{-p_j'} \ln^k t\, \omega'(t) + g(x,t)$ with $g \in \mathcal{K}^{\infty,M}(X^\wedge)$ for arbitrarily large M. As indicated by the use of N' and p_j', there may be more and different exponents in the resulting representation. It is straightforward to see, cf. I.3.2.2, that all p_j' are of the

form $p_k - l$, for a suitable p_k and $l \in \mathbf{N}$. Moreover, if the c_{jk} vary over a finite-dimensional subspace of $C^\infty(X)$, then so will the c'_{jk}.
Spaces with asymptotics are therefore well-defined if we either keep coordinates fixed or else interprete the subscript P associated with an asymptotic type P as an equivalence class of of possible asymptotic types. This is the sense in which all the notation involving asymptotic types should be understood.

Recall that we always have

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_{(-\infty,0]}^{s,\gamma}(X^\wedge) + \mathcal{K}_P^{\infty,\gamma}(X^\wedge).$$

1.5.3 Definition. For $P \in As(X, \mathbf{g})$ and $\mathbf{g} = (\gamma, \Theta)$ let

$$\mathcal{S}_P^\gamma(X^\wedge) = [\omega]\mathcal{K}_P^{\infty,\gamma}(X^\wedge) + [1-\omega]\mathcal{S}(X^\wedge),$$

where $\mathcal{S}(X^\wedge) = \mathcal{S}(\mathbf{R}_+)\hat{\otimes}_\pi C^\infty(\overline{X})$. The definition depends on the choice of Θ.

1.5.4 Lemma. *Let $\tilde{\phi} \in C_0^\infty(\Omega \times \mathbf{R}), \phi = \tilde{\phi}|_{X^\wedge}$. Then the multiplication operator*

$$M_\phi : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

is bounded. If $P \in As(X, \mathbf{g})$ satisfies the "shadow condition" (i.e. given a triple $(p, m, L) \in P$ and $j \in \mathbf{N}$, there is an element $(p-j, m(j), L(j)) \in P$ with $m(j) \geq m, L(j) \supseteq L$) then also

$$M_\phi : \mathcal{K}_P^{s,\gamma}(X^\wedge) \to \mathcal{K}_P^{s,\gamma}(X^\wedge)$$

is continuous.

1.5.5 Remark. Of course, all notions make sense for distributions with values in finite-dimensional Hermitean vector bundles which are smooth up to the boundary with the obvious modifications.

1.6 Green Operators. The Algebras $C_G(X^\wedge, \mathbf{g})$ and $C_G(\mathbb{D}, \mathbf{g})$

1.6.1 Definition. Let $\mathbf{g} = (\gamma, \delta, \Theta)$ with $\gamma, \delta \in \mathbf{R}, \Theta = (\theta, 0], -\infty \leq \theta < 0; \mathbf{g}$ also is called a weight datum. Moreover, let P, Q be two asymptotic types, $P = (P_1, P_2) \in As(X, Y, (\delta, \Theta)), Q = (Q_1, Q_2) \in As(X, Y, (-\gamma, \Theta))$, and $V_1, W_1, \ldots$ smooth Hermitean vector bundles.

(a) Let

$$G \in \bigcap_{s>-1/2} \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-1/2}(Y^\wedge, W_1), \mathcal{K}^{\infty,\delta}(X^\wedge, V_2) \oplus \mathcal{K}^{\infty,\delta-1/2}(Y^\wedge, W_2)).$$

We shall write $G \in C_G^0(X^\wedge, \mathbf{g})_{P,Q}$ if the following holds: for all $s > -1/2$

$$G = \begin{bmatrix} G_G & G_K \\ G_T & G_S \end{bmatrix} : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-1/2}(Y^\wedge, W_1) \end{matrix} \to \begin{matrix} \mathcal{S}_{P_1}^\delta(X^\wedge, V_2) \\ \oplus \\ \mathcal{S}_{P_2}^{\delta-1/2}(Y^\wedge, W_2) \end{matrix} \qquad (1)$$

and

$$G^*: \begin{array}{c} \mathcal{K}^{s,-\delta}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{s,-\delta-1/2}(Y^\wedge, W_2) \end{array} \to \begin{array}{c} \mathcal{S}^{-\gamma}_{Q_1}(X^\wedge, V_1) \\ \oplus \\ \mathcal{S}^{-\gamma-1/2}_{Q_2}(Y^\wedge, W_1) \end{array} \qquad (2)$$

are continuous. In (2), G^* is the formal adjoint of G. It is defined from the duality between $\mathcal{K}^{s,\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{s,\gamma-1/2}(Y^\wedge, W_1)$ and $\mathcal{K}_0^{-s,-\gamma}(X^\wedge, V_1) \oplus \mathcal{K}^{-s,-\gamma-1/2}(Y^\wedge, W_1)$, which comes from an extension of the inner product

$$\begin{aligned} ((f_1, g_1), (f_2, g_2)) &= \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}}} (Mf_1(z), Mf_2(z))_{L^2(X)} dz \\ &+ \frac{1}{2\pi i} \int_{\Gamma_{\frac{n+1}{2}}} (Mg_1(z), Mg_2(z))_{L^2(Y)} dz \end{aligned}$$

on $\mathcal{H}^0(X^\wedge) \oplus \mathcal{H}^{0,-1/2}(Y^\wedge)$. Notice that the second term on the right hand side differs from the standard inner product on $\mathcal{H}^{s,\gamma}(Y^\wedge)$, where the integration is over $\Gamma_{n/2-\gamma}$, for $\dim Y = n-1$. Since $(Mu)(z+1/2) = M(t^{1/2}u)(z)$, this term yields a duality between $\mathcal{H}^{s,\gamma-1/2}(Y^\wedge)$ and $\mathcal{H}^{-s,-\gamma-1/2}(Y^\wedge)$. As before, we will not refer to the bundles in the notation.

(b) $C^0_G(I\!D, \mathbf{g})_{P,Q}$ is the corresponding space with $X^\wedge$ replaced by $I\!D$ and the spaces $\mathcal{S}^{\delta}_{P_1}(X^\wedge, V_2), \ldots, \mathcal{S}^{-\gamma-1/2}_{Q_2}(Y^\wedge, W_1)$ by $\mathcal{H}^{\infty,\delta}_{P_1}(I\!D, V_2), \ldots, \mathcal{H}^{\infty,-\gamma-1/2}_{Q_2}(I\!B, W_1)$. We call the elements of $C^0_G(X^\wedge, \mathbf{g})_{P,Q}$ and $C^0(I\!D, \mathbf{g})_{P,Q}$ the *Green operators of type zero* on $X^\wedge$ and $I\!D$, respectively.

(c) Let $d \in \mathbf{N}$. An operator G acting as in (1) is called a *Green operator of type d*, if it can be written

$$G = \sum_{j=0}^{d} G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix} \qquad (3)$$

with Green operators G_j of type zero. The order s in (1) then is assumed to be $> d - 1/2$. With the replacements in (b) we can use the same definition for operators acting on functions over $I\!D$. In (3), ∂_r denotes the normal derivative defined in a neighborhood of the boundary of the Riemannian manifolds $X^\wedge$ and $I\!D$, respectively, multiplied by a cut-off function, so that it makes sense everywhere.
We shall write

$$G \in C^d_G(X^\wedge, \mathbf{g})_{P,Q} \quad \text{and} \quad G \in C^d_G(I\!D, \mathbf{g})_{P,Q},$$

respectively. Without loss of generality we assume that the asymptotic types P and Q in (1) and (2) are the same for all $G_j, j = 0, \ldots, d$.

(d) The mapping properties (1) and (2) give a natural Fréchet topology for the spaces $C^0_G(X^\wedge, \mathbf{g})_{P,Q}$ and $C^0_G(I\!D, \mathbf{g})_{P,Q}$. The spaces $C^d_G(X^\wedge, \mathbf{g})_{P,Q}$ and $C^d_G(I\!D, \mathbf{g})_{P,Q}$ are topologized as non-direct sums of Fréchet spaces, cf. 1.3.2.

In the following, $\mathbf{g}$, P, Q will denote an arbitrary weight datum and arbitrary asymptotic types. $V_1, W_1, \ldots$ are Hermitean vector bundles smooth up to the boundary.

1.6.2 Theorem.

$$\begin{aligned} & C_G^0(X^\wedge, \mathbf{g})_{P,Q} & (1)\\ \cong\; & \left[\mathcal{S}^\delta_{P_1}(X^\wedge, V_2)\oplus \mathcal{S}^{\delta-1/2}_{P_2}(Y^\wedge, W_2)\right]\hat{\otimes}_\pi \left[\mathcal{S}^{-\gamma}_{\overline{Q}_1}(X^\wedge, V_1)\oplus \mathcal{S}^{-\gamma-1/2}_{\overline{Q}_2}(Y^\wedge, W_1)\right]. \end{aligned}$$

The isomorphism is given by the mapping that associates with G its integral kernel. Here, $\overline{Q} = (\overline{Q}_1, \overline{Q}_2)$ is an asymptotic type in $As(X, Y, \mathbf{g})$. $\overline{Q}_k$ is constructed by replacing each element $(p, m, L) \in Q_k$ by the complex conjugate $(\overline{p}, m, \overline{L}), k = 1, 2$. Similarly,

$$\begin{aligned} & C_G^0(I\!D, \mathbf{g})_{P,Q} & (2)\\ \cong\; & \left[\mathcal{H}^{\infty,\delta}_{P_1}(I\!D, V_2)\oplus \mathcal{H}^{\infty,\delta-1/2}_{P_2}(I\!B, W_2)\right]\hat{\otimes}_\pi \left[\mathcal{H}^{\infty,-\gamma}_{\overline{Q}_1}(I\!D, V_1)\oplus \mathcal{H}^{\infty,-\gamma-1/2}_{\overline{Q}_2}(I\!B, W_1)\right]. \end{aligned}$$

1.6.3 Corollary. (a) Let ϕ_1 and ϕ_2 be excision functions for the singular set of D, and let $G \in C_G^0(I\!D, \mathbf{g})$. Then $\phi_1 G \phi_2$ is a regularizing singular Green operator in Boutet de Monvel's calculus for $I\!D$.

1.6.4 Lemma. *Let $G_1 \in C_G^0(X^\wedge, \mathbf{g})_{P,Q}$ and $G_2 \in C_G^0(I\!D, \mathbf{g})_{P,Q}$. Then the mappings*

$$G_1 : \begin{matrix} \mathcal{K}^{s,\gamma}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma-1/2}(Y^\wedge, W_1) \end{matrix} \to \begin{matrix} \mathcal{K}^{t,\delta}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{t,\delta-1/2}(Y^\wedge, W_2) \end{matrix}$$

and

$$G_2 : \begin{matrix} \mathcal{H}^{s,\gamma}(I\!D, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma-1/2}(I\!B, W_1) \end{matrix} \to \begin{matrix} \mathcal{H}^{t,\delta}(I\!D, V_2) \\ \oplus \\ \mathcal{H}^{t,\delta-1/2}(I\!B, W_2) \end{matrix}$$

are compact for every choice of $s, t > -1/2$.

1.6.5 Lemma. *Let $\mathbf{g}_1 = (\gamma, \delta, \Theta), \mathbf{g}_2 = (\delta, \eta, \Theta)$ be weight data, P, Q, R, and S asymptotic types, let $G_1 \in C_G^d(X^\wedge, \mathbf{g}_1)_{P,Q}$, and $G_2 \in C_G^{d'}(X^\wedge, \mathbf{g}_2)_{R,S}$. Then*

$$G_2 G_1 \in C_G^d(X^\wedge, \mathbf{g}_3)_{R,S'}$$

with $\mathbf{g}_3 = (\gamma, \eta, \Theta)$ and a resulting asymptotic type S'. We tacitly assume that G_1 and G_2 act on vector bundles so that the composition makes sense.
The corresponding result also holds with $X^\wedge$ replaced by $I\!D$.

1.6.6 Definition and Remark. For $\mathbf{g} = (\gamma, \delta, \Theta)$ we let $C_G(X^\wedge, \mathbf{g})$ denote the space of all operators that belong to any one of the families $C_G^d(X^\wedge, \mathbf{g})_{P,Q}$ for arbitrary d, P, Q. In view of Lemma 1.6.5, the elements of $C_G(X^\wedge, \mathbf{g})$ that act on fitting weight data and vector bundles can be composed. The composition is continuous with respect to the corresponding topologies. The corresponding results are true for $C_G(I\!D, \mathbf{g})$.

1.7 Mellin Symbols with Values in Boutet de Monvel's Algebra

The following lemma is easily deduced from the elementary properties of the Mellin transform, see [20] or [40, Appendix 5.1].

1.7.1 Lemma. (a) *Let ω be a cut-off function near 0. Then $M\omega(z) = z^{-1}M(-t\partial_t\omega)(z)$. Since $-t\partial_t\omega \in C_0^\infty(\mathbf{R}_+)$, its Mellin transform is rapidly decreasing on each line Γ_β. If χ is a smooth function on $\mathbf{C}$ which vanishes near zero and is equal to 1 near infinity, then $\chi M\omega$ is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.*

(b) *Given a cut-off function $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(t) \equiv 1$ near zero, $p \in \mathbf{C}$, and $k \in \mathbf{N}$, let*

$$\psi_{p,k}(z) = M_{t\to z}(t^{-p}\ln^k t\,\omega(t))(z) \quad = \quad \frac{d^k}{dz^k}(-z^{-1}M(t\partial_t\omega)(z))(z-p).$$

Here we interprete $M_{t\to z}$ as the weighted Mellin transform M_γ with $\gamma < 1/2 - \operatorname{Re} p$. Then $\psi_{p,k}$ extends to a meromorphic function in $\mathbf{C}$ with a single pole of order $k+1$ in p. If χ is a smooth function on $\mathbf{C}$ which vanishes near p and is equal to 1 outside some compact set, then $\chi\psi_{p,k}$ is rapidly decreasing on each line Γ_β, uniformly for β in compact intervals.

1.7.2 Definition. (a) A *Mellin asymptotic type* is a sequence

$$P = \{(p_j, m_j, L_j)\}_{j\in\mathbf{Z}}$$

with $p_j \in \mathbf{C}$, $\operatorname{Re} p_j \to \pm\infty$ as $j \to \mp\infty$, $m_j \in \mathbf{N}$, and L_j a finite-dimensional subspace of finite-dimensional operators in $\mathcal{B}^{-\infty,d}(X)$.
We denote the collection of all these asymptotic types by $As\left(\mathcal{B}^{-\infty,d}(X)\right)$. Just like in 1.5.1, we let $\pi_{\mathbf{C}}P = \{p_j : j \in \mathbf{Z}\}$.

(b) Let $P \in As\left(\mathcal{B}^{-\infty,d}(X)\right)$, $\mu \in \mathbf{R}, d \in \mathbf{N}$. Then $M_P^{\mu,d}(X)$ denotes the space of all functions

$$a \in \mathcal{A}\left(\mathbf{C}\setminus\pi_{\mathbf{C}}P, \mathcal{B}^{\mu,d}(X)\right) \tag{1}$$

with the following properties

(i) in a neighborhood of $p_j \in \pi_{\mathbf{C}}P$

$$a(z) = \sum_{k=0}^{m_j} \nu_{jk}(z-p_j)^{-k-1} + a_0(z) \tag{2}$$

with $\nu_{jk} \in L_j$, $k = 0, \ldots, m_j$, and a_0 holomorphic near p_j.

(ii) Given $c_1 < c_2$ in $\mathbf{R}$ we can find $\sigma_{jk} \in L_j$ such that, for each $\beta \in [c_1, c_2]$,

$$a(\beta+i\tau) - \sum_{\{j:p_j\in[c_1,c_2]\}} \sum_{k=0}^{m_j} \psi_{p_j,k}(\beta+i\tau)\sigma_{jk} \in \mathcal{B}^{\mu,d}(X;\mathbf{R}_\tau), \tag{3}$$

uniformly for β in $[c_1, c_2]$.

We call the elements of $M_P^{\mu,d}(X)$ Mellin symbols of order μ, type d, with asymptotic type P.
We are assuming in (1) that the vector bundles $a(z)$ is acting on, cf. 1.3.4(1), are independent of z.

(c) $M_{P,cl}^{\mu,d}(X)$ is the corresponding space with $\mathcal{B}^{\mu,d}(X)$ replaced by $\mathcal{B}_{cl}^{\mu,d}(X)$.

(d) If $P = \emptyset$ then we shall write $M_O^{\mu,d}(X)$ and $M_{O,cl}^{\mu,d}(X)$.

1.7.3 Remark. The topology of $M_P^{\mu,d}(X)$ is given by three semi-norm systems

(i) that for the topology of $\mathcal{A}\left(\mathbf{C}\backslash\pi_{\mathbf{C}}P, \mathcal{B}^{\mu,d}(X)\right)$;

(ii) that induced by $a \mapsto \nu_{jk} \in L_j \subseteq \mathcal{B}^{-\infty,d}(X)$, where $a \in M_P^{\mu,d}(X)$ is as in 1.7.2(2), and the Euclidean topologie on L_j;

(iii) that given by 1.7.2(ii).

$M_P^{\mu,d}(X)$ is a Fréchet space in the above topology. $M_P^{-\infty,d}(X) = \bigcap_\mu M_P^{\mu,d}(X)$ is a nuclear Fréchet space.

1.7.4 Theorem. *Let P be a Mellin asymptotic type, $\mu \in \mathbf{R}, d \in \mathbf{N}$. The function $a \in \mathcal{A}(\mathbf{C}\backslash\pi_{\mathbf{C}}P, \mathcal{B}^{\mu,d}(X))$ is a Mellin symbol in $M_P^{\mu,d}(X)$ if and only if it can be written*

$$a(z) = \sum_{k=0}^{d} a_k(z) \begin{bmatrix} \partial_r^k & 0 \\ 0 & I \end{bmatrix} \tag{1}$$

with $a_k \in M_Q^{\mu-k,0}(X)$. Here, ∂_r stands for the operator given by the normal derivative in a neighborhood of the boundary, multiplied by a suitable cut-off function. Q is a slightly modified asymptotic type; it contains the same p_j and m_j, but the L_j are now suitable finite-dimensional spaces of finite-dimensional operators in $\mathcal{B}^{-\infty,0}(X)$.

1.7.5 Proposition. *Let $\mu, \mu' \in \mathbf{Z}$, $d, d' \in \mathbf{N}$, and let $P = \{(p_j, m_j, L_j)\}$, $P' = \{(p_j', m_j', L_j')\}$ be two Mellin asymptotic types. For $a \in M_P^{\mu,d}(X)$ and $b \in M_{P'}^{\mu',d'}(X)$ the function*

$$c(z) = a(z)\, b(z) \tag{1}$$

belongs to $M_{P''}^{\mu'',d''}(X)$, where

- $\mu'' = \mu + \mu'$;
- $d'' = \max\{\mu' + d, d'\}$;
- *P'' is a suitable Mellin asymptotic type that can be determined from a and b; in particular, $\pi_{\mathbf{C}}P'' \subseteq \pi_{\mathbf{C}}P \cup \pi_{\mathbf{C}}P'$.*

We are tacitly assuming that the composition in (1) makes sense, i.e. $a(z)$ and $b(z)$ are acting on appropriately chosen bundles.

1.7.6 Theorem.

$$M_P^{\mu,d}(X) = M_O^{\mu,d}(X) + M_P^{-\infty,d}(X).$$

1.7.7 Definition. Let $\gamma \in \mathbf{R}$, E, F Hilbert spaces.
(a) If f is a function on $U \subseteq \mathbf{C}$, then let $(T^\gamma f)(z) = f(z+\gamma)$ whenever $z + \gamma \in U$.

(b) For a polynomially bounded function g on $\Gamma_{1/2}$ with values in $\mathcal{L}(E,F)$ let $\mathrm{op}_M g : C_0^\infty(\mathbf{R}_+, E) \to C^\infty(\mathbf{R}_+, F)$ be defined by

$$(\mathrm{op}_M g)(u) = M^{-1} g\, M u$$

with the vector-valued Mellin transform $M : L^2(\mathbf{R}_+, E) \to L^2(\Gamma_{1/2}, E)$.
(c) For g defined on $\Gamma_{1/2-\gamma}, \gamma \in \mathbf{R}$, let

$$\mathrm{op}_M^\gamma g = t^\gamma\, \mathrm{op}_M \left(T^{-\gamma} g\right) t^{-\gamma} = M_\gamma^{-1} g M_\gamma,$$

with the weighted Mellin transform M_γ. In particular, $\mathrm{op}_M^0 = \mathrm{op}_M$.

1.7.8 Lemma. *Let $a \in M_P^{\mu,d}(X)$, $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$, $\gamma \in \mathbf{R}$, P a Mellin asymptotic type with $\pi_{\mathbf{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$. Suppose that, for fixed z, $a(z) \in \mathcal{B}^{\mu,d}(X)$ acts on vector bundles as in 1.3.4. Then*

$$\operatorname{op}_M^\gamma a : \begin{array}{c} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} C^\infty(\overline{X}^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2) \end{array} \tag{1}$$

is a continuous operator.

1.7.9 Theorem. *Under the assumptions of 1.7.8, $\operatorname{op}_M^\gamma a$ has a bounded extension*

$$\operatorname{op}_M^\gamma a : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

for all $s \in \mathbf{R}$, $s > d - 1/2$.

1.7.10 Corollary. Let $\omega, \omega' \in C_0^\infty(\overline{\mathbf{R}}_+)$. Under the assumptions of 1.7.8

$$\omega \operatorname{op}_M^\gamma(a)\, \omega' : \begin{array}{c} \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is bounded for all $s \in \mathbf{R}, s > d - 1/2$.

1.7.11 Lemma. *Use the notation of 1.7.8 and assume additionally that $d = 0, s > -1/2$, and $\mu \leq 0$. Then the operator $A = \operatorname{op}_M^\gamma a$ has a formal adjoint A^* with respect to the dualities*

$$\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1), \mathcal{H}_0^{-s,-\gamma-\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{H}^{-s,-\gamma-\frac{n-1}{2}}(Y^\wedge, W_1)$$

and

$$\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \oplus \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2), \mathcal{H}_0^{-s+\mu,-\gamma-\frac{n}{2}}(X^\wedge, V_2) \oplus \mathcal{H}^{-s+\mu,-\gamma-\frac{n-1}{2}}(Y^\wedge, W_2).$$

We have

$$A^* = \operatorname{op}_M^{-\gamma-n} a^{(*)} \quad \text{with} \quad a^{(*)} = a(n+1-\bar{z})^*; \tag{1}$$

the last asterisk indicates the matrix adjoint. The fact that $a \in M_P^{\mu,0}(X)$ implies that $a^{(*)} \in M_Q^{\mu,0}(X)$ for a resulting asymptotic type Q.

1.7.12 Theorem. *Let $a \in M_P^{\mu,d}(X)$, with μ, d, P as in 1.7.8. Moreover, let $\omega, \omega' \in C_0^\infty(\overline{\mathbf{R}}_+)$ and $\mathbf{g} = (\gamma + n/2, \Theta), \Theta = (\theta, 0]$ be a weight datum.*
Then for every asymptotic type $Q = (Q_1, Q_2) \in As(X, Y, \mathbf{g})$ there is an asymptotic type $R = (R_1, R_2) \in As(X, Y, \mathbf{g})$ such that

$$\omega \operatorname{op}_M^\gamma(a)\, \omega' : \begin{array}{c} \mathcal{K}_{Q_1}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}_{Q_2}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{K}_{R_1}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}_{R_2}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is continuous for all $s > d - 1/2$.

1.8 Mellin Operators and Green Operators

1.8.1 Theorem. *Let $a \in M_P^{\mu;d}(X), \mu \in \mathbf{Z}, d \in \mathbf{N}, P$ a Mellin asymptotic type. Moreover, let $\gamma \in \mathbf{R}, \beta \geq 0, \omega, \omega_1 \in C_0^\infty(\overline{\mathbf{R}}_+)$, and suppose that*

$$\pi_{\mathbf{C}} P \cap \Gamma_{1/2-\gamma} = \pi_{\mathbf{C}} P \cap \Gamma_{1/2-\gamma+\beta} = \emptyset.$$

Then

$$\omega\, t^\beta \operatorname{op}_M^\gamma(a)\,\omega_1 - \omega \operatorname{op}_M^\gamma(T^\beta a)\, t^\beta \omega_1 \in C_G^d(X^\wedge, \mathbf{g})_{Q,R} \tag{1}$$

for suitable asymptotic types $Q, R \in As(X, Y, \mathbf{g}), \mathbf{g} = (\gamma + n/2, \gamma + n/2, (-\infty, 0])$, depending on $P; (T^\beta a)(z) = a(z + \beta)$. The operator in (1) has finite-dimensional range. It is given as a contour integral around the finitely many singularities of a in the strip between $\Gamma_{1/2-\gamma}$ and $\Gamma_{1/2-\gamma+\beta}$. In particular, the difference is zero if a has no singularities in the strip $\{1/2 - \gamma \leq \operatorname{Re} z \leq 1/2 - \gamma + \beta\}$.
For $\beta < 0$, the same is true with the weight datum $\mathbf{g} = (\gamma + n/2 - \beta, \gamma + n/2 + \beta, (-\infty, 0])$.

1.8.2 Theorem. *Let $h \in M_P^{-\infty,d}(X)$, $\gamma \in \mathbf{R}$, $\pi_{\mathbf{C}} P \cap \Gamma_{1/2-\gamma} = \emptyset$. Moreover, let $\omega, \omega_1, \omega_2, \omega_3, \omega_4$ be arbitrary cut-off functions near $0 \in \mathbf{R}$, and $\varphi \in C_0^\infty(\mathbf{R}_+)$. Then*

(a) $\omega \operatorname{op}_M^\gamma(h)\, \varphi \in C_G^d(X^\wedge, \mathbf{g})_{Q,O}$.

(b) $\varphi \operatorname{op}_M^\gamma(h)\, \omega \in C_G^d(X^\wedge, \mathbf{g})_{O,R}$.

(c) $\omega_1 \operatorname{op}_M^\gamma(h)\, \omega_2 - \omega_3 \operatorname{op}_M^\gamma(h)\, \omega_4 \in C_G^d(X^\wedge, \mathbf{g})_{Q,R}$.

In (a), (b) *and* (c), *Q and R are suitable asymptotic types in $As(X, Y, \mathbf{g})$; O is the 'zero' asymptotic type, and $\mathbf{g}$ is the weight datum $\mathbf{g} = (\gamma + n/2, \gamma + n/2, (-\infty, 0])$.*

1.8.3 Remark. (a) In the notation of 1.8.1, we have for $f \in C_0^\infty(\overline{X^\wedge})$ and $\beta \in \mathbf{R}$

$$\begin{aligned} t^\beta\, \omega \operatorname{op}_M^{\gamma-\beta}(a)\, \omega_1 f &= t^\beta \omega \int_{\Gamma_{1/2-\gamma+\beta}} t^{-\zeta} a(\zeta)\, M(\omega_1 f)(\zeta) d\zeta \\ &= \omega \int_{\Gamma_{1/2-\gamma}} t^{-z} a(z + \beta) M(\omega_1 f)(z + \beta) dz \\ &= \omega \operatorname{op}_M^\gamma(T^\beta a)\, \omega_1\, t^\beta f. \end{aligned} \tag{1}$$

By 1.8.1, the last operator equals $\omega\, t^\beta \operatorname{op}_M^\gamma(a)\, \omega_1 f$ modulo a Green operator, say G. Here we have assumed that $\Gamma_{1/2-\gamma} \cap \pi_{\mathbf{C}} P = \emptyset = \Gamma_{1/2-\gamma+\beta} \cap \pi_{\mathbf{C}} P$. For every $j \geq \beta$ we therefore have

$$\omega\, t^j \operatorname{op}_M^\gamma(a)\, \omega_1 - \omega\, t^j \operatorname{op}_M^{\gamma-\beta}(a)\, \omega_1 = t^{j-\beta}\, G,$$

which also is a Green operator, namely with respect to $(\gamma + n/2, \gamma + n/2, (-\infty, 0])$, even $(\gamma + n/2, \gamma + n/2 + j - \beta, (-\infty, 0])$ for $\beta \geq 0$ and with respect to $(\gamma + n/2 - \beta, \gamma + n/2 + j, (-\infty, 0])$ for $\beta < 0$.
(b) In view of 1.8.1 and the discreteness of the singularity set we note the following consequence: If $j > 0$, $\Gamma_{1/2-\gamma} \cap \pi_{\mathbf{C}} P = \emptyset$, and $\epsilon > 0$ is sufficiently small, then on $C_0^\infty(\overline{X^\wedge})$

$$\omega\, t^j \operatorname{op}_M^\gamma(a)\, \omega_1 \;=\; \omega\, t^j \operatorname{op}_M^{\gamma-\epsilon}(a)\, \omega_1. \tag{2}$$

Part (a) of this remark is the basis for the proposition below.

1.8.4 Proposition. *Let $\gamma \in \mathbf{R}, j > 0$, and $0 \leq \rho_k, \rho_k' \leq j, k = 1, \ldots, r$. Moreover, let P_k, P_k' be Mellin asymptotic types with $\pi_{\mathbf{C}} P_k \cap \Gamma_{1/2-\gamma+\rho_k} = \emptyset = \pi_{\mathbf{C}} P_k' \cap \Gamma_{1/2-\gamma+\rho_k'}$, and finally let $a_k \in M_{P_k}^{\mu,d}(X), a_k' \in M_{P_k'}^{\mu,d}(X)$. For $\omega, \omega_1 \in C_0^\infty(\overline{\mathbf{R}}_+)$ define*

$$\begin{aligned} A &= \omega\, t^j \sum_{k=1}^{r} \mathrm{op}_M^{\gamma-\rho_k}(a_k)\, \omega_1, \qquad \text{and} \\ A' &= \omega\, t^j \sum_{k=1}^{r} \mathrm{op}_M^{\gamma-\rho_k'}(a_k')\, \omega_1. \end{aligned}$$

Then $A - A' \in C_G^d(X^\wedge, \mathbf{g})_{Q,R}$, whenever $\sum_{k=1}^r a_k(z) = \sum_{k=1}^r a_k'(z)$ for all z.
Here, $\mathbf{g} = (\gamma + n/2, \gamma + n/2, (-\infty, 0])$; Q and R are resulting asymptotic types.

1.9 The Algebras $C_{M+G}(X^\wedge, \mathbf{g})$ and $C_{M+G}(I\!D, \mathbf{g})$.

1.9.1 Definition. Let $\mu, \nu \in \mathbf{R}, \mu - \nu \in \mathbf{N}, d \in \mathbf{N}$, and let $\mathbf{g} = (\gamma + n/2, \gamma + n/2 - \mu, \Theta)$ be a weight datum, $\gamma \in \mathbf{R}$. We suppose that $\Theta = (-N, 0]$, for some $N \in \mathbf{N} \backslash \{0\}$.
For $d \in \mathbf{N}$ we let $C_{M+G}^{\nu,d}(X^\wedge, \mathbf{g})$ denote the space of all operators $A = A_M + A_G$, where

(i) A_M is a Mellin operator of the form $A_M = t^{-\nu} \sum_{j=0}^{N-1} \omega_j\, t^j\, \mathrm{op}_M^{\gamma_j}(h_j)\, \tilde{\omega}_j$ with
   - (i.1) suitable cut-off functions $\omega_j, \tilde{\omega}_j$ near zero,
   - (i.2) $\gamma - (\mu - \nu) - j \leq \gamma_j \leq \gamma$,
   - (i.3) $h_j \in M_{P_j}^{-\infty,d}(X)$, and
   - (i.4) Mellin asymptotic types P_j with $\pi_{\mathbf{C}} P_j \cap \Gamma_{1/2-\gamma_j} = \emptyset$.

(ii) A_G is a Green operator in $C_G^d(X^\wedge, \mathbf{g})_{P,Q}$ for suitable asymptotic types $P, Q \in As(X, Y, \mathbf{g})$.

Clearly, $C_{M+G}^{\nu,d}(X^\wedge, g) \subseteq C_{M+G}^{\mu,d}(X^\wedge, g)$, since

$$t^{-\nu} \sum_{j=0}^{N-1} \omega_j\, t^j\, \mathrm{op}_M^{\gamma_j}(h_j)\, \tilde{\omega}_j = t^{-\mu} \sum_{j=0}^{N-1} \omega_j\, t^{j+\mu-\nu}\, \mathrm{op}_M^{\gamma_j}(h_j)\, \tilde{\omega}_j$$

and $\mu - \nu \in \mathbf{N}$. $C_{M+G}^{\nu,d}(I\!D, \mathbf{g})$ is the corresponding space, where in (ii) we replace $X^\wedge$ by $I\!D$, and in (i) we additionally make the support of $\omega_j, \tilde{\omega}_j$ so small that the operators are well-defined on the cylindrical parts of $I\!D$ close to the singularities. In view of 1.8.2 we might also ask that the cut-off functions ω_j and $\tilde{\omega}_j$ are independent of j.

In the following we will assume that $\gamma, \mu, \nu \in \mathbf{R}, d, N \in \mathbf{N}, \Theta = (-N, 0]$, and the weight datum $\mathbf{g} = (\gamma + n/2, \gamma + n/2 - \mu, \Theta)$ are fixed with the properties in 1.9.1 unless specified otherwise. In order to also fix the notation suppose that A acts on vector bundles $V_1, \ldots, W_2$ in the following way:

$$A : \begin{array}{c} C_0^\infty(X^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \rightarrow \begin{array}{c} C^\infty(X^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2). \end{array}$$

1.9.2 Remark. Using Theorem 1.7.4 and the definition of the Green operators, an operator $A \in C^{\nu,d}_{M+G}(X^\wedge, \mathbf{g})$ can be written

$$A = \sum_{j=0}^{d} A_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & 1 \end{bmatrix}$$

with $A_j \in C^{\nu,0}_{M+G}(X^\wedge, \mathbf{g})$.

1.9.3 Theorem. *For operators $A \in C^{\nu,d}_{M+G}(X^\wedge, \mathbf{g})$ and $B \in C^{\nu,d}_{M+G}(\mathbb{D}, \mathbf{g})$ the mappings*

$$A : \begin{array}{c} \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{K}^{\infty,\gamma+\frac{n}{2}-\mu}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{\infty,\gamma+\frac{n-1}{2}-\mu}(Y^\wedge, W_2) \end{array}$$

and

$$B : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbb{D}, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B}, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\gamma+\frac{n}{2}-\mu}(\mathbb{D}, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B}, W_2) \end{array}$$

are continuous for all $s > d - 1/2$.
If $P = (P_1, P_2) \in As(X, Y, (\gamma + n/2, \Theta))$ is an asymptotic type, then there is a resulting asymptotic type $P' = (P_1', P_2') \in As(X, Y, (\gamma + n/2 - \mu, \Theta))$ such that

$$A : \begin{array}{c} \mathcal{K}^{s,\gamma+\frac{n}{2}}_{P_1}(X^\wedge, V_1) \\ \oplus \\ \mathcal{K}^{s,\gamma+\frac{n-1}{2}}_{P_2}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{K}^{\infty,\gamma+\frac{n}{2}-\mu}_{P_1'}(X^\wedge, V_2) \\ \oplus \\ \mathcal{K}^{\infty,\gamma+\frac{n-1}{2}-\mu}_{P_2'}(Y^\wedge, W_2) \end{array}$$

and

$$B : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}_{P_1}(\mathbb{D}, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}_{P_2}(\mathbb{B}, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\gamma+\frac{n}{2}-\mu}_{P_1'}(\mathbb{D}, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma+\frac{n-1}{2}-\mu}_{P_2'}(\mathbb{B}, W_2) \end{array}$$

are continuous for all $s > d - 1/2$.
Note: Since $\Theta = (-N, 0]$ is a finite weight interval, $\pi_{\mathbb{C}} P_1$ and $\pi_{\mathbb{C}} P_2$ are finite sets in the strip $\{1/2 - \gamma - N < \operatorname{Re} z < 1/2 - \gamma\}$; $\pi_{\mathbb{C}} P_1'$ and $\pi_{\mathbb{C}} P_2$ are finite sets in the strip $\{1/2 + \mu - \gamma - N < \operatorname{Re} z < 1/2 + \mu - \gamma\}$, cf. 1.5.1.

1.9.4 Lemma. *Let $A \in C^{\nu,d}_{M+G}(X^\wedge, \mathbf{g})$ be as above. Given $\alpha, \beta \geq 0$ with $\alpha + \beta \geq N$ we will have*

$$t^\alpha A\, t^\beta \in C^d_G(X^\wedge, \mathbf{g})_{P',Q'}$$

with resulting asymptotic types P' and Q'. In particular, $C^{\nu,d}_{M+G}(X^\wedge, \mathbf{g}) \subset C^d_G(X^\wedge, \mathbf{g})$ for $\mu - \nu \geq N$. Recall that $\Theta = (-N, 0]$.

1.9.5 Definition. Let $A = A_M + A_G \in C^{\nu,d}_{M+G}(X^\wedge, \mathbf{g})$ be as in Definition 1.9.1. Define

$$\sigma_M^{\nu-j}(A) = h_j, \quad j = 0, \ldots, N - (\mu - \nu) - 1,$$

and call $\sigma_M^{\nu-j}(A)$ the conormal symbol of order $\nu - j$ of A.
Note that for $j \geq N - (\mu - \nu)$, the operators $\omega_j\, t^{-\nu+j} \operatorname{op}_M^{\gamma_j}(a_j)\, \tilde{\omega}_j$ are necessarily Green operators.

1.9.6 Remark. We know from Proposition 1.8.4 that two operators in $C_{M+G}^{\nu,d}(X^\wedge,\mathbf{g})$ which have the same conormal symbols of all order differ only by a Green operator, provided the weights γ_j are suitably chosen.
Vice versa, the conormal symbols $\sigma_M^{\nu-j}(A), j=0,\ldots,N-(\mu-\nu)-1$, are also well-defined. This follows from the proposition, below, which is of independent interest.

1.9.7 Proposition. *The operator A in 1.9.3 is a Green operator, if and only if $\sigma_M^{\nu-j}(A)=0, j=0,\ldots,N-(\mu-\nu)-1$.*

1.9.8 Theorem. *Let $A\in C_{M+G}^{\nu,0}(X^\wedge,\mathbf{g}), \mathbf{g}=(\gamma+n/2,\gamma+n/2-\mu,\Theta)$. Then the formal adjoint A^* of A belongs to $C_{M+G}^{\nu,0}(X^\wedge,\mathbf{h}), \mathbf{h}=(-\gamma-n/2+\mu,-\gamma-n/2,\Theta)$.*

1.9.9 Theorem. *Let $A\in C_{M+G}^{\nu,d}(X^\wedge,\mathbf{g}), H\in C_G^{d'}(X^\wedge,\mathbf{h})_{Q,R}, K\in C_G^{d'}(X^\wedge,\mathbf{k})_{S,T}$, where $\mathbf{h}=(\gamma+n/2-\mu,\delta,\Theta), \mathbf{k}=(\delta,\gamma+n/2,\Theta)$, and Q,R,S,T are corresponding asymptotic types. Then*

$$HA\in C_G^{d}(X^\wedge,\mathbf{h}_1)_{Q,\tilde R} \tag{1}$$

$$AK\in C_G^{d'}(X^\wedge,\mathbf{k}_1)_{\tilde Q,R} \tag{2}$$

with $\mathbf{h}_1=(\gamma+n/2,\delta,\Theta), \mathbf{k}_1=(\delta,\gamma+n/2-\mu,\Theta)$ and resulting asymptotic types $\tilde Q,\tilde R$.

1.9.10 Theorem. *Let $A\in C_{M+G}^{\nu,d}(X^\wedge,\mathbf{g})$ and $B\in C_{M+G}^{\nu',d'}(X^\wedge,\mathbf{h})$ with $\mathbf{h}=(\gamma+n/2+\mu',\gamma+n/2,\Theta)$ and $\mathbf{g}=(\gamma+n/2,\gamma+n/2-\mu,\Theta)$. Then $AB\in C_{M+G}^{\nu+\nu',d'}(X^\wedge,\mathbf{k})$ with $\mathbf{k}=(\gamma+n/2+\mu',\gamma+n/2-\mu,\Theta)$. The conormal symbols satisfy the relations*

$$\sigma_M^{\nu+\nu'-r}(AB)=\sum_{p+q=r}\left[T^{\nu'-q}\,\sigma_M^{\nu-p}(A)\right]\,\sigma_M^{\nu'-q}(B).$$

1.9.11 Lemma. *Let P be a Mellin asymptotic type, $d\in\mathbf{N}$, and $h\in M_P^{-\infty,d}(X)$. Then $I+h(z)\in\mathcal{B}^{-\infty,d}(X)$ is an invertible operator on $H^s(X,V_1)\oplus H^s(Y,W_1), s>d-1/2$, for all but countably many $z\in\mathbf{C}$. Moreover, there is a Mellin asymptotic type Q and an $f\in M_Q^{-\infty,d}(X)$ such that*

$$[I+h(z)]^{-1}=I+f(z).$$

2 The Cone Algebra without Asymptotics

2.1 General Mellin Symbols with Values in Boutet de Monvel's Algebra

In Section 4 of Part I we introduced Mellin symbols with asymptotics; they are meromorphic functions on $\mathbf{C}$ with values in Boutet de Monvel's algebra. For the definition of the Mellin operator $\mathrm{op}_M^\gamma a$ associated with the Mellin symbol a, we only need to know a on the line $\Gamma_{1/2-\gamma}$, and we certainly do not need its analyticity. We shall extend the calculus to even larger classes of Mellin symbols by considering the case where the symbols additionally depend on the space variables t and t' – comparable to studying pseudodifferential 'double' symbols $p(x,y,\xi)$ after having treated Fourier multipliers $p(\xi)$.

2.1.1 Notation. In the following let $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$ be fixed. Given $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$ we shall write $f = f(t, t', z)$, where z indicates the variable in $\Gamma_{1/2-\gamma}$. For t, t', z fixed, $f(t, t', z)$ is a boundary value problem in Boutet de Monvel's calculus, so it acts on sections of vector bundles over $\overline{X}$ and Y. In order to fix the notation, assume that

$$f(t,t',z): \begin{array}{c} C^\infty(\overline{X}, V_1) \\ \oplus \\ C^\infty(Y, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{array} \tag{1}$$

with smooth vector bundles V_1, V_2, over $\overline{X}$ and W_1, W_2, over Y.

2.1.2 Definition. Let $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$. For $u \in C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1) = C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$ let

$$[\mathrm{op}_M^\gamma f]u(t) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} \int_0^\infty (t/t')^{-z} f(t,t',z) u(t') \frac{dt'}{t'} dz. \tag{1}$$

The right hand side of (1) is to be understood as an iterated integral. If f is independent of t' or, equivalently, $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$, then (1) reduces to

$$[\mathrm{op}_M^\gamma f]u(t) = \frac{1}{2\pi i} \int_{\Gamma_{1/2-\gamma}} t^{-z} f(t,z) [M_\gamma u](z) dz. \tag{2}$$

We did not specify the variable x in (1) or (2), understanding that, for fixed $t', u(t') = u(\cdot, t')$ is in $C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1)$ and that $f(t, t', z)$ acts as an operator in Boutet de Monvel's calculus with respect to the x-variables.
Like pseudodifferential double symbols, Mellin double symbols are not uniquely determined. It is immediate from integration by parts in (1) that

$$\mathrm{op}_M^{1/2}[\ln^k(t/t') f(t,t',z)] = \mathrm{op}_M^{1/2}[\partial_z^k f(t,t',z)]. \tag{3}$$

For $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$ or $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$ we will have a continuous map

$$\mathrm{op}_M^\gamma f: \begin{array}{c} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \to \begin{array}{c} C^\infty(\overline{X}^\wedge, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2) \end{array}. \tag{4}$$

Smoothness of f up to zero yields continuity of $\mathrm{op}_M^\gamma f$ on the weighted Mellin-Sobolev spaces, cf. Theorem 2.1.3; the preceding relation (3), however, shows that smoothness is not necessary.

2.1.3 Theorem. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma})), \omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$. For $s > d - 1/2$, there is a bounded extension*

$$\omega_1[\mathrm{op}_M^\gamma f]\omega_2: \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}. \tag{1}$$

For $d = 0$ and $s \le -1/2$

$$\omega_1[\mathrm{op}_M^\gamma f]\omega_2 : \begin{array}{c} \mathcal{H}_0^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}_{\{0\}}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array} \tag{2}$$

is continuous. Here, the subscript $\{0\}$ indicates that we are using the $\mathcal{H}_0$-spaces for $s-\mu \le 0$, and the usual $\mathcal{H}$-spaces otherwise.

For the proof of Theorem 2.1.3 we need the following two results.

2.1.4 Lemma. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$ as in 2.1.3. Then there are functions $\varphi_j \in C^\infty(\overline{\mathbf{R}}_+)$, $\psi_j \in C^\infty(\overline{\mathbf{R}}_+)$, $j = 1,2,\ldots$, tending to zero in the topology of $C^\infty(\overline{\mathbf{R}}_+)$, elements $a_j \in \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}), j = 1,2\ldots$, tending to zero in the corresponding topology, and a sequence $\{\lambda_j\} \in l^1$ such that*

$$f(t,t',z) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(t)\psi_j(t')a_j(z) \tag{1}$$

with convergence in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$. Conversely, each series of this type defines an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$.
Similarly, if $g \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$, then there are null sequences $\varphi_j \in C^\infty(\overline{\mathbf{R}}_+)$ and $a_j \in \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$ and a sequence $\{\lambda_j\} \in l^1$ such that

$$g(t,z) = \sum_{j=1}^{\infty} \lambda_j\, \varphi_j(t)\, a_j(z).$$

Again, all series of this form determine elements in $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$.
The same results hold with $\mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$ replaced by $\mathcal{B}^{\mu,d}_{cl}(X;\Gamma_{1/2-\gamma})$.

Proof. By definition

$$C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})) = C^\infty(\mathbf{R} \times \mathbf{R}, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))\,|_{\mathbf{R}_+\times\mathbf{R}_+}\,.$$

In view of the nuclearity of $C^\infty(\mathbf{R})$ we have

$$C^\infty(\mathbf{R} \times \mathbf{R}, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})) = [C^\infty(\mathbf{R})\hat{\otimes}_\pi C^\infty(\mathbf{R})]\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}).$$

Representation (1) then is immediate from the representation of elements in π-tensor products of vector spaces, cf. Treves [51]. ◁

2.1.5 Proposition. *Let $a \in \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$. Then*

$$\mathrm{op}_M^\gamma a : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is bounded for each $s > d - 1/2$. For $d = 0$ and $s \le -1/2$,

$$\mathrm{op}_M^\gamma a : \begin{array}{c} \mathcal{H}_0^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}_{\{0\}}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is continuous. As before, the subscript $\{0\}$ indicates that we are using the $\mathcal{H}_0$-spaces for $s-\mu \le 0$, and the usual $\mathcal{H}$-spaces otherwise.

Proof. For both statements we may use the proof of Theorem I.4.1.11. For the sake of completeness let us repeat the argument. For simplicity we will assume that the vector bundles over X are trivial one-dimensional while those over Y vanish.
Write $a(1/2-\gamma+i\tau)=\sum_{j=0}^{d} a_j(\tau)\partial_r^j+\sum_{j=0}^{d} r_j(\tau)\partial_r^j$, where the a_j are local terms, given by symbols of order $\mu-j$ and type zero, while each $r_j(\tau)$ is an integral operator whose kernel is rapidly decreasing in τ and smooth in the space variables up to the boundary of X.
The normal derivative ∂_r maps $\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge)$ to $\mathcal{H}^{s-1,\gamma+\frac{n}{2}}(X^\wedge)$ for $s>1/2$. Moreover, the integral operators induced by the r_j are continuous on both the spaces $\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge), s>-1/2$ and $\mathcal{H}_0^{s,\gamma+\frac{n}{2}}(X^\wedge), s\le -1/2$.
So we can focus on the first sum and assume that we are dealing with a single parameter-dependent operator $a=a(\tau)$ of order μ and type zero in Boutet de Monvel's algebra on $\mathbf{R}^n_+$, supported by a compact set, uniformly in τ.
Now we reduce the problem to a continuity result for operator-valued pseudodifferential operators: We know from 1.4.4 that $M_\gamma\mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbf{R}^n_+\times\mathbf{R}_+)=\mathcal{F}_{n+1}H^s(\mathbf{R}^n_+\times\mathbf{R})$ and $M_\gamma\mathcal{H}_0^{s,\gamma+\frac{n}{2}}(\mathbf{R}^n_+\times\mathbf{R}_+)=\mathcal{F}_{n+1}H_0^s(\mathbf{R}^n_+\times\mathbf{R})$, identifying $\Gamma_{1/2-\gamma}$ and $\mathbf{R}$. $\mathcal{F}_{n+1}$ denotes the Fourier transform with respect to the last variable. Applying additionally the Fourier transform with respect to the first $n-1$ variables, $\mathcal{F}'$, the space $\mathcal{F}_{n+1}H^s(\mathbf{R}^n_+\times\mathbf{R})$ is mapped to $\mathcal{W}^s(\mathbf{R}^{n-1}\times\mathbf{R},H^s(\mathbf{R}_+))$ and $\mathcal{F}_{n+1}H_0^s(\mathbf{R}^n_+\times\mathbf{R})$ is mapped to $\mathcal{W}^s(\mathbf{R}^{n-1}\times\mathbf{R},H_0^s(\mathbf{R}_+))$.
Hence, for $s>-1/2$,

$$\mathrm{op}_M^\gamma a=M_\gamma^{-1}\mathcal{F}'^{-1}\,\sigma_\wedge(a)\,\mathcal{F}'M_\gamma:\mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbf{R}^n_+\times\mathbf{R}_+)\to\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(\mathbf{R}^n_+\times\mathbf{R}_+)$$

is continuous if and only if

$$\begin{aligned}&\mathrm{op}^+a\\&=(\mathcal{F}'\mathcal{F}_{n+1})^{-1}\,\sigma_\wedge(a)\,(\mathcal{F}'\mathcal{F}_{n+1}):\mathcal{W}^s(\mathbf{R}^n\times\mathbf{R},H^s(\mathbf{R}_+))\to\mathcal{W}^{s-\mu}(\mathbf{R}^n\times\mathbf{R},H^{s-\mu}(\mathbf{R}_+))\end{aligned}$$

is bounded. The latter fact, however, is given in I.2.2.19.
For $s\le -1/2$, the continuity of

$$\mathrm{op}_M^\gamma a=M_\gamma^{-1}\mathcal{F}'^{-1}\,\sigma_\wedge(a)\,\mathcal{F}'M_\gamma:\mathcal{H}_0^{s,\gamma+\frac{n}{2}}(\mathbf{R}^n_+\times\mathbf{R}_+)\to\mathcal{H}_0^{s-\mu,\gamma+\frac{n}{2}}(\mathbf{R}^n_+\times\mathbf{R}_+)$$

is equivalent to that of

$$(\mathcal{F}'\mathcal{F}_{n+1})^{-1}\,\sigma_\wedge(a)\,(\mathcal{F}'\mathcal{F}_{n+1}):\mathcal{W}^s(\mathbf{R}^n\times\mathbf{R},H_0^s(\mathbf{R}_+))\to\mathcal{W}^{s-\mu}(\mathbf{R}^n\times\mathbf{R},H_{\{0\}}^{s-\mu}(\mathbf{R}_+)).$$

Again, I.2.2.19 gives the desired result. Notice that we can omit the subscripts *comp* and *loc*, for $a(\tau)$ is compactly supported. ◁

Proof of Theorem 2.1.3. We first notice that

$$\omega_1[\mathrm{op}_M^\gamma f]\omega_2=\mathrm{op}_M^\gamma g$$

with $g(t,t',z)=\omega_1(t)\omega_2(t')f(t,t',z)$. Using the representation 2.1.4(1) it is sufficient to show the following two facts:

(i) Multiplication by a function $\varphi\in C_0^\infty(\overline{\mathbf{R}}_+)$ is a bounded operator on $\mathcal{H}^{s,\gamma}(X^\wedge,V)$ and $\mathcal{H}^{s,\gamma}(Y^\wedge,W)$ for all s and γ and for arbitrary vector bundles V over X, W over Y; the corresponding operator norms depend continuously on the semi-norms of φ in $C^\infty(\overline{\mathbf{R}}_+)$, keeping the support in a fixed compact set.

