Max-Planck-Institut für Mathematik
Bonn

# On the Newton polygon of a Jacobian mate

by

Leonid Makar-Limanov

Max-Planck-Institut für Mathematik
Preprint Series 2013 (53)

# On the Newton polygon of a Jacobian mate

Leonid Makar-Limanov

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics
Wayne University
Detroit, MI 48202
USA

Department of Mathematics
and Computer Science
The Weizmann Institute of Science
Rehovot 76100
Israel

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
USA

MPIM 13-53

# On the Newton polygon of a Jacobian mate.

Leonid Makar-Limanov

*To the memory of Shreeram Abhyankar*
*one of the champions of the Jacobian Conjecture*

**Abstract**

This note contains an up-to-date description of the "minimal" Newton polygons of the polynomials satisfying the Jacobian condition.

**Mathematics Subject Classification (2000):** Primary 14R15, 12E05; Secondary 12E12.

**Key words:** Jacobian conjecture, Newton polygons

**Introduction.**

Consider two polynomials $f,\ g \in \mathbb{C}[x,y]$ where $\mathbb{C}$ is the field of complex numbers with the Jacobian $\mathrm{J}(f,g) = 1$ and $\mathbb{C}[f,g] \neq \mathbb{C}[x,y]$ i.e. a counterexample to the JC (Jacobian conjecture) which states that $\mathrm{J}(f,g) = 1$ implies $\mathbb{C}[f,g] = \mathbb{C}[x,y]$ (see [K]). This conjecture occasionally becomes a theorem even for many years but today it is a problem.

One of the approaches to this problem which is still popular, is through obtaining information about the Newton polygons of polynomials $f$ and $g$. It is known for many years that there exists an automorphism $\xi$ of $\mathbb{C}[x,y]$ such that the Newton polygon $\mathcal{N}(\xi(f))$ of $\xi(f)$ contains a vertex $v=(m,n)$ where $n>m>0$ and is included in a trapezoid with the vertex $v$, edges parallel to the $y$ axes and to the bisectrix of the first quadrant adjacent to $v$, and two edges belonging to the coordinate axes (see [A], [AO], [H], [L], [M], [MW], [Na1], [Na2], [NN1], [NN2], [O], [R]). This was improved quite recently by Pierrette Cassou-Noguès who showed that $\mathcal{N}(f)$ does not have an edge parallel to the bisectrix (see [CN]). Here a shorter (and more elementary) version of the proof of this fact is suggested. A proof of the "trapezoid" part based on the work [Di] of Dixmier published in 1968 is also included to have all the information on $\mathcal{N}(f)$ in one place with streamlined proofs.

As a byproduct we'll get a proof of the Jung theorem that any automorphism of $\mathbb{C}[x,y]$ is a composition of linear and "triangular" automorphisms.

**Trapezoidal shape.**

In this section, using technique developed by Dixmier in [Di], we will check the claim that if $f\in\mathbb{C}[x,y]$ is a Jacobian mate i.e. when $\mathrm{J}(f,g)=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}=1$ for some $g\in\mathbb{C}[x,y]$ then there exists an automorphism $\xi$ of $\mathbb{C}[x,y]$ such that the Newton polygon $\mathcal{N}(\xi(f))$ of $\xi(f)$ is contained in a trapezoid described in the introduction.

Recall that if $p\in\mathbb{C}[x,y]$ is a polynomial in 2 variables and each monomial of $p$ is represented by a lattice point on the plane with the coordinate vector equal to the degree vector of this monomial then the convex hull $\mathcal{N}(p)$ of the points so obtained is called the Newton polygon. For reasons which are not clear to me Newton included the origin (a non-zero constant term) in his definition.

