THE SIGNATURE AND GENUS OF BRAID POSITIVE LINKS

This is a preprint. I would be grateful for any comments and corrections!

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Abstract. We show that the signature of a braid positive link is minorated by an increasing function of its genus.

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1. Introduction

Braid positive links (see, e. g., [Bu, Cr]) are links which can be represented as the closure of positive braids.[1] Beside because of this intuitive property, braid positive have been considered in knot theory, as they have occured in several contexts, e. g., the theory of dynamical systems [BW] and singularity theory [A, BoW, Ru2].

Motivated by the results in [BW], in [Ru], Lee Rudolph showed that braid positive links have positive signature, a result, subsequently extended to positive links by Cochran-Gompf [CG], Traczyk [Tr], and slightly further by Przytycki-Taniyama and myself [PT, St2].

In this paper we show that in fact the signature, here denoted by σ, grows unboundedly on braid positive links.

Theorem 1.1 There exists a function $f:\mathbb{N}\to\mathbb{N}$ with $\liminf\limits_{n\to\infty} f(n)=\infty$, such that if a link L is braid positive, then $\sigma(L)\geq f(\chi'(L))$, where for a link of $s(L)$ split components, $\chi'(L):=s(L)-\chi(L)$ and $\chi(L)$ is the maximal Euler characteristic of a spanning Seifert surface of the link L.

Note, that by the inequality of Bennequin [Be, theorem 3], and its subsequent generalizations due to Rudolph [Ru2], it is uninteresting to conclude anything on genus, 4-ball genus and unknotting number of braid positive links from theorem 1.1. But exactly the fact that yet σ has not been involved into an inequality of Bennequin type motivated the present result.

Theorem 1.1 is a special case of a (much larger) conjecture, made in [St2], the statement to be true for positive links. This conjecture is motivated by the extendability of Bennequin's inequality (from braid) to arbitrary diagrams and by the resulting applications of Gauß sums and Seifert surfaces as in [St3].

The proof of theorem 1.1 will be a combination of the methods of [Tr] and [St] and use the signature formulas of Murasugi [Mu, §9-11]. Although not formally contained in theorem 1.1, our proof will also contain a reproof of Rudolph's result, basically by applying Traczyk's argument in this special case.

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[1] Such links are elsewhere usually called 'positive braids', being identified, not to my taste, with the objects they arise from under the closure operation, or less common 'positive links', which is confusing, as it is my (and generally more usual) designation of the (much larger) class of links, which have diagrams with all crossings positive.

2. Preliminaries

Here we recall some basic facts and notations.

The n-strand braid group B_n is generated by the elementary (Artin) braids σ_i for $1 \le i \le n-1$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, called henceforth Yang-Baxter (YB) relation, and $[\sigma_i, \sigma_j] = 1$ for $|i-j| > 1$ (the brackets denoting the commutator), called commutativity relation.

We introduce an alternative notation for braid(word)s by replacing the σ_i by their subscripts and their inverses by the negated subscripts, and putting the result into brackets, e. g., $(\sigma_1\sigma_2^2)^5\sigma_1^{-1}\sigma_2 = [(12^2)^5 - 12]$.

By $[\beta]$ we denote the exponent sum of a braid β, that is, the image of β under the homomorphism $B_n \to \mathbb{Z}$ given by $[\sigma_i] := 1$ for any i. By $[\beta]_i$ we denote the exponent sum of the generator σ_i in the braid *word* β, which is clearly not invariant under the YB relation. That is, $[\,.\,]_i$ is a homomorphism of the free group in the σ_j given by $[\sigma_j]_i := \delta_{ij}$ (where δ is the Kronecker delta).

In this paper we shall be more closely concerned with the 3-strand braid group B_3. This group has a distinguished element $\Delta = \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, whose square, Δ^2 is central in B_3 (and in fact generates its center).

We call a braid (word) alternating if it is a word only in σ_1 and σ_2^{-1} (resp. σ_2 and σ_1^{-1}), and positive if it is a word only in σ_1 and σ_2.

The signature σ, has several definitions. The most common one is using Seifert surfaces and linking pairings. See, e. g., [Ro]. It has the following important property: σ changes by 0 or ± 1 when smoothing out a crossing and by 0, 1 or 2 (resp. 0, -1 and -2) when changing a negative crossing to the positive (resp. vice versa)[1].

