On the collisions of singular points of plane curves

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Abstract

We study the generic degeneration of curves with two singular points when the points merge. First the notion of generic degeneration is defined precisely. Then a method to classify the possible results of generic degenerations is proposed in the case of linear singularity types. We discuss possible bounds on the singularity invariants of the resulting type in terms of the initial types. In particular the strict upper bound on the resulting multiplicity is proved.

1 The problem

1.1 Introduction

Let C be a (complex, plane, projective) curve of (high) degree d, with singular points $x, y \in \mathbb{P}^2$ of (embedded topological) types $\mathbb{S}_x, \mathbb{S}_y$. What can be said about the *resulting type of their generic collision*?

To formulate the question precisely, let $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(d)) = \mathbb{P}_f^{N_d}$ be the complete linear system (the parameter space of plane curves of degree $d \gg 0$). Here $N_d = \binom{d+2}{2} - 1$ (the number of monomials of degree d in 3 variables, minus one). Consider the stratum $\Sigma_{\mathbb{S}_x\mathbb{S}_y} \subset \mathbb{P}_f^{N_d}$ of curves with 2 prescribed singularities. The restriction of the topological closure $\overline{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y}$ corresponds to the collision of the singular points.

The notion "generic" in this case is problematic. The restriction above is often reducible, with components of different dimensions (all of which might be important). One often has to consider collisions with additional conditions. Say, the tangents l_i^x to (some of) the branches of $\mathbb{S}_x$ (do not) coincide with (some of) those l_j^y of $\mathbb{S}_y$. Or, they are distinct from the limiting tangent line $l = \overline{xy}$ to the curve $\widehat{xy}$, along which the points collide. In such cases one might be forced to consider a subvariety of an irreducible component of $\overline{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y}$.

Therefore, we accept the following definition. For a given singularity type $\mathbb{S}$, consider the classifying space of the parameters of the singular germ (e.g. the singular point, the lines of the tangent cone, with their multiplicities: $T_C = (l_1^{p_1} ... l_k^{p_k})$ etc.) To a curve with two singular points $\mathbb{S}_x\mathbb{S}_y$ we assign also the line l through the 2 points. All this defines a lifting of the initial stratum to a bigger ambient space:

$$\widetilde{\Sigma}_{\mathbb{S}_x\mathbb{S}_y} = \left\{ \begin{pmatrix} (x, \{l_i^x\}..) & l, C \\ (y, \{l_j^y\}..) & x \neq y \end{pmatrix} \middle| \ C \text{ has } \begin{matrix} \mathbb{S}_x \text{ at } x, \text{ with } T = ((l_1^x)^{p_1} ... (l_{k_x}^x)^{p_{k_x}}), \ .... \\ \mathbb{S}_y \text{ at } y, \text{ with } T = ((l_1^y)^{p_1} ... (l_{k_y}^y)^{p_{k_y}}), \ .... \end{matrix} \ l = \overline{xy} \right\} \subset \begin{matrix} Aux_x \times Aux_y \times \\ \times (\mathbb{P}_l^2)^* \times \mathbb{P}_f^{N_d} \end{matrix} \quad (1)$$

here Aux_i are the classifying spaces (the name is for auxiliary), $(\mathbb{P}_l^2)^*$ is the space of lines in the plane (each is defined by a one-form). The simplest example is the minimal lifting (the universal curve)

$$\widetilde{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}(x, y) = \left\{ \begin{matrix} (x, y, l, C) \\ x \neq y \end{matrix} \middle| C \text{ has } \mathbb{S}_x \text{ at } x \text{ and } \mathbb{S}_y \text{ at } y, \ l = \overline{xy} \right\} \subset \mathbb{P}_x^2 \times \mathbb{P}_y^2 \times (\mathbb{P}_l^2)^* \times \mathbb{P}_f^{N_d} \quad (2)$$

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Definition 1.1 *The collision* $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$ *is called primitive (relatively to the specified lifting) if the stratum* $\widetilde{\Sigma}_{\mathbb{S}_f}$ *is one of the irreducible components of* $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y}$.

The collision is primitive iff it cannot be further factorized (i.e. the chain $\widetilde{\Sigma}_{\mathbb{S}_f} \subsetneq \overline{\widetilde{\Sigma}}_{\mathbb{S}_{f'}} \subsetneq \overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y}$ with irreducible $\overline{\widetilde{\Sigma}}_{\mathbb{S}_{f'}}$ is impossible).

Example 1.2 The cases below can be obtained e.g. by the methods of §3.1.
- In the case $A_1 + A_1$ we do not fix the tangent lines (as there is no preferred choice). So the lifting is minimal (eq. 2) and the result is unique: $\overline{\widetilde{\Sigma}}_{A_{1x}A_{1y}}(x,y)|_{x=y} = \overline{\widetilde{\Sigma}}_{A_3}$.
- In the case $A_2 + A_1$ we fix the tangent line of A_2 (denoted by l_x) and consider the lifting $\overline{\widetilde{\Sigma}}_{A_{2x}A_{1y}}(x, l_x, y)$. Now, two primitive collisions are possible: $A_2 + A_1 \to A_4$ (with $l_x = l$) and $A_2 + A_1 \to D_5$ (with $l_x \neq l$). Naively, the first case ($l_x = l$) could be thought of as being the boundary of the second ($l_x \neq l$), but the actual situation is converse (since $\Sigma_{D_5} \subset \overline{\Sigma}_{A_4}$). For the minimal lifting (neglecting the tangent of A_2): $\overline{\widetilde{\Sigma}}_{A_{2x}A_{1y}}(x,y)|_{x=y} = \overline{\widetilde{\Sigma}}_{A_4}$.

As the primitivity of the collision depends on the type of lifting, we fix the choice for the rest of this paper. In the tangent cone of the singularity $T_C = (l_1^{p_1} ... l_k^{p_k})$, consider the lines appearing with the multiplicity 1. They correspond to smooth branches, not tangent to any other branch of the singularity. We call such branches *free*. Call the tangents to the non-free branches: the *non-free tangents*. Assign to the singularity the non-free tangents:

$$\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y} := \overline{\left\{ \begin{pmatrix} (x,\{l_i^x\}) & l, C \\ (y,\{l_j^y\}) & x \neq y \end{pmatrix} \middle| \begin{matrix} l_i^x \text{ are the non-free tangents of } C \text{ at } x \\ l_j^y \text{ are the non-free tangents of } C \text{ at } y \end{matrix} \quad l = \overline{xy} \right\}} \subset \begin{matrix} Aux_x \times Aux_y \times \\ \times (\mathbb{P}^2_l)^* \times \mathbb{P}^{N_d}_f \end{matrix} \qquad (3)$$

For ordinary multiple points (all the branches are free) this coincides with the minimal lifting.

Remark 1.3
- To specify a collision one should give (at least) the *collision data*. It is a list, specifying the lines among l, l_i^x, l_j^y that coincide. The simplest case is: all the lines are distinct. Note that this (seemingly generic) assumption is often *non-generic* (e.g. for the collision $A_{k>1} + A_1$).
- We specify the singularity types by giving their normal form (cf. §2.1) or by the letters from Arnol'd's tables [AGLV].
- We work mostly with *linear* singularity types (cf. definition 2.4). Typical examples of linear singularities are: $x_1^p + x_2^q$, $p \leq q \leq 2p$, $A_{k\leq 3}$, $D_{k\leq 6}$, $E_{k\leq 8}$ etc. Every linear singularity type is necessarily generalized Newton-non-degenerate (cf. definition 2.2), in particular it has at most two non-free tangents.