Define a weight degree function on $\mathbb{C}[x, y]$ as follows. First, take weights $w(x) = \alpha,\ w(y) = \beta$ where $\alpha, \beta \in \mathbb{Z}$ and put $w(x^i y^j) = i\alpha + j\beta$. For a $p \in \mathbb{C}[x, y]$ denote the support of $p$, i.e. the collection of all monomials appearing in $p$ with non-zero coefficients by $\mathrm{supp}(p)$ and define $w(p) = \max(w(x^i y^j)|x^i y^j \in \mathrm{supp}(p))$. Polynomial $p$ can be written as $p = \sum p_i$ where $p_i$ are forms homogeneous relative to $w$. The leading form $p_w$ of $p$ according to $w$ is the form of the maximal weight in this presentation.

**Lemma on independence.** Take any two algebraically independent polynomials $a,\ b \in \mathbb{C}[x, y]$ and a non-zero weight degree function $w$ on $\mathbb{C}[x, y]$. Then there exists an $h \in \mathbb{C}[a, b]$ for which $\mathrm{J}(a_w, h_w) \neq 0$ i.e. $h_w$ and $a_w$ are algebraically independent.

Proof. A standard proof of this fact would be based on the notion of Gelfand-Kirillov dimension (see [GK]) and is rather well-known. The proof below uses a deficiency function

$$\mathrm{def}_w(a, h) = w(\mathrm{J}(a, h)) - w(h)$$

(somewhat similar to the one introduced in [ML]) and is more question specific. This function is defined and has values in $\mathbb{Z}$ when $\mathrm{J}(a, h) \neq 0$ i.e. $\mathrm{def}_w$ is defined for any $h \in \mathbb{C}[a, b]$ which is algebraically independent with $a$. Observe that $\mathrm{def}_w(a, hr(a)) = \mathrm{def}_w(a, h),\ r(a) \in \mathbb{C}[a] \setminus 0;\ \mathrm{def}_w(a, h) \leq w(a) - w(xy);$

and that $\mathrm{def}_w(a, h^k) = \mathrm{def}_w(a, h)$ since $\mathrm{J}(a, h^k) = kh^{k-1}\mathrm{J}(a, h)$.

If $a_w$ and $b_w$ are algebraically dependent then there exists an irreducible non-zero polynomial $q = \sum_{i=0}^{k} q_i(x)y^i \in F[x, y]$ for which $q(a_w, b_w) = 0$ and all monomials with non-zero coefficients have the same degree relative to the weight $W(x) = w(a)$, $W(y) = w(b)$. Elements $a$, $b' = q(a, b)$ are algebraically independent since $a$ and $b$ are algebraically independent but there is a drop in weight, i.e. $w(b') < w(q_k(a)b^k)$.

We have $\mathrm{def}_w(a, b') = w(J(a, b')) - w(b') = w(\sum_i J(a, q_i(a)b^i)) - w(b') > w(J(a, q_k(a)b^k)) - w(q_k(a)b^k) = \mathrm{def}_w(a, b^k) = \mathrm{def}_w(a, b)$ since $w(b') < w(q_k(a)b^k)$ while $w(J(a, q_k(a)b^k)) = w(kq_k(a)b^{k-1}) + w(J(a, b)) = w(\sum_i iq_i(a)\ b^{i-1}) + w(J(a, b)) = w(\sum_i J(a, q_i(a)b^i))$ because $\sum_i iq_i(a_w)\ b_w^{i-1} \neq 0$ since $q$ is irreducible. If $a_w$, $b'_w$ are algebraically dependent, we repeat the procedure and obtain a pair $a$, $b''$ with $\mathrm{def}_w(a, b'') > \mathrm{def}_w(a, b')$, etc.. Since $\mathrm{def}_w(a, h) \leq w(a) - w(xy)$ for any $h$ and $\mathrm{def}_w(a, h) \in \mathbb{Z}$, the process will stop after a finite number of steps and we will get an element $h \in \mathbb{C}[a, b]$ for which $h_w$ is algebraically independent with $a_w$. $\square$

Now back to our polynomials $f$, $g$ with $\mathrm{J}(f, g) = 1$. These two polynomials are algebraically independent. To prove it consider a derivation $\partial$ given on $\mathbb{C}[x, y]$ by $\partial(h) = \mathrm{J}(f, h)$. When $\partial$ is restricted to $\mathbb{C}[f, g]$ this is the ordinary partial derivative relative to $g$. Hence if $p(f, g) = 0$ then $p_g(f, g) = 0$ and a contradiction is reached if we assume that $p$ is an irreducible dependence.