(ii) For a (t,t')-independent Mellin symbol $a \in \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$, the operator $\mathrm{op}_M^\gamma a$ has the required mapping properties.

Fact (i) is stated and proven in 1.4.3(h), while (ii) is precisely the statement of Proposition 2.1.5. ◁

We now introduce the operator classes with respect to this calculus.

2.1.6 Definition. Let $\gamma \in \mathbf{R}$ be fixed.

(a) $M\mathcal{B}_\gamma^{-\infty,0}(X^\wedge)$ is the set of all linear mappings

$$G \in \mathcal{L}(C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1), C^\infty(X^\wedge, V_2) \oplus C^\infty(Y^\wedge, W_2)) \tag{1}$$

such that, for all $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ and all $s \in \mathbf{R}$,

$$\omega_1 G \omega_2 : \begin{array}{c} \mathcal{H}_{\{0\}}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{\infty,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array} \tag{2}$$

is continuous. The subscript $\{0\}$ indicates that we use the $\mathcal{H}_0$-spaces for $s \le -1/2$, the usual $\mathcal{H}$-spaces otherwise.

(b) For $d \in \mathbf{N}$, $M\mathcal{B}_\gamma^{-\infty,d}(X^\wedge)$ is the space of all operators of the form

$$G = \sum_{j=0}^{d} G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix}$$

with $G_j \in M\mathcal{B}_\gamma^{-\infty,0}(X^\wedge)$. Here ∂_r denotes the normal derivative with respect to $\partial X = Y$, and the matrix refers to the vector bundles the operator acts on, cf. (2).

(c) For $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$, $M\mathcal{B}_\gamma^{\mu,d}(X^\wedge)$ is the space of all operators of the form $\mathrm{op}_M^\gamma f + G$, where $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$ and $G \in M\mathcal{B}_\gamma^{-\infty,d}(X^\wedge)$.

2.1.7 Remark. Let G^* denote the adjoint operator to $G \in M\mathcal{B}_\gamma^{-\infty,0}(X^\wedge)$, taken with respect to the sesquilinear pairing associated with 2.1.6(2), for details cf. 2.3.2, below. The relations $\mathcal{H}_0^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1)' = \mathcal{H}^{-s,-\gamma-\frac{n}{2}}(X^\wedge, V_1)$ and $\mathcal{H}^{\infty,\gamma+\frac{n}{2}}(X^\wedge, V_1) = \mathrm{proj}-\lim_{s\to\infty} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1)$ together with the corresponding results for the spaces over Y imply the following mapping properties:

$$\omega_1 G^* \omega_2 : \begin{array}{c} \mathcal{H}_{\{0\}}^{s,-\gamma-\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,-\gamma-\frac{n+1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{\infty,-\gamma-\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{\infty,-\gamma-\frac{n+1}{2}}(Y^\wedge, W_2) \end{array}.$$

The following lemma and Corollary 2.1.9, below, show that the definitions in 2.1.6(b) and (c) are consistent:

2.1.8 Lemma. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$. Then f can be written*

$$f(t,t',z) = \sum_{j=0}^{d} f_j(t,t',z) \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix}, \tag{1}$$

where $f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-j,0}(X;\Gamma_{1/2-\gamma}))$ and the matrix on the right hand side of (1) refers to the vector bundles $f(t,t',z)$ is acting on, cf. 2.1.1(1).

Proof. This follows from the decomposition 2.1.4(1): Each of the operators $a_j \in \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$ can be written

$$a_j = \sum_{k=0}^{d} a_{jk} \begin{bmatrix} \partial_r^k & 0 \\ 0 & I \end{bmatrix},$$

with $a_{jk} \in \mathcal{B}^{\mu-k,0}(X;\Gamma_{1/2-\gamma})$. We can then rearrange the summation, since it is absolutely convergent in all semi-norms. ◁

2.1.9 Corollary. For $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_{1/2-\gamma}))$ the operator $\mathrm{op}_M^\gamma f$ is an element of $M\mathcal{B}_\gamma^{-\infty,d}(X^\wedge)$.

Proof. This is immediate from 2.1.8 and 2.1.5. ◁

2.1.10 Remark. Note that there are symbols $f \notin C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,0}(X;\Gamma_{1/2-\gamma}))$ such that still $\mathrm{op}_M^\gamma f \in M\mathcal{B}_\gamma^{-\infty,0}(X^\wedge)$: Choose an arbitrary parameter-dependent operator $0 \neq a \in \mathcal{B}^{-\infty,0}(X;\Gamma_{1/2-\gamma})$. Then the symbol f defined by $f(t,z) = t^{1/2}a(z)$ is an element of $C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,0}(X;\Gamma_{1/2-\gamma})) \setminus C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,0}(X;\Gamma_{1/2-\gamma}))$. On the other hand, the fact that $\mathrm{op}_M^\gamma a$ belongs to $M\mathcal{B}_\gamma^{-\infty,0}(X^\wedge)$ by 2.1.9 together with Lemma 2.1.11, below, implies that $\mathrm{op}_M^\gamma f \in M\mathcal{B}_\gamma^{-\infty,0}(X^\wedge)$.

2.1.11 Lemma. *Let $\varphi \in L^\infty(\mathbf{R}_+)$ and suppose that for all $j \in \mathbf{N}$ we have*

$$(t\partial_t)^j \varphi \in L^\infty(\mathbf{R}_+). \tag{1}$$

Then the operator M_φ of multiplication by φ

$$M_\varphi : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge), \text{ and}$$
$$M_\varphi : \mathcal{H}_0^{s,\gamma}(X^\wedge) \to \mathcal{H}_0^{s,\gamma}(X^\wedge)$$

is bounded for all $s, \gamma \in \mathbf{R}$.

Proof. By interpolation and duality we may assume that $s \in \mathbf{N}$. Then $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the space of all functions $u = u(x,t)$ on $X^\wedge$ such that $t^{\frac{n}{2}-\gamma}(t\partial_t)^j D_x u(x,t) \in L^2(X^\wedge)$ whenever $j \le s$ and D is a differential operator of order $\le s-j$. Now (1) together with Leibniz' rule implies the assertion. Since multiplication by φ does not increase the support, the argument for $\mathcal{H}_0^{s,\gamma}(X^\wedge)$ is the same. ◁

2.1.12 Theorem. (Asymptotic Summation) *Let $d \in \mathbf{N}$ be fixed, $\mu_1, \mu_2, \ldots$ a sequence in $\mathbf{Z}$ tending to $-\infty$, $f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu_j,d}(X; \Gamma_{1/2-\gamma}))$, and $\mu = \max \mu_j$. Then there is an*

$$f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$$

such that for any $N \in \mathbf{N}$ there is a J with

$$f - \sum_{j=1}^{J} f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N,d}(X; \Gamma_{1/2-\gamma})). \tag{1}$$

This f is unique modulo $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X; \Gamma_{1/2-\gamma}))$. We shall write $f \sim \sum_{j=0}^{\infty} f_j$. The same result is true with $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$ replaced by $\overline{\mathbf{R}}_+, \mathbf{R}_+ \times \mathbf{R}_+$, or $\mathbf{R}_+$.

Proof. Choose a partition of unity $\{\varphi_k : k = 1, ..., K\}$ and cut-off functions ψ_k on X, subordinate to the coordinate neighborhoods, satisfying $\varphi_k \psi_k = \varphi_k$. Let $\Phi_k \in \mathcal{B}^{0,0}(X)$ denote the operator of multiplication by $\begin{bmatrix} \varphi_k & 0 \\ 0 & \varphi_k|_Y \end{bmatrix}$, similarly for Ψ_k. For $j = 1, 2, \ldots$ consider the (operator-valued) symbols $\Phi_k f_j(t,t',z)\Psi_k$. We have $f_j(t,t',z) - \sum_{k=1}^{K} \Phi_k f_j(t,t',z)\Psi_k \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+; \mathcal{B}^{-\infty,d}(X; \Gamma_{1/2-\gamma}))$ while $\Phi_k f_j(t,t',z)\Psi_k$ is given by a quintuple of symbols in the respective classes.
Now we appeal to the theorems on asymptotic summation in these classes, cf. [30, Section 2.2.5.1 Proposition 3]. In fact, the present situation differs from the case treated in [30] in two respects: (i) we have the parameter $z \in \Gamma_{1/2-\gamma}$ and (ii) we have the additional variables t and t' in which everything is smooth on $\overline{\mathbf{R}}_+$. Inspection of the classical summation procedure, however, shows that neither (i) nor (ii) causes any difficulty: The variable z enters like an additional covariable while t and t' enter like additional space variables, so that the same procedure can be applied. ◁

2.2 The Kernels of Mellin Symbols

2.2.1 Mellin Operators and Kernels. As the exposition in Section 2.1 shows, the theory of general Mellin symbols does not depend on the particular choice of the line $\Gamma_{1/2-\gamma}$. Throughout this section we will therefore assume that $\gamma = 1/2$ and consider the line Γ_0.
We will be interested in the (operator-valued) kernels of Mellin operators with symbols $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X; \Gamma_0))$. As before, $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$ are fixed. According to 2.1.2 we have for $u \in C_0^\infty(X^\wedge)$

$$[\mathrm{op}_M^{1/2} f]u(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^{\infty} \left(\frac{t}{t'}\right)^{-i\tau} f(t,t',i\tau)u(t') \frac{dt'}{t'} d\tau. \tag{1}$$

So, $\mathrm{op}_M^{1/2} f$ is the integral operator with the distributional (operator-valued) kernel

$$k(t,t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{t}{t'}\right)^{-i\tau} f(t,t',i\tau) d\tau = [M_{1/2}^{-1} f(t,t',\cdot)](\frac{t}{t'}) \, . \tag{2}$$

with respect to the density $\frac{dt}{t}$ on $\mathbf{R}_+$. The integral in (2) is to be understood as a distributional integral.

Occasionally we shall need the following consideration.

2.2.2 Lemma. *Let* $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$, *and let* $\phi \in C_0^\infty(\mathbf{R}_+)$. *Then*

$$\mathrm{op}_M^{1/2}[\phi(t/t')f(t,t',z)] = \mathrm{op}_M^{1/2}[M_{1/2,\rho\to z}\{\phi(\rho)M_{1/2,\zeta\to\rho}^{-1}f(t,t',\zeta)\}]. \tag{1}$$

For the moment we assume the expression on the right hand side makes sense. This fact will be shown later in Theorem 2.2.17.

Proof. By 2.2.1, the integral kernel of the operator on the right hand side is

$$\tilde{k}(t,t') = [\phi(\rho)M_{1/2,\zeta\to\rho}^{-1}f(t,t',\zeta)]_{\rho=t/t'} = \phi(t/t')[M_{1/2,\zeta\to\rho}^{-1}f(t,t',\zeta)]_{\rho=t/t'}.$$

Again by 2.2.1, the last expression is the integral kernel of the operator on the left hand side of (1). Hence both operators coincide. ◁

2.2.3 Definition and Remark. $\mathcal{S}(\Gamma_0, \mathcal{B}^{-\infty,d}(X))$ denotes the Schwartz space of all rapidly decreasing functions on Γ_0 with values in $\mathcal{B}^{-\infty,d}(X)$. It coincides with $\mathcal{B}^{-\infty,d}(X;\Gamma_0)$. To these functions we apply the inverse Mellin transform $M_{1/2}^{-1}$ and call $\mathcal{T}$ the resulting space:

$$\mathcal{T} = M_{1/2}^{-1}[\mathcal{S}(\Gamma_0, \mathcal{B}^{-\infty,d}(X))]\ .$$

What is the topology on $\mathcal{T}$? The nuclearity of $\mathcal{S}(\Gamma_0)$ implies that

$$\mathcal{T} = M_{1/2}^{-1}[\mathcal{S}(\Gamma_0)\hat{\otimes}_\pi\mathcal{B}^{-\infty,d}(X)] = M_{1/2}^{-1}\mathcal{S}(\Gamma_0)\hat{\otimes}_\pi\mathcal{B}^{-\infty,d}(X).$$

So the natural topology on $\mathcal{T}$ is the projective limit topology induced via the topologies on $\mathcal{B}^{-\infty,d}(X)$ and on $M_{1/2}^{-1}\mathcal{S}(\Gamma_0)$. The latter in turn simply is the topology carried over from $\mathcal{S}(\Gamma_0)$ by the isomorphism $M_{1/2}^{-1}$. More precisely: the semi-norm system

$$p_{MN}(g) = \left\{\int_{-\infty}^{\infty} |\ D_\tau^M \tau^N g(i\tau)\ |^2\ d\tau\right\}^{1/2}, \tag{1}$$

$M, N \in \mathbf{N}$, on $\mathcal{S}(\Gamma_0)$ induces the system

$$q_{MN}(h) = \left\{\int_0^{\infty} |\ \ln^M \rho\,(\rho\partial_\rho)^N h(\rho)\ |^2\ \frac{d\rho}{\rho}\right\}^{1/2} \tag{2}$$

on $M_{1/2}^{-1}\mathcal{S}(\Gamma_0)$. Here we have employed the fact that

$$M_{\rho\to z}(\ln^M \rho\,(-\rho\partial_\rho)^N h) = \left(\frac{d}{dz}\right)^M z^N(Mh)(z)\ . \tag{3}$$

Summing up we have the following: Let p_j be a semi-norm system for the topology of $\mathcal{B}^{-\infty,d}(X)$. Then the topology on $\mathcal{T}$ is given by the system $\{r_{MNj} : M, N, j \in \mathbf{N}\}$ defined by

$$r_{MNj}(k) = \left\{\int_0^{\infty} p_j(\ln^M \rho(\rho\partial_\rho)^N k(\rho))^2 \frac{d\rho}{\rho}\right\}^{1/2}. \tag{4}$$

2.2.4 Theorem. *Choose* $\psi \in C_0^\infty(\mathbf{R}_+)$ *with* $\psi \equiv 1$ *near* 1, $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ *and let* $k(t,t',\rho) = [M^{-1}_{1/2,z\to\rho} f(t,t',z)](\rho)$. *Then*

$$(1-\psi(\rho))k(t,t',\rho) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{T}) \; . \tag{1}$$

In particular, the singular support of $k(t,t',\cdot)$ *is contained in* $\{\rho = 1\}$ *for all* $t,t' \in \overline{\mathbf{R}}_+$. *Moreover, the mapping* $(\psi, f) \mapsto (1-\psi)k$ *induces a separately continuous operator*

$$C_0^\infty(\mathbf{R}_+) \times C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0)) \to C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{T}). \tag{2}$$

Clearly, the corresponding result holds with $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$ *replaced by* $\overline{\mathbf{R}}_+$.

Proof. Using that $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0)) = C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+) \hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\Gamma_0)$ and $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{T}) = C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+) \hat{\otimes}_\pi \mathcal{T}$ we only have to show the following:
For $a \in \mathcal{B}^{\mu,d}(X;\Gamma_0)$ and $k = M^{-1}_{1/2} a$, we have $(1-\psi)k \in \mathcal{T}$, and the mapping $a \mapsto (1-\psi)k$ induces a continuous operator from $\mathcal{B}^{\mu,d}(X;\Gamma_0)$ to $\mathcal{T}$.
In order to see this let us first check that, for each fixed ρ, the operator $(1-\psi(\rho))k(\rho)$ belongs to $\mathcal{B}^{-\infty,d}(X)$: $1-\psi(\rho)$ vanishes near $\rho = 1$, thus, for each $L \in \mathbf{N}$, the function $\ln^{-L}\rho\,(1-\psi(\rho))$ is smooth on $\mathbf{R}_+$, and we can write

$$\begin{aligned}
(1-\psi(\rho))k(\rho) &= \frac{1}{2\pi}\int (1-\psi(\rho))\rho^{-i\tau} a(i\tau)d\tau \\
&= (1-\psi(\rho))\,\ln^{-L}\rho\,\frac{1}{2\pi}\int (i\partial_\tau)^L \rho^{-i\tau} a(i\tau)d\tau \\
&= (1-\psi(\rho))\,\ln^{-L}\rho\,\frac{1}{2\pi}\int \rho^{-i\tau}(-i\partial_\tau)^L a(i\tau)d\tau
\end{aligned}$$

after integration by parts. Since $a \in \mathcal{B}^{\mu,d}(X;\Gamma_0)$ we conclude that, for fixed ρ, $(1-\psi(\rho))k(\rho) \in \mathcal{B}^{\mu-L,d}(X)$, hence in $\mathcal{B}^{-\infty,d}(X)$.
In a similar way we will now show that the $\mathcal{T}$-semi-norms for $(1-\psi)k$ are finite. Letting $\psi_j(\rho) := (\rho\partial_\rho)^j(1-\psi(\rho))$, we obtain that $\psi_j \in C_b^\infty(\mathbf{R}_+)$ for $j \in \mathbf{N}$. Moreover, we have for arbitrary $L, M, N \in \mathbf{N}$

$$\ln^M\rho\,(\rho\partial_\rho)^N[(1-\psi(\rho))k(\rho)] = \frac{1}{2\pi}\int \ln^M\rho\,(\rho\partial_\rho)^N[\rho^{-i\tau}(1-\psi(\rho))\ln^{-L}\rho]\,(\partial_\tau^L a)(i\tau)d\tau.$$

Leibniz' rule shows that the integral is a linear combination of terms of the form

$$\ln^{M-L-j_3}\rho\;\psi_{j_2}(\rho)\int_{-\infty}^{\infty}\rho^{-i\tau}\tau^{j_1}(\partial_\tau^L a)(i\tau)d\tau, \tag{3}$$

where $j_1 + j_2 + j_3 = N$. We may now choose a semi-norm system $\{p_j : j = 1,2,\ldots\}$ for $\mathcal{B}^{-\infty,d}(X)$ such that each p_j is a semi-norm on $\mathcal{B}^{\mu-j,d}(X)$. Fixing N, M and j, choose $L > M+N+j+2$. Then $M-L-j_3 < 0$; moreover $(1+\tau^2)\tau^{j_1}(\partial_\tau^L a)(i\tau) \in \mathcal{B}^{\mu-j,d}(X;\mathbf{R}_\tau)$, so that

$$p_j\left(\int_{-\infty}^{\infty}\rho^{-i\tau}(\tau^{j_1}\partial_\tau^L a(i\tau))\,d\tau\right) \le C$$

with a constant $C = C(L, j_1, j)$, independent of ρ. We conclude that, in the notation of 2.2.3(4), the semi-norm $r_{MNj}((1-\psi)k)$ can be estimated by finitely many expressions

$$\text{const.}\left\{\int_0^\infty \left|\ln^{M-L-j_3}\rho\,\psi_{j_2}(\rho)\right|^2 \frac{d\rho}{\rho}\right\}^{1/2} < \infty.$$

Thus all the semi-norms in 2.2.3(4) are finite, and they continuously depend on the semi-norms for a in $\mathcal{B}^{\mu,d}(X;\Gamma_0)$ and on the semi-norms for ψ in $\mathcal{D}_K$, $K \subset \mathbf{R}_+$ compact. Here, $\mathcal{D}_K$ denotes those elements in $C_0^\infty(\mathbf{R}_+)$ that have support in a fixed compact $K \subset \mathbf{R}_+$. For details see [31, Theorem 6.6]. This completes the proof. $\triangleleft$

2.2.5 Corollary. Let f, ψ, and k be as in Theorem 2.2.4 and define

$$h(t,t',z) = M_{1/2,\rho\to z}\{(1-\psi(\rho))k(t,t',\rho)\}.$$

Then $h \in \mathbf{C}^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$. Clearly, the same result holds with $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$ replaced by $\overline{\mathbf{R}}_+$.

Proof. This follows from the tensor product representation in Lemma 2.1.4 together with the fact that $M_{1/2}\mathcal{T} = \mathcal{S}(\Gamma_0, \mathcal{B}^{-\infty,d}(X)) = \mathcal{B}^{-\infty,d}(X;\Gamma_0)$. $\triangleleft$

2.2.6 Lemma. *Let $\varphi \in C_0^\infty(\mathbf{R}_+)$ and $a \in \mathcal{B}^{\mu,d}(X;\Gamma_0)$. Then $\varphi(\rho)[M^{-1}_{1/2,z\to\rho}a(z)](\rho)$ defines an element of $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X))$. The mapping*

$$C_0^\infty(\mathbf{R}_+) \times \mathcal{B}^{\mu,d}(X;\Gamma_0) \to \mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X))$$

given by $(\varphi, a) \mapsto \varphi(M^{-1}_{1/2}a)$ is separately continuous.
The Mellin transform of $\varphi(M^{-1}_{1/2}a)$ gives an element of $\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X))$, and the mapping

$$C_0^\infty(\mathbf{R}_+) \times \mathcal{B}^{\mu,d}(X;\Gamma_0) \to \mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X))$$

given by $(\varphi, a) \mapsto M(\varphi(M^{-1}_{1/2}a))$ is separately continuous.

Proof. By definition, $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X)) = \mathcal{L}(C^\infty(\mathbf{R}_+), \mathcal{B}^{\mu,d}(X))$ with the topology of bounded convergence. In order to show that $\varphi\, M^{-1}_{1/2}a \in \mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X))$ let $\psi \in C^\infty(\mathbf{R}_+)$ and denote by $\langle\ ,\ \rangle$ the duality induced by the inner product on $L^2(\mathbf{R}_+, \frac{d\rho}{\rho})$ via $\langle u, \psi\rangle = (u, \overline{\psi})_{L^2(\mathbf{R}_+, \frac{d\rho}{\rho})}$, $u, \psi \in C_0^\infty(\mathbf{R}_+)$. Then

$$\begin{aligned}
\langle \varphi M^{-1}_{1/2,\tau\to\rho}a, \psi\rangle &= \langle \varphi \int_{-\infty}^{\infty} \rho^{-i\tau}a(i\tau)d\tau, \psi\rangle \\
&= \langle \int_{-\infty}^{\infty} \rho^{-1-i\tau}a(i\tau)d\tau, \rho\varphi\psi\rangle \\
&= \langle (-\rho\partial_\rho)^N \int_{-\infty}^{\infty} \rho^{-1-i\tau}(1+i\tau)^{-N}a(i\tau)d\tau, \rho\varphi\psi\rangle \\
&= \int_0^\infty \int_{-\infty}^{\infty} \rho^{-i\tau}(1+i\tau)^{-N}a(i\tau)d\tau\, \rho^{-1}(\rho\partial_\rho)^N(\rho\varphi(\rho)\psi(\rho))\frac{d\rho}{\rho}. \qquad (1)
\end{aligned}$$

The last integral is an L^1-integral with values in $\mathcal{B}^{\mu,d}(X)$, provided N is sufficiently large. This follows from the fact that, for every semi-norm q on $\mathcal{B}^{\mu,d}(X)$, we have $q(a(i\tau)) = O(\langle\tau\rangle^\mu)$.

Moreover: if the semi-norms for ψ in $C^\infty(\mathbf{R}_+)$ tend to zero, then the last integral tends to zero in all semi-norms of $\mathcal{B}^{\mu,d}(X)$. So it indeed defines an element of $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X))$. Now let us show the continuity of the mapping $(\varphi, a) \mapsto \varphi(M^{-1}_{1/2}a)$ from $C_0^\infty(\mathbf{R}_+) \times \mathcal{B}^{\mu,d}(X;\Gamma_0)$ to $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X))$: As ψ varies over a bounded set in $C^\infty(\mathbf{R}_+)$, the integral in (1) can be estimated in terms of finitely many semi-norms for $a \in \mathcal{B}^{\mu,d}(X;\Gamma_0)$ and finitely many semi-norms for $\varphi \in \mathcal{D}_K, K \subset \mathbf{R}_+$ compact. Finally note that the Mellin transform yields a continuous map from $\mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X))$ to $\mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d})$. Indeed, this follows from the fact that

$$\begin{aligned} \mathcal{E}'(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X)) &= \mathcal{E}'(\mathbf{R}_+)\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X) \quad \text{and} \\ \mathcal{A}(\mathbf{C}, \mathcal{B}^{\mu,d}(X)) &= \mathcal{A}(\mathbf{C})\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X), \end{aligned}$$

together with the fact that the Mellin transform maps $\mathcal{E}'(\mathbf{R}_+)$ to $\mathcal{A}(\mathbf{C})$. The latter is well-known. It is easily seen in the following way. For $f \in \mathcal{E}'(\mathbf{R}_+)$ we have $Mf = \langle f, t^{-z}\rangle$. Now $t^{-z} \in C^\infty(\mathbf{R}_+, \mathcal{A}(\mathbf{C})) = C^\infty(\mathbf{R}_+)\hat{\otimes}_\pi \mathcal{A}(\mathbf{C})$, so pairing it with $f \in \mathcal{E}'(\mathbf{R}_+) = C^\infty(\mathbf{R}_+)'$ gives an element of $\mathcal{A}(\mathbf{C})$ in a continuous way. ◁

2.2.7 Proposition. *Let $a \in \mathcal{B}^{\mu,d}(X;\Gamma_0)$ and $\varphi \in C_0^\infty(\mathbf{R}_+)$. Then $M\varphi(M^{-1}_{1/2}a) \in M_O^{\mu,d}(X)$, and the induced mapping*

$$C_0^\infty(\mathbf{R}_+) \times \mathcal{B}^{\mu,d}(X;\Gamma_0) \to M_O^{\mu,d}(X)$$

is separately continuous.

The proof of Proposition 2.2.7 is rather lengthy. We shall give it in several steps stated as independent lemmas. The final conclusion will be obtained in 2.2.16. As a preparation we first recall what is the topology of $\mathcal{B}^{\mu,d}(X;\Gamma_0)$.

2.2.8 The topology of $\mathcal{B}^{\mu,d}(X;\Gamma_0)$. For simplicity we shall assume that the vector bundles V_1 and V_2 are trivial one-dimensional while W_1 and W_2 are 0. We choose on Ω a covering by coordinate neighborhoods, a partition of unity $\{\varphi_j : j = 1, \ldots, J\}$ and cut-off functions $\{\psi_j : j = 1, \ldots, J\}$ subordinate to this partition such that $\varphi_j\psi_j = \varphi_j$. By κ_j denote the corresponding coordinate maps. In a first step we write an operator $A \in \mathcal{B}^{\mu,d}(X;\Gamma_0)$ in the form

$$A(z) = \sum_{j=1}^{J} \varphi_j A(z)\psi_j + R(z).$$

In this representation, we have $R(z) = \sum_{l=0}^{d} R_l(z)\partial_r^l$ where each R_l is an integral operator with a kernel function $k_l(z, x, y) \in \mathcal{S}(\Gamma_{0,z}, C^\infty(X \times X))$, while

$$\varphi_j A(z)\psi_j = \kappa^*(\varphi_{j*}A_j(z)\psi_{j*});$$

here φ_{j*}, ψ_{j*} denote the functions $\varphi_j \circ \kappa_j^{-1}$ and $\psi_j \circ \kappa_j^{-1}$ on $\mathbf{R}^n$, $\kappa^*(\cdot)$ indicates the pull-back operators from Euclidean space to the manifold, and $A_j(z)$ is a suitable parameter-dependent operator on Euclidean space.
The topology on $\mathcal{B}^{\mu,d}(X;\Gamma_0)$ then is that of a non-direct sum of the topologies of $\mathcal{S}(\Gamma_0, C^\infty(X\times X))$ and those of the symbol spaces on relatively open subsets of $\overline{\mathbf{R}}^n_+$. Let us

recall what those are. Let U be an open subset of $\mathbf{R}^{n-1}$. An element $A \in \mathcal{B}^{\mu,d}(U\times\mathbf{R}_+;\Gamma_0)$ has the form

$$A(z) = [\operatorname{op} p(z)]_+ + \sum_{j=0}^{d} \operatorname{op}' g_j(z) \circ \partial^j_{x_n}. \tag{1}$$

We briefly recall the precise meaning of (1), cf. 1.3.3:

(i) $p \in S^{\mu}_{tr}(U\times\mathbf{R}, \mathbf{R}^n \times \mathbf{R}_z)$, $[\operatorname{op} p(z)]_+ = \mathrm{r}^+ \operatorname{op} p(z)\, \mathrm{e}^+$, with e^+ denoting the operator of extension (by zero) from $U\times\mathbf{R}_+$ to $U\times\mathbf{R}$ and r^+ that of restriction to $U\times\mathbf{R}_+$; op is the usual pseudodifferential action with respect to the variables in $\mathbf{R}^n$. Here, we identify $\mathbf{R}_z$ and Γ_0.

(ii) For $j = 0,\dots,d$, g_j is a parameter-dependent and operator-valued singular Green symbol in $S^{\mu-j}(U, \mathbf{R}^{n-1}\times\mathbf{R}_z; \mathcal{S}'(\mathbf{R}_+), \mathcal{S}(\mathbf{R}_+))$, cf. 1.2.2; op′ denotes the pseudodifferential action with respect to the variables in $\mathbf{R}^{n-1}$.

We therefore topologize the operators in $\mathcal{B}^{\mu,d}(U\times\mathbf{R}_+,\Gamma_0)$ as a non-direct sum, cf. 1.3.2, via the topologies on $S^{\mu}_{tr}(U\times\mathbf{R},\mathbf{R}^n\times\mathbf{R}_z)$ and $S^{\mu-j}(U,\mathbf{R}^{n-1}\times\mathbf{R}_z;\mathcal{S}(\mathbf{R}_+),\mathcal{S}(\mathbf{R}_+))$ for $j=0,\dots,d$. Note that the latter is the projective limit $\text{proj-lim}_{\sigma,\tau\to\infty} S^{\mu-j}(U,\mathbf{R}^{n-1}\times\mathbf{R}_z; H_0^{-\sigma,-\tau}(\mathbf{R}_+), H^{\sigma,\tau}(\mathbf{R}_+))$ with the usual weighted Sobolev spaces on $\mathbf{R}_+$, cf. 1.2.1.

2.2.9 Lemma. *Let $\mathcal{E},\mathcal{F}$ and $\mathcal{Y}$ be Fréchet spaces, and assume that $\mathcal{E}$ and $\mathcal{F}$ are embedded in a common vector space $\mathcal{X}$. Suppose $T:\mathcal{E}+\mathcal{F}\to\mathcal{Y}$ is a linear map, and the restrictions*

$$T:\mathcal{E}\to\mathcal{Y},\quad T:\mathcal{F}\to\mathcal{Y}$$

are continuous in the topologies of $\mathcal{E}$ and $\mathcal{F}$. Then

$$T:\mathcal{E}+\mathcal{F}\to\mathcal{Y}$$

is continuous in the topology of the non-direct sum (cf. 1.3.2 for the definition of non-direct sums of Fréchet spaces).

Proof. Let $\{p_1,p_2,\dots\}$, $\{q_1,q_2,\dots\}$, be increasing systems of semi-norms for $\mathcal{E}$ and $\mathcal{F}$ respectively. Denote the translation invariant metric in $\mathcal{Y}$ by d. Then a system of semi-norms for $\mathcal{E}+\mathcal{F}$ is given by $r_j(x) = \inf\{p_j(e)+q_j(f) : e+f=x\}$. So suppose $x_0\in\mathcal{E}+\mathcal{F}$ and $V\subseteq\mathcal{Y}$ is an ε-ball about Tx_0. Then there is a $j\in\mathbf{N}$ and a $\delta>0$ such that $d(Te,0)<\frac{\varepsilon}{2}$ and $d(Tf,0)<\frac{\varepsilon}{2}$, provided that $e\in\mathcal{E}$, $f\in\mathcal{F}$, $p_j(e)<\delta$ and $q_j(f)<\delta$. This implies that $Tx\in V$ for all x with $r_j(x-x_0)<\delta$: In this case we can find $e_1\in\mathcal{E}$, $f_1\in\mathcal{F}$ such that $e_1+f_1=x-x_0$ and $p_j(e_1)+q_j(f_1)<\delta$. Hence $d(Tx,Tx_0)=d(T(x-x_0),0)\le d(T(e_1),0)+d(T(f_1),0)<\varepsilon$. ◁

2.2.10 Outline. We saw in Lemma 2.2.6 that, for $\varphi\in C_0^\infty(\mathbf{R}_+)$ and $a\in\mathcal{B}^{\mu,d}(X;\Gamma_0)$, we have $M(\varphi M^{-1}_{1/2}a)\in\mathcal{A}(\mathbf{C},\mathcal{B}^{\mu,d}(X))$. In order to prove Proposition 2.2.7 we therefore only have to show that

$$M(\varphi M^{-1}_{1/2}a)\in\mathcal{B}^{\mu,d}(X;\Gamma_\beta),$$

for each line Γ_β, uniformly for β in compact intervals, plus the continuity of the corresponding mapping; indeed this is everything Definition 1.7.2(b) requires. Let us make the same simplification as in the proof of Proposition 2.2.8: We assume that the matrices in $\mathcal{B}^{\mu,d}(X;\Gamma_0)$ consist of the left upper corner only, i.e., the operators act on a trivial one-dimensional bundle over X only, the bundles over Y vanish. It will become clear that full matrices can be treated in the same way. Under this assumption Lemma 2.2.9 shows that it is sufficient to prove the following:

(1) For $s = s(z) \in \mathcal{S}(\Gamma_0, C^\infty(X \times X))$ the function $M(\varphi\, M_{1/2}^{-1} s)|_{\Gamma_\beta}$ is an element of $\mathcal{S}(\Gamma_\beta, C^\infty(X \times X))$, uniformly for β in compact intervals, and the corresponding mapping is continuous.

(2) For $p \in S^\mu_{tr}(U \times \mathbf{R}, \mathbf{R}^n \times \Gamma_0)$, the function $M(\varphi\, M_{1/2}^{-1} p)|_{\Gamma_\beta}$ is an element of $S^\mu_{tr}(U \times \mathbf{R}, \mathbf{R}^n \times \Gamma_\beta)$, uniformly for β in compact intervals, and the corresponding mapping is continuous.

(3) For $g \in S^\mu(U, \mathbf{R}^{n-1} \times \Gamma_0; E, F)$, the function $M(\varphi\, M_{1/2}^{-1} g)|_{\Gamma_\beta}$ is an element of $S^\mu(U, \mathbf{R}^{n-1} \times \Gamma_\beta; E, F)$, uniformly for β in compact intervals, and the corresponding mapping is continuous.

Here, as in Proposition 2.2.8, U denotes an open set in $\mathbf{R}^{n-1}$, while $E = H_0^{-\sigma,-\tau}(\mathbf{R}_+)$, $F = H^{\sigma,\tau}(\mathbf{R}_+)$ for arbitrary fixed $\sigma, \tau \geq 0$. Moreover, we can assume in both cases that the symbols vanish outside compact sets in $U \times \mathbf{R}$ and U, respectively.

We will now prove the statements (1), (2), and (3) of 2.2.10, starting with (1). The final conclusion will be reached in 2.2.16.

2.2.11 Lemma. *Let $\varphi \in C_0^\infty(\mathbf{R}_+)$, $h \in \mathcal{S}(\Gamma_0)$, $s \in \mathcal{S}(\Gamma_0, C^\infty(X \times X))$. Then*

(a) *$M_{1/2}^{-1} h \in C^\infty(\mathbf{R}_+)$. The mapping $h \mapsto M_{1/2}^{-1} h$ is continuous from $\mathcal{S}(\Gamma_0)$ to $C^\infty(\mathbf{R}_+)$.*

(b) *$H := M(\varphi M_{1/2}^{-1} h) \in \mathcal{A}(\mathbf{C})$. Moreover, $H|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for every β, with estimates uniformly in β for β in compact intervals. The corresponding induced mapping $(\varphi, h) \mapsto H$ from $C_0^\infty(\mathbf{R}_+) \times \mathcal{S}(\Gamma_0)$ into this subspace of $\mathcal{A}(\mathbf{C})$ is separately continuous.*

(c) *$F := M(\varphi M_{1/2}^{-1} s) \in \mathcal{A}(\mathbf{C}, C^\infty(X \times X))$, $F|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta, C^\infty(X \times X))$ for every β, with estimates uniformly in β for β in compact intervals. The mapping $(\varphi, s) \mapsto F$ is separately continuous from $C_0^\infty(\mathbf{R}_+) \times \mathcal{S}(\Gamma_0, C^\infty(X \times X))$ to this subspace of $\mathcal{A}(\mathbf{C}, C^\infty(X \times X))$.*

Proof. (a) By the Mellin inversion formula, cf. 1.4.1, $(M_{1/2}^{-1} h)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-is} h(is)\,ds$. The integral converges, and we can differentiate under the integral sign for the derivatives.
(b) In view of (a), $\varphi M_{1/2}^{-1} h$ is a function in $C_0^\infty(\mathbf{R}_+)$; its Mellin transform therefore is rapidly decreasing on each line Γ_β, uniformly in β, cf. Theorem I.5.1.7 or [20]. Clearly, the mapping $(\varphi, g) \mapsto \varphi g$ is separately continuous from $C_0^\infty(\mathbf{R}_+) \times C^\infty(\mathbf{R}_+)$ to $C_0^\infty(\mathbf{R}_+)$, and the Mellin transform is continuous from $C_0^\infty(\mathbf{R}_+)$ to the subspace of $\mathcal{A}(\mathbf{C})$ consisting of functions that restrict to $\mathcal{S}(\Gamma_\beta)$, uniformly for β in compact intervals, i.e. the space $M_O^{-\infty}$ for $\dim X = 0$. So the separate continuity follows from (a).
(c) follows from (b), noting that $\mathcal{S}(\Gamma_\beta, C^\infty(X \times X)) = \mathcal{S}(\Gamma_\beta) \hat{\otimes}_\pi C^\infty(X \times X)$ and $\mathcal{A}(\mathbf{C}, C^\infty(X \times X)) = \mathcal{A}(\mathbf{C}) \hat{\otimes}_\pi C^\infty(X \times X)$. For the continuity assertion we use the continuity of the Mellin transform from $C_0^\infty(\mathbf{R}_+, C^\infty(X \times X))$ to the corresponding subspace of $\mathcal{A}(\mathbf{C}, C^\infty(X \times X))$. $\lhd$

We need some preparations for showing statements (2) and (3) of 2.2.10. Lemma 2.2.14, below, contains a technical result relating Mellin and Fourier transform.

2.2.12 Lemma. *Let u be a function on $\mathbf{R}_+$ and $\beta \in \mathbf{R}$. Then*

$$(S_\beta u)(r) = e^{-(1/2-\beta)r} u(e^{-r}) \tag{1}$$

defines a function $S_\beta u$ on $\mathbf{R}$. If, additionally, $u \in t^\beta L^2(\mathbf{R}_+)$, then

$$(M_\beta u)(1/2 - \beta + i\tau) = (\mathcal{F} S_\beta u)(\tau).$$

Here, M_β is the weighted Mellin transform, and $\mathcal{F}$ is the one-dimensional Fourier transform: $\mathcal{F}f(\tau) = \int e^{-i\sigma\tau} f(\sigma)\, d\sigma$. Vice versa, if $h = M_\beta u$, then

$$[M_\beta^{-1} h](t) = u(t) = [S_\beta^{-1} \mathcal{F}^{-1} v](t), \tag{2}$$

with $v(\tau) = h(1/2 - \beta + i\tau)$.

Proof. By a straightforward computation. ◁

2.2.13 Lemma. *Let E be a Banach space and let $\{\kappa_\lambda : \lambda \in \mathbf{R}_+\} \subseteq \mathcal{L}(E)$ be a strongly continuous group action, cf. 1.2.2. Then there are constants c and M such that*

$$\|\kappa_\lambda\|_{\mathcal{L}(E)} \leq c\ \max\{\lambda, \lambda^{-1}\}^M.$$

A proof may be found in [17] or [44]. ◁

We can now prove statement (3) of 2.2.10.

2.2.14 Lemma. *Let E, F be Banach spaces with strongly continuous group actions κ_λ, $\tilde{\kappa}_\lambda$, $\lambda \in \mathbf{R}_+$. Let $\mu \in \mathbf{R}$, $m, k \in \mathbf{N}$, and*

$$a \in S^\mu(\mathbf{R}^m_x, \mathbf{R}^k_\xi \times \Gamma_{0,z}; E, F).$$

For the notation see 1.2.2. Suppose that $a = a(x, \xi, z)$ vanishes for all x outside a compact set, say K. Then for every $\varphi \in C_0^\infty(\mathbf{R}_+)$ the function

$$A(\zeta) = M_{t\to\zeta}(\varphi(t) M^{-1}_{1/2, z\to t} a)$$

is analytic on $\mathbf{C}$ with values in $S^\mu(\mathbf{R}^m, \mathbf{R}^k; E, F)$. Moreover, for all $\beta \in \mathbf{R}$,

$$A\,|_{\Gamma_\beta} \in S^\mu(\mathbf{R}^m, \mathbf{R}^k \times \Gamma_\beta; E, F)\ , \tag{1}$$

uniformly for β in compact intervals, and the mapping $(\varphi, a) \mapsto A$ from $C_0^\infty(\mathbf{R}_+) \times S^\mu_K(\mathbf{R}^m, \mathbf{R}^k \times \Gamma_0; E, F)$ to this Frechet subspace of $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^m, \mathbf{R}^k; E, F))$ is separately continuous. Here the index K of S^μ indicates the space of those elements that vanish for x outside K.

Proof. We have

$$\begin{aligned} S^\mu(\mathbf{R}^m_x, \mathbf{R}^k_\xi \times \Gamma_{0,z}; E, F) &= C^\infty(\mathbf{R}^m_x, S^\mu(\mathbf{R}^0, \mathbf{R}^k_\xi \times \Gamma_{0,z}; E, F)) \\ &= C^\infty(\mathbf{R}^m) \hat{\otimes}_\pi S^\mu(\mathbf{R}^0, \mathbf{R}^k_\xi \times \Gamma_{0,z}; E, F). \end{aligned}$$

Similarly, $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^m, \mathbf{R}^k; E, F)) = C^\infty(\mathbf{R}^m)\hat{\otimes}_\pi \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^0, \mathbf{R}^k; E, F))$. Without loss of generality we may therefore assume $m = 0$, i.e., $a \in S^\mu(\mathbf{R}^k_\xi \times \Gamma_{0,z}; E, F)$ is independent of x. We conclude that

$$\varphi[M^{-1}_{1/2}a] \in \mathcal{E}'(\mathbf{R}_+, S^\mu(\mathbf{R}^k; E, F))$$

applying the same considerations as in equation (1) of the proof of Lemma 2.2.6: A pairing with $\psi \in C^\infty(\mathbf{R}_+)$ and integration by parts gives us an integral that converges in all semi-norms of $S^\mu(\mathbf{R}^k; E, F)$. Moreover, the tensor product argument used there shows that

$$A = M(\varphi M^{-1}_{1/2}a) \in \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^k; E, F)).$$

This proves the first part of the statement.
Now consider $A|_{\Gamma_\beta}$. We may assume $\beta = 0$ in view of a well-known property of the Mellin transform: $(Mf)(z+\beta) = M_{t\to z}(t^\beta f)(z)$, so that replacing $A|_{\Gamma_\beta}$ by $A|_{\Gamma_0}$ corresponds to replacing $\varphi(t)$ by $t^{-\beta}\varphi(t) \in C_0^\infty(\mathbf{R}_+)$. For the analysis of $A|_{\Gamma_0}$ it is more convenient to switch from the Mellin to the Fourier transform. We write the variable in Γ_0 in the form $z = i\tau$, $\tau \in \mathbf{R}$ and let $p(\tau) = a(i\tau)$. According to 2.2.12 we have

$$\begin{aligned}(M_{1/2}\varphi M^{-1}_{1/2}a)(i\tau) &= (\mathcal{F}S_{1/2}\varphi S^{-1}_{1/2}\mathcal{F}^{-1}p)(\tau) \\ &= (\mathcal{F}\varphi(e^{-r})\mathcal{F}^{-1}p)(\tau)\ .\end{aligned}$$

The symbol p is an element of $S^\mu(\mathbf{R}^{k+1}; E, F)$ and $r \mapsto \varphi(e^{-r}) = \psi(r)$ is a function in $C_0^\infty(\mathbf{R})$. So our task is reduced to showing that $q = \mathcal{F}\psi(r)\mathcal{F}^{-1}p \in S^\mu(\mathbf{R}^{k+1}; E, F)$. We abbreviate $\eta = (\xi, \tau)$ and consider a derivative $D^\beta_\eta q = D^{\beta_1}_\xi D^{\beta_2}_\tau q$. We then estimate

$$\begin{aligned}
&\|\tilde\kappa_{\langle\eta\rangle^{-1}}\{D^\beta_\eta[\mathcal{F}_{r\to\tau}\psi(r)\mathcal{F}^{-1}_{\tilde\tau\to r}p](\xi,\tau)\}\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)} \qquad (2)\\
= \;& \|\tilde\kappa_{\langle\eta\rangle^{-1}}D^\beta_\eta(\hat\psi * p)(\xi,\tau)\kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)}\\
= \;& \|\tilde\kappa_{\langle\eta\rangle^{-1}}\int_{-\infty}^{\infty}\hat\psi(\sigma)(D^\beta_\eta p)(\xi,\tau-\sigma)d\sigma\ \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)}\\
= \;& \|\tilde\kappa_{\langle\eta\rangle^{-1}}\int_{-\infty}^{\infty}\hat\psi(\tau-\sigma)(D^{\beta_1}_\xi D^{\beta_2}_\sigma p)(\xi,\sigma)d\sigma\ \kappa_{\langle\eta\rangle}\|_{\mathcal{L}(E,F)}\\
= \;& \|\int_{-\infty}^{\infty}\tilde\kappa_{\langle\xi,\sigma\rangle\langle\eta\rangle^{-1}}\hat\psi(\tau-\sigma)\tilde\kappa_{\langle\xi,\sigma\rangle^{-1}}(D^{\beta_1}_\xi D^{\beta_2}_\sigma p)(\xi,\sigma)\kappa_{\langle\xi,\sigma\rangle}\kappa_{\langle\eta\rangle\langle\xi,\sigma\rangle^{-1}}d\sigma\|_{\mathcal{L}(E,F)}\\
\le \;& \int_{-\infty}^{\infty}\|\tilde\kappa_{\langle\xi,\sigma\rangle\langle\eta\rangle^{-1}}\|_{\mathcal{L}(F)}|\hat\psi(\tau-\sigma)|\|\tilde\kappa_{\langle\xi,\sigma\rangle^{-1}}(D^{\beta_1}_\xi D^{\beta_2}_\sigma p)(\xi,\sigma)\kappa_{\langle\xi,\sigma\rangle}\|_{\mathcal{L}(E,F)}\|\kappa_{\langle\eta\rangle\langle\xi,\sigma\rangle^{-1}}\|_{\mathcal{L}(E)}d\sigma.
\end{aligned}$$

Here we have used the fact that (scalar) multiplication by $\hat\psi(\tau-\sigma)$ commutes with the action of $\tilde\kappa$. According to 2.2.13 there are constants c and M such that

$$\|\tilde\kappa_{\langle\xi,\sigma\rangle\langle\eta\rangle^{-1}}\|_{\mathcal{L}(F)} \le cL(\xi,\sigma,\eta)^M \qquad (3)$$

and

$$\|\kappa_{\langle\xi,\sigma\rangle^{-1}\langle\eta\rangle}\|_{\mathcal{L}(E)} \le cL(\xi,\sigma,\eta)^M, \qquad (4)$$

where $L(\xi,\sigma,\eta) = \max\{\langle\xi,\sigma\rangle^{-1}\langle\eta\rangle, \langle\xi,\sigma\rangle\langle\eta\rangle^{-1}\}$. Peetre's inequality states that $\langle a+b\rangle^s \le c_s\langle a\rangle^s\langle b\rangle^{|s|}$ for arbitrary $a, b \in \mathbf{R}^m$, $m \in \mathbf{N}, s \in \mathbf{R}$. We recall that $\eta = (\xi, \tau)$ and conclude that

$$\langle\xi,\sigma\rangle^{-1}\langle\eta\rangle \le C\langle(\xi,\sigma)-(\xi,\tau)\rangle = C\langle\sigma-\tau\rangle$$

and, by symmetry, $\langle \xi, \sigma \rangle \langle \eta \rangle^{-1} \leq C \langle \sigma - \tau \rangle$ for a suitable constant C. For the last expression in (2) we combine this estimate with (3) and (4). Together with the facts that

$$\|\tilde{\kappa}_{\langle \xi,\sigma\rangle^{-1}}(D_\xi^{\beta_1} D_\sigma^{\beta_2} p)(\xi,\sigma)\kappa_{\langle \xi,\sigma\rangle}\|_{\mathcal{L}(E,F)} = O(\langle \xi,\sigma\rangle^{\mu-|\beta|})$$

and that $\hat{\psi}$ is rapidly decreasing we conclude with Peetre's inequality that the final integral in (2) is $O(\langle \eta \rangle^{\mu-|\beta|})$.
This shows (1). Clearly, all estimates depend continuously on φ and p, thus they depend continuously on a, and the corresponding mapping is separately continuous. $\triangleleft$

2.2.15 Lemma. *Let $p \in S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_0)$, and suppose that p vanishes for x outside a compact set $K \subset \mathbf{R}^n$. Let $\varphi \in C_0^\infty(\mathbf{R}_+)$. Then*

$$q = M(\varphi M^{-1}_{1/2} p) \in \mathcal{A}(\mathbf{C}, S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^n)) \ ; \tag{1}$$

it vanishes for x outside K. Moreover, for every $\beta \in \mathbf{R}$,

$$q|_{\Gamma_\beta} \in S^\mu_{tr}(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_\beta) \ . \tag{2}$$

The corresponding estimates are satisfied uniformly for β in compact intervals. The mapping $(\varphi, p) \mapsto q$ is separately continuous as a map from

$$C_0^\infty(\mathbf{R}_+) \times S^\mu_{tr,K}(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_0) \tag{3}$$

to this Fréchet subspace of $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^{n-1} \times \mathbf{R}_+, \mathbf{R}^n))$. As before, the index K in (3) indicates that the functions vanish for x outside K.

