AN APPLICATION OF THE CANONICAL BUNDLE FORMULA

YURI G. PROKHOROV

Abstract. We prove a part of Shokurov's conjecture on characterization of toric varieties modulo the minimal model program and adjunction conjecture.

1. Introduction

Let X be a proper toric variety and let $D = \sum_{i=1}^{n} D_i$ be the invariant divisor. It is well-known that

$$n = \dim X + \operatorname{rk}(\text{Weil divisors modulo numerical equivalence}),$$

$K_X + D \sim 0$, and the pair (X, D) has only log canonical singularities.

V.V. Shokurov proposed that these properties can characterize toric varieties:

Conjecture 1.1 ([Sh]). *Let $(X/Z \ni o, D = \sum d_i D_i)$ be a log variety such that (X, D) has only log canonical singularities and $-(K_X + D)$ is nef over Z. Then*

$$\sum d_i \leq \sigma(X/Z) + \dim X, \tag{1.2}$$

where $\sigma(X/Z)$ is the rank of the group Weil divisors on X modulo the numerical equivalence over Z. Moreover, if the equality holds, then $(X/Z \ni o, \lfloor D \rfloor)$ is a toric log pair, i.e, $(X/Z \ni o, \lfloor D \rfloor)$ is formally (or analytically) isomorphic to a toric log pair.

Example 1.3. If $X/X \ni 0$ is a singularity germ, then $\sigma(X/X)$ is exactly the rank of the group of Weil divisors modulo $\mathbb{Q}$-Cartier divisors (see [K1]). For example, let $X/X \ni 0$ be the singularity given by the equation $xy + zt = 0$ in $\mathbb{C}^4$ and let D be the divisor cut out by $xy = 0$. Then (X, D) is log canonical, D has exactly four components and $\sigma(X/X) = 1$. Thus we have equality in (1.2). Clearly, the singularity $X \ni 0$ is toric.

Conjecture 1.2 was proved by Shokurov in dimension 2 (see [Sh] and also [P1]). A special case of this conjecture in dimension 3 was verified in [P2]. In this paper we prove a weak form of Shokurov's conjecture modulo the log minimal model program (LMMP) and the adjunction conjecture for fiber spaces. Before stating our main result we introduce notation and definitions.

This work was is carried out under the support of grants RFFI 02-01-00441 and INTAS-OPEN 2000-269.

Notation. We work over an algebraically closed field of characteristic zero. Notation and conventions of the minimal model theory [KMM], [Ut] will be used freely.

Let (X, D) be a proper log pair. Write $D = \sum d_i D_i$ where the D_i are irreducible components. We denote

$\|D\| = \sum d_i$,
$\langle D \rangle$ is the free abelian group generated by the D_i,
$\langle D \rangle^0$ is the subgroup of $\langle D \rangle$ consisting of $\mathbb{Q}$-Cartier numerically trivial divisors,
$\sigma(X, D) = \operatorname{rk}\left(\langle D \rangle / \langle D \rangle^0\right)$.

Let $f \colon X \to Z$ be a fiber type contraction. We say that a prime divisor P is *horizontal* if $f(P) = Z$ and *vertical* if $\dim f(P) < \dim Z$. For any divisor D we have the decomposition $D = D^{\mathrm{hor}} + D^{\mathrm{vert}}$ into the sum of *horizontal* and *vertical parts*.

Definition 1.4. Let (X, D) be a log pair. *A numerical complement of* $K_X + D$ is a log divisor $K_X + D'$ with $D' \geq D$ such that $K_X + D'$ has only log canonical singularities and numerically trivial. We say that a log divisor $K_X + D$ is *numerically complementary* if it has at least one numerical complement.

