An explicit method of constructing pluriharmonic maps from compact complex manifold into complex Grassmann manifold

by

Seiichi UDAGAWA

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Department of Mathematics
School of Medicine
Nihon University
Itabashi-Ku, Tokyo 173
JAPAN

MPI/90-96

A explicit method of constructing pluriharmonic maps from compact complex manifold into complex Grassmann manifold

Seiichi UDAGAWA

0. Introduction.

Let $\varphi : M \longrightarrow N$ be a smooth map from a complex manifold into a Riemannian manifold. Then, φ is called *pluriharmonic* if (0,1)-exterior covariant derivative $D''\partial\varphi$ of the (1,0)-differential $\partial\varphi$ of φ vanishes identically. Let ∇^{φ} be the pull-back connection on the pull-back bundle $\varphi^{-1}TN$. We have

$$(0.1) \quad (D''\partial\varphi)(\bar{X}, Y) = \nabla^{\varphi}_{\bar{X}}\partial\varphi(Y) - \partial\varphi(\bar{\partial}_{\bar{X}}Y), \qquad X, Y \in C^{\infty}(TM^{1,0}),$$

where $TM^{1,0}$ is the holomorphic tangent bundle of M. If $\varphi^{-1}TN^{\mathbf{C}}$ has the Koszul-Malgrange holomorphic structure, that is, (0,1)-part of ∇^{φ} coincides with $\bar{\partial}$-operator, we may say that φ is pluriharmonic if and only if φ sends any holomorphic section of $TM^{1,0}$ to a holomorphic section of $\varphi^{-1}TN^{\mathbf{C}}$. It is easily observed that if φ is holomorphic and N is a Kähler manifold then $\varphi^{-1}TN^{1,0}$ has the Koszul-Malgrange holomorphic structure, hence any holomorphic map is pluriharmonic. Note that anti-holomorphic map is also pluriharmonic if N is a Kähler manifold. Conversely, the existence of the Koszul-Malgrange holomorphic structure on $\varphi^{-1}TN^{\mathbf{C}}$ (resp. $\varphi^{-1}TN^{1,0}$) is ensured if φ is pluriharmonic and N has non-negative or nonpositive curvature operator (resp. and N is Kähler)(cf. [O-U2]). From the point of view of Riemannian geometry, the most important property of pluriharmonic map is that it is a harmonic map with respect to any Kähler metric on M. Therefore, the concept of pluriharmonic maps generalizes that of harmonic maps from Riemann surface. Moreover, when one restricts a pluriharmonic map from M to any holomorphic curve C of M, it induces a harmonic map from C into N.

In [O-U1], the complex-analyticity, constancy and stability (as a harmonic map) of pluriharmonic maps from compact Kähler manifold were investigated in detail. As the consequences, there are so many non ±-holomorphic examples of pluriharmonic maps, where a map is called ±-holomorphic if it is either holomorphic or anti-holomorphic. As a special case, if the target is a complex Grassmann manifold $G_k(\mathbf{C}^n)$ of k-dimensional complex subspaces in $\mathbf{C}^n$, any pluriharmonic map φ from a Kähler manifold M is ±-holomorphic provided $\mathrm{Maxrank}_{\mathbf{R}} d\varphi \geq 2(n-k-1)(k-1)+3$. In case M with $c_1(M) > 0$ and $b_2(M) = 1$, the rank condition of φ may be replaced by $\dim_{\mathbf{C}} M \geq (n-k-1)(k-1)+2$ and this dimension estimate is best possible. On the other hand, the recent works of Ramanathan[Rm], Chern-Wolfson[C-W], Burstall-Wood[B-W], Burstall-Salamon[B-S], Wolfson[Wol] and Wood[Wd1] state that *any harmonic map from Riemann sphere* S^2 into $G_k(\mathbf{C}^n)$ *may be constructed from a holomorphic map* $S^2 \longrightarrow G_t(\mathbf{C}^n)$ *for some* $1 \leq t \leq k$, which originate from the works of Burns[Bn], Din-Zakrewski[D-Z], Glaser-Stora[G-S], Eells-Wood[E-W] with a complex projective space as target. Given a map $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$, we may identify φ with the pull-back of the universal bundle over $G_k(\mathbf{C}^n)$, denoted by $\underline{\varphi}$, which is a complex subbundle of the trivial bundle $M \times \mathbf{C}^n$. We have the sequence of the *∂'-Gauss bundles* by taking the image of the (1,0)-part of the second fundamental form of each subbundle. Wolfson proved that this sequence must terminate if $M = S^2$. In this sense, his method is explicit and the simplest in the form. In general, $\underline{\varphi}$ has the intersection with certain ∂'-Gauss bundle, say $(r+1)$-th ∂'-Gauss bundle, and such least integer r is called *∂'-isotropy order* of φ. A holomorphic map has *infinite* ∂'-isotropy order, hence one tries to increase the ∂'-isotropy order of a given map by certain algebraic replacement. This is a method of Burstall-Wood, which is explicit and the most natural in the idea. From their works, one may expect to establish the explicit method, using the second fundamental forms, of constructing any pluriharmonic map from a compact complex manifold M with $c_1(M) > 0$ into $G_k(\mathbf{C}^n)$. However, there are many difficulties. For example, ∂'-Gauss bundle of φ has non-removable singularities, and its rank may be greater than that of φ, which implies that it is impossible to generalize Wolfson's method to higher dimension. On the other hand, in [O-U2] Ohnita and the present author succeeded in generalizing the method of Burstall-Wood and proved that any pluriharmonic map φ from $M \setminus S_\varphi$ with M as above into $G_k(\mathbf{C}^n)$ with $k = 2, 3$ and $n \leq 12$ may be constructed, using the second fundamental forms, from a rational map $f : M \longrightarrow G_t(\mathbf{C}^n)$ for some t, where S_φ is a certain singularity of codimension at least two (see Definition 2.1). The restriction on k aries from the complicate of Salamon's diagram, which stands for the relations between $\underline{\varphi}$ and its ∂'-Gauss bundles, and even for the case of harmonic maps from S^2 the method is not known for general k. In higer dimension, the most difficulty exists in that one can say nothing about the relation of the ranks between $\underline{\varphi}$ and its ∂'-Gauss bundle, which is the reason for the restriction on n.

In this paper, the concepts of finite and infinite ∂'-isotropy order are important (see section 3). A pluriharmonic map with infinite ∂'-isotropy order is easily reduced to an anti-holomorphic map, hence we may treat only the case of finite ∂'-isotropy order (see Proposition 7.3). In case the target is a complex projective space $\mathbf{C}P^{n-1}$, it turns out that any pluriharmonic map has infinite ∂'-isotropy order, so that reduced to an anti-holomorphic map (Theorem 3.1). Thus, any pluriharmonic map φ from $M \setminus S_\varphi$ with M as above into $\mathbf{C}P^{n-1}$ may be constructed, in a unique way, from a rational map $f : M \longrightarrow \mathbf{C}P^{n-1}$ (Theorem 7.1). This is not the case for the complex Grassmann manifold of higher rank as target. We can prove that any pluriharmonic map φ with finite ∂'-isotropy order from $M \setminus S_\varphi$ with M as above into

$G_2(\mathbf{C}^n)$ with n arbitrary may be constructed, using the second fundamental forms, from a pluriharmonic map into $\mathbf{C}P^{n-1}$ (Theorems 4.1, 7.2). This technique is partially applicable to the cases of rank 3 and 4. We can prove that any pluriharmonic map φ with finite ∂'-isotropy order from $M \setminus S_\varphi$ with M as above into $G_k(\mathbf{C}^n)$ with $k = 3$ (resp. 4) and $n \leq 15$ (resp. 14) may be constructed, using the second fundamental forms, from a pluriharmonic map into $G_t(\mathbf{C}^n)$ for some $1 \leq t \leq k-1$ (Theorems 5.1, 6.1, 7.3). Although it is less explicit than those stated above, we can prove that any non-holomorphic pluriharmonic map φ from $M \setminus S_\varphi$ with M as above into $G_k(\mathbf{C}^n)$ with $k = 3$ (resp. 4) and $n \leq 20$ (resp. 15) may be constructed, using the second fundamental forms, from a rational map $f : M \longrightarrow G_t(\mathbf{C}^n)$ for some t (Theorems 6.2, 7.4), which also improves the result in [O-U2] stated above.

Refer to [E-L] for the recent developments of harmonic map theory, to [B-B-B-R], [B-B], [O-U1,2], [Ud] for the stability and complex-analyticity of pluriharmonic maps, and to [B-R], [Uh], [V], [Wd2] for the construction of harmonic maps from Riemann sphere to Lie group. Finally, we mention that Ohnita and Valli [O-V] generalized the results of [Uh], [V] to the class of meromorphically pluriharmonic maps.

1. Preliminaries.

Let E be a unitary vector bundle over a complex manifold M, that is, E is endowed with a Hermitian fibre metric h and a connection ∇^E compatible with h. Let F be a complex subbundle of E and let S be the Hermitian orthogonal complement of F in E with respect to h. Then, F and S also become the unitary vector bundles with respect to the induced Hermitian structures. Then, the second fundamental forms, $A^{S,F}$ and $A^{F,S}$, are defined by

$$(1.1) \qquad \nabla^E_X v = \nabla^F_X v + A^{F,S}_X(v)\,, \qquad \nabla^E_X w = \nabla^S_X w + A^{S,F}_X(w)$$

for any $X \in C^\infty(TM)$, $v \in C^\infty(F)$, $w \in C^\infty(S)$, where ∇^E, ∇^F and ∇^S are the Hermitian connections of E, F and S, respectively, and $A^{F,S}$ (resp. $A^{S,F}$) is regarded as $\mathrm{Hom}(F,S)$ (resp. $\mathrm{Hom}(S,F)$)-valued 1-form on M. We easily obtain

$$(1.2) \qquad A^{F,S} = -(A^{S,F})^*,$$

where $(\quad)^*$ denotes the adjoint of $(\quad)$ with respect to h. By the complex structure of M, we may decompose $A^{F,S}$ as $A^{F,S} = A^{F,S}_{(1,0)} + A^{F,S}_{(0,1)}$. Let D be the exterior covariant differentiation defined by the induced connection on $\mathrm{Hom}(F,S)$, and D', D'' be the (1,0)-, (0,1)-part of D, that is, $D = D' + D''$. The (0,1)-exterior covariant derivative $D''A^{F,S}_{(1,0)}$ of $A^{F,S}_{(1,0)}$ is defined by

$$(D''A^{F,S}_{(1,0)})(\bar{Z},W) = \nabla^S_{\bar{Z}} \circ A^{F,S}_W - A^{F,S}_W \circ \nabla^F_{\bar{Z}} - A^{F,S}_{\bar{\partial}_Z W}\,, \tag{1.3}$$

where $Z, W \in C^\infty(TM^{1,0})$. Similarly, $D'A^{F,S}_{(0,1)}$ is defined. Now, assume that E has the Koszul-Malgrange holomorphic structure, that is, a holomorphic structure compatible with the Hermitian structure of E, and F is a holomorphic subbundle of E. We may endow S with a holomorphic vector bundle structure by the isomorphism $S \simeq E/F$, which is, in fact, nothing but the Koszul-Malgrange holomorphic structure (cf. [B-S]). Then, $\mathrm{Hom}(F,S)$ also has the Koszul-Malgrange holomorphic structure and a smooth section A of $T^*M^{1,0} \otimes \mathrm{Hom}(F,S)$ is called *holomorphic* if $D''A \equiv 0$.

Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a smooth map from a complex manifold into a complex Grassmann manifold of k-dimensional complex subspaces in $\mathbf{C}^n$. Then, we may identify φ with a complex subbundle $\underline{\varphi}$ of rank k of the trivial bundle $\underline{\mathbf{C}}^n = M \times \mathbf{C}^n$, of which the fibre at $x \in M$ is given by $\underline{\varphi}_x = \varphi(x)$. Note that $\underline{\varphi}$ is the pull-back of the universal bundle T over $G_k(\mathbf{C}^n)$ by φ. Frequently, we write $\underline{\varphi}$ as φ if there is no confusion.

Definition 1.1. Let E be a complex subbundle of $\underline{\mathbf{C}}^n$. We denote by $E^\perp$ the Hermitian orthogonal complement of E in $\underline{\mathbf{C}}^n$ with respect to the standard Hermitian fibre metric on $\underline{\mathbf{C}}^n$. If F is a complex subbundle of E, the Hermitian orthogonal complement of F in E is denoted by $E \ominus F$.

Set

$$A^\varphi = A^{\underline{\varphi},\underline{\varphi}^\perp}\,, \qquad A^{\varphi^\perp} = A^{\underline{\varphi}^\perp,\underline{\varphi}}. \tag{1.4}$$

Then, by (1.2) we obtain

$$A^\varphi_{(1,0)} = -(A^{\varphi^\perp}_{(0,1)})^*\,, \qquad A^\varphi_{(0,1)} = -(A^{\varphi^\perp}_{(1,0)})^*. \tag{1.5}$$

The property of φ may be interpreted by the property of A^φ. For example, we have

Proposition 1.1. *(I) The following statements are mutually equivalent*

(1) φ is holomorphic (resp. anti-holomorphic)

(2) $\underline{\varphi}$ is a holomorphic (resp. an anti-holomorphic) subbundle of $\underline{\mathbf{C}}^n$,

(3) $A^\varphi_{(0,1)} \equiv 0$ (resp. $A^\varphi_{(1,0)} \equiv 0$).

(II) φ is pluriharmonic if and only if $D''A^\varphi_{(1,0)} \equiv 0$, equivalently $D'A^\varphi_{(0,1)} \equiv 0$.

(III) φ is pluriharmonic if and only if $\varphi^\perp$ is pluriharmonic.

In fact, we may say that if φ is pluriharmonic then $A^\varphi_{(1,0)}$ is a holomorphic section of $T^*M^{1,0} \otimes \mathrm{Hom}(\underline{\varphi}, \underline{\varphi}^\perp)$ by Proposition 1.1, (II) and the following

Proposition 1.2 ([O-U2]). *If φ is pluriharmonic, each of $\underline{\varphi}$ and $\underline{\varphi}^\perp$ has the Koszul-Malgrange holomorphic structure. In particular, any holomorphic subbundle of $\underline{\varphi}$ or $\underline{\varphi}^\perp$, and its Hermitian orthogonal complement in $\underline{\varphi}$ or $\underline{\varphi}^\perp$ have the Koszul-Malgrange holomorphic structures.*

If φ is pluriharmonic, by Propositions 1.1, 1.2, we see that $A^{\varphi^\perp}_{(1,0)}$ is also a holomorphic section of $T^*M^{1,0} \otimes \mathrm{Hom}(\underline{\varphi}^\perp, \underline{\varphi})$.

2. A method of constructing pluriharmonic maps.

Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map from a complex manifold. The following proposition gives a general method of constructing new pluriharmonic map from the old.

Proposition 2.1 ([O-U2]). *Define $\widetilde{\varphi}$ by*

$$\underline{\widetilde{\varphi}} = (\underline{\varphi} \ominus \alpha) \oplus \beta \ , \tag{2.1}$$

where α and β satisfy the following conditions :
(2.2) α and β are holomorphic subbundles of $\underline{\varphi}$ and $\underline{\varphi}^\perp$, respectively,

$$A^{\varphi}_{(1,0)}(\alpha) \subset T^*M^{1,0} \otimes \beta \ , \qquad A^{\varphi^\perp}_{(1,0)}(\beta) \subset T^*M^{1,0} \otimes \alpha \ . \tag{2.3}$$

Then, $\tilde{\varphi}$ is also a pluriharmonic map from M into $G_t(\mathbf{C}^n)$ for some t.