Proof. If it were not for the subscript "tr", (1) and (2) would follow from Lemma 2.2.14, because the usual symbol classes correspond to the operator-valued symbols with $E = F = \mathbf{C}$ and trivial group action.
So we only have to show that the transmission property is preserved under the operation in (1). This, however, is simple: a symbol $a \in S^\mu(\mathbf{R}^n, \mathbf{R}^n \times \Gamma_0)$ has the transmission property iff

$$\partial^k_{x_n} a(x', 0, \xi', \langle \xi' \rangle \xi_n, z) \in S^\mu(\mathbf{R}^{n-1}_{x'}, \mathbf{R}^{n-1}_{\xi'} \times \Gamma_{0,z}) \hat{\otimes}_\pi H_{\xi_n}, \tag{4}$$

cf. [30, Section 2.2.2.1, Definition 2]. In the present situation we have

$$\begin{aligned} \partial^k_{x_n} q(\zeta, x', 0, \xi', \langle \xi' \rangle \xi_n) &= M_{t\to\zeta}(\varphi(t) M^{-1}_{1/2, z\to t} \partial^k_{x_n} p(x', 0, \xi', \langle \xi' \rangle \xi_n, z) \\ &\in \mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1})) \hat{\otimes}_\pi H_{\xi_n} \end{aligned} \tag{5}$$

by a tensored version of the argument in 2.2.14. The last space coincides with $\mathcal{A}(\mathbf{C}, S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1}) \hat{\otimes}_\pi H_{\xi_n})$ and (1) is proven. For (2) we can argue in the same way: restricting (5) to Γ_β furnishes an element in $S^\mu(\mathbf{R}^{n-1}, \mathbf{R}^{n-1}) \hat{\otimes}_\pi H_{\xi_n}$. Note that the symbols always vanish for x and x' outside a compact set.
Finally the separate continuity of the mapping follows from the closed graph theorem and the continuity properties established in Lemma 2.2.14, since the topology of the space with the transmission property is finer than the original one. The closed graph theorem indeed can be applied: a mapping $\Lambda : C_0^\infty(\mathbf{R}_+) \to \mathcal{Y}$, $\mathcal{Y}$ a locally convex space, is continuous if and only if its restriction to the Frechet spaces $\mathcal{D}_K$ are continuous. As before, $\mathcal{D}_K$ denotes those elements in $C_0^\infty(\mathbf{R}_+)$ that have support in a fixed compact $K \subset \mathbf{R}_+$. $\triangleleft$

2.2.16 Conclusion. As pointed out in Remark 2.2.10, the assertion of Proposition 2.2.7 follows from Lemma 2.2.14 and Lemma 2.2.15.

2.2.17 Theorem. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$. For $\varphi \in C_0^\infty(\mathbf{R}_+)$ define*

$$h(t,t',\zeta) = M_{\rho\to\zeta}[\varphi(\rho) M^{-1}_{1/2,z\to\rho} f(t,t',z)] \ .$$

Then $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Moreover, the induced mapping

$$C_0^\infty(\mathbf{R}_+) \times C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0)) \to C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$$

is separately continuous. Clearly, the corresponding statement holds with $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$ replaced by $\overline{\mathbf{R}}_+$.

Proof. This is immediate from Proposition 2.2.7 together with the fact that

$$\begin{aligned} C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0)) &= C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)\hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X;\Gamma_0) \ , \\ C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X)) &= C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)\hat{\otimes}_\pi M_O^{\mu,d}(X) \ . \end{aligned}$$

◁

2.2.18 Corollary. We use the notation of Theorem 2.2.17 and assume additionally that $\varphi \equiv 1$ near 1. Then Corollary 2.2.5 implies that

$$f - h = M_{1/2}[(1-\varphi)M^{-1}_{1/2} f] \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X,\Gamma_0)).$$

2.2.19 Corollary. Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$, $\varphi \in C_0^\infty(\mathbf{R}_+)$, and suppose that, for some fixed $N \in \mathbf{N}$, we have $(1-\rho)^{-N}\varphi(\rho) \in C_0^\infty(\mathbf{R}_+)$. Then there is a symbol $f_N \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N,d}(X;\Gamma_0))$ such that $\mathrm{op}_M^{1/2} M\varphi M^{-1}_{1/2} f = \mathrm{op}_M^{1/2} f_N$.

Proof. Clearly, $M\varphi M^{-1}_{1/2} f = M\varphi(\rho)\ln^{-N}\rho \ln^N\rho \ M^{-1}_{1/2,z\to\rho} f$. We obtain the assertion from an application of Theorem 2.2.17 and the fact that $\ln^N\rho\, M^{-1}_{1/2,z\to\rho} f(t,t',z) = M^{-1}_{1/2,z\to\rho}[\partial_z^N f(t,t',z)]$. ◁

2.2.20 Proposition. *Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$, and suppose that, for some fixed $N \in \mathbf{N}$, we have*

$$(t-t')^{-N} f(t,t',z) \in C^\infty(\overline{\mathbf{R}}_{+,t} \times \overline{\mathbf{R}}_{+,t'}, \mathcal{B}^{\mu,d}(X;\Gamma_{0,z})) \ .$$

Then there is a symbol $f_N \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N,d}(X;\Gamma_0))$ such that $\mathrm{op}_M^{1/2} f = \mathrm{op}_M^{1/2} f_N$.

Proof. Let $g(t,t',z) = (t')^N(t-t')^{-N} f(t,t',z)$. Then $g \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$, and $f(t,t',z) = (\frac{t}{t'}-1)^N g(t,t',z)$. Choose $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi \equiv 1$ in a neighborhood of 1. Let

$$h(t,t',\zeta) = M_{1/2,\rho\to\zeta}[\psi(\rho) M^{-1}_{1/2,z\to\rho} f(t,t',z)] \tag{1}$$

and

$$h_0(t,t',\zeta) = M_{1/2,\rho\to\zeta}[(1-\psi)(\rho) M^{-1}_{1/2,z\to\rho} f(t,t',z)] \ .$$

It follows from Corollary 2.2.5 that $h_0 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$. Since $\mathrm{op}_M^{1/2} f = \mathrm{op}_M^{1/2} h + \mathrm{op}_M^{1/2} h_0$, it is sufficient to treat $\mathrm{op}_M^{1/2} h$ for a function h as in (1).
Now $\mathrm{op}_M^{1/2} h$ is the integral operator on $C_0^\infty(\overline{X}^\wedge)$ with the distributional kernel

$$[\psi(\rho) M^{-1}_{1/2, z\to\rho} f(t,t',z)]|_{\rho=\frac{t}{t'}} = [\varphi(\rho)\ln^N\rho\, M^{-1}_{1/2,z\to\rho} g(t,t',z)]|_{\rho=\frac{t}{t'}},$$

where $\varphi(\rho) = (1-\rho)^N \ln^{-N}\rho\,\psi(\rho) \in C_0^\infty(\mathbf{R}_+)$; recall that the density on $\mathbf{R}_+$ was $\frac{dt}{t}$. Furthermore, $\ln^N\rho\, M^{-1}_{1/2,z\to\rho} g(t,t',z) = M^{-1}_{1/2,z\to\rho}[\partial_z^N g(t,t',z)]$. Therefore, according to Lemma 2.2.2,

$$\mathrm{op}_M^{1/2} h(t,t',\zeta) = \mathrm{op}_M^{1/2}[M_{\rho\to\zeta}\varphi(\rho) M^{-1}_{1/2,z\to\rho}\{\partial_z^N g(t,t',z)\}].$$

By Theorem 2.2.17 $M[\varphi M^{-1}_{1/2}(\partial_z^N g)] \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu-N,d}(X))$. This yields the assertion. ◁

2.2.21 Proposition. *For $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0))$ and $\varphi \in C_0^\infty(\mathbf{R}_+)$ we have $M\varphi(M^{-1}_{1/2} f) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M^{\mu,d}_{O,cl}(X))$, and the induced mapping*

$$C_0^\infty(\mathbf{R}_+) \times C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0)) \to C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M^{\mu,d}_{O,cl}(X))$$

is separately continuous.

Proof. We only have to make sure the space on the left hand side is mapped into the space on the right hand side. The separate continuity will then follow from the closed graph theorem applying the argument at the end of the proof of Lemma 2.2.15. In order to see the former statement, we only have to check that, for every $\beta \in \mathbf{R}$, $M\varphi M^{-1}_{1/2} f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_\beta))$, uniformly for β in compact intervals. The tensor product representation in Lemma 2.1.4 allows us to assume that $f(t,t',z) = \alpha(t,t')a(z)$, where $\alpha \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)$ and $a \in \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0)$. Without loss of generality assume $\alpha \equiv 1$. We have, for arbitrary $N \in \mathbf{N}$,

$$\begin{aligned}
& [M\varphi M^{-1}_{1/2} f](t,t',\beta+i\tau) \\
= & [M\varphi M^{-1}_{1/2} a](\beta+i\tau) \\
= & \frac{1}{2\pi i}\int_0^\infty \varphi(\rho)\rho^{\beta+i\tau}\int_{\Gamma_0}\rho^{-z}a(z)dz d\rho \\
= & \frac{1}{2\pi}\int_0^\infty \rho^{\beta+i\tau+1}\varphi(\rho)\int_{-\infty}^\infty \rho^{-(1+i\sigma)}a(i\sigma)d\sigma d\rho \\
= & \frac{1}{2\pi}\int_0^\infty (-\rho\partial_\rho)^N[\rho^{\beta+i\tau+1}\varphi(\rho)]\int_{-\infty}^\infty (1+i\sigma)^{-N}\rho^{-(1+i\sigma)}a(i\sigma)d\sigma d\rho. \qquad (1)
\end{aligned}$$

Since the semi-norms for $a(i\sigma)$ are all $O(\langle\sigma\rangle^\mu)$, the integral converges in $\mathcal{B}^{\mu,d}_{cl}(X)$ for fixed β,τ. Moreover, we now write $u = \rho - 1$ and use the binomial expansion $(1+u)^\beta = \sum_{j=0}^N \binom{\beta}{j} u^j + O(u^{N+1})$ to conclude that

$$\begin{aligned}
[M\varphi M^{-1}_{1/2} a](\beta+i\tau) &= [M\varphi\rho^\beta M^{-1}_{1/2} a](i\tau) \\
&= \sum_{j=0}^N \binom{\beta}{j}[M\varphi(\rho)(\rho-1)^j M^{-1}_{1/2} a](i\tau) \qquad (2) \\
&\quad + [M\tilde{\varphi}_N M^{-1}_{1/2} a](i\tau)
\end{aligned}$$

with a function $\tilde{\varphi}_N \in C_0^\infty(\mathbf{R}_+)$ that has a zero of order $N+1$ in $\rho = 1$. According to Corollary 2.2.19 the remainder term in (2) yields an operator that can be rewritten with a Mellin symbol in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N-1,d}(X;\Gamma_\beta))$ while the other terms yield elements of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-j,d}(X;\Gamma_\beta))$. Hence we obtain an asymptotic expansion for the symbol on Γ_β in terms of the symbol on Γ_0, and it only remains to check the assertion for $\beta = 0$. We use a similar argument as before. First choose a function $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho) \equiv 1$ near $\rho = 1$ and $\psi\varphi = \varphi$. Now use a Taylor expansion for the function $g \in C_0^\infty(\mathbf{R})$ given by $g(u) = \varphi(e^u)$. We have $g(u) = \sum_{j=0}^N g^{(j)}(0)/j!\ u^j + \tilde{g}_N$, where $\tilde{g}_N(u)u^{-N-1}$ is smooth in $u = 0$. Therefore,

$$\varphi(\rho) = \sum_{j=0}^{N} c_j \ln^j \rho + \tilde{\varphi}_N(\rho) \tag{3}$$

where $c_j = g^{(j)}(0)/j!$, and $\tilde{\varphi}_N(\rho) = \tilde{g}_N(\ln\rho)$. In particular, $\tilde{\varphi}_N(\rho)\ln^{-N-1}\rho$ is smooth near $\rho = 1$, and so is $(1-\rho)^{-N-1}\tilde{\varphi}_N(\rho)$. We conclude that

$$\begin{aligned}
[M\varphi M_{1/2}^{-1}f](t,t',i\tau) &= \sum_{j=0}^{N} c_j M_{1/2}[\psi(\rho)\ln^j\rho M_{1/2,z\to\rho}^{-1}a](i\tau) \\
&\quad + M_{1/2}[\psi(\rho)\tilde{\varphi}_N(\rho)M_{1/2,z\to\rho}^{-1}a](i\tau) \\
&= \sum_{j=0}^{N} c_j M_{1/2}[\psi(\rho)M_{1/2,z\to\rho}^{-1}\partial_z^j a](i\tau) \\
&\quad + M_{1/2}[\psi(\rho)\tilde{\varphi}_N(\rho)M_{1/2,z\to\rho}^{-1}a](i\tau).
\end{aligned} \tag{4}$$

Employing Corollary 2.2.19, the last term can be rewritten as a Mellin operator with a Mellin symbol in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N-1,d}(X;\Gamma_0))$, since $\psi\,\tilde{\varphi}_N$ is a function in $C_0^\infty(\mathbf{R}_+)$ which vanishes to order $N+1$ in $\rho = 1$. For the terms under the summation we note that

$$M[\psi(\rho)M_{1/2,z\to\rho}^{-1}\partial_z^j a] = \partial_z^j a - M[(1-\psi(\rho))M_{1/2,z\to\rho}^{-1}\partial_z^j a].$$

The first term on the right hand side is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}_{cl}^{\mu-j,d}(X;\Gamma_0))$ while the second is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$ according to Corollary 2.2.5. Since N was arbitrary, we obtain the desired result. $\triangleleft$

2.2.22 Remark. The last argument in the proof of Proposition 2.2.21 can be used to obtain an asymptotic expansion for the operator $\mathrm{op}_M^{1/2}[M\varphi M_{1/2}^{-1}f]$, independent of the fact whether f is a classical symbol or not:

Let $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$, and let $\varphi \in C_0^\infty(\mathbf{R}_+)$. Write $\varphi(\rho) = \sum_{j=0}^N c_j \ln^j\rho + \tilde{\varphi}_N(\rho)$, where $c_j = [\varphi \circ \exp]^{(j)}(0)/j!$ and $\tilde{\varphi}_N \ln^{-N-1}\rho$ is smooth near $\rho = 1$. Then equation (4) in the proof of Proposition 2.2.21 together with Corollary 2.2.19 shows that

$$\mathrm{op}_M^{1/2}[M\varphi M_{1/2}^{-1}f] = \sum_{j=0}^{N} c_j \mathrm{op}_M^{1/2}\partial_z^j f + \mathrm{op}_M^{1/2}\tilde{f}_N$$

with $\tilde{f}_N \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N-1,d}(X;\Gamma_0))$.

2.2.23 Remark. For several cases we have proved continuity results of the following form: For suitable Fréchet spaces F and G, the bilinear map

$$\Lambda : C_0^\infty(\mathbf{R}_+) \times F \to G$$

is separately continuous. Since $C_0^\infty(\mathbf{R}_+)$ is barrelled, the mapping Λ automatically is hypocontinuous, see Köthe [22, 40.2(5)].

2.3 Compositions and Adjoints

In this section we will show that the operators in the union of the classes $M\mathcal{B}^{\mu,d}_{\gamma}(X^\wedge)$, $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$, form an algebra and that elements of nonpositive order and type zero have adjoints within the calculus.
We will sometimes need the following observation. The proof is straightforward by induction.

2.3.1 Lemma. *For each $1 \le k \in \mathbf{N}$ there are constants c_{kj}, d_{kj} such that*

$$\begin{aligned} (t\partial_t)^k &= \sum_{j=1}^{k} c_{kj} t^j \partial_t^j \quad \text{and} && (1)\\ t^k \partial_t^k &= \sum_{j=1}^{k} d_{kj} (t\partial_t)^j && (2)\end{aligned}$$

The coefficients d_{kj} are easily seen to be the Sterling numbers of the first kind, while the c_{kj} are the Sterling numbers of the second kind [19].

We start with the result on adjoints.

2.3.2 Lemma. *Let $0 \ge \mu \in \mathbf{Z}$ and $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,0}(X;\Gamma_{1/2-\gamma}))$, $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, $s > -1/2$. Then the operator*

$$\omega_1[\mathrm{op}_M^\gamma f]\omega_2 : \begin{matrix} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{matrix} \to \begin{matrix} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{matrix}$$

has a formal adjoint with respect to the pairings between

$$\begin{matrix} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{matrix} \quad \text{and} \quad \begin{matrix} \mathcal{H}_0^{-s,-\gamma-\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{-s,-\gamma-\frac{n+1}{2}}(Y^\wedge, W_1) \end{matrix}$$

on one hand, and

$$\begin{matrix} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{matrix} \quad \text{and} \quad \begin{matrix} \mathcal{H}_0^{-s+\mu,-\gamma-\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{-s+\mu,-\gamma-\frac{n+1}{2}}(Y^\wedge, W_2) \end{matrix}$$

on the other. The adjoint is given by

$$(\omega_1\,[\mathrm{op}_M^\gamma f]\,\omega_2)^* = \overline{\omega}_2\,[\mathrm{op}_M^{-\gamma-n} f^{(*)}]\,\overline{\omega}_1, \tag{1}$$

where

$$f^{(*)}(t,t',z) = f(t',t,n+1-\overline{z})^*, \tag{2}$$

and the asterisk denotes the pointwise formal L^2-adjoint of the operator $f(t,t',n+1-\overline{z})$: $C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1) \to C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2)$ in $\mathcal{B}^{\mu,0}(X)$.

Proof. We use the representation of f established in Lemma 2.1.4. It implies the convergence of the corresponding series for $\omega_1[\mathrm{op}_M^\gamma f]\omega_2$. We therefore have, using the notation of 2.1.4(1),

$$\begin{aligned}(\omega_1\,[\mathrm{op}_M^\gamma f]\,\omega_2)^* &= \sum_{j=1}^{\infty}\lambda_j(\omega_1\varphi_j[\mathrm{op}_M^\gamma a_j]\omega_2\psi_j)^* \\ &= \sum_{j=1}^{\infty}\lambda_j\overline{\omega_2\psi_j}\,[\mathrm{op}_M^{-\gamma-n}a_j^{(*)}]\overline{\omega_1\varphi_j}. \qquad (3)\end{aligned}$$

We have employed a fact from Lemma 1.5.1.10: The adjoint of the Mellin operator $\mathrm{op}_M^\gamma a$, given by the t,t'-independent symbol $a\in\mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$, is the Mellin operator $\mathrm{op}_M^{-n-\gamma}a^{(*)}$ with $a^{(*)}(z)=a(n+1-\overline{z})^*$. Now (3) implies (2) and completes the proof. $\lhd$

Now we shall have a look at compositions. For what follows, the choice of γ is not essential, and we assume $\gamma=1/2$. We will need the following observation.

2.3.3 Theorem. *For $f\in C^\infty(\overline{\mathbf{R}}_+\times\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\Gamma_0))$ there is a Mellin symbol $g\in C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\Gamma_0))$ such that, for arbitrary $N\in\mathbf{N}$,*

$$\mathrm{op}_M^{1/2}f(t,t',z)-\mathrm{op}_M^{1/2}g(t,z)=\mathrm{op}_M^{1/2}h_N(t,t',z) \qquad (1)$$

for suitable $h_N\in C^\infty(\overline{\mathbf{R}}_+\times\overline{\mathbf{R}}_+,\mathcal{B}^{-N,d}(X;\Gamma_0))$. In particular, $\mathrm{op}_M^{1/2}f-\mathrm{op}_M^{1/2}g\in M\mathcal{B}_{1/2}^{-\infty,d}(X^\wedge)$. The Mellin symbol g can be chosen with the asymptotic expansion

$$g(t,z)\sim\sum_{j=0}^{\infty}\frac{1}{j!}(-t'\partial_{t'})^j\partial_z^j f(t,t',z)|_{t'=t}\,. \qquad (2)$$

Proof. A Taylor expansion gives for arbitrary $N\in\mathbf{N}$

$$f(t,t',z)=\sum_{j=0}^{N-1}\frac{1}{j!}(t'-t)^j\partial_{t'}^j f(t,t',z)|_{t'=t}+\tilde f_N(t,t',z)$$

with $(t'-t)^{-N}\tilde f_N(t,t',z)=1/(N-1)!\int_0^1(1-\theta)^{N-1}\partial_{t'}^N f(t,t+\theta(t'-t),z)d\theta\in C^\infty(\overline{\mathbf{R}}_+\times\overline{\mathbf{R}}_+,\mathcal{B}^{\mu,d}(X;\Gamma_0))$.

By Proposition 2.2.20 there is a $\tilde g_N\in C^\infty(\overline{\mathbf{R}}_+\times\overline{\mathbf{R}}_+,\mathcal{B}^{\mu-N,d}(X;\Gamma_0))$ such that $\mathrm{op}_M^{1/2}\tilde f_N=\mathrm{op}_M^{1/2}\tilde g_N$. In order to treat the terms under the summation let

$$\begin{aligned}f_j(t,t',z) &= \frac{(t'-t)^j}{j!}\,\partial_{t'}^j f(t,t',z)|_{t'=t}\ \text{ and}\\ h_j(t,z) &= \frac{t^j}{j!}\,\partial_{t'}^j f(t,t',z)|_{t'=t}.\end{aligned}$$

Choose $\psi\in C_0^\infty(\mathbf{R}_+)$ with $\psi\equiv 1$ near 1. The function φ_j defined by

$$\varphi_j(\rho)=(\rho^{-1}-1)^j\ln^{-j}\rho\;\psi(\rho)$$

is in $C_0^\infty(\mathbf{R}_+)$. Moreover, according to Lemma 2.2.2,

$$\begin{aligned}\mathrm{op}_M^{1/2}f_j &= \mathrm{op}_M^{1/2}[M_{1/2}\varphi_j\ln^j\rho M_{1/2}^{-1}h_j] \qquad (3)\\ &\quad+\mathrm{op}_M^{1/2}[M_{1/2}(1-\psi)M_{1/2}^{-1}f_j].\end{aligned}$$

We have $M(1-\psi)M_{1/2}^{-1}f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$ by Corollary 2.2.5. For the term before, we note that $\ln^j\rho M_{1/2,z\to\rho}^{-1}h_j(t,z) = M_{1/2,z\to\rho}^{-1}(\partial_z^j h_j(t,z))$. Since $\partial_z^j h_j \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu-j,d}(X;\Gamma_0))$, an application of Theorem 2.2.17 yields that

$$\begin{aligned} g_j(t,\zeta) &= M_{1/2,\rho\to\zeta}[\varphi_j(\rho)\ln^j\rho\, M_{1/2}^{-1}h_j] \\ &= M_{1/2,\rho\to\zeta}[\varphi_j(\rho)M_{1/2}^{-1}\partial_z^j h_j] \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu-j,d}(X;\Gamma_0)) . \end{aligned}$$

Since N is arbitrary, we obtain a sequence $\{g_j\}$ of Mellin symbols of decreasing order. According to Theorem 2.1.12 we may choose $g \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ with $g \sim \sum_{j=0}^\infty g_j$, and we will have for any $N \in \mathbf{N}$,

$$\mathrm{op}_M^{1/2} f - \mathrm{op}_M^{1/2} g = \mathrm{op}_M^{1/2} \tilde{g}_N$$

for suitable $\tilde{g}_N \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-N,d}(X;\Gamma_0))$. It remains to show that g can be taken with the asymptotic expansion (2). One expansion is given by the above consideration:

$$g(t,\zeta) \sim \sum_{j=0}^\infty \frac{1}{j!} M_{1/2,\rho\to\zeta}[\varphi_j(\rho) M_{1/2,z\to\rho}^{-1}(t^j\partial_{t'}^j\partial_z^j f(t,t',z)|_{t'=t})] .$$

Using Remark 2.2.22 and Lemma 2.3.1, it may be written in the form

$$\sum_{k,l=0}^\infty d_{kl}(-t'\partial_t')^k\partial_z^l f(t,t',z)|_{t'=t}$$

with suitable constants d_{kl}. The constants d_{kl} are independent of f; they contain information about the functions $(\rho^{-1}-1)^j/\ln^j\rho, j \in \mathbf{N}$, for suitable j, m, and on the coefficients in the conversion formula 2.3.1(2). We may therefore choose a particularly simple f to determine them. For $f(t,t',z) = \varphi(t')z^k, \varphi \in C_0^\infty(\mathbf{R}_+)$, we have

$$\begin{aligned} [\mathrm{op}_M^{1/2} f]u(t) &= (-t\partial_t)^k(\varphi u)(t) \\ &= \sum_{l=0}^k \binom{k}{l} (-t\partial_t)^l\varphi(t)(-t\partial_t)^{k-l}u(t) \\ &= \mathrm{op}_M^{1/2}\left[\sum_{l=0}^k \binom{k}{l} (-t\partial_t)^l\varphi(t)z^{k-l}\right]u(t) \\ &= \mathrm{op}_M^{1/2}\sum_{l=0}^k \frac{1}{l!}(-t'\partial_{t'})^l\partial_z^l f(t,t',z)|_{t=t'}u(t), \end{aligned}$$

noting that $\partial_z^l z^k = k\cdot\ldots\cdot(k-l+1)z^{k-l} = \frac{k!}{(k-l)!}z^{k-l}$. Hence $d_{kl} = \frac{1}{l!}\delta_{kl}$, just as asserted. $\lhd$

2.3.4 Theorem. *Given $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ there is a symbol $g \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ such that, for arbitrary $N \in \mathbf{N}$,*

$$\mathrm{op}_M^{1/2} f(t,t',z) - \mathrm{op}_M^{1/2} g(t',z) = \mathrm{op}_M^{1/2} h_N(t,t',z)$$

with suitable $h_N \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-N,d}(X;\Gamma_0))$. In particular, $\mathrm{op}_M^{1/2} f - \mathrm{op}_M^{1/2} g \in M\mathcal{B}_{1/2}^{-\infty,d}(X^\wedge)$. The symbol g can be taken with the asymptotic expansion

$$g(t',z) \sim \sum_{j=0}^\infty \frac{1}{j!}(-t\partial_t)^j(-\partial_z)^j f(t,t',z)\,|_{t=t'} .$$

Proof. This is analogous to the proof of Theorem 2.3.3 by interchanging the roles of t and t'. ◁

2.3.5 Theorem. *Let $\mu_1, \mu_2 \in \mathbf{Z}$ and $d_1, d_2 \in \mathbf{N}$. Fix $\omega \in C_0^\infty(\mathbf{R}_+)$. Then the composition of operators yields a continuous mapping*

$$M\mathcal{B}_{1/2}^{\mu_1,d_1}(X^\wedge) \times M\mathcal{B}_{1/2}^{\mu_2,d_2}(X^\wedge) \to M\mathcal{B}_{1/2}^{\mu,d}(X^\wedge),$$

given by

$$\left(\mathrm{op}_M^{1/2} f_1 + G_1, \mathrm{op}_M^{1/2} f_2 + G_2\right) \mapsto (\mathrm{op}_M^{1/2} f_1 + G_1)\omega(\mathrm{op}_M^{1/2} f_2 + G_2).$$

Here $\mu = \mu_1 + \mu_2$, and $d = \max\{\mu_2 + d_1, d_2\}$. More precisely we should take into account the vector bundles the operators are acting on and use the following formulation.
Let V_0, V_1, V_2 be vector bundles over X and W_0, W_1, W_2 be vector bundles over Y; let $\mathrm{op}_M^{1/2} f_j + G_j \in M\mathcal{B}_{1/2}^{\mu_j,d_j}(X^\wedge)$, $j = 1,2$ with $f_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu_j,d_j}(X;\Gamma_0))$, $G_j \in M\mathcal{B}_{1/2}^{-\infty,d_j}(X^\wedge)$ having the mapping properties

$$\begin{aligned}
f_2(t,t',z) &\in \mathcal{L}(C^\infty(\overline{X}, V_0) \oplus C^\infty(Y, W_0), C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1)), \\
G_2 &\in \mathcal{L}(C_0^\infty(\overline{X}^\wedge, V_0) \oplus C_0^\infty(Y^\wedge, W_0), C^\infty(\overline{X}^\wedge, V_1) \oplus C^\infty(Y^\wedge, W_1)), \\
f_1(t,t',z) &\in \mathcal{L}(C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1), C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2)), \\
G_1 &\in \mathcal{L}(C_0^\infty(\overline{X}^\wedge, V_1) \oplus C_0^\infty(Y^\wedge, W_1), C^\infty(X^\wedge, V_2) \oplus C^\infty(Y^\wedge, W_2)).
\end{aligned}$$

Then

$$(\mathrm{op}_M^{1/2} f_1 + G_1)\omega(\mathrm{op}_M^{1/2} f_2 + G_2) = \mathrm{op}_M^{1/2} f + G \in M\mathcal{B}_{1/2}^{\mu,d}(X^\wedge)$$

with $\mu = \mu_1 + \mu_2$, $d = \max(\mu_2 + d_1, d_2)$, $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$, and $G \in M\mathcal{B}_{1/2}^{-\infty,d}(X^\wedge)$ such that

$$f(t,t',z) \in \mathcal{L}(C^\infty(\overline{X}, V_0) \oplus C^\infty(Y, W_0), C^\infty(\overline{X}, V_2) \oplus C^\infty(Y, W_2))$$

and

$$G \in \mathcal{L}(C_0^\infty(\overline{X}^\wedge, V_0) \oplus C_0^\infty(Y^\wedge, W_0), C^\infty(\overline{X}^\wedge, V_2) \oplus C^\infty(Y^\wedge, W_2)).$$

Proof. The proof splits into four parts in a natural way:

(i) The composition $G_1\omega G_2$.
(ii) The composition $G_1\omega(\mathrm{op}_M^{1/2} f_2)$.
(iii) The composition $(\mathrm{op}_M^{1/2} f_1)\omega G_2$.
(iv) The composition $(\mathrm{op}_M^{1/2} f_1)\omega(\mathrm{op}_M^{1/2} f_2)$.

Without loss of generality we may assume that the bundles are trivial one-dimensional over X and 0 over Y. So let us show (i) - (iv).

(i) We may write $G_2 = \sum_{k=0}^{d_2} H_k \partial_r^k$ with H_k of type zero. Then $G_1\omega G_2 = \sum [G_1\omega H_k]\partial_r^k$, and it is easily checked that the operators $G_1\omega H_k$ belong to $M\mathcal{B}_{1/2}^{-\infty,0}(X^\wedge)$ for they have the required mapping properties.

(ii) Using Lemma 2.1.8, write $f_2(t,t',z) = \sum_{j=0}^{d} h_j(t,t',z)\partial_r^j$ with $h_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu_2-j,0}(X;\Gamma_0))$. An application of Theorem 2.1.3 shows that for $\omega_1,\omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, the operators $\omega_1 G_1 \omega \mathrm{op}_M^{1/2}(h_j)\omega_2$ have the mapping properties required for an operator in $M\mathcal{B}_{1/2}^{-\infty,0}(X^\wedge)$. Hence $G_1\omega[\mathrm{op}_M^{1/2} f_2] \in M\mathcal{B}_{1/2}^{-\infty,d}(X^\wedge)$.

(iii) is proven in the same way.

(iv) Let $\tilde{f}_1(t,t',z) = \omega(t')f(t,t',z)$. Then $\tilde{f}_1 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu_1,d_1}(X;\Gamma_0))$ and, by Theorem 2.3.4, there is a symbol $g_1 = g_1(t,z) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu_1,d_1}(X;\Gamma_0))$ with $\mathrm{op}_M^{1/2}(f_1)\omega = \mathrm{op}_M^{1/2}\tilde{f}_1 \equiv \mathrm{op}_M^{1/2} g_1 \bmod M\mathcal{B}_{1/2}^{-\infty,d_1}(X^\wedge)$. Similarly there is a $g_2 = g_2(t',z) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu_2,d_2}(X;\Gamma_0))$ such that $\mathrm{op}_M^{1/2} f_2 \equiv \mathrm{op}_M^{1/2} g_2 \bmod M\mathcal{B}_{1/2}^{-\infty,d_2}(X^\wedge)$. Now we apply Lemma 2.1.4, writing

$$\begin{aligned} g_1(t,z) &= \sum_{j=1}^{\infty} \lambda_j \varphi_j(t) a_j(z) \\ g_2(t',z) &= \sum_{j=0}^{\infty} \tilde{\lambda}_j \psi_j(t') b_j(z) \end{aligned}$$

with suitable $\{\lambda_j\},\{\tilde{\lambda}_j\} \in l^1$, and null sequences $\{\varphi_j\},\{\psi_j\} \subseteq C^\infty(\overline{\mathbf{R}}_+)$, $\{a_j\} \subseteq \mathcal{B}^{\mu_1,d_1}(X;\Gamma_0)$, $\{b_j\} \subseteq \mathcal{B}^{\mu_2,d_2}(X;\Gamma_0)$. We note that

$$\mathrm{op}_M^{1/2}[\lambda_j\varphi_j(t)a_j(z)]\ \mathrm{op}_M^{1/2}[\tilde{\lambda}\psi_k(t')b_k(z)] = \mathrm{op}_M^{1/2}[\lambda_j\tilde{\lambda}_k\, \varphi_j(t)\, \psi_k(t')\, c_{jk}(z)],$$

where $c_{jk}(z) = a_j(z) \circ b_k(z)$; here $\circ$ denotes the composition $\mathcal{B}^{\mu_1,d_1}(X) \times \mathcal{B}^{\mu_2,d_2}(X) \to \mathcal{B}^{\mu,d}(X)$. Since this composition is continuous we obtain the assertion from (i), (ii), and (iii). ◁

2.3.6 Remark. In 2.3.5 we may assume that both f_1, and f_2 are independent of t'. Then f can be taken independent of t' with the asymptotic expansion

$$f(t,z) \sim \sum_{j=0}^{\infty} \frac{1}{j!}\partial_z^j f_1(t,z)(-t\partial_t)^j f_2(t,z). \tag{1}$$

In fact this is a consequence of the asymptotic expansion formulae in 2.3.3 and 2.3.4 in addition to the identity $(x+y)^q = \sum_{m+r=q} \frac{q!}{m!r!} x^m y^r$, cf. [49, proof of Theorem 1.3.30].

2.4 Mellin Quantization

2.4.1 Push-forward of Pseudodifferential Operators. Let U, V be open sets in $\mathbf{R}^n$, $\chi : U \to V$ a diffeomorphism. Moreover, let E, F be Banach spaces with group action. Given an operator

$$P : C_0^\infty(U,E) \to C^\infty(U,F),$$

the push-forward $\chi_* P : C_0^\infty(V,E) \to C^\infty(V,F)$ is defined by

$$(\chi_* P)f(x) = [P(f \circ \chi)][\chi^{-1}(x)].$$

If $P = \operatorname{op} p$ for some $p \in S^\mu(U, \mathbf{R}^n; E, F)$ then there is a symbol $q \in S^\mu(V, \mathbf{R}^n; E, F)$ with $\operatorname{op} q = \chi_* P$ modulo regularizing operators, and q is unique up to symbols in $S^{-\infty}(V, \mathbf{R}^n; E, F)$. In this sense χ defines a push-forward also on the symbol level:

$$\chi_* : S^\mu(U, \mathbf{R}^n; E, F)/S^{-\infty}(U, \mathbf{R}^n; E, F) \to S^\mu(V, \mathbf{R}^n; E, F)/S^{-\infty}(V, \mathbf{R}^n; E, F).$$

The mapping is injective; the inverse is induced by the push-forward via χ^{-1}. The same statements are true for symbols with the transmission property.
One way of proving this is to first convert the symbol p to a 'double' symbol $p_1(y, y', \eta)$ by multiplying p by a cut-off function $\phi = \phi(y, y')$ near the diagonal $\{y = y'\}$; $\operatorname{op} p$ and $\operatorname{op} p_1$ only differ by a regularizing operator. Then one can compute a 'double' symbol $q_1 \in S^\mu(V \times V, \mathbf{R}^n; E, F)$ with $\chi_* \operatorname{op} p_1 = \operatorname{op} q_1$ and finally switch to a y'-independent symbol q with $\operatorname{op} q_1 \equiv \operatorname{op} q$ modulo regularizing operators.
In what follows it will often be possible to find a 'double' symbol q_1 with $\chi_* \operatorname{op} p = \operatorname{op} q_1$ by a straightforward substitution in oscillatory integrals. We will then also write $q_1 = \chi_* p$.

2.4.2 Corollary. Let $\chi : U \to V$ be a diffeomorphism of open sets in $\mathbf{R}$, and let $a \in C^\infty(U, \mathcal{B}^{\mu,d}(X; \mathbf{R}))$ induce a pseudodifferential action by

$$\operatorname{op} a(u)(y) = \frac{1}{2\pi} \int\!\!\int_U e^{i(y-y')\eta} a(y, \eta) u(y') dy' d\eta \tag{1}$$

for $u \in C_0^\infty(U, C^\infty(\overline{X}, V_1) \oplus C^\infty(Y, W_1))$. For the push-forward $\chi_* \operatorname{op} a$ we then have

$$\chi_* \operatorname{op} a = \operatorname{op} b + G, \tag{2}$$

where

(i) the symbol b belongs to $C^\infty(V, \mathcal{B}^{\mu,d}(X; \mathbf{R}))$. It is determined via the symbol push-forward of the various local symbols for a. In this sense we shall use the notation $b = \chi_* a$.

(ii) The operator G belongs to $\mathcal{B}^{-\infty,d}(X^\wedge)$. In other words, we can write

$$G = \sum_{j=0}^{\infty} G_j \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix} ; \tag{3}$$

here ∂_r is the normal derivative on X, and each G_j is a matrix of integral operators with kernel functions which are smooth up to the boundary of X.

Proof. We have $C^\infty(U, \mathcal{B}^{\mu,d}(X; \mathbf{R})) = C^\infty(U) \hat{\otimes}_\pi \mathcal{B}^{\mu,d}(X; \mathbf{R})$. Since convergence of the symbols implies convergence of the associated operators, it is sufficient to assume that $a(y, \eta) = \psi(y) A(\eta)$ with $\psi \in C^\infty(U)$ and $A \in \mathcal{B}^{\mu,d}(X; \mathbf{R})$. The assertion is certainly true for regularizing A: In this case, op a already has the form (3); hence the push-forward is of the same type and (2) holds with $b = 0$, for ∂_r is not affected. We can therefore localize with respect to a coordinate neigborhood Ω_j for Ω and assume that A is given locally by a quintuple of parameter-dependent symbols in Boutet de Monvel's calculus, $(p_j, g_j, k_j, t_j, s_j)$, where $p_j = p_j(x, \xi, \eta) \in S^\mu_{tr}(X_j, \mathbf{R}^n_\xi \times \mathbf{R}_\eta)$, $X_j = \Omega_j \cap \overline{X}$, is a pseudodifferential symbol with the transmission property, g is a parameter–dependent singular Green symbol, etc., cf. 1.3.4. We then have to show that their push-forward is preserved.

In order to see this, let us focus on p_j; the arguments for the other symbols are similar. We have

$$S^{\mu}_{tr}(U \times X_j, \mathbf{R}^n \times \mathbf{R}) = C^{\infty}(U)\hat{\otimes}_{\pi} S^{\mu}_{tr}(X_j, \mathbf{R}^n \times \mathbf{R}); \tag{4}$$

thus $\psi(y)p(x,\xi,\eta) \in S^{\mu}_{tr}(U \times X_j, \mathbf{R}^n \times \mathbf{R})$. These spaces are invariant under coordinate transforms, therefore the push-forward $\chi_*[\psi(y)p(x,\xi,\eta)]$ belongs to $S^{\mu}_{tr}(V \times X_j, \mathbf{R}^n \times \mathbf{R})$ modulo $S^{-\infty}(V \times X_j, \mathbf{R}^n \times \mathbf{R})$. We know that the push-forward of the regularizing part is regularizing. Employing now (4) with with U replaced by V plus the fact that $C^{\infty}(V,\mathcal{F}) = C^{\infty}(V)\hat{\otimes}_{\pi}\mathcal{F}$ for every Fréchet space $\mathcal{F}$, we see that $\chi_*[\psi(y)p(x,\xi,\eta)] \in C^{\infty}(V, S^{\mu}_{tr}(X_j, \mathbf{R}^n \times \mathbf{R}))$ may be considered the pseudodifferential part (with transmission property) of a parameter-dependent symbol tuple for an operator in $C^{\infty}(V, \mathcal{B}^{\mu,d}(X_j;\mathbf{R}))$. Applying the same argument for the four other components g_j, k_j, t_j, and s_j we obtain the symbol $b \in C^{\infty}(V, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. $\lhd$

2.4.3 Pseudodifferential and Mellin Symbols. Given $f \in C^{\infty}(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ let

$$b(y,\eta) = f(e^y, -i\eta), \quad y,\eta \in \mathbf{R}. \tag{1}$$

Denoting by exp the diffeomorphism $y \mapsto e^y$ from $\mathbf{R} \to \mathbf{R}_+$ we have

$$\mathrm{op}_M^{1/2} f = \exp_* \mathrm{op}\, b. \tag{2}$$

In more detail: For $u \in C_0^{\infty}(\mathbf{R}_+, C^{\infty}(\overline{X}, V) \oplus C^{\infty}(Y, V_2))$ let $u^*(y) = u(e^y)$; then $[\mathrm{op}_M^{1/2} f(u)](e^y) = [\mathrm{op}\, b(u^*)](y)$. This is a simple consequence of the identity

$$\frac{1}{2\pi i}\int_{\Gamma_0}\int_0^{\infty}\left(\frac{e^y}{t'}\right)^{-z} f(e^y,z)u(t')\frac{dt'}{t'}dz = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(y-y')\eta} f(e^y,-i\eta)u^*(y')dy'd\eta.$$

Equation (1) implies that $b \in C^{\infty}(\mathbf{R}, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. According to Corollary 2.4.2, we will have $\exp_*\mathrm{op}\, b \equiv \mathrm{op}\, a \mod \mathcal{B}^{-\infty,d}(X^{\wedge})$. Hence,

$$\mathrm{op}_M^{1/2} f \equiv \mathrm{op}\, a \mod \mathcal{B}^{-\infty,d}(X^{\wedge}).$$

We shall now analyze the relationship between f and a.

2.4.4 Definition and Remark. For $\mu \in \mathbf{Z}$ and $d \in \mathbf{N}$ let

$$\mathring{M}\mathcal{B}^{\mu,d}(X^{\wedge}) = \{\mathrm{op}_M^{1/2} f + G \;:\; f \in C^{\infty}(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0)),\ G \in \mathcal{B}^{-\infty,d}(X^{\wedge})\}.$$

Analogously, we let

$$\mathring{M}\mathcal{B}^{\mu,d}_{cl}(X^{\wedge}) = \{\mathrm{op}_M^{1/2} f + G \;:\; f \in C^{\infty}(\mathbf{R}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0)),\ G \in \mathcal{B}^{-\infty,d}(X^{\wedge})\}.$$

For $f \in C^{\infty}(\mathbf{R}_+, \mathcal{B}^{-\infty,0}(X;\Gamma_0))$, $\mathrm{op}_M^{1/2} f$ is an integral operator with smooth kernel on $\overline{X}^{\wedge}$. Hence $\mathring{M}\mathcal{B}^{-\infty,d}(X^{\wedge}) := \bigcap_{\mu} \mathring{M}\mathcal{B}^{\mu,d}(X^{\wedge}) = \mathcal{B}^{-\infty,d}(X^{\wedge})$.

The following lemma may be considered a 'coarse' quantization result. It shows that pseudodifferential and Mellin symbols induce the same operators modulo $\mathcal{B}^{-\infty,d}(X^{\wedge})$ as long as we consider symbol classes with arbitrary behavior near $t = 0$.

2.4.5 Lemma.

$$\mathring{M}\mathcal{B}^{\mu,d}(X^\wedge)/\mathring{M}\mathcal{B}^{-\infty,d}(X^\wedge) \cong \mathcal{B}^{\mu,d}(X^\wedge)/\mathcal{B}^{-\infty,d}(X^\wedge), \text{and}$$
$$\mathring{M}\mathcal{B}^{\mu,d}_{cl}(X^\wedge)/\mathring{M}\mathcal{B}^{-\infty,d}(X^\wedge) \cong \mathcal{B}^{\mu,d}_{cl}(X^\wedge)/\mathcal{B}^{-\infty,d}(X^\wedge).$$

The isomorphism is given by $f \mapsto \exp_* b$ with $b(y,\eta) = f(e^y, -i\eta)$; the inverse by $a \mapsto f$ with $f(s,z) = [\ln_* a](\ln s, iz)$.

Proof. By 2.4.2 and 2.4.3 the mapping $f \mapsto \exp_* b$, where $b = f(e^y, -i\eta)$, maps the left hand side to the right hand side injectively. A direct computation then yields the above inverse. ◁

2.4.6 Corollary. If $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ and $\mathrm{op}_M^{1/2} f \in \mathcal{B}^{-\infty,d}(X^\wedge)$, then $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$.

Let us now have a look at a classical element $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0))$. For each $t \in \mathbf{R}_+$ we have the complete parameter-dependent interior symbol $\sigma_\psi(f(t))$ and the complete parameter-dependent boundary symbol $\sigma_\wedge(f(t))$, cf. 1.3.4. Both are smooth up to $t=0$. Thus all the homogeneous components of the local representatives p_j of the interior symbol and the homogeneous components of the elements g_j, k_j, t_j, s_j of the local representatives of the Green, potential, trace and boundary part of the boundary symbol are smooth up to $t = 0$.

2.4.7 Lemma. *If $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0))$ and $\mathrm{op}_M^{1/2} f \in \mathcal{B}^{-\infty,d}(X^\wedge)$, then $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$.*

Proof. In virtue of Corollary 2.4.6 we know that $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$. Thus all homogeneous components of the symbol of f vanish on $\mathbf{R}_+$. Since they are smooth on $\overline{\mathbf{R}}_+$, we obtain the assertion. ◁

For what follows it will be interesting to know more precisely what the push-forward by exp looks like. We start with a formal calculation.

2.4.8 Lemma. *Let $p \in S^\mu(\mathbf{R}_+, \mathbf{R})$. Then $\exp_* \mathrm{op}\, p$ is the pseudodifferential operator with the 'double' symbol*

$$(\exp_* p)(t,t',\tau) = p(\ln t, M(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-1}. \tag{1}$$

Here $M(t,t') = \frac{\ln t - \ln t'}{t-t'}$ is C^∞ and strictly positive on $\mathbf{R}_+ \times \mathbf{R}_+$.

Proof. For $u \in C_0^\infty(\mathbf{R}_+)$, $t' = e^{y'}$, we have

$$\begin{aligned}
[\mathrm{op}\, p(u \circ \exp)](\ln t) &= \frac{1}{2\pi}\int\int e^{i(\ln t - y')\eta} p(\ln t, \eta) u(e^{y'}) dy' d\eta \\
&= \frac{1}{2\pi}\int\int_0^\infty e^{i(\ln t - \ln t')\eta} p(\ln t, \eta) u(t') \frac{dt'}{t'} d\eta \\
&= \frac{1}{2\pi}\int\int_0^\infty e^{i(t-t')M(t,t')\eta} p(\ln t, \eta) u(t') \frac{dt'}{t'} d\eta \\
&= \frac{1}{2\pi}\int\int_0^\infty e^{i(t-t')\tau} p(\ln t, M(t,t')^{-1}\tau) u(t') \frac{1}{t'} M(t,t')^{-1} dt' d\tau.
\end{aligned}$$

This gives (1). The function $M(t,t')$ is smooth and ≥ 0, for ln is monotonely increasing. Moreover, M has no zero since, for $t = t'$, we have $M(t,t) = \frac{1}{t} > 0$. $\triangleleft$

2.4.9 Lemma. (a) $\partial_{t'}^k M(t,t')\,|_{t'=t} = c_k t^{-k-1}$ *for suitable* $c_k \in \mathbf{R}$, $k = 0, 1, \dots$. *In particular,* $(t'\partial_{t'})^k[t'M(t,t')]|_{t'=t}$ *is smooth up to* $t = 0$.
(b) $t^{k-1}\partial_{t'}^k[M(t,t')^{-1}]|_{t'=t}$ *is smooth up to* $t = 0, k = 0, 1, \dots$.

Proof. (a) Let $u, v \in \mathbf{R}_+$. We have for $1 + x = \frac{u}{v}$, $|x| < 1$

$$\ln u - \ln v = \ln(\frac{u}{v}) = \ln(1+x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \frac{(u-v)^j}{v^j},$$

hence

$$M(u,v) = \frac{\ln u - \ln v}{u - v} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \frac{(u-v)^k}{v^{k+1}}.$$

Therefore

$$\partial_u^k M(u,v)\,|_{u=v} = k! \frac{(-1)^k}{k+1} v^{-k-1}.$$

This proves the first statement. Applying Lemma 2.3.1 we obtain the second statement, too.
(b) By induction, $\partial_{t'}^k[M(t,t')^{-1}]$ is a linear combination of terms of the form

$$M(t,t')^{-r-1} \prod_{l=1}^{r} \partial_{t'}^{j_l} M(t,t'),$$

where $r \leq k$ and $\sum_{l=1}^{r} j_l = k$. This implies that $\partial_{t'}^k[M(t,t')^{-1}]\,|_{t'=t}$ is a linear combination of terms $t^{r+1}t^{-r-k}$, $0 \leq r \leq k$. $\triangleleft$

2.4.10 Definition. Let $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$. By $C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ we denote the set of all $a \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$ for which there is a $b \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$ such that

$$a(t,\tau) = b(t, t\tau).$$

We call these operator-valued symbols *totally characteristic* or *Fuchs type symbols.* Analogously we define $C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}_{cl}^{\mu,d}(X;\mathbf{R}))$ as the set of all $a \in C^\infty(\mathbf{R}_+, \mathcal{B}_{cl}^{\mu,d}(X;\mathbf{R}))$ for which there is a $b \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}_{cl}^{\mu,d}(X;\mathbf{R}))$ such that $a(t,\tau) = b(t,t\tau)$.