This derivation is locally nilpotent on $\mathbb{C}[f, g]$, i.e. $\partial^d(h) = 0$ for $h \in \mathbb{C}[f, g]$ and $d = \deg_g(h) + 1$. Therefore $\partial_w$ which is given by $\partial_w(h) = \mathrm{J}(f_w, h)$ on the ring $\mathbb{C}[f, g]_w$ generated by the leading $w$ forms of elements in $\mathbb{C}[f, g]$ is also a locally nilpotent derivation. Indeed a straightforward computation shows that $\mathrm{J}(a,b)_w = \mathrm{J}(a_w, b_w)$ if $\mathrm{J}(a_w, b_w) \neq 0$.

Take a weight degree function for which $w(f) \neq 0$ and a $w$-homogenous form $\chi \in \mathbb{C}[x,y]$ for which $f_w = \chi^d$ where $d$ is maximal possible. Then by Lemma on independence there exists a $\psi \in \mathbb{C}[f,g]_w$ which is algebraically independent with $\chi$ i.e. $\partial_w(\psi) \neq 0$. Take $k$ for which $\partial_w^k(\psi) \neq 0$ and $\partial_w^{k+1}(\psi) = 0$ and denote $\partial_w^{k-1}(\psi)$ by $\omega$. Then $\partial_w^2(\omega) = 0$, $\partial_w(\omega) \neq 0$ and $\partial_w(\omega) = c_1\chi^{d_1}$ since $\chi$ and $\partial_w(\omega)$ are homogeneous. Therefore $\mathrm{J}(\chi^d, \omega) = c_1\chi^{d_1}$ and $\mathrm{J}(\chi, \omega) = c_2\chi^{d_1-d+1}$. For computational purposes it is convenient to introduce $\varsigma = \frac{\omega}{c_2\chi^{d_1-d}} \in \mathbb{C}(x,y)$; then $\mathrm{J}(\chi, \varsigma) = \chi$ and $w(\varsigma) = w(xy)$.

If $w(x) = 0$ then $\chi = y^j p(z)$, $\varsigma = yq(z)$ where $z = x$; if $w(x) \neq 0$ we can write $\chi = x^r p(z)$, $\varsigma = x^s q(z)$ where $z = x^{\frac{\beta}{-\alpha}} y$. In both cases $p(z) \in \mathbb{C}[z]$, $q(z) \in \mathbb{C}(z)$. In the second case $r,\ s \in \mathbb{Q}$ and $w(\chi) = r\alpha$, $w(\varsigma) = s\alpha$. (Recall that $w(x) = \alpha$, $w(y) = \beta$.) In any case the relation $\mathrm{J}(\chi, \varsigma) = \chi$ is equivalent to

$$\tau p' q - \rho p q' = cp \quad (1)$$

where $\rho = w(\chi)$, $\tau = w(\varsigma) = w(xy)$, and $c \in \mathbb{C}^*$.

(1) can be rewritten as $\ln(p^\tau q^{-\rho})' = \frac{c}{q}$ or

$$p^\tau = q^\rho \exp(c \int \frac{dz}{q}). \quad (2)$$

If $\rho\tau > 0$ then $q(z)$ must be a polynomial since a pole of $q(z)$ would induce a pole of $p(z)$ in the same point.