We recall now the notion of a braid scheme, introduced in [St].

Definition 2.1 A braid scheme is a checkerboard diagram with integers putted on the black fields, e. g.

$$\begin{array}{ccccccc} & & -5 & & & & 6 \\ & -1 & & -2 & & 3 & \\ 1 & & 3 & & 2 & & 4 \end{array}\,.$$

$$\sigma_1 \quad \ldots\ldots\ldots\ldots \quad \sigma_7$$

If integers are omitted, they are assumed to be 0. The braid word corresponding to the diagram is the concatenation of powers of interchangingly with the row of the scheme (rows are numbered from bottom to top) odd and even index (Artin) generators, the powers given by the entries in the scheme. E. g., the braid word corresponding to the above scheme is

$$(\sigma_1\sigma_3^3\sigma_5^2\sigma_7^4)\,(\sigma_2^{-1}\sigma_4^{-2}\sigma_6^3)\,(\sigma_3^{-5}\sigma_7^6)\,.$$

Definition 2.2 A reducing move in a braid scheme is the "pulling down" of an entry (in row ≥ 3) two rows below, if one row below its 2 neighbors (of its 1 neighbor, if it is the first or last generator do not occur),

$$\begin{array}{ccc} & x & \\ 0 & & 0 \\ & y & \end{array} \longrightarrow \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & x+y & \end{array}$$

adding it to the below entry (if it is not 0). A scheme is called reduced if it does not admit a reducing move.

Example 2.1

$$\begin{array}{ccccc} & 3 & & -2 & \\ 0 & & 0 & & -4 \\ & 1 & & 0 & \\ 1 & & 2 & & 3 \end{array} \longrightarrow \begin{array}{ccccc} & 4 & & -2 & \\ 1 & & 2 & & -1 \end{array}\,.$$

[1] There is often confusion about the sign of σ. Here, we use the more naturally seeming convention that positive links should have positive and not negative σ.

Clearly, any scheme can be reduced by finitely many reducing moves.

We will in the proof mainly consider schemes with all entries being 1. In this case a fragment of the scheme like

$$\begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & 1 & \end{array} \quad \text{or} \quad \begin{array}{ccc} & 1 & \\ 0 & & 1 \\ & 1 & \end{array} \tag{1}$$

is the position where a YB relation can be applied.

We will denote the coordinates of the entries by the pair

$$(\text{row of the scheme, column} = \text{index of generator}).$$

Clearly, in each scheme only entries with even or odd coordinate sum will occur. Beside the row coordinate we also leave the column coordinate unlimited, according to the inclusions of the braid groups $B_n \hookrightarrow B_{n+1}$. In subsequent drawings we assume the row coordinate to grow in vertical direction, while the column coordinate to grow in horizontal direction (so that pictures differing by a rotation by 90° are not equivalent).

3. The proof

We begin now with the proof of theorem 1.1.

Proof. Fix a sequence (K_i) of braid positive links $K_i = \hat{\alpha}_i$ with α_i positive and $\chi'(K_i) \to \infty$, and assume that $\sigma(K_i) \equiv \sigma_0$. We would like to produce a contradiction out of this assumption.

We can assume K_i to be connected, i. e., $s(K_i) \equiv 1$, as you can replace split union by connected sum (both σ and χ' are additive under both operations), so that $[\alpha_i]_j = 0$ for some $j > 0$ implies $[\alpha_i]_{j'} = 0$ for any $j' \geq j$. Moreover, we can also assume that α_i are irreducible, that is, $[\alpha_i]_j \neq 1$ for any j.

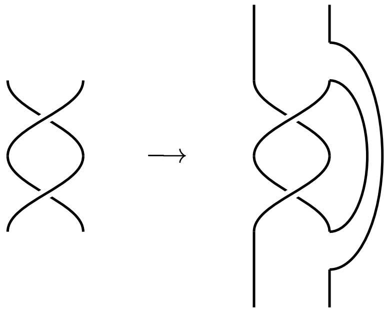
First we put the α_i into the form of a scheme, that is, we write $\alpha_i = \alpha_{i,1} \cdot \ldots \cdot \alpha_{i,n_i}$ such that $\alpha_{i,j}$ and $\alpha_{i,j+1}$ consist just of (powers of) even and odd index generators (the parity for $\alpha_{i,1}$ we set to be this of the first generator index appearing in α_i). Now we use Traczyk's method, considering the closed braid diagram as positive diagram, and try to observe in which cases it can be applied.