2.4.11 Remark. What does this mean for the symbols of the corresponding operators in Boutet de Monvel's calculus? For each fixed $t > 0$, the parameter-dependent operator $a(t) \in \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R})$ has a complete parameter-dependent interior symbol $\sigma_\psi(a(t))$ and a complete parameter-dependent boundary symbol $\sigma_\wedge(a(t))$ see 1.3.4 for details. The fact that $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ implies that both $\sigma_\psi(a(t))(x,\xi,\tau/t)$ and $\sigma_\wedge(a(t))(x',\xi',\tau/t)$ are smooth in t up to $t = 0$. If $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}_{cl}^{\mu,d}(X;\mathbf{R}))$ then
(i) the homogeneous components of the local complete parameter-dependent interior symbols $\sigma_\psi(a(t))(x,\xi,\tau/t)$ are smooth in t up to $t = 0$;

(ii) for the local complete parameter-dependent boundary symbols (now also depending on t),

$$\sigma_\wedge(a(t))_j(x',\xi',\tau) = \begin{pmatrix} \mathrm{op}^+_{x_n} p_j(x,t,\xi',\tau) + g_j(x',t,\xi',\tau) & k_j(x',t,\xi',\tau) \\ t_j(x',t,\xi',\tau) & s_j(x',t,\xi',\tau) \end{pmatrix},$$

all the homogeneous components of the operator-valued symbols

$$g_j(x',t,\xi',\tau/t), k_j(x',t,\xi',\tau/t), t_j(x',t,\xi',\tau/t), s_j(x',t,\xi',\tau/t)$$

are smooth in t up to $t=0$.

2.4.12 Theorem. *For $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ there is an $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ with*

$$\mathrm{op}\, a \equiv \mathrm{op}_M^{1/2} f \mod \mathcal{B}^{-\infty,d}(X^\wedge). \tag{1}$$

If $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0))$, then we can find $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}_{cl}(X;\mathbf{R}))$.

Proof. We know from 2.4.3 that $\mathrm{op}_M^{1/2} f \equiv \mathrm{op}\,(\exp_* p)$ where $p(y,\eta) = f(e^y, -i\eta)$, and, according to Lemma 2.4.8, $c_1(t,t',\tau) = [\exp_* p](t,t',\tau) = p(\ln t, M(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-1}$ $= f(t, -iM(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-1}$. Let us convert the 'double' symbol c_1 to a symbol $c \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$ independent of t':

$$c(t,\tau) \sim \sum_{k=0}^\infty \frac{1}{k!}\partial_{t'}^k D_\tau^k c_1(t,t',\tau)|_{t'=t}. \tag{2}$$

Now

$$\partial_{t'}^k D_\tau^k c_1(t,t',\tau) = \partial_{t'}^k\{(-i)^k(\partial_z^k f)(t, -iM(t,t')^{-1}\tau)\frac{1}{t'}M(t,t')^{-k-1}\}. \tag{3}$$

By induction this is a linear combination of terms of the form

$$(\partial_z^{k+j} f)(t, -iM(t,t')^{-1}\tau)\,\tau^j\, g_{kj}(t,t'), \quad j = 0,\dots,k, \tag{4}$$

where $g_{kj}(t,t')$ is a linear combination of terms of the form

$$(t')^{-1-l_0}\prod_{i=1}^r \partial_{t'}^{l_i}\{M(t,t')^{-1}\}.$$

Here $r = k+1+j$, and $l_0 + \sum_{i=1}^r l_i = k$. Using Lemma 2.4.9 we conclude that $t^{-j}g_{kj}(t,t)$ is smooth up to $t=0$.
Combining (3) and (4) we see that $\partial_{t'}^k D_\tau^k c_1(t,t',\tau)\,|_{t'=t}$ is a linear combination of terms of the form $(\partial_z^{k+j} f)(t, -it\tau)(t\tau)^j s_{kj}(t)$, where s_{kj} is a smooth function on $\overline{\mathbf{R}}_+$. Since $(\partial_z^{k+j} f)(t,-it\tau) \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu-k-j,d}(X;\mathbf{R}))$, we obtain the symbol a by asymptotic summation in $C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$. Note that there is asymptotic summation in this class: Given a sequence $\{a_j\}$ with $a_j \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu-j,d}(X;\mathbf{R}))$ and $a_j(t,\tau) = b_j(t,t\tau)$ for $b_j \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu-j,d}(X;\mathbf{R}))$ choose $b \sim \sum b_j$ and let $a(t,\tau) = b(t,t\tau)$. Then $a - \sum_{j=0}^N a_j \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu-N,d}(X;\mathbf{R})) \subseteq C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu-N,d}(X;\mathbf{R}))$; hence $\mathrm{op}\, c - \mathrm{op}\, a \in \mathcal{B}^{-\infty,d}(X^\wedge)$.
If f is classical, then the construction shows that the resulting a also is classical. ◁

2.4.13 Theorem. (Mellin Quantization) *Let $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$. Then there is an $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ such that*

$$\mathrm{op}_M^{1/2} f \equiv \mathrm{op}\, a \mod \mathcal{B}^{-\infty,d}(X^\wedge). \tag{1}$$

For later use we note that $f(t,i\tau) - a(t,-\tau/t) \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu-1,d}(X,\mathbf{R}_\tau))$.
For $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}_{cl}(X;\mathbf{R}))$ we can choose $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}_{cl}(X;\Gamma_0))$. In that case, f is unique up to an element in $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$ and

$$f(t,i\tau) - a(t,-\tau/t) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu-1,d}_{cl}(X;\Gamma_0)).$$

Proof. We know from Lemma 2.4.5 that $\mathrm{op}\, a \equiv \mathrm{op}_M^{1/2} g$ with

$$g(t,t',z) = [\ln_* a](\ln t, \ln t', iz)\ ; \tag{2}$$

here, we use the 'double' symbol of $[\ln_* a]$ one obtains by straightforward substitution in the oscillatory integral. Given a symbol $q \in S^\mu(\mathbf{R}_+,\mathbf{R})$ a computation similar to that in 2.4.8 shows that

$$(\ln_* q)(y,y',\eta) = q(e^y, M(e^y,e^{y'})\eta)e^{y'}M(e^y,e^{y'})$$

with the function $M(t,t') = \frac{\ln t - \ln t'}{t-t'}$ introduced in 2.4.8. Hence, in our case,

$$g(t,t',i\tau) = a(t, -M(t,t')\tau)\, t'M(t,t'). \tag{3}$$

Now we apply Theorem 2.3.3. We have $\mathrm{op}_M^{1/2} g \equiv \mathrm{op}_M^{1/2} f \mod \mathcal{B}^{-\infty,d}(X^\wedge)$ whenever $f \in C^\infty(\mathbf{R}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ has the asymptotic expansion

$$f(t,z) \sim \sum_{k=0}^{\infty} \frac{1}{k!}(-t'\partial_{t'})^k \partial_z^k g(t,t',z)\ |_{t'=t}\ . \tag{4}$$

By assumption, the symbol $b(t,\tau) = a(t,t^{-1}\tau)$ is an element of $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. Thus $t^{-k}(\partial_\tau^k a)(t,t^{-1}\tau) = \partial_\tau^k b(t,\tau) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu-k,d}(X;\mathbf{R}))$. By Lemma 2.4.9, the function $(t'\partial_{t'})^j(t'M(t,t'))\ |_{t'=t}$ is smooth up to $t=0$ for $j=0,1,\ldots$. So all the terms on the right hand side of (4) are smooth up to $t=0$, and the asymptotic summation can be carried out in $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$.
If a is classical, then the asymptotic expansion (4) produces a classical Mellin symbol f. Suppose we have $f, \tilde{f} \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ with $\mathrm{op}_M^{1/2} f \equiv \mathrm{op}\, a \equiv \mathrm{op}_M^{1/2}\tilde{f} \mod \mathcal{B}^{-\infty,d}(X^\wedge)$. Then $\mathrm{op}_M^{1/2} f - \mathrm{op}_M^{1/2}\tilde{f} \in \mathcal{B}^{-\infty,d}(X^\wedge)$, so $f - \tilde{f} \in C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$ by Corollary 2.4.6. If, in addition, f and $\tilde{f}$ are classical Mellin symbols, then $f - \tilde{f} \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$ by Lemma 2.4.7. From the corresponding identity in the non-classical case we have

$$f(t,i\tau) - a(t,-\tau/t) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu-1,d}_{cl}(X;\Gamma_0)),$$

since the homogeneous components of order μ vanish on $\mathbf{R}_+$ and are smooth up to $t=0$.
◁

2.4.14 Mellin Quantization for Arbitrary Weights. In the previous section we studied the question how to associate to a totally characteristic pseudodifferential symbol $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ a Mellin symbol $f_{1/2} \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ with $\mathrm{op}\, a \equiv \mathrm{op}_M^{1/2} f_{1/2} \mod \mathcal{B}^{-\infty,d}(X^\wedge)$. Given an arbitrary weight $\gamma \in \mathbf{R}$ this result allows us to easily find a Mellin symbol $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$ such that $\mathrm{op}\, a \equiv \mathrm{op}_M^\gamma f_\gamma \mod \mathcal{B}^{-\infty,d}(X^\wedge)$:

2.4.15 Theorem. *For every* $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ *and every* $\gamma \in \mathbf{R}$ *there is an* $f_\gamma \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$ *such that*

$$\mathrm{op}_M^\gamma f_\gamma \equiv \mathrm{op}\, a \mod \mathcal{B}^{-\infty,d}(X^\wedge). \tag{1}$$

The corresponding statement holds for classical symbols.

Proof. The Mellin symbol f_γ can be computed in terms of the function $f = f_{1/2}$ in Theorem 2.4.13. The definition of op_M^γ shows that

$$\mathrm{op}\, a \equiv \mathrm{op}_M^{1/2} f_{1/2} = \mathrm{op}_M^\gamma g_\gamma,$$

where $g_\gamma(t,t',z) = (t/t')^{1/2-\gamma} f_{1/2}(t, z-1/2+\gamma)$. We convert g_γ to a t'-independent symbol f_γ with

$$\begin{aligned} f_\gamma(t,z) &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k \partial_z^k g_\gamma(t,t',z)|_{t'=t} \\ &\sim \sum_{k=0}^\infty \frac{1}{k!}(-t'\partial_{t'})^k (\frac{t}{t'})^{1/2-\gamma}|_{t'=t}\ \partial_z^k f_{1/2}(t, z-1/2+\gamma) \\ &\sim \sum_{k=0}^\infty \frac{1}{k!}(1/2-\gamma)^k \partial_z^k f_{1/2}(t, z-1/2+\gamma), \end{aligned} \tag{2}$$

where we used that $(-t'\partial_{t'})^k (t/t')^{1/2-\gamma}|_{t'=t} = (x\partial_x)^k x^{1/2-\gamma}|_{x=1} = (1/2-\gamma)^k$. Since $f_{1/2}$ is smooth up to $t=0$, the asymptotic summation can be carried out in $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$, and we obtain the assertion.
If a is classical, then so is $f_{1/2}$ by 2.4.13, hence f_γ will be classical. ◁

2.4.16 Remark. In Part I we defined spaces $M_P^{\mu,d}(X)$ of (t,t')-independent Mellin symbols of order μ and type d. They are meromorphic functions on $\mathbf{C}$, their only singularities are poles described in terms of the asymptotic type P. We can then consider the classes $C^\infty(\overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ and the associated Mellin operators. If the singularity set P is empty we shall write $h \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Then $h(t,\cdot)$ is an entire function, and Cauchy's theorem implies that $\mathrm{op}_M^\gamma h = \mathrm{op}_M^{\gamma'} h$ for all $\gamma,\gamma' \in \mathbf{R}$. We now let $f_{1/2} = h|_{\Gamma_0}$. According to Theorem 2.4.15, $\mathrm{op}_M^\gamma f_\gamma \equiv \mathrm{op}_M^\gamma(h|_{\Gamma_{1/2-\gamma}}) \mod \mathcal{B}^{-\infty,d}(X^\wedge)$. Therefore, $f_\gamma - h|_{\Gamma_{1/2-\gamma}} \in C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_{1/2-\gamma}))$. This can be viewed as a slightly different convergence result for the Taylor series on the right hand side of 2.4.15(2).

2.4.17 Corollary. For every $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\mu,d}(X;\mathbf{R}))$ there is an $h_\gamma \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ such that

$$\mathrm{op}_M^\gamma h_\gamma \equiv \mathrm{op}\, a \mod \mathcal{B}^{-\infty,d}(X^\wedge) \tag{1}$$

for every $\gamma \in \mathbf{R}$. The corresponding statement holds for classical symbols.

Proof. According to Theorem 2.4.13 we find $f \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ with $\mathrm{op} a \equiv \mathrm{op}_M^{1/2} f$ mod $\mathcal{B}^{-\infty,d}(X^\wedge)$. Choose $\varphi \in C_0^\infty(\mathbf{R}_+)$ with $\varphi(\rho) \equiv 1$ in a neighborhood of $\rho = 1$. Let

$$h(t,z) = M_{\rho\to z}\varphi(\rho) M^{-1}_{1/2,\zeta\to\rho} f(t,\zeta).$$

Then $h \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ by Theorem 2.2.17 and $f - h \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$ by Corollary 2.2.18. By Cauchy's theorem, cf. Remark 2.4.16,

$$\mathrm{op}_M^\gamma h = \mathrm{op}_M^{1/2} h \equiv \mathrm{op}_M^{1/2} f \equiv \mathrm{op} a \mod \mathcal{B}^{-\infty,d}(X).$$

Given classical f, the Mellin symbol h will be classical by Proposition 2.2.21. ◁

3 The Cone Algebra with Asymptotics

3.1 The Cone Pseudodifferential Operators

3.1.1 Notation and Terminology. Let us recall the 'stretched object' $I\!D$ associated with the manifold with conical singularities D, cf. Definition 1.1.3. It is a manifold with cylindrical ends obtained by replacing, for each singularity v, a neighborhood of v by the cylinder $\overline{X} \times [0,1)$. We assume that the gluing near $\overline{X} \times \{1\}$ between the cylindrical and the remaining part has been performed by a fixed diffeomorphism. We will now construct an algebra of operators which consists of usual operators in Boutel de Monvel's calculus outside any neighborhood $\overline{X} \times \{0\}$, and Mellin operators on functions or sections over $\overline{X} \times [0,1)$. We will patch these operators together to operators acting on functions or sections over $I\!D$. In order to avoid superfluous pull-backs and push-forwards we will identify operators defined over the above cylindrical part of $I\!D$ by operators defined over $\overline{X} \times [0,1)$ or $\overline{X} \times \overline{\mathbf{R}}_+$. We shall say that a function or distribution on $I\!D$ is *supported close to the singular set*, if there is an $\varepsilon > 0$ such that it vanishes outside the sets $\overline{X} \times [0,\varepsilon)$ associated with the singularities. Conversely, we shall say that it is *supported away from the singular set*, if it vanishes on all the sets $\overline{X} \times [0,\varepsilon)$ for suitable $\varepsilon > 0$. We shall also use the terminology *"on the singular part of $I\!D$"* and *"on the regular part of $I\!D$"*. We shall now consider Mellin symbols $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$. Here $\mu \in \mathbf{Z}$, $d \in \mathbf{N}$, and P is a Mellin asymptotic type, cf. 1.7.2(a). For $(t,t') \in \overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$, $h(t,t',\cdot)$ is a meromorphic function on $\mathbf{C}$ with singularities described by P; it takes values in Boutel de Monvel's algebra over X. $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ is endowed with the natural Fréchet topology induced by the Fréchet topology of $M_P^{\mu,d}(X)$. By definition we have $M_P^{\mu,d}(X) \hookrightarrow \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})$, hence

$$C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X)) \hookrightarrow C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma})) \tag{1}$$

provided the line $\Gamma_{1/2-\gamma}$ does not intersect the singularity set $\pi_{\mathbf{C}}P$ of P.

3.1.2 Conventions. For the rest of this section let $\mu \in \mathbf{Z}$, $d \in \mathbf{N}, N \in \mathbf{N}, \gamma \in \mathbf{R}$, and the Mellin asymptotic type P be fixed; $\mathfrak{g}_0$ is the weight datum $(\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-N, 0])$. We shall always assume that

$$\pi_{\mathbf{C}}P \cap \Gamma_{1/2-\gamma} = \emptyset.$$

Moreover we suppose that a generic element $h \in \mathbf{C}^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ acts on vector bundles V_1, V_2 over $\overline{X}$ and W_1, W_2 over Y as in Section 2, cf. 2.1.1(1).

Unless specified otherwise, $\omega, \omega_1, \omega_2, \ldots, \tilde{\omega}, \tilde{\omega}_1, \tilde{\omega}_2, \ldots$ denote functions in $C_0^\infty(\overline{\mathbf{R}}_+)$. When speaking of an element $A \in \mathcal{B}^{\mu,d}(I\!D)$ in Boutet de Monvel's calculus on $I\!D$, we mean an element of Boutet de Monvel's calculus on the open bounded manifold $I\!D \setminus \{t = 0\}$. The following lemma collects a few straightforward results.

3.1.3 Lemma. *For $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ the following assertions hold.*
(a) *There are functions $\varphi_j, \psi_j \in C^\infty(\overline{\mathbf{R}}_+)$, $j = 1, 2, \ldots$, tending to zero in the topology of $C^\infty(\overline{\mathbf{R}}_+)$, elements $a_j \in M_P^{\mu,d}(X)$, $j = 1, 2, \ldots$, tending to zero in the Fréchet topology of $M_P^{\mu,d}(X)$, cf. 1.7.3, and a sequence $\{\lambda_j\} \in l^1$ such that*

$$h(t,t',z) = \sum_{j=1}^{\infty} \lambda_j \varphi_j(t) \psi_j(t') a_j(z) \tag{1}$$

with convergence in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$. *Vice versa, each series with these properties defines an element of* $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$. *A corresponding result holds for* $C^\infty(\overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$.
(b) *h can be written*

$$h(t,t',z) = \sum_{j=1}^{d} h_j(t,t'z) \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix},$$

where $h_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_Q^{\mu-j,0}(X))$, ∂_r *denotes the normal derivative, and* Q *is a slightly modified Mellin asymptotic type with* $\pi_{\mathbf{C}}Q = \pi_{\mathbf{C}}P$, *cf. Theorem* I.4.1.5.
(c)

$$C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X)) = C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X)) + C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$$

is a non-direct sum of Fréchet spaces. This result will be improved by Theorem 3.1.9, *below.*
(d)

$$\mathrm{op}_M^\gamma h : \begin{array}{c} C_0^\infty(\overline{X}^\wedge, V_1) \\ \oplus \\ C_0^\infty(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y^\wedge, W_2) \end{array}$$

is continuous.
(e) *For* $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$

$$\omega_1[\mathrm{op}_M^\gamma h]\omega_2 : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is bounded provided $s > d - 1/2$. *For* $d = 0$ *and* $s \leq -1/2$,

$$\omega_1[\mathrm{op}_M^\gamma h]\omega_2 : \begin{array}{c} \mathcal{H}_0^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}_{\{0\}}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is bounded. The subscript $\{0\}$ *indicates that we are using the* $\mathcal{H}_0$*-spaces for* $s - \mu \leq 0$ *and the usual* $\mathcal{H}$*-spaces otherwise.*

Proof. (a) is immediate from the nuclearity of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+)$, cf. the proof of Lemma 2.1.4.
(b) follows from (a) and the decomposition result for $M_P^{\mu,d}(X)$ in Theorem 1.7.4.
(c) follows from (a) and Theorem 1.7.6.
(d) and (e) follow from the corresponding results in the case without asymptotics, since $h|_{\Gamma_{1/2-\gamma}} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X; \Gamma_{1/2-\gamma}))$ as $\pi_{\mathbf{C}}P \cap \Gamma_{1/2-\gamma} = \emptyset$. ◁

We shall need the following Lemma as a preparation for Theorem 3.1.6, but it also is of independent interest.

3.1.4 Lemma. *Let* $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ *and* $\psi \in C_0^\infty(\mathbf{R}_+)$ *with* $\psi(\rho) \equiv 1$ *for* ρ *close to* 1. *Define* $h_\psi(t,t',z) = M_{\rho\to z}\psi(\rho)M_{\gamma,\zeta\to\rho}^{-1}h(t,t',\zeta)$. *Then*

$$h - h_\psi \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{-\infty,d}(X)).$$

Recall that we showed already that $h - h_\psi \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_{1/2-\gamma})) \cap C^\infty(\overline{\mathbf{R}}_+ \times\overline{\mathbf{R}}_+,, M_O^{\mu,d}(X))$, cf. Theorem 2.2.17 and Corollary 2.2.18.

Proof. Without loss of generality let us assume $\gamma = 1/2$. Choose $\beta \in \mathbf{R}$ and a nonnegative integer $M > \mu+|\beta|+1$. Then $D_z^M h(t,t',\cdot)$ is integrable along Γ_β. Moreover, the analyticity of the function $z \mapsto \rho^{-z} D_z^M h(t,t',z)$ together with Cauchy's Theorem implies that

$$\int_{-\infty}^{\infty} \rho^{-i\tau}(D_z^M h)(t,t',i\tau)d\tau = \int_{-\infty}^{\infty} \rho^{-(\beta+i\tau)}(D_z^M h)(t,t',\beta+i\tau)d\tau\,, \tag{1}$$

so that $(M_{1/2}^{-1}(D_z^M h))(t,t',\rho) = \rho^{-\beta} M_{1/2,\zeta\to\rho}^{-1}(D_z^M h(t,t',\,\zeta+\beta))$. Hence, for $z = \beta + i\tau$,

$$\begin{aligned}
& (h-h_\psi)(t,t',z) & (2)\\
= & \int_0^\infty \rho^{\beta+i\tau-1}(1-\psi(\rho))(M_{1/2}^{-1}h)\,(t,t',\rho)\;d\rho \\
= & \int_0^\infty \rho^{\beta+i\tau-1}(1-\psi(\rho))\ln^{-M}\rho(M_{1/2}^{-1}(D_z^M h))(t,t',\rho)\;d\rho \\
= & \int_0^\infty \rho^{\beta+i\tau-1}(1-\psi(\rho))\ln^{-M}\rho\rho^{-\beta}M_{1/2/,\zeta\to\rho}^{-1}(D_z^M h)(t,t',\,\zeta+\beta)\;d\rho \\
= & \int_0^\infty \rho^{i\tau-1}(1-\psi(\rho))M_{1/2,\zeta\to\rho}^{-1}h(t,t',\,\zeta+\beta)\;d\rho \\
= & \left[M_{1/2,\rho\to z}(1-\psi(\rho))M_{1/2,\zeta\to\rho}^{-1}h(t,t',\,\zeta+\beta)\right](z-\beta). & (3)
\end{aligned}$$

On the other hand, the function $(t,t',z) \mapsto h(t,t',z+\beta)$ is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$; the corresponding symbol estimates hold uniformly for β in compact intervals. Applying Corollary 2.2.18, the function in (2) is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_\beta))$, uniformly for β in compact intervals. ◁

3.1.5 Corollary. For $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ we have $h - h_\psi \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$. The notation is as in Lemma 3.1.4.

Proof. We can write $h = h_0 + h_s$, with $h_0 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ and $h_s \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty}(X))$. Denoting by the subscript ψ the result of the operator $f \mapsto M\psi M_\gamma^{-1} f$ of Lemma 3.1.4 we have

$$h - h_\psi = (h_0 - h_{0\psi}) + h_s - h_{s\psi}.$$

By Lemma 3.1.4, $h_0 - h_{0\psi} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$, while $h_{s\psi} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$ by Theorem 2.2.17. ◁

3.1.6 Corollary. If $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ and $h|_{\Gamma_{1/2-\gamma}} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_{1/2-\gamma}))$ then $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$.

Proof. Denote, as before, by the subscript ψ the result of the operator $f \mapsto M\psi M_\gamma^{-1} f$ of Lemma 3.1.4. Then $h_\psi \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$ by Theorem 2.2.17, while $h - h_\psi \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$ by Corollary 3.1.5. ◁

3.1.7 Theorem. (Asymptotic Summation) *Let* $h_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu_j,d}(X))$, $j = 0, 1, \ldots$, *with* $\mu_j \to -\infty$ *and* $\mu = \max\{\mu_j\}$. *Then there is an* $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ *with* $h \sim \sum_{j=0}^\infty h_j$, *i.e., given* $M \in \mathbf{N}$ *there is a* $J \in \mathbf{N}$ *with*

$$h - \sum_{j=0}^{J} h_j \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu-M,d}(X)). \tag{1}$$

Proof. We have $h_j|_{\Gamma_0} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu_j,d}(X;\Gamma_0))$ so that, by Theorem 2.1.12, there is an $f \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_0))$ with $f \sim \sum h_j|_{\Gamma_0}$ modulo $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_0))$. Let $h = M\psi M_{1/2}^{-1} f$ for some $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho) \equiv 1$ near $\rho = 1$. Then $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Given $M \in \mathbf{N}$ there is a J such that $f_J := f - \sum_{j=0}^J h_j|_{\Gamma_0} \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-M,d}(X;\Gamma_0))$. Then

$$\begin{aligned} h - \sum_{j=0}^{J} h_j &= M\,\psi\, M_{1/2}^{-1} f_J - M\,(1-\psi)\, M_{1/2}^{-1}\left(\sum_{j=0}^{J} h_j|_{\Gamma_0}\right) \\ &\in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu-M,d}(X)) \end{aligned}$$

by Lemma 3.1.4 and Theorem 2.2.17. $\triangleleft$

3.1.8 Lemma. *Given* $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, *and* $Q = (Q_1, Q_2) \in As(X, Y, (\gamma + \frac{n}{2}, (-N, 0]))$, *there is a resulting asymptotic type* $R = (R_1, R_2) \in As(X, Y, (\gamma + \frac{n}{2}, (-N, 0])$ *such that*

$$\omega_1[\mathrm{op}_M^\gamma h]\omega_2 : \begin{array}{c} \mathcal{H}_{Q_1}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}_{Q_2}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}_{R_1}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}_{R_2}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is continuous for each $s > d - 1/2$.

Proof. This follows from the representation in 3.1.3(a): First fix j, and consider $h_j(t, t', z) = \lambda_j \varphi_j(t) \psi_j(t') a_j(z)$ with $\lambda_j \in \mathbf{C}$, $\varphi_j, \psi_j \in C_0^\infty(\overline{\mathbf{R}}_+)$ and $a_j \in M_P^{\mu,d}(X)$. By Lemma 1.5.4, multiplication by ψ_j maps $\mathcal{H}_{Q_1}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{H}_{Q_2}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1)$ to $\mathcal{H}_{Q_1'}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{H}_{Q_2'}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1)$. Here, $Q' = (Q_1', Q_2')$ is the asymptotic type induced by Q with the 'shadows' in 1.5.4 added.
According to Theorem 1.7.12, $\mathrm{op}_M^\gamma a_j$ maps this space to $\mathcal{H}_{R_1'}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \oplus \mathcal{H}_{R_2'}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2)$ for suitable $R' = (R_1', R_2')$. Now the continuity of the multiplication by φ_j shows the continuity of $\mathrm{op}_M^\gamma h_j$.
The topologies on $C_0^\infty(\overline{\mathbf{R}}_+)$ and $M_P^{\mu,d}(X)$ are stronger than the topologies of continuous mappings involved. Therefore the fact that $\sum_{j=0}^\infty h_j$ converges in the symbol topology implies the assertion. $\triangleleft$

3.1.9 Theorem. *For* $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$ *and* $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$,

$$\omega_1[\mathrm{op}_M^\gamma h]\omega_2 \in C_{M+G}^{0,d}(X^\wedge, \mathbf{g}_0).$$

Proof. Write

$$
\begin{aligned}
h(t,t',z) &= \sum_{j=0}^{N} \frac{t^j}{j!}\partial_t^j h(0,t',z) + t^{N+1}\tilde{h}_N(t,t',z) \\
&= \sum_{j,k=0}^{N} \frac{t^j}{j!}\frac{t'^k}{k!}\partial_t^j\partial_{t'}^k h(0,0,z) + t'^{N+1}\tilde{\tilde{h}}_N(t',z) + t^{N+1}\tilde{h}_N(t,t',z) \\
&= \sum_{j,k=0}^{N} t^j t'^k h_{jk}(z) + t'^{N+1}\tilde{\tilde{h}}_N(t',z) + t^{N+1}\tilde{h}_N(t,t',z) \qquad (1)
\end{aligned}
$$

with the obvious notation. Then $h_{jk} \in M_P^{-\infty,d}(X), \tilde{h}_N \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$, and $\tilde{\tilde{h}}_N \in C^\infty(\overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$.
Let us treat the three terms separately starting with $t^j t'^k h_{jk}$. Choose a small $\varepsilon_k \geq 0$ such that $\Gamma_{1/2-\gamma+k-\varepsilon_k} \cap \pi_{\mathbf{C}} P = \emptyset$; $\varepsilon_k = 0$ is allowed. Indeed, for $k=0$, we can always choose $\varepsilon_0 = 0$, since $\Gamma_{1/2-\gamma} \cap \pi_{\mathbf{C}} P = \emptyset$ by assumption. Theorem 1.8.1 shows that

$$
\begin{aligned}
\omega_1[\mathrm{op}_M^\gamma t^j\, t'^k\, h_{jk}]\omega_2 &= \omega_1 t^{j+k} t^{-\varepsilon_k}[\mathrm{op}_M^\gamma(T^{-k+\varepsilon_k}h_{jk})]t^{\varepsilon_k}\omega_2 + G_{jk} \\
&= \omega_1 t^{j+k}[\mathrm{op}_M^{\gamma-\varepsilon_k}(T^{-k}h_{jk})]\omega_2 + G_{jk},
\end{aligned}
$$

with $G_{jk} \in C_G^d(X^\wedge, \mathbf{g}_0)$.
Let us check that $\omega_1 t^{j+k}[\mathrm{op}_M^{\gamma-\varepsilon_k}(T^{-k}h_{jk})]\omega_2$ is a smoothing Mellin operator: For one thing $T^{-k}h_{jk} \in M_{T^{-k}P}^{-\infty,d}(X)$; the notation $T^{-k}P$ indicates that the position of the poles are shifted by k. Moreover, we will have $\gamma - (j+k) \leq \gamma - \varepsilon_k \leq \gamma$, if ε_k is chosen sufficiently small.
Next let us show that $\mathrm{op}_M^\gamma[t'^{N+1}\tilde{\tilde{h}}_N(t',z)]$ is a Green operator in $C_G^d(X^\wedge, \mathbf{g}_0)$. According to Lemma 3.1.3(b), we can write

$$
\tilde{\tilde{h}}_N(t',z) = \sum_{j=0}^{d} h_{N,j}(t',z) \begin{bmatrix} \partial_r^j & 0 \\ 0 & I \end{bmatrix}
$$

with $h_{N,j} \in C^\infty(\overline{\mathbf{R}}_+, M_Q^{-\infty,0}(X))$ for a suitable asymptotic type Q with $\pi_{\mathbf{C}} Q = \pi_{\mathbf{C}} P$. Choose $\nu \in \mathbf{R}$ with $N < \nu < N+1$ and $\pi_{\mathbf{C}} P \cap \Gamma_{1/2-\gamma+\nu} = \emptyset$. By Theorem 1.8.1, $\omega_1[\mathrm{op}_M^\gamma h_{N,j}]t^{N+1}\omega_2 = \omega_1 t^\nu \mathrm{op}_M^\gamma[T^{-\nu}h_{N,j}]t^{N+1-\nu}\omega_2 + G_j t^{N+1-\nu}$ for some $G_j \in C_G^0(X^\wedge, \mathbf{g}_0)_{R,S}$ with appropriate asymptotic types $R = (R_1, R_2), S = (S_1, S_2)$. In view of the factor t^ν we conclude that

$$
\omega_1[\mathrm{op}_M^\gamma h_{N,j}]t^{N+1}\omega_2 : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{S}_{R_1}^{\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{S}_{R_2}^{\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}
$$

is continuous. Its adjoint is the operator $\overline{\omega}_2 t^{N+1}[\mathrm{op}_M^\gamma h_{N,j}]^*\overline{\omega}_1 = \overline{\omega}_2 t^{N+1}\mathrm{op}_M^{-\gamma-n} h_{N,j}^{(*)}\overline{\omega}_1$, cf. Lemma 2.3.2. It maps $\mathcal{H}^{s,-\gamma-\frac{n}{2}}(X^\wedge, V_2) \oplus \mathcal{H}^{s,-\gamma-\frac{n-1}{2}}(Y^\wedge, W_2)$ to $\mathcal{S}_O^{-\gamma-\frac{n}{2}}(X^\wedge, V_1) \oplus \mathcal{S}_O^{-\gamma-\frac{n-1}{2}}(Y^\wedge, W_1)$. Hence the second term in (1) is a Green operator.
In essentially the same way we can treat the third term. This completes the proof. $\triangleleft$

3.1.10 Lemma. *Let $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Then, for all $\beta, \gamma \in \mathbf{R}, \omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ we have*

$$
\omega_1[\mathrm{op}_M^\gamma h]t^\beta\omega_2 = \omega_1 t^\beta[\mathrm{op}_M^\gamma T^{-\beta}h]\omega_2.
$$

Recall that $T^{-\beta}$ is the translation operator by β: $T^{-\beta}h(t,t',z) = h(t,t',z-\beta)$.

Proof. Without loss of generality assume that V_1, V_2 are trivial 1-dimensional while $W_1 = W_2 = 0$. Let $u \in C_0^\infty(\overline{X}^\wedge)$, $x \in X$, $t \in \mathbf{R}_+$ be fixed. It is sufficient to show that

$$\omega_1(t)[\mathrm{op}_M^\gamma h](t^\beta \omega_2 u)(x,t) = \omega_1(t) t^\beta [\mathrm{op}_M^\gamma T^{-\beta} h](\omega_2 u)(x,t) . \tag{1}$$

Let us first suppose that

$$h(t,t',z) = \varphi(t)\,\psi(t')\,a(z) \tag{2}$$

with $\varphi, \psi \in C^\infty(\overline{\mathbf{R}}_+)$, $a \in M_O^{\mu,d}(X)$. Then

$$\begin{aligned} [\mathrm{op}_M^\gamma h](t^\beta \omega_2 u)(t) &= \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} \varphi(t)\,a(z) \int_0^\infty (t/t')^{-z} t'^\beta \psi(t')\,\omega_2(t') u(t')\,\frac{dt'}{t'} dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}} t^\beta \varphi(t)\,a(z) \int_0^\infty (t/t')^{-z-\beta} (\psi\,\omega_2\,u)(t')\,\frac{dt'}{t'} dz \\ &= \frac{1}{2\pi i} t^\beta \varphi(t) \int_{\Gamma_{1/2-\gamma+\beta}} a(w-\beta) \int_0^\infty (t/t')^{-w} (\psi\,\omega_2\,u)(t')\,\frac{dt'}{t'} dw . \end{aligned}$$

The interior integral furnishes a holomorphic function of w which is rapidly decreasing on all lines Γ_δ, uniformly for δ in compact intervals. Since the function $w \mapsto a(w-z)$ is holomorphic with values in $\mathcal{B}^{\mu,d}(X)$ and polynomially bounded, Cauchy's theorem shows that we can replace integration over $\Gamma_{1/2-\gamma+\beta}$ by integration over $\Gamma_{1/2-\gamma}$ in the outer integral. This proves the assertion for h of type (2).

In the general case, Lemma 3.1.3(a) shows that h is a series in terms of this kind which converges in all semi-norms. This implies the convergence of the corresponding operators in, say, $\mathcal{L}(C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X})), C^\infty(\mathbf{R}_+, C^\infty(\overline{X})))$. Hence (1) also holds in the general case. $\lhd$

3.1.11 Theorem. *Given $h = h(t,t',z) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$, there is a $g = g(t,z) \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ such that, for all $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$,*

$$\omega_1[\mathrm{op}_M^\gamma h]\omega_2 - \omega_1[\mathrm{op}_M^\gamma g]\omega_2 \in C_{M+G}^{0,d}(X^\wedge, \mathbf{g}_0) . \tag{1}$$

Similarly, there is an $\tilde{g} = \tilde{g}(t',z) \in C^\infty(\overline{\mathbf{R}}_+, M_P^{\mu,d}(X))$ such that

$$\omega_1[\mathrm{op}_M^\gamma h]\omega_2 - \omega_1[\mathrm{op}_M^\gamma \tilde{g}]\omega_2 \in C_{M+G}^{0,d}(X^\wedge, \mathbf{g}_0). \tag{2}$$

If $P = O$, then we can achieve that the error is in $C_G^d(X, \mathbf{g}_0)$.

Proof. According to Lemma 3.1.3(c), h can be written as a sum of two functions, one in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ and one in $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_P^{-\infty,d}(X))$. From Theorem 3.1.9 we know that the latter induces an operator in $C_{M+G}^{0,d}(X^\wedge, \mathbf{g}_0)$ after multiplication by ω_1 and ω_2. Hence we may assume that $P = O$. Let $m = 2N+2$. A Taylor expansion gives

$$h(t,t',z) = \sum_{j=0}^{m-1} t'^j h_j(t,z) + t'^m h_m(t,t',z) \tag{3}$$

with $h_j(t,z) = \frac{1}{j!}\partial_t^j h(t,0,z) \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, and $h_m \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. By Lemma 3.1.10

$$\omega_1 \mathrm{op}_M^\gamma[t'^j h_j]\omega_2 = \omega_1 \mathrm{op}_M^\gamma[t^j T^{-j} h_j]\omega_2 ,$$

so let us look at $t'^m h_m(t,t',z)$. We apply Theorem 2.3.3 in connection with Theorem 2.2.17 and Corollary 2.2.18: There is a function $g_m = g_m(t,z) \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$ and an $R \in M\mathcal{B}_\gamma^{-\infty,d}(X)$, such that

$$\mathrm{op}_M^\gamma(T^{-N-1}h_m) = \mathrm{op}_M^\gamma(T^{-N-1}g_m) + R.$$

Therefore, applying twice Lemma 3.1.10,

$$\begin{aligned}\omega_1 \mathrm{op}_M^\gamma(h_m(t,t',z)t'^m)\omega_2 &= \omega_1 \mathrm{op}_M^\gamma(t^{N+1}T^{-N-1}h_m(t,t',z)t'^{m-N-1})\omega_2 \\ &= \omega_1 \mathrm{op}_M^\gamma(t^m T^{-m} g_m(t,z))\omega_2 + \omega_1 t^{N+1} R t^{m-N-1}\omega_2\end{aligned}$$

The first summand is of the desired type. In view of the fact that both $N+1$ and $m-N-1$ are $> N$, both $\omega_1 t^{N+1}$ and $\omega_2 t^{m-N-1}$ map $\mathcal{H}^{\infty,\gamma+\frac{n}{2}}(X^\wedge)$ to $\mathcal{S}_O^{\gamma+\frac{n}{2}}(X^\wedge)$. Hence the second summand is an operator in $C_G^d(X^\wedge, \mathbf{g})_{O,O}$. This concludes the proof of the first part; the second part follows in the same way. ◁

3.1.12 Remark. We deduce from Lemma 3.1.4 and Theorems 2.3.3, 2.3.4 that we may choose g and h with the asymptotic expansion

$$\begin{aligned} g(t,z) &\sim \sum_{j=0}^\infty \frac{1}{j!}(-t'\partial_{t'})^j \partial_z^j h(t,t',z)|_{t'=t}, \text{ and} \\ h(t',z) &\sim \sum_{j=0}^\infty \frac{1}{j!}(-t\partial_t)^j(-\partial_z)^j h(t,t',z)|_{t=t'}\end{aligned}$$

modulo $C^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$.

3.1.13 Lemma. *Let $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X)), \omega \in C_0^\infty(\overline{\mathbf{R}}_+)$, and $\varphi \in C_0^\infty(\mathbf{R}_+)$ with $\operatorname{supp}\omega \cap \operatorname{supp}\varphi = \emptyset$. Then*
(a) $\omega[\mathrm{op}_M^\gamma h]\varphi \in C_G^d(X^\wedge, \mathbf{g}')_{O,O}$, *and*
(b) $\varphi[\mathrm{op}_M^\gamma h]\omega \in C_G^d(X^\wedge, \mathbf{g}')_{O,O}$.
Here, $\mathbf{g}' = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-\infty, 0])$.

Proof. (a) Writing

$$h(t,t',z) = \sum_{j=0}^d h_j(t,t',z)\begin{bmatrix}\partial_r^j & 0\\ 0 & I\end{bmatrix},$$

we see that it is no restriction to assume $d=0$. Moreover, we find $\tilde\varphi \in C_0^\infty(\mathbf{R}_+), \tilde\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\varphi\tilde\varphi = \varphi, \omega\tilde\omega = \omega$, and $\operatorname{supp}\tilde\varphi \cap \operatorname{supp}\tilde\omega = \emptyset$. Then $\omega\,[\mathrm{op}_M^\gamma h]\,\varphi = \omega\,\mathrm{op}_M^\gamma\,[\tilde\omega(t)\,h(t,t',z)\,\tilde\varphi(t')]\,\varphi$. In other words: we may assume that there is an $\varepsilon > 0$ such that

$$h(t,t',z) = 0 \text{ whenever } |t-t'| < \varepsilon.$$

Now let $m \in \mathbf{N}$ be arbitary and $\mathbf{g}_m = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-m, 0])$. It is sufficient to show the result for $\mathbf{g}_m$ instead of $\mathbf{g}'$. Applying Lemma 3.1.10

$$\begin{aligned}\omega\,[\mathrm{op}_M^\gamma h]\,\varphi &= \omega\,\mathrm{op}_M^\gamma[h(t,t',z)t'^m]\,[t^{-m}\varphi] \\ &= \omega\,t^m\,[\mathrm{op}_M^\gamma T^{-m}h]\,[t^{-m}\varphi]. \qquad (1)\end{aligned}$$

Note that $\varphi_m = t^{-m}\varphi \in C_0^\infty(\mathbf{R}_+)$ and, for arbitrary $L \in \mathbf{N}$, $(t-t')^{-L}(T^{-m}h)(t,t',z) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu,d}(X;\Gamma_{1/2-\gamma}))$. Combining Proposition 2.2.20 with Theorem 2.1.3, we conclude that, for every $s > -1/2$,

$$\omega[\mathrm{op}_M^\gamma h]\varphi : \begin{array}{c}\mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1)\\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1)\end{array} \to \omega \begin{array}{c}\mathcal{H}^{\infty,\gamma+m+\frac{n}{2}}(X^\wedge, V_2)\\ \oplus \\ \mathcal{H}^{\infty,\gamma+m+\frac{n-1}{2}}(Y^\wedge, W_2)\end{array} \hookrightarrow \begin{array}{c}\mathcal{S}_O^{\gamma+\frac{n}{2}}(X^\wedge, V_2)\\ \oplus \\ \mathcal{S}_O^{\gamma+\frac{n-1}{2}}(Y^\wedge, W_2)\end{array}.$$

Now consider the adjoints. We first note that, indeed, there is an adjoint, since, by Proposition 2.2.20, $\omega[\mathrm{op}_M^\gamma h]\varphi = \omega[\mathrm{op}_M^\gamma h_0]\varphi$ for some h_0 of order ≤ 0. Write

$$[\omega\,[\mathrm{op}_M^\gamma h]\,\varphi]^* = \overline{\varphi}\,[\tilde{\omega}\,[\mathrm{op}_M^\gamma h]\,\tilde{\varphi}]^*\,\overline{\omega}; \tag{2}$$

and apply the above method, then (a) is settled. The proof of (b) is analogous. ◁

3.1.14 Lemma. *Let $R \in \mathcal{B}^{-\infty,d}(X^\wedge)$ and $\varphi, \psi \in C_0^\infty(\mathbf{R}_+)$. Then, for every choice of $\gamma_1, \gamma_2 \in \mathbf{R}$,*

$$\varphi\, R\, \psi \in C_G^d(X^\wedge, \tilde{\mathbf{g}})_{O,O}$$

with $\tilde{\mathbf{g}} = (\gamma_1, \gamma_2, (-\infty, 0])$.

The *proof* is an easier analog of the proof of Lemma 3.1.13, using the fact that, for every choice of $M \in \mathbf{N}$, $t^{-M}\varphi$ and $t^{-M}\psi$ are functions in $C_0^\infty(\mathbf{R}_+)$. ◁

3.1.15 The Cone Algebra. Let $\gamma \in \mathbf{R}$, $\mu, \nu \in \mathbf{Z}$ such that $\mu - \nu \in \mathbf{N}$, $d, N \in \mathbf{N}$, $N > 0$. Write $\mathbf{g} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N, 0])$. Let V_1, V_2 be vector bundles over $I\!D$ and W_1, W_2 vector bundles over $\partial I\!D = I\!B$. In order to connect to the assumption made before assume that, near the conical points, i.e., on $X \times [0,1)$ and $Y \times [0,1)$, respectively, V_1 and V_2 are induced from bundles over X while W_1, W_2 are induced from bundles over Y. By $C_0^\infty(I\!D, V_1)$ denote for the moment the space of all smooth sections of V_1 that vanish near the singular set.
$C^{\nu,d}(I\!D, \mathbf{g})$ is the space of all operators

$$A : \begin{array}{c} C_0^\infty(I\!D, V_1) \\ \oplus \\ C_0^\infty(I\!B, W_1)\end{array} \longrightarrow \begin{array}{c} C^\infty(I\!D, V_2) \\ \oplus \\ C^\infty(I\!B, W_2)\end{array} \tag{1}$$

that can be written in the following form

$$A = A_M + A_\psi + R, \tag{2}$$

where

(i) $A_M = \omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h]\omega_2$, with suitable functions $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ supported in $[0,1)$ and $h \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu,d}(X))$, is a Mellin-type operator acting on functions supported close to the singular part.

(ii) $A_\psi = \psi_1 \tilde{A} \psi_2$, with suitable functions $\psi_1, \psi_2 \in C^\infty(I\!D)$, supported away from the singular set and $\tilde{A} \in \mathcal{B}^{\nu,d}(I\!D)$, is an operator in Boutet de Monvel's calculus on the regular part of $I\!D$.

(iii) $R \in C^{\nu,d}_{M+G}(\mathbb{D}, \mathbf{g})$, is a sum of smoothing Mellin and Green operators, acting close to the singular set, cf. Definition 1.9.1.

We call the entity of all $C^{\nu,d}(\mathbb{D}, \mathbf{g}), \mu \geq \nu \in \mathbf{Z}, d \in \mathbf{N}$ *the cone algebra associated with the weight datum* $\mathbf{g}$.

3.1.16 The Classical Cone Algebra. $C^{\nu,d}_{cl}(\mathbb{D}, \mathbf{g})$ is the space of all operators $A = A_M + A_\psi + R \in C^{\nu,d}(\mathbb{D}, \mathbf{g})$ for which, in the notation 3.1.15,

(i) the Mellin symbol h is an element of $C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M^{\mu,d}_{cl}(X))$, and

(ii) the pseudodifferential operator $\tilde{A}$ is an element of $\mathcal{B}^{\mu,d}_{cl}(\mathbb{D})$.

3.1.17 Remark. By Theorem 2.4.12 we have $C^{\nu,d}(\mathbb{D}, \mathbf{g}) \hookrightarrow \mathcal{B}^{\nu,d}(\mathbb{D})$. Let $\mathbf{g}_1 = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N_1, 0])$, $\mathbf{g}_2 = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N_2, 0])$ with $N_1 \geq N_2$, and use the notation of Definition 3.1.15. Then

$$C^{\nu,d}(\mathbb{D}, \mathbf{g}_1) \hookrightarrow C^{\nu,d}(\mathbb{D}, \mathbf{g}_2).$$

3.1.18 Remark. The boundary $\mathbb{B} = \partial\mathbb{D}$ of $\mathbb{D}$ is a boundaryless manifold with conical singularities, and the corresponding cone algebra $C^{\nu,d}(\mathbb{B}, \mathbf{g}')$ is embedded in $C^{\nu,d}(\mathbb{D}, \mathbf{g})$:

$$C^{\nu,d}(\mathbb{B}, \mathbf{g}') \hookrightarrow C^{\nu,d}(\mathbb{D}, \mathbf{g});$$

here $\mathbf{g}' = (\gamma + \frac{n-1}{2}, \gamma + \frac{n-1}{2} - \mu, (-N, 0])$, and the embedding is given by identifying parameter-dependent operators $A(\lambda)$ of the form

$$A(\lambda) = \begin{pmatrix} 0 & 0 \\ 0 & S(\lambda) \end{pmatrix} : \begin{array}{c} C^\infty(\overline{X}, V_1) \\ \oplus \\ C^\infty(Y, W_1) \end{array} \longrightarrow \begin{array}{c} C^\infty(\overline{X}, V_2) \\ \oplus \\ C^\infty(Y, W_2) \end{array}$$

in Boutet de Monvel's calculus on X with the parameter-dependent operator

$$S(\lambda) : C^\infty(Y, W_1) \to C^\infty(Y, W_2)$$

in the usual pseudodifferential calculus on Y, plus the corresponding identification of the lower right corners of elements in Boutet de Monvel's calculus on $\mathbb{D}$ with pseudodifferential operators on $\mathbb{B}$. The shift in the weight is due to the fact that $\dim \mathbb{B} = n - 1$ and that the dimension determines the positions of singularities, cf. Definition 1.9.1(i.2).

3.1.19 Lemma. *Suppose* $A \in \bigcap_{j=0}^\infty C^{\mu-j,d}(\mathbb{D}, \mathbf{g})$. *Then* $A \in C^d_G(\mathbb{D}, \mathbf{g})$.

Proof. For arbitrary $M \in \mathbf{N}$ write

$$A = t^{M-\mu}\omega_1 \,[\mathrm{op}^\gamma_M h]\, \omega_2 + (1 - \omega_4)A_\psi(1 - \omega_3) + R;$$

here, $h \in C^\infty(\overline{\mathbf{R}}_+, M^{-\infty,d}_O(X)), A_\psi \in \mathcal{B}^{-\infty,d}(X)$, and $R \in \bigcap_{j=0}^\infty C^{\mu-j}_{M+G}(\mathbb{D}, \mathbf{g})$. According to Theorem 3.1.9, $t^{M-\mu}\omega_1 \,[\mathrm{op}^\gamma_M h]\, \omega_2 \in t^M C^{\mu,d}_{M+G}(\mathbb{D}, \mathbf{g}) = C^{\mu-M,d}_{M+G}(\mathbb{D}, \mathbf{g})$. By Lemma 3.1.14, $(1 - \omega_4)A_\psi(1 - \omega_3) \in C^d_G(\mathbb{D}, \mathbf{g})$. The fact that, by Lemma 1.9.4, $C^{\mu-M,d}_{M+G}(\mathbb{D}, \mathbf{g}) \hookrightarrow C^d_G(\mathbb{D}, \mathbf{g})$ whenever $\mu - M \geq N$ completes the argument. ◁

3.1.20 Lemma. *Let $\omega \in C_0^\infty(\overline{\mathbf{R}}_+), \omega(t) \equiv 1$ for t close to 0, $s, \gamma \in \mathbf{R}$. Then multiplication by $(1-\omega(t))t^{-N}$ defines bounded maps*

$$(1-\omega(t))t^{-N} : H^s(\Omega^\wedge) \to \mathcal{H}^{s,\gamma}(\Omega^\wedge), \tag{1}$$

and

$$(1-\omega(t))t^{-N} : H^s(X^\wedge) \to \mathcal{H}^{s,\gamma}(X^\wedge), \tag{2}$$

provided that $N > [s] + 1 + n/2 - \gamma$. Recall that Ω is the boundaryless 'double' of X.

