NONCOMMUTATIVE TWO-DIMENSIONAL TOPOLOGICAL FIELD THEORIES AND HURWITZ NUMBERS FOR REAL ALGEBRAIC CURVES

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Contents

Abstract

It is well-known that classical two-dimensional topological field theories are in one-to-one correspondence with Frobenius algebras. We define topological field theories for non necessary oriented surfaces possibly with a boundary and call them Klein topological field theories. We prove that Klein topological field theories bijectively correspond

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to a more complicated algebraic structure which we call a structure algebra.

A structure algebra is a non-commutative analogue of a Frobenius algebra. Semisimple structure algebras are classified in the work. Starting from an arbitrary finite group we construct a structure algebra and prove that it is semisimple one.

We prove that classical Hurwitz numbers of complex algebraic curves and their analogues for real algebraic curves have natural interpretation in terms of structure constants of a structure algebra for symmetric group.

Key words: topological field theory, Frobenius algebra, Hurwitz numbers.
MSC: 16W, 57M, 81T

1. Introduction

The notion of a topological field theory (TFT) was introduced in works [35, 4, 33]. TFTs play important role both in pure mathematics and in mathematical physics. In particular, two-dimensional topological field theories are used in string theory and brane theory [23], G-flux and K-theories [25], WZW, G/G and CS theories [13]. In this work we deal with two-dimensional TFTs only.

A TFT is based essentially on a functor $\Omega \mapsto V_\Omega$ from a category of topological manifolds with singularities into a category of vector spaces. V_Ω depends on singularities of a manifold Ω. If a functor is fixed, than TFT is defined as an invariant choice of a linear form $\Phi \in V_\Omega^*$ for each manifold Ω.

Two-dimensional topological field theory was defined first for a category of compact oriented surfaces with boundary contours [4] ('closed' theories). Contracting these contours to points we can and will consider closed topological field theories as theories based on the category of compact oriented surfaces without boundaries endowed with finitely many special points.

Later was defined so called 'open-closed' TFT [23, 26]. In our terms open-closed TFTs are based on the category of compact oriented surfaces with finite set of special points lying either in the interior or on the boundary contours of a surface. Formerly, in [19] was outlined the construction of TFT for the category to nonoriented surfaces. In this work (in contrast to previously mentioned ones) partition functions for triangulated surfaces are defined instead of a functor.

Closed 2D TFTs are classified by Frobenius algebras, i.e., associative algebras endowed with invariant nondegenerate scalar products [10, 1, 11, 34]. In [23] is constructed an algebra corresponding to an open-closed TFT. In more explicit form this algebra is described in [26]. In

[26] is claimed (without a proof) that this correspondence is a bijection. An algebraisation is outlined also in [19].

In this work we give a definition of a topological field theory based on a category of arbitrary compact surfaces (i.e., nonoriented surfaces, possibly nonorientable, possibly with a boundary) that are endowed with a finite set of special points lying either in the interior or on the boundary. We call any such TFT *a Klein topological field theory* (KTFT).

We prove that Klein topological field theories correspond bijectively to noncommutative analogues of Frobenius algebras. We call these algebras *structure algebras*.

A structure algebra is an associative (typically noncommutative) algebra endowed with an invariant scalar product and three features: a decomposition into direct sum of a commutative subalgebra and an ideal (reflecting two kinds of special points, interior and boundary ones), an involutive antiautomorphism (reflecting the change of the local orientation at a special point) and an element U (reflecting nonorientability of a surface). Emphasize that KTFT gives an additional structure even for the category of orientable surfaces; it is an involutive antiautomorphism. This involutive antiautomorphism is missed in closed and open-closed TFTs because fixing the orientation of a surface allows to work without it.

We classify all semisimple structure algebras. In terms of TFTs this is the classification of massive Klein topological field theories.

Starting from any finite group we construct structure algebras. (Only one structure algebra corresponds to symmetric group, but it can happen that several structure algebras correspond to the same group G.) These structure algebras are semisimple.

We apply Klein topological field theory to study (generalized) Hurwitz numbers. More than 100 years ago Hurwitz [18] raised the following problem. Let Ω be a complex algebraic curve of genus g, $p_1, \ldots, p_m \in \Omega$ be pairwise distinct points and $\alpha_1, \ldots, \alpha_m$ be partitions n (a decomposition of number n into unordered sums of positive integers). Denote by $S(\alpha_1, ..., \alpha_m)$ the set of classes of birational equivalence of morphisms $\pi : P \to \Omega$ of order n having critical values $p_1, \ldots, p_m$ and for $i = 1, \ldots, m$ local degrees of π at points $\pi^{-1}(p_i)$ are summands from α_i. The problem is to comput the sum $\sum_{\pi \in S(\alpha_1,\ldots,\alpha_m)} \frac{1}{|\operatorname{Aut}(\pi)|}$, where $|\operatorname{Aut}(\pi)|$ is the order of the group of automorphisms of a covering π. These sums are called *Hurwitz numbers*.

The generating function for Hurwitz numbers is the partition function for $2D$ supersymmetric Yang-Mills theory [7]. These numbers are connected with string theory [16], mirror symmetry [9], theory of singularities [3], matrix models [22] and integrable systems [32]. There exist combinatorial, differential and integral formulas for the simplest of Hurwitz numbers. See, for example, [3, 15, 12].

Analog of Hurwitz numbers for morphisms into fixed real algebraic curve is defined in this work. We call these numbers *Hurwitz numbers of a real algebraic curve*, or *generalized Hurwitz numbers*, or just *Hurwitz numbers* for short. Generalized Hurwitz numbers include classical Hurwitz numbers as a particular case.

Generalized Hurwitz numbers can be defined in topological terms as weighted numbers of 'stratified coverings' over a compact stratified surface possibly with boundary (see 5.1).

According to [7] Hurwitz numbers of real algebraic curves are important for string theory and supersymmetric Yang-Mills theory.

Hurwitz numbers of real algebraic curves were investigated in [24, 5] in several special cases.

In this work we prove that Hurwitz numbers of real algebraic curves generate a Klein topological field theory. We call this KTFT by *Hurwitz topological field theory* (HTFT) and prove that its structure algebra is isomorphic to the structure algebra of symmetric group S_n. Using this construction we obtain simple algebraic expressions for all Hurwitz numbers. Namely, any Hurwitz number is a sum of products of Hurwitz numbers of three surfaces: a sphere with three special points, a triangle, i.e., a disc with three boundary special points, and a disc with one boundary and one interior special point. Character formulas for classical Hurwitz numbers [9] are essentially particular cases of our expressions.

The paper is organized as follows.

In section 2 we define a structure algebra, classify semisimple structure algebras and construct semisimple structure algebras for any finite group.

In section 3 we prove a geometrical theorem about 'complete cut systems' of an arbitrary surface with special points. A complete cut system generates the decomposition of a surface. Obtained decompositions are analogs of 'punt' decompositions of a surfaces without a boundary. We define adjacent complete cut systems and prove that two arbitrary complete cut systems of a surface are connected by a chain of adjacent ones.

In section 4 we define a Klein topological field theory. We construct a correspondence between KTFTs and structure algebras and prove that it is bijective correspondence. The theorem about complete cut systems (section 3) is a base of the proof.

In section 5 we construct a topological field theory (HTFT) from generalized Hurwitz numbers. We prove that its structure algebra is isomophic to the structure algebra of symmetric group and give a formula for Hurwitz numbers. At the end of section 5 we note, that there are G-analogs of generalized Hurwitz numbers for a finite group G. They come from coverings with structure group G instead of S_n. The theory of 'G-Hurwitz numbers' is quit similar to the theory described in section 5.

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2. Structure algebra

2.1. **Definition of a structure algebra.** All vector spaces (algebras) in the paper are vector spaces (algebras) over complex numbers. Let X be an associative algebra with unit endowed with a nondegenerate invariant scalar product $(x,y) : X \otimes X \to \mathbb{C}$, i.e., $(x,y) = (y,x)$, $(xy,z) = (x,yz)$ and $(x,a) = 0$ for an arbitrary $x \in X$ implies $a = 0$.

Fix a basis E of X. We denote by $\alpha, \alpha', \dots \in E$ elements of the basis. We use the same letters as indices of tensors. Thus, we denote by $F_{\alpha',\alpha''}$ the matrix of the scalar product in basis E and we denote by $F^{\alpha',\alpha''}$ the tensor of the dual form. Hence, $F_{\alpha',\alpha''}F^{\alpha'',\alpha'''} = \delta_{\alpha'}^{\alpha'''}$. As usual, repeating indices mean summation.

Denote by $\widehat{K}_X$ an element $F^{\alpha',\alpha''}\alpha' \otimes \alpha''$ of the tensor product $X \otimes X$ and by K_X Casimir element. By definition, $K_X = F^{\alpha',\alpha''}\alpha'\alpha'' \in X$. Clearly, elements $\widehat{K}_X$ and K_X have coordinate-independent definitions. There are three operators $X \to X$ associated with Casimir element. They are the left multiplication by K_X, the right multiplication by K_X and operator $x \mapsto F^{\alpha',\alpha''}\alpha' x \alpha''$. The latter we denote by V_{K_X}.

By $x \mapsto x^*$ we denote an involutive antiautomorphism of X, i.e., $(x^*)^* = x$ and $(xy)^* = y^*x^*$. Denote by $K_{X,*}$ twisted Casimir element $F^{\alpha',\alpha''}\alpha'\alpha''^*$. Obviously, it coincides with Casimir element of twisted scalar product $(x,y)_* = (x,y^*)$.

Definition. *A structure algebra* $\mathcal{H} = \{H = A \dotplus B, (x,y), x \mapsto x^*, U\}$ *is a finite dimensional associative algebra* H *endowed with*

a decomposition $H = A \dotplus B$ *of* H *into the direct sum of two vector spaces;*

an invariant symmetric bilinear form $(x,y) : H \otimes H \to \mathbb{C}$*;*

an involutive antiautomorphism $H \to H$*, denoted by* $x \mapsto x^*$*;*

an element $U \in A$*,*

such that the following axioms hold:

1° *A is a subalgebra belonging to the center of algebra H; algebra A has unit $1_A \in A$ and 1_A is also the unit of algebra H;*

2° *B is a two-sided ideal of H (typically noncommutative); algebra B has a unit $1_B \in B$;*

3° *restrictions $(x,y)|_A$ and $(x,y)|_B$ are nondegenerate scalar products on algebras A and B resp.*

4° *$(V_{K_B}(b_1), b_2) = (\widehat{K}_A, b_1 \otimes b_2)$ for arbitrary $b_1, b_2 \in B$ (this axiom reflects Cardy relation* [23]*);*

5° *an involutive antiautomorphism preserves the decomposition* $H = A\dot{+}B$ *and the form* (x,y) *on* H*, i.e.,* $A^* = A$, $B^* = B$, $(x^*, y^*) = (x, y)$*;*
6° $U^2 = K_{A,*}$*;*
7° $(U, b) = (K_{B,*}, b)$ *for any* $b \in B$*;*
8° $(aU)^* = aU$ *for any* $a \in A$*.*

Remark. Forms $(x,y)|_A$ and $(x,y)|_B$ are nondegenerate and we use them to raise or lower indices in tensors. Bilinear form (x,y) on H is not assumed to be nondegenerate. Linear subspaces A and B are not assumed to be orthogonal. Clearly, if an element $a \in A$ is orthogonal to B then $aB = 0$. Therefore, if A is orthogonal to B then $AB = 0$.

Fix a structure algebra $\mathcal{H} = \{H = A \dot{+} B, (x,y), x \mapsto x^*, U\}$. Let a be an element of A. Then formula $(\phi(a), b) = (a, b)$ defines an element $\phi(a) \in B$ because bilinear form $(x,y)|_B$ is nondegenerate.

Lemma 2.1. *The mapping* $\phi : A \to B$ *is a homomorphism of algebra* A *into the center of the algebra* B *such that for any* $a \in A$, $b \in B$ *the equality* $ab = \phi(a)b$ *holds.*

Proof. By invariance of the form (x,y) we have $(ab, b') = (a, bb') = (\phi(a), bb') = (\phi(a)b, b')$. Therefore, $ab = \phi(a)b$, which means that ϕ is a homomorphism. By definition, A lies in the center of H. Hence $(ab, b') = (ba, b') = (b, ab') = (b, \phi(a)b') = (b\phi(a), b')$. Therefore, $\phi(a)b = b\phi(a)$ and $\phi(a)$ belongs to the center of B. □

We shall reformulate the definition of a structure algebra $\mathcal{H} = \{H = A \dot{+} B, (x,y), x \mapsto x^*, U\}$ in a coordinate form. Fix basis E_A of A and E_B of B. We denote by $\alpha, \alpha_1, \dots$ the elements of E_A and by $\beta, \beta_1, \dots$ the elements of E_B. Indices of tensors are denoted below by the same letters $\alpha, \alpha_1, \dots; \beta, \beta_1, \dots$.

Define tensors:
(1) $F_{\alpha_1,\alpha_2} = (\alpha_1, \alpha_2)$;
(2) $F_{\beta_1,\beta_2} = (\beta_1, \beta_2)$;
(3) $R_{\alpha,\beta} = (\alpha, \beta)$;
(4) $S_{\alpha_1,\alpha_2,\alpha_3} = (\alpha_1\alpha_2, \alpha_3)$;
(5) $T_{\beta_1,\beta_2,\beta_3} = (\beta_1\beta_2, \beta_3)$;
(6) $R_{\alpha,\beta_1,\beta_2} = (\alpha\beta_1, \beta_2)$;
(7) $I_{\alpha_1,\alpha_2} = (\alpha_1^*, \alpha_2)$;
(8) $I_{\beta_1,\beta_2} = (\beta_1^*, \beta_2)$;
(9) $D_\alpha = (U, \alpha)$;
(10) $J_\alpha = (1_A, \alpha)$;
(11) $J_\beta = (1_B, \beta)$.

Tensors (1)-(11) completely define a structure algebra. Indeed, tensors (1)-(3) define the bilinear form, tensors (4)-(6) define the multiplication in algebra H, tensors (7)-(8) define the involutive antiautomorphism, tensor (9) defines the element U and tensors (10), (11) define units $1_A \in A$ and $1_B \in B$.

We call tensors (1)-(11) structure constants of structure algebra $\mathcal{H}$ in basis E_A, E_B.

Let us write down exact formulas. Suppose we are given by two linear spaces A and B with basis E_A of A, E_B of B and by *arbitrary* tensors F_{α_1,α_2}, …, J_β that have the same type as tensors from the left-hand sides of equalities (1) - (11) and are denoted by the same letters.

Assume F_{α_1,α_2} and F_{β_1,β_2} are symmetric and nondegenerate. Using F_{α_1,α_2} and F_{β_1,β_2} and dual tensors we will raise or lower indices of tensors. In the case of asymmetric tensors we always raise the last index.

Let $H = A \oplus B$. Define bilinear form on H by formulas (1)-(3). Define a multiplication in H as follows: $\alpha_1\alpha_2 = S^{\alpha}_{\alpha_1,\alpha_2}\alpha$; $\beta_1\beta_2 = T^{\beta}_{\beta_1,\beta_2}\beta$; $\beta\alpha = \alpha\beta = R^{\beta'}_{\alpha,\beta}\beta'$. By definition, A is a subalgebra and B is an ideal of H. Define a linear map $x \mapsto x^*$ as follows: $\alpha^* = I^{\alpha'}_{\alpha}\alpha'$; $\beta^* = I^{\beta'}_{\beta}\beta'$. Define elements $U \in A, 1_A$, $1_B \in B$ by formulas $U = D^\alpha\alpha$, $1_A = J^\alpha\alpha$, $1_B = J^\beta\beta$. Denote by $S_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ the contraction $S^{\alpha'}_{\alpha_1,\alpha_2}S_{\alpha',\alpha_3,\alpha_4}$ and by $T_{\beta_1,\beta_2,\beta_3,\beta_4}$ the contraction $T^{\beta'}_{\beta_1,\beta_2}T_{\beta',\beta_3,\beta_4}$.

Lemma 2.2. *A set of data $\mathcal{H} = \{H = A \dotplus B, (x,y), x \mapsto x^*, U\}$ is a structure algebra if and only if*
(1) F_{α_1,α_2}, F_{β_1,β_2} are symmetric nondegenerate tensors;
(2) $S_{\alpha_1,\alpha_2,\alpha_3}$ and $S_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$ are symmetric tensors;
(3) tensors $T_{\beta_1,\beta_2,\beta_3}$ and $T_{\beta_1,\beta_2,\beta_3,\beta_4}$ are invariant under cyclic permutations;
(4) $R_{\alpha,\beta_1,\beta_2} = R^{\beta'}_{\alpha}T_{\beta',\beta_1,\beta_2}$;
(5) $R^{\alpha'}_{\beta}S_{\alpha',\alpha_1,\alpha_2} = R^{\beta'}_{\alpha_1}R^{\beta''}_{\alpha_2}T_{\beta',\beta'',\beta}$;
(6) $R_{\alpha,\beta_1,\beta_2} = R_{\alpha,\beta_2,\beta_1}$;
(7) $R_{\alpha',\beta_1}F^{\alpha',\alpha''}R_{\alpha'',\beta_2} = T^{\beta''}_{\beta_1,\beta'}T^{\beta'}_{\beta'',\beta_2}$
(8) $I^{\alpha'}_{\alpha_1}I_{\alpha',\alpha_2} = F_{\alpha_1,\alpha_2}$, $I^{\beta'}_{\beta_1}I_{\beta',\beta_2} = F_{\beta_1,\beta_2}$
(9) $I_{\alpha_1,\alpha_2} = I_{\alpha_2,\alpha_1}$, $I_{\beta_1,\beta_2} = I_{\beta_2,\beta_1}$, $I^{\alpha'}_{\alpha}R_{\alpha',\beta} = I^{\beta'}_{\beta}R_{\alpha,\beta'}$
(10) $I^{\alpha'}_{\alpha_1}I^{\alpha''}_{\alpha_2}I^{\alpha'''}_{\alpha_3}S_{\alpha''',\alpha'',\alpha'} = S_{\alpha_1,\alpha_2,\alpha_3}$, $I^{\beta'}_{\beta_1}I^{\beta''}_{\beta_2}I^{\beta'''}_{\beta_3}T_{\beta''',\beta'',\beta'} = T_{\beta_1,\beta_2,\beta_3}$,
(11) $S_{\alpha,\alpha',\alpha''}D^{\alpha'}D^{\alpha''} = S_{\alpha,\alpha',\alpha''}I^{\alpha',\alpha''}$;
(12) $D^{\alpha'}R_{\alpha',\beta} = I^{\beta',\beta''}T_{\beta',\beta'',\beta}$
(14) $D^{\alpha'}S_{\alpha',\alpha_1,\alpha_2} = I^{\alpha'}_{\alpha_1}D^{\alpha''}S_{\alpha',\alpha'',\alpha_2}$
(15) $J^{\alpha'}S_{\alpha',\alpha_1,\alpha_2} = F_{\alpha_1,\alpha_2}$, $J^{\alpha'}R_{\alpha',\beta_1,\beta_2} = F_{\beta_1,\beta_2}$
(16) $J^{\beta'}T_{\beta',\beta_1,\beta_2} = F_{\beta_1,\beta_2}$

Proof. We shall list below the correspondence between relations (1)-(12) and axioms of a structure algebra. All proofs are by direct calculations.

Relation (1) coincides with axiom 3°.

Relation (2) is equivalent to the claim 'A is an associative commutative algebra endowed with an invariant scalar product'.

Relation (3) is equivalent to the claim 'B is an associative algebra endowed with invariant scalar product'.

Define a map $\phi : A \to B$ by $\phi(\alpha) = R_{\alpha}^{\beta}\beta$. Clearly $(\alpha, \beta) = (\phi(\alpha), \beta)$.

Relation (4) is equivalent to the claim $\alpha\beta = \phi(\alpha)\beta$.

Relation (5) is equivalent to the claim 'ϕ is a homomorphism of algebras'.

Relation (6) is equivalent to the claim $\phi(\alpha)\beta = \beta\phi(\alpha)$.

It is easy to show that if ϕ is a homomorphism and its image lies in the center of algebra B then H is an associative algebra and the form (x, y) is invariant and vice versa.

Relation (7) is equivalent to axiom 4°.

Relation (8) is equivalent to the claim '$x \mapsto x^*$ is an involution'.

Relation (9) is equivalent to the claim 'the involution $x \mapsto x^*$ preserves the bilinear form (x, y)'.

Relations (10) is equivalent to the claim 'the involution $x \mapsto x^*$ preserving the form (x, y) is an antiautomorphism'. Indeed, two relations from (10) are the direct reformulation of this fact for subalgebras A and B. Let us prove that $(\alpha\beta)^* = \beta^*\alpha^*$. We have $(\phi(\alpha), \beta) = (\alpha, \beta)$. Therefore, $(\phi(\alpha^*), \beta) = (\alpha^*, \beta) = (\alpha, \beta^*)$. Analogously, $(\phi(\alpha)^*, \beta) = (\phi(\alpha), \beta^*) = (\alpha, \beta^*)$. We obtain that $\phi(\alpha^*) = \phi(\alpha)^*$. Hence, $(\alpha\beta)^* = (\phi(\alpha)\beta)^* = \beta^*\phi(\alpha)^* = \beta^*\phi(\alpha^*) = \beta^*\alpha^*$.

Relation (11) is equivalent to axiom 6°.

Relation (12) is equivalent to axiom 7°.

Relation (14) is equivalent to axiom 8°.

Relations (15), (16) are equivalent to the claims '1_A is a unit of H' and '1_B is a unit of B'.

□

We call (1) - (16) *relations* for structure constants of a structure algebra.

2.2. **Semisimple structure algebras.** A structure algebra $\mathcal{H}$ is called *semisimple* if the algebra H is semisimple.

Let $\mathcal{H}$ be a finite dimensional complex semisimple structure algebra. Then both subalgebras A and B are semisimple algebras. Indeed, B is an ideal of H and A is isomorphic to the factor H/B.

By Wedderburn theorem, B is isomorphic to the direct sum of matrix algebras, $B = \oplus_{i=1}^{k} \mathrm{M}_{n_i}$. A is commutative semisimple algebra; therefore, A is isomorphic to the direct sum of one-dimensional algebras, $A = \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{m}$. Clearly, as an abstract algebra H is isomorphic to the direct sum $A \oplus B$. Despite this fact, the decomposition $H = A \dotplus B$ that is included in the set of data $\mathcal{H}$ typically does not coincide with a decomposition into the direct sum of two-sided ideals.

Let us write down formulas for the multiplication $A \times B \to B$ and for the invariant bilinear form $(x, y) : H \otimes H \to \mathbb{C}$. There is a uniquely defined, up to permutations, complete system $e_1, \dots e_m$ of orthogonal idempotents in A and $A = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_m$. Orthogonal idempotents are orthogonal with respect to the invariant scalar product. Therefore, we have $(e_i, e_j) = \lambda_i \delta_{i,j}$, $\lambda_i \neq 0$.

Similarly, components M_{n_i} of algebra $B = \oplus_{i=1}^k \mathrm{M}_{n_i}$ are orthogonal with respect to any invariant scalar product on B. All invariant scalar products on M_{n_i} are proportional to each other. Hence, if $X, Y \in \mathrm{M}_{n_i}$ then $(X, Y) = \mu_i \operatorname{tr}(XY)$, $\mu_i \neq 0$.

Denote by E_i the unit matrix of M_{n_i}. Clearly, elements $\{E_i | i = 1, \dots, k\}$ form a basis of the center of algebra B and they are orthogonal idempotents. By lemma 2.1, morphism $\phi : A \to B$ is a homomorphism into the center of B. Therefore, the image of a complete system $\{e_i | i = 1, \dots, m\}$ of orthogonal idempotents is a system of orthogonal idempotents in the center of B. Hence, $\phi(e_i) = \sum_{j \in N_i} E_j$, where $N_i \subset \{1, \dots, k\}$ and if $i \neq i'$ then $N_i \cap N_{i'} = \emptyset$. Thus,

$$e_i b = b e_i = (\sum_{j \in N_i} E_j) b \text{ for any } b \in B \quad (2.1)$$

and this formula completely describes the multiplication between elements of A and B. For elementary matrices $E_{s,i,j} \in M_{n_s}$ we get

$$(e_i, E_{s,i,j}) = (\phi(e_i), E_j) = \begin{cases} \mu_s \delta_{i,j} \text{ if } s \in N_i \\ 0 \text{ if } s \notin N_i \end{cases} \quad (2.2)$$

Conversely, let $A = \mathbb{C}^m$ and $B = \oplus_{i=1}^k \mathrm{M}_{n_i}$ be algebras with invariant scalar products defined by constants λ_i, $i = 1, \dots, m$ and μ_j, $j = 1, \dots, k$ resp. Define a multiplication between A and B by formula 2.1 and the scalar product between A and B by formula 2.2. Then axioms 1°-3° of a structure algebra are satisfied. Let us denote such an algebra endowed with an invariant bilinear form by $\widetilde{H} = A \dotplus B$.

Lemma 2.3. *Cardy axiom 4° is satisfied in algebra $\widetilde{H}$ if and only if*
(1) for $i = 1, \dots, m$ a subset N_i of the set $\{1, \dots, k\}$ is either empty set or contains one element;
(2) $\{1, \dots, k\} = \sqcup_i N_i$;
(3) if $N_i = \{j\}$ then $\lambda_i = \mu_j^2$.

Proof. Let us compute element $\widehat{K}_A$ and transformation V_{K_b}. Evidently, $\widehat{K}_A = \sum_{i=1}^m \frac{1}{\lambda_i} e_i \otimes e_i$ and $K_A = \sum_{i=1}^m \frac{1}{\lambda_i} e_i$. Denote by $E_{s,i,j}$ the elementary matrix $E_{i,j} \in \mathrm{M}_{n_s}$. Evidently, all matrices $E_{s,i,j}$ form a basis of algebra B. We obtain by direct calculations that $K_B = \sum_{s=1}^k \frac{n_s}{\mu_s} E_s$ and $V_{K_B}(E_{s,i,j}) = \delta_{i,j} \frac{1}{\mu_s} E_s$. Therefore, putting $b_1 = E_{s,i,j}$, $b_2 = E_{s',i',j'}$ in axiom 4° we obtain that:

a) left side $= \delta_{s,s'} \delta_{i,j} \delta_{i',j'} \frac{1}{m_s} \mu_s = \delta_{s,s'} \delta_{i,j} \delta_{i',j'}$;

b) right side $= \begin{cases} \frac{\mu_s \mu_{s'}}{\lambda_l} \delta_{i,j} \delta i', j' \text{ if } s, s' \in N_l \\ 0 \text{ if } s \text{ and } s' \text{ do not lie in the same set } N_l \end{cases}$

The proof is completed by comparing these formulas.

□

Assume that axiom 4° holds for an algebra $\widetilde{H}$. Then we can and will reorder components of A and B in such a way that

a) for $i = 1, \ldots, k$ we have $\phi(e_i) = E_i$, where E_i is the unit matrix of the component M_{n_i};

b) for $i = k+1, \ldots, m$ we have $\phi(e_i) = 0$.

Let us check axioms related to an involutive antiauthomorphism $x \mapsto x^*$ and an element U for algebra $\widetilde{H}$. Suppose $x \mapsto x^*$ is an involutive transformation satisfying axiom 5°.

Obviously, involutive antiautomorphism $x \mapsto x^*$ can either fix an idempotent e_s or permute two idempotents $e_{s'}$ and $e_{s''}$. If the latter occurs then $\lambda_{s'} = \lambda_{s''}$ because this antiautomorphism preserves the scalar product. Thus, the involution of the set $\{1, \ldots, m\}$ is induced. We denote it by the same sign $s \mapsto s^*$.

Analogously, the involution $x \mapsto x^*$ either preserves a component M_{n_s} or permutes two components M_{n_s} and $\mathrm{M}_{n'_s}$ of algebra B. If the latter occurs then we have $n_s = n_{s'}$ and $\mu_s = \mu_{s'}$. Thus, the involution of the set $\{1, \ldots, k\}$ is induced. This involution coincides with the restriction to $\{1, \ldots, k\}$ of previously defined involution $s \mapsto s^*$ of the set $\{1, \ldots, m\}$ because transformation $x \mapsto x^*$ preserves bilinear form and e_i is orthogonal to M_{n_j} if and only if $i \neq j$.

Obviously, the restriction of the transformation $x \mapsto x^*$ to A is completely defined by the involution $s \mapsto s^*$ of the set $\{1, \ldots, m\}$. There is an ambiguity for the restriction of the involutive transformation to subalgebra B. The possibilities are well known. We describe them below without proofs.

If $s \leq k$ and $s^* \neq s$ then the restriction of $x \mapsto x^*$ to $\mathrm{M}_{n_s} \oplus \mathrm{M}_{n_{s^*}}$ is conjugated to the transformation $\{X, Y\} \to \{Y', X'\}$, where X' denotes the transpose of X.Here $\{X, Y\}$ is an element of $\mathrm{M}_{n_s} \oplus \mathrm{M}_{n_{s^*}}$, i.e., two matricies. Thus, changing the basis of B if necessary, we have $\{X, Y\}^* = \{Y', X'\}$.

If $s \leq k$ and $s^* = s$ then an involutive antiautomorphism of M_{n_s} is conjugated to one of two canonical antiautomorphisms.

The first of them is associated with a symmetric bilinear form and coincides with the transpose of a matrix.