While the collision phenomenon is most natural, it seems to be complicated and not much studied. Listed below are more specific questions (for a given pair $\mathbb{S}_x\mathbb{S}_y$).

1.2 The specific questions and some partial results

1.2.1 A method/algorithm to classify the results of collision.

We propose a method (cf. §3.1) to check explicitly the possible results of a collision, when $\mathbb{S}_x$ is generalized Newton-non-degenerate and $\mathbb{S}_y$ is linear. First we write down the defining equations of the lifted stratum $\widetilde{\Sigma}_{\mathbb{S}_x\mathbb{S}_y}$ (outside the diagonal $x = y$). Then specialize the so obtained ideal to the diagonal $x = y$, thus describing the ideal of the stratum $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y}$. The specialization (the flat limit) is done e.g. by the usual technic of Gröbner basis. The final step is to recognize the singularity type $\mathbb{S}_f$, from the defining ideal of the stratum $\widetilde{\Sigma}_{\mathbb{S}_f}$.

Using the method we discuss in some details the case: $\mathbb{S}_y$ is an ordinary multiple point(§3.1). In particular we list all the possible collision results for the cases:

• $\mathbb{S}_x$ is an ordinary multiple point(i.e. all its branches are free)
• one branch of $\mathbb{S}_x$ is the cusp $(x_1^p + x_2^{p+1})$, all others are free (i.e. smooth and non-tangent).

There is also a geometric method for some collisions, but these seem to be very special (considered shortly in §3.1.5).

1.2.2 Some bounds on the invariants of the resulting types

1.2.2.1 What are the possible values of the resulting multiplicity? Here we have two results (§3.2.2):

Proposition 1.4 *Given two types $\mathbb{S}_x, \mathbb{S}_y$ of multiplicities $m_x \geq m_y$*
• *There always exists a collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$ with $mult(\mathbb{S}_f) = m_x$*
• *Let r_x, r_y be the number of free branches. If $r_x + r_y \geq m_y$ then for any collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$: $m_{\mathbb{S}_f} = m_x$. If $r_x + r_y < m_y$ then for any collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$: $mult(\mathbb{S}_f) \leq m_x - r_x + m_y - r_y$.*

1.2.2.2 Other invariants. Possible bounds for the resulting number of branches or order of determinacy are difficult as these invariants are not semi-continuous (cf. §3.2.3).
• When is the lower bound $\mu_f = \mu_{\mathbb{S}_x} + \mu_{\mathbb{S}_y} + 1$ realized? What is the upper bound for μ_f?
• How to characterize the $\delta = const$ collisions? (They seem to be especially simple.) What is the upper bound for δ_f?

1.2.3 When does the collision commute with degeneration/deformation?

Namely, when the following diagram commutes? Here the degeneration (deformation) in both rows must be of course "of the same nature" (though applied to the different types, e.g. $A_k \to A_{k+1}$, $D_k \to D_{k+1}$). Eventhough we cannot answer this question, the idea itself leads to a useful criterion (§3.2.1).

$$\begin{tikzcd}
\mathbb{S}_x^{def} + \mathbb{S}_y \arrow[d] & \mathbb{S}_x + \mathbb{S}_y \arrow[l, "deform"'] \arrow[r, "degen"] \arrow[d] & \mathbb{S}_x^{deg} + \mathbb{S}_y \arrow[d] \\
\mathbb{S}^{def} & \mathbb{S} \arrow[l, "deform"'] \arrow[r, "degen"] & \mathbb{S}^{deg}
\end{tikzcd}$$

1.2.4 Topological approach

The curve $C_{\mathbb{S}_x,\mathbb{S}_y}$ can be thought of as a partial smoothing of $C_\mathbb{S}$. Correspondingly, in the smoothing of $C_{\mathbb{S}_x,\mathbb{S}_y}$ we can choose vanishing cycles that will form a subset of vanishing cycles of $C_\mathbb{S}$. Which restrictions does this produce? For example, an ADE singularity $\mathbb{S}$ can split to a collection of points of types $\mathbb{S}_i \in ADE$ iff the union of Dynkin diagrams $D_{\mathbb{S}_i}$ can be obtained from $D_\mathbb{S}$ by deletion of some vertices. This solves completely the problem of $ADE + ADE \to ADE$ collisions (cf. §3.3.1). The natural generalization is therefore:
Given the initial types $\mathbb{S}_x, \mathbb{S}_y$ and a type $\mathbb{S}$, whose Dynkin diagram $\mathbb{D}_\mathbb{S}$ (in some basis) contains $D_{\mathbb{S}_x}, D_{\mathbb{S}_y}$ (separated by at least one vertex). Is the collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}$ possible?

We hope to consider this question later.

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2 Some auxiliary notions and notations

When considering the local questions, we work in the local coordinates (x_1, x_2) around the point. Working with the strata we use the homogeneous coordinates (x_0, x_1, x_2) on $\mathbb{P}^2$. A (projective) line through the point

$x \in \mathbb{P}^2$ is defined by a 1-form l (so that $l \in (\mathbb{P}^2)^*,\ l(x) = 0$).

We often work with symmetric $p-$forms $\Omega^p \in S^p(\widehat{\mathbb{P}^2})^*$ (here $(\widehat{\mathbb{P}^2})^*$ is a 3-dimensional vector space of linear forms). Thinking of the form as of a symmetric tensor with p indices ($\Omega^{(p)}_{i_1,\dots,i_p}$), we write $\Omega^{(p)}(\underbrace{x,\dots,x}_{k})$ as a shorthand for the tensor, multiplied k times by the point $x \in \widehat{\mathbb{P}^2}$

$$\Omega^{(p)}(\underbrace{x,\dots,x}_{k}) := \sum_{0\le i_1,\dots,i_k\le 2} \Omega^{(p)}_{i_1,\dots,i_p} x_{i_1}\dots x_{i_k} \tag{4}$$

So, for example, the expression $\Omega^{(p)}(x)$ is a $(p-1)-$form. Unless stated otherwise, we assume the symmetric form $\Omega^{(p)}$ to be generic (in particular non-degenerate, i.e. the corresponding curve $\{\Omega^{(p)}(\underbrace{x,\dots,x}_{p}) = 0\} \subset \mathbb{P}^2$ is smooth).

Symmetric forms typically occur as tensors of derivatives of order p, e.g. $f^{(p)}$. Sometimes, to emphasize the point at which the derivatives are calculated we assign it. So, e.g. $f|^{(p)}_x(\underbrace{y,\dots,y}_{k})$ means: the tensor of derivatives of order p, calculated at the point x, and contracted k times with y.