Remark. If we reverse the orientation of M, we see that we may use $A^{\varphi}_{(0,1)}$, $A^{\varphi^\perp}_{(0,1)}$ in place of $A^{\varphi}_{(1,0)}$, $A^{\varphi^\perp}_{(1,0)}$, in this case, α and β are chosen to be anti-holomorphic subbundles of $\underline{\varphi}$ and $\underline{\varphi}^\perp$, respectively.

To show the examples of α and β which satisfy the conditions (2.2) and (2.3), we consider $A^{\varphi}_{(1,0)}$ as a bundle homomorphism $A^{\varphi}_{(1,0)} : TM^{1,0} \otimes \underline{\varphi} \longrightarrow \underline{\varphi}^\perp$ and set

$$\mathrm{Im} A^{\varphi}_{(1,0)} = \cup_{x \in M} \mathrm{Im}(A^{\varphi}_{(1,0)})_x \ .$$

$\mathrm{Im} A^{\varphi}_{(1,0)}$ is a holomorphic subbundle of $\underline{\varphi}^\perp$ over $M \backslash V$, where V is an analytic subset of M. It can be observed that $\mathrm{Im} A^{\varphi}_{(1,0)}$ extends to a holomorphic subbundle of $\underline{\varphi}^\perp$ over $M \setminus W$, where W is an analytic subset of codimension at least 2, and denote it by $\underline{\mathrm{Im}} A^{\varphi}_{(1,0)}$ (cf. [O-U2]). Similarly, considering $A^{\varphi}_{(1,0)}$ as an another homomorphism $A^{\varphi}_{(1,0)} : \underline{\varphi} \longrightarrow T^*M^{1,0} \otimes \underline{\varphi}^\perp$ we set

$$\mathrm{Ker} A^{\varphi}_{(1,0)} = \cup_{x \in M} \mathrm{Ker}(A^{\varphi}_{(1,0)})_x \ .$$

In the same way as above, $\mathrm{Ker}A^{\varphi}_{(1,0)}$ extends to a holomorphic subbundle of $\underline{\varphi}$ over $M \setminus W'$, which is denoted by $\underline{\mathrm{Ker}}A^{\varphi}_{(1,0)}$, where W' is an analytic subset of codimension at least 2. When we construct the new pluriharmonic map from the old, we have the new singularity set, hence we give the following definition

Definition 2.1. Denote by S_{φ} the singularity set of M with $\mathrm{codim}_{\mathbf{C}} S_{\varphi} \geq 2$ such that φ is a pluriharmonic map from $M \backslash S_{\varphi}$ and S_{φ} is of the form

$$S_{\varphi} = \cup_{j=1}^{k} S_j$$

for some positive integer k and each S_i $(i = 1, \cdots, k)$ is an analytic subset of $M \backslash \cup_{j=1}^{i-1} S_j$ with $\mathrm{codim}_{\mathbf{C}} S_i \geq 2$.

We need the following lemma, which we frequently utilize

Lemma 2.1 **([O-U2]).** *Assume that M is a compact complex manifold with the positive first Chern class, $c_1(M) > 0$. Let E be a Hermitian holomorphic vector bundle over $M \backslash S$, where S is as in Definition 2.1 without the assumption on φ, and let A be a holomorphic multi-differential with values in* $\mathrm{End}(E)$*. Then, A is nilpotent, that is, $A^m \equiv 0$ as a holomorphic multi-differential with values in* $\mathrm{End}(E)$ *for some positive integer* $m \leq \mathrm{rank} E$.

For example, $A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)}$ is a holomorphic quadratic differential with values in $\mathrm{End}(\underline{\varphi})$ over $M \backslash S_{\varphi}$, hence nilpotent by Lemma 2.1 if M is compact and $c_1(M) > 0$. In particular, $A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)}$ has the non-trivial kernel. In this case, any non-zero holomorphic subbundle α of $\underline{\varphi}$ contained in $\mathrm{Ker}(A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)})$ satisfies the conditions (2.2) and (2.3) with $\beta = \underline{\mathrm{Im}}(A^{\varphi}_{(1,0)}\, |_{\alpha})$ (see Lemma 2.2 below for the holomorphicity of $A^{\varphi}_{(1,0)}\, |_{\alpha}$). In summary, we state the following

Proposition 2.2. *Let $\varphi : M \backslash S_{\varphi} \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map. Then, the following map $\widetilde{\varphi}$ defines a pluriharmonic map : $M \backslash S_{\widetilde{\varphi}} \longrightarrow G_t(\mathbf{C}^n)$ for some t :*
(2.4) $\underline{\widetilde{\varphi}} = \underline{\mathrm{Im}} A^{\varphi}_{(1,0)}$ *if* $A^{\varphi}_{(1,0)} \not\equiv 0$.
(2.5) $\underline{\widetilde{\varphi}} = \underline{\varphi} \ominus \underline{\mathrm{Ker}} A^{\varphi}_{(1,0)}$ *if* $\underline{\mathrm{Ker}} A^{\varphi}_{(1,0)} \neq \underline{0}$.
(2.6) $\underline{\widetilde{\varphi}} = (\underline{\varphi} \ominus \alpha) \oplus \underline{\mathrm{Im}}(A^{\varphi}_{(1,0)}\, |_{\alpha})$*, where α is a holomorphic subbundle of $\underline{\varphi}$ contained in* $\mathrm{Ker}(A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)})$*, if $\alpha \neq \underline{0}$, which is satisfied if M is compact and $c_1(M) > 0$.*

However, (2.5) may be considered as a special case of (2.6) because $\underline{\mathrm{Ker}} A^{\varphi}_{(1,0)}$ is contained in $\mathrm{Ker}(A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)})$. Moreover, if M is compact and $c_1(M) > 0$ then (2.4) is also obtained by the successive procedure of type (2.6), which follows from the more general proposition below, Proposition 2.3. For the notational simplicity, we give

Definition 2.2. Set $G'(\varphi) = G^{(1)}(\varphi) = \underline{\mathrm{Im}}A^{\varphi}_{(1,0)}$ and inductively define the *r-th ∂'-Gauss bundle* of φ, $G^{(r)}(\varphi)$, by

$$G^{(i+1)}(\varphi) = G'(G^{(i)}(\varphi)) \qquad \text{for } i = 1, 2, \cdots .$$

Similarly, define the *r-th ∂''-Gauss bundle*, $G^{(-r)}(\varphi)$, by

$$G''(\varphi) = G^{(-1)}(\varphi) = \underline{\mathrm{Im}}A^{\varphi}_{(0,1)} , \quad G^{(-i-1)}(\varphi) = G''(G^{(-i)}(\varphi)) \qquad \text{for } i = 1, 2, \cdots .$$

In particular, set $G'_{\varphi}(\alpha) = \underline{\mathrm{Im}}(A^{\varphi}_{(1,0)} |_{\alpha})$ and $G''_{\varphi}(\gamma) = \underline{\mathrm{Im}}(A^{\varphi}_{(0,1)} |_{\gamma})$ for a holomorphic subbundle α of $\underline{\varphi}$ and an anti-holomorphic subbundle γ of $\underline{\varphi}$, respectively.

We need the following

Lemma 2.2. *Let τ and μ be the Hermitian vector bundles over M with the Koszul-Malgrange holomorphic structures and let A be a holomorphic multi-differential with values in* $\mathrm{Hom}(\tau, \mu)$. *Then, the following statements are true*
(1) If α is a holomorphic subbundle of τ, then $A |_{\alpha}$ is holomorphic.
(2) If β is an anti-holomorphic subbundle of μ and $\pi : \mu \longrightarrow \beta$ is a Hermitian orthogonal projection, then $\pi \circ A$ is holomorphic.
(3) If γ is a subbundle of τ with $\tau \ominus \gamma \subset$ $\mathrm{Ker}A$ *and γ has the Koszul-Malgrange holomorphic structure with respect to the connection induced from τ, then $A |_{\gamma}$ is a holomorphic multi-differential with values in* $\mathrm{Hom}(\gamma, \mu)$.
(4) If δ is a subbundle of μ containing the image of A and δ has the Koszul-Malgrange holomorphic structure with respect to the connection induced from μ, then A is a holomorphic multi-differential with values in $\mathrm{Hom}(\tau, \delta)$.

Proof. Set $A = \sum_{i_1, \cdots, i_k} A_{i_1 \cdots i_k} dz^{i_1} \otimes \cdots \otimes dz^{i_k}$. Then, A is holomorphic if and only if, locally, $A_{i_1 \cdots i_k}$ is holomorphic, that is,

$$\nabla^{\mu}_{\bar{X}} \circ A_{i_1 \cdots i_k} = A_{i_1 \cdots i_k} \circ \nabla^{\tau}_{\bar{X}} \qquad \text{for any } X \in C^{\infty}(TM^{1,0}) .$$

(1) Set $\varepsilon = \tau \ominus \alpha$, then $A^{\alpha,\varepsilon}_{(0,1)} \equiv 0$. Therefore, we have

$$\nabla^{\mu}_{\bar{X}} \circ A_{i_1 \cdots i_k} |_{\alpha} = A_{i_1 \cdots i_k} \circ \nabla^{\tau}_{\bar{X}} |_{\alpha} = A_{i_1 \cdots i_k} \circ (\nabla^{\alpha}_{\bar{X}} + A^{\alpha,\varepsilon}_{\bar{X}}) = A_{i_1 \cdots i_k} \circ \nabla^{\alpha}_{\bar{X}} .$$

(2) Set $\kappa = \mu \ominus \beta$, then $A^{\kappa,\beta}_{(0,1)} \equiv 0$. Denote by π_{β} and π_{κ} the projections $\pi_{\beta} : \mu \longrightarrow \beta$ and $\pi_{\kappa} : \mu \longrightarrow \kappa$, respectively. Then, we have

$$\begin{aligned} \pi_{\beta} \circ A_{i_1 \cdots i_k} \circ \nabla^{\tau}_{\bar{X}} &= \pi_{\beta} \circ \nabla^{\mu}_{\bar{X}} \circ A_{i_1 \cdots i_k} \\ &= \pi_{\beta} \circ (\nabla^{\beta}_{\bar{X}} \circ \pi_{\beta} \circ A_{i_1 \cdots i_k} + A^{\kappa,\beta}_{\bar{X}} \circ \pi_{\kappa} \circ A_{i_1 \cdots i_k}) \\ &= \nabla^{\beta}_{\bar{X}} \circ \pi_{\beta} \circ A_{i_1 \cdots i_k} . \end{aligned}$$

(3) Set $\eta = \tau \ominus \gamma \subset \mathrm{Ker}A$, then $A(\eta) \equiv 0$. We have

$$\nabla^{\mu}_{\bar{X}} \circ A_{i_1 \cdots i_k} |_{\gamma} = A_{i_1 \cdots i_k} \circ \nabla^{\tau}_{\bar{X}} |_{\gamma} = A_{i_1 \cdots i_k} \circ (\nabla^{\gamma}_{\bar{X}} + A^{\gamma,\eta}_{\bar{X}}) = A_{i_1 \cdots i_k} \circ \nabla^{\gamma}_{\bar{X}} \, .$$

(4) Set $\nu = \mu \ominus \delta$, then, since $\mathrm{Im}A \subset \delta$ we obtain

$$A_{i_1 \cdots i_k} \circ \nabla^{\tau}_{\bar{X}} = \nabla^{\mu}_{\bar{X}} \circ A_{i_1 \cdots i_k} = \nabla^{\delta}_{\bar{X}} \circ A_{i_1 \cdots i_k} + A^{\delta,\nu}_{\bar{X}} \circ A_{i_1 \cdots i_k} = \nabla^{\delta}_{\bar{X}} \circ A_{i_1 \cdots i_k} .$$

q.e.d.

Now, we prove the following

Proposition 2.3. *Assume that M is compact and $c_1(M) > 0$. Let $\varphi : M \backslash S_{\varphi} \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map and define $\widetilde{\varphi}$ by $\underline{\widetilde{\varphi}} = (\underline{\varphi} \ominus \alpha) \oplus \beta$, where α and β satisfy the conditions (2.2) and (2.3). Then, there is a finite sequence $\{\varphi_i\}_{i=0}^{N}$ of pluriharmonic maps with (1) $\varphi = \varphi_0$ (2) $\widetilde{\varphi} = \varphi_N$ (3) for $i = 0, 1, \cdots, n-2$, each φ_{i+1} is obtained from φ_i by $\underline{\varphi}_{i+1} = (\underline{\varphi}_i \ominus \alpha_i) \oplus G'_{\varphi_i}(\alpha_i)$, where α_i is a holomorphic subbundle of $\underline{\varphi}_i$ contained in $\mathrm{Ker}(A^{\varphi_i^{\perp}}_{(1,0)} \circ A^{\varphi_i}_{(1,0)})$ such that*
(I) if $\beta = G'_{\varphi}(\alpha)$, φ_N is also obtained from φ_{N-1} by the above procedure (3) for $i = N-1$,
(II) if $\beta \neq G'_{\varphi}(\alpha)$, there is a holomorphic subbundle β_{N-1} of $(\underline{\varphi}_{N-1} \oplus G''(\varphi_{N-1}))^{\perp}$ so that φ_N is obtained from φ_{N-1} by $\underline{\varphi}_N = \underline{\varphi}_{N-1} \oplus \underline{\beta}_{N-1}$.

Proof. First, observe that $A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)} |_{\alpha}$ is a holomorphic quadratic differential with values in $\mathrm{End}(\alpha)$ by Lemma 2.2. Then, it follows from Lemma 2.1 that $(A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)} |_{\alpha})^k \equiv 0$ for some positive integer $k \leq \mathrm{rank}\alpha$. Set $L = A^{\varphi^{\perp}}_{(1,0)} \circ A^{\varphi}_{(1,0)} |_{\alpha}$, and define $\alpha_0, \cdots, \alpha_{k-1}$ and $\beta_0, \cdots, \beta_k$ by

$$\alpha_i = \underline{\mathrm{Im}L^{k-1-i}} \ominus \underline{\mathrm{Im}L^{k-i}} \quad \text{with} \quad \alpha = \underline{\mathrm{Im}L^{0}} \qquad \text{for } i = 0, 1, \cdots, k-1, \tag{2.7}$$

$$\beta_i = G'_{\varphi}(\underline{\mathrm{Im}L^{k-1-i}}) \ominus G'_{\varphi}(\underline{\mathrm{Im}L^{k-i}}) \quad \text{with} \quad \beta = G'_{\varphi}(\underline{\mathrm{Im}L^{-1}}) \tag{2.8}$$

for $i = 0, 1, \cdots, k$. Define a sequence $\{\varphi_i\}_{i=0}^{k}$ by

$$\underline{\varphi}_{i+1} = (\underline{\varphi}_i \ominus \alpha_i) \oplus \beta_i \quad \text{with} \quad \varphi_0 = \varphi \quad \text{for } i = 0, 1, \cdots, k-1. \tag{2.9}$$

By (2.7) and (2.8), we see that for any $i = 0, 1, \cdots, k-1$,

$$\bigoplus_{j=0}^{i} \alpha_j = \underline{\mathrm{Im}L^{k-1-i}} \, , \quad \bigoplus_{j=0}^{i} \beta_j = G'_{\varphi}(\underline{\mathrm{Im}L^{k-1-i}}) \, . \tag{2.10}$$

so that

$$(2.11) \qquad G'_{\varphi}(\alpha_i) \cap (G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i}))^{\perp} = \beta_i \, .$$

We fix any integer i with $0 \leq i \leq k-1$. We show that $G'_{\varphi_i}(\alpha_i) = \beta_i$ and α_i is a holomorphic subbundle of $\underline{\varphi}_i$ contained in $\mathrm{Ker}(A^{\varphi_i^{\perp}}_{(1,0)} \circ A^{\varphi_i}_{(1,0)})$. By (2.9) and (2.10), we have

$$(2.12) \qquad \underline{\varphi}_i = (\underline{\varphi} \ominus \underline{\mathrm{Im}}L^{k-i}) \oplus G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i}) \, .$$