The second of them is associated with a nondegenerate skew-symmetric bilinear form. Hence, it exists only if $n_s = 2r$. Let us identify the set $\{1, \ldots, n_s\}$ with the set $\mathbb{Z}_2 \times \{1, \ldots, r\}$, where $\mathbb{Z}_2$ is the group with two elements $\{0, 1\}$. If $i = (\epsilon, i')$ then put $\epsilon(i) = \epsilon$ and $i^\tau = (\epsilon + 1, i')$. Define a transformation $\tau : \mathrm{M}_{n_s} \to \mathrm{M}_{n_s}$ by formula $(E_{i,j})^\tau = (-1)^{\epsilon(i)+\epsilon(j)} E_{j^\tau, i^\tau}$. One can check directly that τ is the involutive antiautomorphism preserving the invariant scalar product.

Therefore, if $s^* = s$ then in an appropriate basis we have either $E^*_{s,i,j} = E_{s,j,i}$ or $E^*_{s,i,j} = (-1)^{\epsilon(i)+\epsilon(j)} E_{s,j^\tau,i^\tau}$. In order to distinguish these cases let us introduce an invariant $\nu = \nu(s) \in \pm 1$ and put $\nu(s) = 1$ in the former case, $\nu(s) = -1$ in the latter case.

Denote by P a set of fixed points of involution $s \mapsto s^*$ and put $P_0 = P \cap \{1, \dots, k\}$.

As a result of the above considerations, we obtain the following lemma.

Lemma 2.4. *Let $x \mapsto x^*$ be an involutive antiautomorphisms of algebra $\widetilde{H}$. Suppose, $x \mapsto x^*$ satisfies axiom $5°$. Then antiautomorphism $x \mapsto x^*$ induces*

- *involution $s \mapsto s^*$ of set $\{1, \dots, m\}$ such that $(\{1, \dots, k\})^* = \{1, \dots, k\}$ and $\mu_s = \mu_{s^*}$, $\lambda_s = \lambda_{s^*}$;*
- *numbers $\nu(s) \in \{\pm 1\}$ for $s \in P_0$ where $P_0 = \{s | s \le k, s^* = s\}$*

Antiautomorphisms $s \mapsto s^*$ up to inner automorphism of $\widetilde{H}$ are classified by pairs $(s \mapsto s^*; \nu(s))$.

Let us fix additionally an involutive antiautomorphism satisfying axiom $5°$ and check the possibilities of choosing an element $U \in A$ satisfying axioms $6° - 8°$.

Lemma 2.5. *An element $U \in A$ satisfies axioms $6° - 8°$ if and only if $U = \sum_{i \in P_0} \frac{\nu_i}{\mu_i} e_i + \sum_{j \in P \setminus P_0} x^j e_j$, where $(x^j)^2 = \frac{1}{\lambda_j}$.*

Proof. Let $U = \sum_{i=1}^m x^i e_i$ be an element of A satisfying axioms $6° - 8°$. Then $U^2 = \sum_{i=1}^m (x^i)^2 e_i$. Twisted Casimir element $K_{A,*}$ is equal to $\sum_i \frac{1}{\lambda_i} e_i e_{i^*} = \sum_{i \in P} \frac{1}{\lambda_i} e_i$. Therefore, $U = \sum_{i \in P} x^i e_i$ and $(x^i)^2 = \frac{1}{\lambda_i}$ (axiom $6°$) Evidently, $(U, E_{s,i,j}) = \begin{cases} x_s \mu_s \delta_{i,j} \text{ if } s \in P_0 \\ 0 \text{ if } s \notin P_0 \end{cases}$

Compute element $K_{B,*}$. By definition, $K_{B,*} = \sum_{s=1}^k \sum_{i,j} \frac{1}{\mu_s} E_{s,i,j} E^*_{s,j,i}$. If $s^* \neq s$ then $E_{s,i,j} E^*_{s,j,i} = 0$, hence only summands corresponding to $s \in P_0$ are nonzero.

If $\nu_s = 1$ then $\sum_{i,j} \frac{1}{\mu_s} E_{s,i,j} E^*_{s,j,i} = \sum_{i,j} \frac{1}{\mu_s} E_{s,i,j} E_{s,i,j} = \frac{1}{\mu_s} \sum_i E_{s,i,i} = \frac{1}{\mu_s} E_s = \frac{1}{\mu_s} \nu_s E_s$.

If $\nu_s = -1$ then $\sum_{i,j} \frac{1}{\mu_s} E_{s,i,j} E^*_{s,j,i} = \sum_{i,j} \frac{1}{\mu_s} E_{s,i,j} (-1)^{\epsilon(j)+\epsilon(i)} E_{s,i^*,j^*} = \frac{1}{\mu_s} \sum_i E_{s,i,i^*} (-1) E_{s,i^*,i} = \frac{1}{\mu_s} \nu_s E_s$.

Therefore, $K_{B,*} = \sum_{s \in P_0} \frac{1}{\mu_s} \nu_s E_s$ and $(K_{B,*}, E_{s,i,j}) = \begin{cases} \nu_s \delta_{i,j} \text{ if } s \in P_0 \\ 0 \text{ if } s \notin P_0 \end{cases}$

By axiom $7°$ $x_s \mu_s = \nu_s$ for $s \in P_0$. Thus, we prove that $U = \sum_{i \in P_0} \frac{\nu_i}{\mu_i} e_i + \sum_{j \in P \setminus P_0} x^j e_j$, where $(x^j)^2 = \frac{1}{\lambda_j}$.

Conversely, let U is given by the latter formula. Then reversing our steps, we obtain that it satisfies axioms $6°$ and $7°$. By direct calculations we obtain also that U satisfies axioms $8°$. □

Remark. In the case of a semisimple structure algebra axiom 8° follows from axioms 6° and 7°.

Thus, we proved the following theorem.

Theorem 2.1. *Let $\mathcal{H} = \{H = A \dotplus B, (x, y), x \mapsto x^*, U\}$ be a semisimple structure algebra. Then*
(1) $A = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_m$, $B = \mathrm{M}_{n_1} \oplus \cdots \oplus \mathrm{M}_{n_k}$, $k \le m$ and $\phi(e_i) = E_i$ for $i \le k$;
(2) $(X_i, \widetilde{X}_i) = \mu_i \operatorname{tr}(X_i \widetilde{X}_i)$, $\mu_i \ne 0$, for $X_i, \widetilde{X}_i \in M_{n_i}$, $i = 1, \dots, k$, and $(X_i, \widetilde{X}_j) = 0$ for $i \ne j$;
(3) $(e_i, e_i) = \mu_i^2$ for $i = 1, \dots, k$; $(e_i, e_i) = \lambda_i$ for $i > k$; $(e_i, e_j) = 0$ for $i \ne j$;
(4) if $M_{n_i}^* = M_{n_j}$ then $e_i^* = e_j$;
a) if $i \ne j$ then $n_i = n_j$, $\mu_i = \mu_j$ and for $X \in M_{n_i}$ we have $X^* = X' \in M_{n_j}$;
b) if $i = j$ then either $X^* = X'$ or n_i is even and $X^* = X^\tau$; put $\nu_i = 1$ in the first case and $\nu_i = -1$ in the second case;
(5) for $i > k$ we have $e_i^* = e_j$ and if $i \ne j$ then $\lambda_i = \lambda_j$;
(6) $U = \sum_{i \in P_0} \frac{\nu_i}{\mu_i} e_i + \sum_{j \in P \setminus P_0} x^j e_j$, where $P = \{x \in \{1, \dots, m\} | e_i^* = e_i\}$, $P_0 = P \cap \{1, \dots, k\}$ and $(x^j)^2 = \frac{1}{\lambda_j}$.

Conversely, if conditions (1)-(6) hold then $\mathcal{H}$ is a semisimple structure algebra

Corollary 2.1. *If a semisimple associative algebra H, its decomposition $H = A \oplus B$ and bilinear form (x, y) satisfy axioms $1^\circ - 4^\circ$ then there exists at least one involutive antiautomorphism satisfying axiom 5°. For each of them there exists $2^p > 0$, where $p = |P \setminus P_0|$ elements $U \in A$ such that $\mathcal{H} = \{H = A \oplus B, (x, y), x \mapsto x^*, U\}$ is a semisimple structure algebra.*

Proof is evident.

2.3. **Structure algebra of a finite group.** In this section we assign a structure algebra $\mathcal{H} = \mathcal{H}(G)$ to any finite group G. Denote by $|\Lambda|$ the cardinality of a set Λ. Denote by $\mathbb{C}[G]$ the group algebra of group G. Denote by A the center of algebra $\mathbb{C}[G]$. Let us assign an element $E_\alpha = \sum_{g \in \alpha} g \in \mathbb{C}[G]$ to a class α of conjugated elements. It is known, that elements E_α form a basis of A.

Denote by $M(G)$ an algebra of all endomorphisms of vector space $\mathbb{C}[G]$. Elements of G form a basis of $\mathbb{C}[G]$). Therefore, an elementary matrix in $M(G)$ corresponds to an ordered pair (g_1, g_2) of elements of G. Denote an elementary matrix by E_{g_1,g_2}.

Two ordered pairs (g_1, g_2), (g_1', g_2') of elements of G are called conjugated if and only if there exists an element $g \in G$ such that $g g_1 g^{-1} = g_1'$, $g g_2 g^{-1} = g_2'$. Let (s_1, s_2) be an ordered pair of involutive elements of G and β be a class of ordered pairs conjugated to (s_1, s_2). Define an element $E_\beta \in M(G)$ by the formula $E_\beta = \sum_{(s',s'')\in\beta} E_{s',s''}$. Denote by B a linear subspace of $M(G)$ generated by all E_β. Obviously elements E_β form a basis of B. Action of $g \in G$ on group G by conjugation $x \mapsto gxg^{-1}$ defines an element of $M(G)$; we denote it by V_g. Obviously mapping $g \mapsto V_g$ defines the representation $V : \mathbb{C}[G] \to M(G)$.

Lemma 2.6. .
(1) $V(g) = \sum_{h\in G} E_{gh,hg}$.
(2) $V(g)E_{g_1,g_2} = E_{gg_1g^{-1},g_2}$, $E_{g_1,g_2}V(g) = E_{g_1,g^{-1}g_2g}$
(3) $V(g)E_{g_1,g_2}V(g^{-1}) = E_{gg_1g^{-1},gg_2g^{-1}}$

The proof is elementary.

Lemma 2.7. *Linear subspace B coincides with the set of matrices $X \in M(G)$ such that*

a) X is a linear combination of elements E_{s_1,s_2}, where s_1, s_2 are involutive elements of G;

b) X commutes with all V_g, $g \in G$.

The proof follows from lemma 2.6. □

Put $V_\alpha = V(E_\alpha)$. Define the algebra structure on the direct sum $H = A \oplus B$. The multiplication of elements of A and elements of B follows from algebra structure on A and B respectively. Define multiplication of elements $E_\alpha \in A$ and $E_\beta \in B$ by formulas: $E_\alpha E_\beta = V_\alpha E_\beta$, $E_\beta E_\alpha = E_\beta V_\alpha$.

Lemma 2.8. *$H = A \oplus B$ is a semisimple associative algebra, A is a central subalgebra, B is an ideal.*

Proof. Associativity of H and properties of the decomposition $H = A \oplus B$ can be easily checked by direct calculations. A is a semisimple algebra as the center of a group algebra.

Let us proove that B is a semisimple algebra. Denote by S the set of all involutions in G. Clearly, the subspace $\mathbb{C}[S] \subset \mathbb{C}[G]$ is invariant with respect to the representation $V : \mathbb{C}[G] \to M(G)$. By lemma 2.7, B coincides with the centralizer of $V(G) \subset M(S)$. Therefore, B is a semisimple algebra. Hence H is a semisimple algebra. □

There are natural involutive antiautomorphism in both algebras A and B. Namely, denote by "$*$" the linear extension of the involution $g \mapsto g^{-1}$, $g \in G$, to group algebra $\mathbb{C}[G]$. Obviously, $E^*_\alpha = E_{\alpha^*}$, where α^* consists of inverse elements to elements of α and "$*$" is an involutive antiautomorphism of A.

Denote by the same sign "$*$" the transpose of matrices of $M(G)$, it is an involutive antiautomorphism of the algebra. For a class $\beta = [(s_1, s_2)]$ put $\beta^* = [(s_2, s_1)]$. Clearly, $E^*_\beta = E_{\beta^*}$. Therefore, $B^* = B$. So we have defined the involutive transformation of H.

Lemma 2.9. *Involutive transformation $x \mapsto x^*$ is an antiautomorphism of algebra H.*

Proof. We should check that $(E_\alpha E_\beta)^* = E_\beta^* E_\alpha^*$. We have $(E_\alpha E_\beta)^* = (V_\alpha E_\beta)^* = E_\beta^* V_\alpha^*$. By lemma 2.6 $V_g = \sum_{h\in G} E_{gh,hg}$. Therefore, $V_g^* = \sum_{h\in G} E_{hg,gh} = \sum_{h\in G} E_{g^{-1}h',h'g^{-1}}$, where $h' = ghg$. Hence $V_g^* = V_{g^{-1}}$ and $V_\alpha^* = V_{\alpha^*}$. □

It is well-known that any invariant symmetric bilinear form (x,y) is uniquely defined by a linear form $f(x)$ such that $f(xy-yx)=0$ identically. Bilinear form corresponding to $f(x)$ is defined as $(x,y) = f(xy)$. Define a linear form f on H by formulas: $f(E_\alpha) = \frac{1}{|G|}\delta_{\alpha,1}$, $f(E_\beta) = \frac{1}{|G|}\operatorname{tr}(E_\beta)$ (recall that E_β is an element of matrix algebra $M(G)$). Clearly, f defines the invariant symmetric bilinear form (x,y) on H.

The restriction $(x,y)|_A$ coincides with the restriction to the center of the standard invariant form on $\mathbb{C}[G]$ given by formula $(g_1,g_2) = \frac{1}{|G|}\delta_{g_1,g_2^{-1}}$ for $g_1,g_2\in G$. Therefore, $(E_{\alpha_1},E_{\alpha_2}) = \frac{1}{\nu_{\alpha_1}}\delta_{\alpha_1,\alpha_2^*}$, where $\nu_\alpha = \frac{|G|}{|\alpha|}$. Obviously ν_α is equal to the number of elements in the centralizer of any $g\in\alpha$. Note that form $(x,y)|_A$ is nondegenerate.

The restriction $(x,y)|_B$ coincides with the restriction to B of the standard invariant form on $M(G)$ given by formula $(X_1,X_2) = \frac{1}{|G|}\operatorname{tr}(X_1X_2)$. Therefore, $(E_{\beta_1},E_{\beta_2}) = \frac{1}{\nu_{\beta_1}}\delta_{\beta_1,\beta_2^*}$, where $\nu_\beta = \frac{|G|}{|\beta|}$. Obviously ν_β is equal to the number of elements in the stabilizer of any element $(s_1,s_2)\in\beta$. Clearly, the form $(x,y)|_B$ is nondegenerate. Note that scalar products (E_α,E_β) can be nonzero because by definition $(E_\alpha,E_\beta) = f(E_\alpha E_\beta) = \frac{1}{|G|}\operatorname{tr}(V_\alpha E_\beta)$.

Clearly, involution $x\mapsto x^*$ preserves this bilinear form. According to notations of previous subsection put $F_{\alpha_1,\alpha_2} = (E_{\alpha_1},E_{\alpha_2})$, $F_{\beta_1,\beta_2} = (E_{\beta_1},E_{\beta_2})$ and $R_{\alpha,\beta} = (E_\alpha,E_\beta)$.

Lemma 2.10. *Cardy axiom* 4° *holds for algebra* H.

Proof. First, let us compute the left hand side $L = (V_{K_B}(E_{\beta_1}),E_{\beta_2})$ of the identity in axiom 4°. By definition, $V_{K_B}(E_{\beta_1}) = F^{\beta',\beta''}E_{\beta'}E_{\beta_1}E_{\beta''}$. We have $F^{\beta',\beta''} = \nu_{\beta'}\delta^{\beta',\beta''^*}$. Hence $V_{K_B}(E_{\beta_1}) = \sum_\beta \nu_\beta E_\beta E_{\beta_1}E_{\beta^*}$ and $L = \sum_\beta \nu_\beta \frac{1}{|G|}\operatorname{tr}(E_\beta E_{\beta_1}E_{\beta^*}E_{\beta_2}) = \sum_\beta \frac{1}{|\beta|}\operatorname{tr}(E_\beta E_{\beta_1}E_{\beta^*}E_{\beta_2})$.

Each summand in this formula is a product of elementary matrices $E_{x,y}E_{s_1,s_2}E_{u,v}E_{s_3,s_4}$, where $(x,y)\in\beta, (s_1,s_2)\in\beta_1, (u,v)\in\beta^*, (s_3,s_4)\in\beta_2$. If the trace of the summand is nonzero then $y=s_1, u=s_2, v=s_3, x=s_4$. Let this conditions be satisfied. Then the trace is equal to 1 and the pair (s_4,s_1) is conjugated to the pair (s_3,s_2). Therefore,

$$L = \sum_{(s_1,s_2,s_3,s_4,g|(s_1,s_2)\in\beta_1,(s_3,s_4)\in\beta_2,gs_2g^{-1}=s_1,gs_3g^{-1}=s_4,g\in G} \frac{1}{|[(s_4,s_1)]|}\frac{1}{|Z_G(s_4,s_1)|} =$$

$$= \frac{|\{(s_1,s_2,s_3,s_4,g|(s_1,s_2)\in\beta_1,(s_3,s_4)\in\beta_2,g\in G, gs_2g^{-1}=s_1, gs_3g^{-1}=s_4\}|}{|G|}.$$

Let us compute the right hand side $R = (\widehat{K}_A, E_{\beta_1} \otimes E_{\beta_2})$ of the identity from axiom $4°$. By definition,

$R = \sum_{\alpha',\alpha''} F^{\alpha',\alpha''}(E_{\alpha'}, E_{\beta_1})(E_{\alpha''}, E_{\beta_2}) = \sum_\alpha \nu_\alpha (E_\alpha, E_{\beta_1})$ $\cdot (E_{\alpha^*}, E_{\beta_2}) = \sum_\alpha \nu_\alpha \frac{1}{|G|^2} \operatorname{tr}(V_\alpha E_{\beta_1}) \operatorname{tr}(V_{\alpha^*} E_{\beta_2})$.

By lemma 2.6, we have $V_\alpha = \sum_{h\in G, g\in\alpha} E_{gh,hg} = \sum_{x\in G, g\in \alpha} E_{x,g^{-1}xg}$. Analogously, $V_{\alpha^*} = \sum_{y\in G, h\in\alpha} E_{y,hyh^{-1}}$.

Hence,

$R = \sum_\alpha \frac{1}{|G||\alpha|} (\sum_{x\in G, g\in\alpha, (s_1,s_2)\in\beta_1} \operatorname{tr}(E_{x,g^{-1}xg} E_{s_1,s_2}))\cdot$ $\cdot(\sum_{y\in G, h\in\alpha, (s_3,s_4)\in\beta_2} \operatorname{tr}(E_{y,hyh^{-1}} E_{s_3,s_4}))$.

Nonzero summands arise from the cases $x = s_2, g^{-1}xg = s_1, y = s_4, hyh^{-1} = s_3$; in this case a summand is equal to $\frac{1}{\alpha}$.

Therefore,

$R = \sum_\alpha \frac{1}{|G||\alpha|} |\{s_1, s_2, s_3, s_4, g, h | (s_1, s_2) \in \beta_1, (s_3, s_4) \in \beta_2, g \in \alpha, h \in \alpha, s_2 = g s_1 g^{-1}, s_3 = h s_4 h^{-1}\}|$.

Element h runs through the set α of elements conjugated to g. Hence, we can present it as $h = zgz^{-1}$, where z runs through all elements of group G. Each element h has k presentations of this type, where $k = |Z_G(g)|$ is the cardinality of the centralizer of element g. Therefore, $R = \sum_\alpha \frac{1}{|G||\alpha|}\frac{1}{\nu_\alpha} |\{s_1, s_2, s_3, s_4, g, z | (s_1, s_2) \in \beta_1, (s_3, s_4) \in \beta_2, g \in \alpha, z \in G, s_2 = g s_1 g^{-1}, s_3 = (zgz^{-1}) s_4 (zg^{-1}z^{-1}\}|$. Note that $\frac{1}{|G||\alpha|}\frac{1}{\nu_\alpha} = \frac{1}{|G|^2}$ and therefore, this coefficient can be carried out of summing. Therefore, $R = \frac{1}{|G|^2} |\{s_1, s_2, s_3, s_4, g, z | (s_1, s_2) \in \beta_1, (s_3, s_4) \in \beta_2, g \in G, z \in G, s_2 = g s_1 g^{-1}, s_3 = (zgz^{-1}) s_4 (zg^{-1}z^{-1}\}|$.

The equality $s_3 = (zgz^{-1}) s_4 (zg^{-1}z^{-1})$ can be rewritten as $s'_3 = g s'_4 g^{-1}$, where $s'_3 = z^{-1} s_3 z$, $s'_4 = z^{-1} s_4 z$. For two fixed $z = z_1$ and $z = z_2$ the numbers of tuples $(s_1, s_2, s_3, s_4, g, z)$ satisfying conditions are equal since the pairs (s_3, s_4) run through all representatives of the class β_2. Therefore, $R = \frac{1}{|G|^2} |G| |\{s_1, s_2, s_3, s_4, g, z | (s_1, s_2) \in \beta_1, (s_3, s_4) \in \beta_2, g \in G, s_2 = g s_1 g^{-1}, s_3 = g s_4 g^{-1}\}|$.

We get that $L = R$. $\square$

Thus, we have proved all axioms for a structure algebra except those concerning an element $U \in A$. By corollary 2.1, there exists at least one element $U \in A$ satisfying axioms $6° - 8°$. Let us fix it. Therefore, we prove theorem.

Theorem 2.2. *For any finite group G the set of data $\mathcal{H}(G) = (H = A \oplus B, (x, y), x \mapsto x^*, U)$ is a structure algebra.*

Remark. It can be shown that in the case of the symmetric group $G = S_n$ the element U is uniquely determined.

3. Cuts of stratified surfaces

3.1. **Stratified surfaces.** Denote by ∂X a boundary of a topological space X and denote by X° its interior $\overline{X} \setminus \partial X$. We deal below with compact topological manifolds possibly with boundary and call them 'manifolds' for short.

Examples

(1) A connected one-dimensional manifold is homeomorphic either to a circle or to a segment.
(2) A connected orientable two-dimensional manifold is homeomorphic to a sphere with g handles and s holes. We call it *a surface of type* $(g, s, 1)$.
(3) A connected nonorientable two-dimensional manifold is homeomorphic either to a projective plane with a handles and s holes or to a Klein bottle with a handles and s holes. We call it *a surface of type* $(g, s, 0)$, where $g = a + 1/2$ in the former case and $g = a + 1$ in the latter case.

Definition. *Let Λ be a finite set. The decomposition $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ of a topological space Ω into the disjoint union of subspaces (strata) $\Omega_\lambda \subset \Omega$ is called a stratification if and only if*

(1) *each stratum Ω_λ is homeomorphic to a connected manifold without a boundary;*
(2) *each stratum is open in its closure;*
(3) *the boundary $\partial\Omega_\lambda$ of a stratum coincides with the union of strata of less dimension.*

We call topological space Ω itself a base of the stratification. Zero-dimensional strata are called special points.

Remark.

(1) A stratum is not necessarily homeomorphic to an open ball.
(2) Joining of strata induces a partial ordering of set Λ.
(3) There are several different definitions of a stratified manifold (see [17]). All these definitions are equivalent in dimensions 1 and 2. Fortunately, in this work we deal with dimensions 1 and 2 only. $\square$

A homeomorphism $\phi : \Omega \to \Omega'$ of the bases of stratified manifolds $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ and $\Omega' = \coprod_{\lambda\in\Lambda'} \Omega'_\lambda$ is called *an isomorphism of stratifications*, if and only if the restriction of ϕ to any stratum of Ω is the homeomorphism with an appropriate stratum of Ω'. Clearly, an isomorphism of stratifications induces the isomorphism of partial ordered sets Λ and Λ'.

Definition. *A stratification $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ of a manifold Ω is called special stratification if and only if all strata of codimension one belong to the boundary of Ω.*

Examples

(1) A connected specially stratified one-dimensional manifold is isomorphic either to a circle with the unique stratum or to a segment with natural stratification (strata are two end points and the open interval).
(2) Let $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ be a connected specially stratified two-dimensional manifold. We call the set of data $G = (g, \varepsilon, m, m_1, \dots, m_s)$ *a type of* Ω. Here (g, s, ε) is the type of surface Ω, m is the number of interior special points, m_i is the number of special points on i-th boundary contour. Two types that differ only in the order of m_i are considered as equal. It can be easily shown that an isomorphism class connected two-dimensional specially stratified manifolds is uniquely determined by its type.
(3) Let Ω be a stratified surface consisting of connected components Ω_i. Denote by G unordered set of types G^i of surfaces Ω_i. We call G a type of Ω. Clearly, up to isomorphism, Ω is uniquely determined by its type.

Remark. A special stratification of a surface is uniquely defined by a set Ω_0 of its special points.

We call a two-dimensional specially stratified manifold Ω consisting of finitely many connected components *a stratified surface* for short. Let Ω consists of c connected components. Denote by $G^i = (g^i, \varepsilon^i, m^i, m^i_1, \dots, m^i_s)$ $(i = 1, \dots, c)$ a type of i-th connected component Ω^i. Define an invariant $\mu(\Omega)$ by formula

$$\mu(\Omega) = \sum_i 2g^i + \sum_i m^i + \sum_i s^i + \frac{1}{2}\sum_i \sum_j m^i_j - 2c$$

From the definition follows that $\mu(\Omega)$ is half-integer. Clearly, invariant μ is additive with respect to the disjoint union of stratified surfaces:

$$\mu(\Omega_1 \coprod \Omega_2) = \mu(\Omega_1) + \mu(\Omega_2)$$

We say that a connected stratified surface Ω is *a trivial surface* if and only if $\mu(\Omega) \leq 0$. All trivial surfaces can be easily listed.

Lemma 3.1. *Any trivial stratified surface is isomorphic to one of surfaces from the following list:*
(1) sphere S^2 without special points ($\mu = -2$);
(2) projective plane $\mathbb{R}P^2$ without special points ($\mu = -1$);
(3) disc D^2 without special points ($\mu = -1$);
(4) sphere (S^2, p) with a unique interior special point p ($\mu = -1$);
(5) disc (D^2, q) with a unique boundary special point q and without interior special points ($\mu = -\frac{1}{2}$);
(6) sphere (S^2, p_1, p_2) with two interior special points ($\mu = 0$);
(7) projective plane $(\mathbb{R}P^2, p)$ with one interior special point ($\mu = 0$);
(8) torus T^2 without special points ($\mu = 0$);
(9) Klein bottle Kl without special points ($\mu = 0$);

(10) disc (D^2, p) *with a unique interior special point and without boundary special points* ($\mu = 0$)*;*
(11) disc (D^2, q_1, q_2) *with two boundary special points and without interior special points* ($\mu = 0$)*;*
(12) Möbius band Mb *without special points* ($\mu = 0$)*;*
(13) cylinder Cyl *without special points* ($\mu = 0$)*.*

3.2. **Cut systems.** Let Ω be a stratified surface. A generic not self-intersecting curve $\gamma \subset \Omega$ is called *a simple cut*. This implies that γ does not meet any special point and either is (closed) contour consisting purely of interior points, or is a segment, its end points belong to the boundary of Ω and all interior points of the segment are interior points of the surface.

A set Γ of pairwise nonintersecting simple cuts $\gamma \subset \Omega$ is called *a cut system of* Ω (Fig.1a).

Let Ω, Ω' be two stratified surfaces endowed with cut systems Γ, Γ'. We say that an isomorphism $\phi : \Omega \to \Omega'$ of stratified surfaces is *an isomorphism of pairs* (Ω, Γ), (Ω', Γ') if and only if $\phi(\Gamma) = \Gamma'$.

Definition. *A triple* $(\Omega_*, \Gamma_*, \tau)$ *consisting of*

- *a stratified surface* Ω_**;*
- *a subset* $\Gamma_* \subset \partial\Omega_*$ *such that each connected component of* Γ_* *coincides with the closure of a one-dimensional stratum of* Ω_**;*
- *an involutive homeomorphism* $\tau : \Gamma_* \to \Gamma_*$ *having no fixed points*

is called cut surface. In this case pair (Γ_*, τ) *is called a gluing system.*

An isomorphism of cut surfaces $(\Omega_*, \Gamma_*, \tau)$, $(\Omega'_*, \Gamma'_*, \tau')$ *is an isomorphism* $\phi : \Omega_* \to \Omega'_*$ *of stratified surfaces such that* $\phi(\Gamma_*) = \Gamma'_*$ *and* $\tau' \circ \phi = \phi \circ \tau$*.*

Let $(\Omega_*, \Gamma_*, \tau)$ be a cut surface. Gluing points x and $\tau(x)$ we obtain a surface Ω and 'gluing topological map' glue : $\Omega_* \to \Omega$. Clearly, stratification of Ω_* induces the stratification of Ω and the image $\Gamma =$ glue(Γ_*) is a cut system of Ω and glue : $(\Omega_*, \Gamma_*, \tau) \to (\Omega, \Gamma)$ is a functor from the category of cut surfaces to the category of pairs (Ω, Γ) (in both categories morphisms are isomorphisms).

Conversely, if Γ is a cut system of Ω then one can construct cut surface $(\Omega_*, \Gamma_*, \tau)$ as follows. Points of Ω_* are points of $\Omega \setminus \Gamma$ and pairs (x, c), where $x \in \Gamma$ and c is a coorientation of Γ in a neighborhood of point x. The stratification of Ω_* and gluing system (Γ_*, τ) are defined evidently (Fig.1b). Clearly, we constructed a functor cut : $(\Omega, \Gamma) \to (\Omega_*, \Gamma_*, \tau)$.