If

$$\Big| \sum_{j \text{ even}} [\alpha_i]_j - \sum_{j \text{ odd}} [\alpha_i]_j \Big| \tag{2}$$

is unbounded in i (which more precisely should mean that it has an unboundedly growing subsequence), then Traczyk's method of smoothing out all crossings corresponding to even/odd index generators except one for each generator, and using the σ of the remaining connected sum of $(2, [\alpha_i]_j)$ torus links, shows $\sigma(K_i) \to \infty$. Therefore, assume that (2) is bounded by some constant b_0 independent on i.

Assume now the number n_i of generator squares (where σ_j^n is counted as $\lfloor n/2 \rfloor$ squares) in α_i is unbounded (on some subsequence). So in particular there are unboundedly many (at least $n_i/2$) even (or odd) index generator squares. Then choose in Traczyk's method the even (or odd) index generators to be smoothed out. Then for each generator square the smoothing out of both crossings does not augment σ (and if the pair of crossings is the last for the generator, smoothing out one of them reduces σ by 1, as it corresponds to factoring out a positive Hopf link, as in case c) of [Tr, figure 2]). To see this, connect one of the strands by a band (a technique applied in [Mu, §9-11])

 ,

which changes σ by at most ± 1, and factor out a positive Hopf link changing σ by -1. The result is the same as smoothing out both crossings.

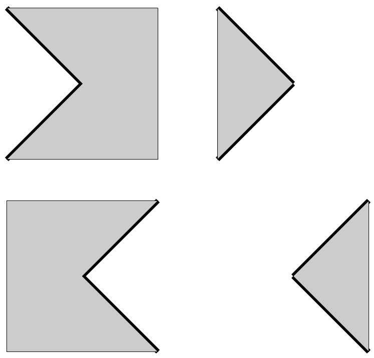
Using now that (2) is bounded, we can choose even to smooth out the generators with indices of the parity $p \in \{0,1\}$ where $\sum_{j \equiv p(2)} [\alpha_i]_j$ is larger, as the defect on σ is bounded, and know that σ does not increase at least $n_i/2$ times, so

$$\sigma(K_i) \geq \frac{n_i}{2} - b_0 \xrightarrow[i \to \infty]{} \infty$$

Therefore, only finitely many of the powers of generators in α_i are higher than one. Modifying the braids α_i at some bounded number of positions, which changes σ just by a bounded quantity, we obtain braids α_i' with all generators appearing without squares. If we now show $\sigma(\hat{\alpha}_i') \to \infty$ then we would also have $\sigma(\hat{\alpha}_i) \to \infty$.

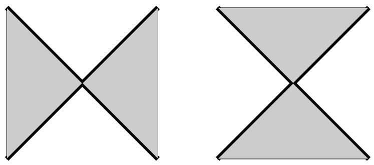
Therefore, we can w.l.o.g. assume that α_i in their scheme form consist only of entries '1'.

We now try to apply the case of a subdiagram admitting a Reidemeister III move (in our case a YB relation) in Traczyk's paper. These are the scheme fragments (1). If the number of such fragments is unbounded, then the number of fragments with sufficiently high distance to each other (so as the Reidemeister III moves to be applicable separately) is still unbounded, and Traczyk's argument in his case a) shows $\sigma(\hat{\alpha}_i) \to \infty$. Therefore, assume that the number of fragments (1) is bounded. This in particular implies that the strand number of α_i grows unboundedly (here in the meaning that *any* subsequence grows unboundedly), as it was observed in [St] that the number of fragments (1) is at least $O_n^{\asymp}\left(\dfrac{1}{n^2}\right)$ of the number of entries for an n-string braid (scheme).



type I type II

Figure 1



type 1 type 2

Figure 2: Singular types

Replace an element in the scheme at position (i,j) by a square with vertices $(i, j\pm 1)$ and $(i\pm 1, j)$ and join all such squares. You obtain a certain subset $R \subset \mathbb{R}^2$ with piecewise linear boundary ∂R. The only kind of self-tangencies ∂R is of type 1 on figure 2; those of type 2 cannot occur, because we assumed the scheme to be reduced. Also, because of this assumption, the region R is connected. The outer component of the boundary ∂R contributes two more type II fragments on figure 1 (local extrema of the second coordinate function on R) than type I (local extrema of the second coordinate function on $\mathbb{R}^2 \setminus R$), and the inner components of ∂R contribute two more type I fragments than type II fragments (including those of type 1). Thus when the number of type I fragments is bounded, so is the number of type II and type 1 fragments.