2.1 On the singularity types

Definition 2.1 *[GLSbook] Let $(C_x, x) \subset (\mathbb{C}^2_x, x)$ and $(C_y, y) \subset (\mathbb{C}^2_y, y)$ be two germs of isolated curve singularities. They are topologically equivalent if there exist a homeomorphism $(\mathbb{C}^2_x, x) \mapsto (\mathbb{C}^2_y, y)$ mapping (C_x, x) to (C_y, y). The corresponding equivalence class is called the (embedded topological) singularity type. The variety of points (in the parameter space $\mathbb{P}^{N_d}_f$), corresponding to curves with singularity of the same (topological) type $\mathbb{S}$ is called the* **equisingular stratum** $\Sigma_{\mathbb{S}}$

The topological type can be specified by a (simple polynomial) representative of the type: the **normal** form. Several simplest types are (all the notations are from [AGLV], we ignore the moduli of analytic classification):

$$\begin{array}{l} A_k : x_2^2 + x_1^{k+1},\ \ D_k : x_2^2 x_1 + x_1^{k-1},\ \ E_{6k} : x_2^3 + x_1^{3k+1},\ \ E_{6k+1} : x_2^3 + x_2 x_1^{2k+1},\ \ E_{6k+2} : x_2^3 + x_1^{3k+2} \\ J_{k\ge1,i\ge0} : x_2^3 + x_2^2 x_1^k + x_1^{3k+i},\ \ Z_{6k-1} : x_2^3 x_1 + x_1^{3k-1},\ \ Z_{6k} : x_2^3 x_1 + x_2 x_1^{2k},\ \ Z_{6k+1} : x_2^3 x_1 + x_1^{3k} \\ X_{k\ge1,i\ge0} : x_2^4 + x_2^3 x_1^k + x_2^2 x_1^{2k} + x_1^{4k+i},\ \ W_{12k} : x_2^4 + x_1^{4k+1},\ \ W_{12k+1} : x_2^4 + x_2 x_1^{3k+1} \end{array} \tag{5}$$

Using the normal form $f = \sum a_{\mathbf{I}} \mathbf{x}^{\mathbf{I}}$ one can draw the *Newton diagram* of the singularity. Namely, one marks the points $\mathbf{I}$ corresponding to non-vanishing monomials in f, and takes the convex hull of the sets $\mathbf{I} + \mathbb{R}^2_+$. The envelope of the convex hull (the chain of segment-faces) is the Newton diagram.

Definition 2.2 *[GLSbook]*
• *The singular germ is called Newton-non-degenerate with respect to its diagram if the truncation of its polynomial to every face of the diagram is non-degenerate (i.e. the truncated polynomial has no singular points in the torus $(C^*)^2$).*
• *The germ is called generalized Newton-non-degenerate if it can be brought to a Newton-non-degenerate form by a locally analytic transformation.*
• *The singular type is called Newton-non-degenerate if it has a (generalized) Newton-non-degenerate representative.*

For Newton-non-degenerate types the normal form is always chosen to be Newton-non-degenerate . So, the Newton-non-degenerate type $\mathbb{S}$ can be specified by giving the Newton diagram of its normal form $\mathbb{D}_{\mathbb{S}}$.

Newton-non-degeneracy implies strong restrictions on the tangent cone:

Proposition 2.3 *Let $T_C = \{(l_1, p_1)...(l_k, p_k)\}$ be the tangent cone of the germ $C = \cup C_j$ (here all the tangents l_i are different, p_i are the multiplicities, so that $\sum_i p_i = mult(C)$). If the germ is generalized Newton-non-degenerate then $p_i > 1$ for at most two tangents l_i.*

So, for a generalized Newton-non-degenerate germ there are at most two distinguished tangents. We always orient the coordinate axes along these tangents.

As we consider the topological types, one could expect that to bring a germ to the Newton diagram of the normal form, one needs local homeomorphisms. However for curves the locally analytic transformation always suffice. In this paper we restrict consideration further to the types for which only linear transformations suffice.

Definition 2.4 *[Ker06] A (generalized Newton-non-degenerate) singular germ is called linear if it can be brought to the Newton diagram of its type by projective transformations only (or linear transformations in the local coordinate system centered at the singular point). A linear stratum is the equisingular stratum, whose open dense part consists of linear germs. The topological type is called linear if the corresponding stratum is linear.*

The linear types happen to be abundant due to the following observation

Proposition 2.5 *[Ker06, section 3.1] The Newton-non-degenerate topological type is linear iff every segment of the Newton diagram has the bounded slope: $\frac{1}{2} \leq tg(\alpha) \leq 2$.*

Example 2.6 The simplest class of examples of linear singularities is defined by the series: $f = x^p + y^q, \ p \leq q \leq 2p$. In general, for a given series only for a few types of singularities the strata can be linear. In the low modality cases the linear types are:
- Simple singularities (no moduli): $A_{1\leq k\leq 3}, \ D_{4\leq k\leq 6}, \ E_{6\leq k\leq 8}$
- Unimodal singularities: $X_9 (= X_{1,0}), \ J_{10} (= J_{2,0}), \ Z_{11\leq k\leq 13}, \ W_{12\leq k\leq 13}$
- Bimodal: $Z_{1,0}, \ W_{1,0}, \ W_{1,1}, \ W_{17}, \ W_{18}$

Most singularity types are nonlinear. For example if a curve has an A_4 point, the best we can do by projective transformations is to bring it to the Newton diagram of A_3 $a_{0,2}x_2^2 + a_{2,1}x_2x_1^2 + a_{4,0}x_1^4$.

This quasi-homogeneous form is degenerated ($a_{2,1}^2 = 4a_{0,2}a_{4,0}$) and by quadratic (nonlinear!) change of coordinates the normal form of A_4 is achieved.

By the finite determinacy theorem the topological type of the germ is fixed by a finite jet of the defining series. Namely, for every type $\mathbb{S}$, there exists k such that for all bigger $n \geq k$: $jet_n(f_1)$ has type $\mathbb{S}$ iff f_1 has type $\mathbb{S}$. The minimal such k is called: *the order of determinacy.* E.g. $o.d.(A_k) = k + 1$, $o.d.(D_k) = k - 1$. The classical theorem is [GLSbook, §I.2.2]: if $m^{k+1} \subset m^2 Jac(f)$ then $o.d.(f) \leq k$.

3 Partial results

3.1 Explicit calculation of collisions

One way of treating the problem could be to consider explicit equations of the stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}$ and then to restrict them to the diagonal $x = y$. But it is difficult to write down the complete set of the generators of the ideal $I(\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y})$. Instead, we start from the ideals $I(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$, $I(\overline{\overline{\Sigma}}_{\mathbb{S}_y})$ of the coordinate ring $K[Aux_x \times Aux_y \times (\mathbb{P}^2_l)^* \times \mathbb{P}^{N_d}_f]$. Their sum $I(\overline{\overline{\Sigma}}_{\mathbb{S}_x}) \oplus I(\overline{\overline{\Sigma}}_{\mathbb{S}_y})$ defines the stratum $\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}$ outside the diagonal. Over the diagonal the sum does not define the stratum (since the intersection $\overline{\overline{\Sigma}}_{\mathbb{S}_x} \cap \overline{\overline{\Sigma}}_{\mathbb{S}_y}$ has residual components of excess dimension).

One way to continue is to take the topological closure: $\overline{\overline{\overline{\Sigma}}_{\mathbb{S}_x} \underset{x\neq y}{\cap} \overline{\overline{\Sigma}}_{\mathbb{S}_y}}$. From the calculational point of view we should take the flat limit of $I(\overline{\overline{\Sigma}}_{\mathbb{S}_x}) \oplus I(\overline{\overline{\Sigma}}_{\mathbb{S}_y})$ as $y = x + \sum \epsilon^i v_i, \ \ \epsilon \to 0$. This is done e.g. by finding the Gröbner basis [Stev-book, section 2].

Thus the problem is reduced (at least theoretically) to the study of ideals $I(\overline{\overline{\Sigma}}_{\mathbb{S}_x})$, $I(\overline{\overline{\Sigma}}_{\mathbb{S}_y})$. For many singularity types the generators of the ideals are known [Ker06] and can be written in a simple form. These types include the *linear* singularities (cf. the definition 2.4). Examples of such types are $A_{k\leq 3}$, $D_{k\leq 6}$, $E_{k\leq 8}$, $x_1^p+x_2^q$, $p\leq q\leq 2p$...

