The classification of Pin_4-bundles over a 4-complex

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Abstract

In this paper we show that the Lie-group Pin_4 is isomorphic to the semidirect product $(SU_2 \times SU_2) \rtimes \mathbf{Z}/2$ where $\mathbf{Z}/2$ operates by flipping the factors. Using this structure theorem we prove a classification theorem for Pin_4-bundles over a finite 4-complex X.

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1 Introduction

Let G be a compact Lie group. The set of isomorphism classes of principal G-bundles over a topological space X is in one-to-one correspondence to free homotopy classes from X to BG. The homotopy type of BG is determined by being the orbit space of a free G action on a contractible space EG. This means that knowing the homotopy type of the $(k+1)$-skeleton of BG translates the classification of principal G-bundles over a finite k-complex X into calculations in obstruction theory.

We now specialize to X being a finite 4-complex. The case $G = SU_2$ is very easy: The fact that the 5-skeleton of BSU_2 is S^4 and Hopf's classification theorem for $[X, S^4]$ imply that SU_2-bundles over a four-complex X are in 1-1 correspondence to $H^4(X;\mathbb{Z})$, the isomorphism given by the second Chern class.

A. Dold and H. Whitney clarified the case $G = SO_n$. In [DW59] a general classification theorem for SO_n-bundles is given in terms of obstruction theory. The three dimensional case takes a particular nice form: SO_3-bundles over a 4-complex are classified by the second Stiefel-Whitney class w_2 and the first Pontrjagin class p_1. Moreover, every pair (w_2, p_1) satisfying $\mathcal{P}w_2 \equiv p_1 \mod 4$, where $\mathcal{P}$ is the Pontrjagin square, is realized as classifying pair for some SO_3-bundle P over X.

Let's move on to the case of a disconnected structure group. The case of $G = O_3$ follows from the SO_3-case since $O_3 = SO_3 \times \mathbb{Z}/2$ and therefore $BO_3 = BSO_3 \times B\mathbb{Z}/2$. In this paper we want to look at the Lie group Pin_4, a double cover of O_4, and will give a classification theorem for Pin_4-bundles over a 4-complex.

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2 Clifford algebras, Pin and Spin

General setup

We'll give a brief review of the basics. A more detailed reference is chapter I in [LM89]. Let (V, q) be a real vector space with quadratic form q. The Clifford algebra $Cl(V, q)$ is the algebra generated by all $v \in V$ and 1 subject to the relations $v \cdot v = -q(v) \cdot 1$. We are particularly interested in the case $V = \mathbb{R}^n$ and $q^{\pm}(v) = \mp|v|^2$, and will write $Cl_n^{\pm}$ for $Cl(\mathbb{R}^n, q^{\pm})$.

$Pin_n^{\pm}$ is the subgroup of the multiplicative group of $Cl_n^{\pm}$ generated by elements $v \in S^{n-1}$. Conjugation with an element $v \in \mathbb{R}^n \subseteq Pin_n^{\pm}$ leaves $\mathbb{R}^n \subseteq Cl_n^{\pm}$ invariant and preserves q. Therefore we get a map

$$\begin{array}{rccc} \widetilde{Ad}: & Pin_n^{\pm} & \to & O_n \\ & \phi & \mapsto & (y \mapsto \alpha(\phi) y \phi^{-1}), \end{array}$$

where $y \in \mathbb{R}^n$ and α is the endomorphism of $Cl_n^{\pm}$ which extends $v \mapsto -v$ on $\mathbb{R}^n$. $\widetilde{Ad}$ is a twofold cover, [LM89, I.2.10]. For $v \in \mathbb{R}^n$ $\widetilde{Ad}(v)$ is just the reflection at the hyperplane perpendicular to v.

The preimage of SO_n under $\widetilde{Ad}$ is called $Spin_n^{\pm}$ and is the subgroup of $Pin_n^{\pm}$ consisting of products of an even number of $v \in S^{n-1}$. Since $\alpha(\phi) = \phi$ for $\phi \in Spin_n$ we see that restricted to $Spin_n$ $\widetilde{Ad}(\phi)$ is just given by conjugation with $\phi \in Spin_n$. We will write Ad for the map $Pin_n^{\pm} \to O_n$ given by conjugation, and therefore $\widetilde{Ad}_{|Spin_n} = Ad_{|Spin_n}$.