Since L^{k-i} and $A^{\varphi}_{(1,0)} \circ L^{k-i}$ are holomorphic, α_i and β_i are anti-holomorphic subbundles of $\underline{\mathrm{Im}}L^{k-1-i}$ and $G'_{\varphi}(\underline{\mathrm{Im}}L^{k-1-i})$, respectively, that is,

$$(2.13) \qquad A^{\alpha_i, \underline{\mathrm{Im}}L^{k-i}}_{(1,0)} \equiv 0 \, , \quad A^{\beta_i, G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i})}_{(1,0)} \equiv 0 \, .$$

It follows from (2.12) and (2.13) that

$$(2.14) \qquad G'_{\varphi_i}(\alpha_i) = G'_{\varphi_i}(\alpha_i) \cap \underline{\varphi}^{\perp} = G'_{\varphi}(\alpha_i) \cap (G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i}))^{\perp} = \beta_i \, .$$

Since $\underline{\mathrm{Im}}L^{k-1-i}$ is a holomorphic subbundle of $\underline{\varphi}$, α_i is a holomorphic subbundle of $\underline{\varphi} \ominus \underline{\mathrm{Im}}L^{k-i}$. Moreover, we have

$$\begin{aligned} \underline{\mathrm{Im}}A^{G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i}), \varphi}_{(1,0)} &= G'_{\varphi^{\perp}}(G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i})) \\ &= \underline{\mathrm{Im}}L^{k+1-i} \, , \end{aligned}$$

which is orthogonal to $\underline{\varphi} \ominus \underline{\mathrm{Im}}L^{k-i}$, hence $\underline{\varphi} \ominus \underline{\mathrm{Im}}L^{k-i}$ is a holomorphic subbundle of $\underline{\varphi}_i$ by (2.12). Thus, α_i is a holomorphic subbundle of $\underline{\varphi}_i$. Finally, by (2.13) and (2.14) we have

$$A^{\varphi_i^{\perp}}_{(1,0)} \circ A^{\varphi_i}_{(1,0)}(\alpha_i) = A^{\varphi_i^{\perp}}_{(1,0)}(\beta_i) \subset (\underline{\mathrm{Im}}L^{k-i} \oplus (\underline{\varphi}^{\perp} \ominus G'_{\varphi}(\underline{\mathrm{Im}}L^{k-i}))) \, ,$$

so that $A^{\varphi_i^{\perp}}_{(1,0)} \circ A^{\varphi_i}_{(1,0)}(\alpha_i) \equiv 0$. If $\beta = G'_{\varphi}(\alpha)$ then $\beta_k = \underline{0}$ and $\alpha = \bigoplus_{j=0}^{k-1} \alpha_j$, $\beta = \bigoplus_{j=0}^{k-1} \beta_j$, hence we obtain (I). For (II), we only have to show that β_k is a holomorphic subbundle of $(\underline{\varphi}_k \oplus G''(\varphi_k))^{\perp}$. $\underline{\varphi}_k$ is given by $\underline{\varphi}_k = (\underline{\varphi} \ominus \underline{\alpha}) \oplus G'_{\varphi}(\alpha)$ and $G'_{\varphi}(\alpha)$ is a holomorphic subbundle of β, hence $A^{G'_{\varphi}(\alpha), \beta_k}_{(0,1)} \equiv 0$ by $\beta = G'_{\varphi}(\alpha) \oplus \beta_k$. Moreover, by the condition (2.3) we have $A^{\beta_k, \underline{\varphi} \ominus \alpha}_{(1,0)} = A^{\varphi^{\perp}, \varphi \ominus \alpha}_{(1,0)}(\beta_k) \equiv 0$. Therefore, $\beta_k \perp G''(\varphi_k)$ and β_k is in $(\underline{\varphi}_k \oplus G''(\varphi_k))^{\perp}$. Since $A^{\alpha, \beta_k}_{(1,0)} = A^{\varphi, \beta_k}_{(1,0)}(\alpha) \equiv 0$ and β_k

is a holomorphic subbundle of $\underline{\varphi}^{\perp} \ominus G'_{\varphi}(\alpha)$, β_k is a holomorphic subbundle of $\underline{\varphi}_k^{\perp}$, hence of $(\underline{\varphi}_k \oplus G''(\varphi_k))^{\perp}$. q.e.d.

We have given the self-contained but lengthy proof. It is easier to understand the reason that Proposition 2.3 holds, if we use the Salamon's diagram, which will be defined and used in the next section.

We call the procedure (2.6) the *forward replacement* of α. When we use $A^{\varphi}_{(0,1)}$ and an anti-holomorphic subbundle γ of $\underline{\varphi}$, we call the corresponding procedure the *backward replacement* of γ.

3. Salamon's diagram and the isotropy order of pluriharmonic map.

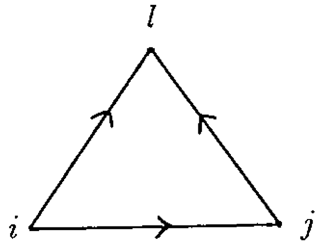
Let $\underline{\mathbf{C}}^n = M \times \mathbf{C}^n$ be the trivial bundle over a complex manifold M with the standard Hermitian fibre metric h_0. Let $\tau_1, \cdots, \tau_k$ be a set of mutually orthogonal subbundles of $\underline{\mathbf{C}}^n$ with respect to h_0 such that each τ_i $(i = 1, \cdots, k)$ has the Koszul-Malgrange holomorphic structure compatible with the Hermitian structure induced from h_0 and $\underline{\mathbf{C}}^n = \bigoplus_{j=1}^{k} \tau_j$. Denote by $A^{\tau_i,\tau_j}_{(1,0)}$ the (1,0)-*second fundamental form* of τ_i in $\tau_i \oplus \tau_j$ for $1 \leq i \neq j \leq k$ (cf. section 1). Following [B-W], we give

Definition 3.1. We mean by a *diagram* $\{\tau_i, A^{\tau_i,\tau_j}_{(1,0)}\}$ the directed graph with vertices $\tau_1, \cdots, \tau_k$ and for each ordered pair (i,j) an edge from τ_i to τ_j representing $A^{\tau_i,\tau_j}_{(1,0)}$. The absence of a given edge in the graph indicates the vanishing of the corresponding (1,0)-second fundamental form.

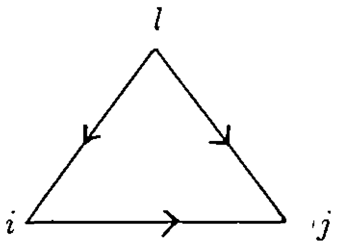
An important use of this diagram is to decide whether a given homomorphism, such as the composition of some (1,0)-second fundamental forms, is holomorphic or not. For this purpose, we need

Proposition 3.1. *Given a diagram* $\{\tau_i, A^{\tau_i,\tau_j}_{(1,0)}\}$, $A^{\tau_i,\tau_j}_{(1,0)} : TM^{1,0} \otimes \tau_i \longrightarrow \tau_j$ *is holomorphic if the diagram contains no configurations of the following forms :*

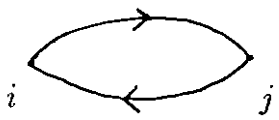
(1)



(2)



(3)



where $1 \leq l \leq k$ *with* $l \neq i, j$.

The proof of Proposition 3.1 is just the same method as in [B-W] (cf. Lemma 2.2). The particularly important case is when Lemma 2.1 is utilized. For example, if

$A^{\tau_i,\tau_i+1}_{(1,0)}$ $(1 \le i \le k-1)$ and $A^{\tau_k,\tau_1}_{(1,0)}$ are all holomorphic, we see that the composition $A^{\tau_k,\tau_1}_{(1,0)} \circ A^{\tau_{k-1},\tau_k}_{(1,0)} \circ \cdots \circ A^{\tau_1,\tau_2}_{(1,0)}$ is a holomorphic section of $\bigotimes^k T^*M^{1,0} \otimes \mathrm{End}(\tau_1)$ by Leibniz' rule, hence nilpotent. We often refer to it as a *holomorphic circuit* and denote it by $\{\tau_1, \tau_2, \cdots, \tau_k, \tau_1\}$ for notational simplicity.

Next, we introduce the concept of isotropy order of a given pluriharmonic map. Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map from a complex manifold. We denote by $G^{(r)}(\varphi)$ the r-th ∂'-Gauss bundle of φ as in section 2.

Definition 3.2. We say that φ has *∂'-isotropy order r* if $\underline{\varphi}$ is orthogonal to each $G^{(i)}(\varphi)$ $(1 \le i \le r)$ and not orthogonal to $G^{(r+1)}(\varphi)$ with respect to h_0. Moreover, we say that φ has *finite* (resp. *infinite*) ∂'-isotropy order if $r < \infty$ (resp. $r = \infty$). Similarly, the corresponding notion of *∂''-isotropy order* for ∂''-Gauss bundles is defined.

Note that $\underline{\varphi} \perp G'(\varphi)$ always holds, so that any φ has ∂'-isotropy order ≥ 1.

Lemma 3.1 ([O-U2]). *If φ has ∂'-isotropy order $\ge r$, then $G^{(i)}(\varphi) \perp G^{(j)}(\varphi)$ for any i, j such that $0 <| i - j |\le r$.*

If φ has ∂'-isotropy order $\ge r$, then by Lemma 3.1 we may set $R = \underline{\varphi}^{\perp} \ominus (\bigoplus_{j=1}^{r} G^{(j)}(\varphi))$. It follows from Proposition 1.2 and Lemma 2.2, (3) that all $\underline{\varphi}$, $G^{(i)}(\varphi)$ $(1 \le i \le r)$ and R have the Koszul-Malgrange holomorphic structures compatible with the Hermitian structures induced from h_0, and all $A^{\varphi, G'(\varphi)}_{(1,0)}$ and $A^{G^{(i)}(\varphi), G^{(i+1)}(\varphi)}_{(1,0)}$ $(1 \le i \le r-1)$ are holomorphic. We often use this fact, without any comment, in the following. If φ is a holomorphic map, $A^{\varphi}_{(0,1)} = -(A^{\varphi^{\perp}}_{(1,0)})^* \equiv 0$, so that $A^{\varphi^{\perp}}_{(1,0)} \equiv 0$ and $\underline{\varphi} \perp G^{(i)}(\varphi)$ for any $i \ge 1$. Therefore, *a holomorphic map has infinite ∂'-isotropy order.* However, since every $G^{(i)}(\varphi)$ is a subbundle of $\underline{\mathbf{C}}^n$, Lemma 3.1 implies that there exists a positive integer s such that $G^{(s)}(\varphi) = \underline{0}$, that is, φ is reduced to an anti-holomorphic map $f : M \backslash S_f \longrightarrow G_t(\mathbf{C}^n)$ for some t. In general, a given pluriharmonic map has finite ∂'-isotropy order. A method for that is to increase the ∂'-isotropy order of a given pluriharmonic map by the successive procedures of type (2.6), that is, the forward replacement, so that it is reduced to an anti-holomorphic map, in case M is compact and $c_1(M) > 0$. However, when the target is a complex projective space $\mathbf{C}P^{n-1}$ with Fubini-Study metric, a given pluriharmonic map turns out to have infinite ∂'-isotropy order. In fact, we have

Theorem 3.1. *Assume that M is compact and $c_1(M) > 0$. Let $\varphi : M \backslash S_{\varphi} \longrightarrow \mathbf{C}P^{n-1}$ be a pluriharmonic map. Then, $G^{(s)}(\varphi) = \underline{0}$ for some positive integer $s \le n-1$. Moreover, if φ is non-holomorphic, each $G^{(i)}(\varphi)$ $(0 \le i \le s-1)$ defines a pluriharmonic map into $\mathbf{C}P^{n-1}$ and φ is reduced to an anti-holomorphic map $\varphi_{s-1} : M \backslash S_{\varphi_{s-1}} \longrightarrow \mathbf{C}P^{n-1}$.*

Theorem 3.1 is a reformulation of Theorem (7.30) in [O-U2]. This theorem plays an important role when we treat the complex Grassmann manifold of higher rank as target. We give here a proof of it using Salamon's diagram.

Proof of Theorem 3.1. If φ is anti-holomorphic, we have nothing to prove, so that we may assume that φ is non anti-holomorphic. Suppose that φ has ∂'-isotropy order $\geq r$. We have a diagram by Lemma 3.1

(3.1)

$$\begin{tikzcd}
\underline{\varphi} \arrow[r] & G'(\varphi) \arrow[r] & \cdots \arrow[r, dashed] & G^{(r)}(\varphi) \arrow[r] \arrow[lll, bend right] & R \arrow[llll, bend right]
\end{tikzcd}$$

where $R = \underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi))$. We show that $A^{G^{(r)}(\varphi),\varphi}_{(1,0)}$ is holomorphic. If $r = 1$, $G^{(r)}(\varphi)$ is a holomorphic subbundle of $\underline{\varphi}^\perp$ and $A^{G^{(r)}(\varphi),\varphi}_{(1,0)} = A^{\varphi^\perp}_{(1,0)}|_{G^{(r)}(\varphi)}$ is holomorphic by Proposition 1.1 and Lemma 2.2. If $r \geq 2$, by Proposition 3.1, $A^{G^{(r)}(\varphi),\varphi}_{(1,0)}$ is holomorphic. Then, we have a holomorphic circuit $\{\underline{\varphi}, G'(\varphi), \cdots, G^{(r)}(\varphi), \underline{\varphi}\}$, and by Lemma 2.1 and rank$\underline{\varphi} = 1$ we have $A^{G^{(r)}(\varphi),\varphi}_{(1,0)} \equiv 0$ because all the other edges in (3.1) are surjective by the definitions. Therefore, $\underline{\varphi} \perp G^{(r+1)}(\varphi)$ and φ has ∂'-isotropy order $\geq r+1$. Thus, φ has infinite ∂'-isotropy order, so that $G^{(s)}(\varphi) = \underline{0}$ for some positive integer $s \leq n-1$. If φ is non $\pm$-holomorphic, by Proposition 3.2 in [O-U1] we have $\mathrm{rank}_{\mathbf{C}}\partial\varphi \leq 1$ on $M \backslash S_\varphi$, which implies rank$G'(\varphi) = 1$ and $G'(\varphi)$ defines a pluriharmonic map into $\mathbf{C}P^{n-1}$, so does $G^{(i)}(\varphi)$ while $G^{(i-1)}(\varphi)$ defines non $\pm$-holomorphic map. If $G^{(r)}(\varphi)$ defines a holomorphic map, then

$$A^{G^{(r-1)}(\varphi),G^{(r)}(\varphi)}_{(1,0)} = A^{G^{(r)}(\varphi)^\perp}_{(1,0)}|_{G^{(r-1)}(\varphi)} \equiv 0 \ ,$$

hence $G^{(r-1)}(\varphi)$ already defines an anti-holomorphic map into $\mathbf{C}P^{n-1}$. q.e.d.

When the target is a complex Grassmann manifold of higher rank, a generic pluriharmonic map surely has finite ∂'-isotropy order. Therefore, in this case we can't expect the result like Theorem 3.1 because the situation of rank$G^{(i+1)}(\varphi) >$ rank$G^{(i)}(\varphi)$ may occur. We use the forward replacement, which is the most basic one by Proposition 2.3, to treat the case of higher rank. We may reverse the procedures in Theorem 3.1 so that any pluriharmonic map can be constructed, using the second fundamental forms, from a holomorphic map or a rational map, which is proved in section 7.