In fact we attack a more general case: when the type $\mathbb{S}_y$ is linear and $\mathbb{S}_x$ is generalized Newton-non-degenerate . Start from a generalized Newton-non-degenerate type $\mathbb{S}_x$, bring the corresponding germ to a Newton-non-degenerate form by a locally analytic transformation. Since the result of collision is invariant under the locally analytic transformations of $\mathbb{C}^2$, can assume that the germ $\mathbb{S}_x$ is brought to its Newton diagram by linear transformations. Consider the corresponding subvariety $\overline{\overline{\Sigma}}^l_{\mathbb{S}_x}\subset\overline{\overline{\Sigma}}_{\mathbb{S}_x}$ consisting of those germs that can be brought to their Newton diagram by *linear* transformations. (In [Ker06] such a subvariety was called: the linear substratum.) If the type $\mathbb{S}_x$ is originally linear then of course $\overline{\overline{\Sigma}}^l_{\mathbb{S}_x}\equiv\overline{\overline{\Sigma}}_{\mathbb{S}_x}$

3.1.1 The collision trajectory

We always keep the point x and at least one of the non-free tangents to (C,x) fixed. In general y approaches x along a (smooth) curve $\widehat{xy}: y=x+\sum\epsilon^i v_i$. To simplify the problem, one would like to rectify the curve into the line $l=\overline{xy}$ (by a locally analytic transformation preserving the tangents). But our method places severe restrictions on the possible transformations. We assume $\mathbb{S}_y$ to be a linear type, while $\mathbb{S}_x$ is generalized Newton-non-degenerate . To be able to write the defining conditions, the germ (C,x) is assumed to possess the Newton diagram of the type $\mathbb{S}_x$. Correspondingly, only the transformations preserving the diagram are allowed.

• If $\mathbb{S}_x$ is linear then all the transformations preserving the tangents are allowed (i.e. $x_i\to x_i+\phi_i,\quad \phi\in m^2$). In particular, the collision can always be assumed to happen along a line.

• If $\mathbb{S}_x$ is not linear (but generalized Newton-non-degenerate), then we have only the transformations allowed by the diagram. So, if the tangent to $\widehat{xy}$ is distinct from all the non-free tangents of $\mathbb{S}_x$, then the curve $\widehat{xy}$ can be rectified to the line $\overline{xy}$. Otherwise, one can only get an upper bound on the degree of the curve $\widehat{xy}$.

3.1.2 The algorithm

The initial data consists of the two strata $\overline{\Sigma}^l_{\mathbb{S}_x},\overline{\Sigma}_{\mathbb{S}_y}$, with known generators of their ideals:

$$I(\overline{\overline{\Sigma}}_{\mathbb{S}_x})=<\{h_i(x)\}_i>,\qquad I(\overline{\overline{\Sigma}}_{\mathbb{S}_y})=<\{g_i(y)\}_j> \tag{6}$$

Here the points x,y are assigned to emphasize the dependence. (Of course, the generators depend on other parameters of the singularity also.) Fix the collision data of the types $\mathbb{S}_x,\mathbb{S}_y$: l^x_i,l^y_j,l.

3.1.2.1 The series. Expand $y=x+\sum_i\epsilon^i v_i$. Here ϵ is an infinitesimal parameter, while the vectors v_i define the way of collision. The collision in general happens along a (smooth) curve and higher order expansion parameters of the curve can be important (e.g. this is the case in A_k+A_1 collision). Expand, all the generators $g_j(y)$ into power series of ϵ, i.e. $g_j(y)=g_j(x)+\epsilon()+....$ Here we take into account the equations of $\mathbb{S}_x$. Depending on the collision data, some additional terms in the series $g(y)$ can vanish.

3.1.2.2 Gröbner basis. Apply now the Gröbner basis procedure for the terms: $f_i(x),g_j(y)$. We work outside the diagonal (in the ring $K[[Aux_x\times Aux_y\times(\mathbb{P}^2_l)^*\times\mathbb{P}^{N_d}_f,\epsilon,\epsilon^{-1}]]$). Therefore, each time we get a series with a common factor of ϵ we divide by ϵ.

The process depends in general on the (non-)coincidence of various tangents to the branches, the collision line $\bar{xy}$ (i.e. the tangent to the collision curve), the conic osculating to the collision curve etc.

Note that the initial system of generators $f_i(x),g_j(y)$ has a lot of structure (cf. the example §), various equations are combined into some symmetric forms. Preserving this structure helps to recognize the resulting types. By the general theory, after a finite number of steps the procedure terminates: the standard basis have been constructed.

Now take the limit $\epsilon\to 0$, omitting all the higher order terms. The so obtained system is the system of generators of the ideal $I(\overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y})$.

3.1.2.3 Recognition of the final singularity type $\mathbb{S}_f$. The resulting equations are in terms of the tangents l^x_i, l^y_j to $\mathbb{S}_x, \mathbb{S}_y$ and parameters of the collision trajectory $y = x + \sum \epsilon^i v_i$. If some of the tangents l^x_i, l^y_j coincide, then we should also consider the way they approach: $l^y_j = l^x_j + \sum \epsilon^i w_i$. Note, that all the initial equations $f_i(x), g_j(y)$ are linear in f, since we work with linear (sub)strata. Therefore all the resulting equations are also linear in f.

- The simplest case is when the initial system involves only l^x_i, l^y_j, v, f (e.g. for $\mathbb{S}_x\mathbb{S}_y$-linear). Then the resulting stratum is *linear*. Thus the singular type is easy to recognize (to write the normal form, to draw the Newton diagram etc.).
- When parameters of the expansions $y = x + \sum \epsilon^i v_i$, $l^y_j = l^x_j + \sum \epsilon^i w_i$ appear explicitly in the equations, the situation is more complicated. One possible way is to fix some specific values of the parameters and construct the resolution tree.

3.1.3 A generalized Newton-non-degenerate singularity $\mathbb{S}_x$ and the ordinary multiple point $\mathbb{S}_y = x_1^{q+1} + x_2^{q+1}$

Here we assume $mult(\mathbb{S}_x) = p + 1 \geq mult(\mathbb{S}_y) = q + 1$ and the collision data is generic, i.e. the curve $\widehat{xy}$ is not tangent to any of the non-free branches of $\mathbb{S}_x$. Thus (cf. §3.1.1) the curve $\widehat{xy}$ can be assumed to be a line: $\widehat{xy} = \overline{xy} = l$.

We should translate the conditions at the point y to conditions at x. Outside the diagonal $x = y$ the stratum is defined by the set of conditions corresponding to $\overline{\overline{\Sigma}}_{\mathbb{S}_x}$, and by the condition $f|^{(q)}_y = 0$. This is the (symmetric) form of derivatives of order q, calculated at the point y (in projective coordinates). In the neighborhood of x expand $y = x + \sum_i \epsilon^i v_i$ (here ϵ is small and v_1 is the direction along the line $l = \bar{xy}$). Since we have assumed that the collision data is generic, in the above expansion we need only the first term: $y = x + \epsilon v$.

To take the flat limit, expand $f|^{(q)}_y$ around x, we get $0 = f|^{(q)}_y = f|^{(q)}_x + .. + \frac{\epsilon^{p-q}}{(p-q)!} f|^{(p)}_x (v..v) + ...$ First several terms in the expansion vanish, up to the multiplicity of $\mathbb{S}_x$. Normalize by common factor of ϵ:

$$\frac{1}{(p-q+1)!} f|^{(p+1)}_x (\underbrace{v..v}_{p+1-q}) + \frac{\epsilon}{(p-q+2)!} f|^{(p+2)}_x (\underbrace{v..v}_{p+2-q}) + \frac{\epsilon^2}{(p-q+3)!} f|^{(p+3)}_x (\underbrace{v..v}_{p+3-q}) + \dots \qquad (7)$$

To take the flat limit, we should find all the syzygies between these series and the equations for $\overline{\overline{\Sigma}}_{\mathbb{S}_x}$. First we find the "internal" syzygies of the series themselves.