$Pin_n^{\pm}$ has a nontrivial one dimensional representation $\chi : Pin_n^{\pm} \to \mathbb{Z}/2$ which is given by extending $V \ni v \mapsto -1$ to all of $Pin_n^{\pm}$. We see that $Ker(\chi) = Spin_n^{\pm}$.

Since SO_n is connected, $\pi_1(SO_n) = \mathbb{Z}/2$ and both of $Spin_n^{\pm}$ are nontrivial coverings, we see that $Spin_n^+$ and $Spin_n^-$ must be isomorphic as groups and coverings of SO_n. Keeping the ambiguity in mind we will from now on drop the superscript and refer only to $Spin_n$.

Spin$_4$ and quaternions

Recall that $\mathbb{H} = \mathbb{R}\langle i, j, k\rangle$ subject to the relations $i^2 = j^2 = k^2 = ijk = -1$. The conjugate of a quaternion $q = a + bi + cj + dk$ is given by $\overline{q} = a - bi - cj - dk$, and $N(q) := q\overline{q}$ defines a norm on $\mathbb{H}$. The group of unit quaternions, i.e. the 3-sphere, can be identified with SU_2, in particular

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Consider the map

$$\begin{array}{rccc} \mu : & SU_2 \times SU_2 & \to & GL_4(\mathbb{R}) \\ & (p,q) & \mapsto & (x \mapsto pxq^{-1}) \end{array}$$

given by quaternionic multiplication. μ maps into SO_4 since $\mu(p,q)$ is norm preserving and has determinant 1. μ defines a double cover of SO_4 and hence there is an isomorphism $\Phi : SU_2 \times SU_2 \to Spin_4$.

The structure of $\mathbf{Pin_n}$

The exact sequence of groups

$$1 \to Spin_n \to Pin_n^{\pm} \xrightarrow{\chi} \mathbb{Z}/2 \to 1$$

with $n \geq 3$ splits via

$$\sigma(-1) = \begin{cases} e_1 & \text{in the } Pin_4^+ \text{ case,} \\ e_1e_2e_3 & \text{in the } Pin_4^- \text{ case.} \end{cases}$$

The center of $Spin_n$ is given by

$$C(Spin_n) = \begin{cases} \mathbb{Z}/2 =< -1 > & n \text{ odd,} \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 =< \omega > \oplus < -\omega > & n \equiv 0 \mod 4 \\ \mathbb{Z}/4 =< \omega > & n \equiv 2 \mod 4, \end{cases}$$

where $\omega = e_1 \dots e_n$ is the volume element. We'll always assume that $(e_1, \dots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$. Moreover if $\sigma^{\pm}$ is any splitting element of the extension above then $\sigma^{-1}\omega\sigma = (-1)^{n-1}\omega$.

Recall that for group extensions of the form

$$1 \to N \to G \to Q \to 1$$

with nonabelian N there may not be -unlike in the case where N is abelian- a well-defined operation of Q on N given by conjugation with a (set theoretic) section $Q \to G$. However any two choices for a section differ by an element in N which induces an inner automorphism of N. This means that there is a well defined homomorphism from Q to $Out(N) = Aut(N)/Inn(N)$.

Since $Spin_4 \cong S^3 \times S^3$ we see that $Out(Spin_4) = \mathbb{Z}/2$. The higher dimensional even $Spin$ groups are simple and looking at their Dynkin diagram we can read of their outer isomorphisms. The list for the dimensions divisible by 4 is

$$.Out(Spin_{4n}) = \begin{cases} \mathbb{Z}/2 & n = 1 \\ S_3 & n = 2 \\ \mathbb{Z}/2 & n \geq 3. \end{cases}$$

Moreover, in all cases the automorphism class is detected by the induced automorphism of the center.

Now extensions $N \to G \to Q$ with fixed homomorphism $\phi : Q \to Out(N)$ are - if there is any - in one-to-one correspondence to $H^2(Q; CN)$, where we view CN as a Q module, see [EM47]. Therefore we calculate

$$H^2(\mathbb{Z}/2; CSpin_n) = \begin{cases} \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \equiv 0 \mod 4 \\ \mathbb{Z}/2 & n \equiv 2 \mod 4. \end{cases}$$

Putting the information together we see

Proposition 2.1 *If $n \equiv 0 \mod 4$ then Pin_n^+ and Pin_n^- are isomorphic as groups.*

In general it might be quite difficult to give a more concrete description of the operation of $\mathbb{Z}/2$ on $Spin_n$ than just saying that is is given by conjugation with a split element. However for $Spin_4$ this is very easy:

Theorem A *Pin_4^+ and Pin_4^- are both isomorphic to the semidirect product*

$$(SU_2 \times SU_2) \ltimes \mathbb{Z}/2$$

where -1 operates by flipping the factors.