Now we are ready to discuss the shape of $\mathcal{N}(f)$. Let $m = \deg_x(f)$, $n = \deg_y(f)$. Assume that $f$ does not contain a monomial $cx^m y^n$. Then $\mathcal{N}(f)$ has a vertex $(m,k)$ where $k < n$ (and maximal possible) and an edge $e$ with the vertex $(m,k)$ and a negative slope. We can find a weight degree function $w$ so that the Newton polygon of the leading form $f_w$ of $f$ relative to $w$ is $e$. Since the slope of $e$ is negative $\rho\tau$ is positive and $\varsigma = x^s q(z)$ is a homogeneous polynomial. Indeed, $w(x) \neq 0$ and we checked above that $\varsigma$ is a polynomial in $z$ and therefore a polynomial in $y$. Since $w(y) \neq 0$ similar considerations show that $\varsigma$ is a polynomial in $x$.

There are just four options for $\mathcal{N}(\varsigma)$ because $w(\varsigma) = w(xy)$. Here is the list of all possibilities: (1) $\varsigma = cxy$; (2) $\varsigma = cx(y + c_1x^k),\ k > 0$; (3) $\varsigma = c(x + c_1y^k)y,\ k > 0$; (4) $\varsigma = c(x + c_1y)(y + c_2x),\ c_1c_2 \neq 0$. In each case there is an automorphism of $\mathbb{C}[x,y]$ which transforms $\varsigma$ into $cxy$ and then the image of $\chi = f_e$ under this automorphism is also a monomial ($\mathrm{J}(\chi, cxy) = \chi$ is satisfied only by monomials $x^iy^j$ where $c(i-j) = 1$ and these monomials have different weights). Hence in the first case $\chi$ is a monomial, in the second case $\chi = c_3x^a(y + c_1x^k)^b$, in the third case $\chi = c_3(x + c_1y^k)^a y^b$, and in the fourth case $\chi = c_3(x + c_1y)^a(y + c_2x)^b$.

Define $A(f) = \deg_x(f)\deg_y(f)$. In each case there is an automorphism $\zeta$ such that $A(\zeta(f)) < A(f)$: in the second and the forth cases we can take $\zeta(x) = x,\ \zeta(y) = y - c_1x^k$ (indeed, $\zeta(x^a(y+c_1x^k)^b) = x^a(y - c_1x^k + c_1x^k)^b = x^ay^b$ and $\deg_x(\zeta(f)) < \deg_x(f),\ \deg_y(\zeta(f)) = \deg_y(f)$) and in the third and the forth cases we can take $\zeta(x) = x - c_1y^k,\ \zeta(y) = y$ (then $\deg_x(\zeta(f)) = \deg_x(f),\ \deg_y(\zeta(f)) < \deg_y(f)$).

Hence if $x^my^n \notin \mathrm{supp}(f)$ one of the automorphisms $\zeta(x) = x,\ \zeta(y) = y - c_1x^k$; $\zeta(x) = x - c_1y^k,\ \zeta(y) = y$ (usually automorphisms $\zeta(x) = x,\ \zeta(y) = y + \phi(x)$ and $\zeta(x) = x + \phi(y),\ \zeta(y) = y$ are called *triangular*) decreases $A(f)$. Since $A$ is a nonnegative integer there is an automorphism $\xi$ which is a composition of triangular automorphisms for which $A(\xi(f))$ is minimal possible and $\mathcal{N}(\xi(f))$ contains a vertex $(\deg_x(\xi(f)), \deg_y(\xi(f)))$.

Replace $f$ by $\xi(f)$ for which $A(\xi(f))$ is minimal. The leading form of $f$, say for a weight $w(x) = 1,\ w(y) = 1$ is $x^m y^n$. The corresponding $\varsigma = cxy$. Since $\mathrm{J}(x^m y^n, cxy) = c_1 x^m y^n$ where $c_1 \neq 0$ we cannot have $m = n$ and an assumption that $n > m$ is not restrictive (if $m > n$ apply an automorphism $\alpha(x) = y,\ \alpha(y) = x$).

If $m = 0$ then $f = f(y)$. Since then $\mathrm{J}(f, g) = -f_y g_x$ this implies that $\deg_y(f) = 1,\ g = g_0(y) + cx$ where $c \in \mathbb{C}^*$ and $\mathbb{C}[f, g] = \mathbb{C}[x, y]$.