The syzygies are obtained as a consequence of the Euler identity for homogeneous polynomial $\sum x_i \partial_i f = deg(f) f$. By successive contraction of the tensor series with x we get the series

$$\begin{array}{l} \frac{1}{(p-q+1)!} f|^{(p+1)}_x (\underbrace{v..v}_{p+1-q}) + \frac{\epsilon}{(p-q+2)!} f|^{(p+2)}_x (\underbrace{v..v}_{p+2-q}) + \frac{\epsilon^2}{(p-q+3)!} f|^{(p+3)}_x (\underbrace{v..v}_{p+3-q}) + \frac{\epsilon^3}{(p-q+4)!} f|^{(p+4)}_x (\underbrace{v..v}_{p+4-q}) +.. \\ \frac{(d-p-2)}{(p-q+2)!} f|^{(p+1)}_x (\underbrace{v..v}_{p+2-q}) + \frac{\epsilon(d-p-3)}{(p-q+3)!} f|^{(p+2)}_x (\underbrace{v..v}_{p+3-q}) + \frac{\epsilon^2(d-p-4)}{(p-q+4)!} f|^{(p+3)}_x (\underbrace{v..v}_{p+4-q}) + \frac{\epsilon^3(d-p-5)}{(p-q+5)!} f|^{(p+4)}_x (\underbrace{v..v}_{p+5-q}) +.. \\ \quad .. \qquad\qquad .. \qquad\qquad .. \qquad\qquad .. \\ \frac{\prod_{i=2}^{q+1}(d-p-i)}{(p+1)!} f|^{(p+1)}_x (\underbrace{v..v}_{p+1}) + \frac{\epsilon \prod_{i=2}^{q+1}(d-p-1-i)}{(p+2)!} f|^{(p+2)}_x (\underbrace{v..v}_{p+2}) + \frac{\epsilon^2 \prod_{i=2}^{q+1}(d-p-2-i)}{(p+3)!} f|^{(p+3)}_x (\underbrace{v..v}_{p+3}) + \frac{\epsilon^3 \prod_{i=2}^{q+1}(d-p-3-i)}{(p+4)!} f|^{(p+4)}_x (\underbrace{v..v}_{p+4}) +.. \end{array} \qquad (8)$$

Here the first row is the initial series, the second is obtained by contraction with x once, the $p+2$'th row is obtained by contracting $(p+1)$ times with x.

Apply now the Gaussian elimination, to bring this system to the upper triangular form.

- Eliminate from the first column all the entries of the rows $2..(p+2)$. For this contract the first row sufficient number of times with v (fix the numerical coefficient) and subtract.
- Eliminate from the second column all the entries of the rows $3..(p+2)$.
- ...

Normalize the rows (i.e. divide by the necessary power of ϵ).

In this way we get the "upper triangular" system of series (we omit the numerical coefficients):

$$\begin{array}{ccccccccc}
f|_x^{(p+1)}(\underbrace{v..v}_{p+1-q}) & + & \epsilon f|_x^{(p+2)}(\underbrace{v..v}_{p+2-q}) & + & \epsilon^2 f|_x^{(p+3)}(\underbrace{v..v}_{p+3-q}) & + & \epsilon^3 f|_x^{(p+4)}(\underbrace{v..v}_{p+4-q}) & +.. & \\
0 & + & f|_x^{(p+2)}(\underbrace{v..v}_{p+3-q}) & + & \epsilon f|_x^{(p+3)}(\underbrace{v..v}_{p+4-q}) & + & \epsilon^2 f|_x^{(p+4)}(\underbrace{v..v}_{p+5-q}) & +.. & \\
0 & + & 0 & + & f|_x^{(p+3)}(\underbrace{v..v}_{p+5-q}) & + & \epsilon f|_x^{(p+4)}(\underbrace{v..v}_{p+6-q}) & +.. & \\
.. & .. & .. & .. & & & & & \\
0 & + & 0 & + & 0 & + & \cdots & & + f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q+1}) +..
\end{array} \tag{9}$$

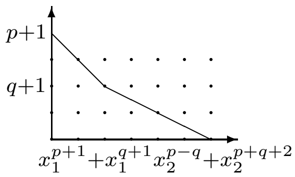
There are no more "internal" syzygies, i.e. we have obtained the Gröbner basis for the initial system (7).

Now the generators of $I(\overline{\widetilde{\Sigma}}_{\mathbb{S}_x})$ should be added and one checks again for the possible syzygies.

Example 3.1 $\mathbb{S}_x = x_1^{p+1} + x_2^{p+1}$ The defining equations of the stratum $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x}(x)$ are: $f|_x^{(p)} = 0$ (as there are no non-free branches the lifting is minimal). Therefore, there are no more syzygies, so just take the limit $\epsilon \to 0$ (i.e. omit the higher order terms in each row). Finally, we get the defining system of equations:

$$f|_x^{(p)} = 0, \quad f|_x^{(p+1)}(\underbrace{v..v}_{p+1-q}) = 0, \quad f|_x^{(p+2)}(\underbrace{v..v}_{p+3-q}) = 0, \quad f|_x^{(p+3)}(\underbrace{v..v}_{p+5-q}) = 0 \ \ .., f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q+1}) = 0 \tag{10}$$

Corollary 3.2 For the lifting $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}(x,y)$ there exists only one primitive collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$ with the final type having the normal form $(x_1^{p-q} + x_2^{p-q})(x_1^{q+1} + x_2^{2q+2})$.



Indeed, as was emphasized, the system is linear in f, so it defines a linear (sub)stratum $\overline{\widetilde{\Sigma}}^l_{\mathbb{S}_f}$. We can obtain the Newton diagram of the resulting type by fixing (in projective coordinates) e.g. $x = (0,0,1)$, $v = (0,1,0)$. Since all the slopes of the diagram are bounded we get that the type is linear and $\overline{\widetilde{\Sigma}}^l_{\mathbb{S}_f} = \overline{\widetilde{\Sigma}}_{\mathbb{S}_f}$.

In several simplest cases we have: $A_1 + A_1 \to A_3$, $D_4 + A_1 \to D_6$, $X_9 + A_1 \to X_{1,2}$, $D_4 + D_4 \to J_{10}$, $X_9 + D_4 \to Z_{13}$.

Remark 3.3 If the curve $\widehat{xy}$ is tangent to one of the non-free branches of $\mathbb{S}_x$, then the system (9) should be re-derived. When $\mathbb{S}_x$ is linear, we can assume that $\widehat{xy} = \overline{xy} = l$, this greatly simplifies the calculations.

Example 3.4 $\mathbb{S}_x = x_1^{p+1} + x_2^{p+2}$ Now the result of collision depends on the (non)coincidence of the line $l = \overline{xy}$ with the tangent line l_x to $\mathbb{S}_x$. The lifted stratum $\overline{\widetilde{\Sigma}}_{\mathbb{S}_x}$ is defined by the condition (cf. [Ker06]) $f|_x^{(p+1)} \sim \underbrace{l_x \times .. \times l_x}_{p+1}$, this can be written also as $f|_x^{(p+1)}(v_x) = 0$.