A word of warning: The isomorphism between Pin_4^+ and Pin_4^- is one of Lie groups and is not compatible with the projection to O_4. Therefore the obstructions for the existence of Pin_4^+ and Pin_4^- structures on a given O_4 bundle are different in general. However, since we are only interested in Pin_4 principal bundles and isomorphisms between them, we can drop the superscript again and refer to Pin_4 as given by the semidirect product above.

3 Bundle theory

3.1 Spin$_4$ bundles

The adjoint representation $Ad : Spin_n \to SO_n$ defines an associated SO_n bundle P_{SO} for every $Spin_n$ bundle P. We denote the Euler and Pontrjagin classes of P_{SO} by $e(P)$ and $p_i(P)$ respectively. The isomorphism $Spin_4 \cong SU_2 \times SU_2$ implies that $BSpin_4 \simeq BSU_2 \times BSU_2 \simeq \mathbb{H}P^\infty \times \mathbb{H}P^\infty$. Recall that $H^4(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}$ is generated by $c_2(\gamma)$ where γ is the universal SU_2-bundle. The homotopy class of a map $f : X \to BSpin_4$ therefore defines an ordered pair $(a, b) \in H^4(X; \mathbb{Z}^2)$ by pulling back $(c_2(\pi_1^*\gamma), c_2(\pi_2^*\gamma))$ where $\pi_{1/2}$ is the projection to the first and second factor. Using the Borel-Hirzebruch formalism for charactersitc classes one calculates (see [HH58])

Lemma 3.1 *The characteristic classes a, b and e, p_1 of a $Spin_4$ bundle are subject to the relations*

$$e = -a + b \qquad p_1 = 2(a + b)$$

Combining Hopf's classification theorem for $[X, S^4]$ with the above lemma we see

Proposition 3.2 *i) Two $Spin_4$ principal bundles over a compact 4-complex X are isomorphic iff their characteristic classes (a, b) in $H^4(X; \mathbb{Z}^2)$ coincide as ordered pairs. Moreover, every ordered pair (a, b) can be realized.*

ii) If $H^4(X; \mathbb{Z})$ has no 2-torsion, then two $Spin_4$ bundles are isomorphic iff their Euler and first Pontrjagin class coincide.

iii) A $Spin_4$ bundle over an oriented 4-manifold is characterized by its Euler- and Pontrjagin number.

Moreover, the pairs (e,p) which can be realized are exactly the ones satisfying $2|p$ and $4|(p + 2e)$.

3.2 Pin_4 bundles

Let us start by fixing some handy notation for weakly associated Pin_4-bundles:

Lemma 3.3 *Any $Spin_4$-principal bundle with characteristic classes (a, b) is isomorphic to $P + Q$, where P and Q are SU_2-bundles with second Chern class equal to a and b, and $P+Q := \Delta^*(P \times Q)$, $\Delta : X \hookrightarrow X^2$ is the diagonal. The weakly associated Pin_4-bundle has the form*

$$(P + Q) \times_{Spin_4} Pin_4 \cong P + Q \coprod Q + P$$

and right multiplication on the right hand side with an element $(\alpha, \beta; \epsilon) \in Pin_4$ is given by

$$(p, q)(\alpha, \beta; \epsilon) = \begin{cases} (p\alpha, q\beta) & \epsilon = 1 \\ (q\beta, p\alpha) & \epsilon = -1 \end{cases}$$

$$(q, p)(\alpha, \beta; \epsilon) = \begin{cases} (q\alpha, p\beta) & \epsilon = 1 \\ (p\beta, q\alpha) & \epsilon = -1 \end{cases}$$

Proof: A typical element of the LHS looks like $[(p, q, a, b, e)]$ with $p \in P$, $q \in Q$ and $(a, b, e) \in Pin_4$. Moreover $[(p, q, a, b, e)] = [(ps^{-1}, qt^{-1}, sa, tb, e)]$ for all $(s, t) \in Spin_4$. An element of the LHS thererefore has a unique representative of the form $[(p, q, 1, 1, e)]$ for which we will write $[p, q, e]$.