Proof. (2) follows from (1). (1) is local in the x-variables. First let $s \in \mathbf{N}$. In order to check that, for $u \in H^s(\Omega^\wedge)$, we have $(1-\omega(t))t^{-N}u \in \mathcal{H}^{s,\gamma}(\Omega^\wedge)$ it is sufficient to check that, given k, α with $|\alpha| + k \le s$,

$$t^{n/2-\gamma}(t\partial_t)^k \partial_x^\alpha \left[(1-\omega(t))t^{-N}u\right] \in L^2(U \times \mathbf{R}_+)$$

for any coordinate neighborhood U for Ω. By Leibniz' rule for differentiation the left hand side is an expression of linear combinations of terms of the form

$$t^{n/2-\gamma+k_0}(1-\omega(t))^{(k_1)} t^{-N-k_2} \partial_t^{k_3} \partial_x^\alpha u\,,$$

where $k_1 + k_2 + k_3 = k_0 \le k$. Since $\partial_t^{k_3}\partial_x^\alpha u \in L^2$ and the preceding factor is a bounded function, we get the desired statement.
For $s \in -\mathbf{N}$, we use the dualities $(H^s(\Omega^\wedge), H_0^{-s}(\Omega^\wedge))$ and $(\mathcal{H}^{s,\gamma}(\Omega^\wedge), \mathcal{H}^{-s,-\gamma}(\Omega^\wedge))$. The subscript "0" indicates that we deal with distributions with support in $\Omega \times \overline{\mathbf{R}}_+$ a fact we need not worry about, for we multiply by a function that vanishes for small t. The corresponding adjoint operator to $(1-\omega(t))t^{-N}$ is $(1-\overline{\omega}(t))t^{-N+n}$; the shift by n is due to the fact that $\mathcal{H}^{0,\frac{n}{2}}(\Omega^\wedge) = L^2(\Omega^\wedge)$. So the task is to show the boundedness of

$$(1-\overline{\omega}(t))t^{-N+n} : \mathcal{H}^{s,\gamma}(\Omega^\wedge) \to H^s(\Omega^\wedge)$$

for $s \in \mathbf{N}$, and this can be done in the same way as before. ◁

3.1.21 Theorem. *Let $g \in C^\infty(\mathbf{R}_+ \times \mathbf{R}_+, M_O^{\mu,d}(X))$, and suppose that, for all $k, l \in \mathbf{N}$,*

$$t^k t'^l \partial_t^k \partial_{t'}^l g \in C_b(\mathbf{R}_+ \times \mathbf{R}_+, M_O^{\mu,d}(X)). \tag{1}$$

Then

$$\mathrm{op}_M^\gamma g : \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(X^\wedge, V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(Y^\wedge, W_1) \end{array} \to \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}}(X^\wedge, V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}}(Y^\wedge, W_2) \end{array}$$

is bounded for each $s > d - 1/2, \gamma \in \mathbf{R}$. For fixed s, γ, we only need to have (1) for finitely many k and l. Note that this Theorem generalizes Lemma 2.1.11

For the proof we need the following results.

3.1.22 Lemma. *Let g be as in Theorem 3.1.21. Then*

$$\mathrm{op}_M^\gamma g = \exp_* \mathrm{op}\, b$$

for $b = b(r, r', \rho) = e^{-(1/2-\gamma)(r-r')} g(e^r, e^{r'}, 1/2 - \gamma - i\rho)$.

Proof. Let $u \in C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}))$. Then

$$\begin{aligned}(\mathrm{op}_M^\gamma g)u(e^r) &= \frac{1}{2\pi i}\int_{\Gamma_{1/2-\gamma}}\int_0^\infty (e^r/t')^{-z}\, g(e^r, t', z)u(t')dt'/t'dz \\ &= \frac{1}{2\pi}\int\int e^{-(r-r')(1/2-\gamma+i\tau)} g(e^r, e^{r'}, 1/2-\gamma+i\tau)(u\circ\exp)(r')dr'd\tau \\ &= (\mathrm{op}\, b)(u\circ\exp)(r)\end{aligned}$$

where $b(r, r', \rho) = e^{-(1/2-\gamma)(r-r')} g(e^r, e^{r'}, 1/2-\gamma-i\rho)$. ◁

3.1.23 Lemma. *For $s \in \mathbf{R}$*

$$\mathcal{H}^{s,\frac{n+1}{2}}(\Omega^\wedge) = \exp_* H^s(\Omega \times \mathbf{R}).$$

Here exp is the mapping $(x, r) \to (x, e^r)$.

Proof. Using a partition of unity on Ω we see that the result is local in the x-variables. We can therefore apply Remark 1.4.4 and immediately get the statement. ◁

3.1.24 Corollary. Restricting to $X^\wedge$, we conclude that

$$\mathcal{H}^{s,\frac{n+1}{2}}(X^\wedge) = \exp_* H^s(X \times \mathbf{R}).$$

Proof of Theorem 3.1.21. It is sufficient to prove the theorem for $\gamma = 1/2$, for

$$\mathrm{op}_M^\gamma g = t^{\gamma-1/2}[\mathrm{op}_M^{1/2} T^{1/2-\gamma} g] t^{1/2-\gamma}.$$

Moreover, we may work with trivial 1-dimensional bundles V_1, V_2, while W_1, W_2 can be assumed to be zero. By 3.1.22, the boundedness of $\mathrm{op}_M^{1/2} T^{1/2-\gamma} g$ is equivalent to the boundedness of

$$\mathrm{op}\, b : H^s(X \times \mathbf{R}) \to H^{s-\mu}(X \times \mathbf{R}),$$

where $b(r, r', \rho) = g(e^r, e^{r'}, 1/2-\gamma-i\rho) \in C_b^\infty(\mathbf{R}\times\mathbf{R}, \mathcal{B}^{\mu,d}(X;\mathbf{R}))$. This boundedness result, however, is well-known. Notice that, for fixed s, we only need finitely many estimates on b for the continuity of $\mathrm{op}\, b$, thus we only need a finite number of the estimates in (1) for g. ◁

We are now able to prove a stronger version of Lemma 3.1.13, namely the following.

3.1.25 Lemma. *Let $g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, and $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega_1\omega_2 = \omega_1$. Then, for every choice of γ, N,*

$$\omega_1\, [\mathrm{op}_M^\gamma g]\, (1-\omega_2) \in C_G^d(X^\wedge, (\gamma, \gamma, (-N, 0]))_{O,O}.$$

Here O refers to the asymptotic type $(\gamma, \gamma, (-N, 0])$.

Proof. Without loss of generality we may assume that $d = 0$ and that V_1, V_2 are trivial one-dimensional while W_1, W_2 vanish. We then have to show that, for each $s > -1/2$,

$$\omega_1 [\mathrm{op}_M^\gamma g] (1-\omega_2) : \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{S}_O^{\gamma+\frac{n}{2}}(X^\wedge) \tag{1}$$

and

$$(\omega_1 [\mathrm{op}_M^\gamma g] (1-\omega_2))^* : \mathcal{K}^{s,-\gamma-\frac{n}{2}}(X^\wedge) \to \mathcal{S}_O^{-\gamma-\frac{n}{2}}(X^\wedge) \tag{2}$$

are continuous. Let us start with (1). In view of the factor ω_1, it is indeed sufficient to show that, for each $M \in \mathbf{N}$,

$$\omega_1 [\mathrm{op}_M^\gamma g] (1-\omega_2) : \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}_O^{s-\mu+M,\gamma+\frac{n}{2}}(X^\wedge) \tag{3}$$

is continuous. Here we let, for the moment, $\mathcal{H}_O^{\tilde{s},\gamma+\frac{n}{2}}(X^\wedge), \tilde{s} \in \mathbf{R}$, denote the space $\bigcap_{0\le c\le N} \mathcal{H}^{\tilde{s},\gamma+\frac{n}{2}+c}(X^\wedge)$, where N refers to the weight datum $(\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-N,0])$. Now $\omega_1 [\mathrm{op}_M^\gamma g] (1-\omega_2) = \mathrm{op}_M^\gamma h$ with

$$h(t,t',z) = \omega_1(t) g(t,z)(1-\omega_2(t')) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,0}(X)).$$

Since we have a zero of infinite order, we can apply Proposition 2.2.20: For each $m \in \mathbf{N}$, we find a symbol $h_m \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, \mathcal{B}^{\mu-m,0}(X;\Gamma_{1/2-\gamma}))$ with $\mathrm{op}_M^\gamma h = \mathrm{op}_M^\gamma h_m$. We need to know more about the precise form of h_m and therefore review the proof of Proposition 2.2.20. We have $h_m = h_0 + \tilde{h}_m$ where

$$\begin{aligned} h_0 &= M(1-\psi(\rho))M_\gamma^{-1} h, \text{ and} \\ \tilde{h}_m &= t'^m (t-t')^{-m} M\varphi(\rho) M_\gamma^{-1}(\partial_z^m h). \end{aligned}$$

Here ψ is an arbitrary $C_0^\infty(\mathbf{R}_+)$ function equal to 1 near $\rho = 1$, and $\varphi \in C_0^\infty(\mathbf{R}_+)$ is a suitable function. Clearly, $h_0 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{-\infty,0}(X))$ and $\tilde{h}_m \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu-m,0}(X))$. Note that, for suitable functions $\tilde{\omega}_1, \tilde{\omega}_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\tilde{\omega}_1(t), \tilde{\omega}_2(t) \equiv 1$ near $t = 0$, we have

$$\begin{aligned} h_0(t,t',z) &= \tilde{\omega}_1(t) h_0(t,t',z)(1-\tilde{\omega}_2(t')), \quad \text{and} \\ \tilde{h}_m(t,t',z) &= \tilde{\omega}_1(t) \tilde{h}_m(t,t',z)(1-\tilde{\omega}_2(t')). \end{aligned}$$

Next apply Lemma 3.1.20: Given $\gamma_1 \in \mathbf{R}$, there is a $k > 0$ such that

$$(1-\tilde{\omega}_2(t))t^{-k} : H^s(X^\wedge) \to \mathcal{H}^{s,\gamma_1}(X^\wedge) \tag{4}$$

is continuous. By Lemma 4.2.2, also

$$(1-\tilde{\omega}_2(t))t^{-k} : \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s,\gamma_1}(X^\wedge) \tag{5}$$

is continuous, provided k is large. We have from Lemma 3.1.10

$$\begin{aligned} \omega_1 [\mathrm{op}_M^\gamma g] (1-\omega_2) &= \mathrm{op}_M^\gamma h_0 + \mathrm{op}_M^\gamma \tilde{h}_m \\ &= \tilde{\omega}_1 [\mathrm{op}_M^\gamma h_0]\, t^k (1-\tilde{\omega}_2) t^{-k} \\ &\quad + \tilde{\omega}_1 \left[\mathrm{op}_M^\gamma \left[(t-t')^{-m} t'^m M\varphi M_\gamma^{-1}(\partial_z^m h)\right] t'^k\right] (1-\tilde{\omega}_2) t^{-k} \\ &= t^k \tilde{\omega}_1 \left[\mathrm{op}_M^\gamma T^{-k} h_0\right] (1-\tilde{\omega}_2) t^{-k} \\ &\quad + t^{m+k} \tilde{\omega}_1 \left[\mathrm{op}_M^\gamma (t-t')^{-m} T^{-k-m} (M\varphi M_\gamma^{-1}(\partial_z^m h))\right] (1-\tilde{\omega}_2) t^{-k}. \end{aligned}$$

Both the symbols $T^{-k}h_0$ and $T^{-k-m}M\varphi M_\gamma^{-1}\partial_z^m h$ satisfy the assumptions of Theorem 3.1.21, since h vanishes for $|t-t'|<\varepsilon$, where ε is a suitable positive number, $(t-t')^{-N} \in C_b^\infty(\mathbf{R}^2\backslash\{|t-t'|<\varepsilon\})$, and h has compact support with respect to t and is independent of t' for large t'. Applying Theorem 3.1.21 and using (4) as well as (5)

$$\omega_1\,[\mathrm{op}_M^\gamma g]\,(1-\omega_2) : \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s-\mu+m,\gamma+\frac{n}{2}}(X^\wedge)$$

is continuous. Now let $0 \le c \le N$ and, in (4), $\gamma_1 = \gamma + c$. Since

$$\mathrm{op}_M^{\gamma+c} g = \mathrm{op}_M^\gamma g$$

on $C_0^\infty(\mathbf{R}_+, C^\infty(\overline{X}))$, which is dense in $\mathcal{H}^{s,\gamma+c}(X^\wedge)$, a second application of Theorem 3.1.21 in connection with (5) shows that also

$$\omega_1\,[\mathrm{op}_M^\gamma g]\,(1-\omega_2) : \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \mathcal{H}^{s-\mu+m,\gamma+\frac{n}{2}+c}(X^\wedge)$$

is bounded. According to (3) this is what we had to show for (1).
Next let us prove (2). Note first that $\mathrm{op}_M^\gamma g$ indeed has an adjoint in the Mellin calculus, since it can be written with a Mellin symbol of negative order. The argument now is simple: we have, by Lemma 2.3.2,

$$[\omega_1\,[\mathrm{op}_M^\gamma g]\,(1-\omega_2)]^* = (1-\overline{\omega}_2)\left[\mathrm{op}_M^{-\gamma-n} g^{(*)}\right]\overline{\omega}_1$$

with $g^{(*)}(t,t',z) = g(t', n+1-\overline{z})^*$. Choose arbitrary m and k in $\mathbf{N}$. Applying an argument analogous to that used in the first part, we may replace $(1-\overline{\omega}_2)[\mathrm{op}_M^{-\gamma-n} g^{(*)}]\overline{\omega}_1$ by

$$t^{-k}(1-\tilde{\omega}_2)\left[\mathrm{op}_M^{-\gamma-n} h_1\right]\tilde{\omega}_1 t^k + t^{-k}(1-\tilde{\omega}_2)\left[\mathrm{op}_M^{-\gamma-n} h_2\right]\tilde{\omega}_1 t^{m+k},$$

where $h_1 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{-\infty,0}(X))$ and $h_2 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu-m,0}(X))$ satisfy the assumption of Theorem 3.1.21, and $\tilde{\omega}_1, \tilde{\omega}_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ are suitable functions. Multiplication by $\tilde{\omega}_1 t^{m+k}$ or $\tilde{\omega}_1 t^k$ maps $\mathcal{K}^{s,-\gamma-\frac{n}{2}}(X^\wedge)$ to $\mathcal{H}^{s,-\gamma-\frac{n}{2}-K}(X^\wedge)$ for arbitrary $K \in \mathbf{N}$. The operators $\mathrm{op}_M^\gamma h_1$ and $\mathrm{op}_M^\gamma h_2$ map this space to $\mathcal{H}^{s+m-\mu,-\gamma-\frac{n}{2}-K}(X^\wedge)$. Finally, multiplication by $1-\tilde{\omega}_2$ sends this space to distributions u that vanish in a fixed neighborhood of $\{t=0\}$ and have the property that

$$t^{n+\gamma+K}(t\partial_t)^j\partial_x^\alpha u \in L^2(X^\wedge), \text{ for } j+|\alpha| \le s+m-\mu.$$

Since m and K were arbitrary, these functions are elements of $\mathcal{S}_O^{-\gamma-\frac{n}{2}}(X^\wedge)$, and the proof is complete. ◁

3.1.26 Lemma. *Let $h = h(t,t',z) \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, and $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$. Then there is a $g = g(t,z) \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, depending on ω_1, ω_2, such that*

$$\omega_1\,[\mathrm{op}_M^\gamma h]\,\omega_2 - \mathrm{op}_M^\gamma g \in C_G^d(X^\wedge, \mathrm{g}_0),$$

$\mathrm{g}_0 = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-N,0])$.

Proof. Let $\tilde{h}(t,t',z) = h(t,t',z)\omega_1(t)\omega_2(t')$, so that $\omega_1[\mathrm{op}_M^\gamma h]\omega_2 = \mathrm{op}_M^\gamma \tilde{h}$. Next choose g for $\tilde{h}$ according to Theorem 3.1.11. By construction, we may assume that $g(t,z) = 0$ whenever $t \notin \mathrm{supp}\, \omega_1$.
Now pick functions $\tilde{\omega}_1, \tilde{\omega}_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ that are equal to 1 near zero and on the support of ω_1 and ω_2, so that $\tilde{\omega}_1\omega_1\,[\mathrm{op}_M^\gamma h]\,\omega_2\tilde{\omega}_2 = \omega_1\,[\mathrm{op}_M^\gamma h]\,\omega_2$. By Theorem 3.1.11,

$$\omega_1\,[\mathrm{op}_M^\gamma h]\,\omega_2 - \tilde{\omega}_1\,[\mathrm{op}_M^\gamma g]\,\tilde{\omega}_2 \in C_G^d(X^\wedge, \mathbf{g}_0).$$

Clearly, $\tilde{\omega}_1[\mathrm{op}_M^\gamma g]\tilde{\omega}_2 = [\mathrm{op}_M^\gamma g]\tilde{\omega}_2$. Moreover, $[\mathrm{op}_M^\gamma g](1-\tilde{\omega}_2) = \tilde{\omega}_1[\mathrm{op}_M^\gamma g](1-\tilde{\omega}_2) \in C_G^d(X^\wedge, \mathbf{g}_0)$ by Lemma 3.1.25, so the proof is complete. ◁

We next analyze the structure of those Mellin operators that induce Green operators. An essential tool is the following result which extends Proposition I.4.3.7 and follows by similar arguments.

3.1.27 Proposition. *Let* $\mathbf{g} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-N, 0]), \mu \in \mathbf{Z}, d \in \mathbf{N}$, *and* $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+), \omega_j(t) \equiv 1$ *for small* t. *Consider an operator of the form*

$$A = \omega_1 \sum_{j=0}^{N-1} t^j [\mathrm{op}_M^{\gamma_j} h_j]\omega_2 + R_N$$

for $R_N = t^N \omega_1[\mathrm{op}_M^\gamma h_N]\omega_2$. *Here,* $h_j \in M_{P_j}^{\mu,d}(X)$, $j = 0, \ldots, N-1$, *for suitable asymptotic types* P_j *with* $\pi_{\mathbf{C}} P_j \cap \Gamma_{1/2-\gamma_j} = \emptyset$, *while* $h_N \in C^\infty(\overline{\mathbf{R}}_+, M_{P_N}^{\mu,d}(X))$ *for an asymptotic type* P_N *with* $\pi_{\mathbf{C}} P_N \cap \Gamma_{1/2-\gamma} = \emptyset$.
Suppose $A \in C_G^d(X^\wedge, \mathbf{g})$. *Then* $h_j = 0$ *for* $j = 1, \ldots, N-1$.

Proof. For simplicity let us deal with a trivial scalar bundle over $X^\wedge$ only. Choose a cut-off function ω near zero and a function $\phi \in C_0^\infty(X)$. For $p \in \mathbf{C}$ with $\mathrm{Re}\, p < 1/2$ let $u_p(t) = t^{\gamma - p}\omega(t)\phi(x) \in \mathcal{K}^{\infty, \gamma + \frac{n}{2}}(X^\wedge)$. By 1.7.1(b) we have $Mu_p(z) = v(z + \gamma - p)$, where

$$v(z) = M\omega(z)\,\phi = (\frac{c}{z} + f(z))\,\phi \qquad (1)$$

with an entire function f and some $c \neq 0$. By assumption, there is a Green operator $G \in C_G(X^\wedge, \mathbf{g})$ such that

$$0 = A + G = \sum_{j=0}^{N-1} \omega_1\, t^j\, \mathrm{op}_M^{\gamma_j}(h_j)\,\omega_2 + R_N + G.$$

We will show that we can recover the functions $h_0, \ldots, h_{N-1}$ by considering the Mellin transform of $(A+G)u_p$. Hence all will have to vanish in order for $A + G$ to be zero.
We may choose ω with support very close to zero. Therefore it is no loss of generality to ask that $\omega_2\,\omega = \omega$; in other words, the function ω_2 can be ignored in our considerations.
Let us now analyze the effects of the various operators. The continuity of $R_N : \mathcal{K}^{s,\gamma+\frac{n}{2}}(X^\wedge) \to \omega\,\mathcal{K}^{s-\mu,\gamma+\frac{n}{2}+N}(X^\wedge)$ for arbitrary $s > d - 1/2$ implies that $R_N u_p \in \mathcal{S}_O^{\gamma+\frac{n}{2}}(X^\wedge)$ for all p. Hence $M_{\tilde{\gamma},t\to z} R_N u_p$ exists for all $\gamma \le \tilde{\gamma} < \gamma + N$. It yields a holomorphic function of $z \in \{1/2 - \gamma - N < \mathrm{Re}\, z < 1/2 - \gamma\}$ for fixed $p \in \{\mathrm{Re}\, p < 1/2\}$ and vice versa of p for fixed z, cf. Theorem I.3.2.8.

The function Gu_p belongs to $\mathcal{S}_R^{\gamma+\frac{n}{2}}(X^\wedge)$ for a finite asymptotic type R independent of p. For fixed p, the Mellin transform $M_{\tilde\gamma,t\to z}Gu_p$ exists for all but finitely many $\tilde\gamma$ in $[\gamma,\gamma+N)$ and yields a holomorphic function of z in the semi-strip

$$S_\sigma = \{z \in \mathbf{C} : 1/2-\gamma-N < \operatorname{Re} z < 1/2-\gamma, \operatorname{Im} z > \sigma\}, \tag{2}$$

for a suitably large constant σ, depending on R. For fixed $z_0 \in S_\sigma$ and p_0, the function $p \mapsto M_{\tilde\gamma,t\to z}Gu_p(z_0)$ is holomorphic in p as p varies over a suitably small neighborhood of p_0 in $\{\operatorname{Re} z < 1/2\}$.
Choose a small $\epsilon > 0$ such that

(i) $\pi_{\mathbf{C}}P_j \cap \Gamma_{1/2-\gamma-\epsilon+j} = \emptyset$ for $j = 0,\dots,N$,

(ii) P_0 has no singularities with real part in $[1/2-\gamma-\epsilon, 1/2-\gamma]$.

Then $\omega_1[\operatorname{op}_M^\gamma h_0]u_p = \omega_1[\operatorname{op}_M^{\gamma+\epsilon}h_0]u_p$. Moreover, there are Green operators $G_j \in C_G^d(X^\wedge,\mathbf{g})$, $j = 1,\dots,N-1$ such that

$$t^j\operatorname{op}_M^{\gamma_j}h_ju_p = t^j\omega_1\operatorname{op}_M^{\gamma+\epsilon-j}h_ju_p + G_ju_p.$$

Fix z_0 with real part $1/2-\gamma-\epsilon$. The operators G_j are independent of ϵ for small $\epsilon > 0$ for they are determined by the singularities of the h_j. We may apply the above statement: $M_{\tilde\gamma}G_ju_p$ exists for all but finitely many $\tilde\gamma$ in $[\gamma,\gamma+N)$; for fixed p_0, z_0 with large $\operatorname{Im} z_0$, the function $p \mapsto MG_ju_p(z_0)$ is holomorphic in p whenever p runs over a small neighborhood U of p_0 in $\{\operatorname{Re} z < 1/2\}$.
Restricting p to U we have $u_p \in \mathcal{H}^{\infty,\gamma+\epsilon-j+\frac{n}{2}}(X^\wedge)$ and therefore $t^j[\operatorname{op}_M^{\gamma+\epsilon-j}h_j]u_p \in \mathcal{H}^{\infty,\gamma+\epsilon+\frac{n}{2}}(X^\wedge)$, uniformly for $p \in U$. In particular, $M_{\gamma+\epsilon}(1-\omega_1)t^j\operatorname{op}_M^{\gamma+\epsilon-j}h_ju_p(z_0)$ exists and extends to a holomorphic function of p on U. Finally,

$$M_{\gamma+\epsilon,t\to z}(t^j[\operatorname{op}_M^\gamma h_j]u_p)(z_0) = M_{\gamma+\epsilon-j,t\to z}([\operatorname{op}_M^\gamma h_j]\,u_p)(z_0+j) = h_j(z_0+j)v(z_0+j+\gamma-p)$$

extends to a meromorphic function of p on $\mathbf{C}$.
Now we fix j_0. By possibly increasing the imaginary part of z_0 we may assume that $M_{\gamma+\epsilon}Gu_p(z_0)$, $M_{\gamma+\epsilon}G_ju_p(z_0)$, and $M_{\gamma+\epsilon}R_Nu_p(z_0)$ are holomorphic in p in a small neighborhood of $p_0 = z_0+\gamma-j_0$ We then integrate $M(A+G)u_p(z_0)$ over a small contour C around p_0. By Cauchy's formula, the holomorphic contributions vanish, and (1) implies that

$$\frac{1}{2\pi i}\int_C M_{\gamma+\epsilon}(A+G)u_p(z_0)\,dp = c\;h_{j_0}(z_0+j_0)\phi.$$

On the other hand, $A+G = 0$, hence $h_{j_0}(z_0+j_0)$ is zero. Since we may vary z_0 slightly and since we know that the h_j are meromorphic functions, we conclude that h_{j_0} and consequently all h_j vanish. ◁

3.1.28 Proposition. *If $g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X)), \omega_1,\omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ equal to 1 near zero, and*

$$\omega_1\,[\operatorname{op}_M^\gamma g]\,\omega_2 \in C_G^d(X^\wedge,\mathbf{g}_0)_{P,Q}$$

for $\mathbf{g}_0 = (\gamma+\frac{n}{2},\gamma+\frac{n}{2},(-N,0])$ and some asymptotic types P and Q, then

$$g(t,z) \in t^NC^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,d}(X)) + \mathcal{S}_O^{1/2}(\mathbf{R}_+, M_O^{-\infty,d}(X)).$$

Moreover, the asymptotic types P and Q both are O.

Proof. For simplicity let us assume that the operators act on a trivial 1-dimensional bundle $V_1 = V_2$ over X while the bundles W_1, W_2 over Y are zero. Also it is no loss of generality to assume $d = 0$. We first write

$$g(t,z) = \sum_{j=0}^{N-1} \frac{t^j}{j!}\partial_t^j g(0,z) + t^N g_N(t,z)$$

with $g_N \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,0}(X))$. According to Proposition 3.1.27, the fact that $\omega_1[\mathrm{op}_M^\gamma g]\omega_2$ is a Green operator implies that $\frac{t^j}{j!}\partial_t^j g(0,z) = 0$ for $j = 0,\dots,N-1$, so that $g \in t^N C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,0}(X))$. Applying now Lemma 3.1.10, we conclude that $\omega_1[\mathrm{op}_M^\gamma g]\omega_2$ has the mapping properties of a Green operator with asymptotic types $P = Q = O$. Next we employ Lemma 3.1.26 and suppose that

$$\omega_1 \, [\mathrm{op}_M^\gamma g] \, \omega_2 = \mathrm{op}_M^\gamma g,$$

for otherwise we might replace g by a function with the same properties. According to Theorem 1.6.2, the Green operator has an integral kernel $k = k(t,t',x,y)$ in

$$\mathcal{S}_O^{\gamma+\frac{n}{2}}(X^\wedge)\hat{\otimes}_\pi \mathcal{S}_O^{-\gamma-\frac{n}{2}}(X^\wedge) \hookrightarrow \mathcal{S}_O^\gamma(\mathbf{R}_+)\hat{\otimes}_\pi \mathcal{S}_O^{-\gamma-n}(\mathbf{R}_+)\hat{\otimes}_\pi C^\infty(X \times X).$$

From the kernel k we can recover the symbol by

$$g(t,z) = \int_0^\infty (t/t')^z k_X(t,t')dt'/t'.$$

Here $k_X(t,t')$ is the operator on, say, $C^\infty(X)$, resulting from the action of the integral operator with kernel $k(t,t',x,y)$ for fixed t,t'. Let us suppose first that

$$k(t,t',x,y) = f(t)g_0(t')l_X(x,y)$$

with $f \in \mathcal{S}_O^\gamma(\mathbf{R}_+), g_0 \in \mathcal{S}_O^{-\gamma-n}(\mathbf{R}_+), l_X \in C^\infty(X \times X)$. Then

$$g(t,z) = t^z f(t)(Mg_0)(z)\, L_X,$$

where L_X is the operator on $C^\infty(X)$ given by the kernel l_X. Thus $L_X \in \mathcal{B}^{-\infty,0}(X)$, and all semi-norms in $\mathcal{B}^{-\infty,0}(X)$ can be estimated in terms of the semi-norms for l_X. Moreover, $M_\gamma g_0(z) \in \mathcal{S}(\Gamma_{1/2-\gamma})$: In fact, on one hand $M_\gamma : \omega\mathcal{H}^{\infty,\gamma}_{(-N,0]}(\mathbf{R}_+) \to \mathcal{S}(\Gamma_{1/2-\gamma})$; on the other hand, $M_\gamma : (1-\omega)\mathcal{S}(\mathbf{R}_+) \to \mathcal{S}(\Gamma_{1/2-\gamma})$. Now we restrict to the line $\Gamma_{1/2-\gamma}$. We have $g|_{\Gamma_{1/2-\gamma}}(t, 1/2+i\tau) = t^{1/2-\gamma+i\tau} f(t)h(\tau)L_X$ with $h(\tau) = (M_\gamma g_0)(1/2-\gamma+i\tau) \in \mathcal{S}(\mathbf{R})$. We shall now prove that

$$g(t,z)|_{\Gamma_{1/2-\gamma}} \in \mathcal{S}_O^{1/2}(\mathbf{R}_+, \mathcal{B}^{-\infty,0}(X;\Gamma_{1/2-\gamma})). \tag{1}$$

In order to see this, we only have to make sure that the function $t^{1/2-\gamma+i\tau}f(t)h(\tau)$ belongs to $\mathcal{S}_O^{1/2}(\mathbf{R}_+, \mathcal{S}(\mathbf{R}))$. This in turn amounts to checking that it satisfies the estimates

$$\| \sup_\tau |\tau^{r_1}\partial_\tau^{r_2}(t\partial_t)^{l_1}\{\langle t\rangle^{l_2}\, t^{-\gamma+i\tau-N+\varepsilon}f(t)h(\tau)\}| \|_{L^2(\mathbf{R}_+)} < \infty$$

for every choice of $r_1, r_2, l_1, l_2 \in \mathbf{N}, \varepsilon > 0$. Using the fact that, for $l \in \mathbf{N}$, we have $(t\partial_t)^l t^{-\gamma+i\tau} = (-\gamma+i\tau)^l$ and $\partial_\tau^l t^{-\gamma+i\tau} = t^{-\gamma+i\tau}\ln^l t$, this follows from the properties of f_1 and h.

In view of the properties of the π tensor product, relation (1) stays true for general k, since the semi-norms for g can be estimated in terms of the semi-norms for f, g_0, and l_X. Now we choose a function $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho) \equiv 1$ for ρ near 1. From Proposition 2.2.7 and the nuclearity of $\mathcal{S}_O^{1/2}(\mathbf{R}_+)$, we conclude that

$$M\psi M_\gamma^{-1} g \in \mathcal{S}_O^{1/2}(\mathbf{R}_+, M_O^{-\infty,0}(X)).$$

On the other hand we know from Lemma 3.1.4 that

$$M(1-\psi)M_\gamma^{-1} g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,0}(X)).$$

We see that $g = g_1 + g_2$ with $g_1 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$ and $g_2 \in \mathcal{S}_O^{1/2}(\mathbf{R}_+, M_O^{-\infty,d}(X))$. Now we again apply Propsition 3.1.27 to see that $\partial_t^j(g_1(0,z) + g_2(0,z)) = 0, j = 1, \ldots, N-1$. Since $\mathcal{K}^{\infty,1/2}_{(-N,0]}(\mathbf{R}_+) = \bigcap_{\epsilon>0} \mathcal{K}^{\infty,1/2+N-\epsilon}$, we deduce from Lemma 1.4.6 that $p_l(g_2(t,z)) = O(t^{N-\epsilon})$ for every $\varepsilon > 0$ and every semi-norm p_l in $M_O^{-\infty,0}(X)$. On the other hand, g_1 is smooth in t; so we find that $\partial_t^j g_1(0,z) = 0, j = 1, \ldots, N-1$ and the proof is complete. ◁

3.1.29 Corollary. (a) If $g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, and $\mathrm{op}_M^\gamma g \in C_G^d(X^\wedge, \mathrm{g})$, then $g \in t^N C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$.
(b) If $g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$, and

$$\mathrm{op}_M^\gamma g \in C_G^d(X^\wedge, (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-\infty, 0]),$$

then $g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$, moreover g vanishes to infinite order at $t = 0$.
(c) If $g \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{\mu,d}(X))$, and $\mathrm{op}_M^\gamma g \in C_G^d(X^\wedge, \mathbf{g})$, then $g \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-\infty,d}(X))$.

Proof. (a) We have

$$g(t,z) = \sum_{j=0}^{N-1} \frac{t^j}{j!} \partial_t^j g(0,z) + t^N g_N(t,z)$$

for suitable $g_N \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Using that elements in $\mathcal{S}_O^{1/2}(\mathbf{R}_+, M_O^{-\infty,d}(X))$ are $O(t^{N-\epsilon})$ for each $\varepsilon > 0$, see Lemma 1.4.6, we conclude that $\partial_t^j g(0,z) = 0$ for $j = 0, \ldots, N-1$.
(b) is immediate from the fact that functions in $\mathcal{S}_O^{1/2}(\mathbf{R}_+, M_O^{-\infty,d}(X))$ are $O(t^{N-\epsilon})$ near $t = 0$.
(c) Since $g \in C^\infty(\mathbf{R}_+, M_{O,cl}^{-\infty,d}(X))$ by Proposition 3.1.28 we obtain that all homogeneous terms of the symbols vanish on $\mathbf{R}_+$. By definition of the topology in $C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{\mu,d}(X))$, all homogeneous symbol terms are smooth up to $t = 0$, so $g \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-\infty,d}(X))$. ◁

3.1.30 Proposition. *Let $h \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\mu,d}(X^\wedge))$, $M = t^{-\mu}\mathrm{op}_M^\gamma h$, and $A \in \mathcal{B}^{\mu,d}(X^\wedge)$ with*

$$M - A \in \mathcal{B}^{-\infty,d}(X^\wedge).$$

Suppose $\omega_1, \omega_2, \omega_3$, and $\tilde\omega_1, \tilde\omega_2, \tilde\omega_3$ are two triples of functions in $C_0^\infty(\overline{\mathbf{R}}_+)$ equal to 1 near zero with $\omega_1\omega_2 = \omega_1, \omega_1\omega_3 = \omega_3$ and $\tilde\omega_1\tilde\omega_2 = \tilde\omega_1, \tilde\omega_1\tilde\omega_3 = \tilde\omega_3$. Then, for every $\chi \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\chi \equiv 1$ on $\operatorname{supp}\omega_2 \cup \operatorname{supp}\tilde\omega_2$,

$$[\omega_1 M \omega_2 + (1-\omega_1)A(1-\omega_3)]\chi \equiv [\tilde\omega_1 M \tilde\omega_2 + (1-\tilde\omega_1)A(1-\tilde\omega_3)]\chi.$$

modulo $C_G^d(X^\wedge, \mathbf{g}), \mathbf{g} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-\infty, 0])$.
Note that

$$[\omega_1 M \omega_2 - (1-\omega_1)A(1-\omega_3)]\chi = \omega_1 M \omega_2 - (1-\omega_1)A(\chi - \omega_3)$$

and

$$[\tilde{\omega}_1 M \tilde{\omega}_2 - (1-\tilde{\omega}_1)A(1-\tilde{\omega}_3)]\chi = \tilde{\omega}_1 M \tilde{\omega}_2 - (1-\tilde{\omega}_1)A(\chi - \tilde{\omega}_3).$$

Proof. Choose a function $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega \equiv 1$ near zero and $\omega\omega_3 = \omega$. In particular, we will then have $\omega\omega_1 = \omega$. Then

$$\begin{aligned} \omega_1 M \omega_2 - \tilde{\omega}_1 M \tilde{\omega}_2 &= [\omega M \omega_2 + (\omega_1 - \omega) M \omega_2] - [\omega M \tilde{\omega}_2 + (\tilde{\omega}_1 - \omega) M \tilde{\omega}_2] \\ &= \omega M(\omega_2 - \tilde{\omega}_2) + (\omega_1 - \omega) M \omega_2 - (\tilde{\omega}_1 - \omega) M \tilde{\omega}_2 \\ &= \omega M(\omega_2 - \tilde{\omega}_2) + (\omega_1 - \omega) M \chi + (\omega_1 - \omega) M(\omega_2 - \chi) \\ &\quad -(\tilde{\omega}_1 - \omega) M \chi - (\tilde{\omega}_1 - \omega) M(\tilde{\omega}_2 - \chi) \\ &= \omega M(\omega_2 - \tilde{\omega}_2) + (\omega_1 - \tilde{\omega}_1) M \chi \\ &\quad +(\omega_1 - \omega) M(\omega_2 - \chi) - (\tilde{\omega}_1 - \omega) M(\tilde{\omega}_2 - \chi). \end{aligned} \tag{1}$$

Now $\operatorname{supp}\omega \cap \operatorname{supp}(\omega_2 - \tilde{\omega}_2) = \emptyset, \operatorname{supp}(\omega_1 - \omega) \cap \operatorname{supp}(\omega_2 - \chi) = \emptyset$, and $\operatorname{supp}(\tilde{\omega}_1 - \omega) \cap \operatorname{supp}(\tilde{\omega}_2 - \chi) = \emptyset$. In view of Lemma 3.1.13 we therefore have

$$\omega M(\omega_2 - \tilde{\omega}_2),\ (\omega_1 - \omega) M(\omega_2 - \chi),\ (\tilde{\omega}_1 - \omega) M(\tilde{\omega}_2 - \chi) \in C_G^d(X^\wedge, \mathbf{g}),$$

and the only term of interest is

$$(\omega_1 - \tilde{\omega}_1) M \chi = (\omega_1 - \tilde{\omega}_1) M \omega + (\omega_1 - \tilde{\omega}_1) M(\chi - \omega). \tag{2}$$

The fact that $\operatorname{supp}(\omega_1 - \tilde{\omega}_1) \cap \operatorname{supp}\omega = \emptyset$ implies that the first summand on the right hand side of (2) is an operator in $C_G^d(X^\wedge, \mathbf{g})$. Now we use that

$$(\omega_1 - \tilde{\omega}_1) M(\chi - \omega) = (\omega_1 - \tilde{\omega}_1) A(\chi - \omega) + (\omega_1 - \tilde{\omega}_1) R(\chi - \omega) \tag{3}$$

with an operator $R \in \mathcal{B}^{-\infty,d}(X^\wedge)$. According to Lemma 3.1.14, the second summand is an operator in $C_G^d(X^\wedge, \mathbf{g})$, so the remaining contribution is $(\omega_1 - \tilde{\omega}_1)A(\chi - \omega)$.
Consider the second difference.

$$\begin{aligned} & (1-\omega_1)A(\chi - \omega_3) - (1-\tilde{\omega}_1)A(\chi - \tilde{\omega}_3) \\ =\ & (1-\omega_1)A(\chi - \omega) + (1-\omega_1)A(\omega - \omega_3) \\ & -(1-\tilde{\omega}_1)A(\chi - \omega) - (1-\tilde{\omega}_1)A(\omega - \tilde{\omega}_3) \\ =\ & -(\omega_1 - \tilde{\omega}_1)A(\chi - \omega) + (1-\omega_1)A(\omega - \omega_3) - (1-\tilde{\omega}_1)A(\omega - \tilde{\omega}_3). \end{aligned} \tag{4}$$

Since $\operatorname{supp}(1-\omega_1) \cap \operatorname{supp}(\omega - \omega_3) = \emptyset$ and $\operatorname{supp}(1-\tilde{\omega}_1) \cap \operatorname{supp}(\omega - \tilde{\omega}_3) = \emptyset$, we may replace A in the last two summands by an operator in $\mathcal{B}^{-\infty,d}(X^\wedge)$. An application of Lemma 3.1.14 then shows that the second and the third operator in (4) belong to $C_G^d(X^\wedge, \mathbf{g})$. Since the operator $-(\omega_1 - \tilde{\omega}_1)A(\chi - \omega)$ cancels the operator remaining from the first difference, the proof is complete. ◁

3.1.31 Proposition. *Every operator in* $C^{\nu,d}(\mathbb{D},\mathbf{g})$ *for* $\mathbf{g}=(\gamma+\frac{n}{2},\gamma+\frac{n}{2}-\mu,(-N,0])$ *can be written in the form*

$$A=\omega_1\, t^{-\nu}[\mathrm{op}_M^\gamma h]\omega_2+(1-\omega_1)A_\psi(1-\omega_3)+R \tag{1}$$

with $h\in C^\infty(\overline{\mathbf{R}}_+,M_O^{\nu,d}(X))$, $A_\psi\in\mathcal{B}^{\nu,d}(\mathbb{D})$, $R\in C^{\nu,d}_{M+G}(\mathbb{D},\mathbf{g})$; *moreover, we can choose* A_ψ *in such a way that, for all functions* $\omega,\tilde\omega\in C_0^\infty([0,1/2))$, *equal to* 1 *near* 0 *and with* $\omega\tilde\omega=\omega$, *we have*

$$\omega t^{-\nu}[\mathrm{op}_M^\gamma h]\tilde\omega-\omega A_\psi\tilde\omega\in\mathcal{B}^{-\infty,d}(X^\wedge). \tag{2}$$

In (1), $\omega_1,\omega_2,\omega_3$ *are functions in* $C_0^\infty(\overline{\mathbf{R}}_+)$ *supported in* $[0,1/2)$ *with* $\omega_1(t)=\omega_2(t)=\omega_3(t)\equiv1$ *for* t *close to zero and* $\omega_1\omega_2=\omega_1,\omega_1\omega_3=\omega_3$. *The representation is independent of the choice of* $\omega_1,\omega_2,\omega_3$ *in the following sense: if we replace* $\omega_1,\omega_2,\omega_3$ *by* $\tilde\omega_1,\tilde\omega_2,\tilde\omega_3$ *with the same properties, we have to modify* R *by an operator in* $C_G^d(\mathbb{D},\mathbf{g})$.

For the proof we need the following lemma.

3.1.32 Lemma. *Let* $A\in\mathcal{B}^{\nu,d}(X^\wedge),\gamma\in\mathbf{R}$, *and let* $\psi_1,\psi_2\in C^\infty(\mathbf{R}_+)$ *be functions with* $\psi_1(t)=\psi_2(t)=0$ *for small* t. *Then there is a Mellin symbol* $h_1\in C^\infty(\overline{\mathbf{R}}_+,M_O^{\nu,d}(X))$ *with*

$$t^{-\nu}\,\mathrm{op}_M^\gamma h_1-\psi_1A\psi_2\in\mathcal{B}^{-\infty,d}(X^\wedge). \tag{1}$$

Proof of Lemma 3.1.32. By Lemma 2.4.5 we can find $f\in C^\infty(\mathbf{R}_+,\mathcal{B}^{\nu,d}(X;\Gamma_{1/2-\gamma}))$ with

$$t^{-\nu}\,\mathrm{op}_M^\gamma f-\psi_1A\psi_2\in\mathcal{B}^{-\infty,d}(X^\wedge). \tag{2}$$

Moreover, $f(t,z)=0$ for small t due to ψ_1, hence $f\in C^\infty(\overline{\mathbf{R}}_+,\mathcal{B}^{\nu,d}(X;\Gamma_{1/2-\gamma}))$. We now apply Theorem 2.2.17 and Corollary 2.2.18 to find $h\in C^\infty(\overline{\mathbf{R}}_+,M_O^{\nu,d}(X))$ with

$$\mathrm{op}_M^\gamma f-\mathrm{op}_M^\gamma h\in M\mathcal{B}_\gamma^{-\infty,d}(X^\wedge)\ \hookrightarrow\ \mathcal{B}^{-\infty,d}(X^\wedge).$$

◁

Proof of Proposition 3.1.31. Clearly, each of the operators of the form (1) is an operator in the cone algebra $C^{\nu,d}(\mathbb{D},\mathbf{g})$. In order to see that each operator can be represented in this way we do the following. Suppose $A\in C^{\nu,d}(\mathbb{D},\mathbf{g})$ is given in the form

$$A=\tilde\omega_1t^{-\nu}[\mathrm{op}_M^\gamma\tilde h]\tilde\omega_2+\psi_1\tilde A_\psi\psi_2+R$$

with $\tilde h\in C^\infty(\overline{\mathbf{R}}_+,M_O^{\nu,d}(X)),\tilde A_\psi\in\mathcal{B}^{\nu,d}(\mathbb{D})$, and $R\in C^{\nu,d}_{M+G}(\mathbb{D},\mathbf{g})$. Applying Theorem 2.4.12 we find $A_1\in\mathcal{B}^{\nu,d}(X^\wedge)$ (even in $C^\infty(\overline{\mathbf{R}}_+,\tilde{\mathcal{B}}^{\nu,d}(X;\mathbf{R}))$ with

$$\tilde\omega_1[\mathrm{op}_M^\gamma\tilde h]\tilde\omega_2-A_1\in\mathcal{B}^{-\infty,d}(X^\wedge).$$

Hence $t^{-\nu}\tilde\omega_1[\mathrm{op}_M^\gamma\tilde h]\tilde\omega_2-t^{-\nu}A_1\in\mathcal{B}^{-\infty,d}(X^\wedge)$, and $t^{-\nu}A_1\in\mathcal{B}^{\nu,d}(X^\wedge)$. Next consider the operator $A_2=\psi_1\tilde A_\psi\psi_2$. Choose functions $\chi_1,\chi_2,\chi_3\in C_0^\infty(\overline{\mathbf{R}}_+)$, supported in $[0,1)$, equal to 1 on $[0,1/2)$, and satisfying $\chi_1\chi_2=\chi_1,\chi_1\chi_3=\chi_3$. Then

$$A_2=\chi_1A_2\chi_2+\chi_1A_2(1-\chi_2)+(1-\chi_1)A_2\chi_3+(1-\chi_1)A_2(1-\chi_3).$$

Notice that $\chi_1A_2(1-\chi_2)$ and $(1-\chi_1)A_2\chi_3$ are regularizing operators. Since $A_2=\psi_1\tilde A_\psi\psi_2$ with functions ψ_1,ψ_2 vanishing near the singular set, both these operators induce Green operators in $C_G^d(X^\wedge,\mathbf{g})$ by Lemma 3.1.14.

Applying Lemma 3.1.32 we find $h_2 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu,d}(X))$ with

$$t^{-\nu}\mathrm{op}_M^\gamma h_2 - \chi_1 A_2 \chi_2 \in \mathcal{B}^{-\infty,d}(X^\wedge). \tag{1}$$

So let

$$\begin{aligned} h_1(t,t',z) &= \tilde\omega_1(t)\tilde h(t,z)\tilde\omega_2(t') + h_2(t,z), \\ A_\psi &= t^{-\nu}A_1 + A_2. \end{aligned}$$

Given $\omega, \tilde\omega$ with support in $[0,1/2)$ we have $\omega\chi_1 = \omega, \tilde\omega\chi_2 = \tilde\omega$, and therefore

$$\begin{aligned} \omega[t^{-\nu}\mathrm{op}_M^\gamma h_1 - A_\psi]\tilde\omega &= \omega[\tilde\omega_1 t^{-\nu}[\mathrm{op}_M^\gamma \tilde h]\tilde\omega_1 - t^{-\nu}A_1]\tilde\omega \\ &\quad + \omega[t^{-\nu}\mathrm{op}_M^\gamma h_2 - \chi_1 A_2\chi_2]\tilde\omega. \end{aligned}$$

Now choose $\omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega_2(t) \equiv 1$ near zero and support in $[0,1/2)$ such that $(X \times \operatorname{supp}\omega_2) \cap [\operatorname{supp}\psi_1 \cup \operatorname{supp}\psi_2] = \emptyset$; moreover choose $\omega_1, \omega_3 \in C^\infty(\overline{\mathbf{R}}_+)$ with ω_1, ω_3 equal to 1 near zero and $\omega_1\omega_2 = \omega_1, \omega_1\omega_3 = \omega_3$.
We then have, with the abbreviation $M = t^{-\nu}\mathrm{op}_M^\gamma \tilde h$,

$$\begin{aligned} & A - R - [\omega_1 t^{-\nu}[\mathrm{op}_M^\gamma h_1]\omega_2 + (1-\omega_1)A_\psi(1-\omega_3)] \\ = {} & \tilde\omega_1\, M\,\tilde\omega_2 - \omega_1\tilde\omega_1\, M\,\tilde\omega_2\omega_2 - \omega_1 t^{-\nu}[\mathrm{op}_M^\gamma h_2]\omega_2 \qquad (2) \\ & + A_2 - (1-\omega_1)[t^{-\nu}A_1 + A_2](1-\omega_3). \qquad (3) \end{aligned}$$

Consider the first difference (2) first. In view of the fact that $\operatorname{supp}\omega_1 \cap \operatorname{supp}(1-\omega_2) = \emptyset$ while $\operatorname{supp}\tilde\omega_2$ is compact, Lemma 3.1.13 implies that

$$\omega_1\tilde\omega_1\, M\,\tilde\omega_2\omega_2 = \omega_1\tilde\omega_1\, M\,\tilde\omega_2 + G_1$$

with an operator $G_1 \in C_G^d(X^\wedge, \mathbf{g})$. Moreover, since $(X \times \operatorname{supp}\omega_1) \cap \operatorname{supp}\psi_1 = \emptyset$, we may assume that $\omega_1(t)h_2(t,z) \equiv 0$; if necessary, we can modify h_2 to achieve this while (1) remains preserved. So (2) equals $(1-\omega_1)\tilde\omega_1\, M\,\tilde\omega_2 + G_1$. Let us now have a look at (3). Since $(X \times \operatorname{supp}\omega_1) \cap \operatorname{supp}\psi_1 = \emptyset$ and $(X \times \operatorname{supp}\omega_3) \cap \operatorname{supp}\psi_2 = \emptyset$, we have $A_2 - (1-\omega_1)A_2(1-\omega_3) = 0$. We introduced A_1 as an operator satisfying

$$t^{-\nu}A_1 - \tilde\omega_1\, M\,\tilde\omega_2 \in \mathcal{B}^{-\infty,d}(X).$$

It is therefore no restriction to assume that $t^{-\nu}A_1 = \chi_1 t^{-\nu}A_1\chi_1$. An application of Lemma 3.1.14 shows that

$$(1-\omega_1)t^{-\nu}A_1(1-\omega_3) = (1-\omega_1)\tilde\omega_1\, M\,\tilde\omega_2(1-\omega_3) + G_2$$

with $G_2 \in C_G^d(X^\wedge, \mathbf{g})$. Next we notice that $\operatorname{supp}\omega_3 \cap \operatorname{supp}(1-\omega_1) = \emptyset$, and therefore

$$(1-\omega_1)\tilde\omega_1\, M\,\tilde\omega_2(1-\omega_3) = (1-\omega_1)\tilde\omega_1\, M\,\tilde\omega_2 + G_3$$

with $G_3 \in C_G^d(X^\wedge, \mathbf{g})$.
Thus the sum of (2) and (3) is a sum of Green operators in $C_G^d(X^\wedge, \mathbf{g})$. We now have proven that each operator in $C_G^d(X^\wedge, \mathbf{g})$ can be represented in the form (1) with $h_1 \in C^\infty(\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+, M_O^{\nu,d}(X))$ and *special* $\omega_1, \omega_2, \omega_3$. Applying Proposition 3.1.30, we see that any other choice of cut-off functions results in a change in $C_G^d(X^\wedge, \mathbf{g})$ only.
Finally, we apply Theorem 3.1.11 in order to see that we may find a function $h \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu,d}(X))$ satisfying

$$\omega_1 t^{-\nu}[\mathrm{op}_M^\gamma h_1]\omega_2 - \omega_1 t^{-\nu}[\mathrm{op}_M^\gamma h]\omega_2 \in C_G^d(X^\wedge, \mathbf{g}).$$

This completes the proof. ◁

3.1.33 Totally Characteristic Symbols. The operator A_ψ in Proposition 3.1.31 is uniquely determined by 3.1.31(1), (2) up to an element of $\mathcal{B}^{-\infty,d}(I\!D)$. Applying the Mellin quantization result 2.4.12, we can find a symbol $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}^{\nu,d}(X;\mathbf{R}))$ such that the operator $t^{-\nu}[\mathrm{op}_M^\gamma h] - t^{-\nu}[\mathrm{op}\, a]$ is an element of $\mathcal{B}^{-\infty,d}(X^\wedge)$; hence

$$\omega_1 t^{-\nu}[\mathrm{op}\, a]\omega_2 - \omega_1 A_\psi \omega_2 \in \mathcal{B}^{-\infty,d}(X^\wedge)$$

for all $\omega_1, \omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, supported in $[0, 1/2)$. We can and will therefore assume that the pseudodifferential part A_ψ of A has a totally characteristic operator-valued symbol.