Proposition 3.5 *For $l \neq l_x$ the (only) resulting type is $(x_1^{p+1-q} + x_2^{p+1-q})(x_1^{q+1} + x_2^{2q+1})$. For $l = l_x$ the (only) resulting type is $(x_1^{p-q} + x_2^{p+1-q})(x_1^{q+1} + x_2^{2q+2})$.*

proof: As $\mathbb{S}_x, \mathbb{S}_y$ are linear, can assume that the trajectory is a line: $l = \overline{xy}$.

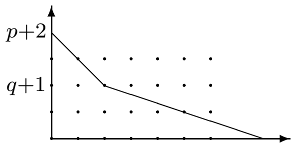
- $l_x \neq l$. Contract the first row of (9) with v_x. The ϵ^0 term vanish and the whole series is divided by ϵ.

So, we get: $0 = f|_x^{(p+2)}(\underbrace{v..v}_{p+2-q}\ v_x) + \ldots$. Contract this series with v and subtract from the third row of (9) (contracted with v_x). Apply the same procedure, up to the last row. Direct check shows that there are no more syzygies, so substitute $\epsilon = 0$ and get

$$\begin{array}{l} f|_x^{(p+1)} \sim \underbrace{l_x \times .. \times l_x}_{p+1}, \quad f|_x^{(p+1)}(\underbrace{v..v}_{p+1-q}) = 0, \quad f|_x^{(p+2)}(\underbrace{v..v}_{p+3-q}) = 0, \quad f|_x^{(p+2)}(\underbrace{v..v}_{p+2-q}, v_x) = 0, \\ f|_x^{(p+3)}(\underbrace{v..v}_{p+5-q}) = 0, \quad f|_x^{(p+3)}(\underbrace{v..v}_{p+4-q}\ v_x) = 0, \quad .., f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q+1}) = f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q}\, v_x) = 0 \end{array} \tag{11}$$

which gives (since $v_x \neq v$ and $l_x(v) \neq 0$):

$$f|_x^{(p+1)} = 0, \quad f|_x^{(p+2)}(\underbrace{v..v}_{p+2-q}) = 0, f|_x^{(p+3)}(\underbrace{v..v}_{p+4-q}) = 0, \ldots, f|_x^{(p+q+1)}(\underbrace{v..v}_{p+q}) = 0 \tag{12}$$

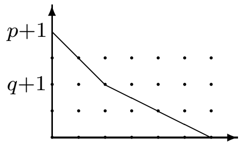


$x_1^{p+2}+x_1^{q+1}x_2^{p+1-q}+x_2^{p+q+2}$

From here we get the normal form: $(x_1^{p+1-q} + x_2^{p+1-q})(x_1^{q+1} + x_2^{2q+1})$.

• $l_x = l$. In this case the system should be re-derived, starting from eq. (7). Everything is just shifted ($p \to p+1$) and we get the equations:

$$\begin{array}{l} f|_x^{(p)} = 0, \quad f|_x^{(p+1)}(\underbrace{v..v}_{p+1-q}) = 0, \quad f|_x^{(p+2)}(\underbrace{v..v}_{p+2-q}) = 0, \\ f|_x^{(p+3)}(\underbrace{v..v}_{p+4-q}) = 0 \quad .., f|_x^{(p+q+2)}(\underbrace{v..v}_{p+q+2}) = 0 \end{array} \tag{13}$$



$x_1^{p+1}+x_1^{q+1}x_2^{p+1-q}+x_2^{p+q+3}$

This gives the normal form of the singularity $(x_1^{p-q} + x_2^{p+1-q})(x_1^{q+1} + x_2^{2q+2})$

3.1.4 More general case

If $\mathbb{S}_x$ is not an ordinary multiple point, then to the conditions of the system (9), one adds the conditions of $\mathbb{S}_x$ and checks for possible additional syzygies.

In some cases there are no new syzygies. For example, let the tangent cone of $\mathbb{S}_x$, with multiplicities be $T_{C_x} = \{l_1^{p_1}..l_k^{p_k}\}$, such that $\forall i: \ p_i \leq p+1-q$. Consider the primitive collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$ such that the collision line l is distinct from all the tangents with $p_i > 1$. Then the defining ideal of the resulting stratum is especially simple:

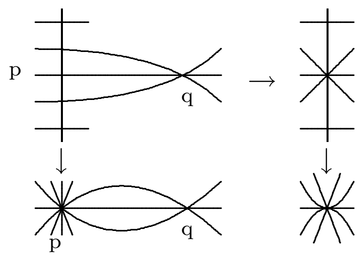
$$I(\overline{\Sigma}_{\mathbb{S}_f}) = < I(\overline{\Sigma}_{\mathbb{S}_x}), \lim_{\epsilon \to 0} I_\epsilon > \tag{14}$$

here I_ϵ is the ideal of the equation (9).

3.1.5 Geometric approach

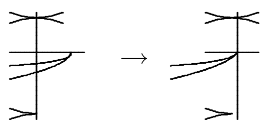
Sometimes the collision can be traced explicitly on the blown up plane. Namely, blow up the plane at x (if needed, do it several times, for example resolve the germ (C, x)). Assume that all the branches of (C, y) intersect one component E of the exceptional divisor.

Then push the point y to E, contracting some parts of the curve. One gets a curve on the blown up plane, with a singular point (of the type $\mathbb{S}_y$) on E. Now, blow down (i.e. contract all the exceptional divisors). This gives the resulting germ.



Example 3.6 The collision of two ordinary multiple points. Suppose, the multiplicities of $\mathbb{S}_x, \mathbb{S}_y$ are $p+1, q+1$ such that $p \geq q$. Blowup at x, push y to the exceptional divisor, then blowdown, as in the picture.

More generally, suppose the number of free branches for the type $\mathbb{S}_x$ is at least the multiplicity of $\mathbb{S}_y$. Use the same procedure as above, to get the final answer. The restrictions of this approach are evident: the primitive collision can be traced for some special types only. In addition, working with real pictures we necessarily loose information.



3.2 Bounds on invariants

3.2.1 Semi-continuity principle

This principle allows to reduce some general questions to the collisions of more restricted types.

Proposition 3.7 *Let inv be an invariant of the singularity type, upper semi-continuous (i.e. non-increasing under the deformations).*

• *Let $S_y \to S'_y$ be a degeneration and $S_x + S'_y \to S'_f$ a primitive collision. Then there exists a primitive collision $S_x + S_y \to S_f$ and a degeneration, such that the diagram commutes. In particular, $inv(S'_f) \geq inv(S_f)$.*

$$\begin{tikzcd} \mathbb{S}_y \arrow[r, "degen"] \arrow[d, dotted, "\mathbb{S}_x+"'] & \mathbb{S}_y{}' \arrow[d, "+\mathbb{S}_x"] \\ \mathbb{S}_f \arrow[r, dotted, "degen"] & \mathbb{S}_f{}' \end{tikzcd}$$