Theorem 1.5. *Let (X, D) be a proper log variety such that $K_X + D$ is numerically complementary. Assume that in dimension* $\dim X$ *both the LMMP and Weak Adjunction Conjecture* 2.1 *hold. Then*

$$\|D\| \leq \sigma(X, D) + \dim X. \tag{1.6}$$

Moreover, if the equality holds, then X is rational. We can omit the LMMP and Conjecture 2.1 *as assumptions when* $\dim X \leq 3$.

Corollary 1.7. *Let X be a non-singular proper variety of dimension ≤ 3 and let D be a reduced simple normal crossing divisor on X such that $K_X + D \equiv 0$. Then $\|D\| \leq \rho(X) + \dim X$, where $\rho(X)$ is the Picard number. Moreover, if the equality holds, then X is rational.*

Remark. According to recent preprint [A2] Weak Adjunction Conjecture 2.1 hold in the case when $K_X + D$ has a Kawamata log terminal numerical complement.

After finishing the main part of this work the author was informed that similar and even more general results were obtained by J. McKernan [M]. His proofs are completely different, much easier, and mainly do not use the LMMP.

Acknowledgments. The work has been completed during my stay at Max-Planck-Institut für Mathematik in August 2002. I would like to thank MPIM for hospitality and support. I also would like to thank Professor V.V. Shokurov for some remarks and Dr. O. Fujino, who informed me about [M].

2. Preliminary facts

Adjunction (see [K2], [A1], [F]). Let $f\colon X \to Z$ be a contraction and let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on X such that $d_i \le 1$ whenever $f(D_i) = Z$.

For a prime divisor $W \subset Z$, define a number c_W as the log canonical threshold:

$$c_W = \sup \{c \mid (X, D + cf^*W) \text{ is log canonical over the generic point of } W\}.$$

Then the $\mathbb{Q}$-divisor

$$D_Z := \sum_W (1 - c_W)W$$

is called the *discriminant* of $K_X + D$.

Note that the definition of the discriminant D_Z is a codimension one construction, so computing D_Z we can systematically remove codimension two subvarieties in Z and pass to generic hyperplane sections $f_H\colon X \cap f^{-1}(H) \to Z \cap H$. In particular, f^*W is well-defined.

Adjunction Conjecture 2.1 (weak form)**.** *Let $f\colon X \to Z$ be a fiber space and let D be a $\mathbb{Q}$-divisor on X such that*

(i) *(X, D) is log canonical,*
(ii) *$K_X + D$ is $\mathbb{Q}$-linearly trivial over Z.*

Then there is an effective $\mathbb{Q}$-divisor M_Z on Z such that $K_Z + D_Z + M_Z$ is log canonical and

$$K_X + D \sim_{\mathbb{Q}} f^*(K_Z + D_Z + M_Z). \tag{2.2}$$

According to Kawamata [K2] this conjecture is true for contractions of relative dimension one.

Properties of σ.

Lemma 2.3. *Let $g\colon X' \to X$ be a birational contraction, let D' be the proper transform of D, and let $E = \sum E_i$ be the exceptional divisor. Then*

$$\sigma(X', D' + E) \le \sigma(X, D) + \|E\|. \tag{2.4}$$

Moreover, if g is a contraction of an extremal face in some LMMP, then in (2.4) the equality holds.

Proof. The surjective map $g_*\colon \langle D' + E\rangle \to \langle D\rangle$ induces the exact sequence

$$0 \longrightarrow \langle E\rangle \longrightarrow \langle D' + E\rangle / g^*\langle D\rangle^0 \longrightarrow \langle D\rangle/\langle D\rangle^0 \longrightarrow 0$$

Since $g^*\langle D\rangle^0 \subset \langle D' + E\rangle^0$, this gives us the desired inequality. □

Corollary 2.5. *Let $g\colon X' \to X$ be a contraction of an extremal ray and let D be a divisor on X'. Then*

$$\sigma(X', D) - 1 \le \sigma(X, g_*D) = \sigma(X', D + E) - 1 \le \sigma(X', D).$$

Furthermore, if E is a component of D, then $\sigma(X, g_*D) = \sigma(X', D) - 1$.