Lemma 3.2. *Functors* glue *and* cut *establish the equivalence of categories of cut surfaces (with morphisms defined as isomorphisms of cut surfaces) and pairs* (Ω, Γ) *(with morphisms defined as isomorphisms of pairs).*

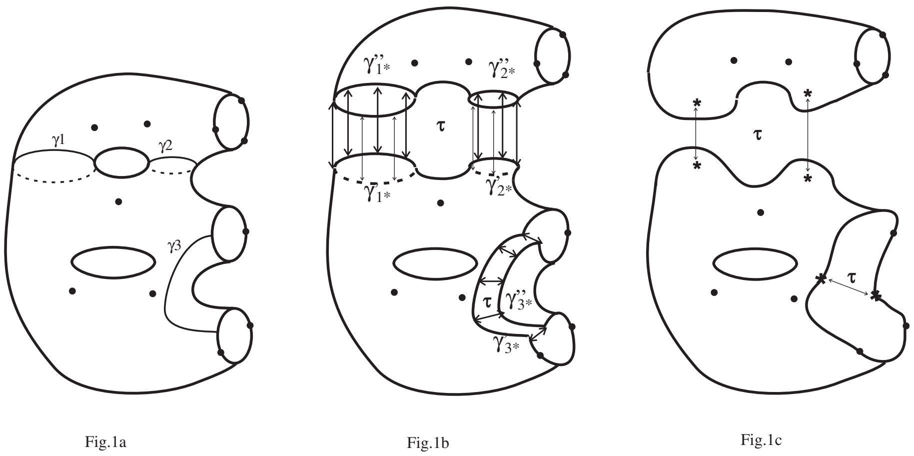

FIGURE 1. The construction of a contracted cut surface.
Fig.1a. Cut system $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ of surface Ω.
Fig.1b. Cut surface $(\Omega_*, \Gamma_*, \tau)$ obtained from cut system Γ. Here $\Gamma_* = \{\gamma'_{1*}, \gamma''_{1*}, \gamma'_{2*}, \gamma''_{2*}, \gamma'_{3*}, \gamma''_{3*}\}$.
Fig.1c. Contracted cut surface $\Omega_\# = \Omega/\Gamma$. Special points obtained as contractions of the connected components of Γ_* are marked by $*$.

Proof is skipped.

Let $(\Omega_*, \Gamma_*, \tau)$ be a cut surface. Denote by $\Omega_\#$ a surface obtained by contracting each connected component of Γ_* to a point. Clearly, the stratification of Ω_* induces the stratification of $\Omega_\#$. Special points of this stratification are images of special points of Ω_* and points that are obtained as contracted connected components of Γ_* (Fig.1c). Homeomorphism τ induces the involution of the set of special points of $\Omega_\#$ coming from contracted connected components. We denote this involution by the same letter τ. Clearly, we have constructed a functor $\mathsf{contr} : (\Omega_*, \Gamma_*, \tau) \to \Omega_\#$ from category of cut surfaces to a category of stratified surfaces.

Therefore, we can assign a stratified surface $\Omega_\# = \mathsf{contr}(\mathsf{cut}(\Omega, \Gamma))$ to any pair (Ω, Γ), where Γ is a cut system of a stratified surface Ω. We call $\Omega_\#$ *a contracted cut surface* and denote it by Ω/Γ.

Remark. The set $(\Omega_\#)_0$ of special points of a contracted cut surface $\Omega_\# = \Omega/\Gamma$ carries an additional structure. Namely, 1) $(\Omega_\#)_0$ contains the distingushed subset Ω_0 consisting of special points that are images of special points of Ω; 2) there is fixed involution $\tau : (\Omega_\#)_0 \setminus \Omega_0 \to (\Omega_\#)_0 \setminus \Omega_0$ (τ may have fixed points).

Let us classify simple cuts of a surface as follows. Denote by γ an arbitrary simple cut of a connected stratified surface Ω of type $G = (g, \varepsilon, m, m_1, \ldots, m_s)$. Two situations may occur. First, contracted cut surface $\Omega/\{\gamma\}$ is a connected surface. Denote its type by $G_\# = (g_\#, \varepsilon_\#, m_\#, m_{\#1}, \ldots, m_{\#s})$. Second, $\Omega/\{\gamma\}$ consists of two connected components. Then type $G_\#$ of $\Omega/\{\gamma\}$ is the set of types of its connected components. Denote types of components by $G^i_\# = (g^i_\#, \varepsilon^i_\#, m^i_\#, m^i_{\#1}, \dots, m^i_{\#s^i})$, where $i = 1, 2$. In this case the image of a special point of Ω in $\Omega_\#$ belongs to one of components. Thus, γ induces a division $\Omega_0 = (\Omega_0)_1 \sqcup (\Omega_0)_2$ of set Ω_0 of special points of Ω into two subsets.

For a cut system γ consisting of one simple cut we use notations $(\Omega_*, \gamma_*, \tau)$ for cut surface obtained by cutting stratified surface Ω along γ.

Lemma 3.3. *Let Ω be a stratified surface. Then*

(1) *any simple cut γ of Ω belongs to just one of classes $1-9$ from the list below;*
(2) *the identities between types $G_\#$ and G that are written in each item of the list holds.*

List of classes of simple cuts

A. *In classes 1-4 a simple cut γ is supposed to be homeomorphic to a circle*

Class 1 consists of *separating contours* (Fig.2a). A simple cut γ is called a separating contour if and only if $\gamma_* \subset \Omega_*$ consists of two contours and surface $\Omega_\#$ consists of two connected components,

$G = (g^1_\# + g^2_\#, \varepsilon^1_\#\varepsilon^2_\#, m^1_\# + m^2_\# - 2, m^1_{\#1}, \dots, m^1_{\#s_1}, m^2_{\#1}, \dots, m^2_{\#s_2})$

Class 2 consists of *cuts of a handle* (Fig.2b). A simple cut γ is called a cut of a handle if and only if $\gamma_* \subset \Omega_*$ consists of two contours, $\Omega_\#$ is connected surface and $\varepsilon_\# = \varepsilon$,

$G = (g_\# + 1, \varepsilon_\#, m_\# - 2, m_{\#1}, \dots, m_{\#s})$.

Class 3 consists of *cuts of a neck of Klein bottle* (Fig.2c). A simple cut γ is called a cut of a neck of Klein bottle if and only if $\gamma_* \subset \Omega_*$ consists of two contours, $\Omega_\#$ is connected surface , $\varepsilon = 0$ and $\varepsilon_\# = 1$,

$G = (g_\# + 1, 0, m_\# - 2, m_{\#1}, \dots, m_{\#s})$.

Class 4 consists of *Möbius cuts* (Fig.2d). A simple cut γ is called a Möbius cut if and only if $\gamma_* \subset \Omega_*$ is connected contour. In this case $\varepsilon = 0$, $\Omega_\#$ is connected surface , $\varepsilon_\#$ is equal either to 0 or to 1,

$G = (g_\# + \frac{1}{2}, 0, m_\# - 1, m_{\#1}, \dots, m_{\#s})$.

B. *In classes 5-9 a simple cut γ is supposed to be homeomorphic to a segment.* Hence, γ_* consists of two disjoint segments.

Class 5 consists of *cuts between two holes* (Fig.2e). A simple cut γ is called a cut between two holes if and only if end points of γ belong to different boundary contours. In this case $\Omega_\#$ is a connected surface and in an appropriate numeration of boundary contours we obtain the identity $G_\# = (g, \varepsilon, m, m_1 + m_2 + 2, m_3, \dots, m_s)$.

C. *In classes 6-9 both boundary points of a segment γ are supposed to belong to the same boundary contour ω.* Denote by ω_* the preimage of ω in cut surface Ω_* and by $\omega_\#$ the image of ω_* in contracted cut surface $\Omega_\#$.

Class 6 consists of *separating segments* (Fig.2f). A simple cut γ is called a separating segment if and only if $\Omega_\#$ consists of two connected components. In this case $\omega_\#$ consists of two contours and in an appropriate numeration of boundary contours of $\Omega_\#$ we obtain the identity

$G = (g^1_\# + g^2_\#, \varepsilon^1_\# \varepsilon^2_\#, m^1_\#, +m^2_\#, m^1_{\#1} + m^2_{\#1} - 2, m^1_{\#2}, \dots, m^1_{\#s_1}, m^2_{\#2}, \dots, m^2_{\#s_2})$.

Class 7 consists of *cuts of a handle through a hole* (Fig.2g).

A simple cut γ is called a cut of a handle through a hole, if and only if $\Omega_\#$ is connected surface, $\omega_\#$ consists of two contours and $\varepsilon_\# = \varepsilon$. In this case in an appropriate numeration of boundary contours of $\Omega_\#$ we obtain the identity

$G = (g_\# + 1, \varepsilon_\#, m_\#, m_{\#1} + m_{\#2} - 2, m_{\#3}, \dots, m_{\#s})$.

Class 8 consists of *cuts of a neck of a Klein bottle through a hole* (Fig.2h). A simple cut γ is called a cut of a neck of a Klein bottle through a hole if and only if $\Omega_\#$ is connected surface, $\omega_\#$ consists of two contours, $\varepsilon = 0$ and $\varepsilon_\# = 1$.

In this case in an appropriate numeration of boundary contours of $\Omega_\#$ we obtain the identity

$G = (g_\# + 1, 0, m_\#, m_{\#1} + m_{\#2} - 2, m_{\#3}, \dots, m_{\#s})$.

Class 9 consists of *cuts across Möbius band* (Fig.2i). A simple cut γ is called a cut across Möbius band if and only if boundary contour $\omega_\#$ is connected contour.

In this case $\Omega_\#$ is a connected surface, $\varepsilon = 0$, $\varepsilon_\#$ equals either to 0 or to 1 and in an appropriate numeration of boundary contours of Ω we obtain the identity

$G = (g_\# + \frac{1}{2}, 0, m_\#, m_{\#1} - 2, m_{\#2}, \dots, m_{\#s})$.

Lemma 3.4. *Let Γ be a cut system of a stratified surface Ω. Then $\mu(\Omega) = \mu(\Omega_\#)$, where $\Omega_\# = \Omega/\Gamma$ is the contracted cut surface.*

Proof. The equality can be easily checked for every class of simple cuts. □

Definition. *Two cut systems Γ' and Γ'' of Ω are called equivalent if there exists an isomorphism $\phi : \Omega \to \Omega$ of the stratified surfaces conserving all special points, all one-dimensional strata and their orientations, and such that $\phi(\Gamma') = \Gamma''$.*

Remark. In this definition the requirement of conserving one-dimensional strata of a stratified surface Ω is essential only for boundary contours with zero, one or two special points. If a boundary contour contains more that two special points then any isomorfism of Ω conserving all

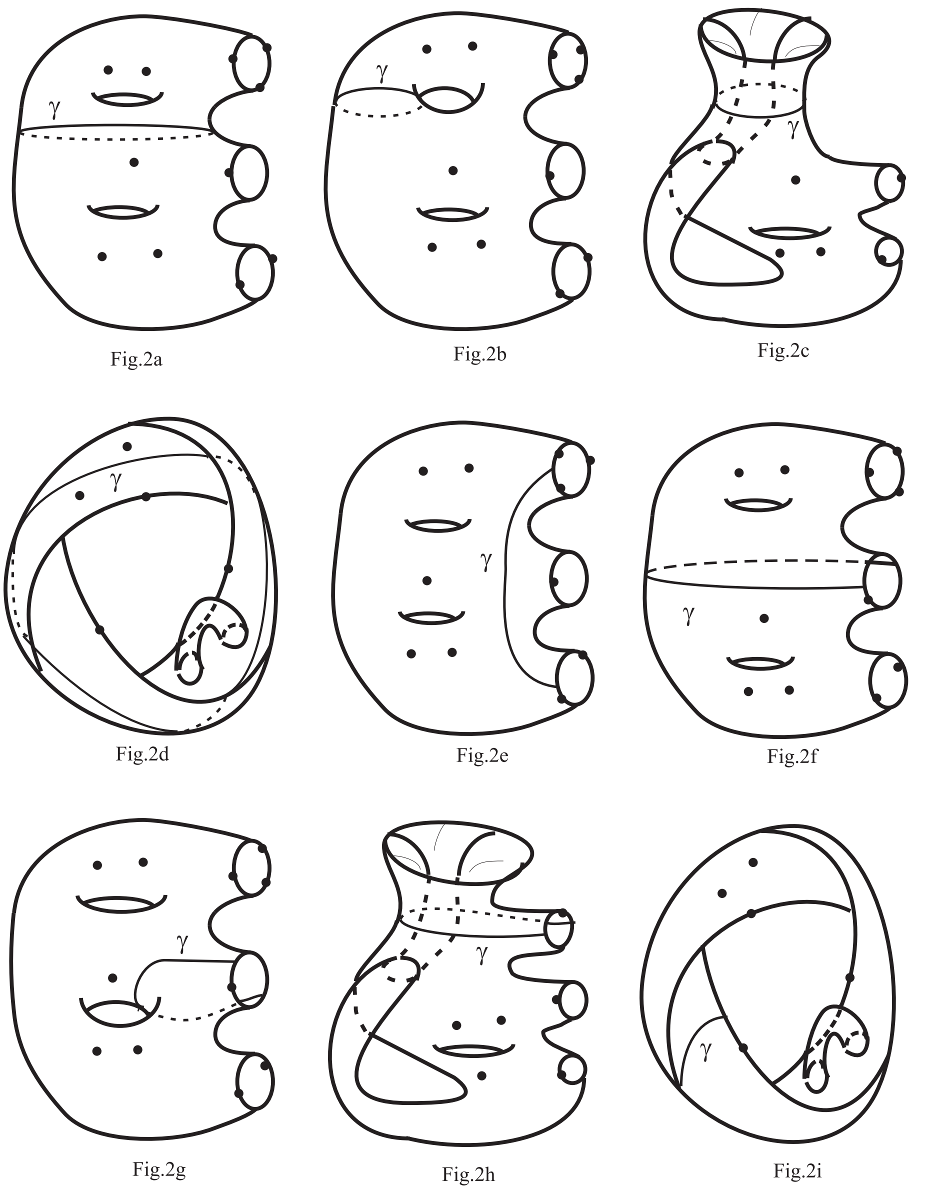

FIGURE 2. Examples of cuts of classes 1-9. Each cut is marked by γ. Fig.2a. A separating contour. Fig.2b. A cut of a handle. Fig.2c. A cut of a neck of Klein bottle. Fig.2d. A Möbius cut. Fig.2e. A cut between two holes. Fig.2f. A separating segment. Fig.2g. A cut of a handle through a hole. Fig.2h. A cut of a neck of a Klein bottle through a hole. Fig.2i. A cut across Möbius band

special points conservses also all one-dimentional strata on the contour and their orientations.

Lemma 3.5. *Two simple cuts γ' and γ'' of a stratified surface Ω are equivalent if and only if*

(1) *γ' and γ'' belong to the same class;*
(2) *types of contracted cut surfaces $\Omega'_{\#} = \Omega/\gamma'$ and $\Omega''_{\#} = \Omega/\gamma''$ coincide;*
(3) *if γ' and γ'' are separating contours or separating segments the divisions of the set of special points of Ω, induced by γ' and γ'', coincide;*
(4) *if γ' and γ'' are segments then their end points belong to the same one-dimensional strata.*

Proof. Obviously, if γ' and γ'' are equivalent cuts of a stratified surface Ω then conditions (1)–(4) are satisfied.

Conversely, let γ' and γ'' be simple cuts satisfying conditions (1) – (4). Denote by $(\Omega'_*, \gamma'_*, \tau'_*)$ and $(\Omega''_*, \gamma''_*, \tau''_*)$ cut surfaces obtained by cutting Ω along γ' and γ'' resp. Conditions (1) – (4) provide the equality of types of stratified surfaces Ω'_* and Ω''_*. Therefore, there exists an isomorphism $\phi : \Omega'_* \to \Omega''_*$ of stratified surfaces. Moreover, it can be easily shown for every class of simple cuts that one can choose ϕ satisfying the properties:

- for any special point $r \in \Omega_0$ of Ω isomorphism ϕ brings the image r' of r in Ω'_* to the image r'' of r in Ω''_*
- $\phi(\gamma'_*) = \gamma''_*$
- ϕ is the isomorphism of cut surfaces, i.e., $\phi \circ \tau' = \tau'' \circ \phi$.

Clearly, ϕ generates an homeomorphism $\overline{\phi} : \Omega \to \Omega$ such that $\overline{\phi}$ preserves all special points and $\overline{\phi}(\gamma') = \gamma''$. □

Corollary 3.1. *Let Γ' and Γ'' be two cut systems of a stratified surface Ω. Suppose Γ' contains a simple cut γ' and Γ'' contains a simple cut γ'' such that γ' and γ'' are of the same class, types of Ω/γ' and Ω/γ'' coincide, γ' and γ'' induces the same division of the set Ω_0 (if applicable) and end points of γ' and γ'' belong to the same one-dimensional stratum (if applicable). Then there exists a cut system $\overline{\Gamma}$ such that it is equivalent to Γ'' and contains cut γ'.*

Proof is evident.

Lemma 3.6. *Let Γ' and Γ be two equivalent cut systems of a stratified surface Ω. Fix local orientations in a small neighborhoods of all special points. Then there exists an isomorphism $\phi : \Omega \to \Omega$ of the stratified surface such that $\phi(\Gamma') = \Gamma$, ϕ fixes all special points of Ω and preserves an orientation of a small neighborhood of any special point.*

Proof is skipped.

Lemma 3.7. *Let Γ' and Γ'' be two cut systems of a stratified surface Ω. Suppose both Γ' and Γ'' contain the same simple cut γ. If cut systems $\Gamma' \setminus \gamma$ and $\Gamma'' \setminus \gamma$ of contracted cut surface Ω/γ are equivalent then Γ' and Γ'' are equivalent cut systems of Ω.*

Proof. Suppose, cut systems $\widetilde{\Gamma}' = \Gamma' \setminus \gamma$ and $\widetilde{\Gamma}'' = \Gamma'' \setminus \gamma$ of Ω/γ are equivalent. Let $(\Omega_*, \gamma_*, \tau_*)$ be the cut surface, corresponding to the pair (Ω, γ). Denote by ω_* the image of γ_* in Ω/γ. Clearly, ω_* consists of either two or one special point of contracted cut surface Ω/γ. Denote by U the joint of neighborhoods of points from ω_* such that $U \cap (\widetilde{\Gamma}' \cup \widetilde{\Gamma}'') = \emptyset$. By lemma 3.6, there exists a homeomorphism $\widetilde{\phi} : \Omega/\gamma \to \Omega/\gamma$, that is identical on U and such that $\widetilde{\phi}(\widetilde{\Gamma}') = \widetilde{\Gamma}''$. Thus there exist a homeomorphism $\phi_* : \Omega_* \to \Omega_*$ such that $\phi_*(\widetilde{\Gamma}') = \widetilde{\Gamma}''$ and $\tau_*\phi_* = \phi_*\tau_*$. Gluing by τ_* gives a homeomorphism $\phi : \Omega \to \Omega$ such that $\phi(\gamma) = \gamma$, $\phi(\widetilde{\Gamma}') = \widetilde{\Gamma}''$ and $\widetilde{\phi}$ fixes all special points. □

Corollary 3.2. *Let Γ' and Γ'' be two cut systems of a stratified surface Ω. Suppose Γ° is a cut system such that $\Gamma^\circ \subset \Gamma'$ and $\Gamma^\circ \subset \Gamma''$. If cut systems $\Gamma' \setminus \Gamma^\circ$ and $\Gamma' \setminus \Gamma^\circ$ of Ω/Γ° are equivalent cut systems of $\Omega_\# = \Omega/\Gamma_0$ then Γ' and Γ'' are equivalent cut systems of Ω.*

3.3. **Basic and simple surfaces.**

Definition. *A stratified surface Ω is called a stable surface if and only if any connected component Ω_i of Ω is a nontrivial surface (i.e., $\mu(\Omega_i) > 0$)*

A cut system $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ of a stable stratified surface Ω is calledd a stable cut system if and only if the contracted cut surface $\Omega_\# = \Omega/\Gamma$ is also stable.

A connected stratified surface Ω is called a basic surface if and only if there is no nonempty stable cut systems of Ω.

A cut system Γ of a stable stratified surface Ω is called a complete cut system if and only if any connected component of the contracted cut surface $\Omega_\# = \Omega/\Gamma$ is a basic surface.

Let us formulate several elementary statements. A complete cut system is a stable cut system. Let $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a cut system of a stratified surface Ω. Choose a subset $\Gamma' = \{\gamma_{i_1}, \ldots, \gamma_{i_l}\}$ of Γ. Then Γ' is a cut system of Ω and the image of $\Gamma \setminus \Gamma'$ in the contracted cut surface $\Omega_\# = \Omega/\Gamma'$ is a cut system of $\Omega_\#$. If Γ is a stable (complete) cut system then $\Gamma \setminus \Gamma'$ is a stable (complete) cut system of $\Omega_\#$. Conversely, a cut system Γ'' on the contracted cut surface $\Omega_\# = \Omega/\Gamma'$ can be lifted to a cut system $\Gamma = \Gamma' \sqcup \overline{\Gamma}''$, where $\overline{\Gamma}'' \subset \Omega$ is obtained from Γ'' in two steps: first, take the preimage of Γ'' in cut surface Ω_*; second, take image of the preimage under glueing map $\mathsf{glue} : \Omega_* \to \Omega$. If Γ'' is a stable (complete) cut system then Γ is a stable (complete) cut system of $\Omega_\#$.

Lemma 3.8. *A basic stratified surface Ω is isomorphic to one of the three stratified surfaces:*
(1) sphere (S^2, p_1, p_2, p_3) with three interior special points p_1, p_2, p_3 ($\mu = 1$);

(2) triangle, i.e., disc (D^2, q_1, q_2, q_3) *with three boundary special points* q_1*,* q_2*,* q_3 $(\mu = \frac{1}{2})$*;*
(3) disc (D, p, q) *with one interior special point* p *and one boundary special point* q $(\mu = \frac{1}{2})$*.*

Proof is elementary

Lemma 3.9. *For any stable stratified surface* Ω *there exists a complete cut system of* Ω*.*

Proof. It is sufficient to prove lemma for connected surfaces. If genus g of a stratified surface Ω is not zero, then there exist a cut of a handle or a Möbius cut. Using one of this cuts we reduce the question to a surface with less genus and, by inductive arguments, to a surface of genus zero. After it one can use inductive arguments with respect to the number of special points plus number of boundary contours. For example, if Ω contains at least two interior special points and is not a basic surface then a contour γ separating a disc with two of these points off the rest of surface is a stable cut and contracted cut surface Ω/γ is the disjoint union of a sphere with three interior points and a stable surface with less parameter of induction. We skip elementary details. □

Let us describe isomorphism class of complete cut systems for several stratified surfaces.

Lemma 3.10. *Denote by* N *be the number of isomorphism classes of complete cut systems of a stratified surfaces. Then*
(1) $N = 3$ *for a sphere with four interior special points (Fig.3a);*
(2) $N = 2$ *for a Klein bottle with one interior special point (Fig.3b);*
(4) $N = 2$ *for a disc with four boundary special points (Fig.3c);*
(5) $N = 2$ *for a disc with two boundary special points and one interior special point (Fig.3d);*
(6) $N = 3$ *for a disc with one boundary special point and two interior special points (Fig.3e);*
(7) $N = 2$ *for a Möbius band with one boundary special point (Fig.3f);*
(8) $N = 2$ *for a cylinder having one boundary special point on each boundary contour, two boundary special points in total (Fig.3g).*

Proof. (1) Let Ω be a sphere with four interior special points. Clearly, a complete cut system Γ of Ω consists of one separating contour that induces the division of Ω_0 into two pairs of special points. By lemma 3.5 two contours inducing the same division are equivalent. Therefore, there are three nonequivalent complete cut systems Γ_d marked by a partition d of the set $\{1, 2, 3, 4\}$ into two-element subsets.

(2) Let Ω be a Klein bottle with one interior special point. Any complete cut system of Ω contains a simple cut γ such that the genus of Ω/γ is less than 1. Clearly, only two cases are possible:

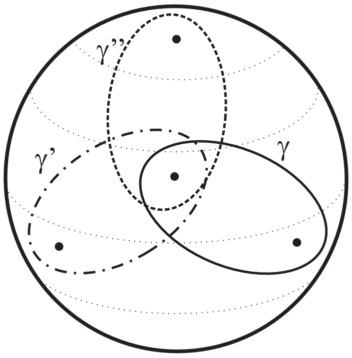

Fig.3a

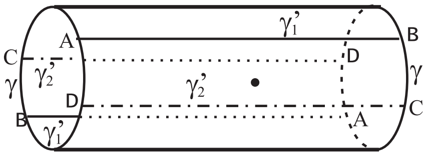

Fig.3b

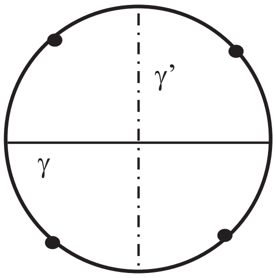

Fig.3c

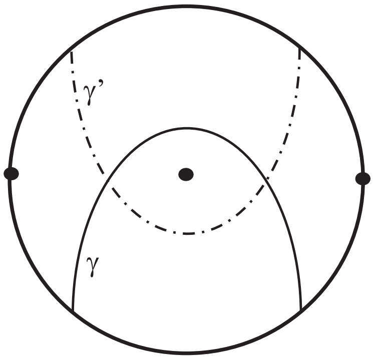

Fig.3d

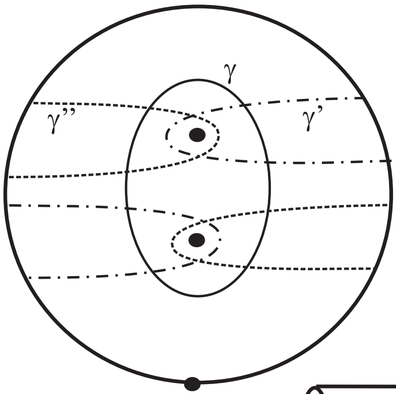

Fig.3e

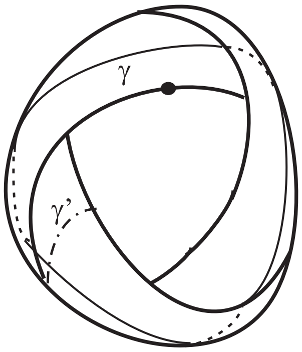

Fig.3f

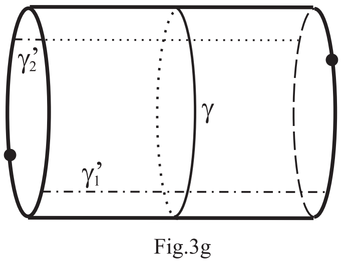

Fig.3g

FIGURE 3. Non-equivalent complete cut systems on simple stratified surfaces.
Different complete cut systems are drawn by diffent line types. Cuts in a cut system are denoted by γ with the same number of accents and are distinguished by lower indeces. Surfaces are numbered according to lemma 3.10. Fig.3b presents Klein bottle cutted by cut γ forming one cut system.

i) $\gamma = \gamma_k$, where γ_k denotes a cut of the neck of the Klein bottle. Clearly, $\Gamma_1 = \{\gamma_k\}$ is a complete cut system.

ii) $\gamma = \gamma_m$, where γ_m is a Möbius cut. Clearly, there is a complete cut system $\Gamma_2 = \{\gamma_m, \gamma'_m\}$ consisting of two Möbius cuts and all complete cut systems consisting of two Möbius cuts are equivalent.

(3) Let Ω be a surface of genus $g = \frac{3}{2}$ (i.e., a torus with a hole glued by a Möbius band) having no special points. There are the following stable simple cuts (up to equivalence) of Ω.

- Möbius cut γ_{m1} such that Ω/γ_{m1} is a torus with one interior special point;
- Möbius cut γ_{m2} such that Ω/γ_{m2} is a Klein bottle with one interior special point;
- cut of a handle γ_h; in this case Ω/γ_h is $\mathbb{R}P^2$ with two interior special points.

Clearly, any complete cut system Γ of Ω contains simple cut γ reducing the genus of Ω by $\frac{1}{2}$ (i.e., genus of Ω/γ is equal to 1). Hence, γ is a Möbius cut.

Suppose, $\gamma = \gamma_{m1}$. Then, evidently, $\Gamma = \Gamma_1$, where $\Gamma_1 = \{\gamma_{m1}, \gamma_h\}$ and all complete cut systems containing γ_{m1} are equivalent.

Suppose, $\gamma = \gamma_{m2}$. Then $\Gamma \setminus \gamma$ is a complete cut system of a Klein bottle Ω/γ having one interior special point. By (2) there are two isomorphism classes of complete cut systems of Ω/γ. Therefore, either $\Gamma = \Gamma_2$, where $\Gamma_2 = \{\gamma_{m2}, \gamma'_{m2}, \gamma''_{m2}\}$ or $\Gamma = \Gamma_3$, where $\Gamma_3 = \{\gamma_{m2}, \gamma_h\}$ (lifting of a cut of the neck of Klein bottle Ω/γ to Ω is a cut of a handle). The latter is not equivalent to γ_1 because Möbius cuts in them are not equivalent.

(3') Let Ω be a surface of genus $(g = \frac{3}{2})$ (i.e., a torus with a hole glued by a Möbius band) with one interior special point. There are the following stable simple cuts (up to equivalence) of Ω.

- Möbius cut γ_{m1} such that Ω/γ_{m1} is a torus with two interior special points;
- Möbius cut γ_{m2} such that Ω/γ_{m2} is a Klein bottle with two interior special points;
- separating contour γ_{s1} such that Ω/γ_{s1} consists of two connected components, namely, a torus with one interior special point and $\mathbb{R}P^2$ with two interior special points.
- separating contour γ_{s2} such that Ω/γ_{s2} consists of two connected components, namely, a Klein bottle with one interior special point and $\mathbb{R}P^2$ with two interior special points;
- cut of a handle γ_h; in this case Ω/γ_h is $\mathbb{R}P^2$ with three interior special points.