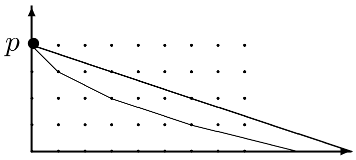
• *Under the assumptions above, suppose for every primitive collision $S_x + S'_y \to S'_f$ the bound $inv(S'_f) \leq a$ is satisfied. Then for any primitive collision $S_x + S_y \to S_f$: $inv(S_f) \leq a$.*

proof: • We should prove the existence of a type $\mathbb{S}_f$ such that $\Sigma_{\mathbb{S}_f{}'} \subset \overline{\Sigma}_{\mathbb{S}_f}$. But this is obvious due to the inclusion of ideals: $I(\mathbb{S}_x) \oplus I(\mathbb{S}_y) \subset I(\mathbb{S}_x) \oplus I(\mathbb{S}_y{}')$.
• Let the degeneration $\overline{\overline{\Sigma}}_{\mathbb{S}_y} \to \overline{\overline{\Sigma}}_{\mathbb{S}'_y}$ be done by applying the equations $\{f_i\}$. So, $I(\overline{\overline{\Sigma}}_{\mathbb{S}'_y}) =< I(\overline{\overline{\Sigma}}_{\mathbb{S}_y}), f_1 ... >$. Every collision $S_x + S'_y \to S'_f$ is described by the flat limit: $< I(\overline{\overline{\Sigma}}_{\mathbb{S}_x}), I(\overline{\overline{\Sigma}}_{\mathbb{S}_y}), f_1 ... > \underset{y\to x}{\longrightarrow} I(\overline{\overline{\Sigma}}_{\mathbb{S}'_f})$. The collision $S_x + S_y \to S_f$ is described by $< I(\overline{\overline{\Sigma}}_{\mathbb{S}_x}), I(\overline{\overline{\Sigma}}_{\mathbb{S}_y}) > \underset{y\to x}{\longrightarrow} I(\overline{\overline{\Sigma}}_{\mathbb{S}_f})$. This gives the inclusion $\overline{\overline{\Sigma}}_{\mathbb{S}'_f} \subset \overline{\overline{\Sigma}}_{\mathbb{S}_f}$ for at least one type $\mathbb{S}'_f$. The proposition follows by the semi-continuity of the invariant. ■
A useful consequence is the possibility to consider only linear sub-strata. Namely, let $\Sigma^{(l)}_{\mathbb{S}_x\mathbb{S}_y} \subset \Sigma_{\mathbb{S}_x\mathbb{S}_y}$ be a linear substratum. Then $\overline{\overline{\Sigma}}^{(l)}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y} \subset \overline{\overline{\Sigma}}_{\mathbb{S}_x\mathbb{S}_y}|_{x=y}$ and all the lower bounds for semi-continuous invariants of $\Sigma^{(l)}_{\mathbb{S}_x\mathbb{S}_y}$ are satisfied for $\Sigma_{\mathbb{S}_x\mathbb{S}_y}$.

3.2.2 Multiplicity

Proposition 3.8 *For any initial types $\mathbb{S}_x, \mathbb{S}_y$ there exists a primitive collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$ with the resulting multiplicity: $mult(\mathbb{S}_f) = max(mult(\mathbb{S}_x), mult(\mathbb{S}_y))$.*

proof: Use the semi-continuity principle. First degenerate each of $\mathbb{S}_x, \mathbb{S}_y$ to a uni-branched Newton-non-degenerate type (preserving multiplicities). This can always be done as follows.
Force all the tangents of a given germ to coincide. If the so obtained germ is not Newton-non-degenerate with respect to its Newton diagram , kill all the necessary monomials, preserving the multiplicity. (This is always possible by standard arguments from [AGLV, section III.3]). If the so-obtained germ is not semi-quasi-homogeneous remove the necessary monomials, preserving x_1^p.



So, we have arrived to the semi-quasi-homogeneous germs, of the types $\mathbb{S}_x{}'$: $x_1^{p_x} + x_2^{q_x}$, and $\mathbb{S}_y{}'$: $x_1^{p_y} + x_2^{q_y}$. Collide them such that all the tangents coincide (i.e. $l_x = l = l_y$). Immediate application of the collision algorithm gives that the multiplicity of the resulting type is $max(mult(\mathbb{S}_x{}'), mult(\mathbb{S}_y{}'))$. Now invoke the semi-continuity principle. ■
In general the situation is much more complicated, multiplicity can jump significantly. This happens when the collision line l and all the non-free tangents are distinct. However there is always the following bound:

Proposition 3.9 *Let the initial type $\mathbb{S}_x\mathbb{S}_y$ have the multiplicities m_x, m_y and the numbers of free branches r_x, r_y. If $r_x + r_y \geq m_y$, then for any collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$: $m_{\mathbb{S}_f} = m_x$. If $r_x + r_y < m_y$, then for any collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$: $mult(\mathbb{S}_f) \leq m_x - r_x + m_y - r_y$.*

proof: We degenerate the types (preserving the multiplicities and the number of free branches) and then apply the semi-continuity principle.

• Degenerate both $\mathbb{S}_x$ and $\mathbb{S}_y$ to generalized Newton-non-degenerate types;

$$\mathbb{S}_x \to x_1^{m_x} + x_1^{m_x - r_x} x_2^{r_x} + x_2^{N_x}, \quad N_x \gg 0, \qquad \mathbb{S}_y \to x_1^{m_y} + x_1^{m_y - r_y} x_2^{r_y} + x_2^{N_y}, \quad N_y \gg 0 \tag{15}$$

• Degenerate $\mathbb{S}_y$ to the form $x_1^{m_x} + x_1^{m_y - r_y} x_2^{m_x - m_y + r_y} + x_2^{N_y}$

• By the semi-continuity, can assume both of the degenerated germs to be linear, i.e. we consider the linear substrata $\Sigma^{(l)}_{\mathbb{S}_x'} \subset \Sigma_{\mathbb{S}_x'}$ and $\Sigma^{(l)}_{\mathbb{S}_y'} \subset \Sigma_{\mathbb{S}_y'}$. Thus can write the defining conditions of the stratum $\widetilde{\Sigma}^{(l)}_{\mathbb{S}_x\mathbb{S}_y}$ (outside the diagonal $x = y$) explicitly:

$$f|_x^{(m_x+k)} \sim (A_{r_x+k+\delta^x_k}, \underbrace{l_x.l_x}_{m_x - r_x - \delta^x_k}), \quad k = 0,1...N_x f|_y^{(m_x+k)} \sim (A_{r_y+m_x-m_y+k+\delta^y_k}, \underbrace{l_y.l_y}_{m_y - r_y - \delta^y_k}), \quad k = 0,1...N_y \tag{16}$$

So, if $r_x + r_y \geq m_y$, the conditions for $k = 0$ can be resolved without increasing the multiplicity: $f|_x^{(m_x)} \sim (A_{r_x+r_y-m_y}, \underbrace{l_x.l_x}_{m_x - r_x}, \underbrace{l_y.l_y}_{m_y - r_y - \delta^y_k})$. From the equation above it is seen that all further conditions (with $k > 0$) do not increase the multiplicity. So the final multiplicity is m_x.

If $r_x + r_y < m_y$ then necessarily $f|_x^{(m_x)} = 0 = f|_x^{(m_x+1)} = ... = f|_x^{(m_x+m_y-r_x-r_y-1)}$, while the conditions for $f|_x^{(m_x+m_y-r_x-r_y)}$ can be resolved in the form $f|_x^{(m_x+m_y-r_x-r_y)} \sim (A_{**}, \underbrace{l_x.l_x}_{**}, \underbrace{l_y.l_y}_{***})$. As previously, it follows that all the higher order conditions can be resolved also. ■

Note that this bound is sharp, e.g. it is realized in the collision of $x_1^{m_x} + x_1^{m_x - r_x} x_2^{r_x} + x_2^{N_x}$ and $x_1^{m_x} + x_1^{m_y - r_y} x_2^{m_x - m_y + r_y} + x_2^{N_y}$ (as in the proof), with N_x, N_y big enough. But it is not the best possible, e.g. when there are distinct non-free tangents, the bound probably could be improved.