Easy results in the local case.

Proposition 2.6 ([Ut, 18.22-23]). *Let $(X \ni o, B)$ be a log canonical singularity such that every component of B is $\mathbb{Q}$-Cartier. Then*

$$\|B\| \le \dim X.$$

Moreover, if the equality holds, then $X \ni o$ is a quotient of a smooth point X' by an abelian group $\mathfrak{A}$ acting on X' free in codimension one. The proper transform B_i' of each component $B_i \subset \operatorname{Supp}(B)$ is $\mathfrak{A}$-stable. Therefore, $(X, \lfloor B \rfloor)$ is a $\mathbb{Q}$-factorial toric log pair.

Corollary 2.7. *Let (X, B) be a log canonical log pair such that each component B_i of B is $\mathbb{Q}$-Cartier. Then $\|B\| \le \operatorname{codim} \cap B_i$.*

Corollary 2.8 (cf. [Ut, Corollary 18.24]). *Let V be a projective $\mathbb{Q}$-factorial log terminal variety with $\rho(V) = 1$ and let D be a boundary on V such that $-(K_V + D)$ is nef and (V, D) is log canonical. Then $\|D\| \le \dim V + 1$. Moreover, if the equality holds, then $(V, \lfloor D \rfloor)$ is a toric log variety. If V and all the D_i are defined over a non-closed field $\Bbbk$, then V is $\Bbbk$-rational.*

Proof. Assume that

$$\|D\| \ge \dim V + 1 \tag{2.9}$$

Take an embedding $V \subset \mathbb{P}^n$ so that V is projectively normal and take a projective cone $X \subset \mathbb{P}^{n+1}$ over V. Let $B = \sum b_i B_i$ be the corresponding cone over D. Then X is normal and $\mathbb{Q}$-factorial. Let $\sigma \colon \tilde{X} \to X$ be the blow up of the vertex, let $\tilde{B}$ be the proper transform of B, and let S be the exceptional divisor. We can write

$$\sigma^*(K_X + B) = K_{\tilde{X}} + \tilde{B} - a(S, B)S.$$

It is clear that S is a Cartier divisor and $(S, \tilde{B}|_S) \simeq (V, D)$. Since $K_V + D \equiv 0$, we have $a(S, B) = -1$. Thus $\sigma^*(K_X + B) = K_{\tilde{X}} + \tilde{B} + S$ and by the Inversion of Adjunction [Ut, 17.7] the pair $(\tilde{X}, \tilde{B} + S)$ is log canonical. Hence so is (X, B). Moreover, $(\tilde{X}, S)$ is plt (because V is log terminal). Now take (X', B') and $\mathfrak{A}$ such as in Proposition 2.6. Consider the diagram

$$\begin{array}{ccc} \tilde{X} & \xleftarrow{\ \tilde{\pi}\ } & \tilde{X}' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ X & \xleftarrow{\ \pi\ } & X' \end{array}$$