Let Γ be a complete cut system of Ω.

Suppose, Γ contains a separating contour γ_{s1}. Then Ω/γ_{s1} is the disjoint union of a torus with one interior special point and $\mathbb{R}P^2$ with two interior special points. Clearly, there is only one equivalent class of complete cut systems of Ω/γ_{s1}. The lifting Γ_1 of a complete cut system of Ω/γ_{s1} to Ω consists of the following simple cuts: $\Gamma_1 = \{\gamma_{m1}, \gamma_{s1}, \gamma_h\}$.

Suppose, Γ contains a separating contour γ_{s2}. Then Ω/γ_{s2} is the disjoint union of a a Klein bottle with one interior special point and $\mathbb{R}P^2$ with two interior special points. By (2) there are two equivalency classes of complete cut systems of Ω/γ_{s2}. Their liftings to Ω are as follows: $\Gamma_2 = \{\gamma_{m2}, \gamma_{s2}, \gamma_h\}$ and $\Gamma_3 = \{\gamma_{m2}, \gamma'_{m2}, \gamma''_{m2}, \gamma_{s2}\}$.

Suppose Γ does not contain a separating contour and contains a simple cut γ_{m1}. Therefore, Ω/γ_{m1} is a torus with two interior special points. It can be shown that in this case $\Gamma = \Gamma_4$, where $\Gamma_4 = \{\gamma_{m1}, \gamma_h, \gamma'_h\}$.

Suppose, Γ contains a simple cut γ_{m2}. Therefore, Ω/γ_{m1} is a Klein bottle with two interior special points. It can be shown that in this case $\Gamma = \Gamma_5$, where $\Gamma_5 = \{\gamma_{m2}, \gamma_h, \gamma'_h\}$.

(4) Let Ω be a disc with four boundary special points. Clearly, any complete cut system of Ω consists of a single simple cut γ, which is separating segment inducing the partition of Ω_0 into two pairs of consecutive boundary points. Mark boundary special points by $0, 1, 2, 3$ going around boundary contour. There are two isomorphism classes of γ, namely, $\Gamma_1 = \{\gamma_{0,1}\}$, where $\gamma_{0,1}$ separates points $0, 1$ of $2, 3$, and $\Gamma_2 = \{\gamma_{1,2}\}$, where $\gamma_{1,2}$ separates points $1, 2$ of $3, 0$.

(5) Let Ω be a a disc with two boundary special points and one interior special point. Clearly, any complete cut system of Ω consists of a single simple cut γ, which is separating segment, its end points belong to the same segment of the boundary and γ surrounds the interior special points. Thus, there are two isomorphism classes of complete cut systems of Ω, namely, $\Gamma_1 = \{\gamma_l\}$ and $\Gamma_2 = \{\gamma_r\}$, where end points of γ_l (resp. γ_r) belong to the 'left' (resp. 'right') segment of the boundary contour.

(6) Let Ω be a a disc with one boundary special point and two interior special points. Clearly, there are three isomorphism classes of complete cut systems. First of them, denoted by Γ_1, consists of a single separating contour γ_o. Each of two other complete cut systems, Γ_2 and Γ_3 consist of two separating segments γ_s, γ'_s, such that end points of any segment belong to the same segment of the boundary and the segment surrounds one interior special points. The only difference is the order of end points of two segments with respect to a fixed orientation of the boundary contour.

(7) Let Ω be a a Möbius band with one boundary special point. Clearly, there are two isomorphism classes of complete cut systems of Ω. They are as follows.

$\Gamma_1 = \{\gamma_m\}$, where γ_m is a Möbius cut of Ω.

$\Gamma_2 = \{\gamma_{mh}\}$, where γ_{mh} is a cut across Möbius band.

(8) Let Ω be a a cylinder having one boundary special point on each boundary contour, two boundary special points in total. Clearly, there

are two isomorphism classes of complete cut systems of Ω. They are as follows.

$\Gamma_1 = \{\gamma_s\}$, where γ_s is a separating contour.

$\Gamma_2 = \{\gamma_{sh}\}$, where γ_{sh} is a cut between two holes.

□

Definition. *Stratified surfaces (1) – (8) from list in lemma 3.10 are called simple stratified surfaces.*

3.4. **Neighboring complete cut systems.** In this subsection we deal with complete cut systems of a fixed stratified surface. It is supposed that any boundary contour of Ω contains at least one special point.

We call a pair of complete cut systems Γ_1 and Γ_2 of Ω *an adjacent pair* if the following condition holds. Denote by Γ_0 the joint of all simple cuts that belongs to Γ_1 and Γ_2 simultaneously and denote by $\Omega_\#$ the contracted cut surface Ω/Γ_0. Both $\Gamma_1 \setminus \Gamma_0$ and $\Gamma_2 \setminus \Gamma_0$ give rise to the complete cut systems of $\Omega_\#$. The condition is that cut systems $\Gamma_1 \setminus \Gamma_0$ and $\Gamma_2 \setminus \Gamma_0$ belong to the same connected component of $\Omega_\#$ and this component is a simple surface.

We call isomorphism classes C_1 and C_2 of complete cut systems *an adjacent classes* if and only if there exists representatives $\Gamma_1 \in C_1$ and $\Gamma_2 \in C_2$ such that Γ_1 and Γ_2 are adjacent complete cut systems.

We call two isomorphism classes C', C'' of complete cut systems *a neighboring classes* if and only if there exists a sequence of isomorphism class $C_1, \dots, C_k$ such that $C_1 = C'$, $C_k = C''$ and C_i is adjacent to C_{i+1} for $i = 1, \dots, k-1$. We call two complete cut systems from neighboring isomorphism classes *neighbors.*

We call a stable surface *an almost simple surface* if and only if it is isomorphic to one of the following stratified surfaces:

(1) a disc with two or less interior special points and an arbitrary number of boundary special points;

(2) a cylinder with at most one interior special points and an arbitrary number of boundary special points;

(3) a Möbius band with at most one interior special points and an arbitrary number of boundary special points.

(4) a surface of genus ($g = \frac{3}{2}$) (i.e., a torus with a hole glued by a Möbius band) having no special points or with one interior special point.

Lemma 3.11. *All complete cut systems of an almost simple stratified surface Ω are neighbors.*

Proof. If Ω is a disc without interior special points then the statement of lemma can be reformulated in terms of triangulations of a polygon and follows from the connectedness of an associhedron [8].

Let Ω be a disc with one interior special point. Then the statement of lemma in this case follows from the following claims.

1) Any complete cut system of Ω contains a unique simple cut γ such that Ω/γ consists of two connected components and one of components is a disc with one interior and one boundary special points. Let us call a cut of this type a type H cut. Obviously end points of γ belong to the same boundary segment.

2) All complete cut systems of Ω containing fixed cut γ of type H are neighbours.

3) A complete cut system Γ containing γ of type H is adjacent to a complete cut system Γ' containing a cut γ' of type H such that its end points lie on the neighbor segment to a segment containing end points of γ.

Let Ω be a disc with two interior special point. In this case the following claims leads to the proof of lemma.

1) A complete cut system Γ contains either two cuts of type H or a contour separating a disc with two interior special points from the rest of surface. We call the last cut *a cut of type* P.

2) A complete cut system containing a cut of type H is a neighbor of a complete cut system containing a cut of type P.

3) All complete cut systems containing a fixed cut γ of type H are neighbors.

4) A complete cut system Γ containing γ of type H is adjacent to a complete cut system Γ' containing a cut γ' of type H such that its end points lie on the neighbor segment to a segment containing end points of γ.

5)Let Ω be a surface of genus $g = \frac{3}{2}$ (i.e., a torus with a hole glued by a Möbius band) having no special points. We will prove that there are just 3 nonequivalent complete cut systems of Ω and all of them are neighbours.

There are the following stable simple cuts (up to equivalence) of Ω.

- Möbius cut γ_{m1} such that Ω/γ_{m1} is a torus with one interior special point;
- Möbius cut γ_{m2} such that Ω/γ_{m2} is a Klein bottle with one interior special point;
- cut of a handle γ_h; in this case Ω/γ_h is $\mathbb{R}P^2$ with two interior special points.

Clearly, any complete cut system Γ of Ω contains simple cut γ reducing the genus of Ω by $\frac{1}{2}$ (i.e., genus of Ω/γ is equal to 1). Hence, γ is a Möbius cut.

Suppose, $\gamma = \gamma_{m1}$. Then, evidently, $\Gamma = \Gamma_1$, where $\Gamma_1 = \{\gamma_{m1}, \gamma_h\}$ and all complete cut systems containing γ_{m1} are equivalent.

Suppose, $\gamma = \gamma_{m2}$. Then $\Gamma \setminus \gamma$ is a complete cut system of a Klein bottle Ω/γ having one interior special point. By (2) there are two isomorphism classes of complete cut systems of Ω/γ. Therefore, either $\Gamma = \Gamma_2$, where $\Gamma_2 = \{\gamma_{m2}, \gamma'_{m2}, \gamma''_{m2}\}$ or $\Gamma = \Gamma_3$, where $\Gamma_3 = \{\gamma_{m2}, \gamma_h\}$ (lifting of a cut of the neck of Klein bottle Ω/γ to Ω is a cut of a handle).

The latter is not equivalent to γ_1 because Möbius cuts in them are not equivalent.

Finally, it can be easely shown, that Γ_2 is a neighbor of Γ_3 and Γ_1 is a neighbor of Γ_3. Indeed, both cut systems in a pair have equivalent simple cut.

(5') Let Ω be a surface of genus ($g = \frac{3}{2}$) (i.e., a torus with a hole glued by a Möbius band) with one interior special point.

We will prove that there are just 5 nonequivalent complete cut systems of Ω and all of them are neighbours.

There are the following stable simple cuts (up to equivalence) of Ω.

- Möbius cut γ_{m1} such that Ω/γ_{m1} is a torus with two interior special points;
- Möbius cut γ_{m2} such that Ω/γ_{m2} is a Klein bottle with two interior special points;
- separating contour γ_{s1} such that Ω/γ_{s1} consists of two connected components, namely, a torus with one interior special point and $\mathbb{R}P^2$ with two interior special points.
- separating contour γ_{s2} such that Ω/γ_{s2} consists of two connected components, namely, a Klein bottle with one interior special point and $\mathbb{R}P^2$ with two interior special points;
- cut of a handle γ_h; in this case Ω/γ_h is $\mathbb{R}P^2$ with three interior special points.

Let Γ be a complete cut system of Ω.

Suppose, Γ contains a separating contour γ_{s1}. Then Ω/γ_{s1} is the disjoint union of a torus with one interior special point and $\mathbb{R}P^2$ with two interior special points. Clearly, there is only one equivalent class of complete cut systems of Ω/γ_{s1}. The lifting Γ_1 of a complete cut system of Ω/γ_{s1} to Ω consists of the following simple cuts: $\Gamma_1 = \{\gamma_{m1}, \gamma_{s1}, \gamma_h\}$.

Suppose, Γ contains a separating contour γ_{s2}. Then Ω/γ_{s2} is the disjoint union of a a Klein bottle with one interior special point and $\mathbb{R}P^2$ with two interior special points. By (2) there are two equivalency classes of complete cut systems of Ω/γ_{s2}. Their liftings to Ω are as follows: $\Gamma_2 = \{\gamma_{m2}, \gamma_{s2}, \gamma_h\}$ and $\Gamma_3 = \{\gamma_{m2}, \gamma'_{m2}, \gamma''_{m2}, \gamma_{s2}\}$.

Suppose Γ does not contain a separating contour and contains a simple cut γ_{m1}. Therefore, Ω/γ_{m1} is a torus with two interior special points. It can be shown that in this case $\Gamma = \Gamma_4$, where $\Gamma_4 = \{\gamma_{m1}, \gamma_h, \gamma'_h\}$.

Suppose, Γ contains a simple cut γ_{m2}. Therefore, Ω/γ_{m1} is a Klein bottle with two interior special points. It can be shown that in this case $\Gamma = \Gamma_5$, where $\Gamma_5 = \{\gamma_{m2}, \gamma_h, \gamma'_h\}$.

Like in the case (5), it can be easely shown, that Γ_1 is a neighbor of Γ_2, Γ_1 is a neighbor of Γ_4 and Γ_2, Γ_3, Γ_5 are neighbors. Indeed, both cut systems in each pair have equivalent simple cut.

6) For the rest of almost simple surfaces the proof is by similar arguments, we skip them here.

□

A pair of complete cut systems is said to be parallel if they contain at least one common simple cut. A pair of isomorphism classes of complete cut systems is said to be parallel if there exists parallel representatives of the classes.

Lemma 3.12. *All complete cut systems of an orientable surface Ω without a boundary are neighbors.*

Proof. We use inductive arguments with respect to genus g of the surface Ω. The statement of lemma is known for the sphere [21]. If $g > 0$ then any complete cut system Γ contains a cut of a handle. By lemmas 3.5 and corollary 3.1, any pair of isomorphism classes of complete cut systems is parallel. The proof follows from lemma 3.7 and the inductive arguments. □

A simple cut γ of Ω is called *projective* if it is a Möbius cut and Ω/γ is a nonorientable surface.

Lemma 3.13. *All complete cut systems of a nonorientable surface Ω without a boundary are neighbors.*

Proof. We use inductive arguments with respect to genus g of the surface Ω. Let Γ be a complete cut system of Ω.

If $g = \frac{1}{2}$ then any complete cut system of Ω contains a Möbius cut. By lemma 3.12 and 3.7, we obtain that all complete cut systems are neighbors. Similar arguments are valid for $g = 1$. Using a slightly more complicated reasoning one can prove lemma for $g = \frac{3}{2}$.

Let $g > \frac{3}{2}$ and lemma is proven for smaller genera. It can be shown, that any complete cut system Γ contains a simple cut γ such that genus $g_\#$ of Ω/γ is less than the genus of Ω and Ω/γ is nonorientable. By inductive arguments, $\Gamma \setminus \gamma$ is equivalent to a complete cut system containing a projective cut. By lemma 3.7, Γ is equivalent to a complete cut system containing a projective cut. Therefore, all isomorphism classes of complete cut systems of Ω are parallel. The proof of lemma is completed by inductive arguments using lemma 3.7. □

A stable cut γ of a connected stratified surface Ω is called *normal* if it is either a contour or a separating segment such that one of the connected components of a contracted cut surface is homeomorphic to a disk without interior special points (no restrictions for boundary special points). If Ω is disconnected stratified surface then γ is said normal if it is a normal cut of a connected component of Ω.

Clearly, any simple cut of a stratified surface without boundary is normal.

We call a simple cut, that is not normal cut, a *special* cut.

Definition. *A complete cut system Γ is called normal cut system if Γ consists of normal simple cuts only.*

For example, any complete cut system of a stratified surface without boundary is normal.

Clearly, if Γ is a normal cut system of Ω and $\Gamma_0 \subset \Gamma$ is a subsystem of cuts then $\Gamma \setminus \Gamma_0$ is normal cut system of Ω/Γ_0.

Lemma 3.14. *Any stable surface admits a normal complete cut system.*

Proof. The proof is by inductive arguments with respect to the genus and the number of boundary contours. The step of induction is from Ω to Ω/γ where γ is a normal cut. $\square$

Lemma 3.15. *All normal cut systems of a stratified surface Ω are neighbors.*

Proof. If Ω is a surfaces without a boundary then the lemma follows from lemmas 3.12 and 3.13.

If Ω is a disk without interior special points then the lemma follows from lemma 3.11.

In order to prove the lemma for an arbitrary Ω we use inductive arguments with respect to the number of boundary contours. Let Γ' and Γ'' be normal cut systems of Ω. Let ω be a boundary contour of Ω. It can be easily shown that any normal cut system contains a separating oval γ with the following properties. The connected component of Ω/γ that contains ω is homeomorphic to a disc with one interior special point (coming from γ). The homotopy class of γ does not depend on the choice of a normal cut system. Therefore, all isomorphism classes of normal cut systems of Ω are parallel. The proof of the lemma is completed by inductive arguments. $\square$

Lemma 3.16. *Let Ω be a stable surface and γ be a stable special cut of Ω. Suppose, Ω is not a simple or almost simple surface. Then there exists a normal cut γ' homeomorphic to an oval such that $\{\gamma, \gamma'\}$ is a stable cut system of Ω.*

Proof. If contracted cut surface $\Omega_\# = \Omega/\gamma$ admits a stable simple cut homeomorphic to an oval then the claim of lemma is obviously true. Suppose, $\Omega_\#$ does not admit a simple cut with this property. Then any connected component of $\Omega_\#$ is either a basic surface or a disk with no more than one interior special point. It can be proven by exhaustion that Ω is either a simple or almost simple surface. $\square$

Let Γ be a complete cut system of Ω. Denote by Γ_n the subset of all normal cuts from Γ. Γ is called *an almost normal* if one connected component of Ω/Γ_n is either a simple or an almost simple surface and all other connected components are basic surfaces.

By lemmas 3.11 and 3.14, we obtain the following statement.

Lemma 3.17. *Any almost normal cut system has a neighbor that is a normal cut system.*

Lemma 3.18. *Any special stable cut γ can be included in an almost normal complete cut system.*

Proof. Use inductive arguments with respect to genus, number of boundary contours and number of interior special points.

If Ω is either a simple or an almost simple surfaces then the claim of lemma is trivial.

Otherwise choose a normal simple cut γ' provided by lemma 3.16. By inductive hypothesis, there exists an almost normal cut system $\Gamma_\#$ of the contracted cut surface Ω/γ' such that $\gamma \subset \Omega_\#$. Clearly, $\Gamma_\# \sqcup \gamma'$ is an almost simple cut system of Ω. □

Theorem 3.1. *Let Ω be an arbitrary stable stratified surface. Then all complete cut systems of Ω are neighbors.*

Proof. By lemma 3.15, it is sufficient to prove that any complete cut system Γ is the neighbor of a normal cut system. Use inductive arguments with respect to the number of special cuts. Suppose, Γ contains $n > 0$ special cuts. Let $\gamma \in \Gamma$ be a special cut. By lemma 3.18, there exists an almost normal cut system Γ' such that $\gamma \subset \Omega'$.

By inductive hypothesis, complete cut systems $\Gamma \setminus \gamma$ and $\Gamma' \setminus \gamma$ of the contracted cut surface Ω/γ are neighbors. Therefore, Γ and Γ' are neighbors. By lemma 3.17, Γ' is the neighbor of a normal cut system. Hence, Ω is the neighbor of a normal cut system. □

4. Two-dimensional topological field theory

4.1. **Definition of Klein topological field theory.** First, we shall fix a tensor category of surfaces (a 'basic' category) and a functor from it to the tensor category of vector spaces.

A set of local orientations. For any special point $r \in Q$ fix an orientation o_r of its small neighborhood. Denote the set of these local orientations by $\mathcal{O}$. Let Ω be a connected stratified surface. A set $\mathcal{O}$ of local orientations is called *admissible* if and only if either Ω is orientable surface and all local orientations are induced by an orientation of Ω (Fig.4a) or Ω is a nonorientable surface and local orientations at all special points from any boundary contour ω_i are compatible with one of orientations of ω_i (Fig.4b). Thus, there are two admissible sets of local orientations in the former case and there are 2^s, where s is a number of connected components of $\partial\Omega$, admissible sets of local orientations in the latter case.

A set of local orientations for disconnected Ω is said admissible if and only if it is admissible for each connected component of Ω.

Lemma 4.1. *Let $(\Omega', \mathcal{O}')$ and $(\Omega'', \mathcal{O}'')$ be two pairs consisting of stratified surfaces and admissible sets of local orientations at special points. If stratified surfaces Ω' and Ω'' are isomorphic then there exists an isomorphism $\phi : \Omega' \to \Omega''$ such that $\phi(\mathcal{O}') = \mathcal{O}''$.*

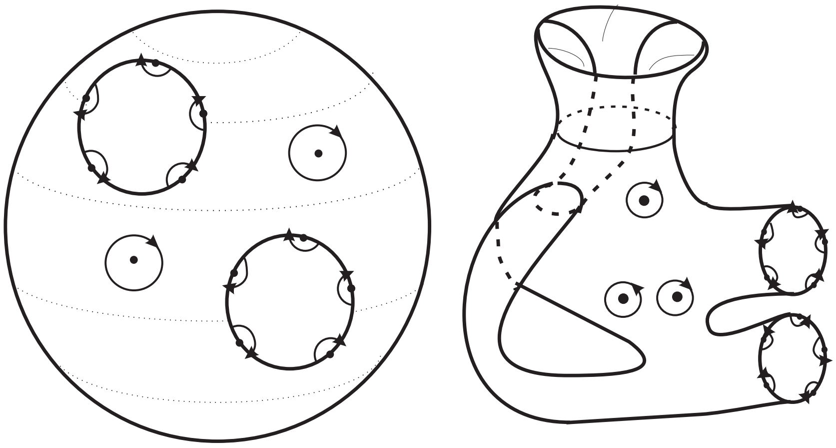

Fig.4a Fig.4b

FIGURE 4. Admissible set of local orientations of (4a) oriented surface; (4b) non-oriented surface.

Proof follows from standard properties of surfaces.

Let Ω be a stratified surface. Denote by Ω_a the set of all interior special points and by Ω_b the set of all boundary special points. Put also $\Omega_0 = \Omega_a \sqcup \Omega_b$. Fix a local orientation o_r in a small neighborhood of any special point $r \in \Omega_0$ and denote by $\mathcal{O}$ the set of all these local orientations. Pairs $(\Omega, \mathcal{O})$ are *objects of the basic category. Morphisms* are any combinations of morphisms of types 1)-4).

1) *Isomorphism* $\phi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$. By definition, ϕ is an isomorphism $\phi : \Omega \to \Omega'$ of stratified surfaces compatible with local orientations at special points.

2) *Changing of local orientations* $\psi : (\Omega, \mathcal{O}) \to (\Omega, \mathcal{O}')$. Thus, there is one such morphism for any pair $(\mathcal{O}, \mathcal{O}')$ of sets of local orientations on a stratified surface Ω.

3) *Adding a special point* $\xi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$. Morphism ξ depends on a point $r \in \Omega \backslash \Omega_0$ endowed with a local orientation o_r. By definition, topological surface Ω' coincides with surface Ω. The stratification of Ω' is defined as a refinement of the stratification of Ω by additional point r. Therefore, $\Omega'_0 = \Omega_0 \sqcup \{r\}$. Local orientation o_r completes the set of local orientations, i.e., $\mathcal{O}' = \mathcal{O} \sqcup \{o_r\}$.

4) *Cutting* $\eta : (\Omega, \mathcal{O}) \to (\Omega_\#, \mathcal{O}_\#)$. Morphism η depends on a cut system Γ endowed with orientations of all simple cuts $\gamma \in \Gamma$. $\Omega_\#$ is defined as contracted cut surface Ω/Γ. Stratified surface $\Omega_\#$ inherits special points of Ω and local orientations at any of them. A special point $r \in (\Omega_\#)_0 \setminus \Omega_0$ appears as contracted connected component of a cut contour γ_* in cut manifold Ω_*. In obvious way the orientation of simple cut γ induces the local orientation at r. The set of all these local orientations at all points of $(\Omega_\#)_0$ form the set $\mathcal{O}_\#$.

Define *'tensor product'* $\theta : (\Omega', \mathcal{O}') \times (\Omega'', \mathcal{O}'') \to (\Omega, \mathcal{O})$ of two pairs $(\Omega', \mathcal{O}')$ and $(\Omega'', \mathcal{O}'')$ as their disjoint union $(\Omega, \mathcal{O}) = (\Omega' \sqcup \Omega'', \mathcal{O}' \sqcup \mathcal{O}'')$.

We shall define below a functor $(\Omega, \mathcal{O}) \to V(\Omega, \mathcal{O})$ from the basic category of surfaces to the category of vector spaces.

Let $\{X_m | m \in M\}$ be a finite set of $n = |M|$ vector spaces X_m. The action of symmetric group S_n on the set $\{1, \ldots, n\}$ induces its action on the linear space $\big(\oplus_{\sigma:\{1,\ldots,n\} \leftrightarrows M} X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}\big)$, an element $s \in S_n$ takes $X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(n)}$ to $X_{\sigma(s(1))} \otimes \cdots \otimes X_{\sigma(s(n))}$. Denote by $\otimes_{m \in M} X_m$ the subspace of all invariants of this action.

Vector space $\otimes_{m \in M} X_m$ is canonically isomorphic to a tensor product of all X_m in any fixed order, the isomorphism is a projection of vector space $\otimes_{m \in M} X_m$ to the summand that is equal to the tensor product of X_m in a given order. Assume that all X_m are equal to a fixed vector space X. Then any bijection $M \leftrightarrow M'$ of sets induces the isomorphism $\otimes_{m \in M} X_m \leftrightarrow \otimes_{m' \in M'} X_{m'}$.

Let A and B be finite dimensional vector spaces over complex numbers endowed with involutive transformations $A \to A$ and $B \to B$, which we denote by $x \mapsto x^*$ $(x \in A)$ and $y \mapsto y*$ $(y \in B)$ resp.

Let $(\Omega, \mathcal{O})$ be a pair consisting of a stratified surface Ω and a set $\mathcal{O}$ of local orientations at its special points. Assign a copy A_p of a vector space A to any point $p \in \Omega_a$ and a copy B_q of a vector space B to any point $q \in \Omega_b$. Elements of A_p (resp. B_q) are called interior (resp. boundary) fields. Put $V_\Omega = (\otimes_{p \in \Omega_a} A_p) \otimes (\otimes_{q \in \Omega_b} B_q)$.

For any morphism of pairs $(\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ define a morphism of vector spaces $V_\Omega \to V_{\Omega'}$ as follows.

1) An isomorphism $\phi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ induces the isomorphism $\phi_* : V_\Omega \to V_{\Omega'}$ because ϕ generates the bijections $\Omega_a \leftrightarrows \Omega'_a$ and $\Omega_b \leftrightarrows \Omega'_b$ of sets of special points.

2) For a changing of local orientations $\psi : (\Omega, \mathcal{O}) \to (\Omega, \mathcal{O}')$ define a linear map $\psi_* : V_\Omega \to V_\Omega$ as $(\otimes_{r \in \Omega_0} \psi_r)$, where for any $r \in \Omega_0$

$$\psi_r(x) = \begin{cases} x & \text{if } o_r = o'_r \\ x^* & \text{if } o_r = -o'_r \end{cases}$$

3) In order to define a morphism $\xi_* : V_\Omega \to V_{\Omega'}$ for adding a special point $\xi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ we need to fix elements $1_A \in A$ and $1_B \in B$. These notations of elements are motivated in the frame of a Klein topological field theory, where they are 'trivial fields'.

In this case there exists a canonical isomorphism $V_{\Omega'} = V_\Omega \otimes X$, where X is equal either to A (if adding special point r belongs to the interior of Ω) or to B (if adding special point r belongs to the boundary of Ω). For $x \in V_\Omega$ put $\xi_*(x) = x \otimes 1_X$, where 1_X is either 1_A or 1_B resp.

4) In order to define a morphism $\eta_* : V_\Omega \to V_{\Omega_\#}$ for any cutting morphism $\eta : (\Omega, \mathcal{O}) \to (\Omega_\#, \mathcal{O}_\#)$ we need to fix elements $\widehat{K}_{A,*} \in A \otimes A$, $\widehat{K}_{B,*} \in B \otimes B$ and $U \in A$. (Notations of them will be clear from the sequel.)

Evidently, it is sufficient to define η_* for an arbitrary oriented simple cut $\gamma \subset \Omega$. In this case we have a canonical isomorphism $V_{\Omega_\#} = V_\Omega \otimes X$, where

$$X = \begin{cases} A \otimes A & \text{if } \gamma \text{ is an contour and is not Möbius cut} \\ B \otimes B & \text{if } \gamma \text{ is a segment} \\ A & \text{if } \gamma \text{ is a Möbius cut} \end{cases}$$

For $x \in V_\Omega$ put $\eta_*(x) = x \otimes z$, where z is either $\widehat{K}_{A,*}$, or $\widehat{K}_{B,*}$, or U resp.

Finally, for a 'tensor product' $\theta : (\Omega', \mathcal{O}') \times (\Omega'', \mathcal{O}'') \to (\Omega' \sqcup \Omega'', \mathcal{O}' \sqcup \mathcal{O}'')$ there is evident canonical linear map $\theta_* : V_{\Omega'} \otimes V_{\Omega''} \to V_{\Omega' \sqcup \Omega''}$

We say that a set of data $\mathcal{T} = (A, x \mapsto x^*, B, y \mapsto y^*, \{\Phi_{(\Omega,\mathcal{O})}\})$, where A and B are linear spaces, $x \mapsto x^*$ and $y \mapsto y^*$ are involutive linear transformations of A and B resp., $\{\Phi_{(\Omega,\mathcal{O})}\}$ is a system of linear forms $\Phi_{(\Omega,\mathcal{O})} : V_\Omega \to \mathbb{C}$, is a *Klein topological field theory* if the following axioms are satisfied.