4. Pluriharmonic maps into $G_2(\mathbf{C}^n)$.

In this section, we give an explicit method of constructing any pluriharmonic map $\varphi : M\backslash S_\varphi \longrightarrow G_2(\mathbf{C}^n)$, where M is a compact complex manifold with $c_1(M) > 0$. If φ has infinite ∂'-isotropy order, then $G^{(s)}(\varphi) = \underline{0}$ for some positive integer s, hence φ is reduced to an anti-holomorphic map $\varphi_{s-1} : M\backslash S_{\varphi_{s-1}} \longrightarrow G_t(\mathbf{C}^n)$ for some t. Thus, we may assume that φ has finite ∂'-isotropy order. We prove

Theorem 4.1. *Let $\varphi : M\backslash S_\varphi \longrightarrow G_2(\mathbf{C}^n)$ be a pluriharmonic map. Assume that φ has finite ∂'-isotropy order. Then, there is a sequence $\{\varphi_i\}_{i=0}^N$ of pluriharmonic maps such that*
(1) $\varphi_0 = \varphi$, (2) $\varphi_N : M\backslash S_{\varphi_N} \longrightarrow \mathbf{C}P^{n-1}$,
(3) for $i = 0, 1, \cdots, N-1$, each φ_i has ∂'-isotropy order $r+i$, where r is the ∂'-isotropy order of φ_0, and φ_{i+1} is obtained from φ_i by the forward replacement of α^i, where $\alpha^i = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+i)}(\varphi_i),\varphi_i}$, which is a holomorphic subbundle of $\underline{\varphi}_i$ contained in $\mathrm{Ker}(A_{(1,0)}^{\varphi_i^\perp} \circ A_{(1,0)}^{\varphi_i})$.

Proof. Let r be the ∂'-isotropy order of φ. As in the proof of Theorem 3.1, we see that $A_{(1,0)}^{G^{(r)}(\varphi),\varphi}$ is holomorphic. Set $A_{r,\varphi} = A_{(1,0)}^{G^{(r)}(\varphi),\varphi} \circ A_{(1,0)}^{G^{(r-1)}(\varphi),G^{(r)}(\varphi)} \circ \cdots \circ A_{(1,0)}^{\varphi,G'(\varphi)}$. By Lemma 2.1, $A_{r,\varphi}^2 \equiv 0$, so that $\alpha^0 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r)}(\varphi),\varphi} \subset \mathrm{Ker}A_{r,\varphi} \subset \mathrm{Ker}(A_{(1,0)}^{\varphi^\perp} \circ A_{(1,0)}^{\varphi})$ and $\mathrm{rank}\alpha^0 = 1$. Set $\alpha_0^0 = \alpha^0$, $\alpha_i^0 = G_\varphi^{(i)}(\alpha_0^0)$ for $i = 1, \cdots, r$, and set $\gamma_0^0 = \underline{\varphi} \ominus \alpha_0^0$, $\gamma_i^0 = G^{(i)}(\varphi) \ominus \alpha_i^0$ for $i = 1, \cdots, r$. We have a diagram

(4.1)
$$\begin{tikzcd}
\gamma_0^0 \arrow[r] \arrow[ddr] & \gamma_1^0 \arrow[r] \arrow[ddr] & {} \arrow[r, dashed] \arrow[ddr] & \gamma_r^0 \arrow[rrdd] \arrow[dddlll] & {} & \gamma_0^0 \\
& & & & R & \\
\alpha_0^0 \arrow[r] \arrow[uu] & \alpha_1^0 \arrow[r] \arrow[uu] & {} \arrow[r, dashed] & \alpha_r^0 \arrow[uu] \arrow[rruu] & {} & \alpha_0^0 \arrow[uu]
\end{tikzcd}$$

where $R = \underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi))$. By Proposition 3.1, $A_{(1,0)}^{\alpha_i^0,\alpha_{i+1}^0}$, $A_{(1,0)}^{\gamma_i^0,\gamma_{i+1}^0}$ $(0 \leq i \leq r-1)$, $A_{(1,0)}^{\gamma_r^0,\alpha_0^0}$ and $A_{(1,0)}^{\alpha_r^0,R}$ are all holomorphic. Further, set $\alpha_{r+1}^0 = \underline{\mathrm{Im}}A_{(1,0)}^{\alpha_r^0,R}$ and $R_0' = R \ominus \alpha_{r+1}^0$. Again, we have a diagram

(4.2)
$$\begin{tikzcd}
\gamma_0^0 \arrow[r] \arrow[dr] & \gamma_1^0 \arrow[r] \arrow[dr] & {} \arrow[r, dashed] \arrow[dr] & \gamma_r^0 \arrow[r] \arrow[dr] \arrow[dlll] & R_0' \arrow[r] \arrow[dr] & \gamma_0^0 \\
\alpha_0^0 \arrow[r] \arrow[u] & \alpha_1^0 \arrow[r] \arrow[u] & {} \arrow[r, dashed] & \alpha_r^0 \arrow[r] \arrow[u] & \alpha_{r+1}^0 \arrow[r] \arrow[u] \arrow[ur] & \alpha_0^0 \arrow[u]
\end{tikzcd}$$

By (4.2) and Proposition 3.1, we see that $A^{\alpha^0_{r+1},\gamma^0_0}_{(1,0)}$ is also holomorphic. We have a holomorphic circuit $\{\alpha^0_0,\alpha^0_1,\cdots,\alpha^0_{r+1},\gamma^0_0,\gamma^0_1,\cdots,\gamma^0_r,\alpha^0_0\}$, which must vanish by Lemma 2.1. However, $A^{\gamma^0_i,\gamma^0_{i+1}}_{(1,0)}$ $(0\leq i\leq r-1)$ and $A^{\gamma^0_r,\alpha^0_0}_{(1,0)}$ are all surjective. Since $\mathrm{rank}\gamma^0_0=1$, we obtain $A^{\alpha^0_{r+1},\gamma^0_0}_{(1,0)}\equiv 0$. Hereafter, we use the convention that if $\alpha^0_i=\underline{0}$ for some $1\leq i\leq r+1$ we understand that $A^{\alpha^0_{r+1},\gamma^0_0}_{(1,0)}\equiv 0$ is trivially satisfied. Set $\underline{\varphi}_1=(\underline{\varphi}\ominus\alpha^0_0)\oplus\alpha^0_1$. If $\alpha^0_1=\underline{0}$, then $\mathrm{rank}\underline{\varphi}_1=1$ and φ_1 is a pluriharmonic map into $\mathbf{C}P^{n-1}$. Hence, assume that $\alpha^0_1\neq\underline{0}$. Then, from (4.2) we have

$$\underline{\varphi}_1=\gamma^0_0\oplus\alpha^0_1\ ,\ \ G^{(i)}(\varphi_1)=\gamma^0_i\oplus\alpha^0_{i+1}\ (1\leq i\leq r),\ \ G^{(r+1)}(\varphi_1)\subset R'_0\oplus\alpha^0_0\ ,$$

so that φ_1 has ∂'-isotropy order $r+1$. To continue this procedure, we investigate the properties of φ_1 further. Setting $R_1=(R'_0\oplus\alpha^0_0)\ominus G^{(r+1)}(\varphi_1)$, we have a diagram

$$\tag{4.3}\begin{tikzcd}
\underline{\varphi}_1 \arrow[r] & G'(\varphi_1) \arrow[r] & \cdots \arrow[r, dashed] & G^{(r+1)}(\varphi_1) \arrow[r] \arrow[lll, bend right] & R_1 \arrow[llll, bend right]
\end{tikzcd}$$

By (4.3) and Proposition 3.1, we see that $A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$ is holomorphic. We have a holomorphic circuit $\{\underline{\varphi}_1,G'(\varphi_1),\cdots,G^{(r+1)}(\varphi_1),\underline{\varphi}_1\}$, and setting

$$A_{r+1,\varphi_1}=A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}\circ A^{G^{(r)}(\varphi_1),G^{(r+1)}(\varphi_1)}_{(1,0)}\circ\cdots\circ A^{\varphi_1,G'(\varphi_1)}_{(1,0)}$$

we see that A_{r+1,φ_1} is nilpotent. Set $\alpha^1_0=\underline{\mathrm{Im}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$ then $\mathrm{rank}\alpha^1_0\leq\mathrm{rank}\underline{\varphi}_1-1$. Let $\widetilde{P}^1:G^{(r+1)}(\varphi_1)\longrightarrow\alpha^0_0$ and $P_1:\alpha^1_0\longrightarrow\alpha^0_1$ be the Hermitian orthogonal projections. It follows from the surjectivity of $A^{\gamma^0_r,\alpha^0_0}_{(1,0)}$ and the fact $G^{(r)}(\varphi_1)=\gamma^0_r\oplus\alpha^0_{r+1}$ that $\widetilde{P}^1$ is surjective. Since $(R'_0\oplus\alpha^0_0)\perp\alpha^0_1$, $G^{(r+1)}(\varphi_1)\subset R'_0\oplus\alpha^0_0$ and $A^{R'_0,\alpha^0_1}_{(1,0)}\equiv 0$ by (4.2), we obtain

$$\tag{4.4} P_1\circ A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}(v)=A^{G^{(r+1)}(\varphi_1),\alpha^0_1}_{(1,0)}(v)=A^{\alpha^0_0,\alpha^0_1}_{(1,0)}\circ\widetilde{P}^1(v)\ ,$$

where $v\in C^\infty(G^{(r+1)}(\varphi_1))$, which, together with the surjectivities of $\widetilde{P}^1$ and $A^{\alpha^0_0,\alpha^0_1}_{(1,0)}$, implies that P_1 is surjective. Therefore, we have $\mathrm{rank}\alpha^1_0\geq\mathrm{rank}\alpha^0_1=\mathrm{rank}\underline{\varphi}_1-1$, which, together with the above rank inequality, implies that $\mathrm{rank}\alpha_0^1 = \mathrm{rank}\alpha_1^0 = \mathrm{rank}\underline{\varphi}_1 - 1$ and P_1 is an isomorphism, where we note that P_1 is holomorphic by Lemma 2.2. We show that $A^2_{r+1,\varphi_1} \equiv 0$. Set $\alpha_i^1 = G^{(i)}_{\varphi_1}(\alpha_0^1)$, which is a holomorphic subbundle of $G^{(i)}(\varphi_1)$, for $i = 1, \cdots, r+1$. If $\widetilde{P}^1 \,|_{\alpha^1_{r+1}} : \alpha^1_{r+1} \longrightarrow \alpha_0^0$ is surjective, by (4.4) we see that $P_1(\underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} \,|_{\alpha^1_{r+1}})) = P_1(\alpha_0^1)$, hence $\underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} \,|_{\alpha^1_{r+1}}) = \alpha_0^1$, which contradicts the nilpotency of A_{r+1,φ_1}. Therefore, $\widetilde{P}^1 \,|_{\alpha^1_{r+1}} \equiv 0$ by $\mathrm{rank}\alpha_0^0 = 1$, and hence $\alpha^1_{r+1} \subset \mathrm{Ker}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$ by (4.4) and the isomorphicity of P_1. Thus, we have proved that $A^2_{r+1,\varphi_1} \equiv 0$. Moreover, we obtain $\alpha^1_{r+1} \subset R'_0$ and $\alpha^1_{r+2} = \underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi_1)}_{(1,0)} \,|_{\alpha^1_{r+1}}) \subset R_1 \subset R'_0 \oplus \alpha_0^0$. Finally, set

$$R'_1 = R_1 \ominus \alpha^1_{r+2} = ((R'_0 \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1)) \ominus \alpha^1_{r+2}$$

then by (4.2) we see that $R'_1 \perp \alpha_1^0$ and $A^{R'_1,\alpha_1^1}_{(1,0)} \equiv 0$.

We claim that

Claim. For each $i = 0, 1, \cdots$, if, φ_i has ∂'-isotropy order $r+i$, $A^{G^{(r+i)}(\varphi_i),\varphi_i}_{(1,0)}$ is holomorphic and $A^2_{r+i,\varphi_i} \equiv 0$, then define φ_{i+1} by $\underline{\varphi}_{i+1} = (\underline{\varphi}_i \ominus \alpha_0^i) \oplus \alpha_1^i$, where $\alpha_1^i = G'_{\varphi_i}(\alpha_0^i)$, $\alpha_0^i = \underline{\mathrm{Im}}A^{G^{(r+i)}(\varphi_i),\varphi_i}_{(1,0)} \subset \mathrm{Ker}A_{r+i,\varphi_i}$ and $\mathrm{rank}\alpha_0^i = \mathrm{rank}\underline{\varphi}_i - 1$. Then, either φ_{i+1} is a pluriharmonic map into $\mathbf{C}P^{n-1}$ or, φ_{i+1} has ∂'-isotropy order $r+i+1$ and has the following properties :
(1) $A^{G^{(r+i+1)}(\varphi_{i+1}),\varphi_{i+1}}_{(1,0)}$ is holomorphic and $A^2_{r+i+1,\varphi_{i+1}} \equiv 0$,
(2) set $\alpha_0^{i+1} = \underline{\mathrm{Im}}A^{G^{(r+i+1)}(\varphi_{i+1}),\varphi_{i+1}}_{(1,0)} \subset \mathrm{Ker}A_{r+i+1,\varphi_{i+1}}$, then $\mathrm{rank}\alpha_1^i = \mathrm{rank}\alpha_0^{i+1} = \mathrm{rank}\underline{\varphi}_{i+1} - 1$ and the Hermitian orthogonal projection $P_{i+1} : \alpha_0^{i+1} \longrightarrow \alpha_1^i$ is a holomorphic isomorphism,
(3) $G^{(r+s)}(\varphi_{i+1}) \subset R'_{s-1} \oplus \alpha_0^{s-1}$ $(1 \leq s \leq i+1)$,
(4) set $\alpha_j^{i+1} = G^{(j)}_{\varphi_{i+1}}(\alpha_0^{i+1})$ for $j = 1, \cdots, r+i+2$, then $\alpha^{i+1}_{r+s} \subset R'_{s-1}$ $(1 \leq s \leq i+1)$ and $\alpha^{i+1}_{r+i+2} \subset R'_i \oplus \alpha_0^i$,
(5) set $R'_{i+1} = ((R'_i \oplus \alpha_0^i) \ominus G^{(r+i+1)}(\varphi_{i+1})) \ominus \alpha^{i+1}_{r+i+2}$, then $R'_{i+1} \perp \alpha_1^i$ and $A^{R'_{i+1},\alpha_1^{i+1}}_{(1,0)} \equiv 0$.