Consider again a weight given by $w(x) = 1,\ w(y) = 1$. Then $f_w = x^m y^n$. As we observed above $\partial_w$ defined by $\partial_w(h) = \mathrm{J}(f_w, h)$ is locally nilpotent on $\mathbb{C}[f, g]_w$. If $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ then $\mathbb{C}[f, g]_w = \mathbb{C}[x, y]_w = \mathbb{C}[x, y]$. Hence if $\mathbb{C}[f, g] = \mathbb{C}[x, y]$ then $\partial(h) = \mathrm{J}(x^m y^n, h)$ is a locally nilpotent derivation on $\mathbb{C}[x, y]$. If $m > 0$ then $\partial^j(y) = \frac{m(m+d)\dots(m+(j-1)d)}{j!} x^{j(m-1)} y^{j(n-1)+1}$ where $d = n - m > 0$ is never zero and $\mathbb{C}[f, g] \neq \mathbb{C}[x, y]$.

These observations prove a theorem of Jung (see [J]) that any automorphism is a composition of triangular and linear automorphisms. If $\alpha$ is an automorphism of $\mathbb{C}[x, y]$ then $f = \alpha(x)$ is a Jacobian mate since by the chain rule $\mathrm{J}(\alpha(x), \alpha(y)) = c \in \mathbb{C}^*$. As we saw we can apply several triangular automorphisms after which the image of $f$ is a polynomial which is linear either in $x$ or $y$ (since both cases $n > m$ and $m > n$ are possible). After that an additional triangular automorphism reduce $(f,\ g)$ to either $(c_1 x,\ c_2 y + g_1(x))$ or $(c_1 y,\ c_2 x + g_1(y))$ and another triangular automorphism to $(c_1 x,\ c_2 y)$ or $(c_1 y,\ c_2 x)$. Finally a linear automorphism reduces the images to $(x,\ y)$.

From now on assume that $m > 0$. Then there are two edges containing

$v = (m, n)$ as a vertex, the edge $e$ which is either horizontal or below the horizontal line and the edge $e'$ which is either vertical or to the left of the vertical line.

Consider the edge $e$ and the weight $w$ for which $\mathcal{N}(f_w) = e$. If the slope of $e$ is less than 1 then $\rho\tau > 0,\ \varsigma$ is a polynomial and $w(\varsigma) = w(xy)$. In the case $e$ is horizontal $\varsigma = yq(x)$ where $q(x)$ is a polynomial and after an appropriate automorphism $x \to x - c,\ y \to y$ we may assume that $q(0) = 0$. If $w(x) \neq 0$ and $w(y) \neq 0$ then $\varsigma(0, 0) = 0$ because of the shape of $\mathcal{N}(\varsigma)$. If $\varsigma = cxy$ then $e$ is a vertex contrary to our assumption. If $\varsigma = c_1xy + \ldots + c_2x^iy^j$ where $c_2 \neq 0$ and $i > 1$ then $j = \mu(i - 1) + 1$ where $\mu$ is the slope and $\mathrm{J}(x^my^n, x^iy^j) = (mj - ni)x^{m+i-1}y^{n+j-1} \neq 0$ since $mj - ni = (m\mu - n)(i - 1) + m - n < 0$ (recall that $n > m$ and $0 \leq \mu < 1$). But then $\deg_x(\mathrm{J}(f_w, \varsigma)) > \deg_x(f_w)$ and $\mathrm{J}(f_w, \varsigma) \neq cf_w$, a contradiction.

Therefore the slope of $e$ is at least 1. If slope is 1 we cannot get a contradiction using only $\mathrm{J}(f_w, \varsigma) = f_w$ since $\mathrm{J}(y^kh(xy), xy) = -ky^kh(xy)$.