We now define Φ : LHS $\rightarrow$ RHS by

$$\Phi([p, q, a, b, e]) := \begin{cases} (pa, qb) & e = 1, \\ (qb, pa) & e = -1. \end{cases}$$

This is well-defined, and one checks that Φ is equivariant with respect to the right multiplication with elements of Pin_4 on both sides. □

Lemma 3.4 *The classifying space $BPin_4$ is homotopy equivalent to the $\mathbb{H}P^\infty \times \mathbb{H}P^\infty$ bundle over $\mathbb{R}P^\infty$ given by the quotient*

$$(\mathbb{H}P^\infty \times \mathbb{H}P^\infty \times S^\infty)_{/(\bar{x},\bar{y},z)\sim(\bar{y},\bar{x},-z)}.$$

Proof: It suffices to give a free Pin_4 right operation on a contractible space with quotient as claimed. Think of S^∞ to be the unit sphere in $\mathbb{H}^\infty$. Now define an action

$$\begin{array}{ccccc} (S^\infty \times S^\infty \times S^\infty) & \times & Pin_4 & \to & S^\infty \times S^\infty \times S^\infty \\ (x,y,z) & \cdot & (\alpha,\beta,\epsilon) & = & \begin{cases} (x\alpha^{-1}, y\beta^{-1}, z) & \epsilon = 1 \\ (y\alpha^{-1}, x\beta^{-1}, -z) & \epsilon = -1. \end{cases} \end{array}$$

One easily checks that this is a free action with the right quotient. □

Before we can prove our classification result for Pin_4 bundles we need two technical lemmas:

Lemma 3.5 *Let X and Y be two connected CW complexes with basepoints x_0 and y_0. $\tilde{Y} \xrightarrow{\pi} Y$ the universal covering. Identify $\pi_1(Y, y_0)$ with the group of covering transformations of $\tilde{Y}$. Let $\tilde{f}, \tilde{g} : X \to \tilde{Y}$, $f := \pi \circ \tilde{f}$ and $g := \pi \circ \tilde{g}$.*

Then: $f \simeq g$ iff $\tilde{f} \simeq \alpha \circ \tilde{g}$ for some $\alpha \in \pi_1 Y$.

This can quickly be proved using covering space theory. On the algebraic side we have:

Lemma 3.6 *Let X be a finite CW complex of dimension n, $w : G := \pi_1 X \to \mathbb{Z}/2$ a nonzero map, $H := Ker(w)$ and $\Lambda := \mathbb{Z}[G/H]$ considered as a G-module. Let furthermore $\pi : X^w \to X$ be the twofold covering associated to w with covering transformation τ. Then there is an isomorphism*

$$\Phi^* : H^*(X;\Lambda) \to H^*(X^w;\mathbb{Z}).$$

Since $Res_H \Lambda \cong \mathbb{Z} \oplus \mathbb{Z}$ is a trivial H module, the map in cohomology induced by π is given by

$$\begin{array}{ccccc} H^*(X;\Lambda) & \xrightarrow{\pi^*} & H^*(X^w; Res_H\Lambda) & = & H^*(X^w; \mathbb{Z}\oplus\mathbb{Z}) \\ x & \mapsto & & & (\Phi^* x, \tau^*\Phi^* x.) \end{array}$$

In particular π^* is injective.

Proof: Recall some facts from algebra: If $H \subset G$ is a subgroup of finite index, M some H-module, N some G-module, then there is a natural isomorphism

$$\Phi : Hom_G(N, \underbrace{Hom_H(\mathbb{Z}G, M)}_{=:\mathrm{Coind}_H^G M}) \cong Hom_H(N, M),$$

see [Bro82, p.64]. In our situation $\Lambda = \mathrm{Coind}_H^G \mathbb{Z}$, where $\mathbb{Z}$ is the trivial H-module. Now let $N_i := C_i(\widetilde{X})$ be the chain complex of the universal covering of X, thought of as a G- and H-module. Since the isomorphism Φ is natural it commutes with differentials, and so induces an isomorphism

$$\Phi^* : H^*(X;\Lambda) \to H^*(X^w;\mathbb{Z})$$

as claimed.