where $\tilde{X}'$ is the normalization of $\tilde{X}$ in the function field of X'. Let $S' = \tilde{\pi}^{-1}(S)$. Then $\tilde{\pi}$ is finite and $\tilde{\pi}|_{\tilde{X}' \setminus S'}$ is étale in codimension 1. By [Ut, 20.3] the pair $(\tilde{X}', S')$ is plt and $(\tilde{X}', \tilde{B}' + S')$ is log canonical, where $\tilde{B}' = \tilde{\pi}^{-1}(\tilde{B})$. In particular, S' is irreducible and normal. Clearly each component of $\tilde{B}'$ is Cartier. Put $\Delta = \tilde{B}'|_{S'}$ and $\Delta_i = \tilde{B}_i'|_{S'}$. Then $K_{S'} + \Delta$ is log canonical and numerically trivial, $\Delta = \sum b_i \Delta_i$, $\sum b_i \ge \dim S' + 1$, and the Δ_i are ample numerically proportional Cartier divisors. Hence S' is a Fano variety of (Fano) index $\geq \dim S' + 1$ with only log terminal singularities. It is well-known (see e.g. [IP, Th. 3.1.4]) that in this situation $S' \simeq \mathbb{P}^N$, $\sum b_i = \dim S' + 1$, and all the Δ_i are hyperplanes. This gives us the equality in (2.9). Further, $V \simeq S = \mathbb{P}^N/\mathfrak{A}$, where $\mathfrak{A}$ acts on $S' = \mathbb{P}^N$ so that all the Δ_i are stable. By Corollary 2.7 we have $\cap \Delta_i = \varnothing$ and $\lfloor \Delta \rfloor$ is a normal crossing divisor. This gives us that $(V, \lfloor D \rfloor)$ is a toric pair. For the last statement, we note that our construction is defined over $\Bbbk$. Since $\cap \Delta_i = \varnothing$, we can take $\Bbbk$-coordinates on $\mathbb{P}^N$ so that the action of $\mathfrak{A}$ is monomial. The statement is obvious in this case. □

Lemma 2.10. *Let $f\colon X \to Z \ni o$ be the contraction of an extremal ray R and let $D = \sum d_i D_i$ be a boundary on X such that all the components D_i are $\mathbb{Q}$-Cartier and $-(K_X + D)$ is f-nef. Assume that X is $\mathbb{Q}$-factorial and log terminal, and (X, D) is log canonical. Put $D^{\mathrm{hor}} = \sum_{D_i \cdot R > 0} d_i D_i$ and $D^{\mathrm{vert}} = \sum_{D_i \cdot R \leq 0} d_i D_i$. Then $\|D^{\mathrm{vert}}\| \leq \operatorname{codim} f^{-1}(o)$. Moreover,*

- (i) *if f is divisorial, then $\|D^{\mathrm{hor}}\| \leq \operatorname{codim} f(\operatorname{Exc}(f))$;*
- (ii) *if f is flipping and the flip $\chi\colon X \dashrightarrow X^+$ exists, then $\|D^{\mathrm{hor}}\| \leq \operatorname{codim} f^{+-1}(o)$ and $\|D\| \leq \dim X + 1$;*
- (iii) *if f is of fiber type and if the LMMP holds in dimensions $< \dim X$, then $\|D^{\mathrm{hor}}\| \leq \dim X/Z + 1$. If furthermore the equality holds, then a generic fiber F is $\mathbb{Q}$-factorial, $\rho(F) = 1$, the pair $(F, \lfloor D|_F \rfloor)$ is toric, and $X_{\mathcal{K}}$ is $\mathcal{K}$-rational, where $\mathcal{K} = \mathcal{K}(Z)$ is the function field of Z.*

Proof. (i) follows by Corollary 2.7.

(ii) Since $\chi(D^{\mathrm{hor}}) \leq \chi(D)^{\mathrm{vert}}$, we have $\|D^{\mathrm{hor}}\| \leq \operatorname{codim} f^{+-1}(o)$. Thus $\|D\| \leq \operatorname{codim} f^{-1}(o) + \operatorname{codim} f^{+-1}(o)$. On the other hand, by [KMM, Lemma 5-1-17] we have $\operatorname{codim} \operatorname{Exc}(f) + \operatorname{codim} \operatorname{Exc}(f^+) \leq \dim X + 1$. This gives us $\|D\| \leq \dim X + 1$.