1° *Topological invariance.*

For any isomorphism of pairs $\phi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ the following identity holds

$$\Phi_{(\Omega,\mathcal{O})}(x) = \Phi_{(\Omega',\mathcal{O}')}(\phi_*(x).)$$

2° *Invariance of a change of local orientations.*

For any change of local orientations $\psi : (\Omega, \mathcal{O}) \to (\Omega, \mathcal{O}')$ the following identity holds

$$\Phi_{(\Omega,\mathcal{O})}(x) = \Phi_{(\Omega,\mathcal{O}')}(\psi_*(x)).$$

3° *Nondegeneracy.*

Define first a bilinear form $(x, x')_A$ on vector space A. Namely, let $(\Omega, \mathcal{O})$ be a pair, where Ω is a sphere with 2 interior special points p, p' and the set $\mathcal{O} = \{o_p, o_{p'}\}$ is such that local orientations o_p, $o_{p'}$ induce the same global orientations of the sphere. Put $(x, x')_A = \Phi_{(\Omega,\mathcal{O})}(x_p \otimes x'_{p'})$, where x_p and $x'_{p'}$ are images of $x \in A$ and $x' \in A$ in A_p and $A_{p'}$ resp. The correctness of this definition follows from axioms 1° and 2°. Evidently, $(x, x')_A$ is a symmetric bilinear form.

Similarly, define a bilinear form $(y, y')_B$ on vector space B using a disc with 2 boundary special points q, q' instead of a sphere with two interior special points p, p'. As in the previous case local orientations o_q, $o_{q'}$ must induce the same global orientations of the disc. Evidently, $(y, y')_B$ is a symmetric bilinear form.

Axiom is that both forms $(x, x')_A$ and $(y, y')_B$ are nondegenerate.

4° *Invariance of adding unit fields.*

Axioms $1°-3°$ allows us to choose elements $1_A \in A$, $1_B \in B$. Indeed, let Ω be a sphere with one special point p and o_p be a local orientation at p. Then linear form $\Phi_{(\Omega,\{o_p\})}$ is an element of the vector space dual to A. Identify dual vector space with A by means of nondegenerate bilinear form $(x,y)_A$. Thus, we obtain an element of A, which we denote by 1_A. Analogously, starting from a disc with one boundary special point we obtain an element of B, which we denote by 1_B. An element 1_A (resp. 1_B) is called trivial interior (resp. boundary) field. We shall use just these elements in the definition of morphisms ξ_* of type 3).

Axiom is that for any adding of special point $\xi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ the following identity holds

$$\Phi_{(\Omega,\mathcal{O})}(x) = \Phi_{(\Omega',\mathcal{O}')}(\xi_*(x)).$$

$5°$ *Cut invariance.*

Axioms $1° - 3°$ allows us to choose elements $\widehat{K}_{A,*} \in A \times A$, $\widehat{K}_{B,*} \in B \times B$ and $U \in A$. Indeed, any nondegenerate bilinear form on a vector space X canonically defines an element $\widehat{K}_X \in X \otimes X$. Taking forms $(x,x')_{A,*} = (x, x'^*)_A$ and $(y,y')_{B,*} = (y, y'^*)_B$ we obtain elements $\widehat{K}_{A,*}$ and $\widehat{K}_{B,*}$.

Linear form $\Phi_{(\Omega,\mathcal{O})}$ for a projective plane Ω with one interior special point is an element of the vector space dual to A. We denote by U the image of this element in A under the isomorphism induced by nondegenerate bilinear form $(x,x')_A$.

We shall use just these elements for morphisms η_* of type 4).

Axiom is that for any cut system Γ endowed with orientations of all simple cuts the following identity is required

$$\Phi_{(\Omega,\mathcal{O})}(x) = \Phi_{(\Omega_\#,\mathcal{O}_\#)}(\eta_*(x))$$

(here η_* depends on Γ, see 4) above)

$6°$ *Multiplicativity.*

Axiom is that the product $\theta : (\Omega', \mathcal{O}') \times (\Omega'', \mathcal{O}'') \to (\Omega' \sqcup \Omega'', \mathcal{O}' \sqcup \mathcal{O}'')$ of any two pairs $(\Omega', \mathcal{O}')$ and $(\Omega'', \mathcal{O}'')$ the following identity holds

$$\Phi_{(\Omega',\mathcal{O}')}(x_1)\Phi_{(\Omega'',\mathcal{O}'')}(x_2) = \Phi_{(\Omega'\sqcup\Omega'',\mathcal{O}'\sqcup\mathcal{O}'')}(\theta_*(x_1 \otimes x_2))$$

The example of a Klein topological field theory is given in section 5.

4.2. **Correlators of Klein topological field theory.** Axioms of a topological field theory can be reformulated in terms of so called 'systems of correlators'. In terms of correlators the dependence of a linear form $\Phi_{(\Omega,\mathcal{O})}$ on an individual stratified surface Ω endowed with a set of local orientations $\mathcal{O}$ is reduced to the dependence on a topological type $G = (g, \varepsilon, m_1, \ldots, m_s)$ of a surface.

In this section we present the reformulation of a Klein topological field theory in terms of correlators. Let A and B be two linear spaces, $x \mapsto x^*$ and $y \mapsto y^*$ be fixed involutive transformations of A and B resp.

Denote by $G = (g, \varepsilon, m, m_1, \dots, m_s)$ a data that are the topological type of a connected stratified surface.

We assume throughout this section that

- $m_i > 0$ for $i = 1, \dots, s$ (m may be zero; s may be zero)
- $m + \sum m_i > 0$
- numbers m_i are given in ascending ordered $m_1 \leq m_2 \leq \cdots \leq m_s$.

Denote by V_G tensor product $A^{\otimes m} \otimes B^{\otimes(m_1+\cdots+m_s)}$ of vector spaces.

For any G fix one linear function $f_G : V_G \to \mathbb{C}$. According to the style of notations used in field theories we denote corresponding polylinear function as follows:

$$\langle x_1, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s)\rangle_{(g,\varepsilon)} := f_G(z),$$

where $z = x_1 \otimes \cdots \otimes x_m \otimes y_1^1 \otimes \cdots \otimes y_{m_1}^1 \otimes \cdots \otimes y_1^s \otimes \cdots \otimes y_{m_s}^s \in V_G$.

Note that vector spaces V_G and $V_{G'}$ coincide for some $G \neq G'$ but corresponding polylinear functions 'remember' all data from $G = (g, \varepsilon, m, m_1, \dots, m_s)$ due to bracketing of arguments. We call a bracket $(y_1^j, \dots, y_{m_j}^j)$ *a block* of arguments.

We use capital letters X, X_1 etc. to denote subsets consisting of several x_i and several blocks of arguments. For example, $\langle x_1, X\rangle_{g,\varepsilon}$, where $X = (x_2, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s))$ denotes a function $\langle x_1, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s)\rangle_{g,\varepsilon}$.

We write down $\langle X\rangle_G$ instead of $\langle X\rangle_{g,\varepsilon}$ in order to emphasize explicitly the dependence on a type G.

We denote by X_a the set of all x_i in a correlator and we denote by X_b the set of all blocks of arguments. Thus, we write down

$\langle X\rangle_G = \langle X_a, X_b\rangle_G = \langle x_1, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s)\rangle_{g,\varepsilon}$

We use capital letters Y, Y_1, etc. to denote sequential sets of arguments in a block. For example, $\langle X, (y_1^s, Y)\rangle_{g,\varepsilon}$, where

$X = (x_1, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^{s-1}, \dots, y_{m_{s-1}}^{s-1}))$ and $Y = (y_2^s, \dots, y_{m_s}^s)$, denotes polylinear function

$\langle x_1, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s)\rangle_{g,\varepsilon}$.

For an arbitrary $Y = (y_1, \dots, y_k)$ put $Y^* = (y_k^*, \dots, y_1^*)$.

Choose a basis E_A of A and E_B of B and denote basic elements by $\alpha, \alpha', \dots \in E_A$ and $\beta, \beta', \dots \in E_B$. Then we can present f_G as a tensor

$$\langle \alpha_1, \dots, \alpha_m, (\beta_1^1, \dots, \beta_{m_1}^1), \dots, (\beta_1^s, \dots, \beta_{m_s}^s)\rangle_{(g,\varepsilon)}$$

Let $\mathcal{C} = (A, x \mapsto x^*, B, y \mapsto y^*, \{\langle X\rangle_G\})$ be a set of data consisting of

- linear spaces A and B,
- involutive linear transformations $x \mapsto x^*$ ($x \in A$) and $y \mapsto y^*$ ($y \in B$) of A and B resp.,
- polylinear function $\langle X\rangle_G$ for any topological type $G = (g, \varepsilon, m, m_1, \dots, m_s)$ of a stratified surface .

Definition. *A set of data $\mathcal{C} = (A, x \mapsto x^*, B, y \mapsto y^*, \{\langle X\rangle_G\})$ is called a system of correlators and any function $\langle X\rangle_G$ is called a correlator if and only if the following axioms hold.*

(i) Any correlator $\langle x_1, \dots, x_m, (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s)\rangle_{g,\varepsilon}$ is invariant with respect to a permutation of x_i, a permutation of blocks $(y_1^i, \dots, y_{m_i}^i)$ of equal size, a cyclic permutations of arguments inside each block $(y_1^i, \dots, y_{m_i}^i)$.

(ii) If $\varepsilon = 1$ then a correlator is invariant with respect to replacing of x_i by x_i^* and block $Y^j = (y_1^j, \dots, y_{m_j}^j)$ by $(Y^j)^*$ for all $i = 1, \dots, m$, $j = 1, \dots, s$ simultaneously.

If $\varepsilon = 0$ then a correlator is invariant with respect to replacing of x_i by x_i^* for any fixed i and with respect to replacing a block $Y^j = (y_1^j, \dots, y_{m_j}^j)$ by $(Y^j)^*$ for any fixed j.

(iii) Bilinear maps $\langle x_1, x_2\rangle_{0,1}$ and $\langle (y_1, y_2)\rangle_{0,1}$ are nondegenerate bilinear forms on A and B resp.; below we denote them by $(x_1, x_2)_A$ and $(y_1, y_2)_B$ resp.

(iv) Fix a basis $E_A = \{\alpha\}$ of A and a basis $E_B = \{\beta\}$ of B. Denote by $F^{\alpha',\alpha''}$ the tensor dual to the tensor $F_{\alpha',\alpha''}$ of bilinear form (α', α''), by $F^{\beta',\beta''}$ the tensor dual to the tensor $F_{\beta',\beta''} := (\beta', \beta'')$ and by D^α the tensor obtained by lifting the index of tensor $D_\alpha = \langle\alpha\rangle_{\frac{1}{2},0}$. Axiom is that for arbitrary sets X, X_1, X_2, Y_1, Y_2 of arguments denoted according to the conventions described above the relations $(1) - (9)$ hold.

$$(1) \quad \langle X_1, X_2\rangle_{g_1+g_2,\varepsilon_1\varepsilon_2} = \sum_{\alpha',\alpha''\in E_A} \langle X_1, \alpha'\rangle_{g_1,\varepsilon_1} F^{\alpha',\alpha''} \langle \alpha'', X_2\rangle_{g_2,\varepsilon_2}$$

$$(2) \quad \langle X\rangle_{g+1,\varepsilon} = \sum_{\alpha',\alpha''\in E_A} \langle X, \alpha', \alpha''\rangle_{g,\varepsilon} F^{\alpha',\alpha''}$$

$$(3) \quad \langle X\rangle_{g+1,0} = \sum_{\alpha',\alpha''\in E_A} \langle X, \alpha', \alpha''^*\rangle_{g,1} F^{\alpha',\alpha''}$$

$$(4) \quad \langle X\rangle_{g+\frac{1}{2};0} = \sum_{\alpha'\in E_A} \langle X, \alpha'\rangle_{g,\varepsilon} D^{\alpha'}$$

$$(5) \quad \langle X, (Y_1), (Y_2)\rangle_{g,\varepsilon} = \sum_{\beta',\beta''\in E_B} \langle X, (Y_1, \beta', Y_2, \beta'')\rangle_{g,\varepsilon} F^{\beta',\beta''}$$

$$(6) \quad \langle X_1, X_2, (Y_1, Y_2)\rangle_{g_1+g_2,\varepsilon_1\varepsilon_2} = \sum_{\beta',\beta''\in E_B} \langle X_1, (Y_1, \beta')\rangle_{g_1,\varepsilon_1} F^{\beta',\beta''} \langle X_2, (Y_2, \beta'')\rangle_{g_2,\varepsilon_2}$$

$$(7)\quad \langle X,(Y_1,Y_2)\rangle_{g+1,\varepsilon} = \sum_{\beta',\beta''\in E_B} \langle X,(Y_1,\beta'),(Y_2,\beta'')\rangle_{g,\varepsilon} F^{\beta',\beta''}$$

$$(8)\quad \langle X,(Y_1,Y_2)\rangle_{g+1,0} = \sum_{\beta',\beta''\in E_B} \langle X,(Y_1,\beta'),(Y_2^*,\beta''^*)\rangle_{g,1} F^{\beta',\beta''}$$

$$(9)\quad \langle X,(Y_1,Y_2)\rangle_{g+\frac{1}{2},0} = \sum_{\beta',\beta''\in E_B} \langle X,(Y_1,\beta',Y_2^*,\beta''^*)\rangle_{g,\varepsilon} F^{\beta',\beta''}$$

Remark. We pay no attention to the order in which blocks of arguments are written in a correlator because due to (i) of the definition of systems of correlators all orderings of blocks in ascending order of their size give the same function.

Remark. We shall define a correlator for a type $\overline{G}$ of disconnected surfaces. First, fix any ordering of all types G of connected stratified surfaces (use for example the lexicographic ordering). A type $\overline{G}$ of disconnected surfaces is a set $\overline{G} = (G_1,\dots,G_k)$ of types $G_i = (g^{(i)},\varepsilon^{(i)},m^{(i)},m_1^{(i)},\dots,m_{s_i}^{(i)})$ of connected stratified surface. We require types G_i to be numbered according the ordering.

Put $V_{\overline{G}} = A^{\otimes(\sum m^{(i)})}\otimes B^{\otimes(\sum m_j^{(i)})}$. Define polylinear function $\langle X\rangle_{\overline{G}}$ on $V_{\overline{G}}$ as a product of correlators

$$\langle X\rangle_{\overline{G}} = \langle X^{(1)}\rangle_{G_1}\dots\langle X^{(k)}\rangle_{G_k}$$

Here $X = (X_a^{(1)},\dots,X_a^{(k)},X_b^{(1)},\dots,X_b^{(k)})$ and $X^{(i)} = (X_a^{(i)},X_b^{(i)})$.

A type of an arbitrary stratified surface possibly disconnected will be denoted below by G.

Fix a basis E_A of A and a basis E_B of B and use nondegenerate tensors $F_{\alpha',\alpha''} = (\alpha',\alpha'')_A$, $F_{\beta',\beta''} = (\beta',\beta'')_B$ and dual to lower and raise indices.

Denote by 1_A the element $\langle\alpha'\rangle_{0,1}F^{\alpha',\alpha''}\alpha''\in A$. Denote by 1_B the element $\langle\beta'\rangle_{0,1}F^{\beta',\beta''}\beta''\in B$. Denote by U the element $\langle\alpha'\rangle_{\frac{1}{2},0}F^{\alpha',\alpha''}\alpha''\in A$.

Lemma 4.2. *The following identities hold for any acceptable sets of arguments X, Y.*
(1) $\langle 1_A,X\rangle_{g,\varepsilon} = \langle X\rangle_{g,\varepsilon}$
(2) $\langle X,(1_B,Y)\rangle_{g,\varepsilon} = \langle X,(Y)\rangle_{g,\varepsilon}$
(3) $\langle U_A,X\rangle_{g,\varepsilon} = \langle X\rangle_{g+\frac{1}{2},0}$

Proof. (1) Use identity (1) from the definition 4.2 and put in it $X_1 = \emptyset, g_1 = 0, \varepsilon_1 = 1, X_2 = X, g_2 = g, \varepsilon_2 = \varepsilon$.

(2) Use identity (6) from the definition 4.2 and put in it $X_1 = \emptyset, Y_1 = \emptyset, g_1 = 0, \varepsilon_1 = 1, X_2 = X, Y_2 = Y, g_2 = g, \varepsilon_2 = \varepsilon$.

(3) Use identity (1) from the definition and put in it $X_1 = \emptyset, g_1 = \frac{1}{2}, \varepsilon_1 = 0, X_2 = X, g_2 = g, \varepsilon_2 = \varepsilon$. $\square$

Remark. Let γ be an unstable simple cut of a stratified surface Ω. Then one of the connected components of contracted cut surface $\Omega_\#$ is a trivial surface. This component is one of the surfaces (4), (5), (6), (7), (11) from the list of trivial surfaces given in lemma 3.1 due to the assumption that any component of boundary of Ω contains at least one special point. In cases (4), (5), (7) simple cut γ induces one of the relations from lemma 4.2. In cases (6), (11) simple cut γ induces trivial relation, following from the definition of bilinear forms $(x', x'')_A$ and $(y', y'')_B$.

We shall assign a system of correlators to a Klein topological field theory.

Let $\mathcal{T} = (A, x \mapsto x^*, B, y \mapsto y^*, \{\Phi_{(\Omega,\mathcal{O})}\})$ be a Klein topological field theory. For any type $G = (g, \varepsilon, m, m_1, \dots, m_s)$ choose a connected stratified surface Ω of type G and an admissible set of local orientations $\mathcal{O}$. Note that for any boundary contour ω_i admissible set of local orientations generates a cyclic ordering of special points from ω_i.

Denote by M the set of multipliers of the tensor product $V_G = A^{\otimes m} \otimes B^{\otimes(m_1+\dots+m_s)}$. A bijection $\mathcal{N} : M \leftrightarrow \Omega_0$ between set M and set Ω_0 of special points on a stratified surface Ω endowed with an admissible set of local orientations $\mathcal{O}$ is called *an admissible bijection* if

- multipliers A correspond to interior special points;
- first m_1 multipliers B correspond to special points from boundary contour ω_1 such that $|\omega_1| = m_1$, next m_2 multipliers B correspond to special points from boundary contour ω_2 such that $|\omega_2| = m_2$ etc.;
- the linear order of special points on any ω_i induced by the order of multipliers B in V_G is compatible with the cyclic order induced by $\mathcal{O}$.

Fix an admissible bijection $\mathcal{N} : M \leftrightarrow \Omega_0$. Denote by $\phi_\mathcal{N} : V_G \to V_{(\Omega,\mathcal{O})}$ the isomorphism of vector spaces induced by this bijection.

For any element $z = x_1 \otimes \cdots \otimes x_m \otimes y_1^1 \otimes \cdots \otimes y_{m_1}^1 \otimes \cdots \otimes y_1^s \otimes \cdots \otimes y_{m_s}^s \in V_G$ put

$$\langle x_1, \dots, x_m (y_1^1, \dots, y_{m_1}^1), \dots, (y_1^s, \dots, y_{m_s}^s) \rangle_G = \Phi_{(\Omega,\mathcal{O})}(\phi_\mathcal{N}(z)).$$

Thus, we obtain a linear form $\langle X \rangle_G$ for any topological type G of a stratified surface.

Lemma 4.3. *Linear form $\langle X \rangle_{g,\varepsilon}$ does not depend on the choice of a stratified surface Ω, an admissible set of local orientations $\mathcal{O}$ and an admissible bijection $\mathcal{N}$.*

Proof. Lemma follows immediately from topological invariance of Klein field theory. $\square$

Denote by $\mathcal{C}(\mathcal{T})$ the set of data $(A, x \mapsto x^*, B, y \mapsto y^*, \{\langle X \rangle_G\})$.

Lemma 4.4. *The set of data* $\mathcal{C}(\mathcal{T})$ *is a system of correlators.*

Proof. Identities (i) - (iii) follows from appropriate axioms of a Klein topological field theory.

We will pay more attention to identity (iv). Let γ be a simple cut of a stratified surface Ω of type $G = (g, \varepsilon, m, m_1, \dots, m_s)$ with $m_i > 0$. Then contracted cut surface $\Omega_\# = \Omega/\gamma$ is either a connected surface or consists of two connected components $(\Omega_\#)_1$, $(\Omega_\#)_2$. Denote by $G_\#$ the type of $\Omega_\#$. Thus, $G_\# = ((G_\#)_1, (G_\#)_2)$ in the latter case.

Cut invariance of a Klein topological field theory leads to the relation between correlators $\langle X \rangle_G$ and $\langle X \rangle_{G_\#}$.In order to establish this relation we need to fix

admissible sets of local orientations $\mathcal{O}$ on Ω and $\mathcal{O}'_\#$ on $\Omega_\#$;

admissible bijections $\mathcal{N} : M \leftrightarrow \Omega_0$ and $\mathcal{N}_\# : M_\# \leftrightarrow (\Omega_\#)_0$.

(Note that a cut morphism $\eta : (\Omega, \mathcal{O}) \to (\Omega_\#, \mathcal{O}_\#)$ may bring an admissible set of local orientations $\mathcal{O}$ to nonadmissible set of local orientations $\mathcal{O}_\#$.)

We have the chain of morphisms

$V_G \xrightarrow{\phi_{\mathcal{N}}} V_{(\Omega,\mathcal{O})} \xrightarrow{\eta_*} V_{\Omega_\#} \xrightarrow{\phi^{-1}_{\mathcal{N}'}} V_{\Omega'_\#} \to V_{G_\#}$.

Using cut invariance of linear forms $\Phi_{(\Omega,\mathcal{O})}$ and the definition of $\langle X \rangle_{g,\varepsilon}$ we obtain the identity, namely, tensor $\langle X \rangle_G$ is equal to a contraction of tensor $\langle X_\# \rangle_{G_\#}$.

We did an appropriate choice of admissible sets of local orientations $\mathcal{O}$ and $\mathcal{O}'$ separately for all 9 classes of simple cuts. The resulting relations for correlators just form a list of relations in the definition of a sysem of correlators.

$\square$

Conversely, assume we are given a system of correlators $\mathcal{C} = (A, x \mapsto x^*, B, y \mapsto y^*, \langle X \rangle_G\})$.

First, define a linear form $\Phi_{(\Omega,\mathcal{O})}$ for stratified surfaces of type $G = (g, \varepsilon, m, m_1, \dots, m_s)$ such that $m_i > 0$ for $i = 1, \dots, s$ and an admissible set of local orientations $\mathcal{O}$. For this purpose use an admissible bijection $\mathcal{N} : M \to \Omega_0$ and invert the above consideration. The function $\Phi_{(\Omega,\mathcal{O})}$ is well defined because the correlators are symmetric.

Second, using axioms of a Klein topological field theory extend the definition of linear forms $\Phi_{(\Omega,\mathcal{O})}$ to all pairs $(\Omega, \mathcal{O})$, i.e., to arbitrary sets of local orientations $\mathcal{O}$ (using axiom 2°) and to surfaces with boundaries having no special points (using axiom 4°). It can be easily checked that constructed set of data $\mathcal{T}(\mathcal{C}) = (A, x \mapsto x^*, B, y \mapsto y^*, \{\Phi_{(\Omega,\mathcal{O})}\})$

satisfies all axioms of a Klein topological field theory. From definitions it follows that the correspondence $\mathcal{C} \rightsquigarrow \mathcal{T}(\mathcal{C})$ is inverse to the correspondence $\mathcal{T} \rightsquigarrow \mathcal{C}(\mathcal{T})$. Thus

Theorem 4.1. *Correspondence $\mathcal{T} \rightsquigarrow \mathcal{C}(\mathcal{T})$ is a bijection between Klein topological field theories and systems of correlators.*

4.3. **From system of correlators to structure algebras.** Fix a system of correlators $\mathcal{C} = (A, x \mapsto x^*, B, y \mapsto y^*, \{\langle X\rangle_G\})$. In this subsection we consider correlators as tensors in basis E_A and E_B of vector spaces A and B resp. Thus, in notation $\langle X\rangle_G$ of a correlator we assume that X is a set of basic elements $\alpha_i \in E_A$, $\beta_i^j \in E_B$. An index with star like α^* means the contraction with the matrix $I_{\alpha'}^{\alpha''}$ of involutive transformation $x \mapsto x^*$. For example, $\langle \alpha^*, X\rangle_{(g,\varepsilon)} = I_\alpha^{\alpha'}\langle \alpha', X\rangle_{(g,\varepsilon)}$

We deal here with correlators corresponding to a type G of not necessarily connected stratified surface. It was shown above that any simple cut γ of a stratified surface Ω of type G generates an identity with tensor $\langle X\rangle_G$ in the left hand side and a contraction of tensor $\langle X_\#\rangle_{G_\#}$ in the right hand side. Here $G_\#$ is the type of the contracted cut surface $\Omega_\# = \Omega/\gamma$. The explicit formulas for these identities corresponding to simple cuts of all classes are those that are listed in the definition of systems of correlators.

Identities for correlators that differ only by symmetries (i)-(ii), we consider as equal. Accepting this agreement we may claim, that the identity corresponding to a simple cut $\gamma \subset \Omega$ does not depend on the choice of Ω and an admissible set of local orientations $\mathcal{O}$. The identities corresponding to isomorphic pairs (Ω, γ) and (Ω', γ') are equal.

We denote this identity as follows:

$$\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\gamma)}(\langle X_\#\rangle_{G_\#}).$$

Here $X_\#$ is tensor of a type encoded by the type $G_\#$ of contracted cut surface $\Omega_\# = \Omega/\gamma$, namely, if $G_\# = (g_\#, \varepsilon_\#, m_\#, m_{\#1}, \dots, m_{\#s_\#})$ then $X_\#$ has $m_\#$ first indices from A and $\sum m_{\#i}$ remaining indices from B. Right hand side expression is a contraction of this tensor by means of tensors $F^{\alpha',\alpha''}$, $F^{\beta',\beta''}$, D^α and possibly by their contractions with matrices of star involutions $I_{\alpha'}^\alpha$, $I_{\beta'}^\beta$. The explicit form of the contraction is defined by a pair (Ω, γ) and is encoded in the notation of operator $\mathsf{Contr}_{(\Omega,\gamma)}(\dots)$. Indices of $X_\#$ that are not involved into contraction are just indices of tensor X. The type of X is encoded by G.

A cut system $\Gamma = \{\gamma_1, \dots, \gamma_k\}$ generates a chain of identities and finally we obtain the identity as follows:

$$\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\gamma_1)}(\mathsf{Contr}_{(\Omega/\{\gamma_1\},\gamma_2)}(\dots \mathsf{Contr}_{(\Omega/\{\gamma_1,\dots,\gamma_{k-1}\},\gamma_k)}(\langle X_\#\rangle_{G_\#})\dots)),$$

where $G_\#$ is topological type of contracted cut surface $\Omega_\# = \Omega/\Gamma$. Clearly, the result does not depend on the ordering of simple cuts. We denote this identity as

$$\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\Gamma)}(\langle X_\#\rangle_{G_\#}).$$

Isomorphic pairs (Ω,Γ) and (Ω',Γ') generate equal identities. Note that operator $\mathsf{Contr}_{(\Omega,\Gamma)}(\dots)$ can be applied to any tensor $\overline{X}_\#$ of type corresponding to the type $G_\#$ of the contracted cut surface Ω/Γ.

The following identities are evident.

Lemma 4.5. *(1) If $\Gamma^\circ \subset \Gamma$ is a subsystem of a cut system Γ then* $\mathsf{Contr}_{(\Omega,\Gamma)}(\overline{X}_\#) = \mathsf{Contr}_{(\Omega,\Gamma^\circ)}(\mathsf{Contr}_{(\Omega/\Gamma^\circ,\Gamma\setminus\Gamma^\circ)}(\overline{X}_\#))$.
(2) Let Γ' be a cut system on Ω', Γ_1 and Γ_2 be two cut systems on Ω° and $\overline{X}^\circ_\#$ (resp. $\overline{X}'_\#$) be a tensor of type corresponding to the type $G^\circ_\#$ (resp. $G'_\#$) of the contracted cut surface Ω'/Γ' (resp.$\Omega^\circ/\Gamma^\circ$). If $\mathsf{Contr}_{(\Omega^\circ,\Gamma_1)}(\overline{X}_\#) = \mathsf{Contr}_{(\Omega^\circ,\Gamma_2)}(\overline{X}_\#)$ *then* $\mathsf{Contr}_{(\Omega^\circ\sqcup\Omega',\Gamma^\circ\sqcup\Gamma_1)}(\overline{X}^\circ_\#\cdot\overline{X}'_\#) = \mathsf{Contr}_{(\Omega^\circ\sqcup\Omega',\Gamma^\circ\sqcup\Gamma_2)}(\overline{X}^\circ_\#\cdot\overline{X}'_\#)$.

Correlators corresponding to types G of trivial surfaces are called *trivial correlators*. There are five of them; they were used above for definitions of bilinear forms $(\alpha',\alpha'')_A$, $(\beta',\beta'')_B$ and elements $1_A \in A$, $1_B \in B$, $U \in A$.

Correlators corresponding to types G of basic surfaces are called *basic correlators*. Therefore, there are three basic correlators

$$\langle x_1,x_2,x_3\rangle_{0,1},\ \langle (y_1,y_2,y_3)\rangle_{0,1},\ \langle x,(y)\rangle_{0,1}.$$

Lemma 4.6. *The correlator $\langle X\rangle_G$ for the type G of any connected stable stratified surface Ω is equal to a contraction of a product of basic correlator.*

Proof. Indeed, take a stratified surface Ω of type G and a complete cut system Γ of Ω. Then we get the identity $\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\Gamma)}(\langle X_\#\rangle_{G_\#})$. By definition, connected components of $\Omega_\#$ are basic surfaces. Therefore, correlator $\langle X_\#\rangle_{G_\#}$ is a product of basic correlators. □

Let Γ' and Γ'' be two complete cut systems of a stable connected stratified surface Ω. Then we have two identities $\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\Gamma')}(\langle X'_\#\rangle_{G'_\#})$ and $\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\Gamma'')}(\langle X''_\#\rangle_{G''_\#})$. Hence

$$\mathsf{Contr}_{(\Omega,\Gamma')}(\langle X'_\#\rangle_{G'_\#}) = \mathsf{Contr}_{(\Omega,\Gamma'')}(\langle X''_\#\rangle_{G''_\#}),$$

and we obtain the relation between basic correlators. Obviously, replacing Γ' or Γ'' by an equivalent complete cut system brings this relation to the equal one. Moreover, if two triples $(\Omega_1,\Gamma'_1,\Gamma''_1)$ and $(\Omega_2,\Gamma'_2,\Gamma''_2)$ are equivalent, i.e., there exists an isomorphism $\phi:\Omega_1\to\Omega_2$ such that $\phi(\Gamma'_1)=\Gamma'_2$, $\phi(\Gamma''_1)=\Gamma''_2$ (or $\phi(\Gamma'_1)=\Gamma''_2$, $\phi(\Gamma''_1)=\Gamma'_2$), then corresponding relations are equal.