To see the second claim note that on cochain level the map induced by π is given by

$$\begin{array}{ccc} Hom_G(C_i\widetilde{X},\Lambda) & \to & Hom_H(C_i\widetilde{X^w};\mathbb{Z}\oplus\mathbb{Z}) \\ \alpha & \mapsto & (\Phi\alpha, \Phi\alpha\circ\tau). \end{array}$$

This implies the lemma since the differential on the right hand side respects the direct sum decomposition given by $\mathbb{Z}\oplus\mathbb{Z}$. □

The one dimensional nontrivial representation of Pin_4 defines a map w_1 from $BPin_4$ to $\mathbb{R}P^\infty$. The corresponding characteristic class $w_1(P)$ of a Pin_4 bundle P is equal to the first Stiefel-Whitney class of the associated O_4 bundle P_O. Using the cell decomposition of $BPin_4$ given by 3.4 one calculates that $H^4(BPin;\mathbb{Z}^-) =<\tilde{e}>\cong\mathbb{Z}$. If $f_P : X \to BPin_4$ is the classifying map of P we set $\tilde{e}(P) := f_P^*(\tilde{e}) \in H^4(X;\mathbb{Z}^{w_1P})$ and call it the twisted Euler class of P.

Now let X be a compact 4-complex and $P \to X$ a Pin_4 principal bundle with $w_1(P) =: w$. Let $\pi : X^w \to X$ be the twofold covering corresponding to w, Then π^*P lifts to a $Spin_4$ bundle $\tilde{P}$ and hence, by propsition 3.2, there are two classes (a,b) in $H^4(X^w;\mathbb{Z})$ which we will call the classifying pair of P.

Theorem B *Let P and Q be two Pin_4 bundles over a compact 4-complex X with $w_1(P) = w_1(Q) =: w$. Then P is isomorphic to Q iff their classifying pairs (a_P, b_P) and (a_Q, b_Q) coincide as unordered pairs in $H^4(X^w;\mathbb{Z})$.*

- *If $w = 0$ then every unordered pair (a,b) in $H^4(X;\mathbb{Z})$ is realized.*
- *If $w \neq 0$ let $\tau : X^w \to X^w$ be the covering transformation. Then every unordered pair (a,τ^*a), $a \in H^4(X^w;\mathbb{Z})$ is realized.*

The classifying classes a, b of P are related to Euler- and Pontrjagin class via

$$e(\tilde{P}) = -a + b, \qquad p_1(\tilde{P}) = 2(a+b).$$

Proof: We do the case $w = 0$ first: In this case the classifying maps f_P and f_Q lift to $BSpin$. The model for $BPin$ implies that the covering transformation ϕ operates as 'flip' on $H^4(BSpin;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}$. Therefore if $f : X \to BPin_4$ lifts to $BSPin_4$ the classifying classes coincide as unordered tupel for every possible lift of f. Now apply Lemma 3.5 and Proposition 3.2.

Now let w be nonzero. The fibration $BSpin_4 \to BPin_4 \to \mathbb{R}P^\infty$ inplies a commutative diagram

$$\begin{array}{ccc} & & BPin_4 \longleftarrow BSpin_4 \\ & \overset{f_P, f_Q}{\nearrow} & \downarrow w_1 \\ X & \underset{w_1 Q}{\overset{w_1 P}{\longrightarrow}} & \mathbb{R}P^\infty . \end{array}$$

Since $w_1(P) = w_1(Q)$ there is a homotopy $H : X \times I \to \mathbb{R}P^\infty$ connecting $w_1 \circ f_P$ and $w_1 \circ f_Q$. We try to lift this homotopy to $BPin_4$. The obstructions for doing this on the i-skeleton of X are

$$\mathfrak{o}_i \in H^i(X, \{\pi_i(BSpin_4)\}).$$

Since $\pi_i(BSpin_4) = 0$ for $i \leq 3$ we see that H can be lifted over the 3-skeleton $X^{(3)}$.