This *Claim* is already established for $i = 0$. Assume that *Claim* is true for $0 \leq i \leq k$ and $\underline{\varphi}_{i+1}$ $(0 \leq i \leq k)$ does not define a map into $\mathbf{C}P^{n-1}$, so that each φ_{i+1} $(0 \leq i \leq k)$ has the properties (1) $\sim$ (5). Then, we may define φ_{k+2} by $\underline{\varphi}_{k+2} = (\underline{\varphi}_{k+1} \ominus \alpha_0^{k+1}) \oplus \alpha_1^{k+1}$. If $\alpha_1^{k+1} = \underline{0}$, then $\mathrm{rank}\underline{\varphi}_{k+2} = 1$ by (2) for φ_{k+1}, and φ_{k+2} is a pluriharmonic map into $\mathbf{C}P^{n-1}$. Hence, assume that $\alpha_1^{k+1} \neq \underline{0}$. First, we draw the diagram for φ_{i+1} $(0 \leq i \leq k)$. Set $\gamma_0^{i+1} = \underline{\varphi}_{i+1} \ominus \alpha_0^{i+1}$ and

$\gamma_j^{i+1} = G^{(j)}(\varphi_{i+1}) \ominus \alpha_j^{i+1}$ for $j = 1, \cdots, r+i+1$. By the properties (1) $\sim$ (5) for φ_{i+1}, we have a diagram

(4.5)
$$\begin{tikzcd}
\gamma_0^{i+1} \arrow[r] \arrow[dr] & \gamma_1^{i+1} \arrow[r, dashed] \arrow[dr] & {} \arrow[r, dashed] \arrow[dr] & \gamma_{r+1}^{i+1} \arrow[r, dashed] \arrow[dr] & {} \arrow[r, dashed] \arrow[dr] & \gamma_{r+i+1}^{i+1} \arrow[r] \arrow[dr] \arrow[dlllll] & R'_{i+1} \arrow[r] \arrow[dr] & \gamma_0^{i+1} \\
\alpha_0^{i+1} \arrow[r] \arrow[u] & \alpha_1^{i+1} \arrow[r, dashed] \arrow[u] & {} \arrow[r, dashed] & \alpha_{r+1}^{i+1} \arrow[r, dashed] \arrow[u] & {} \arrow[r, dashed] & \alpha_{r+i+1}^{i+1} \arrow[r] \arrow[u] & \alpha_{r+i+2}^{i+1} \arrow[r] \arrow[u] \arrow[ur] & \alpha_0^{i+1} \arrow[u]
\end{tikzcd}$$

In particular, when $i = k$, we have a holomorphic circuit

$$\{\alpha_0^{k+1}, \alpha_1^{k+1}, \cdots, \alpha_{r+k+2}^{k+1}, \gamma_0^{k+1}, \gamma_1^{k+1}, \cdots, \gamma_{r+k+1}^{k+1}, \alpha_0^{k+1}\},$$

which is nilpotent. Since $\operatorname{rank}\gamma_0^{k+1} = 1$ by (2) for φ_{k+1}, and $A_{(1,0)}^{\gamma_j^{k+1},\gamma_{j+1}^{k+1}}$ $(0 \le j \le r+k)$ and $A_{(1,0)}^{\gamma_{r+k+1}^{k+1},\alpha_0^{k+1}}$ are all surjective, we obtain $A_{(1,0)}^{\alpha_{r+k+2}^{k+1},\gamma_0^{k+1}} \equiv 0$. It follows from (4.5) that

$$(4.6)\quad \underline{\varphi}_{k+2} = \gamma_0^{k+1} \oplus \alpha_1^{k+1}, \quad G^{(j)}(\varphi_{k+2}) = \gamma_j^{k+1} \oplus \alpha_{j+1}^{k+1}\ (1 \le j \le r+k+1),$$
$$G^{(r+k+2)}(\varphi_{k+2}) \subset R'_{k+1} \oplus \alpha_0^{k+1}.$$

Then, φ_{k+2} has ∂'-isotropy order $r+k+2$. By (3) and (4) for φ_{k+1} and the definition of R'_s we obtain

$$\gamma_{r+s}^{k+1} \subset G^{(r+s)}(\varphi_{k+1}) \subset R'_{s-1} \oplus \alpha_0^{s-1}\ (1 \le s \le k+1),$$
$$\alpha_{r+s+1}^{k+1} \subset R'_s \subset R'_{s-1} \oplus \alpha_0^{s-1}\ (1 \le s \le k),\quad \alpha_{r+k+2}^{k+1} \subset R'_k \oplus \alpha_0^k,$$

which, together with (4.6), yield

$$(4.7)\qquad G^{(r+s)}(\varphi_{k+2}) \subset R'_{s-1} \oplus \alpha_0^{s-1}\quad (1 \le s \le k+2).$$

Set $R_{k+2} = \underline{\varphi}_{k+2}^{\perp} \ominus (\bigoplus_{j=1}^{r+k+2} G^{(j)}(\varphi_{k+2}))$. We have a diagram

(4.8)
$$\begin{tikzcd}
\underline{\varphi}_{k+2} \arrow[r] & G'(\varphi_{k+2}) \arrow[r] & {} \arrow[r, dashed] & G^{(r+k+2)}(\varphi_{k+2}) \arrow[r] \arrow[lll, bend right] & R_{k+2} \arrow[llll, bend right]
\end{tikzcd}$$

By (4.8) and Proposition 3.1, $A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}}$ is holomorphic. Set $\alpha_0^{k+2} = \underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}}$ and $\alpha_j^{k+2} = G_{\varphi_{k+2}}^{(j)}(\alpha_0^{k+2})$ for $j = 1, \cdots, r+k+2$. Set

$$A_{r+k+2,\varphi_{k+2}} = A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}} \circ A_{(1,0)}^{G^{(r+k+1)}(\varphi_{k+2}),G^{(r+k+2)}(\varphi_{k+2})} \circ \cdots$$
$$\cdots \circ A_{(1,0)}^{\varphi_{k+2},G'(\varphi_{k+2})},$$

which is nilpotent. Hence, $\mathrm{rank}\alpha_0^{k+2} \leq \mathrm{rank}\underline{\varphi}_{k+2} - 1$. Let $\widetilde{P}^{k+2} : G^{(r+k+2)}(\varphi_{k+2}) \longrightarrow \alpha_0^{k+1}$ and $P_{k+2} : \alpha_0^{k+2} \longrightarrow \alpha_1^{k+1}$ be the Hermitian orthogonal projections. It follows from the surjectivity of $A_{(1,0)}^{\gamma_{r+k+1}^{k+1},\alpha_0^{k+1}}$ and the fact $G^{(r+k+1)}(\varphi_{k+2}) = \gamma_{r+k+1}^{k+1} \oplus \alpha_{r+k+2}^{k+1}$ (see (4.5) and (4.6)) that $\widetilde{P}^{k+2}$ is surjective. Since $(R'_{k+1} \oplus \alpha_0^{k+1}) \perp \alpha_1^{k+1}$, $G^{(r+k+2)}(\varphi_{k+2}) \subset R'_{k+1} \oplus \alpha_0^{k+1}$ and $A_{(1,0)}^{R'_{k+1},\alpha_1^{k+1}} \equiv 0$ by (4.5) for $i = k$, we obtain

$$(4.9) \qquad P_{k+2} \circ A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}}(v)$$
$$= A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\alpha_1^{k+1}}(v) = A_{(1,0)}^{\alpha_0^{k+1},\alpha_1^{k+1}} \circ \widetilde{P}^{k+2}(v),$$

where $v \in C^\infty(G^{(r+k+2)}(\varphi_{k+2}))$, which, together with the surjectivities of $\widetilde{P}^{k+2}$ and $A_{(1,0)}^{\alpha_0^{k+1},\alpha_1^{k+1}}$, implies that P_{k+2} is surjective. Therefore, we have $\mathrm{rank}\alpha_0^{k+2} \geq \mathrm{rank}\alpha_1^{k+1} = \mathrm{rank}\underline{\varphi}_{k+2} - \mathrm{rank}\gamma_0^{k+1} = \mathrm{rank}\underline{\varphi}_{k+2} - 1$, where the last equality follows from (2) for φ_{k+1}. Consequently, we see that $\mathrm{rank}\alpha_0^{k+2} = \mathrm{rank}\alpha_1^{k+1} = \mathrm{rank}\underline{\varphi}_{k+2} - 1$ and P_{k+2} is an isomorphism. We show that $A_{r+k+2,\varphi_{k+2}}^2 \equiv 0$. By (4.7) we have

$$(4.10) \qquad \alpha_{r+s}^{k+2} \subset G^{(r+s)}(\varphi_{k+2}) \subset R'_{s-1} \oplus \alpha_0^{s-1} \quad (1 \leq s \leq k+2).$$

First, we must show that $\alpha_{r+s}^{k+2} \subset R'_{s-1}$ $(1 \leq s \leq k+2)$. Let $p^s : \alpha_{r+s}^{k+2} \longrightarrow \alpha_0^{s-1}$, $q^s : \alpha_{r+s}^{k+2} \longrightarrow R'_{s-1}$ $(1 \leq s \leq k+2)$ and $\tau^s : \alpha_{r+s+1}^{k+2} \longrightarrow \alpha_1^{s-1}$ $(1 \leq s \leq k+1)$ be the Hermitian orthogonal projections. Take any $v \in C^\infty(\alpha_{r+s}^{k+2})$. By (4.10), we may set $v = p^s(v) + q^s(v)$, and for $1 \leq s \leq k+1$ we have

$$(4.11) \qquad \tau^s \circ A_{(1,0)}^{\alpha_{r+s}^{k+2},\alpha_{r+s+1}^{k+2}}(v)$$
$$= A_{(1,0)}^{\alpha_0^{s-1},\alpha_1^{s-1}}(p^s(v)) + A_{(1,0)}^{R'_{s-1},\alpha_1^{s-1}}(q^s(v)) = A_{(1,0)}^{\alpha_0^{s-1},\alpha_1^{s-1}} \circ p^s(v),$$

where we have used the facts $(R'_{s-1} \oplus \alpha_0^{s-1}) \perp \alpha_1^{s-1}$ (see (4.5)) and (5) for φ_{s-1}. If p^s is surjective, then, since $A_{(1,0)}^{\alpha_0^{s-1},\alpha_1^{s-1}}$ is surjective, (4.11) implies that τ^s is surjective, where we note that $\alpha_0^{s-1} \neq \underline{0}$ and $\alpha_1^{s-1} \neq \underline{0}$ $(1 \leq s \leq k+1)$ because neither φ_{s-1} nor φ_s defines a map into $\mathbf{C}P^{n-1}$ by the assumption. Since $R'_s \perp \alpha_1^{s-1}$, $\mathrm{rank}\alpha_0^s = \mathrm{rank}\alpha_1^{s-1}$ and $P_s : \alpha_0^s \longrightarrow \alpha_1^{s-1}$ is an isomorphism by (2), (5) for φ_s, the surjectivity of τ^s implies that $p^{s+1} : \alpha_{r+s+1}^{k+2} \longrightarrow \alpha_0^s$ is also surjective. Now, suppose that p^1 is surjective. Then, it follows that each p^s $(1 \leq s \leq k+2)$ is surjective. In particular, $p^{k+2} : \alpha_{r+k+2}^{k+2} \longrightarrow \alpha_0^{k+1}$ is surjective. Note that $p^{k+2} = \widetilde{P}^{k+2} \mid_{\alpha_{r+k+2}^{k+2}}$. Then, it follows from the surjectivity of p^{k+2} and (4.9) that $P_{k+2}(\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}} \mid_{\alpha_{r+k+2}^{k+2}})) = P_{k+2}(\alpha_0^{k+2})$ and hence $\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}} \mid_{\alpha_{r+k+2}^{k+2}}) = \alpha_0^{k+2}$, which contradicts the nilpotency of $A_{r+k+2,\varphi_{k+2}}$. Therefore, we have proved that p^1 is not surjective and $p^1 \equiv 0$ by $\mathrm{rank}\alpha_0^0 = 1$. For any fixed s $(1 \leq s \leq k+1)$, if $p^s \equiv 0$, then by (4.11) and the surjectivity of $A_{(1,0)}^{\alpha_{r+s}^{k+2},\alpha_{r+s+1}^{k+2}}$ for $1 \leq s \leq k+1$ we see that $\tau^s \equiv 0$, where we note that if $\alpha_{r+s+1}^{k+2} = \underline{0}$ then $\tau^s \equiv 0$ is trivially satisfied. Since P_s is an isomorphism, it follows from $\tau^s \equiv 0$ that $p^{s+1} \equiv 0$. Thus, we have proved that $p^s \equiv 0$ $(1 \leq s \leq k+2)$, which, together with (4.10), yields

$$\alpha_{r+s}^{k+2} \subset R'_{s-1} \ (1 \leq s \leq k+2)\,. \tag{4.12}$$

Moreover, the fact $p^{k+2} \equiv 0$, the isomorphicity of P_{k+2} and (4.9) imply that $\alpha_{r+k+2}^{k+2} \subset \mathrm{Ker}A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2}),\varphi_{k+2}}$, so that $A_{r+k+2,\varphi_{k+2}}^2 \equiv 0$. Set

$$\alpha_{r+k+3}^{k+2} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+k+2)}(\varphi_{k+2})} \mid_{\alpha_{r+k+2}^{k+2}}) \subset R_{k+2} \subset R'_{k+1} \oplus \alpha_0^{k+1}\,,$$

and set

$$R'_{k+2} = R_{k+2} \ominus \alpha_{r+k+3}^{k+2} = ((R'_{k+1} \oplus \alpha_0^{k+1}) \ominus G^{(r+k+2)}(\varphi_{k+2})) \ominus \alpha_{r+k+3}^{k+2}\,.$$

By (4.5) for $i = k$, we see that $R'_{k+2} \perp \alpha_1^{k+1}$ and $A_{(1,0)}^{R'_{k+2},\alpha_1^{k+2}} \equiv 0$. In this way, *Claim* is established.

Now, let N be any positive integer and suppose that each φ_i $(0 \leq i \leq N)$ does not define a map into $\mathbf{C}P^{n-1}$. Then, by *Claim* we see that φ_N has ∂'-isotropy order $r+N$. However, this is impossible because the ∂'-isotropy order $r+N$ must be less than n. Therefore, there exists a positive integer N such that φ_N is a pluriharmonic map from $M \backslash S_{\varphi_N}$ into $\mathbf{C}P^{n-1}$, which, together with *Claim*, yields the statements (1) $\sim$ (3) of Theorem 4.1. q.e.d.

The inverse procedures of Theorem 4.1 is also proved in section 7.

5. Pluriharmonic maps into $G_3(\mathbf{C}^n)$.

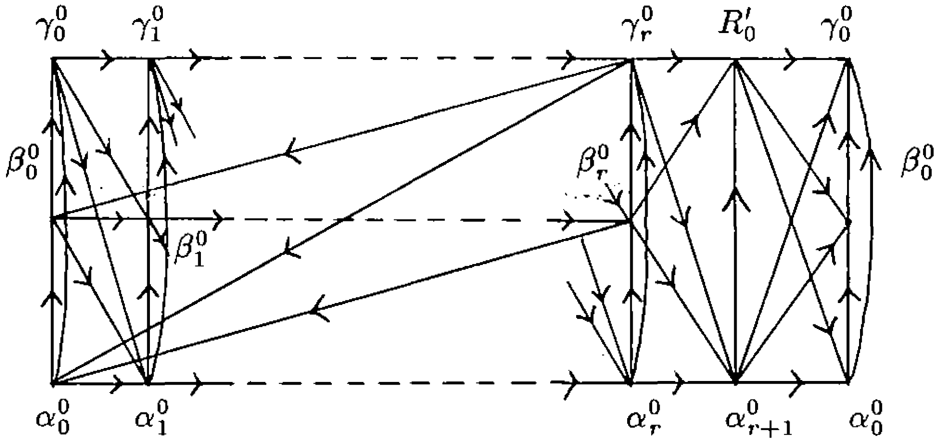
Let $\varphi : M \backslash S_\varphi \longrightarrow G_3(\mathbf{C}^n)$ be a pluriharmonic map, where M is a compact complex manifold with $c_1(M) > 0$. As in the case of $G_2(\mathbf{C}^n)$, we may assume that φ has finite ∂'-isotropy order. Define $A_{r,\varphi}$ as in section 4, where r is the ∂'-isotropy order of φ, then $A_{r,\varphi}$ is nilpotent. There are two possibilities :

(I) $A^3_{r,\varphi} \equiv 0$ and $A^2_{r,\varphi} \not\equiv 0$, (II) $A^2_{r,\varphi} \equiv 0$.

We treat these two cases separately. Although we don't get the result such as Theorem 4.1 because of the complicate of the sequence of pluriharmonic maps into $G_3(\mathbf{C}^n)$, we may increase the ∂'-isotropy order by two, so that we can construct any pluriharmonic map into $G_3(\mathbf{C}^n)$ under the restriction on n. Set $\varphi_0 = \varphi$.