3.1.34 Totally Characteristic Symbols in the Classical Case. For $A \in C_{cl}^{\nu,d}(I\!D, \mathbf{g})$ written in the form of Proposition 3.1.31 there is an $a \in C^\infty(\overline{\mathbf{R}}_+, \tilde{\mathcal{B}}_{cl}^{\nu,d}(X;\mathbf{R}))$ with

$$t^{-\nu}\,[\mathrm{op}_M^\gamma h] - t^{-\nu}\,[\mathrm{op}\ a] \in \mathcal{B}^{-\infty,d}(X^\wedge)$$

and

$$t^{-\nu}\,[\mathrm{op}\ a] - A_\psi \in \mathcal{B}^{-\infty,d}(X^\wedge).$$

Indeed, we then have $h \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{\nu,d}(X))$ and $A_\psi \in \mathcal{B}_{cl}^{\nu,d}(I\!D)$. By Theorem 2.4.12 the associated totally characteristic symbol a is also classical.
Note that the parameter-dependent homogeneous components of order ν both of h and a in the sense of 2.4.11 are uniquely determined by A. Those of h are smooth in t up to $t = 0$, while the homogenous components of $\sigma_\psi(a(t))(x, \xi, \tau/t)$ and $\sigma_\wedge(a(t))(x', \xi', \tau/t)$ are smooth in t up to $t = 0$.
For fixed t, the function $h^\vee(t, \tau) = h(t, -i\tau) \in \mathcal{B}^{\nu,d}(X; \mathbf{R}_\tau)$ is a parameter-dependent operator. By Theorem 2.4.13, its principal symbols are locally near $t = 0$ related to that of a by

$$\begin{aligned} \sigma_\psi^\nu(a(t))(x, \xi, \tau/t) &= \sigma_\psi^\nu(h^\vee(t, \cdot))(x, \xi, \tau); \\ \sigma_\wedge^\nu(a(t))(x', \xi', \tau/t) &= \sigma_\wedge^\nu(h^\vee(t, \cdot))(x', \xi', \tau). \end{aligned}$$

Similarly we have the following relation between the symbols of A_ψ and a from 3.1.33:

$$\begin{aligned} \sigma_\psi^\nu(a(t))(x, \xi, \tau) &= t^\nu \sigma_\psi^\nu(A_\psi)(x, t, \xi, \tau); \\ \sigma_\wedge^\nu(a(t))(x', \xi', \tau) &= t^\nu \sigma_\wedge^\nu(A_\psi)(x', t, \xi', \tau). \end{aligned}$$

In particular,

$$\begin{aligned} t^\nu \sigma_\psi^\nu(A_\psi)(x, t, \xi, \tau/t)|_{t=0} &= \sigma_\psi^\nu(h^\vee(0, \cdot))(x, \xi, \tau), \text{ and} \\ t^\nu \sigma_\wedge^\nu(A_\psi)(x', t, \xi', \tau/t)|_{t=0} &= \sigma_\wedge^\nu(h^\vee(0, \cdot))(x, \xi, \tau). \end{aligned}$$

3.1.35 Theorem. *Let $A \in C^{\nu,d}(I\!D, \mathbf{g})$ for $g = (\gamma = \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N, 0])$ have two representations as in Proposition 3.1.31:*

$$\begin{aligned} A &= \omega_1 t^{-\nu}[\mathrm{op}_M^\gamma h]\omega_2 + (1 - \omega_1)A_\psi(1 - \omega_3) + R \\ &= \tilde\omega_1 t^{-\nu}[\mathrm{op}_M^\gamma \tilde h]\tilde\omega_2 + (1 - \tilde\omega_1)\tilde A_\psi(1 - \tilde\omega_3) + \tilde R \end{aligned}$$

with $R, \tilde R \in C_{M+G}^{\nu,d}(I\!D, \mathbf{g})$ of the form

$$R = \omega_1 \sum_{j=0}^{N-1} t^{j-\nu}[\mathrm{op}_M^{\gamma_j} h_j]\omega_2 + G$$

$$\tilde{R} = \tilde{\omega}_1 \sum_{j=0}^{N-1} t^{j-\nu}[\text{op}_M^{\tilde{\gamma}_j}\tilde{h}_j]\tilde{\omega}_2 + \tilde{G}.$$

in the obvious notation. Then

$$(h + h_0) - (\tilde{h} + \tilde{h}_0) \in tC^\infty(\overline{\mathbf{R}}_+, M_O^{\nu,d}(X)) \cap C^\infty(\mathbf{R}_+, M_O^{-\infty,d}(X)); \tag{1}$$

$$A_\psi - \tilde{A}_\psi \in \mathcal{B}^{-\infty,d}(\mathbb{D}); \quad \textit{and} \tag{2}$$

$$\frac{1}{j!}\partial_t^j h(0,z) + h_j(z) = \frac{1}{j!}\partial_t^j \tilde{h}(0,z) + \tilde{h}_j(z),\ j = 0, \dots, N - (\mu - \nu) - 1. \tag{3}$$

Notice that other values of j are irrelevant. In fact, we may also limit the upper index in the summation for R and $\tilde{R}$ to $N - (\mu - \nu) - 1$.

Proof. The fact that, for $\psi_1, \psi_2 \in C_0^\infty(\mathbf{R}_+)$, we have $\psi_1(R - \tilde{R})\psi_2 \in C_G^d(\mathbb{D}, \mathbf{g}) \hookrightarrow \mathcal{B}^{-\infty,d}(\mathbb{D})$ together with the required compatibility of h and A_ψ on one hand and $\tilde{h}$ and $\tilde{A}_\psi$ on the other, cf. Condition 3.1.31(2), implies that $A_\psi - \tilde{A}_\psi \in \mathcal{B}^{-\infty,d}(\mathbb{D})$. This proves (2).
In order to see (3), choose a function $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ such that $\omega\omega_3 = \omega, \omega\tilde{\omega}_3 = \omega$. Then $\omega(1 - \omega_1) = 0 = \omega(1 - \tilde{\omega}_1)$, $\omega\omega_1 = \omega$, $\omega\tilde{\omega}_1 = \omega$. Hence $\omega A = \omega t^{-\nu}[\text{op}_M^\gamma h]\omega_2 + \omega R = \omega t^{-\nu}[\text{op}_M^\gamma \tilde{h}]\tilde{\omega}_2 + \omega\tilde{R}$. We next recall from Lemma 3.1.13 and Theorem 1.8.2 that $\omega t^{-\nu}[\text{op}_M^\gamma \tilde{h}](\tilde{\omega}_2 - \omega_1)$ and $\omega \sum_{j=0}^{N-1} t^{j-\nu}[\text{op}_M^{\tilde{\gamma}_j}\tilde{h}](\tilde{\omega}_2 - \omega_1)$ both are elements in $C_G^d(X^\wedge, \mathbf{g})$. We deduce that

$$\omega t^{-\nu}[\text{op}_M^\gamma(h - \tilde{h})]\omega_2 + \omega t^{-\nu} \sum_{j=0}^{N-1} t^j \left[\text{op}_M^{\gamma_j} h_j - \text{op}_M^{\tilde{\gamma}_j}\tilde{h}_j\right]\omega_2 \in C_G^d(X^\wedge, \mathbf{g}). \tag{4}$$

We next use a Taylor expansion and write

$$(h - \tilde{h})(t,z) = \sum_{j=0}^{N-1} \frac{t^j}{j!}\left(\partial_t^j h(0,z) - \partial_t^j \tilde{h}(0,z)\right) + t^N f(t,z)$$

with $f \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X))$. Then we apply Proposition 3.1.27 for the uniqueness of the Mellin symbols and conclude that identity (3) holds.
Finally, we consider the operator with respect to the simplified weight datum $\tilde{\mathbf{g}} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-1, 0])$, so that the operators $\omega_1 t^{-\nu+j}[\text{op}_M^{\gamma_j} h_j]\omega_2$ and $\tilde{\omega}_1 t^{-\nu+j}[\text{op}_M^{\tilde{\gamma}_j}\tilde{h}_j]\tilde{\omega}_2$ are Green operators for all $j > 0$. We conclude that

$$0 = \omega t^{-\nu}[\text{op}_M^\gamma(h + h_0) - \text{op}_M^\gamma(\tilde{h} + \tilde{h}_0)]\omega_2 + G_1,$$

where $G_1 \in C_G^d(X^\wedge, \tilde{\mathbf{g}})$. From (3) we know that $h_0 - \tilde{h}_0 = h(0,\cdot) - \tilde{h}(0,\cdot)$, so

$$\omega t^{-\nu}\text{op}_M^\gamma[h - \tilde{h} + h(0,\cdot) - \tilde{h}(0,\cdot)]\omega_2 \in C_G^d(X^\wedge, \tilde{\mathbf{g}}).$$

We conclude from Proposition 3.1.28 that

$$h - \tilde{h} - h(0,\cdot) - h(0,\cdot) \in tC^\infty(\overline{\mathbf{R}}_+, M_O^{\nu,d}(X)).$$

Relation (1) follows. ◁

3.1.36 Symbol Levels in the Cone Algebra. Let A be an operator in $C^{\nu,d}(\mathbb{D},\mathbf{g})$, $\mathbf{g} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N, 0])$ represented in the form of Proposition 3.1.31:

$$A = \omega_1 t^{-\nu}[\mathrm{op}_M^\gamma h]\omega_2 + (1-\omega_1)A_\psi(1-\omega_3) + R,$$

where $\omega_1, \omega_2, \omega_3 \in C_0^\infty([0,1/2)), \omega_1, \omega_2, \omega_3 \equiv 1$ near zero, $\omega_1\omega_2 = \omega_1, \omega_1\omega_3 = \omega_3$, and

$$\omega[\mathrm{op}_M^\gamma h]\tilde{\omega} - \omega A_\psi \tilde{\omega} \in \mathcal{B}^{-\infty,d}(X^\wedge)$$

for all $\omega, \tilde{\omega} \in C_0^\infty([0,1/2))$. Then we define the following:
(a) The *pseudodifferential symbol of* A is the pair $\{\sigma_\psi(A_\psi), \sigma_\wedge(A_\psi)\}$ in the sense of 1.3.4.
(b) As a preparation for the definition of the conormal symbol write the operator R in the form (cf. 1.9.1)

$$R = t^{-\nu}\,\omega \sum_{j=0}^{N-1} t^j [\mathrm{op}_M^{\gamma_j} h_j]\tilde{\omega} + G$$

with $\omega, \tilde{\omega} \in C_0^\infty([0,1/2))$ equal to 1 near zero, $\gamma_j \in \mathbf{R}$ satisfying $\gamma - (\mu - \nu) - j \le \gamma_j \le \gamma$, $h_j \in M_{P_j}^{-\infty}(X), \pi_{\mathbf{C}} P_j \cap \Gamma_{1/2-\gamma_j} = \emptyset$, and $G \in C_G^d(\mathbb{D},\mathbf{g})$.
Then the *conormal symbol of order* $\nu - j$ *of* A, $\sigma_M^{\nu-j}(A)$, is given by

$$\sigma_M^{\nu-j}(A) = \frac{1}{j!}h^{(j)}(0) + h_j, \quad j = 0, \ldots, N - (\mu - \nu) - 1.$$

3.1.37 Principal Symbols in the Classical Case. Using the notation of 3.1.36, let $A \in C_{cl}^{\nu,d}(\mathbb{D},\mathbf{g})$. We then can define the principal symbol of A. It is a triple $\{\sigma_\psi^\nu(A), \sigma_\wedge^\nu(A), \sigma_M^\nu(A)\}$, where

- $\sigma_\psi^\nu(A) := \sigma_\psi^\nu(A_\psi)$ is the principal pseudodifferential symbol of A_ψ in the sense of 1.3.4;
- $\sigma_\wedge^\nu(A) := \sigma_\wedge^\nu(A_\psi)$ is the principal boundary symbol of A_ψ in the sense of 1.3.4, and
- $\sigma_M^\nu(A)$ is the conormal symbol of A in the sense of 3.1.36.

3.1.38 Theorem. *Let* $A \in C^{\nu,d}(\mathbb{D},\mathbf{g}), \mathbf{g} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N, 0]), s > d - 1/2$. *Then* A *has continuous extensions*

$$A: \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2) \end{array} \tag{1}$$

and

$$A: \begin{array}{c} \mathcal{H}_{P_1}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \\ \oplus \\ \mathcal{H}_{P_2}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}_{Q_1}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \\ \oplus \\ \mathcal{H}_{Q_2}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2) \end{array}. \tag{2}$$

For $\nu > \mu$ *the mapping* (1) *is compact. In* (2), $P = (P_1, P_2) \in As(X, Y, (\gamma + \frac{n}{2}, (-N, 0]))$ *is a given asymptotic type, while* $Q = (Q_1, Q_2) \in As(X, Y, (\gamma + \frac{n}{2} - \mu, (-N, 0]))$ *is a resulting asymptotic type.*

Proof. By Definition 3.1.15, we can write A a sum of operators $A = A_M + A_\psi + R$, where A_M is a Mellin operator supported close to the singular set, A_ψ is a pseudodifferential operator supported on the regular part, and R is an operator in $C_{M+G}(I\!D, \mathbf{g})$. So it is sufficient to prove the result for the three operators separately. For A_M it has been established in Lemma 3.1.3(e) and Lemma 3.1.8. The pseudodifferential operator is supported away from the singular set. There $\mathcal{H}^{s,\gamma}(I\!D, V_1)$ coincides with $H^s_{loc}(\text{int}I\!D, V_1)$. Therefore neither the weight γ nor the asymptotic types play a role; the assertion follows from the usual mapping properties of A_ψ:

$$A_\psi : \begin{array}{c} H^s_{comp}(\text{int}I\!D, V_1) \\ \oplus \\ H^s_{comp}(\text{int}I\!B, W_1) \end{array} \longrightarrow \begin{array}{c} H^{s-\mu}_{loc}(\text{int}I\!D, V_2) \\ \oplus \\ H^{s-\mu}_{loc}(\text{int}I\!B, W_2) \end{array}. \tag{3}$$

For the operator R we employ Theorem 1.9.3.
Finally, to see the compactness for $\mu > \nu$ we notice that the range of A_M and A_ψ is in fact contained in $\mathcal{H}^{s-\nu,\gamma+\frac{n}{2}-\nu}(I\!D, V_2) \oplus \mathcal{H}^{s-\nu,\gamma+\frac{n-1}{2}-\nu}(I\!B, W_2)$, which is compactly embedded in $\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\mu}(I\!D, V_2) \oplus \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(I\!B, W_2)$, since $\mu > \nu$. Writing $R = \sum R_j + G$, where each R_j is of the form $\omega_1 t^{j-\nu}[\text{op}_M^{\gamma_j} h_j]\omega_2$ with $h_j \in M^{-\infty,d}_{P_j'}(X)$ and $G \in C^d_G(I\!D, \mathbf{g})$, we know that each R_j indeed maps continuously to $\mathcal{H}^{\infty,\gamma+\frac{n}{2}-\nu}(I\!D, V_2) \oplus \mathcal{H}^{\infty,\gamma+\frac{n-1}{2}-\nu}(I\!B, W_2)$, so we can also use the compact embedding argument. For G at last we know from 1.6.4 that it yields a compact mapping between the spaces in (1). ◁

3.2 The Algebra Structure

3.2.1 Outline. It is the purpose of this section to show that the operators in the cone algebra can be composed without leaving the class; it is obvious that addition and scalar multiplication can be performed within the calculus.
So let $\gamma \in \mathbf{R}$, $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbf{Z}$, $d_1, d_2 \in \mathbf{N}$, $0 < N \in \mathbf{N}$, $\mu_1 - \nu_1, \mu_2 - \nu_2 \in \mathbf{N}$. Moreover let

$$\begin{aligned} \mathbf{g}_1 &= \left(\gamma + \tfrac{n}{2}, \gamma + \tfrac{n}{2} - \mu_1, (-N, 0]\right), \\ \mathbf{g}_2 &= \left(\gamma + \tfrac{n}{2} - \mu_1, \gamma + \tfrac{n}{2} - \mu_1 - \mu_2, (-N, 0]\right), \\ A_1 &= A_{1M} + A_{1\psi} + M_1 + G_1 \in C^{\nu_1,d_1}(I\!D, \mathbf{g}_1), \\ A_2 &= A_{2M} + A_{2\psi} + M_2 + G_2 \in C^{\nu_2,d_2}(I\!D, \mathbf{g}_2). \end{aligned}$$

We assume that the operators A_1 and A_2 act on vector bundles which 'fit together', cf. e.g. the assumptions in Theorem 2.3.5. We know from Theorem 3.1.38 that the composition A_2A_1 is defined as an operator on weighted Mellin Sobolev spaces. We shall see now that $A_2A_1 \in C^{\nu_3,d_3}(I\!D, \mathbf{g}_3)$ with

$$\mathbf{g}_3 = \left(\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu_1 - \mu_2, (-N, 0]\right),\ \nu_3 = \nu_1 + \nu_2,\ d_3 = \max\{\nu_1 + d_2, d_1\}.$$

In other words: The composition of operators defines a continuous multiplication

$$C^{\nu_2,d_2}(I\!D, \mathbf{g}_2) \times C^{\nu_1,d_1}(I\!D, \mathbf{g}_1) \to C^{\nu_3,d_3}(I\!D, \mathbf{g}_3). \tag{1}$$

The subspaces of smoothing Mellin and Green operators form two-sided ideals within this setting; the above mapping has the following continuous restrictions.

$$C^{\nu_2,d_2}_{M+G}(I\!D, \mathbf{g}_2) \times C^{\nu_1,d_1}(I\!D, \mathbf{g}_1) \ \to\ C^{\nu_3,d_3}_{M+G}(I\!D, \mathbf{g}_3), \tag{2}$$

$$C^{\nu_2,d_2}(I\!D,\mathbf{g}_2)\times C^{\nu_1,d_1}_{M+G}(I\!D,\mathbf{g}_1) \rightarrow C^{\nu_3,d_1}_{M+G}(I\!D,\mathbf{g}_3), \quad (3)$$

$$C^{d_2}_G(I\!D,\mathbf{g}_2)\times C^{\nu_1,d_1}(I\!D,\mathbf{g}_1) \rightarrow C^{d_3}_G(I\!D,\mathbf{g}_3), \quad (4)$$

$$C^{\nu_2,d_2}(I\!D,\mathbf{g}_2)\times C^{d_1}_G(I\!D,\mathbf{g}_1) \rightarrow C^{d_1}_G(I\!D,\mathbf{g}_3). \quad (5)$$

Note the 'd_1' at the right hand side of (3) and (5). Since both A_1 and A_2 are sums of four terms, we will have to deal with 16 terms, and this makes the exposition a little lengthy. The line of thought is as follows. We shall first show the ideal property of $C_G(I\!D,\cdot)$, thus dealing with seven compositions, see Lemma 3.2.2. We then prove the ideal property of $C_{M+G}(I\!D,\cdot)$, see Proposition 3.2.3. This leaves us with four compositions. One of them is trivial: the composition $A_{2\psi}A_{1\psi}$ is an operator in $\mathcal{B}^{\nu_3,d_3}(I\!D)$. We shall treat the others in Lemma 3.2.4 and 3.2.5. Notice that the composition $A_{2M}A_{1M}$ is a Mellin operator, $A_{2\psi}A_{1\psi}$ is a pseudodifferential operator, while $A_{2M}A_{1\psi}$ and $A_{2\psi}A_{1M}$ each are a sum of a Green operator and a pseudodifferential operator supported away from $\{t=0\}$.
In particular, since the Green operators and the smoothing Mellin operators are smoothing on the regular part of $I\!D$, we have

- The pseudodifferential symbol $\{\sigma_\psi(A_2A_1),\sigma_\wedge(A_2A_1)\}$ of A_2A_1 is the Leibniz product of the pseudodifferential symbols of A_1 and A_2. Locally,

$$\sigma_\psi(A_2A_1)(\tilde{x},\tilde{\xi}) = (\sigma_\psi(A_2)\#\sigma_\psi(A_1))(\tilde{x},\tilde{\xi}) \sim \sum_\alpha \frac{1}{\alpha!}\partial^\alpha_{\tilde{\xi}}\sigma_\psi(A_2)(\tilde{x},\tilde{\xi})D^\alpha_{\tilde{x}}\sigma_\psi(A_1)(\tilde{x},\tilde{\xi}),$$

  and

$$\sigma_\wedge(A_2A_1)(\tilde{x}',\tilde{\xi}') = (\sigma_\wedge(A_2)\#\sigma_\wedge(A_1))(\tilde{x}',\tilde{\xi}').$$

- The conormal symbol of A_2A_1 is given by

$$\sigma_M^{\nu_1+\nu_2-j}(A_2A_1) = \sum_{p+q=j}\{T^{\nu_1-q}\sigma_M^{\nu_2-p}(A_2)\}\sigma_M^{\nu_1-q}(A_1), \quad (6)$$

  $j=0,\dots,N-(\mu_1+\mu_2-\nu_1-\nu_2)-1$, with the conormal symbols $\sigma_M^{\nu_2-p}(A_2)$ and $\sigma_M^{\nu_1-q}(A_1)$ of A_1 and A_2, respectively.

For simplicity we shall keep the notation $A_1, A_{1M},\dots,G_2,\nu_1,\dots,d_3$ fixed.

3.2.2 Lemma. *The Green operators have the mapping properties* 3.2.1(4) *and* (5).

Proof. From Theorem 1.6.2 we know the kernels of Green operators. In particular, we note that, for $j\in\mathbf{N}$, the composition with the normal derivative $\partial^j_r G$ is a Green operator of type zero whenever G is. From the mapping properties in Theorem 3.1.36 we therefore obtain the required mapping properties of Green operators. ◁

3.2.3 Proposition. *The smoothing Mellin operators have properties* 3.2.1(2) *and* (3).

Proof. We know from Theorem 1.9.10 that the smoothing Mellin operators form an algebra. We therefore have to consider the compositions

(i) $A_{2M}M_1$.
(ii) M_2A_{1M}.
(iii) $A_{2\psi}M_1$.

(iv) $M_2A_{1\psi}$.

Let us begin with (i). We have to deal with a composition

$$t^{-\nu_2}\omega_1\left[\mathrm{op}_M^{\gamma-\mu_1}h_2\right]\omega_2t^{j-\nu_1}[\mathrm{op}_M^{\gamma_1}h_1]\omega_3$$

with $h_2 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu_2,d_2}(X))$, $\omega_1,\omega_2,\omega_3 \in C_0^\infty(\overline{\mathbf{R}}_+)$, $j \in \mathbf{N}$, $h_1 \in M_P^{-\infty,d_1}(X)$, P a Mellin asymptotic type, and $\gamma - j \le \gamma_1 \le \gamma$, $\pi_{\mathbf{C}}P \cap \Gamma_{1/2-\gamma_1} = \emptyset$.
We first apply Lemma 3.1.10 to commute $t^{j-\nu_1}$ to the left, replacing h_2 by $T^{\nu_1-j}h_2$; next use Theorem 3.1.11 and Lemma 3.1.25 to find $h_3 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu_2}(X))$ with

$$t^{-\nu_2}\omega_1\left[\mathrm{op}_M^{\gamma-\mu_1}h_2\right]\omega_2t^{j-\nu_1} = t^{j-\nu_1-\nu_2}\omega_1\mathrm{op}_M^{\gamma-\mu_1}h_3 + t^{j-\nu_1-\nu_2}R$$

with suitable $R \in C_G^{d_2}(I\!D, \mathbf{g}_0)$ with $\mathbf{g}_0 = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-N,0])$. We know already from Theorem 1.9.10 that

$$t^{j-\nu_1-\nu_2}R[\mathrm{op}_M^{\gamma_1}h_1]\omega_3 \in C_G^{d_1}(I\!D, \mathbf{g}_3).$$

The analyticity of h_3 implies that $\mathrm{op}_M^{\gamma-\mu_1}h_3 = \mathrm{op}_M^{\gamma_1}h_3$, hence $[\mathrm{op}_M^{\gamma-\mu_1}h_3][\mathrm{op}_M^{\gamma_1}h_1] = \mathrm{op}_M^{\gamma_1}(h_3h_1)$. An application of Proposition 1.7.5 shows that

$$h_3h_1 \in C^\infty(\overline{\mathbf{R}}_+, M_{P'}^{-\infty}(X))$$

for a suitable asymptotic type P'. Now Theorem 3.1.9 gives the assertion.
The composition (ii) can be treated in a similar way: we consider

$$t^{j-\nu_2}\omega_1\left[\mathrm{op}_M^{\gamma_2}h_2\right]\omega_2t^{-\nu_1}[\mathrm{op}_M^{\gamma}h_1]\omega_3$$

with $j \in \mathbf{N}, \gamma-\mu_1-(\mu_2-\nu_2)-j \le \gamma_2 \le \gamma-\mu_1, h_2 \in M_P^{-\infty,d_2}(X), \pi_{\mathbf{C}}P \cap \Gamma_{1/2-\gamma_2} = \emptyset, h_1 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu_1,d_1}(X))$. We may commute $t^{-\nu_1}$ to the left, noting that

$$t^{j-\nu_2}\omega_1\left[\mathrm{op}_M^{\gamma_2}h_2\right]\omega_2t^{-\nu_1} = t^{j-\nu_1-\nu_2}\omega_1\mathrm{op}_M^{\gamma_2-\nu_1}\left[T^{\nu_1}h_2\right]\omega_2.$$

Now let $h_3(t,z) = \omega_2(t)h_1(t,z)$, find a symbol $h_4 = h_4(t',z)$ with $[\mathrm{op}_M^{\gamma}h_3]\omega_3 = [\mathrm{op}_M^{\gamma}h_4]\omega_3 + R$, $R \in C_G^{d_1}(I\!D, \mathbf{g}_0)$, and proceed as before.
Our next goal is to show that the composition in (iii) and (iv) furnishes Green operators. In view of the cut-off functions associated with M_1 and M_2, and the fact that $A_{1\psi}$ as well as $A_{2\psi}$ are operators supported away from $\{t=0\}$, we have

$$A_{2\psi}M_1 = A_{2\psi}\varphi_1M_1, \text{ and } M_2A_{1\psi} = M_2\varphi_2A_{1\psi}$$

for suitable functions $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R}_+)$. Hence Theorem 1.8.2 and Lemma 3.2.2 yield the assertion. ◁

3.2.4 Lemma. $A_{2M}A_{1M} \in C^{\nu_3,d_3}(I\!D, \mathbf{g}_3)$.

Proof. We have to consider an operator of the form

$$t^{-\nu_2}\omega_1[\mathrm{op}_M^{\gamma-\mu_1}h_2]\omega_2t^{-\nu_1}[\mathrm{op}_M^{\gamma}h_1]\omega_3$$

with $\omega_1, \omega_2, \omega_3 \in C_0^\infty(\overline{\mathbf{R}}_+)$, $h_j \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu_j,d_j}(X))$, $j = 1,2$. Using the analyticity of h_2 we may rewrite it as $t^{-\nu_1-\nu_2}\omega_1[\mathrm{op}_M^\gamma (T^{\nu_1}h_2)]\omega_2[\mathrm{op}_M^\gamma h_1]\omega_3$. We let $h_3(t,z) = \omega_2(t)h_1(t,z)$. By Theorem 3.1.11 in connection with Lemma 3.1.25 we can find $h_4 = h_4(t',z)$ such that

$$\omega_2[\mathrm{op}_M^\gamma h_1]\omega_3 = [\mathrm{op}_M^\gamma h_3]\omega_3 = \mathrm{op}_M^\gamma h_4 + R$$

with $R \in C_G^{d_1}(I\!D, \mathbf{g}_0)$. We know from Proposition 3.2.3 that $C_G(I\!D, \cdot)$ is an ideal, hence the fact that

$$\omega_1[\mathrm{op}_M^\gamma T^{\nu_1}h_2][\mathrm{op}_M^\gamma h_4]\omega_3 = \omega_1[\mathrm{op}_M^\gamma((T^{\nu_1}h_2)h_4)]\omega_3$$

concludes the proof. $\triangleleft$

3.2.5 Lemma. *$A_{2M}A_{1\psi}$ and $A_{2\psi}A_{1M}$ belong to $C^{\nu_3,d_3}(I\!D, \mathbf{g}_3)$. In fact both are sums of a Green operator and a pseudodifferential operator supported away from $\{t = 0\}$.*

Proof. Let $A_{2M} = t^{-\nu_2}\omega_1\left[\mathrm{op}_M^{\gamma-\mu_1}h_2\right]\omega_2$, $A_{1\psi} = \psi_1 B\psi_2$ with $h_2 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\nu_2,d_2}(X))$, $B \in \mathcal{B}^{\nu_1,d_1}(I\!D)$, $\omega_1,\omega_2 \in C_0^\infty(\overline{\mathbf{R}}_+)$, while $\psi_1,\psi_2 \in C^\infty(I\!D)$ both vanish for $t < 2\varepsilon$, $\varepsilon > 0$. Choose a smooth function ω_3 supported in $[0,\varepsilon)$, equal to 1 close to $t = 0$, and a smooth function ω_4 with $\omega_3\omega_4 = \omega_4$. Then

$$\begin{aligned} A_{2M}A_{1\psi} &= A_{2M}\omega_3A_{1\psi} + \omega_4A_{2M}(1-\omega_3)A_{1\psi} + (1-\omega_4)A_{2M}(1-\omega_3)A_{1\psi} \\ &= 0 + C_1 + C_2 \end{aligned}$$

with the obvious notation. Now

$$\omega_4A_{2M}(1-\omega_3) = t^{-\nu_1}\omega_1\mathrm{op}_M^{\gamma-\nu_1}\left[\omega_4(t)h_1(t,z)(1-\omega_3(t'))\right]\omega_2 \in C_{M+G}^{\nu_2,d_2}(I\!D, \mathbf{g}_2)$$

by Theorem 3.1.9 in connection with Theorem 3.1.11 and Remark 3.1.12. Knowing this we may apply Theorem 1.8.2 to conclude that it even is an element of $C_G^{d_2}(I\!D, \mathbf{g}_2)$. So Proposition 3.2.3 implies that C_1 is a Green operator. The operator $(1-\omega_4)A_{2M}(1-\omega_3)$ is supported away from the boundary. It therefore coincides with an operator in $\mathcal{B}^{\nu_2,d_2}(I\!D)$ supported in the interior; hence composition with $A_{1\psi}$ furnishes an element of $\mathcal{B}^{\nu_3,d_3}(I\!D)$, supported away from $\{t = 0\}$. The argument for $A_{2\psi}A_{1M}$ is the same. $\triangleleft$

3.2.6 Formal Neumann Series. Suppose we are given an $R \in C^{-1,d}(I\!D, \mathbf{g})$, where $\mathbf{g} = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-N, 0])$. Then the inverse to the operator $I - R$ is formally given by $\sum_{j=0}^\infty R^j$. Although this series will in general not be convergent, we shall use it in the following sense. Let

$$R = t\,\omega_1[\mathrm{op}_M^\gamma r]\omega_2 + (1-\omega_1)R_\psi(1-\omega_3) + \omega_1\sum_{k=0}^{N-1} t^{1+k}[\mathrm{op}_M^{\gamma_k}r_k]\omega_2 + G$$

with $r \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-1,d}(X))$, $R_\psi \in \mathcal{B}^{-1,d}(I\!D)$, $r_k \in M_{P_k}^{-\infty,d}(X)$, and $G \in C_G^d(I\!D, \mathbf{g})$. According to 3.2.1 we can compute from these data $r^{[j]} \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-j,d}(X))$, $R_\psi^{[j]} \in \mathcal{B}^{-j,d}(I\!D)$, $r_k^{[j]} \in M_{P_k^{[j]}}^{-\infty,d}(X)$, $j = 0,1,\ldots$, such that

$$R^j - t^j\omega_1[\mathrm{op}_M^\gamma r^{[j]}]\omega_2 - (1-\omega_1)R_\psi^{[j]}(1-\omega_3) - \omega_1\sum_{k=0}^{N-1} t^{j+k}[\mathrm{op}_M^{\gamma_k^{[j]}} r_k^{[j]}]\omega_2 \in C_G^d(I\!D, \mathbf{g}).$$

Notice that, for $j \geq N$, we can take $r_k^{[j]} = 0, k = 0, \ldots, N-1$, since the induced operators are Green operators by Lemma 1.9.4. Next we take the asymptotic sum of these symbols: we let

$$s \sim \sum_{j=0}^{\infty} t^j r^{[j]}, S_\psi \sim \sum_{j=0}^{\infty} R_\psi^{[j]}, s_k = \sum_{j=0}^{N-1} t^j r_k^{[j]},$$

where the first is an asymptotic sum in $C^\infty(\overline{\mathbf{R}}_+, M_O^{0,d}(X))$, cf. Theorem 3.1.7, the second an asymptotic sum in $\mathcal{B}^{0,d}(I\!D)$, while the third is finite. Set

$$S = \omega_1[\mathrm{op}_M^\gamma s]\omega_2 + (1-\omega_1)S(1-\omega_3) + \omega_1 \sum_{j=0}^{N-1} [\mathrm{op}_M^{\tilde{\gamma}_k} s_k]\omega_2.$$

We then have

$$S(I-R) - I \in C_G^d(I\!D, \mathbf{g}). \tag{1}$$

In order to see this notice that $S_M := S - \sum_{j=0}^{M} R^j \in C^{-M,d}(I\!D, \mathbf{g})$ by construction. Therefore

$$\begin{aligned} S(I-R) &= \textstyle\sum_{j=0}^{M} R^j(I-R) + S_M(I-R) \\ &= I - R^{M+1} + S_M(I-R) \in I + C^{-M,d}(I\!D, \mathbf{g}). \end{aligned}$$

Since M was arbitrary and $\bigcap_{M\in\mathbf{N}} C^{-M,d}(I\!D, \mathbf{g}) = C_G^d(I\!D, \mathbf{g})$ by Lemma 3.1.19, we get the desired result.

3.3 Ellipticity, Parametrices, and the Fredholm Property

3.3.1 Definition. Let $\mathbf{g} = \left(\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N, 0]\right), \mu \in \mathbf{Z}, d = \mu_+ \in \mathbf{N}$, and

$$A = \omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h]\omega_2 + (1-\omega_1)A_\psi(1-\omega_3) + \omega_1 \sum_{j=0}^{N-1} t^{j-\mu}\left[\mathrm{op}_M^{\gamma_j} h_j\right]\omega_2 + G \in C^{\mu,d}(I\!D, \mathbf{g}) \tag{1}$$

with the usual convention (i.e., $\omega_1, \omega_2, \omega_3 \equiv 1$ near 0, $\omega_1\omega_2 = \omega_1, \omega_1\omega_3 = \omega_3, h \in C^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X)), A_\psi \in \mathcal{B}^{\mu,d}(I\!D), h_j \in M_{P_j}^{-\infty,d}(X), \gamma - j \leq \gamma_j \leq \gamma, G \in C_G^d(I\!D, \mathbf{g})$). Recall that $\mu_+ = \max\{\mu, 0\}$.

We shall say that A is elliptic of order μ, provided that the following holds:

(i) A_ψ is an elliptic element of $\mathcal{B}^{\mu,d}(I\!D)$, i.e., there is a $B_\psi \in \mathcal{B}^{-\mu,d'}(I\!D), d' = (-\mu)_+$, such that for all $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega(t) \equiv 1$ near $t = 0$

$$\begin{aligned} &A_\psi(1-\omega)B_\psi - (1-\omega)I \in \mathcal{B}^{-\infty,d'}(I\!D), \quad \text{and} \\ &B_\psi(1-\omega)A_\psi - (1-\omega)I \in \mathcal{B}^{-\infty,d}(I\!D). \end{aligned}$$

In other words, A_ψ is an elliptic element of Boutet de Monvel's calculus for the interior of $I\!D$ in the standard sense.

(ii) h is elliptic in the following sense: there is a $g \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\mu,d'}(X; \Gamma_{1/2-\gamma}))$ and a function $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$, equal to 1 near $t = 0$, such that

$$\begin{aligned} &\omega(hg - I) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d'}(X; \Gamma_{1/2-\gamma})), \quad \text{and} \\ &\omega(gh - I) \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d}(X; \Gamma_{1/2-\gamma})). \end{aligned}$$

(iii) For each $z \in \Gamma_{1/2-\gamma}$, the operator $\sigma_M^\mu(A)(z) = h(0,z) + h_0(z) \in \mathcal{B}^{\mu,d}(X)$ is invertible by an element in $\mathcal{B}^{-\mu,d'}(X)$.

3.3.2 Remark. (a) In view of the spectral invariance of Boutet de Monvel's algebra, see Corollary 4.1.3(b), we can replace condition 3.3.1(iii) by the invertibility of

$$h(0,z)+h_0(z): H^{\mu_+}(X,V_1)\oplus H^{\mu_+}(Y,W_1)\to H^{\mu_+-\mu}(X,V_2)\oplus H^{\mu_+-\mu}(Y,W_2).$$

(b) It follows from Theorem 3.3.1(i) and 3.3.1(ii) that the conditions in 3.3.1 are independent of the representation of A with suitable h, A_ψ, h_j, G. Condition 3.3.1(ii) reflects the Fuchs type ellipticity of A. As we shall see in 3.3.11, it also is independent of the representation.

3.3.3 Definition. Ellipticity in the Classical Case. Let $A\in C_{cl}^{\mu,d}(\mathbb{D},\mathbf{g})$ be written in the form of Definition 3.3.1 with $A_\psi\in\mathcal{B}_{cl}^{\mu,d}(\mathbb{D})$ and $h\in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{\mu,d}(X))$. We say that A is elliptic of order μ, if
(i) $\sigma_\psi^\mu(A)$ is invertible on $T^*(\text{int }\mathbb{D})\backslash 0$.
(ii) $\sigma_\wedge^\mu(A)$ is an invertible operator family on $T^*(\text{int }\mathbb{B})\backslash 0$.
(iii) $\sigma_M^\mu(A)$ is invertible by an element in $\mathcal{B}^{-\mu,d'}(X;\Gamma_{1/2-\gamma})$.
As before, $d'=(-\mu)_+$.

3.3.4 Proposition. *For $A\in C_{cl}^{\mu,d}(\mathbb{D},g)$, the ellipticity in the sense of Definiton 3.3.3 implies ellipticity in the sense of Definition 3.3.1.*

For the proof we need the following lemma.

3.3.5 Lemma. *Let $\mathcal{A}$ be a unital algebra, $\mathcal{A}^{-1}$ its group of invertible elements. Let $\mathcal{A}_1,\mathcal{A}_2$ be subsets of $\mathcal{A}^{-1}$, endowed with Frechet topologies, and suppose that inversion*

$$(\cdot)^{-1}:\mathcal{A}_1\to\mathcal{A}_2$$

is continuous. Moreover assume that J is a bounded closed interval in $\mathbf{R}$ and $F\in C^\infty(J,A_1)$. Then the function $G: J\to\mathcal{A}_2$ defined by $G(t)=F(t)^{-1}$ is an element of $C^\infty(J,\mathcal{A}_2)$.

Proof. The continuity of inversion implies that G is continuous. For $t,t_0\in J$ we have

$$\frac{G(t)-G(t_0)}{t-t_0}=-G(t)\frac{F(t)-F(t_0)}{t-t_0}G(t_0).$$

Hence the limit $t\to t_0$ exists; it equals $-G(t_0)F'(t_0)G(t_0)$ and also is continuous. Iteration completes the argument. ◁

Proof of Proposition 3.3.4. By 2.44, the invertibility of $\sigma_\psi^\mu(A)$ and $\sigma_\wedge^\mu(A)$ implies the existence of a parametrix to A_ψ, so condition (i) of 3.3.1 is satisfied. Condition 3.3.1(iii) clearly is weaker than 3.3.3(iii), because in 3.3.3(iii) we ask parameter-dependent invertibility. It remains to check 3.3.1(ii). We focus on a neighborhood of $t=0$, so that we only have to deal with $X\times\mathbf{R}_+$. For all fixed $(x,\xi,\tau)\in(T^*X\times\mathbf{R})\backslash 0, t>0$

$$t^\mu\sigma_\psi^\mu(A)(x,t,\xi,\tau/t): V_{1x}\to V_{2x}$$

is invertible as a consequence of the interior ellipticity. Similarly

$$t^\mu\sigma_\wedge^\mu(A)(x',t,\xi',\tau/t)\in\mathcal{B}_{cl}^{\mu,d}(\mathbf{R}_+)$$

is invertible for all $(x', \xi', \tau) \in (T^*Y \times \mathbf{R}) \setminus 0, t > 0$, by an element in $\mathcal{B}_{cl}^{-\mu,d'}(\mathbf{R}_+), d' = (-\mu)_+$. Here $\mathcal{B}_{cl}^{\mu,d}(\mathbf{R}_+)$ denotes the classical elements of order μ and type d in Boutet de Monvel's calculus, acting from $V_{1x'} \otimes \mathcal{S}(\mathbf{R}_+) \oplus W_{1x'}$ to $V_{2x'} \otimes \mathcal{S}(\mathbf{R}_+) \oplus W_{2x'}$. What about $t = 0$? We introduce the notation $h^\vee(t,\tau) = h(t, -i\tau)$. Condition 3.3.3(iii) in connection with 2.4.15(2) shows that $h(0,\tau) \in \mathcal{B}^{\mu,d}(X; \Gamma_{0,\tau})$ is elliptic. Hence, by 3.1.34,

$$t^\mu \sigma_\psi^\mu(A)(x,t,\xi,\tau/t)|_{t=0} = \sigma_\psi^\mu(h^\vee(0,\cdot))(x,\xi,\tau) : V_{1x} \to V_{2x}$$

is invertible for $(x,\xi,\tau) \in (T^*X \times \mathbf{R})\backslash 0$, and

$$t^\mu \sigma_\wedge^\mu(A)(x',t,\xi',\tau/t)|_{t=0} = \sigma_\wedge^\mu(h^\vee(0,\cdot))(x',\xi',\tau) \in \mathcal{B}_{cl}^{\mu,d}(\mathbf{R}_+)$$

is invertible for $(x',\xi',\tau) \in (T^*Y \times \mathbf{R})\backslash 0$.
We recall that $t^\mu \sigma_\psi^\mu(A)(x,t,\xi,\tau/t)$ and $t^\mu \sigma_\wedge^\mu(x',t,\xi',\tau/t)$ are smooth in x,t,ξ,τ up to $t = 0$. Localizing on $(T^*X \times \mathbf{R})\backslash 0$ and $(T^*Y \times \mathbf{R})\backslash 0$ to neighborhoods over which the vector bundles are trivial, we may consider $t^\mu \sigma_\psi^\mu(A)(x,t,\xi,\tau/t)$ a smooth map on a bounded closed interval in $\mathbf{R}^{2n+2}_{x,t,\xi,\tau}$ with values in matrices of finite size, which in addition is pointwise invertible. By Lemma 3.3.5, the inverse $b_\psi = b_\psi(x,t,\xi,\tau)$ given by $b(x,t,\xi,\tau) = [t^\mu \sigma_\psi^\mu(A)(x,t,\xi,\tau/t)]^{-1}$ is a smooth function, since inversion is continuous on matrices. Moreover, the inverse is a homogeneous function in (ξ,τ) of degree $-\mu$.
With the same localization, we can consider $t^\mu \sigma_\wedge^\mu(A)(x',t',\xi',\tau)$ a smooth function on a closed interval in $\mathbf{R}^{2n}_{x',t,\xi',\tau}$, taking values in $\mathcal{B}_{cl}^{\mu,d}(\mathbf{R}_+)$. Since inversion

$$(\cdot)^{-1} : \mathcal{B}_{cl}^{\mu,d}(\mathbf{R}_+)^{-1} \to \mathcal{B}_{cl}^{-\mu,d'}(\mathbf{R}_+)$$

is continuous, we may again apply Lemma 3.3.5: The inverse $b_\wedge = b_\wedge(x',t',\xi',\tau)$ also is a smooth function, homogeneous of degree $-\mu$ in (ξ',τ).
Next we consider b_ψ and $b_\wedge$ as (t,τ)-dependent symbols and define from them a family of operators $B(t,\tau)$ by applying the pseudodifferential action with respect to the x-variables. The homogeneity and smoothness imply that

$$B \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}_{cl}^{-\mu,d'}(X; \mathbf{R}_\tau)).$$

Let $g_0(t,-i\tau) = B(t,\tau)$ so that $g_0 \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}_{cl}^{-\mu,d'}(X;\Gamma_0))$. Now take an arbitrary function $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho) \equiv 1$ near $\rho = 1$ and define

$$g_1(t,z) = M_{\rho\to z}\psi(\rho) M^{-1}_{1/2,\tau\to\rho} g_0(t,z).$$

We have $g_1 \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-\mu,d'}(X))$, $g_1 - g_0 \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}_{cl}^{-\infty,d'}(X;\Gamma_0))$, and

$$\sigma_\psi^{-\mu}(g_1(t,-i\cdot)) = \sigma_\psi^{-\mu}(B(t,\cdot)) = \left[\sigma_\psi^\mu(h(t,-i\cdot))\right]^{-1},$$
$$\sigma_\wedge^{-\mu}(g_1(t,-i\cdot)) = \sigma_\wedge^{-\mu}(B(t,\cdot)) = [\sigma_\wedge^\mu(h(t,-i\cdot))]^{-1}.$$

This implies that, for $\omega_1, \omega_2 \in C_0^\infty(\overline{R}_+)$ with $\omega_1(t) \equiv \omega_2(t) \equiv 1$ for small t and $\omega_1\omega_2 = \omega_1$,

$$\begin{aligned} \omega_1[\mathrm{op}_M^{1/2} g_1]\omega_2[\mathrm{op}_M^{1/2} h] - \omega_1 I &= \mathrm{op}_M^{1/2} r_l, \\ \omega_1[\mathrm{op}_M^{1/2} h]\omega_2[\mathrm{op}_M^{1/2} g_1] - \omega_1 I &= \mathrm{op}_M^{1/2} r_r \end{aligned}$$

for suitable $r_l \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-1,d}(X)), r_r \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-1,d'}(X))$. From the asymptotic expansion formulas for the symbols we conclude that

$$\omega_1(g_1 h - I) \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-1,d}(X)), \text{ and}$$
$$\omega_1(h g_1 - I) \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-1,d'}(X)).$$

The standard iteration process yields a $g \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-\mu,d'}(X))$ satisfying

$$\omega_1(gh - I) \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-\infty,d}(X)),$$
$$\omega_1(hg - I) \in C^\infty(\overline{\mathbf{R}}_+, M_{O,cl}^{-\infty,d'}(X)).$$

In particular, the conditions of 3.3.1(ii) are satisfied. ◁

3.3.6 Theorem. *Let* $\mathbf{g}= (\gamma + \frac{n}{2}, \gamma + \frac{n}{2} - \mu, (-N, 0]), \mu \in \mathbf{Z}, d = \mu_+ \in \mathbf{N}$, *and* $A \in C^{\mu,d}(\mathbb{D}, \mathbf{g})$. *If* A *is elliptic, then there is a parametrix* $B \in C^{-\mu,d'}(\mathbb{D}, \tilde{\mathbf{g}}), d' = (-\mu)_+, \tilde{\mathbf{g}} = (\gamma + \frac{n}{2} - \mu, \gamma + \frac{n}{2}, (-N, 0])$, *such that*

$$\begin{aligned} AB - I &\in C_G^{d'}(\mathbb{D}, \mathbf{g}_1), \text{ and} \\ BA - I &\in C_G^{d}(\mathbb{D}, \mathbf{g}_2) \end{aligned}$$

with $\mathbf{g}_1 = (\gamma + \frac{n}{2} - \mu, \gamma + \frac{n}{2} - \mu, (-N, 0]), \mathbf{g}_2 = (\gamma + \frac{n}{2}, \gamma + \frac{n}{2}, (-N, 0])$.