We now apply a modification of Traczyk's idea and produce a contradiction by Murasugi's formulas.

We take $k \in \{0,1,2\}$, where $\sum\limits_{j\equiv k\,(3)} [\alpha_i]_j$ is minimal, and smooth out all crossings corresponding to σ_{3m+k} except one for each generator.

According to Traczyk's idea, we sacrifice at most $1/3\chi'$. Then from the remaining link $L'$with $\chi'(L') \geq 2/3\chi'(L)$ we need to show that $\sigma(L') \geq 1/2\chi'(L') + \varepsilon_n$, where ε_n growing when the strand number n of L' (and L) grows. But now L' is a connected sum of (positive) closed 3-braid links L_i. Thus, it suffices to show that for these links

$$\sigma(L_i) \geq 1/2\chi'(L_i) + \varepsilon \tag{3}$$

for some $\varepsilon > 0$ independent on i and L_i.

We shall show this for $\varepsilon = 1/2$. We have $L = \hat{\beta}_i$ with positive $\beta_i \in B_3$. But in fact we can say something more on β_i.

Except at positions which lie near fragments of type 1, I or II in R, all β_i are words in $x_1 = [(12)^n 1]$, $x_2 = [(21)^n 1]$ for $n > 0$, $[12]$ and $[21]$. At the finite number of places (finitely many for *all* β_i altogether, not just for any single β_i!), we can insert appropriately some generator so as to "repair" the local shape of β_i to be a word in the above braids, at the cost of changing $\sigma(L')$ (and therefore $\sigma(L)$) by some bounded quantity (the bound depending just on the number of fragments of type I). The statement $\sigma \to \infty$ before and after this correction is equivalent, so we can assume w.l.o.g. that all β_i are words in the above braids.

Then, by applying the YB relation at the last 3 letters in x_1 and x_2 and conjugating by Δ (interchanging σ_1 and σ_2), we find that (up to this conjugacy) $\beta := \beta_i$ or $\beta_i\sigma_1^{-1}$ is a word in $s_1 := [12]$ and $s_2 := [21]$.

We now need the following theorem of Murasugi.

Theorem 3.1 (Murasugi [Mu]) If a 3-braid β_1 is conjugate to a braid of the form

$$\beta = \Delta^{2n}\beta_2 \tag{4}$$

with β_2 an alternating braid, then

$$\sigma(\hat{\beta}_1) \geq 4n + [\beta_2] - 2\,. \tag{5}$$

This theorem has been proved in several parts in [Mu, §9-11]. For the case $\beta_2 = Id$ this is proposition 9.1. Else you have one of Murasugi's normal forms $\Omega_{4,5,6}$. For $\beta_2 \in \Omega_4$ the inequality is shown in proposition 10.1, for $\beta_2 \in \Omega_5$ the same statement follows analogously, and for $\beta_2 \in \Omega_6$ the formula follows from proposition 11.1 and the following remark.

We now show how our $\beta := \beta_i$ is conjugate to a braid of the from (4), and keep track of the contributions of the single factors to both hand sides of (5) to deduce (3).

First we do it for β (rather than $\beta\sigma_1^{-1}$) being (conjugate by Δ to) a braid word in s_1 and s_2. Then (modulo conjugacy)

$$\beta = s_1^{a_1} s_2^{b_1} \dots s_1^{a_n} s_2^{b_n} \tag{6}$$

with $a_i, b_i > 0$. If some a_i or b_i is at least 3, then we factor out one $\Delta^2 = s_1^3 = s_2^3$. By (5) this contributes 4 to σ and 6 to $[\beta]$. Factoring out all such copies of Δ^2, we may assume that all a_i, $b_i < 3$. Then we use

$$\begin{aligned} s_1^2 s_2 &= [1-2]\cdot\Delta^2 \\ s_1 s_2^2 &= [-21]\cdot\Delta^2 \\ s_1^2 s_2^2 &= [11]\cdot\Delta^2 \\ s_1 s_2 &= [-2-2]\cdot\Delta^2\,. \end{aligned}$$

The left hand-sides of the first 3 equalities consume 6 resp. 8 of β and contribute 4 resp. 6 to $\sigma(\hat\beta)$.