(I) Set $R_0 = \underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^r G^{(r)}(\varphi))$, and set $\tau_0 = \underline{\mathrm{Im}} A_{(1,0)}^{G^{(r)}(\varphi),\varphi}$, $\tau_i = G_\varphi^{(i)}(\tau_0)$ for $i = 1, \cdots, r$ and $\gamma_0^0 = \underline{\varphi} \ominus \tau_0$, $\gamma_i^0 = G^{(i)}(\varphi) \ominus \tau_i$ for $i = 1, \cdots, r$. Since $A^2_{r,\varphi} \not\equiv 0$, set $\alpha_0^0 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi),\varphi}|_{\tau_r})$ and $\alpha_i^0 = G_\varphi^{(i)}(\alpha_0^0)$ for $i = 1, \cdots, r$. Then, we see that $\mathrm{rank}\tau_0 = 2$ and $\mathrm{rank}\alpha_0^0 = 1$. Moreover, set $\beta_0^0 = \tau_0 \ominus \alpha_0^0$, $\beta_i^0 = \tau_i \ominus \alpha_i^0$ for $i = 1, \cdots, r$, $\alpha_{r+1}^0 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi)}|_{\alpha_r^0}) \subset R_0$ and $R_0' = R_0 \ominus \alpha_{r+1}^0$. Then, we have a diagram

(5.1)



By (5.1), we have a holomorphic circuit

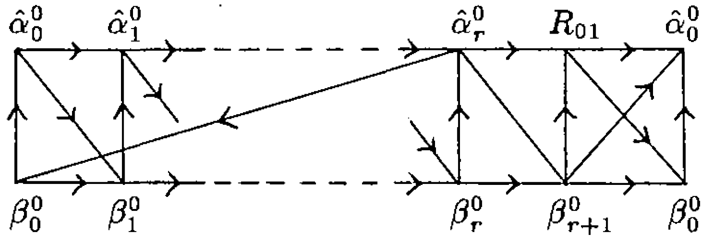
$$\{\alpha_0^0, \alpha_1^0, \cdots, \alpha_{r+1}^0, \gamma_0^0, \gamma_1^0, \cdots, \gamma_r^0, \beta_0^0, \beta_1^0, \cdots, \beta_r^0, \alpha_0^0\} ,$$

which must vanish. Since $A_{(1,0)}^{\gamma_i^0,\gamma_{i+1}^0}$, $A_{(1,0)}^{\beta_i^0,\beta_{i+1}^0}$ $(0 \le i \le r-1)$, $A_{(1,0)}^{\gamma_r^0,\beta_0^0}$ and $A_{(1,0)}^{\beta_r^0,\alpha_0^0}$ are all surjective, we have $A_{(1,0)}^{\alpha_{r+1}^0,\gamma_0^0} \equiv 0$. Set $\underline{\varphi}_0^1 = (\underline{\varphi}_0 \ominus \alpha_0^0) \oplus \alpha_1^0$ then we have

$$\text{(5.2)} \qquad \underline{\varphi}_0^1 = \gamma_0^0 \oplus \beta_0^0 \oplus \alpha_1^0 \ , \ \ G^{(i)}(\varphi_0^1) = \gamma_i^0 \oplus \beta_i^0 \oplus \alpha_{i+1}^0 \ (1 \le i \le r) \ ,$$
$$G^{(r+1)}(\varphi_0^1) \subset R_0' \oplus \beta_0^0 \oplus \alpha_0^0 \ ,$$

so that $\beta_0^0 = \underline{\mathrm{Im}} A_{(1,0)}^{G^{(r)}(\varphi_0^1),\varphi_0^1}$, $\beta_i^0 = G^{(i)}_{\varphi_0^1}(\beta_0^0)$ $(1 \le i \le r)$ and $\mathrm{Im}(A_{(1,0)}^{G^{(r)}(\varphi_0^1)}|_{\beta_r^0}) \subset R_0' \oplus \alpha_0^0$. Note that if $\alpha_1^0 = \underline{0}$ then φ_0^1 is a pluriharmonic map into $G_2(\mathbf{C}^n)$, so that we may assume that $\alpha_1^0 \neq \underline{0}$. Set $\beta_{r+1}^0 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi_0^1)}|_{\beta_r^0})$ and $R_{01} = (R_0' \oplus \alpha_0^0) \ominus \beta_{r+1}^0$ then we have a diagram

(5.3)

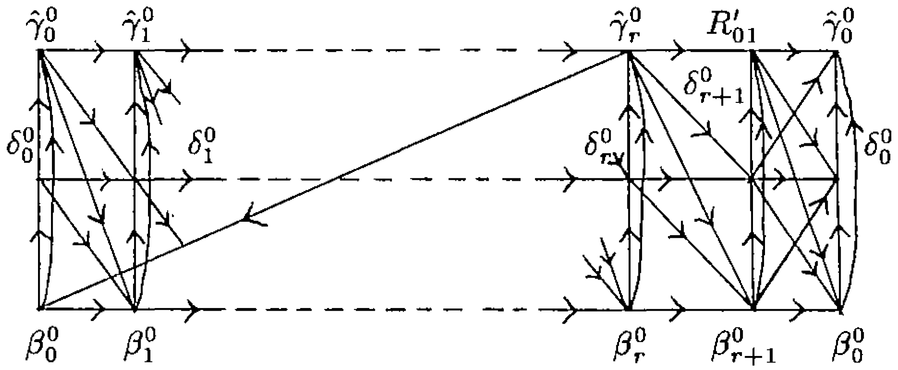


where we have put $\hat{\alpha}_i^0 = \gamma_i^0 \oplus \alpha_{i+1}^0$ $(0 \le i \le r)$. By (5.3), we easily see that $A_{(1,0)}^{\beta_{r+1}^0,\hat{\alpha}_0^0} : \beta_{r+1}^0 \longrightarrow \hat{\alpha}_0^0$ can not be surjective and $\underline{\mathrm{Im}} A_{(1,0)}^{\beta_{r+1}^0,\hat{\alpha}_0^0}$ is contained in the kernel of $A_{(1,0)}^{\hat{\alpha}_r^0,\beta_0^0} \circ A_{(1,0)}^{\hat{\alpha}_{r-1}^0,\hat{\alpha}_r^0} \circ \cdots \circ A_{(1,0)}^{\hat{\alpha}_0^0,\hat{\alpha}_1^0}$, so that $\mathrm{rank}\underline{\mathrm{Im}} A_{(1,0)}^{\beta_{r+1}^0,\hat{\alpha}_0^0} \le \mathrm{rank}\hat{\alpha}_0^0 - 1$. Set $\delta_0^0 = \underline{\mathrm{Im}} A_{(1,0)}^{\beta_{r+1}^0,\hat{\alpha}_0^0}$. Denote by $P_0 : \delta_0^0 \longrightarrow \alpha_1^0$ and $P^0 : \beta_{r+1}^0 \longrightarrow \alpha_0^0$ the Hermitian orthogonal projections. In the same way as (4.4), by (5.1) we obtain

$$P_0 \circ A_{(1,0)}^{\beta_{r+1}^0,\hat{\alpha}_0^0}(v) = A_{(1,0)}^{\alpha_0^0,\alpha_1^0} \circ P^0(v) , \qquad v \in C^\infty(\beta_{r+1}^0) . \tag{5.4}$$

Since $A_{(1,0)}^{\beta_r^0,\alpha_0^0}$ is surjective, P^0 is surjective, which, together with (5.4) and the surjectivity of $A_{(1,0)}^{\alpha_0^0,\alpha_1^0}$, implies that P_0 is surjective, hence $\mathrm{rank}\delta_0^0 \ge \mathrm{rank}\alpha_1^0 = \mathrm{rank}\hat{\alpha}_0^0 - 1$. Thus, we see that $\mathrm{rank}\delta_0^0 = \mathrm{rank}\hat{\alpha}_0^0 - 1$ and P_0 is an isomorphism. Set $\delta_i^0 = G^{(i)}_{\varphi_0^1}(\delta_0^0) \cap \hat{\alpha}_i^0$ for $i = 1, \cdots, r$, and set $\hat{\gamma}_0^0 = \hat{\alpha}_0^0 \ominus \delta_0^0$, $\hat{\gamma}_i^0 = \hat{\alpha}_i^0 \ominus \delta_i^0$ for $i = 1, \cdots, r$. Since $\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi_0^1)}|_{(\beta_r^0 \oplus \delta_r^0)}) \subset R_0' \oplus \alpha_0^0$, set $\delta_{r+1}^0 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi_0^1)}|_{(\beta_r^0 \oplus \delta_r^0)}) \ominus \beta_{r+1}^0 \subset R_{01}$ and $R_{01}' = R_{01} \ominus \delta_{r+1}^0$. We have a diagram

(5.5)



By (5.5) we have a holomorphic circuit

$$\{\beta_0^0, \beta_1^0, \cdots, \beta_{r+1}^0, \delta_0^0, \delta_1^0, \cdots, \delta_{r+1}^0, \hat{\gamma}_0^0, \hat{\gamma}_1^0, \cdots, \hat{\gamma}_r^0, \beta_0^0\},$$

which must vanish. Hence, we get $A_{(1,0)}^{\delta_{r+1}^0, \hat{\gamma}_0^0} \equiv 0$ because rank$\hat{\gamma}_0^0 = 1$, where, as in section 4, we understand that this equation is trivially satisfied if $\delta_i^0 = \underline{0}$ for some $1 \leq i \leq r+1$. Therefore, set $\underline{\varphi}_1 = (\underline{\varphi}_0^1 \ominus (\delta_0^0 \oplus \beta_0^0)) \oplus (\delta_1^0 \oplus \beta_1^0)$ then

$$(5.6) \qquad \underline{\varphi}_1 = \hat{\gamma}_0^0 \oplus \delta_1^0 \oplus \beta_1^0, \qquad G^{(i)}(\varphi_1) = \hat{\gamma}_i^0 \oplus \delta_{i+1}^0 \oplus \beta_{i+1}^0 \ (1 \leq i \leq r),$$
$$G^{(r+1)}(\varphi_1) \subset R'_{01} \oplus \delta_0^0 \oplus \beta_0^0,$$

so that φ_1 has ∂'-isotropy order $\geq r+1$. We remark that $\delta_0^0 \neq \underline{0}$ if $\alpha_1^0 \neq \underline{0}$. First, we show that $A_{r+1,\varphi_1}^3 \equiv 0$. Set $\mu_0^1 = \underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, $\mu_i^1 = G_{\varphi_1}^{(i)}(\mu_0^1)$ for $i = 1, \cdots, r+1$. Denote by $P_1^0 : \mu_0^1 \longrightarrow \delta_1^0 \oplus \beta_1^0$ and $\widetilde{P}^1 : G^{(r+1)}(\varphi_1) \longrightarrow \delta_0^0 \oplus \beta_0^0$ the Hermitian orthogonal projections, which are holomorphic. By (5.5), we see that $\widetilde{P}^1$ is surjective. We have

$$(5.7)\ P_1^0 \circ A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}(v) = A_{(1,0)}^{\delta_0^0 \oplus \beta_0^0, \delta_1^0 \oplus \beta_1^0} \circ \widetilde{P}^1(v), \qquad v \in C^\infty(G^{(r+1)}(\varphi_1)).$$

It follows from (5.7) that P_1^0 is surjective, which, together with the nilpotency of A_{r+1,φ_1}, implies that rank$\mu_0^1 =$ rank$\underline{\varphi}_1 - 1$ and P_1^0 is an isomorphism. Hence, $\widetilde{P}^1 \mid_{\mu_{r+1}^1} : \mu_{r+1}^1 \longrightarrow \delta_0^0 \oplus \beta_0^0$ can not be surjective, and set $\hat{\mu}_{r+1} = \underline{\mathrm{Im}}(\widetilde{P}^1 \mid_{\mu_{r+1}^1})$, $\hat{\nu}_{r+1} = (\delta_0^0 \oplus \beta_0^0) \ominus \hat{\mu}_{r+1}$. Since P_1^0 is surjective, the Hermitian orthogonal projection $P_1^r : \mu_r^1 \longrightarrow \delta_{r+1}^0 \oplus \beta_{r+1}^0$ is also surjective, and denoting by $\widetilde{q}^1 : G^{(r+1)}(\varphi_1) \longrightarrow \delta_0^0$ the Hermitian orthogonal projection we have

$$(5.8) \quad \widetilde{q}^1 \circ A_{(1,0)}^{G^{(r)}(\varphi_1), G^{(r+1)}(\varphi_1)}(w) = A_{(1,0)}^{\delta_{r+1}^0 \oplus \beta_{r+1}^0, \delta_0^0} \circ P_1^r(w), \qquad w \in C^\infty(\mu_r^1).$$

By the definition, $A_{(1,0)}^{\beta_{r+1}^0, \delta_0^0}$ is surjective, hence, by (5.8) we see that $\widetilde{q}^1 \mid_{\mu_{r+1}^1} : \mu_{r+1}^1 \longrightarrow \delta_0^0$ is surjective, which implies that rank$\hat{\mu}_{r+1} =$ rankδ_0^0 and rank$\hat{\nu}_{r+1} = 1$. Set $\alpha_0^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} \mid_{\mu_{r+1}^1})$, $\alpha_i^1 = G_{\varphi_1}^{(i)}(\alpha_0^1)$ for $i = 1, \cdots, r+1$. Recall that

$$\alpha_r^1 \subset G^{(r)}(\varphi_1) \subset \gamma_r^0 \oplus \alpha_{r+1}^0 \oplus R'_0 \oplus \alpha_0^0,$$
$$\alpha_{r+1}^1 \subset G^{(r+1)}(\varphi_1) \subset R'_{01} \oplus \delta_0^0 \oplus \beta_0^0 \quad \text{and} \quad \delta_0^0 \subset \gamma_0^0 \oplus \alpha_1^0.$$

Denote by $\widetilde{r}^1 : G^{(r+1)}(\varphi_1) \longrightarrow \alpha_1^0$ and $r_1 : \alpha_r^1 \longrightarrow \alpha_0^0$ the Hermitian orthogonal projections. Then, by (5.1) we have

$$\widetilde{r}^1 \circ A_{(1,0)}^{G^{(r)}(\varphi_1),G^{(r+1)}(\varphi_1)}(w) = A_{(1,0)}^{\alpha_0^0,\alpha_1^0} \circ r_1(w)\,, \qquad w \in C^\infty(\alpha_r^1)\,. \tag{5.9}$$

Suppose that r_1 is surjective. Then, it follows from (5.9) that $\widetilde{r}^1 |_{\alpha_{r+1}^1} : \alpha_{r+1}^1 \longrightarrow \alpha_1^0$ is surjective, which, together with the isomorphicity of $P_0 : \delta_0^0 \longrightarrow \alpha_1^0$ and the fact $(R'_{01} \oplus \beta_0^0) \perp \alpha_1^0$, implies that $\widetilde{q}^1 |_{\alpha_{r+1}^1} : \alpha_{r+1}^1 \longrightarrow \delta_0^0$ is surjective, hence $\mathrm{rank}\widetilde{P}^1(\alpha_{r+1}^1) = \mathrm{rank}\hat{\mu}_{r+1}$ and $\widetilde{P}^1(\alpha_{r+1}^1) = \hat{\mu}_{r+1}$. Then, it follows from (5.7) that $P_1^0(\alpha_0^1) = P_1^0(\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} |_{\alpha_{r+1}^1}))$, hence $\alpha_0^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} |_{\alpha_{r+1}^1})$ because P_1^0 is an isomorphism. However, this contradicts the nilpotency of A_{r+1,φ_1}. Therefore, we have proved that r_1 is not surjective, hence $r_1 \equiv 0$ by $\mathrm{rank}\alpha_0^0 = 1$. By (5.9), we obtain $\widetilde{r}^1 |_{\alpha_{r+1}^1} \equiv 0$, which, together with the facts that $P_0 : \delta_0^0 \longrightarrow \alpha_1^0$ is an isomorphism and $(R'_{01} \oplus \beta_0^0) \perp \alpha_1^0$, yields $\widetilde{q}^1 |_{\alpha_{r+1}^1} \equiv 0$, hence $\widetilde{P}^1(\alpha_{r+1}^1) \subset \beta_0^0$. However, since $\widetilde{P}^1(\alpha_{r+1}^1) \subset \hat{\mu}_{r+1}$, and $\hat{\mu}_{r+1}$ does not have β_0^0 as a proper subbundle by the facts that $\mathrm{rank}\beta_0^0 = 1$, $\mathrm{rank}\hat{\mu}_{r+1} = \mathrm{rank}\delta_0^0$ and $\widetilde{q}^1 |_{\mu_{r+1}^1}$ is surjective, we see that $\widetilde{P}^1(\alpha_{r+1}^1) \equiv 0$. Therefore, it follows from (5.7) and the isomorphicity of P_1^0 that $\alpha_{r+1}^1 \subset \mathrm{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, so that $A_{r+1,\varphi_1}^3 \equiv 0$. We treat two possibilities of φ_1 separately.