3.3.7 Outline. The proof of Theorem 3.3.7 will take up a large part of this section. In order to avoid unnecessary repetitions, we shall assume that A has the form and the properties in Definition 3.3.1. The notation $\mu, d, d', \mathbf{g}, \tilde{\mathbf{g}}, \mathbf{g}_1, \mathbf{g}_2, \gamma, N, g, h, A_\psi, h_0, \ldots, h_{N-1}, G$ will be fixed. We shall start with a preparatory proposition illuminating conditions 3.3.1(ii) and 3.3.1(iii). Then the parametrix construction will be carried out in several steps. Corollary 3.3.10, below, will complete the proof.

3.3.8 Proposition. *There is a Mellin asymptotic type* Q, *a function* $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$, *equal to 1 near* $t = 0$, *and a Mellin symbol* $\tilde{g} \in C^\infty(\overline{\mathbf{R}}_+, M_Q^{-\mu,d'}(X))$ *such that*

$$\omega(t)(h(t,z) + h_0(z))\tilde{g}(t,z) - \omega(t)I \in C^\infty(\overline{\mathbf{R}}_+, M_{Q_1}^{-\infty,d'}(X)), \tag{1}$$
$$\omega(t)\tilde{g}(t,z)(h(t,z) + h_0(z)) - \omega(t)I \in C^\infty(\overline{\mathbf{R}}_+, M_{Q_1}^{-\infty,d}(X)), \tag{2}$$
$$(h(0,z) + h_0(z))\tilde{g}(0,z) = I, \tag{3}$$
$$\tilde{g}(0,z)(h(0,z) + h_0(z)) = I, \tag{4}$$

with suitable Mellin asymptotic types Q_1 *and* Q_2. *Note that, according to Proposition 3.1.3(c),* $\tilde{g} = \tilde{g}_{ana} + \tilde{g}_{sing}$ *with* $\tilde{g}_{ana} \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-\mu,d'}(X))$ *and* $\tilde{g}_{sing} \in C^\infty(\overline{\mathbf{R}}_+, M_Q^{-\infty,d'}(X))$. *Moreover, (1) together with (3), and (2) together with (4) imply that, for suitable* $s_r \in C^\infty(\overline{\mathbf{R}}_+, M_{Q_r}^{-\infty,d}(X))$ *and* $s_l \in C^\infty(\overline{\mathbf{R}}_+, M_{Q_l}^{-\infty,d'}(X))$, *we have*

$$\begin{aligned} \omega(t)(h(t,z) + h_0(z))\tilde{g}(t,z) - \omega(t)I &= ts_r(t,z), \text{ and} && (5) \\ \omega(t)\tilde{g}(t,z)(h(t,z) + h_0(z)) - \omega(t)I &= ts_l(t,z). && (6) \end{aligned}$$

Proof. Choose a function $\psi \in C_0^\infty(\mathbf{R}_+)$ with $\psi(\rho) \equiv 1$ for ρ close to 1, and let

$$g_0(t,z) = M_{\rho\to z}\psi(\rho)M^{-1}_{\gamma,\zeta\to\rho}g(t,\zeta).$$

Then $g_0|_{\Gamma_{1/2-\gamma}} - g \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\infty,d'}(X;\Gamma_{1/2-\gamma}))$, and $g_0 \in C^\infty(\overline{\mathbf{R}}_+, M_O^{-\mu,d'}(X))$ by Proposition 2.2.21. Moreover, Corollary 3.1.6 in connection with the ellipticity properties 3.3.1(ii) implies that relations corresponding to (1) and (2) hold for g_0 instead of $\tilde{g}$. We also know that

$$\begin{aligned} r_1(z) &= g_0(0,z)(h(0,z)+h_0(z)) - I \in M_{P_1}^{-\infty,d}(X), \text{ and} \\ r_2(z) &= (h(0,z)+h_0(z))g_0(0,z) - I \in M_{P_2}^{-\infty,d'}(X) \end{aligned}$$

for suitable Mellin asymptotic types P_1 and P_2. Now we apply Lemma 1.9.11 to see that there are Mellin asymptotic types Q_3, Q_4, and $r_3 \in M_{Q_3}^{-\infty,d'}(X), r_4 \in M_{Q_4}^{-\infty,d}(X)$ such that

$$\begin{aligned} (I+r_1)^{-1} &= I + r_3, \quad \text{and} \\ (I+r_2)^{-1} &= I + r_4. \end{aligned}$$

Then

$$\begin{aligned} (I+r_3)(z)g_0(0,z)(h(0,z)+h_0(z)) &= I, \quad \text{and} \\ (h(0,z)+h_0(z))g_0(0,z)(I+r_4)(z) &= I \end{aligned}$$

in the sense of meromorphic operator-valued functions. In particular, we have for $z \in \Gamma_{1/2-\gamma}$

$$(I+r_3)(z)g_0(0,z) = g_0(0,z)(I+r_4)(z) = [h(0,z)+h_0(z)]^{-1},$$

since $h(0,z)+h_0(z)$ by assumption is invertible on this line.
According to Proposition 1.7.5 we know that $g_0(0,\cdot)(I+r_4)$ is an element of $M_Q^{-\mu,d'}(X)$ for a suitable Mellin asymptotic type Q. Finally we let

$$\tilde{g}(t,z) = g_0(t,z)(I+r_4)(z).$$

Since $\tilde{g}(t,z) - g_0(t,z) = g_0(t,z)r_4(z) \in C^\infty(\overline{\mathbf{R}}_+, M_Q^{-\infty,d'}(X))$, relations (1) and (2) will be satisfied, while by construction we have (3) and (4). Relations (5) and (6) follow from (1), (2), (3), and (4) with Taylor's formula. ◁

3.3.9 Proposition. *There is a* $B_l \in C^{-\mu,d'}(\mathbb{D},\tilde{\mathbf{g}})$ *such that*

$$B_lA - I \in C^{-1,d}(\mathbb{D},\mathbf{g}_2).$$

In the same way there is a $B_r \in C^{-\mu,d'}(\mathbb{D},\mathbf{g})$ *such that*

$$AB_r - I \in C^{-1,d'}(\mathbb{D},\mathbf{g}_1).$$

Here, $\tilde{\mathbf{g}}, \mathbf{g}_1$ *and* $\mathbf{g}_2$ *are as in Theorem* 3.3.6.

Proof. Let B_ψ be the parametrix to A_ψ of Definition 3.3.1(i). Related to the functions $\omega_1, \omega_2, \omega_3$ used in the representation of A choose $\omega_4, \omega_5, \omega_6 \in C_0^\infty(\overline{\mathbf{R}}_+)$ with the following properties: $\omega_5\omega_3 = \omega_5, \omega_4\omega_5 = \omega_4, \omega_4\omega_6 = \omega_6$. We also assume that $\omega_6\omega = \omega_6$ for the function ω in 3.3.1(ii). Next let

$$B_l = \omega_4 t^\mu[\mathrm{op}_M^{\gamma-\mu}T^{-\mu}\tilde{g}]\omega_5 + (1-\omega_4)B_\psi(1-\omega_6).$$

Here, $\tilde{g}$ is the meromorphic Mellin symbol of Proposition 3.3.8. This indeed is an operator in $C^{-\mu,d'}(I\!D, \mathbf{g})$, since we can decompose $\tilde{g} = \tilde{g}_{ana} + \tilde{g}_{sing}$ as in 3.3.8 and apply Theorem 3.1.9. In order to show the desired result, we may forget about the terms $h_1, \ldots, h_{N-1}$ and G of A, since they contribute errors in $C^{-1,d}_{M+G}(I\!D, \mathbf{g})$ by 3.2.1(3). So we let $h_c(t,z) = h(t,z) + h_0(z)$ and have

$$A = \omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h_c]\omega_2 + (1-\omega_1)A_\psi(1-\omega_3).$$

Then

$$\begin{aligned} B_l A &= [\omega_4 t^\mu[\mathrm{op}_M^{\gamma-\mu}T^{-\mu}\tilde{g}]\omega_5][\omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h_c]\omega_2] \\ &\quad +[\omega_4 t^\mu[\mathrm{op}_M^{\gamma-\mu}T^{-\mu}\tilde{g}]\omega_5][(1-\omega_1)A_\psi(1-\omega_3)] \\ &\quad +[(1-\omega_4)B_\psi(1-\omega_6)][\omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h_c]\omega_2] \\ &\quad +[(1-\omega_4)B_\psi(1-\omega_6)][(1-\omega_1)A_\psi(1-\omega_3)] \\ &= T_1 + T_2 + T_3 + T_4 \end{aligned}$$

with the obvious notation. Let us consider these terms separately, starting with T_1. The identity $\omega_1\omega_3 = \omega_3$ implies that $\omega_5 = \omega_5\omega_1$. Hence, noting that $[\mathrm{op}_M^{\gamma-\mu}T^{-\mu}\tilde{g}]t^{-\mu} = t^{-\mu}\mathrm{op}_M^\gamma \tilde{g}$,

$$\begin{aligned} T_1 &= \omega_4\,[\mathrm{op}_M^\gamma \tilde{g}]\,\omega_5\,[\mathrm{op}_M^\gamma h_c]\,\omega_2 \\ &\equiv \omega_4[\mathrm{op}_M^\gamma \tilde{g}h_c]\omega_2 \qquad \bmod C^{-1,d}(I\!D, \mathbf{g}_2) \\ &= \omega_4 I + \omega_4 t[\mathrm{op}_M^\gamma s_l]\omega_2. \end{aligned}$$

Here s_l is the Mellin symbol introduced in Proposition 3.3.8, and we have used 3.2.1(6). The term T_2 is zero, for $\omega_5(1-\omega_1) = \omega_5 - \omega_5\omega_1 = 0$. In order to treat T_3, we first note that $(1-\omega_6)\,[\omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h_0]\omega_2]$ is a Green operator by Theorem 1.8.2. We now choose a function ω_7 such that $\omega_6\omega_7 = \omega_7$ and obtain from Lemma 3.1.13 that

$$\begin{aligned} T_3 &\equiv (1-\omega_4)B_\psi(1-\omega_6)\omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h]\omega_2 \qquad \bmod C^d_G(I\!D, \mathbf{g}_2) \\ &\equiv (1-\omega_4)B_\psi(1-\omega_6)\omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h]\omega_2(1-\omega_7) \quad \bmod C^d_G(I\!D, \mathbf{g}_2). \end{aligned}$$

By assumption, $\omega_1 t^{-\mu}[\mathrm{op}_M^\gamma h]\omega_2 - \omega_1 A_\psi\omega_2 \in \mathcal{B}^{-\infty,d}(X^\wedge)$. The multiplications by $1-\omega_6$ and $1-\omega_7$ in connection with Lemma 3.1.14 then imply that

$$\begin{aligned} T_3 &\equiv (1-\omega_4)B_\psi(1-\omega_6)\omega_1 A_\psi\omega_2(1-\omega_7) \quad \bmod C^d_G(I\!D, \mathbf{g}_2) \\ &\equiv (1-\omega_4)B_\psi(1-\omega_6)\omega_1 A_\psi(1-\omega_7) \qquad \bmod C^d_G(I\!D, \mathbf{g}_2). \end{aligned}$$

In the last equivalence we have used that the supports of ω_1 and $1-\omega_2$ are disjoint. This is good enough for our purposes and we turn our attention to T_4. Employing the fact that the supports of $1-\omega_1$ and $\omega_3-\omega_7$ do not intersect, we obtain that

$$T_4 \equiv (1-\omega_4)B_\psi(1-\omega_6)(1-\omega_1)A_\psi(1-\omega_7) \quad \bmod C^d_G(I\!D, \mathbf{g}_2).$$

From 3.3.1(i) we conclude that

$$\begin{aligned} T_3 + T_4 &\equiv (1-\omega_4)B_\psi(1-\omega_6)A_\psi(1-\omega_7) \quad \bmod C^d_G(I\!D, \mathbf{g}_2) \\ &\equiv (1-\omega_4)I \qquad \bmod C^d_G(I\!D, \mathbf{g}_2). \end{aligned}$$

Again we have used Lemma 3.1.14. Hence $B_l A = T_1 + T_2 + T_3 + T_4 \equiv I \bmod C^{-1,d}(I\!D, \mathbf{g}_2)$. The construction of B_r is analogous. ◁

3.3.10 Corollary. In the notation of Proposition 3.3.9, there is an operator $B \in C^{-\mu,d'}(I\!D, \tilde{\mathbf{g}})$ such that

$$BA - I \in C_G^d(I\!D, \mathbf{g}_2)$$

and

$$AB - I \in C_G^{d'}(I\!D, \mathbf{g}_1).$$

Proof. Let $R_l = I - B_l A$ and $S_l = \sum_{j=0}^{\infty} R_l^j$ be the formal Neumann series, cf. 3.2.6. Then $S_l B_l \in C^{-\mu,d'}(I\!D, \tilde{\mathbf{g}})$, and, by 3.2.6(1),

$$G_l = S_l B_l A - I \in C_G^d(I\!D, \mathbf{g}_1).$$

Similarly we can let $R_r = I - AB_l$ and $S_r = \sum_{j=0}^{\infty} R_r^j$ as a formal Neumann series. Then $B_r S_r \in C^{-\mu,d'}(I\!D, \tilde{\mathbf{g}})$, and

$$G_r = AB_r S_r - I \in C_G^{d'}(I\!D, \mathbf{g}_2).$$

This implies that

$$B_r S_r = (S_l B_l A - G_l) B_r S_r = S_l B_l (I + G_r) - G_l B_r S_r = S_l B_l + G,$$

where, according to 3.2.1, $G = S_l B_l G_r - G_l B_r S_r \in C_G^{d''}(I\!D, \tilde{\mathbf{g}})$ with $d'' = \max\{d', d-\mu\} = d'$. We can therefore let $B = S_l B_l$ (or $B = B_r S_r$) and obtain the desired result. ◁

3.3.11 Fuchs Type Ellipticity and Mellin Symbols. The proof of Proposition 3.3.8 shows that $h(0,z) + h_0(z)$ is invertible as a meromorphic function on $\mathbf{C}$, so it is rare for it *not* to be invertible on $\Gamma_{1/2-\gamma}$. This, however, is the point where choice of h_0 is important. The relation

$$\omega(t)\tilde{g}(t,z)(h(t,z) + h_0(z)) = \omega(t)I + ts_l(t,z)$$

implies that, for small t, $\omega(t)I + ts_l(t,z) = I + ts_l(t,z)$ is invertible in $\mathbf{C} + \mathcal{B}^{-\infty,d'}(X; \Gamma_{1/2-\gamma})$. The latter space is a Ψ^*-algebra according to Theorem 4.4.4. In particular, inversion is continuous by Theorem 4.4.2. Using Proposition 3.3.5,

$$(I + ts_l(t,z))^{-1} \in C^\infty(\overline{\mathbf{R}}_+, \mathbf{C} + \mathcal{B}^{-\infty,d'}(X; \Gamma_{1/2-\gamma})).$$

Replacing ω by a function $\omega_1 \in C_0^\infty(\overline{\mathbf{R}}_+)$ with $\omega_1(t) \equiv 1$ near zero and supported in a small neighborhood of $t = 0$ and replacing $\tilde{g}$ by $g_1(t,z) = (I + ts_l(t,z))^{-1}\tilde{g}(t,z)$ we then have

$$\omega_1(t) g_1(t,z)(h(t,z) + h_0(z)) = \omega_1(t) I, \qquad z \in \Gamma_{1/2-\gamma}.$$

In the same way we obtain the relation

$$\omega_1(t)(h(t,z) + h_0(z)) g_1(t,z) = \omega_1(t) I, \qquad z \in \Gamma_{1/2-\gamma}.$$

Hence $h(t,z) + h_0(z)$ has an inverse in $C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\mu,d'}(X; \Gamma_{1/2-\gamma}))$ for small t.
Next suppose we have another representation of the operator A as in Theorem 3.1.35 with Mellin symbols $\tilde{h}, \tilde{h}_0$. We then know that

$$(h(t,z) + h_0(z)) - (\tilde{h}(t,z) - \tilde{h}_0(z)) \in tC^\infty(\overline{\mathbf{R}}_+, M_O^{\mu,d}(X)) \cap C^\infty(\mathbf{R}_+, M_O^{-\infty,d}(X)).$$

Thus

$$\omega_1(t)g_1(t,z)(\tilde{h}(t,z)-\tilde{h}_0(z)) = I + tr(t,z)$$

with $r \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{0,d}(X;\Gamma_{1/2-\gamma})) \cap C^\infty(\mathbf{R}_+, \mathcal{B}^{-\infty,d}(X;\Gamma_{1/2-\gamma}))$. For each small t, we can therefore invert $I+tr(t,\cdot)$ in $\mathbf{C}+\mathcal{B}^{-\infty,d}(X;\Gamma_{1/2-\gamma})$ by Theorem 4.4.4. On the other hand we know that $\mathcal{B}^{0,d}(X;\Gamma_{1/2-\gamma})$ has continuous inversion by Theorem 4.4.2. Hence Proposition 3.3.5 shows that $(I+tr(t,z))^{-1} \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{0,d}(X;\Gamma_{1/2-\gamma}))$. Consequently we find $\omega_2 \in C^\infty(\overline{\mathbf{R}}_+)$ with $\omega_2(t) \equiv 1$ near $t=0$ and $g_2 = (I+tr)^{-1}g_1 \in C^\infty(\overline{\mathbf{R}}_+, \mathcal{B}^{-\mu,d'}(X;\Gamma_{1/2-\gamma}))$ with

$$\begin{aligned} \omega_2(t)g_2(t,z)(\tilde{h}(t,z)+\tilde{h}_0(z)) &= \omega_2 I, \text{ and} \\ \omega_2(t)(\tilde{h}(t,z)+\tilde{h}_0(z))g_2(t,z) &= \omega_2 I. \end{aligned}$$

In particular, the Fuchs type ellipticity condition 3.3.1(ii) also holds for the other representation.

3.3.12 Theorem. *Let $A \in C^{\mu,d}(\mathbb{D},\mathbf{g})$ be elliptic, $\mathbf{g} = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}-\mu, (-N,0])$, $\mu \in \mathbf{Z}, d \in \mathbf{N}, d \le \mu_+ = \max\{\mu, 0\}$. Then*

$$A: \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2) \end{array}$$

is a Fredholm operator. Given $f \in \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \oplus \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2)$ or $g \in \mathcal{H}_{P_1}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \oplus \mathcal{H}_{P_2}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2)$ for some fixed asymptotic type (P_1,P_2) and any solutions $u,v \in \mathcal{H}^{t,\gamma+\frac{n}{2}}(\mathbb{D},V_1)\oplus\mathcal{H}^{t,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1)$ of the equations $Au=f$ or $Av=g$ with $t > (-\mu)_+ - 1/2$, we may conclude that

$$\begin{aligned} u &\in \mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \oplus \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1) \quad \text{and} \\ v &\in \mathcal{H}_{Q_1}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \oplus \mathcal{H}_{Q_2}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1) \end{aligned}$$

for a suitable asymptotic type (Q_1,Q_2).

Proof. By Corollary 3.3.10 there is an operator $B \in C^{-\mu,d'}(\mathbb{D},\tilde{\mathbf{g}}), \tilde{\mathbf{g}} = (\gamma+\frac{n}{2}-\mu, \gamma+\frac{n}{2}, (-N,0]), d' = (-\mu)_+ = \max\{-\mu,0\}$ such that

$$\begin{aligned} R_r &= AB - I \in C_G^{d'}(\mathbb{D},\mathbf{g}_1), \text{ and} \\ R_l &= BA - I \in C_G^{d}(\mathbb{D},\mathbf{g}_2) \end{aligned}$$

with $\mathbf{g}_1 = (\gamma+\frac{n}{2}-\mu, \gamma+\frac{n}{2}-\mu, (-N,0]), \mathbf{g}_2 = (\gamma+\frac{n}{2}, \gamma+\frac{n}{2}, (-N,0])$. The operator B induces a bounded map

$$B: \begin{array}{c} \mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \\ \oplus \\ \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2) \end{array} \longrightarrow \begin{array}{c} \mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \\ \oplus \\ \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1), \end{array}$$

by Theorem 3.1.38 while, by Lemma 1.6.4,

$$\begin{aligned} R_r &\in \mathcal{K}\left(\mathcal{H}^{s-\mu,\gamma+\frac{n}{2}-\mu}(\mathbb{D},V_2) \oplus \mathcal{H}^{s-\mu,\gamma+\frac{n-1}{2}-\mu}(\mathbb{B},W_2)\right), \\ R_l &\in \mathcal{K}\left(\mathcal{H}^{s,\gamma+\frac{n}{2}}(\mathbb{D},V_1) \oplus \mathcal{H}^{s,\gamma+\frac{n-1}{2}}(\mathbb{B},W_1)\right). \end{aligned}$$

So the first statement is immediate. Supposing that $Au = f$ or $Av = g$ we conclude that $Bf = BAu = (I + R_l)u$, hence $u = Bf - R_l u$; similarly $v = Bg - R_l v$. Since B maps spaces with and without asymptotics, cf. Theorem 3.1.38, and since R_l maps any space $\mathcal{H}^{t,\gamma+\frac{n}{2}}(I\!D, V_1) \oplus \mathcal{H}^{t,\gamma+\frac{n-1}{2}}(I\!B, W_1)$, $t > d' - 1/2$, to $\mathcal{H}_{Q_3}^{\infty,\gamma+\frac{n}{2}}(I\!D, V_1) \oplus \mathcal{H}_{Q_4}^{\infty,\gamma+\frac{n-1}{2}}(I\!B, W_1)$ for suitable asymptotic type (Q_3, Q_4), we get the assertion. $\triangleleft$

4 Appendix

4.1 A Theorem on Analytic Fredholm Families

In this section we shall prove a result on the invertibility of analytic Fredholm families. It is a variant of a classical theorem on analytic Fredholm families, with a long history. First steps are due to Tamarkin 1927, [50] progress was made by Atkinson, Gohberg [12], Sz.-Nagy, and Gramsch, see [14] for more details. What we are mainly interested in is Theorem 4.1.6, below. Once Theorem 4.1.1 and Lemma 4.1.5 have been proven, Theorem 4.1.6 follows from a result of Gramsch and Kaballo [15, Proposition 1.6]. For the completeness of the exposition we give a proof. It is based on [49, Section 2.2.5].
We consider an analytic family $\{A(z) : z \in \mathbb{C}\}$ of operators in $\mathcal{B}^{0,0}(X)$, acting on sections of vector bundles V over X and W over $\partial X = Y$, respectively:

$$A(z) : \begin{array}{c} C^\infty(X,V) \\ \oplus \\ C^\infty(Y,W) \end{array} \rightarrow \begin{array}{c} C^\infty(X,V) \\ \oplus \\ C^\infty(Y,W) \end{array}.$$

$A(z)$ extends to a bounded linear map

$$A(z) : \begin{array}{c} H^0(X,V) \\ \oplus \\ H^0(Y,W) \end{array} \rightarrow \begin{array}{c} H^0(X,V) \\ \oplus \\ H^0(Y,W) \end{array}.$$

For simplicity we shall use the notation

$$\begin{aligned} H &= H^0(X,V) \oplus H^0(Y,W), \\ C^\infty &= C^\infty(X,V) \oplus C^\infty(Y,W). \end{aligned}$$

We shall denote by $N(A)$ and $R(A)$ the kernel and range of an operator A. Three facts will play an important role.

4.1.1 Theorem. (Schrohe [34, 37, 39]) *We consider $\mathcal{B}^{0,0}(X)$ a subalgebra of $\mathcal{L}(H)$. The symbol topology in $\mathcal{B}^{0,0}(X)$ is stronger than that of $\mathcal{L}(H)$, so the embedding $\mathcal{B}^{0,0}(X) \hookrightarrow \mathcal{L}(H)$ is continuous. $\mathcal{B}^{0,0}(X)$ is symmetric: For $A \in \mathcal{B}^{0,0}(X)$ the $\mathcal{L}(H)$-adjoint A^* also belongs to $\mathcal{B}^{0,0}(X)$. Finally, if $A \in \mathcal{B}^{0,0}(X)$, and $A : H \to H$ is invertible, then $A^{-1} \in \mathcal{B}^{0,0}(X)$. In other words, $\mathcal{B}^{0,0}(X)$ is a Ψ^*-subalgebra of $\mathcal{L}(H)$.*

Ψ^*-algebras were defined by Gramsch [13, Definition 5.1]:

4.1.2 Definition. Let $\mathcal{A}$ be a Fréchet subalgebra of a unital C^*-algebra $\mathcal{C}$ with the same unit. $\mathcal{A}$ is called a Ψ^*-subalgebra of $\mathcal{C}$ if it is continuously embedded, symmetric, and spectrally invariant, i.e., $\mathcal{A} \cap \mathcal{C}^{-1} = \mathcal{A}^{-1}$.

Here, $\mathcal{A}^{-1}$ and $\mathcal{C}^{-1}$ denote the groups of invertible elements in the respective algebras.

4.1.3 Corollary. (a) The identity $\mathcal{B}^{0,0}(X) \cap \mathcal{L}(H)^{-1} = \mathcal{B}^{0,0}(X)^{-1}$ implies that $\mathcal{B}^{0,0}(X)$ has an open group of invertible elements. Hence a theorem by Waelbroeck, see Theorem 4.1.4, below, implies that inversion is continuous on $\mathcal{B}^{0,0}(X)$.
(b) Let $A \in \mathcal{B}^{\mu,d}(X), \mu \in \mathbf{Z}, d = \mu_+ = \max\{\mu, 0\}$. If V_1, V_2, and W_1, W_2, are vector bundles over X and Y, respectively, and

$$A : \begin{array}{c} H^{\mu_+}(X, V_1) \\ \oplus \\ H^{\mu_+}(Y, W_1) \end{array} \longrightarrow \begin{array}{c} H^{\mu_+ - \mu}(X, V_2) \\ \oplus \\ H^{\mu_+ - \mu}(Y, W_2) \end{array}$$

is an isomorphism, then $A^{-1} \in \mathcal{B}^{-\mu,d'}(X), d' = (-\mu)_+$. In fact this is straightforward from Theorem 4.1.1 using order reductions; see Schrohe [37].

4.1.4 Theorem. **(Waelbroeck, [55, Chapter VII, Proposition 2])** *Let $\mathcal{A}$ be a Fréchet algebra, $\mathcal{A}^{-1}$ its group of invertible elements. Inversion $x \mapsto x^{-1}$ is continuous in $\mathcal{A}^{-1}$ if and only if the invertible element form a G_δ subset of $\mathcal{A}$.*

4.1.5 Lemma. *Let $A \in \mathcal{B}^{0,0}(X)$, and suppose $A : H \to H$ has finite rank. Then $A \in \mathcal{B}^{-\infty,0}(X)$.*

Proof. (cf. Schrohe [37, Lemma 4.3]) The fact that C^∞ is dense in H implies that $A(C^\infty)$ is dense in the finite-dimensional range of A. Since $A(C^\infty) \subseteq C^\infty$, the range of $A : H \to H$ is contained in C^∞. Let $f_1, \ldots, f_k$ be an orthonormal basis. Then

$$Af = \sum_{j=1}^{k} (f, u_j) f_j$$

for suitable $u_j \in H$. The operator A^* also has finite rank. The same argument applies. So we conclude that A is an integral operator with a kernel in $C^\infty \otimes C^\infty$ (the algebraic tensor product). $\lhd$

4.1.6 Theorem. *Let U be a domain in $\mathbf{C}$ and*

$$A : U \to \mathcal{B}^{0,0}(X)$$

an analytic family of elliptic operators acting on H. Assume that there is a $\tilde{z} \in U$ such that $A(\tilde{z})$ is invertible in $\mathcal{L}(H)$. Then $A(z)$ is invertible in $\mathcal{B}^{0,0}(X)$ for all z outside a countable set D with no accumulation point in U. The function $z \mapsto A(z)^{-1}$ is a meromorphic function with values in $\mathcal{B}^{0,0}(X)$; in $z \in D$, $A(z)^{-1}$ has a pole, the coefficients of the principal part of the Laurent series being finite rank operators in $\mathcal{B}^{-\infty,0}(X)$.

Proof. *Step* 1. The embedding $\mathcal{B}^{0,0}(X) \hookrightarrow \mathcal{L}(H)$ is continuous, and we may consider A a mapping $A : U \to \mathcal{L}(H)$. Since $A(z)$ is elliptic, A is an analytic Fredholm family. Let $V \subset\subset U$ be an open set. We shall say that a function is analytic on $\overline{V}$ if it extends to an open neighborhood of V as an analytic function. Consider

$$A : \overline{V} \to \mathcal{L}(H).$$

For each $z \in \overline{V}$ we obtain that $R(A(z))^{\perp} = N(A^*(z))$ is a finite-dimensional subspace of C^∞ since $A(z)$ is elliptic. We may choose finitely many C^∞ functions generating it, say $f_{1z}, \dots, f_{Nz}$. Define the operator $k_z : \mathbf{C}^N \to H$ by $k_z(c_1, \dots, c_N) = \sum_{j=1}^N c_j f_{jz}$. Then

$$(A(z) \;\; k_z) : \begin{matrix} H \\ \oplus \\ \mathbf{C}^N \end{matrix} \to H$$

is surjective. By continuity, (A, k_z) will also be surjective in a neighborhood of z. In view of the compactness of $\overline{V}$ we may find finitely many functions $f_1, \dots, f_M$, define the operator $k : \mathbf{C}^M \to H$ by $k(c_1, \dots, c_M) = \sum_{j=1}^M c_j f_j$ and achieve that

$$(A(z) \;\; k) : \begin{matrix} H \\ \oplus \\ \mathbf{C}^M \end{matrix} \to H$$

is surjective for all $z \in \overline{V}$.

Step 2. In particular: Let p be the orthogonal projection onto $F = \text{span } \{f_1, \dots, f_M\}$ in H and $q = I - p$. Then

$$q(A \;\; k) = (qA(z) \;\; 0) : \begin{matrix} H \\ \oplus \\ \mathbf{C}^M \end{matrix} \to qH$$

is surjective, and so is $qA : H \to qH$. Notice that $qH = (I - p)H = F^{\perp}$ is finite codimensional. Without loss of generality we may assume that $f_1, \dots, f_M$ are orthonormal so that

$$p(f) = \sum_{j=1}^{M} (f, f_j) f_j \, .$$

Therefore p is a finite rank operator in $\mathcal{B}^{-\infty,0}(X)$, while $q \in \mathcal{B}^{0,0}(X)$ is a Fredholm operator of index 0.

Step 3. For each z, the kernel of $qA(z) : H \to H$ is a finite-dimensional subspace L_z of H, consisting of C^∞ functions. The orthogonal projection p_z onto L_z is given as a resolvent integral

$$p_z = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I - A(z)^* A(z))^{-1} d\lambda.$$

Hence $z \mapsto p_z$ is a holomorphic function with values in $\mathcal{B}^{0,0}(X)$. Here we are using the fact that $\mathcal{B}^{0,0}(X)$ is spectrally invariant in $\mathcal{L}(H)$. For each fixed z, $p_z \in \mathcal{B}^{-\infty,0}(X)$, since it is a finite rank operator. Hence $q_z := I - p_z$ is a holomorphic family of Fredholm operators of index 0. Now we may identify A with the matrix function

$$A(z) = \begin{pmatrix} B(z) & K(z) \\ T(z) & Q(z) \end{pmatrix} : \begin{matrix} L_z^{\perp} \\ \oplus \\ L_z \end{matrix} \to \begin{matrix} F^{\perp} \\ \oplus \\ F \end{matrix} \; ; \; z \in \overline{V}.$$

Here $B(z) = qA(z)q_z$, $K(z) = qA(z)p_z$, $T(z) = pA(z)q_z$, $Q(z) = pA(z)q_z$. Note that $B(z) \in \mathcal{B}^{0,0}(X)$, while $K(z), T(z), Q(z) \in \mathcal{B}^{-\infty,0}(X)$. All these functions are analytic $\mathcal{B}^{0,0}(X)$-valued functions. By construction, $B(z) : L_z^{\perp} \to F^{\perp}$ is invertible for each $z \in \overline{V}$.

Step 4. L_{z_0} is a vector bundle over V: For each point $z_0 \in V$, there is an open neighborhood V_{z_0} and an analytic map

$$R : V_{z_0} \to \mathcal{L}(L_{z_0}, H)$$

so that $R(z)$ is an isomorphism between L_{z_0} and L_z for each $z \in V_{z_0}$. In order to see this, choose an orthonormal basis $\{l_1, \ldots, l_N\}$ of L_{z_0}. We have $\det(p_{z_0}l_j, p_{z_0}l_k) = \det(l_j, l_k) = 1$. The continuity of $z \mapsto p_z$ implies that the determinant stays nonzero as z varies over a neighborhood V_{z_0} of z_0. The vectors $p_z l_j$ will therefore form a basis of L_z. The mapping R then is defined by

$$R(z)l_j = p_z l_j \;; \quad j = 1, \ldots, M.$$

Step 5. Given $z_0 \in V$, there is an analytic family $\{B^-(z) : z \in V_{z_0}\} \subset \mathcal{B}^{0,0}(X)$ such that $B^-(z)$ inverts $B(z)$ as an operator in $\mathcal{L}(L_z^\perp, F^\perp)$: Define $S(z) : H \to H$ by

$$S(z)f = \sum_{j=1}^{M}(f, R(z)l_j)f_j, \quad z \in V_{z_0}.$$

Then $S(z) : L_z \to F$ is an isomorphism for all $z \in V_{z_0}$: The matrix $((R(z)l_k, R(z)l_j))_{j,k=1,\ldots,M}$ is invertible, so $\{R(z)l_k : k = 1, \ldots, M\}$ is a basis of L_z with

$$S(z)(R(z)l_k) = \sum_{j=1}^{M}(R(z)l_k, R(z)l_j)f_j, \; k = 1, \ldots, M,$$

being linearly independent.
Clearly, $z \mapsto B(z) + S(z)$ is an analytic family on V_{z_0} with values in $\mathcal{B}^{0,0}(X)$, moreover,

$$B(z) + S(z) : H \to H$$

is invertible for all $z \in V_{z_0}$. Theorem 4.1.1 in connection with Corollary 4.1.3 implies that $z \mapsto C(z) := (B(z) + S(z))^{-1} \in \mathcal{A}(V_{z_0}, \mathcal{B}^{0,0}(X))$. Moreover, it is easily checked that $C(z) = q_z C(z) q + p_z C(z) p$ and that $z \mapsto B^-(z) := q_z C(z) q \in \mathcal{A}(V_{z_0}, \mathcal{B}^{0,0}(X))$ is the desired family.

Step 6. Let

$$J_1(z) = \begin{pmatrix} I & 0 \\ -T(z)B^-(z) & I \end{pmatrix} : \begin{matrix} F^\perp \\ \oplus \\ F \end{matrix} \to \begin{matrix} F^\perp \\ \oplus \\ F \end{matrix}$$

and

$$J_1(z) = \begin{pmatrix} I & -B^-(z)K(z) \\ 0 & I \end{pmatrix} : \begin{matrix} L_z^\perp \\ \oplus \\ L_z \end{matrix} \to \begin{matrix} L_z^\perp \\ \oplus \\ L_z \end{matrix}.$$

Both J_1 and J_2 are analytic $\mathcal{B}^{0,0}(X)$-valued functions. In fact, both J_1 and J_2 are elements of $\mathcal{B}^{0,0}(X)^{-1}$, hence

$$J_1^{-1}, J_2^{-1} : V_{z_0} \to \mathcal{B}^{0,0}(X)$$

are analytic. A computation shows that

$$J_1 A J_2 = \begin{pmatrix} B & 0 \\ 0 & Q - TB^-K \end{pmatrix}. \tag{1}$$

Hence $A(z)$ is an isomorphism for precisely those z where

$$E(z) := (Q - TB^- K)(z) \ : \ L_z \to F$$

is an isomorphism. E is a family of operators acting between finite dimensional spaces both of dimension M. The space F is independent of z; we therefore pick an arbitrary isomorphism $J : F \to \mathbf{C}^M$.

Step 7. Let $R(z)^{-1}$ be the inverse to the isomorphism $R(z) : L_{z_0} \to L_z$. This also is an analytic family of operators on V_{z_0}: For $l \in L_z$ and the above fixed vectors l_j let

$$\tilde{S}(z)\, l = \sum_{k=1}^{M} (l, p_z l_k)\, l_k.$$

Clearly $\widetilde{S}$ is analytic. For each $z \in V_{z_0}$, $\tilde{S}(z) : L_z \to L_{z_0}$ is a linear map. We have

$$\tilde{S}(z)(p_z l_m) = \sum_{j=1}^{M} (p_z l_m, p_z l_k) l_k.$$

Hence $\tilde{S}(z)R(z)$ acts as the invertible matrix function $z \mapsto ((p_z l_j, p_z l_k))_{j,k}$ on the basis $\{l_1, \ldots, l_M\}$ of L_{z_0}, so that $R(z)^{-1}$ is analytic. Identifying L_{z_0} and $\mathbf{C}^M$, we may consider $R(z)^{-1}$ an invertible element of $\mathcal{L}(L_z, \mathbf{C}^M)$.

Step 8. We now have an analytic mapping

$$V_{z_0} \ni z \mapsto H(z) := JE(z)R(z) \in \mathcal{L}(\mathbf{C}^M).$$

By assumption, there is at least one point $\tilde{z}$, where $A(\tilde{z})$ is invertible. Apply the above construction to the corresponding neighborhood $V_{\tilde{z}}$. Then $E(\tilde{z})$ is invertible and so is $H(\tilde{z})$. Since H is a matrix-valued analytic function, it will be invertible on all of $V_{\tilde{z}}$ except for a discrete set of singularities. According to Cramer's rule, the singularities are poles.

Step 9. Since U is a domain, the above considerations in connection with identity (1) show that $A(z)$ is invertible for all $z \in U$ except for a countable subset. The spectral invariance of $\mathcal{B}^{0,0}(X)$ in $\mathcal{L}(H)$ implies that $A(z)^{-1} \in \mathcal{B}^{0,0}(X)$ whenever it exists. Moreover, the fact that inversion is continuous in $\mathcal{B}^{0,0}(X)$ shows the analyticity of $z \mapsto A(z)^{-1}$ outside the singularities. Since the singularities of $A(z)^{-1}$ are precisely those of $E(z)^{-1}$, and, locally

$$E(z) = J^{-1}H(z)R(z)^{-1}$$

we conclude that the singularities of E^{-1} are poles and that the coefficients of the principal parts are finite rank operators. The coefficients of the principal part of the Laurent series for $A(z)^{-1}$ in a pole can be computed by Cauchy's integral theorem from the values of $A(z)^{-1}$ on a contour around the pole. The fact that $A(z)^{-1}$ is an analytic $\mathcal{B}^{0,0}(X)$-valued function implies that also the coefficients are operators in $\mathcal{B}^{0,0}(X)$. On the other hand we know that they are finite rank operators as elements of $\mathcal{L}(H)$. We conclude from Lemma 4.1.5 that they even belong to $\mathcal{B}^{-\infty,0}(X)$. ◁

4.2 The Cone Sobolev Spaces H^s_{cone}

As in Section 1, X denotes the open interior of a compact manifold with boundary, $\overline{X}$, which is embeddded in a closed compact manifold Ω.

4.2.1 The Spaces H^s_{cone}. Let $\{\Omega_j\}_{j=1}^J$ be a finite covering of Ω by open sets, $\kappa_j : \Omega_j \to U_j$ the coordinate maps onto bounded open sets in $\mathbf{R}^n$, and $\{\varphi_j\}_{j=1}^J$ a subordinate partition of unity. The maps κ_j induce a push-forward of functions and distributions: For a function u on Ω_j

$$(\kappa_{j*}u)(x) = u(\kappa_j^{-1}(x)), \quad x \in U_j; \tag{1}$$

for a distribution u ask that

$$(\kappa_{j*}u)(\varphi) = (u, \varphi \circ \kappa_j), \quad \varphi \in C_0^\infty(U_j). \tag{2}$$

For $j = 1, \ldots, J$ consider the diffeomorphisms

$$\chi_j : U_j \times \mathbf{R} \to \{(x\langle t\rangle, t) : x \in U_j, t \in \mathbf{R}\} =: C_j \subset \mathbf{R}^{n+1}.$$

given by $\chi_j(x,t) = (x\langle t\rangle, t)$. Its inverse is $\chi_j^{-1}(y,t) = (y/\langle t\rangle, t)$. For $s \in \mathbf{R}$ we define $H^s_{cone}(\Omega\times\mathbf{R})$ as the set of all $u \in H^s_{loc}(\Omega\times\mathbf{R})$ such that, for $j = 1, \ldots, J$, the push-forward $(\chi_j\kappa_j)_*(\varphi_j u)$, which may be regarded as a distribution on $\mathbf{R}^{n+1}$ after extension by zero, is an element of $H^s(\mathbf{R}^{n+1})$. The space $H^s_{cone}(\Omega \times \mathbf{R})$ is endowed with the corresponding Hilbert space topology. We let

$$\begin{aligned}
H^s_{cone}(X \times \mathbf{R}) &= \{u|_{X\times\mathbf{R}} : u \in H^s_{cone}(\Omega \times \mathbf{R})\},\\
H^s_{0,cone}(X \times \mathbf{R}) &= \{u \in H^s_{cone}(\Omega \times \mathbf{R}) : \operatorname{supp} u \subseteq \overline{X} \times \mathbf{R}\},\\
H^s_{cone}(\Omega^\wedge) &= \{u|_{\Omega\times\mathbf{R}_+} : u \in H^s_{cone}(\Omega \times \mathbf{R})\},\\
H^s_{cone}(X^\wedge) &= \{u|_{X\times\mathbf{R}_+} : u \in H^s_{cone}(X \times \mathbf{R})\},\\
H^s_{0,cone}(X^\wedge) &= \{u|_{\overline{X}\times\mathbf{R}_+} : u \in H^s_{0,cone}(X \times \mathbf{R})\}.
\end{aligned}$$

For $s = 0$, we have $u \in H^0_{cone}(\Omega\times\mathbf{R})$ if and only if $\chi_{j*}\kappa_{j*}(\varphi_j u) \in L^2(\mathbf{R}^{n+1})$ for $j = 1, \ldots, J$. In view of the identities

$$\begin{aligned}
\int_{C_j} |\chi_{j*}\kappa_{j*}(\varphi_j u)(y,t)|^2 \, dy\, dt &= \int_{C_j} |\kappa_{j*}(\varphi_j u)(y/\langle t\rangle, t)|^2 \, dy\, dt\\
&= \int_{U_j}\int_{\mathbf{R}} \langle t\rangle^n \, |\kappa_{j*}(\varphi_j u)(x,t)|^2 \, dt\, dx
\end{aligned}$$

this is the case if and only if $\langle t\rangle^{n/2}\, \kappa_{j*}(\varphi_j u) \in L^2(U_j \times \mathbf{R})$. Moreover, supposing $w = \kappa_{j*}(\varphi_j u)$ is sufficiently smooth, we have

$$\begin{aligned}
\frac{\partial \chi_{j*}w}{\partial y_k}(y,t) &= \frac{\partial}{\partial y_k}\left[w(y/\langle t\rangle, t)\right]\\
&= \langle t\rangle^{-1} \frac{\partial w}{\partial x_k}(y/\langle t\rangle, t) = \langle t\rangle^{-1} \chi_{j*}\left(\frac{\partial w}{\partial x_k}\right)(y,t), && (3)\\
\frac{\partial \chi_{j*}w}{\partial t}(y,t) &= -\frac{t}{\langle t\rangle^2}\sum_{k=1}^n \frac{y_k}{\langle t\rangle}\frac{\partial w}{\partial x_k}(y/\langle t\rangle, t) + \frac{\partial w}{\partial t}(y/\langle t\rangle, t)\\
&= -\frac{t}{\langle t\rangle^2}\sum_{k=1}^n \chi_{j*}\left(x_k\frac{\partial w}{\partial x_k}\right)(y,t) + \chi_{j*}\left(\frac{\partial w}{\partial t}\right)(y,t). && (4)
\end{aligned}$$

For $s \in \mathbf{N}$, the fact that $u \in \langle t\rangle^{-n/2} H^s(\Omega \times \mathbf{R})$ implies that $\langle t\rangle^{n/2} \frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^k}{\partial t^k} w \in L^2(U_j \times \mathbf{R})$ whenever $k + |\alpha| \le s, j = 1, \dots, J$. Then we have $\chi_{j*}(\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^k}{\partial t^k})w \in L^2(\mathbf{R}^{n+1})$. Using (3) and (4) we conclude that $u \in H^s_{cone}(\Omega \times \mathbf{R})$, so that

$$\langle t\rangle^{-n/2} H^s(\Omega \times \mathbf{R}) \hookrightarrow H^s_{cone}(\Omega \times \mathbf{R}).$$

Conversely, let $v = (\chi_j\kappa_j)_*(\varphi_j u)$. Then

$$\begin{aligned}
\frac{\partial(\chi_j^{-1})_* v}{\partial x_k}(x,t) &= \frac{\partial}{\partial x_k}[v(x\langle t\rangle, t)] = \langle t\rangle \frac{\partial v}{\partial y_k}(x\langle t\rangle, t) + \frac{\partial v}{\partial t}(x\langle t\rangle, t) \\
&= \langle t\rangle (\chi_j^{-1})_* \left(\frac{\partial v}{\partial y_k}\right)(x,t) + (\chi_j^{-1})_* \left(\frac{\partial v}{\partial t}\right)(x,t), \qquad (5) \\
\frac{\partial(\chi_j^{-1})_* v}{\partial t}(x,t) &= \frac{t}{\langle t\rangle}\sum_{k=1}^{n} x_k \frac{\partial v}{\partial y_k}(x\langle t\rangle, t) + \frac{\partial v}{\partial t}(x\langle t\rangle, t) \\
&= \frac{t}{\langle t\rangle} x_k (\chi_j^{-1})_* \left(\frac{\partial v}{\partial y_k}\right)(x,t) + (\chi_j^{-1})_* v(x,t). \qquad (6)
\end{aligned}$$

If $u \in H^s_{cone}(\Omega \times \mathbf{R}), s \in \mathbf{N}$, then $\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^k}{\partial t^k} v \in L^2(\mathbf{R}^{n+1})$, $|\alpha| + k \le s$, $j = 1, \dots, J$. Hence $(\chi_j^{-1})_* \left[\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^k}{\partial t^k} v\right] \in \langle t\rangle^{-n/2} L^2(U_j \times \mathbf{R})$. Applying (5) and (6), $u \in \langle t\rangle^{s-\frac{n}{2}} H^s(\Omega \times \mathbf{R})$. We conclude that

$$H^s_{cone}(\Omega \times \mathbf{R}) \hookrightarrow \langle t\rangle^{s-\frac{n}{2}} H^s(\Omega \times \mathbf{R}).$$

4.2.2 Lemma. (a) *For $s \in \mathbf{R}$, the dual space $(H^s_{cone}(\Omega \times \mathbf{R}))'$ to $H^s_{cone}(\Omega \times \mathbf{R})$ can be identified with $H^{-s}_{cone}(\Omega \times \mathbf{R})$. Here the duality is with respect to the $L^2(\mathbf{R}^{n+1})$ inner product induced via the maps χ_j in 4.2.1, equivalently with respect to the inner product in $\langle t\rangle^{-n/2} L^2(\Omega \times \mathbf{R})$.*
(b) *For $s \ge 0$ choose $s \le s' \in \mathbf{N}$. Then*

$$\begin{array}{rcccl}
\langle t\rangle^{-n/2} H^s(\Omega \times \mathbf{R}) & \hookrightarrow & H^s_{cone}(\Omega \times \mathbf{R}) & \hookrightarrow & \langle t\rangle^{s'-n/2} H^s(\Omega \times \mathbf{R}), \\
\langle t\rangle^{-n/2} H^s(X \times \mathbf{R}) & \hookrightarrow & H^s_{cone}(X \times \mathbf{R}) & \hookrightarrow & \langle t\rangle^{s'-n/2} H^s(X \times \mathbf{R}), \\
\langle t\rangle^{-n/2} H^s(\Omega^\wedge) & \hookrightarrow & H^s_{cone}(\Omega^\wedge) & \hookrightarrow & \langle t\rangle^{s'-n/2} H^s(\Omega^\wedge), \\
\langle t\rangle^{-n/2} H^s(X^\wedge) & \hookrightarrow & H^s_{cone}(X^\wedge) & \hookrightarrow & \langle t\rangle^{s'-n/2} H^s(X^\wedge), \\
\langle t\rangle^{-n/2-s'} H^{-s}(\Omega \times \mathbf{R}) & \hookrightarrow & H^{-s}_{cone}(\Omega \times \mathbf{R}) & \hookrightarrow & \langle t\rangle^{-n/2} H^{-s}(\Omega \times \mathbf{R}), \\
\langle t\rangle^{-n/2-s'} H^{-s}(X \times \mathbf{R}) & \hookrightarrow & H^{-s}_{cone}(X \times \mathbf{R}) & \hookrightarrow & \langle t\rangle^{-n/2} H^{-s}(X \times \mathbf{R}), \\
\langle t\rangle^{-n/2-s'} H^{-s}(\Omega^\wedge) & \hookrightarrow & H^{-s}_{cone}(\Omega^\wedge) & \hookrightarrow & \langle t\rangle^{-n/2} H^{-s}(\Omega^\wedge), \\
\langle t\rangle^{-n/2-s'} H^{-s}(X^\wedge) & \hookrightarrow & H^{-s}_{cone}(X^\wedge) & \hookrightarrow & \langle t\rangle^{-n/2} H^{-s}(X^\wedge).
\end{array}$$

(c) *There is a natural $\mathbf{R}_+$-action on $\Omega \times \mathbf{R}$, inducing a corresponding action on functions or distributions, $\kappa_\lambda, \lambda \in \mathbf{R}_+$, by*

$$(\kappa_\lambda u)(x,t) = \lambda^{\frac{n+1}{2}} u(x, \lambda t).$$

For fixed λ, κ_λ is an element of $\mathcal{L}(H^s_{cone}(\Omega \times \mathbf{R})), s \in \mathbf{R}$. The mapping $\lambda \mapsto \kappa_\lambda$ is strongly continuous (The action κ_λ should not be confused with the coordinate maps κ_j.)

Proof. (a) is immediate from the definition of the duality in connection with the fact that $H^s(\mathbf{R}^{n+1})' = H^{-s}(\mathbf{R}^{n+1})$.