Now let X^w be the two fold covering of X associated to w. Under these conditions there exist lifts $f_{\tilde{P}}$ and $f_{\tilde{Q}}$ of $\pi^* f_P$ and $\pi^* f_Q$ to $BSpin$.

$$\begin{array}{ccc} & & BSpin_4 \\ & \overset{f_{\tilde{P}}, f_{\tilde{Q}}}{\nearrow} & \downarrow \\ X^w & \underset{\pi^* f_Q}{\overset{\pi^* f_P}{\longrightarrow}} & BPin \\ \downarrow \pi & & \| \\ X & \xrightarrow{f_P, f_Q} & BPin. \end{array}$$

The preceeding case implies that $\pi^* f_P \simeq \pi^* f_Q$ iff the classifying pairs of P and Q coincide as unordered pairs. The assumptions therefore imply that $\pi^* \mathfrak{o}_4 = 0 \in H^4(X^w; \pi^* \pi_4 BSpin_4)$. But $\pi_4 BSpin_4$ is as $\pi_1 X$-module isomorphic to $\mathbb{Z}[\pi_1 X / \pi_1 X^w]$. Therefore lemma 3.6 implies that

$$\pi^* : H^4(X; \pi_4 BSpin_4) \to H^4(X^w; \pi^* \pi_4 BSpin_4)$$

is injective. Hence $\mathfrak{o}_4 = 0 \in H^4(X; \pi_4 BSpin_4)$, and therefore $f_P \simeq f_Q$.

To see that $a_P = \tau^* b_P$ observe that the diagram

$$\begin{array}{ccc} X^w & \xrightarrow{f_{\tilde{P}}} & BSpin_4 \\ \tau \downarrow & & \downarrow T \\ X^w & \xrightarrow{f_{\tilde{P}}} & BSpin_4, \end{array}$$

where t is the covering transformation of $BSPin_4$ over $BPin_4$, is commutative since the 2-fold covering $X^w \to X$ is the pullback of $BSpin \to BPin$ under f_P. Since T^* flips the chosen generators a, b of $H^4(BSpin_4; \mathbb{Z})$ this shows that

$$(\tau^* \tilde{f}^* a, \tau^* \tilde{f}^* b) = (\tilde{f}^* b, \tilde{f}^* a).$$

Since for $w = 0$ the existence follows from the existence part of 3.2 we can restrict ourself to the case $w \neq 0$. To see that every pair $\{a, \tau^*a\}$, $a \in H^4(X^w; \mathbb{Z})$, is realized for some Pin_4-bundle over X let P_a be the SU_2-bundle with $c_2 = a$ and form the $Spin_4$ bundle $P_a + P_{\tau^*a}$. According to lemma 3.3 the weakly associated Pin_4 bundle $\hat{P} := (P_a + P_{\tau^*a}) \times_{Spin_4} Pin_4$ is equal to $P_a + P_{\tau^*a} \coprod P_{\tau^*a} + P_a$. Since $P_{\tau^*a} = \tau^* P_a$ as SU_2-bundles there is a map $\tau' : P_{\tau^*a} \to P_a$ covering τ. To simplify notation we'll write τ' also for $(\tau')^{-1} : P_a \to P_{\tau^*a}$.

Having the notation set up we define an involution $\hat{\tau}$ on $\hat{P}$ by

$$\begin{array}{cccccccc} \hat{\tau}: & P_a + P_{\tau^*a} & \coprod & P_{\tau^*a} + P_a & \to & P_a + P_{\tau^*a} & \coprod & P_{\tau^*a} + P_a \\ & (p,q) & & & \mapsto & & & (\tau'p, \tau'q) \\ & & & (q,p) & \mapsto & (\tau'q, \tau'p) & & \end{array}$$

This map is a Pin_4 equivariant involution on $\hat{P}$ covering τ on X^w and flipping the components of $\hat{P}$. The quotient $\hat{P}/\hat{\tau}$ is therefore a Pin_4 bundle over X with data (w, a, τ^*a).

The relation between (a, b) and Euler and Pontrjagin class follows from Lemma 3.1. □

Corollary 3.7 *Let $H^4(X^w; \mathbb{Z})$ contain no 2-torsion.*

i) Two $Spin_4$ bundles are isomorphic as Pin_4 bundles iff p_1 coincides and e coincides up to sign.

ii) Two Pin_4 bundles with $w_1 \neq 0$ are isomorphic iff they have the same w_1 and their twisted Euler classes coincide up to sign.

Corollary 3.8 *Pin_4 bundles over a nonorientable 4-manifold X with $w_1 = w_1(X)$ are classified by the absolute value of their Euler number.*

Moreover, every number $k \geq 0$ is realized as Euler number for some Pin_4 bundle P.

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