We call the relations corresponding to nonequivalent pairs of complete cut systems on simple surfaces *defining relations*. The relations corresponding to unstable cuts of simple and trivial surfaces we call *defining relations* too.

Using the list of simple surfaces and the classification of complete cut systems on them (see lemma 3.10) we obtain the following list of defining relations between basic correlators. In the list we omit relations that are consequence of given ones. The enumeration of relations corresponds to the enumeration of simple surfaces. In order to simplify formulas we use nonbasic correlators $\langle\alpha,\alpha',\alpha'',\alpha'''\rangle_{0,1}$ and $\langle(\beta,\beta',\beta'',\beta''')\rangle_{0,1}$ because they have clear expression via basic correlators.

Defining relations corresponding to complete cut systems of simple surfaces

(1) $\langle\alpha_1,\alpha_2,\alpha'\rangle_{0,1}F^{\alpha',\alpha''}\langle\alpha'',\alpha_3,\alpha_4\rangle_{0,1} =$
$= \langle\alpha_1,\alpha_3,\alpha'\rangle_{0,1}F^{\alpha',\alpha''}\langle\alpha'',\alpha_2,\alpha_4\rangle_{0,1}$

(2) $\langle\alpha,\alpha',\alpha''^*\rangle_{0,1}F^{\alpha',\alpha''} = \langle\alpha,\alpha',\alpha''\rangle_{0,1}D^{\alpha'}D^{\alpha''}$

(4) $\langle(\beta_1,\beta_2,\beta')\rangle_{0,1}F^{\beta',\beta''}\langle(\beta'',\beta_3,\beta_4)\rangle_{0,1} =$
$= \langle(\beta_2,\beta_3,\beta')\rangle_{0,1}F^{\beta',\beta''}\langle(\beta'',\beta_4,\beta_1)\rangle_{0,1}$

(5) $\langle(\beta_1,\beta_2,\beta')\rangle_{0,1}F^{\beta',\beta''}\langle\alpha,(\beta'')\rangle_{0,1} = \langle(\beta_2,\beta_1,\beta')\rangle_{0,1}F^{\beta',\beta''}\langle\alpha,(\beta'')\rangle_{0,1}$

(6) $\langle\alpha_1,\alpha_2,\alpha'\rangle_{0,1}F^{\alpha',\alpha''}\langle\alpha'',(\beta)\rangle_{0,1} =$
$= \langle(\beta,\beta',\beta'')\rangle_{0,1}F^{\beta',\beta'''}\langle\alpha_1,(\beta''')\rangle_{0,1}F^{\beta'',\beta^{(iv)}}\langle\alpha_2,(\beta^{(iv)})\rangle_{0,1}$

(7) $\langle\alpha',(\beta)\rangle_{0,1}D^{\alpha'} = \langle(\beta,\beta',\beta''^*)\rangle_{0,1}F^{\beta',\beta''}$

(8) $\langle\alpha',(\beta_1)\rangle_{0,1}F^{\alpha',\alpha''}\langle\alpha'',(\beta_2)\rangle_{0,1} = \langle(\beta_1,\beta',\beta_2,\beta'')\rangle_{0,1}F^{\beta',\beta''}$

Defining relations completely describe properties of systems of correlators. Precise statements are formulated and proved in sequel subsection using structure algebras.

Denote trivial and basic correlators as follows.
(1) $F_{\alpha_1,\alpha_2} = \langle\alpha_1,\alpha_2\rangle_{0,1}$
(2) $F_{\beta_1,\beta_2} = \langle(\beta_1,\beta_2)\rangle_{0,1}$;
(3) $R_{\alpha,\beta} = \langle\alpha,(\beta)\rangle_{0,1}$;
(4) $S_{\alpha_1,\alpha_2,\alpha_3} = \langle\alpha_1,\alpha_2,\alpha_3\rangle_{0,1}$;
(5) $T_{\beta_1,\beta_2,\beta_3} = \langle(\beta_1,\beta_2,\beta_3)\rangle_{0,1}$;
(6) $R_{\alpha,\beta_1,\beta_2} = \langle\alpha,(\beta_1,\beta_2)\rangle_{0,1}$;
(7) $I_{\alpha_1,\alpha_2} = \langle\alpha_1^*,\alpha_2\rangle_{0,1}$;
(8) $I_{\beta_1,\beta_2} = \langle(\beta_1^*,\beta_2)\rangle_{0,1}$;
(9) $D_\alpha = \langle\alpha\rangle_{\frac{1}{2},0}$;
(10) $J_\alpha = \langle\alpha\rangle_{0,1}$;
(11) $J_\beta = \langle(\beta)\rangle_{0,1}$.

Theorem 4.2. *Tensors (1)-(11) are structure constants of a structure algebra, which we denote by* $\mathcal{H}(\mathcal{C})$.

Proof. By lemma 2.2 it is sufficient to verify relations (1)–(16) for these tensors. These relations either follow from the symmetries of correlators or coincide with defining relations for basic correlators (1)-(8). □

4.4. **From structure algebra to Klein topological field theory.** Let $\mathcal{H} = (H = A \oplus B, (x, y), x \mapsto x^*, U \in A)$ be a structure algebra. We shall construct a Klein topological field theory $\mathcal{T}(\mathcal{H}) = (A, x \mapsto x^*, B, y \mapsto y^*, \{\Phi_{(\Omega,\mathcal{O})}\})$ with the same vector spaces A and B and involutive transformations $x \mapsto x^*$ and $y \mapsto y^*$ which are restrictions of involutive antiautomorphism of algebra H to its summands.

First, define linear forms $\Phi_{(\Omega,\mathcal{O})} : V_\Omega \to \mathbb{C}$ for trivial and basic surfaces Ω. Choose basis E_A of vector space A and E_B of vector space B. It is sufficient to define a linear form on basic elements $(\otimes_{p\in\Omega_a}\alpha_p)\otimes(\otimes_{q\in\Omega_b}\beta_q)$ of vector space V_Ω. Here Ω_a (resp. Ω_b) is the set of all interior (resp. boundary) special points of Ω, $\alpha_p \in E_A$, $\beta_q \in E_B$.

For example, let Ω be isomorphic to a disc with three boundary special points q_1, q_2, q_3. Choose an admissible set $\mathcal{O}$ of local orientations and fix an admissible ordering $\mathcal{N}$ of special points. In this case $\mathcal{O}$ and $\mathcal{N}$ are uniquely defined by fixing an orientation of the disc and choosing a first special point on its boundary. The order of special points defines the isomorphism $V_\Omega \approx B_1 \otimes B_2 \otimes B_3$, where B_i is a copy of vector space B and i means that the isomorphism brings a vector β_i assigned to i-th special point q_i to this multiplier ($i = 1, 2, 3$). For basic element $\otimes_{i=1}^3\beta_i \in V_\Omega$ put $\Phi_{(\Omega,\mathcal{O})}(\otimes_{i=1}^3\beta_i) = T_{\beta_1,\beta_2,\beta_3}$, where tensor $T_{\beta_1,\beta_2,\beta_3}$ belongs to structure constants of structure algebra $\mathcal{H}$ (see subsection 2.1). For a nonadmissible set $\mathcal{O}$ of local orientations define $\Phi_{(\Omega,\mathcal{O})}$ by invariance of changing of local orientations (axiom 2° from the definition of Klein topological field theory). Properties of structure constants of a structure algebra (see lemma 2.2) provide the correctness of the definition of $\Phi_{(\Omega,\mathcal{O})}$, its topological invariance and invariance with respect to changing of local orientations.

Analogous reasoning leads to correct definitions of $\Phi_{(\Omega,\mathcal{O})}$ for all trivial and basic surfaces. Key formulas for a surface Ω, an admissible set $\mathcal{O}$ of local orientations and an admissible numbering $\mathcal{N}$ of special points are as follows. Right-hand sides of these formulas are expressed via structure constants of $\mathcal{H}$.

Trivial surfaces

(1) Ω is a sphere S^2 without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = J^\alpha J_\alpha$.
(2) Ω is a projective plane $\mathbb{R}P^2$ without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = J_\alpha D^\alpha$.
(3) Ω is a disc D^2 without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = J^\beta J_\beta$.

(4) Ω is a sphere S^2 with an interior special point p: $V_\Omega = A$, $\Phi_{(\Omega,\mathcal{O})}(\alpha) = J_\alpha$.
(5) Ω is a disc D^2 with a boundary special point q: $V_\Omega = B$, $\Phi_{(\Omega,\mathcal{O})}(\beta) = J_\beta$.
(6) Ω is a sphere S^2 with two interior special points p_1, p_2: $V_\Omega \approx A \otimes A$, $\Phi_{(\Omega,\mathcal{O})}(\alpha_1 \otimes \alpha_2) = F_{\alpha_1,\alpha_2}$.
(7) Ω is a projective plane $\mathbb{R}P^2$ with an interior special point p: $V_\Omega = A$, $\Phi_{(\Omega,\mathcal{O})}(\alpha) = D_\alpha$.
(8) Ω is a torus T^2 without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = F_{\alpha',\alpha''}F^{\alpha',\alpha''} = \dim A$.
(9) Ω is a Klein bottle Kl without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = I_{\alpha',\alpha''}F^{\alpha',\alpha''} = \mathrm{tr}(x \mapsto x^*)$.
(10) Ω is a disc D^2 with an interior special point p: $V_\Omega = A$, $\Phi_{(\Omega,\mathcal{O})}(\alpha) = J^\beta R_{\alpha,\beta}$.
(11) Ω is a disc D^2 with two boundary special points q_1, q_2: $V_\Omega \approx B \otimes B$, $\Phi_{(\Omega,\mathcal{O})}(\beta_1 \otimes \beta_2) = F_{\beta_1,\beta_2}$.
(12) Ω is a Möbius band Mb without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = J^\beta D^\alpha R_{\alpha,\beta}$.
(13) Ω is a cylinder Cyl without special points: $V_\Omega = \mathbb{C}$, $\Phi_{(\Omega,\mathcal{O})}(1) = J^{\beta'}R_{\alpha',\beta'}F^{\alpha',\alpha''}J^{\beta''}R_{\alpha'',\beta''}$.

Basic surfaces

(1) Ω is a sphere S^2 with three interior special points p_1, p_2, p_3: $V_\Omega \approx A \otimes A \otimes A$, $\Phi_{(\Omega,\mathcal{O})}(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) = S_{\alpha_1,\alpha_2,\alpha_3}$.
(2) Ω is a disc D^2 with three boundary special points q_1, q_2, q_3: $V_\Omega \approx B \otimes B \otimes B$, $\Phi_{(\Omega,\mathcal{O})}(\beta_1 \otimes \beta_2 \otimes \beta_3) = T_{\beta_1,\beta_2,\beta_3}$.
(3) Ω is a disc D^2 with a boundary special point q and an interior special point p: $V_\Omega = A \otimes B$, $\Phi_{(\Omega,\mathcal{O})}(\alpha \otimes \beta) = R_{\alpha,\beta}$.

Multiplicativity (axiom 6° from the definition of Klein topological field theory) allows us to define correctly linear forms $\Phi_{(\Omega,\mathcal{O})}$ also for stratified surfaces Ω that are disjoint union of several trivial and basic surfaces.

Lemma 4.7. *The already defined linear forms $\Phi_{(\Omega,\mathcal{O})}$ for trivial and basic surfaces and their disjoint unions satisfy topological invariance axiom, invariance of a change of local orientations axiom, nondegeneracy axiom, invariance of adding trivial field axiom (when applicable), cut invariance axiom (for cuts of trivial and basic surfaces) and multiplicativity axiom.*

Proof. Topological invariance and invariance of a change of local orientations follows from the definitions of functions $\Phi_{(\Omega,\mathcal{O})}$ as it was shown above for a disc with three boundary special points. Nondegeneracy follows from nondegeneracy of bilinear forms $(x', x'')|_A$ and $(y', y'')|_B$ in algebra $\mathcal{H}$. Invariance of adding a trivial field and cut invariance for trivial and basic surfaces are checked for all such surfaces. Any verification turns into a simple identity for structure constants of $\mathcal{H}$. We skip the details. Multiplicativity follows from the definition of $\Phi_{(\Omega,\mathcal{O})}$ for disjoint unions of trivial and basic surfaces. $\square$

Define $\Phi_{(\Omega,\mathcal{O})}$ for a connected stable surface Ω. Choose a complete cut system Γ of Ω and orientations of all simple cuts $\gamma\subset\Gamma$. Denote by $\eta:(\Omega,\mathcal{O})\to(\Omega_\#,\mathcal{O}_\#)$ cut morphism corresponding to Γ (see subsection 4.1). By definition of a complete cut system, all connected components of contracted cut surface $\Omega_\#$ are basic surfaces. Therefore, function $\Phi_{(\Omega_\#,\mathcal{O}_\#)}$ is already defined. Put $\Phi_{(\Omega,\mathcal{O},\Gamma)}(x)=\Phi_{(\Omega_\#,\mathcal{O}_\#)}(\eta_*(x))$. We have to prove that this function does not depend on the choice of Γ.

Lemma 4.8. *Let Γ and Γ' be equivalent complete cut systems endowed with orientations of all simple cuts of a stable surface Ω. Then $\Phi_{(\Omega,\mathcal{O},\Gamma)}(x)=\Phi_{(\Omega,\mathcal{O},\Gamma')}(x)$. Particularly, $\Phi_{(\Omega,\mathcal{O},\Gamma)}(x)$ does not depend on a choice of orientations of simple cuts.*

Proof. Suppose, we change an orientation of a simple cut $\gamma\subset\Gamma$ of a complete cut system Γ. Denote the same complete cut system with changed orientation of the cut by Γ'. We have two cut morphisms $\eta_\Gamma:(\Omega,\mathcal{O})\to(\Omega_\#,\mathcal{O}_\#)$ and $\eta_{\Gamma'}:(\Omega,\mathcal{O})\to(\Omega_\#,\mathcal{O}'_\#)$ and changing local orientations morphism $\psi:(\Omega_\#,\mathcal{O}_\#)\to(\Omega_\#,\mathcal{O}'_\#)$. Vector space $V_{\Omega_\#}$ is isomorphic to $V_\Omega\otimes X$, where vector space X depends on Γ. By definition, linear map $\eta_*:V_\Omega\to V_{\Omega_\#}$ brings element $x\in V_\Omega$ to $x\otimes z$, where z is a tensor product of several elements $K_{A,*}\in A$, $K_{B,*}\in B$ and $U\in A$ (see subsection 2.1). It can be easily checked that these elements of structure algebra are invariants of involution $x\mapsto x^*$. Therefore, $(\eta_{\Gamma'})_*(x)=(\eta_\Gamma)_*(\psi_*(x))=(\eta_\Gamma)_*(x)$ and $\Phi_{(\Omega,\mathcal{O},\Gamma)}(x)=\Phi_{(\Omega,\mathcal{O},\Gamma')}(x)$.

The equality $\Phi_{(\Omega,\mathcal{O},\Gamma)}=\Phi_{(\Omega,\mathcal{O},\Gamma')}$ for different but equivalent complete cut systems Γ and Γ' follows from the topological invariance for trivial and basic surfaces and their disjoint unions applied to linear forms $\Phi_{(\Omega/\Gamma,\mathcal{O}_\#)}$ and $\Phi_{(\Omega/\Gamma',\mathcal{O}'_\#)}$. $\square$

Lemma 4.9. *If Ω is a simple stratified surface then $\Phi_{(\Omega,\mathcal{O},\Gamma)}=\Phi_{(\Omega,\mathcal{O},\Gamma')}$ for any pair Γ, Γ' of complete cut systems of Ω.*

Proof. Consider separately simple surfaces (1) - (8) (see lemma 3.10). For example, let Ω be a disc with two interior special points p_1, p_2 and one boundary special point q. By lemma 4.8, it is sufficient to consider only nonequivalent complete cut systems Γ and Γ'. There are two nonequivalent complete cut systems Γ and Γ' of Ω. One of them, Γ, consists of one separating contour γ and Ω/Γ is isomorphic to the disjoint union of a sphere with three interior special points p_1,p_2,p' and a disc with one interior special point p'' and one boundary special point q. Another one, Γ', consists of two separating segments cut γ_1, γ_2 and Ω/Γ' is isomorphic to the disjoint union of three components:

∘ a disc with interior special point p_1 and boundary special point q'_1,

∘ a disc with interior special point p_2 and boundary special point q_2',
∘ a disc with three boundary special points q, q_1'', q_2''.

From definitions of $\Phi_{(\Omega,\mathcal{O},\Gamma)}$ and $\Phi_{(\Omega,\mathcal{O},\Gamma')}$ we obtain that the equality $\Phi_{(\Omega,\mathcal{O},\Gamma)} = \Phi_{(\Omega,\mathcal{O},\Gamma')}$ is equivalent to the equality

$$R_{\alpha_1,\beta_1'} F^{\beta_1',\beta_1''} R_{\alpha_2,\beta_2'} F^{\beta_2',\beta_2''} T_{\beta'\beta''\beta} = S_{\alpha_1\alpha_2\alpha'} F^{\alpha',\alpha''} R_{\alpha'',\beta}$$

for structure constants. The latter coincides with relation (5) for structure constants (see lemma 2.2) of structure algebra $\mathcal{H}$.

Analogous reasoning are applicable in all cases. Equalities for structure constants derived from equalities $\Phi_{(\Omega,\mathcal{O},\Gamma)} = \Phi_{(\Omega,\mathcal{O},\Gamma')}$ for complete cut systems of simple surfaces are just relations for structure constants of a structure algebra. We omit the details. □

It follows from the lemma that formula $\Phi_{(\Omega,\mathcal{O})} = \Phi_{(\Omega,\mathcal{O},\Gamma)}$ correctly defines a function $\Phi_{(\Omega,\mathcal{O})}$ for all simple surfaces. Now, we can define by multiplicativity linear form $\Phi_{(\Omega,\mathcal{O})}$ for any surface that is the disjoint union of trivial, basic and simple surfaces. Topological invariance for these forms and invariance of change of local orientations can be easily checked.

Lemma 4.10. *Let Γ and Γ' be adjacent complete cut systems of a stable stratified surface Ω. Then $\Phi_{(\Omega,\mathcal{O},\Gamma)} = \Phi_{(\Omega,\mathcal{O},\Gamma')}$*

Proof. Let Γ and Γ' be two nonequivalent adjacent complete cut systems of Ω. By lemma 4.8, the replacement of Γ by equivalent cut system leads to the same linear form. Hence, we may assume that the set Γ° of cuts common to Γ and Γ' has the following property. Contracted cut surface Ω/Γ° is the disjoint union of a simple surface Ω° and several basic surfaces and cut systems $\overline{\Gamma} = \Gamma \setminus \Gamma^\circ$, $\overline{\Gamma'} = \Gamma' \setminus \Gamma^\circ$ belong to Ω°; therefore $\overline{\Gamma}$ and $\overline{\Gamma'}$ are complete cut systems of simple surface Ω°.

We have that linear forms $\Phi_{(\Omega/\Gamma^\circ,\mathcal{O}_\#^\circ)}$ are already correctly defined for any set $\mathcal{O}_\#^\circ$ of local orientations. From *transitivity property* of cut morphisms we obtain that $\Phi_{(\Omega,\mathcal{O},\Gamma)}(x) = \Phi_{(\Omega/\Gamma^\circ,\mathcal{O}_\#^\circ)}(\eta^\circ(x))$ and $\Phi_{(\Omega,\mathcal{O},\Gamma')}(x) = \Phi_{(\Omega/\Gamma^\circ,\mathcal{O}_\#^\circ)}(\eta^\circ(x))$, where $\eta^\circ : V_\Omega \to V_{\Omega/\Gamma^\circ}$ is linear map induced by cut morphism $\eta^\circ : (\Omega, \mathcal{O}) \to (\Omega/\Gamma^\circ, \mathcal{O}_\#^\circ)$. Therefore, $\Phi_{(\Omega,\mathcal{O},\Gamma)} = \Phi_{(\Omega,\mathcal{O},\Gamma')}$. □

By theorem 3.1 any pair of complete cut systems of Ω is connected by a chain of adjacent complete cut systems. Thus, formula $\Phi_{(\Omega,\mathcal{O})} = \Phi_{(\Omega,\mathcal{O},\Gamma)}$ correctly defines function $\Phi_{(\Omega,\mathcal{O})}$ and we define linear forms $\Phi_{(\Omega,\mathcal{O})}$ for all connected stratified surfaces. Define $\Phi_{(\Omega,\mathcal{O}})$ for disconnected surfaces by multiplicativity.

Theorem 4.3. *A tuple of data $\mathcal{T}(\mathcal{H}) = (A, x \mapsto x^*, B, y \mapsto y^*, \{\Phi_{(\Omega,\mathcal{O})}\})$ is a Klein topological field theory.*

Proof. Let us verify axioms of a Klein topological field theory.

1° It is sufficient to check topological invariance for connected stable surfaces that are not trivial or basic surfaces. Let $\phi : (\Omega, \mathcal{O}) \to (\Omega', \mathcal{O}')$ be an isomorphism. Choose a complete cut system Γ of Ω and denote by Γ' complete cut system $\phi(\Gamma)$ of Ω'. The isomorphism ϕ induces the isomorphism $\phi_{\#} : (\Omega_{\#}, \mathcal{O}_{\#}) \to (\Omega'_{\#}, \mathcal{O}'_{\#})$ of contracted cut surfaces $\Omega_{\#} = \Omega/\Gamma$ and $\Omega'_{\#} = \Omega'/\Gamma'$. The topological invariance with respect of ϕ follows from the topological invariance of $\phi_{\#}$. The latter was proven in lemma 4.7.

2° Invariance of changing of local orientations is verified analogously to 1°.

3° Nondegeneracy follows from nondegeneracy of bilinear forms $(x', x'')|_A$ and $(y', y'')_B$.

4° Invariance of adding trivial field follows from the same statement proved for basic and trivial surfaces (lemma 4.7). Indeed, let Ω be a stable surface and r be a nonspecial point. Choose complete cut system Γ that does not meet r. Then the image of r in contracted cut surface belongs to a connected component $\Omega_{\# i}$, which is basic surface. Invariance of adding new special point r follows from the same statement for its image in $\Omega_{\# i}$.

5° It is sufficient to check cut invariance for a simple cut γ of a connected stable surface Ω that is not trivial or basic surface.

Suppose that γ is a stable cut. Then it can be included in a complete cut system Γ of Ω. By properties of cut morphisms $(\eta_\Gamma)_* = (\eta_{\Gamma\setminus\gamma})_* \circ (\eta_\gamma)_*$, where $(\eta_\Gamma)_* : V_\Omega \to V_{\Omega/\Gamma}$, $(\eta_{\Gamma\setminus\gamma})_* : V_{\Omega/\gamma} \to V_{\Omega/\Gamma}$, $(\eta_\gamma)_* : V_\Omega \to V_{\Omega/\gamma}$ are linear maps corresponding to cut morphisms. Therefore, $\Phi_{(\Omega,\mathcal{O})}(x) = \Phi_{(\Omega/\gamma,\mathcal{O}_{\#})}(\eta_\gamma)_*(x))$.

Suppose that γ is trivial (unstable) cut. Then there exists a complete cut system Γ such that $\Gamma \cap \gamma = \emptyset$. Hence, the image of γ in contracted cut surface is a simple cut of a connected component $\Omega_{\# i}$, which is basic surface. The invariance with respect to cut γ follows from the invariance with respect of its image in the basic surface, which was proved in lemma 4.7.

6° Multiplicativity is clear because linear forms $\Phi_{(\Omega,\mathcal{O})}$ for disconnected surfaces are defined using this axiom.

□

Theorem 4.4. (Main) *The correspondences $\mathcal{T} \rightsquigarrow \mathcal{C}(\mathcal{T}) \rightsquigarrow \mathcal{HC}(\mathcal{T}))$ and $\mathcal{H} \rightsquigarrow \mathcal{T}(\mathcal{H})$ are reciprocal bijections between Klein topological field theories $\mathcal{T} = \{A, x \mapsto x^*, B, y \mapsto y^*, \Phi_{(\Omega,\mathcal{O})}\}$ and isomorphism classes of structure algebras $\mathcal{H} = \{H = A \dotplus B, (x, y), x \mapsto x^*, U\}$. The correlators for Klein topological theory $\mathcal{T}(\mathcal{H})$ have the following expressions in terms of structure algebra $\mathcal{H}$:*

$$\langle x_1, \dots, x_m, ({y_1}^1, \dots, y^1_{m_1}), \dots, (y^s_1, \dots, y^s_{m_s})\rangle_{g,1} =$$
$$= (x_1 \dots x_m (y^1_1 \dots y^1_{m_1}) V_{K_B}(y^2_1 \dots y^2_{m_1}) \dots V_{K_B}(y^s_1 \dots y^s_{m_s}), K^g_A)$$

and

$$\langle x_1, \ldots, x_m, (y_1^1, \ldots, y_{m_1}^1), \ldots, (y_1^s, \ldots, y_{m_s}^s)\rangle_{g,0} =$$
$$= (x_1 \ldots x_m(y_1^1 \ldots y_{m_1}^1)V_{K_B}(y_1^2 \ldots y_{m_1}^2) \ldots V_{K_B}(y_1^s \ldots y_{m_s}^s), U^{2g})$$

Proof. The first statement follows from the exact constructions of the correspondences $\mathcal{T} \rightsquigarrow \mathcal{C}(\mathcal{T})$, $\mathcal{C} \rightsquigarrow \mathcal{H}(\mathcal{C})$, $\mathcal{H} \rightsquigarrow \mathcal{T}(\mathcal{H})$ and from theorems 4.1, 4.2, 4.3. The second statement follows from a representation $\langle X\rangle_G = \mathsf{Contr}_{(\Omega,\Gamma)}(\langle X_\#\rangle_{G_\#})$, where Γ is a specially chosen complete cut system. Namely, Γ contains a system of cuts Γ° between holes 1st and 2nd, 2nd and 3d, etc. and $\Gamma \setminus \Gamma^\circ$ is a normal cut system of Ω/Γ°. $\square$

Let $\mathcal{H}$ be a Klein topological field theory. One can restrict basic category of surfaces to the category of oriented surfaces with local orientations at all special points induced by a global orientation of a surface. It can be shown that this restriction is equivalent to an open-closed topological field theory as it is defined in [23], [26].

It is convenient to discuss the correspondence between variants of TFTs in terms of algebras. Note, that both involutive antiautomorphism $x \mapsto x^*$, responsible for changes of local orientations, and element U, responsible for gluing a hole by Möbius band, are not in use in oriented case.

The following claim is clear from Main theorem and results of section 2.

Corollary 4.1. *Any massive (corresponding to a semisimple algebra) open-closed topological field theory can be extended to a Klein topological field theory. Equivalent classes of extensions are in the bijection with pairs, consisting of an involutive antiautomorphism $x \mapsto x^*$ and an element U satisfying axioms of a structure algebra.*

Number of extensions of an open-closet TFT to KTFT can be easely computed by results of subsection 2.2

5. Hurwitz topological field theory

5.1. **Stratified coverings.** Fix a stratified topological space $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$. Let $\pi : P \to \Omega$ be a continuous epimorphic map of a topological space P onto Ω. Denote by P_λ the preimage of a stratum Ω_λ and by $\pi_\lambda : P_\lambda \to \Omega_\lambda$ the restriction of π to P_λ. Analogously, for an arbitrary subset $X \in \Omega$ denote by $\pi_X : P_X \to X$ the restriction of π to $P_X = \pi^{-1}(X)$.

Definition. *A continuous map $\pi : P \to \Omega$ is called a covering over the stratification (or a stratified covering) if and only if for any stratum λ the restriction $\pi_\lambda : P_\lambda \to \Omega_\lambda$ is a local homeomorphism, i.e., a covering.*

A stratified covering $\pi : P \to \Omega$ is called special coverings if and only if

(1) Ω is a specially stratified manifold (i.e., any codimension 1 stratum

belongs to the boundary of Ω*);*
(2) P *is a manifold, possibly with a boundary.*

In this work we deal with specially stratified coverings over surfaces and call them *coverings* for short.

A covering is called *n-sheeted covering* if and only if the full preimage of any point of a stratum of maximal dimension consists of n points. n-sheeted coverings over disconnected base can (and sometimes, will) be considered. In general, it is not assumed that a cover space P is connected.

An *isomorphism* of two coverings $\pi_1 : P_1 \to \Omega$ and $\pi_2 : P_2 \to \Omega$ over the same base is a homeomorphism $\phi : P_1 \to P_2$ such that $\pi_2 \circ \phi = \pi_1$. By $\mathcal{C}ov_n(\Omega)$ we denote the category of n-sheeted stratified coverings over Ω with morphisms being isomorphisms of coverings. By $\mathrm{Cov_n}(\Omega)$ is denoted the set of isomorphism classes in category $\mathcal{C}ov_n(\Omega)$. Clearly, $\mathrm{Cov_n}(\Omega)$ is a finite set.

Lemma 5.1. *An isomorphism* $\phi : \Omega \to \Omega'$ *of stratified manifolds induces the bijection between sets* $\mathrm{Cov_n}(\Omega)$ *and* $\mathrm{Cov_n}(\Omega')$.