(1) *The case of* $A_{r+1,\varphi_1}^2 \equiv 0$. If $\mu_0^1 = \underline{0}$, by the isomorphicity of P_1^0, we have $\delta_1^0 \oplus \beta_1^0 = \underline{0}$, hence φ_1 is a pluriharmonic map into $\mathbf{C}P^{n-1}$. Hence, we may assume that $\mu_0^1 \neq \underline{0}$. Since $\mu_{r+1}^1 \subset \mathrm{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, set $\mu_{r+2}^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1)} |_{\mu_{r+1}^1})$ then $\mu_{r+2}^1 \subset (R'_{01} \oplus \delta_0^0 \oplus \beta_0^0) \ominus G^{(r+1)}(\varphi_1)$. Set $R'_1 = ((R'_{01} \oplus \delta_0^0 \oplus \beta_0^0) \ominus G^{(r+1)}(\varphi_1)) \ominus \mu_{r+2}^1$. We have a diagram

$$\begin{tikzcd}
\gamma_0^1 \arrow[r] \arrow[dr] & \gamma_1^1 \arrow[rr, dashed] \arrow[dr] & & \gamma_{r+1}^1 \arrow[r] \arrow[dr] \arrow[dlll] & R'_1 \arrow[r] \arrow[dr] & \gamma_0^1 \\
\mu_0^1 \arrow[r] \arrow[u] & \mu_1^1 \arrow[rr, dashed] \arrow[u] & {} \arrow[r] & \mu_{r+1}^1 \arrow[r] \arrow[u] & \mu_{r+2}^1 \arrow[r] \arrow[u] \arrow[ur] & \mu_0^1 \arrow[u]
\end{tikzcd} \tag{5.10}$$

where $\gamma_i^1 = G^{(i)}(\varphi_1) \ominus \mu_i^1$ $(0 \leq i \leq r+1)$. Recall that $\mathrm{rank}\gamma_0^1 = 1$. Hence, by (5.10) we see that $A_{(1,0)}^{\mu_{r+2}^1,\gamma_0^1} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1 \ominus \mu_0^1) \oplus \mu_1^1$ then

$$\underline{\varphi}_2 = \gamma_0^1 \oplus \mu_1^1 \ , \quad G^{(i)}(\varphi_2) = \gamma_i^1 \oplus \mu_{i+1}^1 \ (1 \leq i \leq r+1), \quad G^{(r+2)}(\varphi_2) \subset R_1' \oplus \mu_0^1 \ ,$$

thus, φ_2 has ∂'-isotropy order $\geq r+2$. Note that rank$\underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+2)}(\varphi_2),\varphi_2} = \mathrm{rank}\underline{\varphi}_2 - 1$.

(2) *The case of* $A^2_{r+1,\varphi_1} \not\equiv 0$. Set $\gamma_i^1 = G^{(i)}(\varphi_1) \ominus \mu_i^1$ and $\beta_i^1 = \mu_i^1 \ominus \alpha_i^1$ for $i = 0, 1, \cdots, r+1$, where $G^{(0)}(\varphi_1) = \underline{\varphi}_1$. Since $\alpha^1_{r+1} \subset \mathrm{Ker} A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, set $\alpha^1_{r+2} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1)} |_{\alpha^1_{r+1}})$ then $\alpha^1_{r+2} \subset (R'_{01} \oplus \delta_0^0 \oplus \beta_0^0) \ominus G^{(r+1)}(\varphi_1)$. Set $R_1' = ((R'_{01} \oplus \delta_0^0 \oplus \beta_0^0) \ominus G^{(r+1)}(\varphi_1)) \ominus \alpha^1_{r+2}$. We have the same diagram as (5.1), where we must replace the upper index 0 by 1, r by $r+1$ and R_0' by R_1', and we denote by $(5.1)_1$ the new diagram. Since rank$\gamma_0^1 = 1$, we obtain $A_{(1,0)}^{\alpha^1_{r+2},\gamma_0^1} \equiv 0$. Set $\underline{\varphi}_1^1 = (\underline{\varphi}_1 \ominus \alpha_0^1) \oplus \alpha_1^1$, then by $(5.1)_1$ we have

$$(5.11) \quad \underline{\varphi}_1^1 = \gamma_0^1 \oplus \beta_0^1 \oplus \alpha_1^1 \ , \qquad G^{(i)}(\varphi_1^1) = \gamma_i^1 \oplus \beta_i^1 \oplus \alpha^1_{i+1} \ (1 \leq i \leq r+1),$$
$$G^{(r+2)}(\varphi_1^1) \subset R_1' \oplus \beta_0^1 \oplus \alpha_0^1 \ ,$$

so that

$$\beta_0^1 = \underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+1)}(\varphi_1^1),\varphi_1^1} \ , \qquad \beta_i^1 = G^{(i)}_{\varphi_1^1}(\beta_0^1) \ (1 \leq i \leq r+1)$$
$$\text{and} \qquad \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1^1)} |_{\beta^1_{r+1}}) \subset R_1' \oplus \alpha_0^1 \ .$$

Set $\beta^1_{r+2} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1^1)} |_{\beta^1_{r+1}})$ and $R_{11} = (R_1' \oplus \alpha_0^1) \ominus \beta^1_{r+2}$. Again, we have the same diagram as (5.3), where we must replace the upper index 0 by 1, r by $r+1$ and R_{01} by R_{11}, and we denote by $(5.3)_1$ the new diagram. By $(5.3)_1$, we see that $A_{(1,0)}^{\beta^1_{r+2},\hat{\alpha}_0^1} : \beta^1_{r+2} \longrightarrow \hat{\alpha}_0^1$ can not be surjective, hence rank$\underline{\mathrm{Im}} A_{(1,0)}^{\beta^1_{r+2},\hat{\alpha}_0^1} \leq \mathrm{rank}\hat{\alpha}_0^1 - 1$. Set $\delta_0^1 = \underline{\mathrm{Im}} A_{(1,0)}^{\beta^1_{r+2},\hat{\alpha}_0^1}$ and $\delta_i^1 = G^{(i)}_{\varphi_1^1}(\delta_0^1) \cap \hat{\alpha}_i^1$ for $i = 1, \cdots, r+1$. Denote by $P_1 : \delta_0^1 \longrightarrow \alpha_1^1$ and $P^1 : \beta^1_{r+2} \longrightarrow \alpha_0^1$ the Hermitian orthogonal projections. We obtain

$$(5.12) \qquad P_1 \circ A_{(1,0)}^{\beta^1_{r+2},\hat{\alpha}_0^1}(v) = A_{(1,0)}^{\alpha_0^1,\alpha_1^1} \circ P^1(v) \ , \qquad v \in C^\infty(\beta^1_{r+2}) \ .$$

Since $A_{(1,0)}^{\beta^1_{r+1},\alpha_0^1}$ is surjective, it follows that P^1 is surjective, which, together with (5.12) and the surjectivity of $A_{(1,0)}^{\alpha_0^1,\alpha_1^1}$, implies that P_1 is surjective, hence rank$\delta_0^1 \geq$

rankα_1^1 = rank$\hat{\alpha}_0^1 - 1$. Thus, we have proved that rankδ_0^1 = rank$\hat{\alpha}_0^1 - 1$ and P_1 is an isomorphism. By (5.6), we see that

$$\delta_{r+1}^1 \subset \gamma_{r+1}^1 \oplus \alpha_{r+2}^1 \subset R'_{01} \oplus \delta_0^0 \oplus \beta_0^0 \ .$$

Recall that $\delta_0^0 \oplus \beta_0^0 = \hat{\mu}_{r+1} \oplus \hat{\nu}_{r+1}$, rank$\hat{\nu}_{r+1} = 1$, $\mu_0^1 = \alpha_0^1 \oplus \beta_0^1$ and $P_1^0 : \mu_0^1 \longrightarrow \delta_1^0 \oplus \beta_1^0$ is a holomorphic isomorphism. Note that $\hat{\mu}_{r+1}$ is a holomorphic subbundle of $\delta_0^0 \oplus \beta_0^0$. By (5.7), we have $P_1^0(\alpha_0^1) = \underline{\mathrm{Im}}(A_{(1,0)}^{\delta_0^0 \oplus \beta_0^0, \delta_1^0 \oplus \beta_1^0} \mid_{\hat{\mu}_{r+1}})$, which is a holomorphic subbundle of $\delta_1^0 \oplus \beta_1^0$. Set $\hat{\beta}_0^1 = (\delta_1^0 \oplus \beta_1^0) \ominus P_1^0(\alpha_0^1)$, and denote by $\hat{P}_1^0 : \beta_0^1 \longrightarrow \hat{\beta}_0^1$ the composition of $P_1^0 \mid_{\beta_0^1} : \beta_0^1 \longrightarrow \delta_1^0 \oplus \beta_1^0$ and the Hermitian orthogonal projection $: \delta_1^0 \oplus \beta_1^0 \longrightarrow \hat{\beta}_0^1$. Then, $\hat{P}_1^0$ is a holomorphic isomorphism. Moreover, we see that $A_{(1,0)}^{\delta_0^0 \oplus \beta_0^0, \hat{\beta}_0^1} \mid_{\hat{\nu}_{r+1}} : \hat{\nu}_{r+1} \longrightarrow \hat{\beta}_0^1$ is holomorphic and surjective. We obtain

$$\hat{P}_1^0 \circ A_{(1,0)}^{\delta_{r+1}^1, \beta_0^1}(v) = A_{(1,0)}^{\delta_0^0 \oplus \beta_0^0, \hat{\beta}_0^1} \circ \widetilde{P}_{\hat{\nu}}^1(v) \ , \qquad v \in C^\infty(\delta_{r+1}^1) \ , \tag{5.13}$$

where $\widetilde{P}_{\hat{\nu}}^1 : \delta_{r+1}^1 \longrightarrow \hat{\nu}_{r+1}$ is the Hermitian orthogonal projection. Now, suppose that $\widetilde{P}_{\hat{\nu}}^1$ is surjective. Then, (5.13), together with the isomorphicity of $\hat{P}_1^0$ and the surjectivity of $A_{(1,0)}^{\delta_0^0 \oplus \beta_0^0, \hat{\beta}_0^1} \mid_{\hat{\nu}_{r+1}}$, implies that $A_{(1,0)}^{\delta_{r+1}^1, \beta_0^1} : \delta_{r+1}^1 \longrightarrow \beta_0^1$ is surjective. However, by $(5.3)_1$ we see that $A_{(1,0)}^{\delta_{r+1}^1, \beta_0^1}$ can not be surjective, hence a contradiction. Therefore, we have $\hat{P}_{\hat{\nu}}^1 \equiv 0$, and by (5.13) we obtain $A_{(1,0)}^{\delta_{r+1}^1, \beta_0^1} \equiv 0$. Hence, set $\delta_{r+2}^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1^1)} \mid_{\beta_{r+1}^1 \oplus \delta_{r+1}^1}) \ominus \beta_{r+2}^1 \subset R_{11}$ and $R'_{11} = R_{11} \ominus \delta_{r+2}^1$. Then, we have the same diagram as (5.5), where we must replace the upper index 0 by 1, r by $r+1$ and R'_{01} by R'_{11}, and we denote by $(5.5)_1$ the new diagram. Since rank$\hat{\gamma}^1$ = rank$\hat{\alpha}_0^1$ − rank$\delta_0^1 = 1$, it follows from $(5.5)_1$ that $A_{(1,0)}^{\delta_{r+2}^1, \hat{\gamma}_0^1} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1^1 \ominus (\delta_0^1 \oplus \beta_0^1)) \oplus (\delta_1^1 \oplus \beta_1^1)$, then by $(5.5)_1$ we have

$$\underline{\varphi}_2 = \hat{\gamma}_0^1 \oplus \delta_1^1 \oplus \beta_1^1 \ , \qquad G^{(i)}(\varphi_2) = \hat{\gamma}_i^1 \oplus \delta_{i+1}^1 \oplus \beta_{i+1}^1 \ (1 \leq i \leq r+1),$$
$$G^{(r+2)}(\varphi_2) \subset R'_{11} \oplus \delta_0^1 \oplus \beta_0^1 \ ,$$

so that φ_2 has ∂'-isotropy order $\geq r+2$. Note that rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}(\varphi_2), \varphi_2}$ = rank$\underline{\varphi}_2 - 1$.

Next, we treat the second possibility (II).

(II) In this case, we apply the same methods as in section 4 and (I), (2). We frequently utilize them without details. Since $A_{r,\varphi}^2 \equiv 0$, setting $\alpha_0^0 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r)}(\varphi), \varphi}$,

$\alpha_i^0 = G_\varphi^{(i)}(\alpha_0^0)$ for $i = 1, \cdots, r+1$, $\gamma_0^0 = \underline{\varphi} \ominus \alpha_0^0$, $\gamma_i^0 = G^{(i)}(\varphi) \ominus \alpha_i^0$ for $i = 1, \cdots, r$ and $R_0' = (\underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi))) \ominus \alpha_{r+1}^0$, we obtain the diagram (4.2). There are already three possibilities : (II-1) rank$\alpha_0^0 = 2$, (II-2) rank$\alpha_0^0 = 1$ and $A_{(1,0)}^{\alpha_{r+1}^0, \gamma_0^0} \equiv 0$, (II-3) rank$\alpha_0^0 = \text{rank}\underline{\text{Im}} A_{(1,0)}^{\alpha_{r+1}^0, \gamma_0^0} = 1$.