(iii) Let F be a generic fiber of f. Assume that $\|D^{\mathrm{hor}}\| \geq \dim X/Z + 1 = \dim F + 1$. Denote $\Delta = D|_F$ and $\Delta_i = D_i|_F$. Then all the Δ_i are ample and numerically proportional. We claim that F is $\mathbb{Q}$-factorial and $\rho(F) = 1$. Indeed, let $g\colon \tilde{F} \to F$ be a small $\mathbb{Q}$-factorialization and let $\tilde{\Delta} = g^*\Delta$. Then $-(K_{\tilde{F}} + \tilde{\Delta})$ is nef, $(\tilde{F}, \tilde{\Delta})$ is log canonical and $\tilde{F}$ has only log terminal singularities. Since $\tilde{\Delta} \neq 0$, there is a $K_{\tilde{F}}$-negative extremal ray R. Then $\tilde{\Delta}_i \cdot R > 0$ for all $\tilde{\Delta}_i = g^*\Delta_i$ (here we do not assume that $\tilde{\Delta}_i$ is irreducible). If $\rho(\tilde{F}) > 1$, this contradicts (i), (ii), or the inductive hypothesis. By Corollary 2.8 we have

$$\|D^{\mathrm{hor}}\| = \|D|_F\| = \dim F + 1$$

and the pair $(F, \lfloor D|_F \rfloor)$ is toric. Since $\|D^{\mathrm{hor}}\| = \|D|_F\|$, all the components of D are defined over $\mathcal{K}$. So the last assertion follows by Corollary 2.8. □

3. Proof of Theorem 1.5

Let (X, D) be a log pair such that

$$\|D\| \geq \sigma(X, D) + \dim X. \tag{3.1}$$

Assume that $K_X + D$ has a numerical complement $K_X + G$. Replace (X, G) with its minimal $\mathbb{Q}$-factorial log terminal modification (blow-up divisors with discrepancy $a(\cdot, G) = -1$) and D with the sum of its proper transform and the reduced exceptional divisor. Thus X is $\mathbb{Q}$-factorial and log terminal. Run K_X-MMP. By Lemma 2.3 and Corollary 2.5 this preserves (3.1). All the divisorial contractions are positive with respect to G. Therefore, we cannot contract a connected component of G and $K \equiv -G$ cannot be nef. At the end we get a fiber type extremal G-positive contraction $f\colon X \to Z$. Note that our new X is $\mathbb{Q}$-factorial and log terminal. It is sufficient to prove our theorem for this new X.

If Z is a point, then $\rho(X) = 1$ and X is a log terminal Fano variety . By Corollary 2.8, $G = D$ and we have the equality in (3.1). Moreover, $(X, \lfloor D \rfloor)$ is a toric pair.

Consider the case when $\dim Z > 0$. On this step we use the canonical bundle formula (2.2).

Proposition 3.2. *Assumptions as in Theorem* 1.5. *Assume additionally that X is $\mathbb{Q}$-factorial, log terminal, and there exists a fiber type extremal contraction $f\colon X \to Z$ with* $\dim Z > 0$. *Then the inequality* (1.6) *holds. Furthermore, if the equality holds, then $K_X + D$ is numerically trivial over Z, a generic fiber F is $\mathbb{Q}$-factorial, $\rho(F) = 1$, the pair $(F, \lfloor D|_F \rfloor)$ is toric, and both X and Z are rational.*

Proof. Assume (3.1). By Lemma 2.10 we have $\|D^{\mathrm{hor}}\| \leq \dim X/Z + 1$. Since $f\colon X \to Z$ is an extremal contraction, $D^{\mathrm{vert}} = f^*\Delta$ for some effective $\mathbb{Q}$-Cartier divisor Δ. Write $D_Z = D'_Z + D''_Z$, where D'_Z and D''_Z are effective divisors without common components, $\operatorname{Supp} D'_Z \subset \operatorname{Supp} \Delta$ and none of the components of D''_Z is contained in $\operatorname{Supp} \Delta$.

Claim 3.3. $\|D^{\mathrm{vert}}\| \leq \|D'_Z\|$.