Indeed, the bijection is induced by morphism $\pi \mapsto \phi \circ \pi$.

Definition. *Let* $\pi : P \to \Omega$ *and* $\pi' : P' \to \Omega'$ *be two coverings over stratified manifolds* Ω *and* Ω'*. A pair of homeomorphisms* $\hat{\phi} : P \to P'$ *and* $\phi : \Omega \to \Omega'$ *is called a topological equivalence of coverings if and only if* $\pi' \circ \hat{\phi} = \phi \circ \pi$*. A class of topologically equivalent coverings is called a topological type of coverings.*

Obviously, two isomorphic coverings over a stratified manifold Ω are of the same topological type. The converse is not true: there exist nonisomorphic coverings over a surface that are of the same topological type.

Let $\mathrm{Aut}\,\Omega$ be the group of all homeomorphisms of Ω, that preserve the stratification of Ω (a permutation of strata is allowed). The group $\mathrm{Aut}\,\Omega$ acts on the set $\mathrm{Cov}(\Omega)$: if $\phi \in \mathrm{Aut}\,\Omega$ and $\pi \in \mathrm{Cov}(\Omega)$ then $\phi(\pi) = \phi \circ \pi$. Clearly, two coverings $\pi, \pi' \in \mathrm{Cov}(\Omega)$ are of the same topological type if and only if they lie in the same orbit of the group $\mathrm{Aut}(\Omega)$.

Note, that one can correctly assign a topological type of a covering $\pi : P \to \Omega$ to a topological type of Ω. In contrast, the isomorphism class $[\pi]$ of covering π can be associated with just manifold Ω but not with its topological type because one cannot choose an isomorphism between two different (but isomorphic) bases Ω and Ω' canonically.

For an oriented base Ω we define an *oriented topological type* of a covering with respect to homeomorphisms preserving the orientation.

There exist only two, up to isomorphism, compact connected special stratified one-dimensional manifolds, a circle S^1 and a closed segment

I. It is convenient to consider also one noncompact manifold, namely, a ray R (with two obvious strata).

5.1.1. *Coverings over a circle.* A covering $\pi : P \to S^1$ over a circle has no special points, i.e., it is a nonramified covering. Isomorphism classes of n-sheeted coverings over S^1 are in one-to-one correspondence with unordered partitions of the number n into a sum $n = n_1 + \cdots + n_s$. If $\pi : P \to S^1$ is a covering then s is the number of connected components of P and n_i is the degree of the restriction of π to i-th connected component of P.

Equivalently, the isomorphism class of a covering over S^1 can be presented 1) as a Young diagram of order n; 2) as a conjugacy class of a symmetric group S_n. We denote by α a partition of n, as well, as corresponding Young diagram of order n and a conjugacy class of S_n. Group $\operatorname{Aut} S^1$ acts trivially on the set $\mathrm{Cov_n}(S^1)$ of isomorphism classes of coverings. Therefore, topological type of a covering can be also presented by the partition α.

5.1.2. *Coverings over a ray.* Let $\pi : P \to R$ be an n-sheeted covering over a ray. Denote by M the set of connected components of the preimage of an open interval $R^\circ = R - \partial R$. Obviously, $|M| = n$ and the restriction of π to any component is a homeomorphism. Any point p' of the preimage of the end-point $p \in R$ belong to the closure of either just one or just two components of M. Therefore, a special covering defines an involutive element s of group $\operatorname{Aut} M$ of all bijections $M \to M$. Group $\operatorname{Aut} M$ is isomorphic to symmetric group S_n; there is no canonical isomorphism, therefore, involution $s \in M$ correctly defines only conjugasy class of its image in S_n.

Evidently, isomorphism classes of coverings over the ray R are in one-to-one correspondence with conjugacy classes of involutions in S_n. On other hand, conjugacy classes of involutions are in one-to-one correspondence with decompositions of n into the sum $n = k + 2l$.

Group $\operatorname{Aut} R$ acts trivially on the set $\mathrm{Cov_n}(R)$ of isomorphism classes of coverings. Therefore, topological type of a covering can be also presented as a conjugacy class of an involutive permutation.

5.1.3. *Coverings over a segment.* (Fig.5)

Formal description of isomorphism classes of n-sheeted coverings over an interval I is as follows. Denote by p_1 and p_2 end points of I. Let $\pi : P \to I$ be an n-sheeted covering. Denote by M the set of connected components of the preimage of I° in P. Obviously, $\#M = n$ and the restriction of π to any component is a homeomorphism.

A point p_i $(i = 1, 2)$ defines an involutive permutation of set M, which we denote by s_i. Thus, to each special covering π we assign the ordered pair (s_1, s_2) of involutions $s_1, s_2 \in \operatorname{Aut} M \approx S_n$. Two pairs

(s_1, s_2) and (s_1', s_2') are called conjugated if there exists $g \in S_n$ such that $s_1' = gs_1g^{-1}$ and $s_2' = gs_2g^{-1}$.

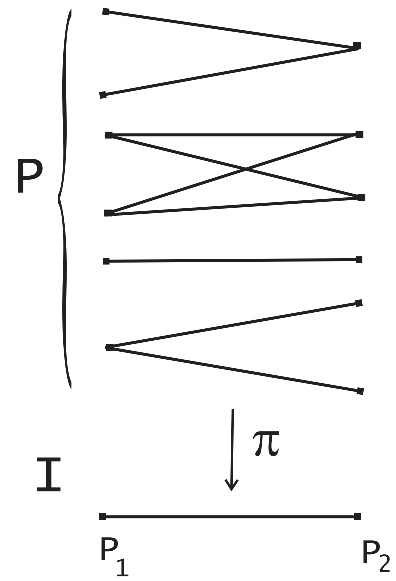

FIGURE 5. Stratified covering π over segment I.

Lemma 5.2. *There exists a natural one-to-one correspondence between set* $\mathrm{Cov_n}(I)$ *of isomorphism classes of n-sheeted coverings over an interval I and the set of conjugacy classes of pairs of involutions in S_n.*

Proof is skipped.

5.1.4. *Analogue of Young diagram for pairs of involutions.* The analogue of Young diagram can be defined for an ordered pair of involutions. Let $D_\infty = \langle \Sigma_1, \sigma_2 | \sigma_1^2 = \sigma_2^2 = 1\rangle$ be the infinite dihedral group and (s_1, s_2) be an ordered pair of involutive permutations of the symmetric group S_n acting by permutations of the set $M = \{1, \dots, n\}$. Then the correspondence $\sigma_i \mapsto s_i$ can be uniquely extended to the representation ρ of D_∞ by permutations of the set M. Thus, M can be decomposed into the orbits $M_1, \dots, M_l$ of D_∞. Any orbit can be considered as a transitive representation of D_∞.

Let M_i be an orbit of D_∞. Then it can be of one of four types:

- (type 1) σ_1 and σ_2 has no fixed points in M_i;
- (type 2) σ_1 has one fixed point and σ_2 has one fixed point;
- (type 3) σ_1 has two fixed point and σ_2 has no fixed point;
- (type 4) σ_1 has no fixed points and σ_2 has two fixed points.

An orbit of type 1 consists of even number $2n_i$ of elements and any two orbits of type 1 of $2n_i$ elements are equivalent as representations of D_∞. An orbit of type 2 consists of odd number $2n_i - 1$ of elements and any two orbits of type 2 of $2n_i - 1$ elements are equivalent as representations of D_∞. An orbit of type 3 consists of even number $2n_i$ of elements and any two orbits of type 3 of $2n_i$ elements are equivalent as representations of D_∞. An orbit of type 4 consists of even number $2n_i$ of elements and any two orbits of type 4 of $2n_i$ elements are equivalent as representations of D_∞.

Thus, to each pair (s_1, s_2) we can assign a decomposition of n into the sum of natural numbers; the sum is separated into four blocks as follows: $n = (2n^1_1 + \cdots + 2n^1_{s_1}) + ((2n^2_1 - 1) + \cdots + (2n^2_{s_2} - 1)) + (2n^3_1 + \cdots + 2n^3_{s_3}) + (2n^4_1 + \cdots + 2n^4_{s_4})$ Blocks correspond to types of orbits, summands in a block are unordered. We denote this decomposition by a single letter β and call it a *dihedral decomposition of* n.

Lemma 5.3. *Let* (s_1, s_2) *and* (s'_1, s'_2) *be two ordered pairs of involutive elements of* S_n *and* β, β' *be corresponding dihedral decompositions of* n. *Then* (s_1, s_2) *is conjugated to* (s'_1, s'_2) *if and only if* $\beta = \beta'$.

Proof is skipped.

Thus, dihedral decompositions of n are in bijection with conjugacy classes of ordered pairs of involutions. Obviously, one can present β also as the set of four Young diagrams with easily derived special properties. We call this presentation a dihedral Young diagram of a pair (s_1, s_2). In sequel we consider terms 'a dihedral decomposition of n' and 'a dihedral Young diagram of order n' as synonyms. Let β be the dihedral Young diagram of a pair (s_1, s_2). Denote by β^* a dihedral Young diagram of the pair (s_2, s_1). The operation $\beta \mapsto \beta^*$ defines an involutive automorphism of the set $\mathrm{Cov_n}(I)$, which we denote by the same sign "*".

Lemma 5.4. *Dihedral Young diagram* β^* *can be obtained from* β *by replacing blocks of summands of types 3 and 4.*

Proof is elementary.

Assign to each conjugacy class $[(s_1, s_2)]$ conjugacy classes of involutions $[s_1]$ and $[s_2]$ and denote them by $\iota_1(\beta)$ and $\iota_2(\beta)$ respectively.

A conjugacy classes of a pair $[(s, s)]$, $s \in S_n$, is called *a trivial class*; corresponding dihedral Young diagram is said to be trivial. Evidently, they are in one-to-one correspondence with conjugacy classes of involutive elements in S_n. The dihedral Young diagram β of the class $[(s, s)]$ is $n = (2 + \cdots + 2) + (1 + \cdots + 1)$ (blocks of summands of types 3 and 4 are empty).

Fix an orientation of interval I. Then homeomorphisms of I preserving the orientation fixes points p_1 and p_2. These homeomorphisms generate the index two subgroup $\mathrm{Aut}^+ I$ of automorphism group $\mathrm{Aut}\, I$. Evidently, $\mathrm{Aut}^+ I$ acts trivially on the set $\mathrm{Cov}(I)$. Thus, oriented topological types of special coverings over a segment I are in one-to-one correspondence with dihedral Young diagrams.

Any element from $\mathrm{Aut}\, I \setminus \mathrm{Aut}^+ I$ acts on the set $\mathrm{Cov}(I)$ as $(s_1, s_2) \mapsto (s_2, s_1)$. Thus, coverings with dihedral Young diagrams β and β^* are of the same topological type. Therefore, topological type and isomorphism class of coverings do not coincide for the segment.

5.1.5. *Dimension 2.* (Fig.6)

Let $\pi : P \to \Omega$ be a covering over a stratified surface $\Omega = \coprod_{\lambda\in\Lambda} \Omega_\lambda$ and $p \in \Omega$ be any point of base. Topological invariants of the restriction $\pi_U : P_U \to U$ of the covering to a sufficiently small neighborhood U of p depends only on a stratum containing p. More precisely, if p belongs to the generic stratum, then π_U is trivial covering, i.e., π_U mappings n copies of U onto U. Thus, n is a unique invariant at a generic point.

If p is an interior special point, then one can choose U isomorphic to an open disc. In this case π_U is defined, up to topological equivalence, by the topological type of its restriction to the boundary ∂U (it is a circle) [20]. Thus, a Young diagram describes the covering in the neighborhood of p.

If p belongs to a one-dimensional boundary stratum E, then it is convenient to choose U homeomorphic to the direct product of a neighborhood U_E of p in E and a ray R [29]. Clearly, topological type of the covering π_U is uniquely defined by the topological type of its restriction to the ray R. Therefore, the covering in the neighborhood of $p \in E$ is described by a conjugacy class of an involutive element of S_n.

If q is a boundary special point, then it is convenient to choose a neighborhood U homeomorphic to an open cone $\mathcal{C}^\circ(I)$ over a closed interval I. Clearly, isomorphism classes of coverings over U are in one-to-one correspondence with isomorphism classes of coverings over I. As it was mentioned above, topological type of a covering over I can include more than one isomorphism classes of coverings over I. In order to control isomorphism classes of coverings over U, let us fix an admissible set of local orientations at special points of the surface. Hence, to each boundary special point q we can assign an isomorphism class of coverings over the segment, i.e., dihedral Young diagram. This diagram describes the covering in the neighborhood of q.

Clearly, local invariants of a covering coincide for all points from the same stratum. Thus, to each stratum of Ω we assigned a combinatorial invariant.

5.2. **Cut coverings.** Recall that a cut surface $\Omega_* = (\Omega_*, \Gamma_*, \tau)$ is a triple, consisting of

a stratified surface Ω_*,

a set $\Gamma_* \subset \Omega_*$ such that Γ_* coincides with the joint of several closed pairwise nonintersecting one-dimensional boundary strata,

an involutive homeomorphism $\tau : \Gamma_* \to \Gamma_*$ having no fixed points (see section 3.2).

Fix a cut surface $\Omega_* = (\Omega_*, \Gamma_*, \tau)$.

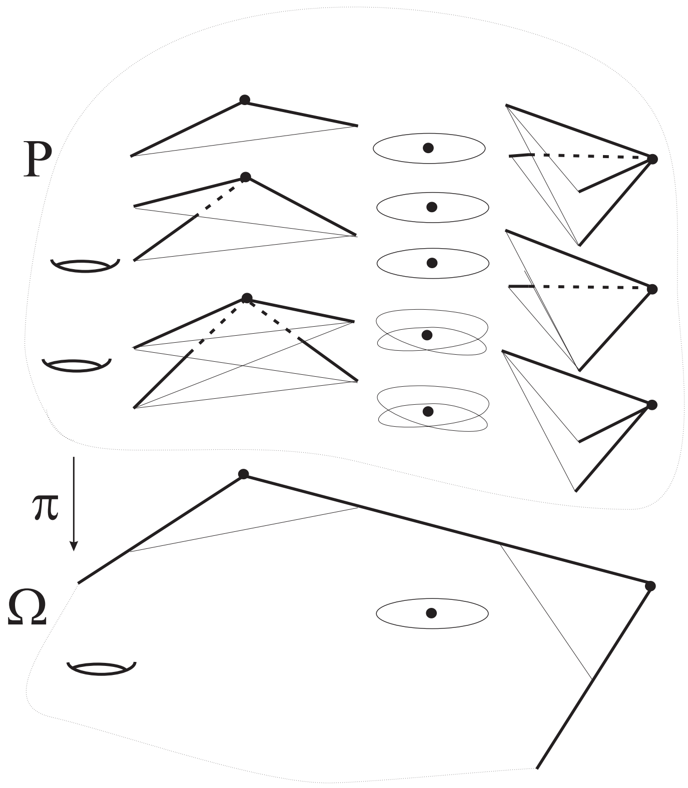

FIGURE 6. Stratified covering π over two-dimensional stratified surface Ω.

Let $\pi_* : P_* \to \Omega_*$ be an n-sheeted covering. Denote by $\widehat{\Gamma}_*$ the preimage of Γ_* in P_*. Suppose that the following condition holds:

$$(*) \qquad \widehat{\Gamma}_* \subset \partial P_*$$

In sequel we deal with coverings π_* satisfying this condition.

Denote by $\xi : \widehat{\Gamma}_* \to \Gamma_*$ the restriction of π_* to $\widehat{\Gamma}_*$. Due to $(*)$ the mapping $\xi : \widehat{\Gamma}_* \to \Gamma_*$ is an n-sheeted covering over Γ_*.

An involutive homeomorphism $\widehat{\tau} : \widehat{\Gamma}_* \to \widehat{\Gamma}_*$ is called *an admissible involution* if $\pi_* \circ \widehat{\tau} = \tau \circ \pi_*$.

Definition. *Let $\pi_* : P_* \to \Omega_*$ be a covering over a cut surface Ω_*. Suppose condition $(*)$ is satisfied. Let $\widehat{\tau} : \widehat{\Gamma}_* \to \widehat{\Gamma}_*$ be an admissible involution. Then pair $(\pi_*, \widehat{\tau})$ is called a cut covering.*

An isomorphism of two cut coverings $(\pi_*, \widehat{\tau})$ and $(\pi'_*, \widehat{\tau}')$ is an isomorphism $\phi : P_* \to P'_*$ of coverings such that $\widehat{\tau}' \circ \phi = \phi \circ \widehat{\tau}$.

Denote by $\mathcal{C}utCov(\Omega_*)$ the category of cut coverings over a cut surface $\Omega_* = (\Omega_*, \Gamma_*, \tau)$ with morphisms being isomorphisms of cut coverings; denote by $\mathrm{CutCov}(\Omega_*)$ the set of isomorphism classes of cut coverings. In sequel, we shall use the same difference in fonts in order to distinguish a category and the set of isomorphism classes of objects in it and in other cases.

Clearly, $\mathrm{CutCov}(\Omega_*)$ is a finite set.

Let $\pi : P \to \Omega$ be a covering over stratified surface Ω, $\Gamma \subset \Omega$ be a cut system, $(\Omega_*, \Gamma_*, \tau)$ be a cut surface obtained by cutting Ω along Γ and $\mathsf{glue} : \Omega_* \to \Omega$ be gluing map.

Clearly, the preimage $\widehat{\Gamma} = \pi^{-1}(\Gamma)$ is a cut system of P. Denote by $(P_*, \widehat{\Gamma}_*, \widehat{\tau})$ the cut surface obtained by cutting P along $\widehat{\Gamma}$ and denote by $\widehat{\mathsf{glue}}$ the glueing map $P_* \to P$. Obviously, there is a natural covering $\pi_* : P_* \to \Omega_*$ such that the following diagram is commutative.

$$\begin{array}{ccc} P_* & \xrightarrow{\widehat{\mathsf{glue}}} & P \\ {\scriptstyle \pi_*}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ \Omega_* & \xrightarrow[\mathsf{glue}]{} & \Omega \end{array}$$

Evidently, covering π_* satisfies $(*)$ and the correspondence $\pi \mapsto \pi_*$ is a functor $\mathsf{cut} : \mathcal{C}ov(\Omega) \to \mathcal{C}ov(\Omega_*)$. This functor generates a map $\mathsf{cut} : \mathrm{Cov}(\Omega) \to \mathrm{Cov}(\Omega_*)$ of isomorphism classes of coverings. Functor cut is a superposition of two functors

(1) $\mathcal{C}ov(\Omega) \to \mathcal{C}ut\mathcal{C}ov(\Omega_*)$ and

(2) $\mathcal{C}ut\mathcal{C}ov(\Omega_*) \to \mathcal{C}ov(\Omega_*)$,

where (1) is induced by cutting along Γ and (2) is just "forgetting" $\widehat{\tau}$.

Lemma 5.5. *Functor $\mathcal{C}ov(\Omega) \to \mathcal{C}ut\mathcal{C}ov(\Omega_*)$ induces the equivalence of categories.*

Proof. One can easily construct an inverse functor $\mathcal{C}ut\mathcal{C}ov(\Omega_*) \to \mathcal{C}ov(\Omega)$. It 'glues' a cut covering $(\pi_*, \widehat{\tau})$ into a covering $\pi : P_*/\widehat{\tau} \to \Omega$. □

Let $(\Omega_*, \Gamma_*, \tau)$ be a cut surface. Denote by $\mathcal{C}ov(\Omega_*, \Gamma_*)$ the subcategory of $\mathcal{C}ov(\Omega_*)$, that consists of coverings $\pi_* : P_* \to \Omega_*$ over Ω_* satisfying condition $(*)$.

Fix a covering $\pi_* \in \mathcal{C}ov(\Omega_*, \Gamma_*)$; denote by $[\pi_*]$ its class. Denote by ξ the restriction $\pi_*|_{\widehat{\Gamma}_*}$.

Clearly, an automorphism of covering π_* induces the automorphism of covering ξ. Hence, we obtain a homomorphism $\phi : \mathrm{Aut}\,\pi_* \to \mathrm{Aut}\,\xi$. The preimage of a generic point of Γ_* consists of n elements where n is the degree of π_*. Therefore, ϕ is a monomorphism. We shall consider group $\mathrm{Aut}\,\pi_*$ as a subgroup of $\mathrm{Aut}\,\xi$.

Denote by $T(\pi_*)$ the set of all admissible involutions $\widehat{\tau} : \widehat{\Gamma}_* \to \widehat{\Gamma}_*$.

Clearly, the composition $\widehat{\tau} \circ \widehat{\tau}'$ of two admissible involutions is an automorphism of covering ξ and vice versa, the composition of an admissible involution and an automorphism of ξ is an admissible involution. Hence, the number of admissible involutions $|T(\pi_*)|$ coincides with the order of group $\mathrm{Aut}\,\xi$.

Group $\operatorname{Aut}\xi$ acts on $T(\pi_*)$ as follows: $g(\widehat{\tau})(x) = g\circ\widehat{\tau}\circ g^{-1}(x)$, where $g\in\operatorname{Aut}\xi$, $\widehat{\tau}\in T(\pi_*)$, $x\in\widehat{\Gamma}_*$.

Denote by $\operatorname{CutCov}(\Omega_*)_{[\pi_*]}$ the preimage of $[\pi_*]$ under the map $\operatorname{CutCov}(\Omega_*)\to\operatorname{Cov}(\Omega_*)$ (coming from functor (2) above).

Lemma 5.6. *There exists a bijection between the set* $\operatorname{CutCov}(\Omega)_{[\pi_*]}$ *and the set of orbits* $T(\pi_*)/\operatorname{Aut}(\pi_*)$.

Proof. Choose a representative $\pi_*\in[\pi_*]$. Any class from $\operatorname{CutCov}(\Omega_*)_{[\pi_*]}$ contains a representative $(\pi_*,\widehat{\tau})$ that includes just the covering π_*. Clearly, cut coverings $(\pi_*,\widehat{\tau})$ and $(\pi_*,\widehat{\tau}')$ with the same π_* are isomorphic, if and only if $\widehat{\tau}$ and $\widehat{\tau}'$ belong to the same orbit of $\operatorname{Aut}\pi_*$. $\square$

Evidently, isomorphic coverings have isomorphic automorphism groups. Therefore, we can assign the isomorphism class of groups $\operatorname{Aut}[\pi]$ to an isomorphism class of coverings $[\pi]$. Analogously, to $[\pi_*]$ we assign a class of isomorpic sets of admissible involutions $T([\pi])$.

By lemma 5.5 there is a bijection between sets $\operatorname{CutCov}(\Omega_*)$ and $\operatorname{Cov}(\Omega)$. Denote by $\operatorname{Cov}(\Omega)_{[\pi_*]}$ the image of $\operatorname{CutCov}(\Omega_*)_{[\pi_*]}$ in $\operatorname{Cov}(\Omega)$.

Theorem 5.1. *Let* Γ *be a cut system of a stratified surface* Ω. *Denote by* (Ω_*,Γ_*,τ) *a cut surface obtained by cutting* Ω *along* Γ. *Let* $[\pi_*]$ *be an arbitrary class of isomorphic coverings* $\pi_*:P_*\to\Omega_*$ *satisfying* $(*)$. *Then*

$$\sum_{[\pi]\in\operatorname{Cov}(\Omega)_{[\pi_*]}}\frac{1}{|\operatorname{Aut}[\pi]|} = |T([\pi_*])|\cdot\frac{1}{|\operatorname{Aut}[\pi_*]|}.$$

Proof. Clearly, the stabilizer of $\widehat{\tau}\in T(\pi_*)$ in group $\operatorname{Aut}\pi_*$ coincides with the group of all automorphisms of cut covering $(\pi_*,\widehat{\tau})$. By lemma 5.5, there is the bijection between cut coverings over cut surface Ω_* and coverings over Ω. Let $\pi\in\mathcal{C}ov(\Omega)$ corresponds to cut covering $(\pi_*,\widehat{\tau})$. Therefore, the stabilizer of $\widehat{\tau}$ in $\operatorname{Aut}\pi_*$ is isomorphic to $\operatorname{Aut}\pi$. Denote by $\Delta([\pi])$ the orbit in $T([\pi_*])$ corresponding to $[\pi]$ (see lemma 5.6). Hence $|\Delta([\pi])| = \frac{|\operatorname{Aut}[\pi_*]|}{|\operatorname{Aut}[\pi]|}$.

In order to complete the proof substitute the latter equality into the equality $|T(\pi_*)| = \sum_{[\pi]\in\operatorname{Cov}(\Omega)_{[\pi_*]}}|\Delta(\pi)|$ which reflects the decomposition of $T(\pi_*)$ into orbits of $\operatorname{Aut}\pi_*$. The claim of theorem coincides with obtained equality up to algebraic transformations. $\square$

Remark. Theorem 5.1 can be easily generalized to stratified coverings in any dimension.

Let Γ be a cut system of a stratified surface Ω. Denote by (Ω_*,Γ_*,τ) a cut surface obtained by cutting Ω along Γ and denote by $\Omega_\#$ the contracted cut surface.

Let $\pi: P\to\Omega$ be a stratified covering. Denote by $(\pi_*,\widehat{\tau})$ the cut covering corresponding to π. By definition, $\pi_*\in\mathcal{C}ov(\Omega_*,\Gamma_*)$. Clearly, covering $\pi_*: P_*\to\Omega_*$ induces the covering $\pi_\#: P_\#\to\Omega_\#$ where

$P_{\#}$ is the surface obtained by contracting to a point each connected component of Γ_*.

Lemma 5.7. *The correspondence $\pi_* \to \pi_{\#}$ induces the bijection between sets* $\mathrm{Cov}(\Omega_*, \Gamma_*)$ *and* $\mathrm{Cov}(\Omega_{\#})$ *of classes of equivalent coverings.*

Proof is skipped.

Let Γ be a cut system of a stratified surface Ω. The composition of maps $\mathrm{Cov}(\Omega) \to \mathrm{CutCov}(\Omega_*) \to \mathrm{Cov}(\Omega_*, \Gamma_*) \to \mathrm{Cov}(\Omega_{\#})$ defines the map $\mathrm{Cov}(\Omega) \to \mathrm{Cov}(\Omega_{\#})$. By theorem 5.1 and lemma 5.7, there is a relation between number $\frac{1}{|\mathrm{Aut}[\pi_{\#}]|}$ where $[\pi_{\#}] \in \mathrm{Cov}(\Omega_{\#})$ and numbers $\frac{1}{|\mathrm{Aut}[\pi]|}$ for all preimages $[\pi] \in \mathrm{Cov}(\Omega)$ of $[\pi_{\#}]$. We shall rewrite them below in slightly different form for simple cuts of all classes.

Let γ be a simple cut of Ω. Thus, in our notations $\Gamma = \{\gamma\}$. We will write γ, γ_*, etc. instead of Γ, Γ_* etc.

Clearly, there are three possibilities for γ_*: (1) γ_* consists of two ovals; (2) γ_* consists of two segments; (3) γ_* consists of one oval.

Denote by r', r'' two special points of $\Omega_{\#}$ coming from components of γ_* in cases (1) and (2) and by r the analogous point in case (3).

We have to fix local orientations at points r', r'' or at r. It is required that these local orientations can be completed to an admissible set of local orientations at all special points of $\Omega_{\#}$. In any case there are local orientations at r', r'' or r (case (3)) induced by an orientation of simple cut γ. Note, that these local orientations may not satisfy the requirement. We correct them by the following rule. If γ is a simple cut of class 1, 2, 5, 6, 7 than change the local orientation at point r'' by opposite one. Fix obtained local orientations and denote them by $\wr', \wr''$ or $\wr$ (case (3)). It can be checked that these orientations satisfy the requirement.

Let π be a covering over Ω and γ'_* be a connected component of γ_*. There is an orientation of γ'_* that is compatible with fixed local orientation at point $r' \in \Omega_{\#}$ coming from the contraction of γ'_*. Denote by ξ' the restriction of covering π_* to the preimage $\pi_*^{-1}(\gamma'_*)$. Thus, we have that ξ' is defined, up to isomorphism, either by a Young diagram or by a dihedral Young diagram.

Therefore, we obtain a local invariant of covering $\pi_{\#} : P_{\#} \to \Omega_{\#}$ at this point as either a Young diagram or a dihedral Young diagram.

Let $\pi_{\#}$ be an arbitrary covering over $\Omega_{\#}$. Denote by α', α'' the Young diagram describing local invariants of $\pi_{\#}$ at points r', r'' resp. in case (1). Analogously, denote by β', β'' dihedral Young diagrams in case (2) and by α Young diagram in case (3).

Denote by $\mathrm{Cov}(\Omega)_{[\pi_{\#}]}$ the preimage of $[\pi_{\#}] \in \Omega_{\#}$ in $\mathrm{Cov}(\Omega)$.

Put $\alpha^* = \alpha$ for a Young diagramm α. We use α^* below in order to obtain forumas that will be correct also for coverings with structure group G different from S_n.