**Edge with slope one.**

Newton introduced the polygon which we call the Newton polygon in order to find a solution $y$ of $f(x, y) = 0$ in terms of $x$ (see [Ne]). Here is the process of obtaining such a solution. Consider an edge $e$ of $\mathcal{N}(f)$ which is not parallel to the $x$ axes and take a weight $w(x) = \alpha,\ w(y) = \beta$ which corresponds to $e$ (the choice of weight is unique if we assume that $\alpha,\ \beta \in \mathbb{Z},\ \alpha > 0$ and $(\alpha, \beta) = 1$ ). Then the leading form $f_w$ allows to determine the first summand of the solution as follows. Consider an equation $f_w = 0$. Since

$f_w$ is a homogeneous form and $\alpha \neq 0$ solutions of this equation are $y = c_i x^{\frac{\beta}{\alpha}}$ where $c_i \in \mathbb{C}$. Choose any $c_i$ and replace $f(x,y)$ by $f_1(x,y) = f(x, c_i x^{\frac{\beta}{\alpha}} + y)$ which is not necessarily a polynomial in $x$ but is a polynomial in $y$, and consider the Newton polygon of $f_1$. This polygon contains the *degree* vertex $v$ of $e$, i.e. the vertex with $y$ coordinate equal to $\deg_y(f_w)$ and an edge $e'$ which is a modification of $e$ ($e'$ may collapse to $v$). Take the other vertex $v_1$ of $e'$ (if $e' = v$ take $v_1 = v$). Use the edge $e_1$ for which $v_1$ is the degree vertex to determine the next summand and so on. After possibly a countable number of steps we obtain a vertex $v_\mu$ and the edge $e_\mu$ for which $v_\mu$ is not the degree vertex, i.e. either $e_\mu$ is horizontal or the degree vertex of $e_\mu$ has a larger $y$ coordinate than the $y$ coordinate of $v_\mu$. It is possible only if $\mathcal{N}(f_\mu)$ does not have any vertices on the $x$ axis. Therefore $f_\mu(x,0) = 0$ and a solution is obtained.

The process of obtaining a solution is more straightforward then it may seem from this description. The denominators of fractional powers of $x$ (if denominators and numerators of these rational numbers are assumed to be relatively prime) do not exceed $\deg_y(f)$. Indeed, for any initial weight there are at most $\deg_y(f)$ solutions while a summand $cx^{\frac{M}{N}}$ can be replaced by $c\varepsilon^M x^{\frac{M}{N}}$ where $\varepsilon^N = 1$ and hence at least $N$ solutions can be obtained (also see [P] for a more elaborate explanation).

If $\mathcal{N}(f)$ has an edge which is parallel to the bisectrix of the first quadrant, i.e. the edge with the slope 1 we can start the resolution process with the weight $w(x) = 1,\ w(y) = -1$. If we choose a non-zero root of the equation $f_w = 0$ then a solution $y = cx^{-1} + \sum_{i=1}^{\infty} c_i x^{\frac{r_i}{N}}$ where $c \in \mathbb{C}^*$ and $-1 < \frac{r_1}{N} <$

$\frac{r_2}{N} < \ldots$ will be obtained.

It is time to recall our particular situation. We have two polynomials $f,\ g \in \mathbb{C}[x, y]$ with $\mathrm{J}(f, g) = 1$ and the Newton polygon of $f$ supposedly contains an edge with slope 1. David Wright observed in [W] that the differential form $ydx - g(x, y)df(x, y)$ is exact if and only if $\mathrm{J}(f, g) = 1$ (a calculus exercise) and therefore

$$ydx - g(x, y)df(x, y) = dH(x, y) \quad (3)$$

where $H \in \mathbb{C}[x, y]$ (see the proof of theorem 3.3 in [W]). By the chain rule $dH(x, \phi(x)) = \phi(x)dx - g(x, \phi(x))df(x, \phi(x))$ for any expression $\phi(x)$ for which the derivative $\frac{d}{dx}$ is defined.

Take for $\phi(x)$ a solution $y = cx^{-1} + \sum_{i=1}^{\infty} c_i x^{\frac{r_i}{N}}$ for $f(x, y) = 0$.