Using now that $\chi'(\hat\beta)=[\beta]-2$ (as $\hat\beta$ is connected), we see that

$$\sigma(\hat\beta)\,\geq\,\frac{\chi'(\hat\beta)}{2}+1\,, \tag{7}$$

so (3) is satisfied even with $\varepsilon=1$, except for one of the following three cases.

- The word (6) is a power of s_1s_2 and there is no factor Δ^2 coming from a_i, $b_i\geq 3$.
- The word (6) is a power of s_1s_2 except for one square, that is, all $a_i=b_i=1$ except one a_i or b_i, which is 2. Modulo conjugacy by Δ we can assume it to be a square of s_1 and again modulo conjugacy we can put it to be the first, so $a_1=2$.
- There is exactly one factor Δ^2 coming from a_i, $b_i\geq 3$ and the word (6) has only $a_i=b_i=1$, that is, is a power of s_1s_2.

We now show (7) for these cases by direct calculation. Denote $\beta_1\succeq\beta_2$, if $\sigma(\hat{\beta_1})\geq\sigma(\hat{\beta_2})$ (which in particular occurs when obtaining β_2 by removing squares of σ_1 or σ_2 in β_1). Then we have (where equalities are meant up to conjugacy)

$$(s_1s_2)^n\,=\,[(1221)^n]\,=\,[(2^21^2)^n]\,\succeq\,[2^21^{2n}]\,,$$

and $\sigma((2^21^{2n})\hat{})=2n$, while $\chi'/2(((s_1s_2)^n)\hat{})=2n-1$.

For the second case, assuming $n>0$ because of the irreducibility of the braid, we get

$$\begin{aligned} s_1(s_1s_2)^n &= [121221(1221)^{n-1}] \\ &= [212221(1221)^{n-1}] \\ &= [121222(1122)^{n-1}] \\ &= [212222(1122)^{n-1}] \\ &\succeq [212^{2n+2}]\,, \end{aligned}$$

and $\sigma([212^{2n+2}]\hat{})=2n+2$, while $\chi'/2((s_1(s_1s_2)^n)\hat{})=2n$.

Finally, assuming again $n>0$ (as the case $\beta=\Delta^2$ has $\sigma=4$),

$$\begin{aligned} \Delta^2(s_1s_2)^n &= [212212(1221)^n] \\ &= [2122121(2211)^n-1] \\ &= [2122212(2211)^n-1] \\ &= [-12122212(2211)^n] \\ &= [212212(2211)^n] \\ &= [112112(2211)^n] \\ &= [2112(2211)^n11] \\ &\succeq [2112221^{2n+2}] \\ &\succeq [2^41^{2n+2}], \end{aligned}$$

and $\sigma([2^41^{2n+2}]\hat{})=2n+4$, while $\chi'/2((\Delta^2(s_1s_2)^n)\hat{})=2n+2$.

This establishes (3) with $\varepsilon=1$ if β is conjugate to a word in s_1 and s_2.

Now we consider the case where $\beta\sigma_1^{-1}$ is (up to conjugacy by Δ) a word in s_1 and s_2. Then we have

$$\beta\,=\,s_1^{a_1}s_2^{b_1}\,\dots\,s_1^{a_n}s_2^{b_n}s_1^{a_{n+1}}\sigma_1 \tag{8}$$

or

$$\beta\,=\,s_2^{b_1}s_1^{a_1}\,\dots\,s_2^{b_n}s_1^{a_n}s_2^{b_{n+1}}\sigma_1 \tag{9}$$

with $a_i, b_i > 0$ and a_{n+1}, $b_{n+1} \geq 0$. For the first $2n$ factors as in (6) we obtain again

$$\sigma \geq \frac{[\beta]-2}{2} + 1\,, \qquad (10)$$

with σ denoting the contribution to $\sigma(\hat{\beta})$ of these factors according to (5). Thus we are interested just in the contribution of the last two factors in (8) resp. (9) to $\sigma(\hat{\beta})$ and $[\beta]$.