3.2.3 Not semi-continuous invariants

For such invariants the semi-continuity principle is not valid, so it is difficult to give any bounds.

Number of branches

For the lower bound we can only propose the conjecture:

for a primitive collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f$: $r_{\mathbb{S}_f} \geq \min(r_{\mathbb{S}_x}, r_{\mathbb{S}_y})$ and $r_{\mathbb{S}_f} \geq |r_{\mathbb{S}_x} - r_{\mathbb{S}_y}|$.

Regarding the upper bound we give an example.

Example 3.10 Consider the primitive collision of two *uni-branched* germs $(x_1^p + x_2^{p+1})$, $(x_1^2 + x_2^3)$, with all 3 lines different $l_x \neq l \neq l_y$. The resulting type is $(x_1^{p-2} + x_2^{p-2})(x_1^3 + x_2^4)$ (with $p - 1$ branches).

So a possible bound on the number of branches should necessarily involve the multiplicities.

Order of determinacy. We only can propose a natural conjecture: $\delta_{\mathbb{S}_f} \leq \delta_{\mathbb{S}_x} + \delta_{\mathbb{S}_y}$

3.2.4 $\delta = const$ collisions

Proposition 3.11

• *Suppose there exists a $\delta = const$ collision $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f'$. Then there exists a primitive $\delta = const$ collision that factors the original: $\mathbb{S}_x + \mathbb{S}_y \to \mathbb{S}_f \stackrel{degeneration}{\to} \mathbb{S}_f'$*

• *Let r_x, r_y be the (total) number of branches of $\mathbb{S}_x \mathbb{S}_y$. For a $\delta = const$ collision: $r_{\mathbb{S}_f} = r_x + r_y - (\mu_{\mathbb{S}_f} + 1 - \mu_x - \mu_y)$. In particular, $r_{\mathbb{S}_f} \le r_{\mathbb{S}_x} + r_{\mathbb{S}_y} - 2$.*

proof: The first statement follows from semi-continuity. The second from the classical formula $\delta = \frac{\mu+r-1}{2}$ and the necessary inequality $\mu_{\mathbb{S}_f} \ge \mu_{\mathbb{S}_x} + \mu_{\mathbb{S}_y} + 1$. ■

3.3 Examples

3.3.1 ADE+ADE→ADE

By the analysis of Dynkin diagrams and by applying the above algorithm we get the following collisions:

$$
\begin{tikzcd}
A_k + A_l \arrow[r] \arrow[dr] & A_{k+l+1} \arrow[d, squiggly] \\
 & D_{k+l+2}
\end{tikzcd}
\quad
\begin{tikzcd}
A_k + A_3 \arrow[r] \arrow[d] \arrow[dr] & A_{k+4} \\
E_{k+4} & D_{k+4}
\end{tikzcd}
\quad
\begin{tikzcd}
A_k + A_1 \arrow[r] \arrow[dr] & A_{k+2} \\
 & E_{k+2}
\end{tikzcd}
\quad
\begin{tikzcd}
A_3 + A_2 \arrow[r] \arrow[d] \arrow[dr, squiggly] & A_6 \arrow[d, squiggly] \\
D_6 \arrow[r, squiggly] & E_7
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_4 + A_2 \arrow[r] \arrow[d] \arrow[dr, squiggly] & A_7 \arrow[d, squiggly] \\
E_7 \arrow[r, squiggly] & D_8
\end{tikzcd}
\quad
\begin{tikzcd}
D_5 + A_k \arrow[r] \arrow[d] & D_{5+k+1} \\
E_{5+k+1} &
\end{tikzcd}
\quad
\begin{tikzcd}
E_6 + A_1 \arrow[r] & E_8 \\
D_k + A_l \arrow[r] & D_{k+l+1}
\end{tikzcd}
\tag{17}
$$

The collisions corresponding to the straight arrows are generic (this can be seen e.g. by codimension or Milnor number). Wavy arrows indicate the non-generic collision or degeneration. For the types E_k, we assume $6 \le k \le 8$

3.3.2 The D_k collisions for some lower cases

$$
D_4 + D_4 \to J_{10}, \quad
\begin{tikzcd}
D_4 + D_5 \arrow[r] \arrow[dr] & X_{1,2}, \quad \mu = 11 \\
 & J_{2,1}, \quad \mu = 11
\end{tikzcd}
\quad
\begin{tikzcd}
D_4 + D_6 \arrow[r] \arrow[dr] & X_{1,2}, \quad \mu = 11 \\
 & J_{2,2}, \quad \mu = 12
\end{tikzcd}
\tag{18}
$$

3.3.3 Some other results

	$\mathbb{S}_x$	$\mathbb{S}_y$	$\mathbb{S}_f$	$\mathbb{S}_f$
	$x_1^{p+1}+x_2^{p+1}$	$x_1^{q+1}+x_2^{q+1}$	$(x_1^{p-q}+x_2^{p-q})(x_1^{q+1}+x_2^{2q+2})$	
μ	p^2	q^2	p^2+q^2+q	
δ	$\frac{p^2+p}{2}$	$\frac{q^2+q}{2}$	$\frac{p(p+1)+q(q+1)}{2}$	
	$x_1^{p}+x_2^{p+1}$	$x_1^{q+1}+x_2^{q+1}$	$(x_1^{p-1-q}+x_2^{p-q})(x_1^{q+1}+x_2^{2q+2})$	$(x_1^{p-q}+x_2^{p-q})(x_1^{q+1}+x_2^{2q+1})$
μ	p^2-p	q^2	$p^2-p+(q+1)^2$	p^2+q^2
δ	$\frac{p^2-p}{2}$	$\frac{q^2+q}{2}$	$\frac{p(p-1)+(q+1)(q+2)}{2}$	$\frac{p(p+1)+q(q-1)}{2}$
	$\prod_{i=1}^r l_i(x_1^p+x_2^{p+1})$	$x_1^{q+1}+x_2^{q+1}$	$p\le q:$ $(x_1^{p+r-q-1}+x_2^{p+r-q-1})(x_1^{q+1}+x_2^{2q+2})$	$q\ge r:$ $(x_1^{p+r-q}+x_2^{p+r-q})(x_1^{q-r+1}+x_2^{2q-2r+1})(x_1^r+x_2^{2r})$ $q<r:$ $(x_1^{p+1}+x_2^{p})(x_1^{r-q}+x_2^{p+r+1-q})(x_1^q+x_2^{2q})$ $(p+r)^2+q^2-r$
μ	p^2-p-1	q^2		
δ	$\frac{p^2-p}{2}$	$\frac{q^2+q}{2}$		
$p\ge 3$	$(x_1+x_2^2)(x_1^{p-2}+x_2^p)$	$x_1^2+x_2^2$	$x_2(x_1^2+x_2^4)(x_1^{p-3}+x_2^{p-2})$	$x_2(x_1+x_2^2)(x_1^{p-1}+x_2^{p-1})$
μ	$(p-1)^2+1$	q^2		
δ	$\lceil\frac{p^2}{2}\rceil-p+1$	$\frac{q^2+q}{2}$		
	$(x_1+x_2^2)(x_1^{p-1}+x_2^p)$	$x_1^2+x_2^2$	$(x_1^{p-2}+x_2^{p-1})(x_1^2+x_2^5)$	$x_2(x_1^{p-2}+x_2^{p-2})(x_1^2+x_2^3)$
μ	p^2-p+1	q^2		
δ	$\frac{p^2-p+2}{2}$	$\frac{q^2+q}{2}$		

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