Then $f(x, \phi(x)) = 0$ and $dH(x, \phi(x)) = \phi(x)dx$ or

$$\frac{dH(x, \phi(x))}{dx} = \phi(x). \quad (4)$$

Since $\phi$ contains $x^{-1}$ with a non-zero coefficient $H(x, \phi(x))$ should contain $\ln x$ with a non-zero coefficient which is clearly not possible. $\square$

We see that on a smooth curve $\gamma$ given by $f(x, y) = 0$ the differential form $ydx$ is exact. This is a very strong restriction on $\gamma$. If $\gamma$ is a rational curve and we do not mind logarithms $ydx$ on $\gamma$ is exact but the exactness of the restriction of $ydx$ on $\gamma$ does not imply that the genus of $\gamma$ is zero (even if logarithms are forbidden). E. g. for $\varphi = x^k y^{2k}(y^k - 1)^{k-1},\ \psi = xy(y^k - 1)$ we have $\mathrm{J}(\varphi, \psi) = k\varphi$ and $ydx - \frac{\psi}{k\varphi}d\varphi = d[xy(2 - y^k)]$. Hence $ydx = d[xy(2 - y^k)]$ on $\varphi = 1$. This curve is birationally equivalent to the $k$th Fermat curve: $x^k y^{2k}(y^k - 1)^{k-1} = 1$, hence $x^k y^{2k}(y^k - 1)^k = y^k - 1$ and $[xy^2(y^k - 1)]^k = y^k - 1$.

Apparently a description of curves on which the form $ydx$ is exact is not known and possibly is rather complicated. I do not have a conjectural description of these curves but to find one seems to be very interesting.

**Conclusion.**

A reader may ask if it is possible to extract more information from (1) and (2). For example when $\rho\tau > 0$ it is easy to observe that all roots of $q$ must be of multiplicity 1; that all roots of $p$ are also roots of $q$; that $\varsigma = xyh(x^a y^b)$ where $a,\ b$ are relatively prime integers and $h$ is a polynomial and hence $m = l(1+ka),\ n = l(1+kb)$ (e. g. when the right leading edge is vertical then $a = 0$ and $m$ divides $n$); that there is a root of $p$ with multiplicity larger than $\frac{\rho}{\tau}$, this observation was made by Nagata in [Na1] and Vinberg (private communication); and possibly something else which eludes me. The problem is that there are plenty of polynomial solutions even for a more restrictive Davenport equation $ap'r - bpr' = 1$ where $a,\ b$ are positive relatively prime integers both larger than 1 (see [Da], [Sh], [St], [Z]). Similarly there are plenty of forms which satisfy the Dixmier equation (2) when $\rho$ and $\tau$ have different signs. So we cannot eliminate additional edges of $\mathcal{N}(f)$ using only this approach. It is not very surprising, everybody who thought about JC knows of its slippery nature! Clearly a description of curves on which $ydx$ is exact will help, but this question is possibly harder than JC.

**Acknowledgements.**

The author is grateful to the Max-Planck-Institut für Mathematik in Bonn, Germany where he is presently a visitor. While working on this project I was supported by an NSA grant H98230-09-1-0008, by an NSF grant DMS-0904713, a Fulbright fellowship awarded by the United States-Israel Educational Foundation, and a FAPESP grant 2011/52030-5 awarded by the State of São Paulo, Brazil.

Finally, a suggestion by a friendly referee to include a remark on the Jung theorem, which is accepted with gratitude helped to improve the paper.

## References Sited

[A] S. S. Abhyankar, Lectures On Expansion Techniques In Algebraic Geometry, Tata Institute of Fundamental Research, Bombay, 1977.

[AO] H. Appelgate and H. Onishi, The Jacobian conjecture in two variables, J. Pure Appl. Algebra 37 (1985), no. 3, 215-227.

[CN] P. Cassou-Nogus, Newton trees at infinity of algebraic curves. Affine algebraic geometry, 1-19, CRM Proc. Lecture Notes, 54, Amer. Math. Soc., Providence, RI, 2011. (The Russell Festschrift.)