(b) Let us consider the first identity. For $s \in \mathbf{N}$, the embeddings have been shown in 4.2.1. We have interpolation both for $H^s(\Omega \times \mathbf{R})$ and $H^s_{cone}(\Omega \times \mathbf{R})$, so we obtain the assertion for $s \geq 0$. The following three embeddings are immediate by restriction. For the fifth embedding, we note that the dual space of $\langle t\rangle^\sigma H^s(\Omega \times \mathbf{R}), \sigma \in \mathbf{R}$, with respect to the duality induced by $\langle t\rangle^{-n/2}L^2(\Omega \times \mathbf{R})$ is $\langle t\rangle^{-\sigma-n}H^{-s}(\Omega \times \mathbf{R})$; it then follows from the first by duality. The final three embeddings again follow by restriction.
(c) For $u \in H^s_{cone}(\Omega \times \mathbf{R})$, let us first check that $\kappa_\lambda u \in H^s_{cone}(\Omega \times \mathbf{R})$. We may suppose u has support in a single coordinate chart and show the statement in local coordinates. We suppose that u is supported in $U \times \mathbf{R}$, for some open set U in $\mathbf{R}^n$, and consider the push-forward of u under $\chi = \chi_j$. By assumption, this is an element of $H^s(\mathbf{R}^{n+1})$. The push-forward of $\kappa_\lambda u$ is

$$\chi_*(\kappa_\lambda u)(y,t) = (\kappa_\lambda u)\left(\frac{y}{\langle t\rangle}, t\right) = \lambda^{\frac{n+1}{2}} u\left(\frac{y}{\langle t\rangle}, \lambda t\right) = \lambda^{\frac{n+1}{2}}\left[\Phi_{\lambda *}^{-1}\chi_*(u)\right](y,t).$$

Here, $\Phi_\lambda : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ is the diffeomorphism

$$\Phi_\lambda(y,t) = \left(\lambda y \frac{\langle \lambda t\rangle}{\lambda\langle t\rangle}, \lambda t\right) = \left(y\frac{\langle \lambda t\rangle}{\langle t\rangle}, \lambda t\right).$$

It is easily checked that $D^\alpha \Phi_\lambda = O(1)$ for all $\alpha \neq 0$. Indeed, this is a consequence of the fact that, for each $\lambda > 0$,

$$t \mapsto \frac{\langle \lambda t\rangle^2}{\lambda^2\langle t\rangle^2} = \frac{1+\lambda^2t^2}{\lambda^2+\lambda^2t^2} = \frac{\lambda^{-2}+t^2}{1+t^2}$$

is a positive function in $C^\infty_b(\mathbf{R})$, bounded away from zero. Its inverse is given by the C^∞_b function $\Phi_\lambda^{-1}(w,r) = (w\frac{\langle r/\lambda\rangle}{\langle r\rangle}, r/\lambda) = \Phi_{1/\lambda}(w,r)$. Therefore, the push-forward by Φ_λ not only leaves $H^s(\mathbf{R}^{n+1})$ invariant, so that $\Phi_{\lambda *}^{-1}\chi_* u \in H^s(\mathbf{R}^{n+1})$; the mapping $\Phi_{\lambda *}^{-1}$ even is bounded on $H^s(\mathbf{R}^{n+1})$ for every $s \in \mathbf{R}$. In order to see the strong continuity of κ_λ, we note that

$$\chi_*(\kappa_\lambda u)(y,t) - \chi_*(\kappa_\mu(y,t)) = \lambda^{\frac{n+1}{2}}\left(\Phi_{\lambda *}^{-1}\chi_* u\right)(y,t) - \mu^{\frac{n+1}{2}}\left(\Phi_{\mu *}^{-1}\chi_* u\right)(y,t).$$

Suppose first that $s = 0$. Let us show that $\|\chi_*\kappa_\lambda u - \chi_*\kappa_\mu u\|_{L^2} \to 0$ as $\lambda \to \mu$. For $u \in C^\infty_0(\mathbf{R}^{n+1})$, supported in $U \times \mathbf{R}$,

$$\begin{aligned} &\int |\lambda^{\frac{n+1}{2}}(\chi_* u)(\Phi_\lambda(x)) - \mu^{\frac{n+1}{2}}(\chi_* u)(\Phi_\mu(x))|^2 dx \\ = &\int |\lambda^{\frac{n+1}{2}}(\chi_* u)\left(\Phi_\lambda\Phi_\mu^{-1}(\tilde{x})\right) - \mu^{\frac{n+1}{2}}\chi_* u(\tilde{x})|^2 J_{\Phi_{1/\mu}}(\tilde{x})\, d\tilde{x} \end{aligned}$$

tends to zero as $\lambda \to \mu$ by Lebesgue's theorem on dominated convergence, since $\Phi_\lambda\Phi_\mu^{-1}(w,r) = \Phi_\lambda(w\frac{\langle r/\mu\rangle}{\langle r\rangle}, r/\mu) = (w\frac{\langle r\lambda/\mu\rangle}{\langle r\rangle}, r\frac{\lambda}{\mu})$. Now choose a sequence $\{u_m\} \subset C^\infty_0(U \times \mathbf{R})$ with $\chi_* u_m \to \chi_* u$ in $L^2(\mathbf{R}^{n+1})$. Then, for $m \in \mathbf{N}$,

$$\begin{aligned} &\|\chi_*\kappa_\lambda u - \chi_*\kappa_\mu u\|_{L^2} \\ \leq\ &\|\chi_*\kappa_\lambda u - \chi_*\kappa_\lambda u_m\|_{L^2} + \|\chi_*\kappa_\lambda u_m - \chi_*\kappa_\mu u_m\|_{L^2} + \|\chi_*\kappa_\mu u_m - \chi_*\kappa_\mu u\|_{L^2} \\ \leq\ &\left(\lambda^{\frac{n+1}{2}}\|\Phi_{\lambda *}^{-1}\|_{\mathcal{L}(L^2)} + \mu^{\frac{n+1}{2}}\|\Phi_{\mu *}^{-1}\|_{\mathcal{L}(L^2)}\right)\|\chi_* u - \chi_* u_m\|_{L^2} + \|\chi_*\kappa_\lambda u_m - \chi_*\kappa_\mu u_m\|_{L^2} \\ \to\ &0 \quad \text{as} \quad m \to \infty, \lambda \to \mu. \end{aligned}$$

The case $s \in \mathbf{N}$ is only slightly more complicated. This shows the strong continuity of κ_λ for $s \in \mathbf{N}$. By interpolation the result extends to $s \geq 0$. The equation

$$\int_{\mathbf{R}^{n+1}} u(\Phi_\lambda(x))\overline{v(x)}\,dx = \int_{\mathbf{R}^{n+1}} u(y)\overline{v(\Phi_{1/\lambda}(y))}J_{\Phi_{1/\lambda}}(y)\,dy$$

shows that, up to a multiplier, the adjoint to κ_λ is $\kappa_{1/\lambda}$. Hence the assertion holds for $s < 0$ by duality. ◁

4.2.3 Remark. (a) We clearly can replace the function $\langle t\rangle$ by any C^∞ function $0 < f$ satisfying $f^{(k)}(t) = O(\langle t\rangle^{1-k})$, e.g. a smooth function $[\cdot] : \mathbf{R} \to \mathbf{R}_+$ with $[t] = |t|$ for $t \geq c > 0$.
(b) The spaces $H^s_{cone}(\Omega \times \mathbf{R})$ coincide with the Sobolev spaces $H^{(s,0)}(\Omega \times \mathbf{R})$ if $\Omega \times \mathbf{R}$ is considered a manifold with two cylindrical ends and SG coordinates are introduced on the ends, cf. [32, Example 3.4, Definition 4.4].

4.2.4 Remark. The subscript "cone" has the following motivation. Let u be a function on $\Omega_j \times \mathbf{R}_+$ with support in $\Omega_j \times (1, \infty)$ with $\Omega_j \subseteq S^n$ a coordinate neighborhood. Define the function U on $\mathbf{R}^{n+1}$ by $U(\tilde{x}) = u(\tilde{x}/|\tilde{x}|, |\tilde{x}|)$. Then $u \in H^s_{cone}(S^n \times \mathbf{R}_+)$ if and only if $U \in H^s(\mathbf{R}^{n+1})$. Notice that $U(\tilde{x}) = 0$ for $|\tilde{x}| \leq 1$.
In order to see this we first let $\kappa : \Omega_j \subseteq S^n \to U_j \subseteq \mathbf{R}^n$ be the coordinate map, and define

$$\chi : U_j \times (1, \infty) \to C_j := \{(y, t) \in \mathbf{R}^{n+1} : y = xt, x \in U_j, t > 1\}$$

For $s \geq 0$ we then have $u \in H^s_{cone}(S^n \times \mathbf{R}_+)$ if and only if $(\chi\kappa)_* u \in H^s(\mathbf{R}^{n+1})$, i.e., $(y, t) \mapsto u(\kappa^{-1}(y/t), t) \in H^s(\mathbf{R}^{n+1})$. Now $u(\kappa^{-1}(y/t), t) = U(\kappa^{-1}(y/t)t) = U \circ \Phi(y, t)$, where $\Phi(y, t) = \kappa^{-1}(y/t)t$. Note that $(y, t) \in C_j$ and that

$$\Phi : C_j \to \Phi(C_j) =: D_j$$

is a diffeomorphism; its inverse being given by $\Phi^{-1}(\tilde{x}) = (\kappa(\tilde{x}/|\tilde{x}|)|\tilde{x}|, |\tilde{x}|)$. We know that the Sobolev spaces on $\mathbf{R}^{n+1}$ are invariant under all diffeomorphisms Φ such that, for every $k > 0$, the k-th total derivative $D^k\Phi$ is bounded and $D\Phi^{-1}$ is bounded.
In order to see the former, we compute that

$$\frac{\partial \Phi}{\partial y_k}(y, t) = \frac{\partial \kappa^{-1}}{\partial x_k}\left(\frac{y}{t}\right); \; \frac{\partial \Phi}{\partial t}(y, t) = -\sum_{l=1}^{n} \frac{\partial \kappa^{-1}}{\partial x_l}\left(\frac{y}{t}\right)\frac{y_l}{t} + \kappa^{-1}\left(\frac{y}{t}\right);$$

iteration then yields the assertion. For the latter statement, we use that

$$\frac{\partial \Phi^{-1}}{\partial \tilde{x}_k}(\tilde{x}) = \left(\sum_{l=1}^{n} \frac{\partial \kappa}{\partial x_l}\left(\frac{\tilde{x}}{|\tilde{x}|}\right)\left[\frac{\delta_{kl}}{|\tilde{x}|} - \frac{\tilde{x}_k\tilde{x}_l}{|\tilde{x}|^2}\right] + \kappa\left(\frac{\tilde{x}}{|\tilde{x}|}\right)\frac{\tilde{x}_l}{|\tilde{x}|}, \frac{\tilde{x}_k}{|\tilde{x}|}\right).$$

4.3 Spectral Invariance of Parameter-Dependent Pseudodifferential Operators

Let Ω be an n-dimensional closed compact manifold. By $L^\mu(\Omega; \mathbf{R}^l) = \operatorname{op} S^\mu(\Omega, \mathbf{R}^{n+l})$, $\mu \in \mathbf{R}$, denote the space of all parameter-dependent pseudodifferential operators of order

μ on Ω with parameter space $\mathbf{R}^l$. The point-wise composition of operators yields a continuous multiplication:

$$L^\mu(\Omega;\mathbf{R}^l) \times L^\nu(\Omega;\mathbf{R}^l) \to L^{\mu+\nu}(\Omega;\mathbf{R}^l),$$

$$(A(\tau), B(\tau)) \mapsto A(\tau)B(\tau), \qquad \tau \in \mathbf{R}^l.$$

Here the spaces $L^\mu(\Omega;\mathbf{R}^l)$, $L^\nu(\Omega;\mathbf{R}^l)$, and $L^{\mu+\nu}(\Omega;\mathbf{R}^l)$ are endowed with the canonical Fréchet topologies of 1.2.4. In particular, $L^0(\Omega;\mathbf{R}^l)$ is a Fréchet algebra. It is continuously embedded in the algebra $C_b(\mathbf{R}^l, \mathcal{L}(L^2(\Omega)))$ of bounded continuous functions on $\mathbf{R}^l$ with values in $\mathcal{L}(L^2(\Omega))$. The following lemma is obvious, noting that $\sup_{\tau\in\mathbf{R}^l}\|A(\tau)^*A(\tau)\| = \sup_{\tau\in\mathbf{R}^l}\|A(\tau)\|^2 = (\sup_{\tau\in\mathbf{R}^l}\|A(\tau)\|)^2$.

4.3.1 Lemma. *$L^0(\Omega;\mathbf{R}^l)$ is a symmetric subalgebra of the C^*-algebra $C_b(\mathbf{R}^l, \mathcal{L}(L^2(\Omega)))$ with respect to the $*$-operation induced by taking pointwise adjoints and the norm $\|A\| = \sup_{\tau\in\mathbf{R}^l}\|A(\tau)\|_{\mathcal{L}(L^2(\Omega))}$.*

We will now show the following theorem.

4.3.2 Theorem. *$L^0(\Omega;\mathbf{R}^l)$ is a Ψ^*-subalgebra of $C_b(\mathbf{R}^l, \mathcal{L}(L^2(\Omega)))$.*

All we have to show is the spectral invariance, i.e., the relation

$$L^0(\Omega;\mathbf{R}^l) \cap C_b(\mathbf{R}^l, \mathcal{L}(L^2(\Omega)))^{-1} = L^0(\Omega;\mathbf{R}^l)^{-1},$$

cf. 4.1.2. This will take up the rest of this section. In part we shall rely on material from [37] and [39]. We first reduce the task a little.

4.3.3 Lemma. *Let $\mathcal{C}$ be a C^*-algebra with unit e. Let $\mathcal{A}$ be a Fréchet subalgebra with a stronger topology, $e \in \mathcal{A}$, and $\mathcal{A}^* = \mathcal{A}$. Suppose there is an $\epsilon > 0$ such that*

$$a^{-1} \in \mathcal{A} \quad \textit{for all} \quad a \in \mathcal{A} \text{ with } \|e - a\|_{\mathcal{C}} < \epsilon. \tag{1}$$

Then $\mathcal{A}$ is a Ψ^*-subalgebra of $\mathcal{C}$.

Proof. We only have to check spectral invariance. So suppose that $a \in \mathcal{A}$ and that there is a $c \in \mathcal{C}$ with $ac = e$. Denote by $\mathcal{B}$ the C^*-closure of $\mathcal{A}$, i.e. the intersection of all closed C^*-subalgebras of $\mathcal{C}$ containing $\mathcal{A}$. Since $\mathcal{A}$ is a symmetric algebra, we simply have $\mathcal{B} = \overline{\mathcal{A}}$, the closure of $\mathcal{A}$ in $\mathcal{C}$.
Now $a \in \mathcal{B} \cap \mathcal{C}^{-1}$, thus $c \in \mathcal{B}$ by a well-known theorem . The continuity of the multiplication implies the existence of some $b \in \mathcal{A}$ with $\|ab - e\|_{\mathcal{C}} < \epsilon$, so $a^{-1} = b(ab)^{-1} \in \mathcal{A}$. $\lhd$

4.3.4 Reduction. By Lemma 4.3.3 it is sufficient to find an $\varepsilon > 0$ such that $(I - A)^{-1} \in L^0(\Omega;\mathbf{R}^l)$ whenever $A \in L^0(\Omega;\mathbf{R}^l)$ and $\sup_{\tau\in\mathbf{R}^l}\|A(\tau)\|_{\mathcal{L}(L^2(\Omega))} < \varepsilon$.

4.3.5 Lemma. *The algebra $\mathbf{C} + L^{-\infty}(\Omega;\mathbf{R}^l) = \{zI + A : z \in \mathbf{C}, A \in L^{-\infty}(\Omega;\mathbf{R}^l)\}$ is a Ψ^*-subalgebra of $C_b(\mathbf{R}^l, \mathcal{L}(L^2(\Omega)))$.*

Proof. Applying Reduction 4.3.4, we have to show that, for $A \in L^{-\infty}(\Omega;\mathbf{R}^l)$, the operator $zI + A$ is invertible in $\mathbf{C} + L^{-\infty}(\Omega;\mathbf{R}^l)$ whenever $|z-1|$ and $\|A\|$ are small. It is no loss of generality to assume $z = 1$. Clearly $L^{-\infty}(\Omega;\mathbf{R}^l) = \mathcal{S}(\mathbf{R}^l, L^{-\infty}(\Omega))$. Now recall that $P \in L^{-\infty}(\Omega)$ if and only if

$$P : H^{-k}(\Omega) \to H^k(\Omega)$$

is continuous for every $k \in \mathbf{N}$. Assuming that $\|A(\tau)\|_{\mathcal{L}(L^2(\Omega))} < 1$, we know that $I - A(\tau)$ is invertible. The identity

$$(I - A(\tau))^{-1} = I + A(\tau) + A(\tau)(I - A(\tau))^{-1}A(\tau) \tag{1}$$

shows that $(I - A(\tau))^{-1} \in I + L^{-\infty}(\Omega)$ for each $\tau \in \mathbf{R}^l$. Moreover the fact that

$$\partial_{\tau_j}(I - A(\tau))^{-1} = (I - A(\tau))^{-1}\partial_{\tau_j}A(\tau)(I - A(\tau))^{-1}$$

in connection with (1) shows that all the countably many semi-norms for $\tau^\alpha\partial_\tau^\beta(I - A(\tau))^{-1}, \alpha, \beta \in \mathbf{N}^l$, in $\mathcal{L}(\mathcal{D}'(\Omega), \mathcal{D}(\Omega))$ are uniformly bounded in τ. Hence $(I - A(\tau))^{-1} \in \mathcal{S}(\mathbf{R}^l, \mathcal{L}(\mathcal{D}'(\Omega), \mathcal{D}(\Omega)) = \mathcal{S}(\mathbf{R}^l, L^{-\infty}(\Omega))$. $\lhd$

Lemma 4.3.5 essentially will allow us to localize the result and to work on Euclidean space, where spectral invariance is well-known.

4.3.6 Notation and Remarks. Let $\{\Omega_j\}_{j=1}^J$ be an open covering of Ω, $\{\varphi_j\}_{j=1}^J$ a subordinate partition of unity, and $\psi_j \in C_0^\infty(\Omega_j), j = 1, \dots, J$, be functions with $\varphi_j\psi_j = \varphi_j$. By $\kappa_j : \Omega_j \to U_j \subseteq \mathbf{R}^n$ denote the coordinate maps. We then have $A \in L^\mu(\Omega;\mathbf{R}^l)$ if and only if

(i) $\kappa_{j*}(\varphi_j A(\cdot)\psi_j) \in \operatorname{op} S^\mu_{1,0}(\mathbf{R}^n, \mathbf{R}^{n+l})$, and

(ii) $\varphi_j A(1 - \psi_j) \in L^{-\infty}(\Omega;\mathbf{R}^l)$.

Here, κ_{j*} is the push-forward of operators defined by

$$(\kappa_{j*}A)f(x) = A(f \circ \kappa_j)(\kappa_j^{-1}(x)), \qquad f \in C_0^\infty(U_j), x \in U_j,$$

and the subscript '0, 1' indicates the symbol classes with uniform estimates with respect to x, cf. Kumano-go [23, Chapter 2, Definition 1.1]. The push-forward of $\varphi_j A(\cdot)\psi_j$ has a uniquely defined symbol $p_j = p_j(x, \xi, \tau) \in S^\mu_{1,0}(\mathbf{R}^n_x, \mathbf{R}^{n+l}_{\xi,\tau})$. Given $A \in L^0(\Omega;\mathbf{R}^l)$ we can regard $\kappa_{j*}(\varphi_j A\psi_j)$ either as an element of $C_b(\mathbf{R}^l, \mathcal{L}(L^2(\mathbf{R}^n)))$ or an element of $\mathcal{L}(L^2(\mathbf{R}^{n+l}))$ in view of the embedding

$$S^\mu_{1,0}(\mathbf{R}^n, \mathbf{R}^{n+l}) \hookrightarrow S^\mu_{1,0}(\mathbf{R}^{n+l}, \mathbf{R}^{n+l}).$$

The following lemma states that there is no difference between both points of view.

4.3.7 Lemma. *The embeddings $S^0_{1,0}(\mathbf{R}^n, \mathbf{R}^{n+l}) \hookrightarrow \mathcal{L}(L^2(\mathbf{R}^{n+l}))$ and $S^0_{1,0}(\mathbf{R}^n, \mathbf{R}^{n+l}) \hookrightarrow C_b(\mathbf{R}^l, \mathcal{L}(L^2(\mathbf{R}^n)))$ are equivalent: Writing the variables in $\mathbf{R}^{n+l}$ as (x,t), the covariables as (ξ, τ) we have for $a, b \in S^0_{1,0}(\mathbf{R}^n, \mathbf{R}^{n+l})$*

$$(a\#_{(x,t)}b)(x, \xi, \tau) = (a(\tau)\#_x b(\tau))(x, \xi), \tag{1}$$

where $\#_{(x,t)}$ is the Leibniz product of symbols with respect to $(x,t) \in \mathbf{R}^{n+l}$ while $\#_x$ is the Leibniz product with respect to x only. Moreover,

$$\|\operatorname{op}_{x,t} a\|_{\mathcal{L}(L^2(\mathbf{R}^{n+l}))} = \|\operatorname{op}_x a(\tau)\|_{C_b(\mathbf{R}^l, \mathcal{L}(L^2(\mathbf{R}^n)))}. \tag{2}$$

Proof. We have by Kumano-go [23, Chapter 2, Theorem 2.6]

$$\begin{aligned}
&(a\#_{(x,t)}b)(x,t,\xi,\tau) \\
= \; & Os - \int\int e^{-ix'\xi'-it'\tau'}a(x,\xi+\xi',\tau+\tau')b(x+x',\xi,\tau)\,dx'\,dt'\,đ\xi'\,đ\tau' \\
= \; & Os - \int\int e^{-ix'\xi'}a(x,\xi+\xi',\tau)b(x+x',\xi,\tau)\,dx'\,đ\xi' \\
= \; & (a\#_x b)(x,\xi,\tau)
\end{aligned}$$

with $đ\xi' = (2\pi)^{-n}d\xi'$, $đ\tau = (2\pi)^{-l}d\tau$. Notice that the integrals do not depend on t; in the second equality we have used that, for $f \in C_b^\infty(\mathbf{R}^l)$,

$$f(\tau) = Os - \int\int e^{-it'\tau'}f(\tau+\tau')dt'\,đ\tau'.$$

Now let us have a look at (2). We may consider the elements of $C_b(\mathbf{R}^l, \mathcal{L}(L^2(\mathbf{R}^n)))$ as 'multipliers' on $L^2(\mathbf{R}^l, L^2(\mathbf{R}^n))$. The operator $A(\tau) \in C_b(\mathbf{R}^l, \mathcal{L}(L^2(\mathbf{R}^n)))$ maps $f \in L^2(\mathbf{R}^l, L^2(\mathbf{R}^n))$ to the function g given by $g(\tau) = A(\tau)f(\tau)$. Clearly we can identify $L^2(\mathbf{R}^l, L^2(\mathbf{R}^n))$ and $L^2(\mathbf{R}^{n+l})$. For $f_1 \in L^2(\mathbf{R}^n), f_2 \in L^2(\mathbf{R}^l)$ let $f(x,t) = f_1(x)f_2(t) = f_1 \otimes f_2(x,t)$. Then, given $a \in S^0(\mathbf{R}^n, \mathbf{R}^{n+l})$ we have

$$\begin{aligned}
\|(\text{op}_{x,t}a)f\|^2_{L^2(\mathbf{R}^{n+l})} &= \int_{\mathbf{R}^l}\int_{\mathbf{R}^n} |\mathcal{F}^{-1}_{\tau\to t}[\text{op}_x a(\tau)f_1(x)(\mathcal{F}_{t'\to\tau}f_2)(\tau)](t)|^2\,dx\,dt \\
&= \int_{\mathbf{R}^l}\int_{\mathbf{R}^n} |\text{op}_x a(\tau)f_1(x)(\mathcal{F}f_2)(\tau)|^2\,dx\,d\tau \\
&= \int_{\mathbf{R}^l}\int_{\mathbf{R}^n} |\text{op}_x a(\tau)f_1(x)|^2\,dx\;|\mathcal{F}f_2(\tau)|^2\,d\tau \\
&= \int_{\mathbf{R}^l} \|\text{op}_x a(\tau)f_1\|^2_{L^2(\mathbf{R}^n)}|(\mathcal{F}f_2)(\tau)|^2\,d\tau \\
&= \|(\text{op}_x a(\cdot)f_1)\mathcal{F}f_2\|^2_{L^2(\mathbf{R}^l, L^2(\mathbf{R}^n))}. \qquad (3)
\end{aligned}$$

Here we have used Plancherel's identity. We conclude that

$$\|(\text{op}_{x,t}a)f\|_{L^2(\mathbf{R}^{n+l})} \le \sup_{\tau\in\mathbf{R}^l}\|\text{op}_x a(\tau)\|_{\mathcal{L}(L^2(\mathbf{R}^n))}\|f_1\|_{L^2(\mathbf{R}^n)}\|\mathcal{F}f_2\|_{L^2(\mathbf{R}^l)}.$$

Hence, in view of the identity $\|f\|_{L^2(\mathbf{R}^{n+l})} = \|f_1\|_{L^2(\mathbf{R}^n)}\|\mathcal{F}f_2\|_{L^2(\mathbf{R}^l)}$ and the fact that $L^2(\mathbf{R}^{n+l}) = L^2(\mathbf{R}^n) \otimes_H L^2(\mathbf{R}^l)$,

$$\|\text{op}_{x,t}a\|_{\mathcal{L}(L^2(\mathbf{R}^{n+l}))} \le \sup_{\tau\in\mathbf{R}^l}\|\text{op}_x a(\tau)\|_{\mathcal{L}(L^2(\mathbf{R}^n))}.$$

Conversely fix $f_1 \in L^2(\mathbf{R}^n)$. Given $\varepsilon > 0$, we can find a bounded interval J such that

$$\|\text{op}_x a(\tau')f_1\|_{L^2(\mathbf{R}^n)} \ge \sup_{\tau\in\mathbf{R}^l}\|\text{op}_x a(\tau)f_1\| - \varepsilon$$

for all $\tau' \in J$. Let $f_2 = |J|^{-1/2}\mathcal{F}^{-1}\chi_J$. As usual, $|J|$ is the length of the interval J and χ_J its characteristic function. Then we deduce from the above identity (3) that

$$\begin{aligned}
\sup_{\tau\in\mathbf{R}^l}\|\text{op}_x a(\tau)f_1\|^2_{L^2(\mathbf{R}^n)} - \varepsilon &\le |J|^{-1}\int_{\mathbf{R}^l}\|\text{op}_x a(\tau)f_1\|^2_{L^2(\mathbf{R}^n)}\chi_J(\tau)\,d\tau \\
&= \int_{\mathbf{R}^l}\|\text{op}_x a(\tau)f_1\|^2_{L^2(\mathbf{R}^n)}|\mathcal{F}f_2(\tau)|^2\,d\tau \\
&= \|\text{op}_{x,t}a(f_1\otimes f_2)\|^2_{L^2(\mathbf{R}^{n+l})} \\
&\le \|\text{op}_{x,t}a\|^2_{\mathcal{L}(L^2(\mathbf{R}^{n+l}))}\|f_1\otimes f_2\|^2_{L^2(\mathbf{R}^{n+l})}.
\end{aligned}$$

But $\|f_1 \otimes f_1\|^2_{L^2(\mathbf{R}^{n+l})} = \|f_1\|^2_{L^2(\mathbf{R}^n)}\|f_2\|^2_{L^2(\mathbf{R}^l)}$ and, by Plancherel, $\|f_2\|_{L^2(\mathbf{R}^l)} = |J|^{-1/2}\|\chi_J\|_{L^2(\mathbf{R}^l)} = 1$. This shows that

$$\sup_{\tau\in\mathbf{R}^l}\|\mathrm{op}_x a(\tau)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \|\mathrm{op}_{x,t}a\|_{\mathcal{L}(L^2(\mathbf{R}^{n+l}))}.$$

◁

We shall make use of the following theorem.

4.3.8 Theorem. *Let $a \in S^0_{1,0}(\mathbf{R}^{n+l},\mathbf{R}^{n+l})$ and suppose that*

$$\mathrm{op}\, a : L^2(\mathbf{R}^{n+l}) \to L^2(\mathbf{R}^{n+l})$$

is invertible. Then there is a symbol $b \in S^0_{1,0}(\mathbf{R}^{n+l},\mathbf{R}^{n+l})$ with $(\mathrm{op}\, a)^{-1} = \mathrm{op}\, b$. If $a \in S^0_{1,0}(\mathbf{R}^n,\mathbf{R}^{n+l})$, then $b \in S^0_{1,0}(\mathbf{R}^n,\mathbf{R}^{n+l})$.

Proof. The first part of Theorem 4.3.8 is a remarkable result due to R. Beals [2]; for a different proof see Schrohe [39] or Ueberberg [53]. In order to check that the symbol b of the inverse is independent of t we note that b satisfies the equation

$$Os - \int\int e^{-ix'\xi'-it'\tau'}b(x,t,\xi+\xi',\tau+\tau')a(x+x',\xi,\tau)\,dx'\,dt'\,đ\xi'\,d\tau' \equiv 1.$$

Since b is uniquely determined, and the above equation holds for all t, we conclude that $b(x,t,\xi,\tau) = b(x,0,\xi,\tau)$. ◁

4.3.9 Lemma. *Let $A \in L^0(\Omega;\mathbf{R}^l)$ and suppose $\|A\|_{C_b(\mathbf{R}^l,\mathcal{L}(L^2(\Omega)))}$ is small. Then, for each $\tau \in \mathbf{R}^l$,*

$$I - \varphi_j A(\tau)\psi_j : L^2(\Omega) \to L^2(\Omega)$$

is invertible. Defining the operator $C(\tau) = (I - \varphi_j A(\tau)\psi_j)^{-1}$ we obtain $C \in L^0(\Omega;\mathbf{R}^l)$.

Proof. We may assume that $\|A\|$ is so small that

$$\|\kappa_{j*}(\varphi_j A\psi_j)\|_{\mathcal{L}(L^2(\mathbf{R}^{n+l}))} = \|\kappa_{j*}(\varphi_j A\psi_j)\|_{C_b(\mathbf{R}^l,\mathcal{L}(L^2(\mathbf{R}^n)))} < 1.$$

Then $I - \kappa_{j*}(\varphi_j A\psi_j)$ is an invertible pseudodifferential operator on $\mathbf{R}^{n+l}$. We find a symbol $a = a(x,\xi,\tau) \in S^0(\mathbf{R}^n,\mathbf{R}^{n+l})$ such that

$$I - \kappa_{j*}(\varphi_j A\psi_j) = I - \varphi_{j*}[\mathrm{op}\, a]\psi_{j*}$$

with $\varphi_{j*} = \kappa_{j*}\varphi_j, \psi_{j*} = \kappa_{j*}\psi_j$. Applying Theorem 4.3.8, there is a symbol $b \in S^0_{1,0}(\mathbf{R}^n,\mathbf{R}^{n+l})$ such that $[I - \kappa_{j*}(\varphi_j A\psi_j)]^{-1} = I + \mathrm{op}\, b$. In view of the fact that

$$\begin{aligned} &(I - \varphi_{j*}[\mathrm{op}\, a]\psi_{j*})^{-1} \\ = \;& I + \varphi_{j*}[\mathrm{op}\, a]\psi_{j*} + \varphi_{j*}[\mathrm{op}\, a]\psi_{j*}(I - \varphi_{j*}[\mathrm{op}\, a]\psi_{j*})^{-1}\varphi_{j*}[\mathrm{op}\, a]\psi_{j*}\end{aligned}$$

b is supported in $U_j \times \mathbf{R}^n \times \mathbf{R}^l$, and we may define

$$D(\tau) = I + \kappa_j^*[\mathrm{op}\, b](\tau).$$

Being the pull-back of an operator in $\mathrm{op}\, S^0(\mathbf{R}^n,\mathbf{R}^{n+l})$, clearly $D \in L^0(\Omega;\mathbf{R}^l)$. On the other hand, $D(\tau)$ inverts $I - \varphi_j A(\tau)\psi_j$, hence $D = C$. ◁

4.3.10 Conclusion. Let $A \in L^0(\Omega; \mathbf{R}^l)$ and suppose $\|A\|_{C_b(\mathbf{R}^l, \mathcal{L}(L^2(\mathbf{R}^n)))}$ is small. Then $I - A(\tau) : L^2(\Omega) \to L^2(\Omega)$ is invertible for all $\tau \in \mathbf{R}$, and the family C defined by

$$C(\tau) = (I - A(\tau))^{-1}$$

is an element of $L^0(\Omega; \mathbf{R}^l)$.

Proof. Write

$$\begin{aligned} I - A(\tau) &= I - \varphi_1 A(\tau) - \varphi_2 A(\tau) - \ldots - \varphi_J A(\tau) \\ &= [I - \varphi_2 A(\tau)(I - \varphi_1 A(\tau))^{-1} \ldots - \varphi_J A(\tau)(I - \varphi_1 A(\tau))^{-1}](I - \varphi_1 A(\tau)). \end{aligned}$$

Now $I - \varphi_1 A(\tau) = (I - \varphi_1 A(\tau)\psi_1)(I - (I - \varphi_1 A \psi_1)^{-1} \varphi_1 A(I - \psi_1))$. Assuming that $\|A\|$ is small, the product is invertible in $L^0(\Omega; \mathbf{R}^l)$ by Lemmata 4.3.5 and 4.3.9. Iteration completes the proof. ◁

4.4 $\mathcal{B}^{0,d}(X; \mathbf{R}^l)$ Has an Open Group of Invertible Elements

4.4.1 Notation. By $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ we denote the space of all parameter-dependent boundary value problems in Boutel de Monvel's calculus as introduced in 1.3.4. The parameter space is here $\mathbf{R}^l$. We will primarily consider the elements of $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ as operator families $\{A(\tau) : \tau \in \mathbf{R}^l\}$. The pointwise composition yields a continuous multiplication

$$\mathcal{B}^{\mu,d}(X; \mathbf{R}^l) \times \mathcal{B}^{\mu',d'}(X; \mathbf{R}^l) \to \mathcal{B}^{\mu+\mu',d''}(X; \mathbf{R}^l)$$

where $d'' = \max\{d', \mu' + d\}$. In particular, $\mathcal{B}^{0,d}(X; \mathbf{R}^l)$. is a Fréchet algebra. The operators in $\mathcal{B}^{\mu,d}(X; \mathbf{R}^l)$ will in general act on spaces of sufficiently smooth sections in vector bundles over X and Y respectively. We will fix bundles V over X and W over Y. For each $\tau \in \mathbf{R}^l$, an operator $A \in \mathcal{B}^{0,d}(X; \mathbf{R}^l)$ will furnish a bounded map

$$A(\tau) : \begin{array}{c} H^s(X, V) \\ \oplus \\ H^s(Y, W) \end{array} \to \begin{array}{c} H^s(X, V) \\ \oplus \\ H^s(Y, W) \end{array}, \tag{1}$$

provided that $s > d - 1/2$. We can therefore, similarly as in Section 4.2, view $\mathcal{B}^{0,d}(X; \mathbf{R}^l)$ as a continuously embedded subalgebra of the C^*-algebra $C_b(\mathbf{R}^l, \mathcal{L}(H^s))$, with the Hilbert space $H^s = H^s(X, V) \oplus H^s(Y, W)$, $s > d - 1/2$.
It is a natural conjecture that $\mathcal{B}^{0,d}(X; \mathbf{R}^l)$ is a Ψ^* subalgebra of $C_b(\mathbf{R}^l, \mathcal{L}(H^s))$. For the definition of Ψ^*-algebras see 4.1.2. A Ψ^* subalgebra of a C^*-algebra always has an open group of invertible elements, cf. 4.1.3. This is what we will show for $\mathcal{B}^{0,d}(X; \mathbf{R}^l)$. The proof relies on techniques used in [37] and [39].

4.4.2 Theorem. *$\mathcal{B}^{0,d}(X; \mathbf{R}^l)$ has an open group of invertible elements. In particular, inversion is continuous on $\mathcal{B}^{0,d}(X; \mathbf{R}^l)$ by Theorem 4.1.4.*

We start the proof with two observations concerning the elements of order $-\infty$:

4.4.3 Lemma. *A family of operators $\{G(\tau) : \tau \in \mathbf{R}^l\}$ acting on vector bundles as in 4.4.1(1) is an element of $\mathcal{B}^{-\infty,0}(X; \mathbf{R}^l)$ if and only if for all multi-indices α, β and all $N \in \mathbf{N}$ the extension*

$$\tau^\alpha D_\tau^\beta G(\tau) : H_0^{-N}(X, V) \oplus H^{-N}(Y, W) \to H^N(X, V) \oplus H^N(Y, W)$$

exists and is uniformly bounded with respect to τ.

Proof. By definition, $\{G(\tau) : \tau \in \mathbf{R}^l\} \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$ if and only if it is an integral operator with a smooth kernel density, $\gamma(x,\tilde{x},\tau)$ such that $\tau \mapsto \gamma(\cdot,\cdot,\tau)$ is rapidly decreasing with respect to all C^∞ semi-norms. In the proof of 4.3.5, on the other hand, we have seen how the kernel semi-norms can be controlled in terms of the mapping properties. ◁

4.4.4 Theorem. *Let $G \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$, and suppose that for given $s \in \mathbf{R}, s > d - \frac{1}{2}$,*

$$I + G(\tau) : H^s \to H^s$$

is invertible for all τ. Then there is an $H \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$ such that $(I+G)^{-1} = I + H$. In particular, $\mathbf{C} + \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$ is a Ψ^*-subalgebra of $C_b(\mathbf{R}^l, \mathcal{L}(H^s))$ for every $s > d - 1/2$.

Proof. For simplicity consider the case where G consists only of the singular Green part, i.e. $W = 0$; moreover, we will assume that G is scalar, i.e. $V = \mathbf{C}$.
Write $G = \sum_{j=0}^{d} G_j \partial_r^j$, where $G_j \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$ and ∂_r denotes the normal derivative, defined in a neighborhood of the boundary. We now use the fact that

$$\begin{aligned}[][I+G]^{-1} &= I - G + G[I+G]^{-1}G \\ &= I - \sum_{j=0}^{d}(G_j - G[I+G]^{-1}G_j)\partial_r^j.\end{aligned}$$

In view of 4.4.3, all we have to check is that for all α, β, N

$$\lambda^\alpha D_\lambda^\beta (G_j(\lambda) - G(\lambda)[I + G(\lambda)]^{-1}G_j(\lambda)) : H_0^{-N}(X) \to H^N(X)$$

is uniformly bounded. This, however, is immediate from the corresponding properties of the G_j. ◁

4.4.5 Reduction. All we have to show for Theorem 4.4.2 is the following: There is a neighborhood U of zero in $\mathcal{B}^{0,d}(X;\mathbf{R}^l)$ such that, for all $A \in U$, we have $I + A : H^s \to H^s$ invertible, and $(I+A)^{-1} \in \mathcal{B}^{0,d}(X;\mathbf{R}^l)$. In order to see this we may assume that $W = 0$. In fact, let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Supposing that $I + A_{11}$ is invertible within $\mathcal{B}^{0,d}(X;\mathbf{R}^l)$ (as an operator acting on $H^s(X,V)$) we write

$$I + A = \begin{bmatrix} I & 0 \\ A_{21}(I+A_{11})^{-1} & I \end{bmatrix} \begin{bmatrix} I + A_{11} & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & (I+A_{11})^{-1}A_{12} \\ 0 & 0 \end{bmatrix}$$

with $B = (I + A_{22}) - A_{21}(I+A_{11})^{-1}A_{12}$. Since the outer matrices can be inverted within $\mathcal{B}^{0,d}(X;\mathbf{R}^l)$, we only have to consider the middle one. Now, the calculus shows that

$$A_{22} - A_{21}(I+A_{11})^{-1}A_{12} \in L^0(Y;\mathbf{R}^l)$$

with the algebra $L^0(Y;\mathbf{R}^l)$ of parameter-dependent pseudodifferential operators on $Y = \partial X$ as introduced in Section 4.3. Moreover, the corresponding semi-norms are small, so that B is invertible in $L^0(Y,\mathbf{R}^l)$ by Theorem 4.3.2.
This shows that we can focus on the invertibility of $I + A_{11}$. Finally we can apply Theorem 4.4.4 and the decomposition trick in Conclusion 4.3.10 to see that the result essentially is local in X, so that we may assume V to be trivial one-dimensional. In the following we shall therefore consider A an operator family

$$A(\tau) : H^s(X) \to H^s(X), \quad \tau \in \mathbf{R}^l.$$

4.4.6 Proposition. *There is a neighborhood U of zero in $\mathcal{B}^{0,d}(X;\mathbf{R}^l)$ such that $I+A$ is parameter-elliptic for all $A \in U$.*

Proof. $\mathcal{B}^{0,d}(X;\mathbf{R}^l)$ is topologized as a non-direct sum of Fréchet spaces; for details see 2.2.8. What we have to show is that $I+A$ is a parameter-elliptic wherever finitely many of the symbol semi-norms for A are small. We can write A in the form $A = P_X + G$; here P is a parameter-dependent pseudodifferential operator in $L^0(\Omega;\mathbf{R}^l)$, G is a parameter dependent singular Green operator, and Ω denotes the compact "double" of X. By definition, we can find representatives P and G such that
(i) suitably many of the symbol semi-norms for P in $L^0(\Omega;\mathbf{R}^l)$ are small, and
(ii) suitably many of the symbol semi-norms for G in $B^{0,d}(X;\mathbf{R}^l)$ are small.
Condition (i) will ensure the invertibility of $I+P$ by an operator $I+Q$ with $Q \in L^0(\Omega;\mathbf{R}^l)$. In addition, suitably many of the symbol semi-norms for Q will be small. We then have

$$(I+Q_X)(I+P_X) = I+G_1,$$

where G_1 has also small semi-norms in $\mathcal{B}^{0,0}(X;\mathbf{R}^l)$. We can therefore invert the operator-valued singular Green symbols. By the standard process, we find a parametrix to $I+G_1$ thus a parametrix to $I+P_X$, in $\mathcal{B}^{0,0}(X;\mathbf{R}^l)$. Let as rewrite it in the form $I+R, R \in \mathcal{B}^{0,0}(\mathbf{R}^l)$. The parametrix construction process is continuous with respect to the symbol topology, hence R can be assumed to have suitably many semi-norms small. Multiplying $I+A$ by $I+R$ we have

$$\begin{aligned}(I+R)(I+A) &= (I+R)(I+P_X)+(I+R)G \\ &= I+S+G+RG\end{aligned}$$

with $S \in \mathcal{B}^{-\infty,0}(X;\mathbf{R}^l)$. Both G and RG have suitably many symbol semi-norms small, so we can construct a parametrix to $I+G+RG$. This completes the proof. $\triangleleft$

4.4.7 Corollary. Under the assumptions of Proposition 4.4.6 we can first find a parametrix B to $I+A$ such that $B(I+A) \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$. B then necessarily is a Fredholm family of index zero. Next we use a construction due to Gramsch and Kaballo [15], which can also be found in the proof of Theorem I.4.3.18: We can modify B by a finite rank operator so that it becomes an invertible element of $\mathcal{B}^{0,d}(X;\mathbf{R}^l)$ and, simultaneously, $B(I+A) \in I+S, S \in \mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$. But then $I+S$ is necessarily invertible in $\mathbf{C}+\mathcal{B}^{-\infty,d}(X;\mathbf{R}^l)$. The inverse by Theorem 4.4.2 also belongs to this algebra. We therefore obtain the assertion: $(I+S)^{-1}B \in \mathcal{B}^{0,d}(X;\mathbf{R}^l)$ is an inverse to $I+A$. $\triangleleft$

4.5 List of Misprints for Part I

We would like to make a few changes with respect to the material in Part I of this paper. For one thing, we would like to modify the definition of the spaces $\mathcal{K}^{s,\gamma}$ in I.3.1.18. Instead of letting them be isomorphic to $t^{-n/2}H^s$ near infinity we would like them to be isomorphic to the cone-like Sobolev spaces H^s_{cone}, introduced in 4.2.1. The new definition can be found in 1.4.14. Let us point out that all the results of Part I are valid with both definitions. This is essentially a consequence of the fact that, for every $\omega \in C_0^\infty(\overline{\mathbf{R}}_+)$, we have

$$[\omega]\mathcal{K}^{s,\gamma}_{old} \cong [\omega]\mathcal{K}^{s,\gamma}_{new}.$$

Notice that the proof of Theorem I.4.3.17 does not carry over literally with the new definition. Due to the above isomorphism, however, the result trivially holds for the modified spaces.
The definition of the spaces $M_P^{\mu,d}(X)$ of meromorphic Mellin symbols with values in Boutet de Monvel's algebra in I.4.1.1 should be replaced by the one given here in Definition 1.7.2. Again, all results of Part I are valid in the new formulation.
Proposition I.4.3.7 could be generalized; at the same time the proof became somewhat more transparent. It is included here as Proposition 3.1.27.
Finally, we use the opportunity to make the following minor corrections.

p.	l.	is	should be
109	12	(c)	(d)
111	6	$\subseteq$	$\subset\subset$
131	2	2.2.10	2.2.11
135	17	that $A(\lambda)$ is invertible	that $I + A(\lambda)$ is invertible
142	2	op $\langle \xi, \tau \rangle$	op $\langle \xi, \tau \rangle^{\mu}$
142	14		*add:* Here, $f(x,t) = t^{-\frac{n+1}{2}+\gamma} u(x, \ln t)$.
142	17	$\mathcal{F}_{s\to t} H^s(\mathbf{R}^n_+ \times \mathbf{R})$	$\mathcal{F}_{t\to\tau} H^s(\mathbf{R}^n_+ \times \mathbf{R}_t)$
151	14		*add:* (f) Let $H_P^{s,\gamma}(\mathbb{D})$ denote the space of all distributions in $H^s_{loc}(\mathrm{int}\,\mathbb{D})$ which, close to each singularity v, are elements of $\mathcal{K}_P^{s,\gamma}(X_v^\wedge)$.
156	1	C^∞	C_0^∞
157	19	$\geq k$	$> k - 1/2$
157	-1	3.1.18(a)	3.1.19
158	6	$\ldots V_1 \ldots V_3 \ldots V_2 \ldots V_4$	$\ldots V_2 \ldots V_4 \ldots V_1 \ldots V_3$
158	10	$\ldots V_1 \ldots V_3 \ldots V_2 \ldots V_4$	$\ldots V_2 \ldots V_4 \ldots V_1 \ldots V_3$
159	3	$s, t \geq 0$	$s, t > -1/2$
160	-8	P, Q, R	P, Q, R, S
160	-7	$\ldots)_{Q,R}$	$\ldots)_{R,S}$
160	-6	$\ldots)_{P,R'}$	$\ldots)_{R,R'}$
160	-5	G_2 and R	G_1 and G_2
160	-4	$R' = R$	$R' = Q$
161	12	(γ, γ, Θ)	(γ, δ, Θ)
161	13	$C^d(\ldots)$	$C^d_G(\ldots)$
161	19	$\geq k$	$> k - 1/2$
162	14	$I + H_0$	$I + H_0 + P$
165	4	topology of $\mathcal{B}^{-\infty,d}$	Euclidean topology on L_j

p.	l.	is	should be
165	-8	$\ldots\mu.$	$\ldots,\mu,$ be differential operators.
166	15	$\mathcal{B}^{\mu,0}(X)$	$\mathcal{B}^{-\infty,0}(X)$
172	-3	$s \geq 0$	$s > -1/2, \mu \leq 0.$
173	2	$\mathcal{H}_{\{0\}}^{-s+\mu,-\gamma+\frac{n}{2}}(X^\wedge, V_2)$	$\mathcal{H}^{-s+\mu,-\gamma+\frac{n}{2}}(X^\wedge, V_2).$
173	4,5		*delete:* Here, ..., cf. 3.3.1.
173	15	$s \geq d$	$s > d - 1/2$
177	-8	$C_G^0(\ldots)$	$C_G^d(\ldots)$
178	5,6		*delete:* Note ... type d.
179	16	weight data G_1	weight data for G_1
179	-1	$\Gamma_{\frac{1}{2}-\gamma}$	$\Gamma_{\frac{n+1}{2}-\gamma}$
181	8	$\pi_{\mathbf{C}} P_2$	$\pi_{\mathbf{C}} P_2'$
188	4	$< \delta.$	$< \delta,$ unless F is nowhere invertible.
188	10	$\mathcal{L}(E) \to \mathcal{L}(E)$	$E \to E$
188	12	A_{N-k}	A_k
188	18	$s \geq d.$	$s > d - 1/2.$
189	16	4.3.14 we may assume that $\gamma = \frac{n}{2}.$	4.3.16 we may assume that $\gamma = 0.$
189	17	$= L^2(X, V_1) \oplus L^2(Y, V_3)$	$= \mathcal{H}^{0,0}(X^\wedge, V_1) \oplus \mathcal{K}^{0,-\frac{1}{2}}(Y^\wedge, V_3)$
189	-9	$L^2(X^\wedge)$ (3 times)	$\mathcal{H}^{0,0}(X^\wedge)$ (3 times)
190	all	$\mathcal{K}^{0,\frac{n}{2}}(X^\wedge)$	$\mathcal{K}^{0,0}(X^\wedge)$
190	all	$\frac{1}{2} + i\ldots$	$\frac{n+1}{2} + i\ldots$
190	all	$\Gamma_{\frac{1}{2}}$	$\Gamma_{\frac{n+1}{2}}$
190	5	by $\frac{1}{2}$	by $\frac{n+1}{2}$
190	10	$\mathcal{K}^{0,\frac{n}{2}}(X^\wedge) = L^2(X^\wedge)$	$\mathcal{K}^{0,0}(X^\wedge) = \mathcal{H}^{0,0}(X^\wedge)$
191	all	$\frac{1}{2} + i\ldots$	$\frac{n+1}{2} + i\ldots$
193	-7	$\rho \in \mathbf{C}$	$\rho > 0$
196	3	$\mu \in \mathbf{Z}$	$0 \geq \mu \in \mathbf{Z}, s > -1/2$
196	9	$\mathcal{H}_{\{0\}}^{-s+\mu,-\gamma+\frac{n}{2}}(X^\wedge, V_2)$	$\mathcal{H}^{-s+\mu,-\gamma+\frac{n}{2}}(X^\wedge, V_2).$
196	10,11		*delete:* Here, ..., cf. 3.3.1.
204	12	C_0^∞	C^∞
205	2	$\langle\xi\rangle^{k-1}$	$\langle\xi\rangle^{k-2}$
205	-1	$\mathrm{O}(\langle\xi'\rangle^{k-2})$	$\mathrm{O}(\langle\xi\rangle^{k-2})$
206	14	$+\chi^{(N+1)}(\theta)$	$+b_N\chi^{(N+1)}(\theta)$

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