Proof. Let $W \subset Z$ be a prime divisor and let $S = f^{-1}(W)_{\mathrm{red}}$. Since $\rho(X/Z) = 1$, S is irreducible. Let d be the coefficient of S in D^{vert}. Write $f^*W = kS$, where $k \in \mathbb{N}$. Then $d + c_W k \leq 1$ (because $(X, D + c_W f^*W)$ is log canonical over the generic point of W). Hence, $d \leq 1 - c_W$. This proves the statement. □

Assume that $D^{\mathrm{hor}} = 0$. Then $D^{\mathrm{vert}} = D$ and

$$\|D^{\mathrm{vert}}\| \geq \sigma(X, D) + \dim X = \sigma(Z, \Delta) + \dim X.$$

On the other hand, by (2.2) we have that $K_Z + G_Z + M_Z$ is log canonical and numerically trivial. It is clear that $D'_Z \leq D_Z \leq G_Z$, i.e., $K_Z + G_Z + M_Z$

is a numerical complement of $K_Z + D'_Z$. Thus by the inductive hypothesis and Claim 3.3,

$$\|D^{\text{vert}}\| \le \|D'_Z\| \le \sigma(Z, D'_Z) + \dim Z,$$

a contradiction.

Now assume that $D^{\text{hor}} \neq 0$. Then $\sigma(X, D) \ge \sigma(X, D^{\text{vert}}) + 1$ and $\|D^{\text{hor}}\| \le \dim X/Z + 1$ (see Lemma 2.10). Hence

$$\|D^{\text{vert}}\| \ge \sigma(X, D) + \dim Z - 1 \ge \sigma(X, D^{\text{vert}}) + \dim Z \ge \sigma(Z, \Delta) + \dim Z. \tag{3.4}$$

As above we have

$$\|D^{\text{vert}}\| \le \|D'_Z\| \le \sigma(Z, D'_Z) + \dim Z. \tag{3.5}$$

This gives us the equalities in (3.4), (3.5) and (3.1). In particular, $\|D'_Z\| = \sigma(Z, D'_Z) + \dim Z$. Since D'_Z is numerically complementary, the inductive hypothesis give us that Z is rational.

Finally, by Lemma 2.10 the pair $(F, \lfloor D|_F \rfloor)$ is toric and X is rational. $\square$

Since by [K2] Conjecture 2.1 holds when $\dim X/Z = 1$, to prove the last part of the theorem we have to consider only the case when $\dim X = 3$ and Z is a curve. In this case, Theorem 1.5 is an immediate consequence of the following.

Proposition 3.6. *Let $f \colon X \to Z$ be an extremal contraction to a curve, where X is $\mathbb{Q}$-factorial and log terminal, and let D be a boundary on X such that $K_X + D$ is numerically complementary. Then $\|D\| \le \dim X + \sigma(X, D)$. Furthermore, if the equality holds, then $\sigma(X, D) = 2$, $K_X + D \equiv 0$, and X is rational.*

Proof. Let $K_X + G$ be a numerical complement. Assume that

$$\|D\| \ge \sigma(X, D) + \dim X. \tag{3.7}$$

Since Z is a curve, $\rho(X) = 2$ and $\sigma(X, D) \le 2$. By Lemma 2.10,

$$\|D^{\text{hor}}\| \le \|G^{\text{hor}}\| \le \dim X, \qquad \|D^{\text{vert}}\| \ge \sigma(X, D). \tag{3.8}$$

As $\rho(X) = 2$, the Mori cone $\overline{NE}(X)$ is generated by two extremal rays, say R and Q. Let R is the ray corresponding to the curves in fibers of f.