In the non-exceptional cases, we established (10) without further conjugation, so we don't need to conjugate the last two factors either.

Again, by factoring out Δ^2, we can assume that all a_i, $b_i < 3$. Then there are 5 choices for the last two factors in (8) resp. (9).

1. [1]. The contributions to both $\sigma(\hat{\beta})$ and $[\beta]$ are 1.
2. [121]. Smoothing out the '2' we get contribution 2 to σ, so the contribution before the smoothing out was at least one. The contribution to $[\beta]$ is 3.
3. [12121]. Adding a '2' at the end we obtain a new copy of Δ^2 with contribution 4 to σ, so the contribution before adding the '2' was at least 3. The contribution to $[\beta]$ is 5.
4. [211]. Same argument as for [121].
5. $[21211] = [12111]$. Smoothing out the '2' we get contribution 4, so the contribution to σ is at least 3, while this to $[\beta]$ is 5.

Now, we need to consider 15 more braids, as in the 3 exceptional cases we conjugated β to $\alpha^{-1}\beta\alpha$ with $\alpha = \sigma_1$, σ_1^{-1} and σ_1^3 resp., so we need to make the same calculation for these three conjugated versions of each of the above 5 braids, but the same contributions to σ are established by analogous and simple calculations.

Combining all 5 cases with (10), we find that at least still

$$\sigma \geq \frac{\chi'}{2} + \frac{1}{2}\,,$$

which is (3), and we are through (by the above outlined contradiction). $\square$

4. Some corollaries

Corollary 4.1 There exist only finitely many braid positive (or n-almost braid positive[1]) links without trivial split components of given signature.

Proof. This is a consequence of theorem 1.1 and the fact that there are only finitely many braid positive links with given χ'. This fact follows from Bennequin's inequality or from Cromwell's work [Cr2] on homogeneous fibred links. See [St4]. $\square$

Corollary 4.2 Let D be a positive diagram of a given number n of Seifert circles. Then

$$\sigma(D) \geq O_n^{\asymp}\left(\frac{1}{n^2}\right) \cdot \chi'(D) - O_n(n^2)\,,$$

where $O_n^{\asymp}$ and O_n denotes the asymptotical behaviour of the expression as $n \to \infty$. For braid diagrams the inequality holds with the O_n term being zero.

[1] closures of braids with exactly n negative crossings

Proof. For braid diagrams this follows from the above mentioned observation of [St] that the number of fragments of the scheme like in (1) (but now with the entries '1' possibly replaced by some higher integers) is at least $O_n^{\asymp}\left(\frac{1}{n^2}\right)$ of the number of entries of the scheme. Each such fragment allows to apply at least one of Traczyk's cases c) or a).

For arbitrary diagrams the statement can be lead back to braid diagrams by Vogel's algorithm [Vo]. Vogel gave an estimate for the number of moves needed to obtain a braid diagram out of a diagram of n Seifert circles. As each such move generates one negative crossing, we obtain the second correction term. □

Corollary 4.3 If (D_i) is a sequence of pairwise distinct n-almost positive diagrams of given σ, then the number of Seifert circles of D_i grows unboundedly, when $i \to \infty$.

Proof. This is a straightforward consequence of the previous corollary. □

If the conjecture of [St2] is true in full generality, we would have that in fact all D_i arise from finitely many diagrams by the so-called $\bar{t}_2'$ moves introduced in [St3] (replacement of a crossing by three antiparallel half-twists).

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*To prevent regular criticism about not quoting some specific result, I declare that in putting together my reference lists I follow the principle of an ingenious mathematician (whose wise words I quote): "This is a partial bibliography of papers" related to the topic of the present one. "It aims to be a complete listing of all the papers on the subject, but at this time, it" may "still fail. If your paper (or a paper you are aware of) is omitted (or abused in any other way), it is simply because of the haphazard way by which this bibliography is organized, and not because of a secret plan of mine to ruin your (or anybody else's) reputation" (except possibly if you are one of the very few persons who behave in my feeling very incorrectly towards me). "Let me know (please..., I want this bibliography to be as complete as possible!) and it will be fixed." Please, "allow me just one selfish rule: I will not enter any paper into this bibliography unless I have (or I am given) a copy" or a published reference "of it."