[Da] H. Davenport, On $f^3(t) - g^2(t)$, Norske Vid. Selsk. Forh. (Trondheim) 38 (1965), 86-87.

[Di] J. Dixmier, Sur les algebres de Weyl, Bull. Soc. Math. France 96 (1968), 209–242.

[GK] I. Gelfand, A. Kirillov, Sur les corps lis aux algbres enveloppantes des algbres de Lie. (French) Inst. Hautes tudes Sci. Publ. Math. No. 31 1966 5–19.

[H] R. Heitmann, On the Jacobian conjecture, J. Pure Appl. Algebra 64 (1990), 35–72.

[J] H. W. E. Jung, Über ganze birationale Transformationen der Ebene, J. reine angew. Math., 184 (1942), 161–174.

[K] O.H. Keller, Ganze Cremona-Transformationen, Monatsh. Math. Physik 47 (1939) 299–306.

[L] J. Lang, Jacobian pairs II, J. Pure Appl. Algebra 74 (1991), 61–71.

[ML] L. Makar-Limanov, Commutativity of certain centralizers in the rings $R_{n,k}$. Funk. An. i ego Pril. 4 (1970), 332–33.

[MW] J. McKay, S. Wang, A note on the Jacobian condition and two points at infinity. Proc. Amer. Math. Soc. 111 (1991), no. 1, 35-43.

[M] T. Moh, On the Jacobian conjecture and the configuration of roots, J. reine angew. Math., 340 (1983), 140–212.

[Na1] M. Nagata, Two-dimensional Jacobian conjecture. Algebra and topology 1988 (Taejon, 1988), 77-98, Korea Inst. Tech., Taejon, 1988.

[Na2] M. Nagata, Some remarks on the two-dimensional Jacobian conjecture. Chinese J. Math. 17 (1989), no. 1, 1-7.

[Ne] I. Newton, De methodis serierum et fluxionum, in D. T Whiteside (ed.), The Mathematical Papers of Isaac Newton, Cambridge University Press, Cambridge, vol. 3, 1967-1981, 32–353; pages 43–71.

[NN1] A. Nowicki, Y. Nakai, On Appelgate-Onishi's lemmas. J. Pure Appl. Algebra 51 (1988), no. 3, 305-310.

[NN2] A. Nowicki, Y. Nakai, Correction to: "On Appelgate-Onishi's lemmas" [J. Pure Appl. Algebra 51 (1988), no. 3, 305-310]; J. Pure Appl. Algebra 58 (1989), no. 1, 101.

[O] M. Oka, On the boundary obstructions to the Jacobian problem. Kodai Math. J. 6 (1983), no. 3, 419-433.

[P] V. A. Puiseux, Recherches sur les fonctions algebriques, Journal of Mathematics Pure and Applied 15 (1850) 365–480.

[R] R. Rentschler, Opérations du groupe additif sur le plan affine. (French) C. R. Acad. Sci. Paris Sér. A-B 267 1968 A384-A387.

[Sh] T. Shioda, Elliptic surfaces and Davenport-Stothers triples. Comment. Math. Univ. St. Pauli 54 (2005), no. 1, 49–68.

[St] W. Stothers, Polynomial identities and Hauptmoduln, Quart. J. Math. Oxford Ser. (2) 32 (1981), no. 127, 349-370.

[W] D. Wright, Affine surfaces fibered by affine lines over the projective line, Illinois J. Math. 41 (1997), no. 4, 589–605.

[Z] U. Zannier, On Davenport's bound for the degree of $f^3-g^2$ and Riemann's existence theorem. Acta Arith. 71 (1995), no. 2, 107-137.

Department of Mathematics, Wayne State University, Detroit, MI 48202, USA;

Max-Planck-Institut für Mathematik, 53111 Bonn, Germany;

Department of Mathematics & Computer Science, the Weizmann Institute of Science, Rehovot 76100, Israel;

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA.

*E-mail address*: lml@math.wayne.edu