Claim 3.9. *There is a boundary $\Delta \le G$ such that (X, Δ) is a Kawamata log terminal log Fano variety.*

Proof. Obviously, $G^{\text{vert}} \cdot Q > 0$. Thus $(K + G - \varepsilon G^{\text{vert}}) \cdot Q < 0$ for $\varepsilon > 0$. Since $(K + G - \varepsilon G^{\text{vert}}) \cdot R = 0$, for $0 < \varepsilon_1 \ll \varepsilon$, we have $(K + G - \varepsilon G^{\text{vert}} - \varepsilon_1 G^{\text{hor}}) \cdot R < 0$ and $(K + G - \varepsilon G^{\text{vert}} - \varepsilon_1 G^{\text{hor}}) \cdot Q < 0$. Put $\Delta = G - \varepsilon G^{\text{vert}} - \varepsilon_1 G^{\text{hor}}$. Then $K + \Delta$ is anti-ample and Kawamata log terminal. $\square$

By the Cone Theorem the ray Q is contractible (i.e., there is a D^{vert}-positive extremal contraction $g\colon X \to W$). This implies that $Z \simeq \mathbb{P}^1$. It is clear that g cannot have fibers of dimension ≥ 2. Since all the components of G^{vert} are strictly positive with respect to Q, by Lemma 2.10 we have

$$\|D^{\mathrm{vert}}\| \leq \|G^{\mathrm{vert}}\| \leq 2 \tag{3.10}$$

If $\sigma(X, D) = 1$, then all the components of D are numerically proportional. By (3.8) and (3.7), $D \neq D^{\mathrm{hor}}$. Hence $D = D^{\mathrm{vert}}$. This contradicts (3.10). Thus $\sigma(X, D) = 2$ and $\|D\| \geq \dim X + 2$. Combining (3.8), (3.10) and (3.7) we get $\|D^{\mathrm{vert}}\| = 2$, $\|D^{\mathrm{hor}}\| = \dim X$, $\|D\| = \dim X + 2$, $D = G$ and $K_X + D \equiv 0$. The rest follows by Lemma 2.10. □

In conclusion, note that our method allow us to prove Conjecture 1.1 in the general form (at least in dimension three). We do not include these results because now there is much better approach [M].

References

[A1] Ambro F. *The adjunction conjecture and its applications,* PhD thesis, The Johns Hopkins University (1999), e-print math.AG/9903060

[A2] Ambro F. *Shokurov's boundary property*, e-print math.AG/0210271

[F] Fujino O. *Application of Kawamata's positivity theorem*, Proc Japan. Acad. **75**, Ser. A (1999) 75–79

[IP] V. A. Iskovskikh and Yu. G. Prokhorov, *Fano Varieties,* Encyclopaedia Math. Sci. **47**, Springer-Verlag, Berlin, 1999.

[K1] Kawamata Y. *Crepant blowing-up of* 3*-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. (2) **127** (1988) 93–163

[K2] Kawamata Y. *Subadjunction of log canonical divisors for a subvariety of codimension* 2*,* in: "Birational Algebraic Geometry", Proc. of Conf. in Memory of Wei-Liang Chow (1996), Contemporary Math. AMS **207**, 79–88

[KMM] Kawamata Y., Matsuda K., Matsuki K. *Introduction to the minimal model problem,* Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math., **10**, North-Holland, Amsterdam, 1987

[Ut] Kollár J. et al. *Flips and abundance for algebraic threefolds,* in "A summer seminar at the University of Utah, Salt Lake City, 1991". Astérisque. **211** (1992)

[M] McKernan J. *A simple characterisation of toric varieties*, Proc. Algebraic Geom. Symp. Kinosaki (2001) 59–72

[P1] Prokhorov Yu. G. *Lectures on complements on log surfaces*, Mem. Japan. Math. Soc. **10**, 2001

[P2] Prokhorov Yu. G. *On a conjecture of Shokurov: Characterization of toric varieties*, Tohoku Math. J. **53** (2001) no. 4, 581–592

[Sh] Shokurov V. V. *Complements on surfaces,* J. Math. Sci. **102** (2000) 3876–3932

Department of Algebra, Faculty of Mathematics, Moscow State Lomonosov University, Moscow 117234, Russia

Max-Plank-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany

E-mail address: prokhoro@mech.math.msu.su