Corollary 5.1. *Let γ be a simple cut of a stratified surface Ω and $\pi_\#$ be a covering over contracted cut surface $\Omega_\#$. Then the following identities hold:*
(1) If γ belongs to one of classes 1, 2 then

$$\sum_{[\pi]\in\mathrm{Cov}(\Omega)_{[\pi_\#]}} \frac{1}{|\operatorname{Aut}[\pi]|} = \delta_{\alpha',\alpha''^*}|\operatorname{Aut}\alpha'|\cdot\frac{1}{|\operatorname{Aut}[\pi_\#]|}$$

(2) If γ belongs to class 3 then

$$\sum_{[\pi]\in\mathrm{Cov}(\Omega)_{[\pi_\#]}} \frac{1}{|\operatorname{Aut}[\pi]|} = \delta_{\alpha',\alpha''}|\operatorname{Aut}\alpha'|\cdot\frac{1}{|\operatorname{Aut}[\pi_\#]|}$$

(3) If γ belongs to one of classes 5,6,7 then

$$\sum_{[\pi]\in\mathrm{Cov}(\Omega)_{[\pi_\#]}} \frac{1}{|\operatorname{Aut}[\pi]|} = \delta_{\beta',\beta''^*}|\operatorname{Aut}\beta'|\cdot\frac{1}{|\operatorname{Aut}[\pi_\#]|}$$

(4) If γ belongs to one of classes 8,9 then

$$\sum_{[\pi]\in\mathrm{Cov}(\Omega)_{[\pi_\#]}} \frac{1}{|\operatorname{Aut}[\pi]|} = \delta_{\beta',\beta''}|\operatorname{Aut}\beta'|\cdot\frac{1}{|\operatorname{Aut}[\pi_\#]|}$$

(5) If γ belongs to class 4 then

$$\sum_{[\pi]\in\mathrm{Cov}(\Omega)_{[\pi_\#]}} \frac{1}{|\operatorname{Aut}[\pi]|} = d_\alpha\cdot\frac{1}{|\operatorname{Aut}[\pi_\#]|}$$

where d_α is a number of involutions in symmetric group S_n having no fixed points and commuting with an element $g\in S_n$ that belongs to conjugated class corresponding to Young diagram α.

Proof follows from theorem 5.1 and lemma 5.7.

5.3. **Hurwitz topological field theory.** Denote by $\mathcal{A}=\mathcal{A}_n$ the set of Young diagrams of order n and denote by A a vector space of formal linear combinations of Young diagrams. Denote by $\mathcal{B}=\mathcal{B}_n$ the set of dihedral Young diagrams of order n and denote by B a vector space of formal linear combinations of dihedral Young diagrams. Define an involutive linear transformation $^*: A\to A$ as identical map. Define an involutive linear transformation $^*: B\to B$ as linear continuation of the map $\beta\mapsto\beta^*$ for dihedral Young diagram β.

Let $(\Omega,\mathcal{O})$ be a pair consisting of a stratified surface and a set of local orientations $\mathcal{O}$ at special points of Ω. Denote by Ω_a the set of interior special points of Ω and by Ω_b the set of boundary special points.

Let $\alpha:\Omega_a\to\mathcal{A}$ and $\beta:\Omega_b\to\mathcal{B}$ be two maps. Denote by α_p (resp., β_q) the image of an interior special point p (resp., boundary special point q) in set $\mathcal{A}$ (resp.,$\mathcal{B}$). Denote by $\mathrm{Cov}(\Omega,\{\alpha_p\},\{\beta_q\})$ the set of isomorphism classes of coverings having a local invariant α_p at each point $p \in \Omega_a$ and a local invariant β_q at each point $q \in \Omega_b$.

We shall define linear function $H_{(\Omega,\mathcal{O})} : V_{(\Omega,\mathcal{O})} \to \mathbb{C}$, where $V_{(\Omega,\mathcal{O})} = (\otimes_{p\in\Omega_a} A_p) \otimes (\otimes_{q\in\Omega_b} B_q)$.

Assume first that $\mathcal{O}$ is an admissible set of local orientations. Then for element $(\otimes_{p\in\Omega_a}\alpha_p) \otimes (\otimes_{q\in\Omega_b}\beta_q) \in V_{(\Omega,\mathcal{O})}$ put

$$H_{(\Omega,\mathcal{O})}((\otimes_{p\in\Omega_a}\alpha_p) \otimes (\otimes_{q\in\Omega_b}\beta_q)) = \sum_{[\pi]\in \mathrm{Cov}(\Omega,\{\alpha_p\},\{\beta_q\})} \frac{1}{|\operatorname{Aut}[\pi]|}.$$

Elements $(\otimes_{p\in\Omega_a}\alpha_p) \otimes (\otimes_{q\in\Omega_b}\beta_q)$ form a basis of $V_{(\Omega,\mathcal{O})}$, hence we can expand $H_{(\Omega,\mathcal{O})}$ by linearity. For nonadmissible $\mathcal{O}$ define linear form $H_{(\Omega,\mathcal{O})}$ in such a way that axiom 2° of Klein field theory is satisfied. This rule correctly defines $H_{(\Omega,\mathcal{O})}$ in all cases because $\sum_{[\pi]\in \mathrm{Cov}(\Omega,\{\alpha_p\},\{\beta_q\})} \frac{1}{|\operatorname{Aut}\pi|}$ does not depend on the choice of an admissible set of local orientations.

Theorem 5.2. *The set of data* $\mathcal{H} = \{A,^* : A \to A, B,^* : B \to B, H_{(\Omega,\mathcal{O})}\}$ *is a Klein topological field theory. We call it Hurwitz topological field theory of degree* n.

Proof. Evidently, axioms 1°, 2°, 3° and 4° are satisfied. Bilinear forms can be easely computed. We obtain equalities: $(\alpha', \alpha'')_A = \delta_{\alpha',\alpha''^*} \frac{1}{|\operatorname{Aut}\alpha'|}$ and $(\beta', \beta'')_B = \delta_{\beta',\beta''^*} \frac{1}{|\operatorname{Aut}\beta'|}$. Element $U \in A$ is equal to $\sum_\alpha d_\alpha (\alpha, \alpha)_A$ where numbers d_α are defined in corollary 5.1. Cleary, there are the following identities for tensors corresponding to bilinear forms and U: $F^{\alpha',\alpha''} = \delta_{\alpha',\alpha''^*}|\operatorname{Aut}\alpha'|$, $F^{\beta',\beta''} = \delta_{\beta',\beta''^*}|\operatorname{Aut}\beta'|$, $D^\alpha = d_\alpha$.

Axiom 5° essentially follows from theorem 5.1 and corollary 5.1.

For example, let us verify the axiom for γ of class 5.

Let Ω be a stratified surface, $\mathcal{O}$ be a set of local orientations at special points and and γ be a simple cut of Ω of class 5, i.e., a cut between two holes. Denote by $\Omega_\#$ the contracted cut surface.

The set $(\Omega_\#)_0$ of all special points of $\Omega_\#$ consists of the image of set Ω_0 and two additional boundary points q', q'' coming from connected components of γ_*. Choose a set of local orientations $\mathcal{O}$ such that the following conditions are satisfied. First, $\mathcal{O}$ is an admissible set of local orientations. Second, induced set of local orientations $\mathcal{O}_\#$ at special points of $\Omega_\#$ coincides with an admissible set of local orientations $\mathcal{O}_{\#adm}$ in all points except q''. Clearly, for any Ω and γ of class 5 it is possible to choose $\mathcal{O}$ satisfying these conditions. (Note, that there is no admissible set of local orientations $\mathcal{O}$ such that $\mathcal{O}_\#$ is also admissible set of local orientations.)

Fix an interior primary field α_p at each interior special point p of Ω and a dihedral primary field β_q at each boundary special point q of Ω. Denote by $x = (\otimes_{p\in\Omega_a}\alpha_p) \otimes (\otimes_{q\in\Omega_b}\beta_q))$ the element of vector space $V_{(\Omega,\mathcal{O})}$. Then, by definition, element $\eta_*(x) \in V_{(\Omega,\mathcal{O})}$ is equal to $y = \sum_{\beta',\beta''} F^{\beta'\beta''}(\otimes_{p\in\Omega_a}\alpha_p) \otimes (\otimes_{q\in\Omega_b}\beta_q) \otimes \beta' \otimes \beta''^*$.

Sustitute the values $F^{\beta'\beta''}$: $y=\sum_\beta |\operatorname{Aut}\beta|(\otimes_{p\in\Omega_a}\alpha_p)\otimes(\otimes_{q\in\Omega_b}\beta_q))\otimes\beta\otimes\beta$.

Introduce notation:

$R=H_{(\Omega_\#,\mathcal{O}_\#)}(\eta_*(x))$

Substitute $y=\eta_*(x)$:

$R=\sum_\beta |\operatorname{Aut}\beta|H_{(\Omega_\#,\mathcal{O}_\#)}((\otimes_{p\in\Omega_a}\alpha_p)\otimes(\otimes_{q\in\Omega_b}\beta_q)\otimes\beta\otimes\beta)$

By (already established) invariance of change of local orientations, we obtain:

$R=\sum_\beta |\operatorname{Aut}\beta|H_{(\Omega_\#,\mathcal{O}_{\# adm})}((\otimes_{p\in\Omega_a}\alpha_p)\otimes(\otimes_{q\in\Omega_b}\beta_q)\otimes\beta\otimes\beta^*)$

By definition of $H_{(\Omega_\#,\mathcal{O}_{\# adm})}$,

$R=\sum_\beta |\operatorname{Aut}\beta|\sum_{[\pi_\#]\in\operatorname{Cov}(\Omega_\#,\{\alpha_p\},\{\{\beta_q\},\beta,\beta^*\})}\frac{1}{|\operatorname{Aut}\pi_\#|}$

By theorem 5.1 and corollary 5.1 we have

$R=\sum_\beta |\operatorname{Aut}\beta|\sum_{[\pi_\#]\in\operatorname{Cov}(\Omega_\#),\{\alpha_p\},\{\{\beta_q\},\beta,\beta^*\})}\frac{1}{|\operatorname{Aut}\beta|}\sum_{[\pi]\in\operatorname{Cov}(\Omega)_{[\pi_*]}}\frac{1}{|\operatorname{Aut}\pi|}=$
$=\sum_\beta\sum_{[\pi_\#]\in\operatorname{Cov}(\Omega_\#),\{\alpha_p\},\{\{\beta_q\},\beta,\beta^*\})}\sum_{[\pi]\in\operatorname{Cov}(\Omega)_{[\pi_*]}}\frac{1}{|\operatorname{Aut}\pi|}=$
$=\sum_{[\pi]\in\operatorname{Cov}(\Omega)}\frac{1}{|\operatorname{Aut}\pi|}=H_{(\Omega,\mathcal{O})}(x)$.

For other class of simple cuts axiom 5° can be verified quit similar.

Axiom 6° is satisfied because the sum $\sum_{[\pi]\in\operatorname{Cov}(\Omega,\{\alpha_p\},\{\beta_q\})}\frac{1}{|\operatorname{Aut}[\pi]|}$ is multiplicative with respect to disjoint union of surfaces.

□

Structure algebra of Hurwitz topological field theory is a vector space $H=A\oplus B$ with multiplication constructed by tensors from $\mathcal{C}(\mathcal{H})$.

Let $\bar H=\bar A\oplus\bar B$ be a structure algebra of symmetric group. Note that Young diagram α corresponds to a class of conjugated elements of S_n. This class we denote by the same sign α. Define a linear map $\phi: A\to\bar A$ as $\phi(\alpha)=E_\alpha$, where $E_\alpha=\sum_{g\in\alpha}g$ is an element of the center of group algebra $\mathbb{C}[S_n]$. Analogously, dihedral Young diagram β corresponds to a conjugacy class of ordered pairs of involution in S_n. Put $\phi(\beta)=E_\beta$, where $E_\beta=\sum_{(s_1,s_2)\in\beta}E_{s_1,s_2}$ and $E_{s_1,s_2}\in M(S_n)$ is an elementary matrix.

Theorem 5.3. *The linear map ϕ is an isomorphism between the structure algebra of Hurwitz topological field theory of degree n and the structure algebra of symmetric group S_n.*

Proof. Obviously, ϕ defines an isomorphism of linear spaces H and $\bar H$. Moreover, ϕ preserves scalar products on A and B and commutes with '$*$' involutions.

In order to prove that ϕ is an isomorphism of structure algebras it is sufficient to check the following equalities:

(1) $S_{\alpha_1,\alpha_2,\alpha_3}=S_{E_{\alpha_1},E_{\alpha_2},E_{\alpha_3}}$,
(2) $T_{\beta_1,\beta_2,\beta_3}=T_{E_{\beta_1},E_{\beta_2},E_{\beta_3}}$,
(3) $R_{\alpha,\beta}=R_{E_\alpha,E_\beta}$.

Proof of (1). After Hurwitz, it is known that computation of weighted numbers of isomorphism classes of coverings over two-dimensional sphere is equivalent to the solution of factorization problem in S_n (see [14]).

Therefore, $S_{\alpha_1,\alpha_2,\alpha_3} = |(\{(a_1, a_2, a_3)|a_i \in \alpha_i, a_1a_2a_3 = 1\}/S_n)|$, where $g \in S_n$ acts as $(a_1, a_2, a_3) \mapsto (ga_1g^{-1}, ga_2g^{-1}, ga_3g^{-1})$. By direct calculations, we obtain that $E_{\alpha_1}E_{\alpha_2} = S^{\alpha_3}_{\alpha_1,\alpha_2}E_{\alpha_3}$, where $S^{\alpha_3}_{\alpha_1,\alpha_2} = S_{\alpha_1,\alpha_2,\alpha_3}|\operatorname{Aut}\alpha_3|$. Hence, $S_{\alpha_1,\alpha_2,\alpha_3} = S_{E_{\alpha_1},E_{\alpha_2},E_{\alpha_3}}$.

Proof of (2). By definition $T_{\beta_1,\beta_2,\beta_3}$ is a number of equivalent classes of coverings over triangle Δ with local invariants at vertices q_i equal to β_i $(i = 1, 2, 3)$.

Let $\pi : P \to \Delta$ be a stratified covering. Evidently, preimage of Δ° consists of n connected components and each one if homeomorphic to Δ°. An edge $e_{i,j}$ of Δ defines a involutive permutation $s_{i,j}$ of these preimages. Therefore, the covering is uniquely defined by triple of involutions $(s_{0,1}, s_{1,2}, s_{2,0})$ of symmetric group S_n up to equivalence $(s_{0,1}, s_{1,2}, s_{2,0}) \sim (gs_{0,1}g^{-1}, gs_{1,2}g^{-1}, gs_{2,0}g^{-1})$. Conversely an equivalent class $[(s_{0,1}, s_{1,2}, s_{2,0})]$ of triple of involutions defines an equivalent class of coverings over Δ. Local invariant at vertex q_i is a dihedral Young diagram β_i corresponding to a conjugacy class $[(s_{i-1,i}, s_{i,i+1})]$ (where $i-1$ and $i+1$ are taken modulo 3) of the pair of involutions.

Hence, $T_{\beta_1,\beta_2,\beta_3} = |(\{(s_{0,1}, s_{1,2}, s_{2,0})|s_{i,i+1} \in S_n, s^2_{i,i+1} = 1, [(s_{i-1,i}, s_{i,i+1})] = \beta_i\}/\sim)|$. By direct calculations we obtain that $E_{\beta_1}E_{\beta_2} = T_{\beta_1,\beta_2,\beta_3}|\operatorname{Aut}\beta_3|E_{\beta_3^*}$. Therefore, $T_{\beta_1,\beta_2,\beta_3} = T_{E_{\beta_1},E_{\beta_2},E_{\beta_3}}$

Proof of (3). By definition $R_{\alpha,\beta}$ is the number of equivalent classes of coverings over disc (D, p, q) with local invariant α at interior special point p and local invariant β at boundary special point q.

Let $\pi : P \to D$ be a stratified covering. Let us connect points p and q by a segment δ and denote by D° the set of all interior point except δ. Evidently, the preimage of D° consists of n connected components and each one is homeomorphic to D°.

Segment δ defines a permutation a of these preimages and the boundary circle defines an involutive permutation s of these preimages. (If X is a connected component of the preimage of D° then $Y = s(X)$ is another component such that $\overline{X} \cap \overline{Y}$ is equal to a connected component of the preimage of a boundary without point q; if there no such component then $s(X) = X$). Therefore, the covering is uniquely defined by a pair (a, s) up to equivalence $(a, s) \sim (gag^{-1}, gsg^{-1})$.

Conversely an equivalent class $[(a, s)]$ defines an equivalent class of coverings over D. Local invariant at vertex p is Young diagram $\alpha = [a]$ and local invariant at vertex q is dihedral Young diagram β corresponding to a pair of involutions (s, asa^{-1}). Hence, $R_{\alpha,\beta} = |(\{(a, s)|a, s \in S_n, a \in \alpha, (s, asa^{-1}) \in \beta\}/\sim)|$. By direct calculations we obtain that $(E_\alpha, E_\beta) = R_{\alpha,\beta}$ (see subsection 2.3)

□

5.4. **Hurwitz numbers.** Classical Hurwitz numbers are weighted numbers of algebraic maps $f : P \to Q$ to Riemann sphere $Q = S^2$ having prescribed branchings at finite number of fixed points $z_1, \dots, z_m \in Q$. Young diagramms $\alpha_1, \dots, \alpha_m$ are fixed. It is required that $z_1, \dots, z_m$ are critical values of f and the branching of f at z_i is described by Young diagramm α_i.

We use term 'generalized Hurwitz numbers' or 'Hurwitz numbers' for short for morphisms of both complex and real algebraic curves, an arbitrary base Q and an arbitrary local types of branchings. Definitions are given below.

A real algebraic curve is a pair (P, τ), where P is a complex algebraic curve, i.e., a compact Riemann surface, and $\tau : P \to P$ is an antiholomorphic involution [2, 30]. Involution τ is called complex conjugation. Fixed points of τ are called real points of the real curve (P, τ). An algebraic map $f : (P, \tau) \to (Q, \omega)$ of real algebraic curves is defined as a holomorphic map $f : P \to Q$ such that $f\tau = \omega f$.

Algebraic maps $f : (P, \tau) \to (Q, \omega)$ of a fixed degree n to a fixed real algebraic curve form a category $\mathcal{M}ap(Q, \omega)$; a morphism $\phi : f \to f'$, where $f : (P, \tau) \to (Q, \omega)$ and $f' : (P', \tau') \to (Q, \omega)$ are two algebraic maps to (Q, ω), is defined as a holomorphic map $\phi : P' \to P$ such that $\phi\tau' = \tau\phi$ and $f' = f\phi$.

Any complex algebraic curve P_c generates real algebraic curve (P, τ). Namely, $P = P_c \coprod \bar{P}_c$ where $\bar{P}_c$ is P_c endowed with conjugated complex structure and τ is just permutation of components P_c and $\bar{P}_c$. Thus, complex algebraic curves can be considered as a particular case of real algebraic curves.

Denote by Ω the factor-space Q/τ. It is known that Ω carries the structure of Klein surface [2, 27]. This implies that Ω is nooriented, possibly nonorientable, possibly with boundary, surface endowed with an atlas such that any transaction function is dianalitic, i.e., either holomorphic or antiholomorphic function . Morphisms of Klein surfaces are defined as dianalitic functions. According to [2] the category of real algebraic curves is isomorphic to the category of Klein surfaces.

Fix a real curve (Q, ω) and put $\Omega = Q/\tau$. Any morphism $f : (P, \tau) \to (Q, \omega)$ of real algebraic curves generates the dianalytic map $\pi(f) : P/\tau \to \Omega$. Denote by $p_1, \dots, p_m \in \Omega$ critical values of $\pi(f)$ that are interior points, and denote by $q_1^j, \dots, q_{m_j}^j$ critical values of $\pi(f)$ that belong to j-th boundary contour of Ω. Clearly, points p_i, q_i^j come from critical values $z_0, \dots, z_t \in Q$ of f: an interior point p_i is the image of two points z_k, z_l, a boundary point q_i^j is the image of one point z_r and z_r is a fixed point of τ. Points p_i, q_i^j induce the stratification of Ω. Forgetting dianalitic structures on P/τ and Ω we obtain that $\pi(f)$ is a stratified covering, i.e., an element of $\mathcal{C}ov(\Omega)$.

Fix the stratification of Ω with special points p_i, q_i^j. According to [28] forgetting dianalitic structures induces the bijection between the set of isomorphism classes of dianalitic maps to Ω, such that all critical values are special points of the stratification, and the set $\mathrm{Cov}(\Omega)$ of isomorphism classes of stratified covering over Ω. Thus, up to isomorphisms, algebraic maps to (Q,τ) can be identified with stratified coverings over Ω.

The latter fact allows us to define Hurwitz numbers for real algebraic curves in topological terms. Namely, define *Hurwitz numbers* for Ω as

$$\mathrm{Hurw}_n(\Omega, \{\alpha_p\}, \{\beta_q\}) = \sum_{[\pi]\in \mathrm{Cov}(\Omega,\{\alpha_p\},\{\beta_q\})} \frac{1}{|\operatorname{Aut}\pi|}$$

Here α_p (resp., β_q) is a Young diagram (resp., a dihedral Young diagram) of degree n assigned to an iterior special point p (resp., to a boundary special point q) of Ω.

By previous considerations, these Hurwitz numbers are weighted numbers of classes of alrebraic maps with prescribed branchings to real algebraic curve (Q,τ).

For $G=(0,1)$ our definition gives weighted numbers of meromorphic functions with fixed critical values of fixed topological types. They includes numbers that were first introduced by Hurwitz [18].

By the results of previous subsection, Hurwitz numbers are equal to correlators for Hurwitz topological field theory. Thus, from theorem 5.3 and theorem 4.4 we get the following theorem.

Theorem 5.4. *Let Ω be a stratified surface of type G. Fix an admissible set of local orientations at special points of Ω. Then*
(1)

$$\mathrm{Hurw}_n(\Omega, \{\alpha_p\}, \{\beta_q\}) = \langle \alpha_1, \dots, \alpha_m, ({\beta_1}^1, \dots, \beta^1_{m_1}), \dots, (\beta^s_1, \dots, \beta^s_{m_s})\rangle_G$$

(2) If $G=(g,1,m,m_1,\dots,m_s)$ then

$$\mathrm{Hurw}_n(\Omega, \{\alpha_p\}, \{\beta_q\}) = \\ = (\alpha_1 \dots \alpha_m(\beta^1_1 \dots \beta^1_{m_1}) V_{K_B}(\beta^2_1 \dots \beta^2_{m_1}) \dots V_{K_B}(\beta^s_1 \dots \beta^s_{m_s}), K^g_A)$$

(3) If $G=(g,0,m,m_1,\dots,m_s)$ then

$$\mathrm{Hurw}_n(\Omega, \{\alpha_p\}, \{\beta_q\}) = \\ = (\alpha_1 \dots \alpha_m(\beta^1_1 \dots \beta^1_{m_1}) V_{K_B}(\beta^2_1 \dots \beta^2_{m_1}) \dots V_{K_B}(\beta^s_1 \dots \beta^s_{m_s}), U^{2g})$$

In (2) and (3) right hand sides are expressions in the structure algebra associated with symmetric group S_n (see subsection 2.3).

Remark.

Let G be a finite group and Ω be a stratified surface. Denote by $\mathcal{C}ov_G(\Omega)$ the caregory of stratified coverings with structure group G

(G-coverings). A definition of a G-covering in the case of stratified surfaces possibly with boundary is a generalization of a standard one and should be explained.

Fix a generic point $x \in \Omega$. Let $\delta \subset \Omega$ be a path with ends at x. Path δ is called generic if it does not cross any special point and cross the boundary of Ω in finitely many points.

Let $\pi : P \to \Omega$ be a stratified covering and $\hat{x} \in P$ be one of preimages of x. Lift δ to $\hat{\delta} \subset P$ using the following rule of lifting through a point $\hat{y}$ such that $\pi(y) \in \delta \cap \partial\Omega$: if the covering $\pi_U : \hat{U} \to U$ of a neighborhood $\hat{U}$ of point $\hat{y}$ over $U = \pi(\hat{U})$ is two-sheeted covering then $\hat{\delta}$ must go from one sheet of $\hat{U}$ to another sheet through point $\hat{y}$.

This rule allows to define a fundamental group, a monodromy group etc. for stratified surfaces and stratified covering. All theory of fundamental group is consistent for this generalization. Note that these definitions are valid for all dimensions in the case of specially stratified manifolds and specially stratified coverings (see definition 5.1). (Recall that in this work we deal with specially stratified surfaces and specially stratified coverings only).

Fix an action of group G on the set of preimages of x. A covering π is said to be G-covering if (generalized) monodromy group at x is contained in G. All definitions and considerations of this section, particularly, the definition of Hurwitz numbers, can be applied to G-coverings without changes. Thus, we can construct 'G-Hurwitz topological field theory'. Structure algebra of it is isomorphic to the structure algebra of group G. G-Hurwitz numbers have just the same representation as in theorem 5.4; these formulas coincide with correlators for G-Hurwitz topological field theory.

G-Hurwits numbers for oriented surfaces without a boundary were considered in [9].

References

[1] Abrams L, Two-dimensional topological quantum field theory and frobenius algebras / Jornal of Knot and Its Ramifications v.5 No.5 (1995), 569-587

[2] Alling N.L., Greenleaf N., Foundations of the theory of Klein surfaces / Lecture Notes in Math. N 219 // Berlin – Heideb. – N.Y.: Springer – Verlag. – 1971. p. 117.

[3] Arnold V., Topological classification of complex trigonometric polynomials and the combinatorics of graphs with the same number of edges and vertices, Funct. Anal. and Appl. 30 (1996), 1, 1-17.

[4] Atiyah M., Topological Quantum Field Theories, Inst. Hautes Etudes Sci. Publ. Math., 68 (1988), 175-186.

[5] Barannikov S.A., About space of real polynoms without multiple critical values, Funktsional. Anal.i Prilozhen., 26 (2) (1992), 10-17.

[6] Cardy J. L., Boundary conditions, fusion rules and the Verlinde formula, Nucl. Phys. B 324 (1989), 581-596.

[7] Cordes S., Moore G., Ramgoolam S., Large N 2D Yang–Mills theory and topological string theory, Commun. Math. Phys., 185 (1997), 543-619.
[8] Devadoss S.L., Tessellations of moduli spaces and the mosaic operad, Contemp. Math. 239 (1999), 91-114.
[9] Dijkgraaf R., Mirror symmetry and elliptic curves, The moduli spaces of curves, Progress in Math., 129 (1995), 149-163, Brikhäuser.
[10] Dijkgraaf R., Geometrical Approach to Two-Dimesional Conforal Field Theory, Ph.D.Thesis(Utrecht,1989)
[11] Dubrovin B., Geometry of $2D$ topological field theories, in: Lecture Notes in Math., 1620, Springer, Berlin (1996), 120-348.
[12] Ekedahl T., Lando S,K., Shapiro M., Vainshtein A., On Hurwitz numbers and Hodge integrals, C.R.Acad. Sci. Paris S/'er. I Math. 328 (1999), 1175-1180.
[13] Gawedzki K., Boundary WZW, G/H, G/G and CS theories. arXiv: hep-th/0108044.
[14] Goulden I.P., Jackson D.M., Transitive factorizations into transpositions and holomorphic mappings on the sphere, Proc. Amer. Math. Soc. 125 (1997), 51-60.
[15] Goulden I.P., Jackson D.M., The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group, Europ. J. Combinatorics, 13 (1992), 357-365.
[16] Gross D.J., Taylor IV W., Twists and Wilson loops in the string theory of two dimensional QCD, Nuclear Phys., B 403 (1993), N 1-2, 395-449.
[17] Hughes B., Weinberger S., Surgery and stratified spaces. Cappell, Sylvain (ed.) et al., Surveys on surgery theory. Vol. 2: Princeton, NJ: Princeton University Press. Ann. Math. Stud. 149 (2001), 319-352.
[18] Hurwitz A., Über Riemann'sche Flächen mit gegeben Verzweigungspunkten, Math. Ann., (1891), Bn.39, 1-61.
[19] Karimipour V., Mostafazadeh A., Lattice topological field theory on nonorientable surfaces, J. Math. Phys. 38 (1) (1997), 49-66.
[20] Kerekjarto B., Vorlesungen über Topologie. I. Flächentopologie. – Berlin, Springer. – 1923.
[21] Keel S., Intersection theory of moduli space of stable N-pointed curves of genus zero, Transactions of the American Mathematical Society, 330:2 (1992), 545-574.
[22] Kostov I.K., Staudacher M., Wynter T., Complex matrix models and statistics of branched coverings of 2D surfaces, Commun. Math. Phys. 191 (1998), 283-298.
[23] Lazaroiu C.I., On the structure of open-closed topological field theory in two-dimensions, Nucl. Phys. B 603 (2001), 497-530.
[24] Mednykh A.D., Pozdnjakova G.G., On the number of nonequivalent coverings over compact nonorientable surface, Sib. Mat. Zb. 27 (1) (1986), 123-131.
[25] Moore G., Some comments on branes, G-flux and K-theory, Int. J. Mod. Phys. A16 (2001), 939-944.
[26] Moore G., D-Branes, RR-Fields and K-Theory, http://online.itp.ucsb.edu/online/mp01/moore2/
[27] Natanzon S.M., Klein surfaces, Russian Math. Surveys, 45:6 (1990), 53-108.
[28] Natanzon S.M., Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves, Selecta Mathematica, v.12, N 3 (1993), 251-291.
[29] Natanzon S.M., Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves. Sel. Math. Sov., 12:3 (1993), 251-291.

[30] Natanzon S.M., Moduli of real algebraic surfaces, and their superanalogues. Differentials, spinors and Jacobians of real curves, Russian Math. Surveys, 54:6 (1999), 1091-1147.
[31] Natanzon S., Turaev V., Systems of correlators and solutions of the WDVV equations, Commun. Math. Phys. 196 (1998), 399-410.
[32] Okounkov A., Pandharipande R., Gromov–Witten theory, Hurwitz numbers and matrix models, I. math.AG/0101147.
[33] Segal G.B., Two–dimensional conformal field theory and modular functions, Swansea Proceedings, Mathematical Physics, 1988, 22-37.
[34] Turaev V., Homotopy field theory in dimension 2 and group-algebras, arXiv:math.QA/9910010
[35] Witten E., Some geometrical applications of quantum field theory, In IXth International Congress on Mathematical Physics(1988) 77-116.

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