(II-1) Since rank$\gamma_0^0 = 1$, we have $A_{(1,0)}^{\alpha_{r+1}^0, \gamma_0^0} \equiv 0$. Set $\underline{\varphi}_1 = (\underline{\varphi} \ominus \alpha_0^0) \oplus \alpha_1^0$, then

$$\underline{\varphi}_1 = \gamma_0^0 \oplus \alpha_1^0 , \qquad G^{(i)}(\varphi_1) = \gamma_i^0 \oplus \alpha_{i+1}^0 \ (1 \le i \le r), \qquad G^{(r+1)}(\varphi_1) \subset R_0' \oplus \alpha_0^0 ,$$

hence φ_1 has ∂'-isotropy order $\ge r+1$. We show that $A_{r+1,\varphi_1}^3 \equiv 0$. Set $\mu_0^1 = \underline{\text{Im}} A_{(1,0)}^{G^{(r+1)}(\varphi_1), \varphi_1}$, $\mu_i^1 = G_{\varphi_1}^{(i)}(\mu_0^1)$ for $i = 1, \cdots, r+1$. Denote by $P_1^0 : \mu_0^1 \longrightarrow \alpha_1^0$ and $\widetilde{P}^1 : G^{(r+1)}(\varphi_1) \longrightarrow \alpha_0^0$ the Hermitian orthogonal projections. We may use (4.4), where we must replace P_1 by P_1^0. (4.4) and the nilpotency of A_{r+1,φ_1} imply that rank$\mu_0^1 = \text{rank}\alpha_1^0 = \text{rank}\underline{\varphi}_1 - 1$ and $\widetilde{P}^1 |_{\mu_{r+1}^1} : \mu_{r+1}^1 \longrightarrow \alpha_0^0$ can not be surjective. Set $\hat{\mu}_{r+1} = \widetilde{P}^1(\mu_{r+1}^1)$ then rank$\hat{\mu}_{r+1} \le 1$. Set $\alpha_0^1 = \underline{\text{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1), \varphi_1} |_{\mu_{r+1}^1})$, $\alpha_i^1 = G_{\varphi_1}^{(i)}(\alpha_0^1)$ for $i = 1, \cdots, r+1$. If $\widetilde{P}^1(\alpha_{r+1}^1) = \hat{\mu}_{r+1}$, by (4.4) we have $P_1^0(\alpha_0^1) = P_1^0(\underline{\text{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1), \varphi_1} |_{\alpha_{r+1}^1}))$, which, together with the isomorphicity of P_1^0, yields $\alpha_0^1 = \underline{\text{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1), \varphi_1} |_{\alpha_{r+1}^1})$, which contradicts the nilpotency of A_{r+1,φ_1}. Thus, we obtain $\widetilde{P}^1 |_{\alpha_{r+1}^1} \equiv 0$, hence by (4.4) we see that $\alpha_{r+1}^1 \subset \text{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1), \varphi_1}$, that is, $A_{r+1,\varphi_1}^3 \equiv 0$. Again, we have two possibilities.

(1) *The case of* $A_{r+1,\varphi_1}^2 \equiv 0$. If $\mu_0^1 = \underline{0}$, then $\alpha_1^0 = \underline{0}$ and φ_1 is a pluriharmonic map into $\mathbf{C}P^{n-1}$. Hence, we may assume that $\mu_0^1 \ne \underline{0}$. Since $\mu_{r+1}^1 \subset \text{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1), \varphi_1}$, set

$$\begin{aligned} \mu_{r+2}^1 &= \underline{\text{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1)} |_{\mu_{r+1}^1}) \subset (R_0' \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1) , \\ R_1' &= ((R_0' \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1)) \ominus \mu_{r+2}^1 . \end{aligned}$$

Then, we have the diagram (5.10), where rank$\gamma_0^1 = 1$. By (5.10), we see that $A_{(1,0)}^{\mu_{r+2}^1, \gamma_0^1} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1 \ominus \mu_0^1) \oplus \mu_1^1$ then

$$\underline{\varphi}_2 = \gamma_0^1 \oplus \mu_1^1 , \qquad G^{(i)}(\varphi_2) = \gamma_i^1 \oplus \mu_{i+1}^1 \ (1 \le i \le r+1), \qquad G^{(r+2)}(\varphi_2) \subset R_1' \oplus \mu_0^1 ,$$

hence φ_2 has ∂'-isotropy order $\ge r+2$. Note that rank$\underline{\text{Im}} A_{(1,0)}^{G^{(r+2)}(\varphi_2), \varphi_2} = \text{rank}\underline{\varphi}_2 - 1$.

(2) *The case of* $A^2_{r+1,\varphi_1} \not\equiv 0$. Recall that $\widetilde{P}^1 |_{\alpha^1_{r+1}} \equiv 0$ and $\alpha^1_{r+1} \subset \mathrm{Ker} A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$, so that $\alpha^1_{r+1} \subset R'_0$ and $\alpha^1_{r+2} = \underline{\underline{\mathrm{Im}}}(A^{G^{(r+1)}(\varphi_1)}_{(1,0)} |_{\alpha^1_{r+1}}) \subset R'_0 \oplus \alpha^0_0$. Set $\gamma^1_i = G^{(i)}(\varphi_1) \ominus \mu^1_i$, $\beta^1_i = \mu^1_i \ominus \alpha^1_i$ for $i = 0, 1, \cdots, r+1$ and $R'_1 = ((R'_0 \oplus \alpha^0_0) \ominus G^{(r+1)}(\varphi_1)) \ominus \alpha^1_{r+2}$, where $G^{(0)}(\varphi_1) = \underline{\varphi}_1$. Then, we have the diagram $(5.1)_1$. Recall that $\mathrm{rank}\gamma^1_0 = \mathrm{rank}\underline{\varphi}_1 - \mathrm{rank}\mu^1_0 = 1$. Hence, by $(5.1)_1$ we obtain $A^{\alpha^1_{r+2},\gamma^1_0}_{(1,0)} \equiv 0$. Set $\underline{\varphi}^1_1 = (\underline{\varphi}_1 \ominus \alpha^1_0) \oplus \alpha^1_1$, then we have

$$\underline{\varphi}^1_1 = \gamma^1_0 \oplus \beta^1_0 \oplus \alpha^1_1 , \qquad G^{(i)}(\varphi^1_1) = \gamma^1_i \oplus \beta^1_i \oplus \alpha^1_{i+1} \; (1 \le i \le r+1),$$
$$G^{(r+2)}(\varphi^1_1) \subset R'_1 \oplus \beta^1_0 \oplus \alpha^1_0 ,$$

so that $\beta^1_0 = \underline{\mathrm{Im}} A^{G^{(r+1)}(\varphi^1_1),\varphi^1_1}_{(1,0)}$, $\beta^1_i = G^{(i)}_{\varphi^1_1}(\beta^1_0)\,(1 \le i \le r+1)$ and $\beta^1_{r+2} = \underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi^1_1)}_{(1,0)} |_{\beta^1_{r+1}}) \subset R'_1 \oplus \alpha^1_0$. Moreover, set $R_{11} = (R'_1 \oplus \alpha^1_0) \ominus \beta^1_{r+2}$. Then, we have the diagram $(5.3)_1$. Set $\delta^1_0 = \underline{\mathrm{Im}} A^{\beta^1_{r+2},\hat{\alpha}^1_0}_{(1,0)}$, then $\mathrm{rank}\delta^1_0 \le \mathrm{rank}\hat{\alpha}^1_0 - 1$. Since $P^1 : \beta^1_{r+2} \longrightarrow \alpha^1_0$ is surjective, it follows that $\mathrm{rank}\delta^1_0 = \mathrm{rank}\alpha^1_1 = \mathrm{rank}\hat{\alpha}^1_0 - 1$ and $P_1 : \delta^1_0 \longrightarrow \alpha^1_1$ is an isomorphism. Set $\delta^1_i = G^{(i)}_{\varphi^1_1}(\delta^1_0) \cap \hat{\alpha}^1_i$ and $\hat{\gamma}^1_i = \hat{\alpha}^1_i \ominus \delta^1_i$ for $i = 0, 1, \cdots, r+1$. We show that $A^{\delta^1_{r+1},\beta^1_0}_{(1,0)} \equiv 0$. We may verify that

$$\delta^1_{r+1} \subset \gamma^1_{r+1} \oplus \alpha^1_{r+2} \subset R'_0 \oplus \alpha^0_0 .$$

Set $\alpha^0_0 = \hat{\mu}_{r+1} \oplus \hat{\nu}_{r+1}$, where $\hat{\mu}_{r+1} = \widetilde{P}^1(\mu^1_{r+1})$ and $\mathrm{rank}\hat{\mu}_{r+1} = \mathrm{rank}\hat{\nu}_{r+1} = 1$ in this case. We have $P^0_1(\alpha^1_0) = \underline{\mathrm{Im}}(A^{\alpha^0_0,\alpha^0_1}_{(1,0)} |_{\hat{\mu}_{r+1}})$ which is a holomorphic subbundle of α^0_1. Set $\hat{\beta}^1_0 = \alpha^0_1 \ominus P^0_1(\alpha^1_0)$, and denote by $\hat{P}^0_1 : \beta^1_0 \longrightarrow \hat{\beta}^1_0$ the composition of $P^0_1 |_{\beta^1_0} : \beta^1_0 \longrightarrow \alpha^0_1$ and the Hermitian orthogonal projection : $\alpha^0_1 \longrightarrow \hat{\beta}^1_0$. Then, $\hat{P}^0_1$ is a holomorphic isomorphism. Moreover, we see that $A^{\alpha^0_0,\hat{\beta}^1_0}_{(1,0)} |_{\hat{\nu}_{r+1}} : \hat{\nu}_{r+1} \longrightarrow \hat{\beta}^1_0$ is holomorphic and surjective. We have

$$(5.14) \qquad \hat{P}^0_1 \circ A^{\delta^1_{r+1},\beta^1_0}_{(1,0)}(v) = A^{\alpha^0_0,\hat{\beta}^1_0}_{(1,0)} \circ \widetilde{P}^1_{\hat{\nu}}(v) , \qquad v \in C^\infty(\delta^1_{r+1}) ,$$

where $\widetilde{P}^1_{\hat{\nu}} : \delta^1_{r+1} \longrightarrow \hat{\nu}_{r+1}$ is the Hermitian orthogonal projection. Now, suppose that $\widetilde{P}^1_{\hat{\nu}}$ is surjective. Then, (5.14), together with the isomorphicity of $\hat{P}^0_1$ and the surjectivity of $A^{\alpha^0_0,\hat{\beta}^1_0}_{(1,0)}$, implies that $A^{\delta^1_{r+1},\beta^1_0}_{(1,0)} : \delta^1_{r+1} \longrightarrow \beta^1_0$ is surjective. However, by

$(5.3)_1$ we see that $A_{(1,0)}^{\delta_{r+1}^1,\beta_0^1}$ can not be surjective, hence a contradiction. Thus, we have proved that $\widetilde{P}_{\hat{\nu}}^1 \equiv 0$ and $A_{(1,0)}^{\delta_{r+1}^1,\beta_0^1} \equiv 0$. Therefore, set

$$\delta_{r+2}^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1^1)} |_{\beta_{r+1}^1 \oplus \delta_{r+1}^1}) \ominus \beta_{r+2}^1 \subset R_{11} \ , \qquad R_{11}' = R_{11} \ominus \delta_{r+2}^1 \ .$$

We have the diagram $(5.5)_1$. By $(5.5)_1$, we see that $A_{(1,0)}^{\delta_{r+2}^1,\hat{\gamma}_0^1} \equiv 0$, because $\mathrm{rank}\hat{\gamma}_0^1 = \mathrm{rank}\hat{\alpha}_0^1 - \mathrm{rank}\delta_0^1 = 1$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1^1 \ominus (\delta_0^1 \oplus \beta_0^1)) \oplus (\delta_1^1 \oplus \beta_1^1)$, then we have

$$\underline{\varphi}_2 = \hat{\gamma}_0^1 \oplus \delta_1^1 \oplus \beta_1^1 \ , \qquad G^{(i)}(\varphi_2) = \hat{\gamma}_i^1 \oplus \delta_{i+1}^1 \oplus \beta_{i+1}^1 \ (1 \le i \le r+1),$$
$$G^{(r+2)}(\varphi_2) \subset R_{11}' \oplus \delta_0^1 \oplus \beta_0^1 \ ,$$

hence φ_2 has ∂'-isotropy order $\ge r+2$. Note that $\mathrm{rank}\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}(\varphi_2),\varphi_2} = \mathrm{rank}\underline{\varphi}_2 - 1$.

(II-2) Set $\underline{\varphi}_1 = (\underline{\varphi} \ominus \alpha_0^0) \oplus \alpha_1^0$ then

$$\underline{\varphi}_1 = \gamma_0^0 \oplus \alpha_1^0 \ , \qquad G^{(i)}(\varphi_1) = \gamma_i^0 \oplus \alpha_{i+1}^0 \ (1 \le i \le r), \qquad G^{(r+1)}(\varphi_1) \subset R_0' \oplus \alpha_0^0 \ ,$$

hence φ_1 has ∂'-isotropy order $\ge r+1$. Set $\mu_0^1 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, $\mu_i^1 = G_{\varphi_1}^{(i)}(\mu_0^1)$ for $i = 1, \cdots, r+1$. The nilpotency of A_{r+1,φ_1} yields $\mathrm{rank}\mu_0^1 \le \mathrm{rank}\underline{\varphi}_1 - 1$. We may use (4.4). It follows from (4.4) that $P_1^0 : \mu_0^1 \longrightarrow \alpha_1^0$ is surjective, hence $\mathrm{rank}\mu_0^1 \ge \mathrm{rank}\alpha_1^0 = \mathrm{rank}\underline{\varphi}_1 - 2$. If $\mu_0^1 = \underline{0}$ then φ_1 is a pluriharmonic map into $\mathbf{C}P^{n-1}$ or $G_2(\mathbf{C}^n)$, hence we may assume that $\mu_0^1 \ne \underline{0}$. Thus, we have $\mathrm{rank}\mu_0^1 = m-1, m-2$, where $m = \mathrm{rank}\underline{\varphi}_1$. We show that $A_{r+1,\varphi_1}^3 \equiv 0$. Set

$$\alpha_0^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} |_{\mu_{r+1}^1}) \ , \qquad \alpha_i^1 = G_{\varphi_1}^{(i)}(\alpha_0^1) \quad \text{for } \ i = 1, \cdots, r+1 \ .$$

First, assume that $\mathrm{rank}\mu_0^1 = m-1$. If $\widetilde{P}^1 |_{\mu_{r+1}^1} : \mu_{r+1}^1 \longrightarrow \alpha_0^0$ is surjective, then by (4.4) we see that $P_1^0 |_{\alpha_0^1}$ is surjective, which implies that $\mathrm{rank}\alpha_0^1 = m-2$ and $P_1^0 |_{\alpha_0^1}$ is an isomorphism. Moreover, if $\widetilde{P}^1 |_{\alpha_{r+1}^1} : \alpha_{r+1}^1 \longrightarrow \alpha_0^0$ is also surjective, by (4.4) we have $P_1^0(\underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} |_{\alpha_{r+1}^1})) = P_1^0(\alpha_0^1)$, which is a contradiction. Hence, $\widetilde{P}^1 |_{\alpha_{r+1}^1} \equiv 0$ by $\mathrm{rank}\alpha_0^0 = 1$, and $\alpha_{r+1}^1 \subset \mathrm{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$ by (4.4), so that $A_{r+1,\varphi_1}^3 \equiv 0$. If $\widetilde{P}^1 |_{\mu_{r+1}^1} \equiv 0$, by (4.4) we get $P_1^0 |_{\alpha_0^1} \equiv 0$. Since μ_0^1 does not have γ_0^0 as a proper subbundle, we conclude that $\mathrm{rank}\alpha_0^1 \le \mathrm{rank}\gamma_0^0 - 1 = 1$. Hence, we must have $\alpha^1_{r+1} \subset \mathrm{Ker} A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$, that is, $A^3_{r+1,\varphi_1} \equiv 0$. Next, assume that $\mathrm{rank}\mu^1_0 = m-2$. In this case, obviously, $\widetilde{P}^1\mid_{\mu^1_{r+1}}$ can not be surjective, hence $\widetilde{P}^1\mid_{\mu^1_{r+1}} \equiv 0$, which yields $P^0_1\mid_{\alpha^1_0} \equiv 0$. However, since $\mathrm{rank}\mu^1_0 = \mathrm{rank}\alpha^0_1$, it follows that P^0_1 is an isomorphism, so that $\alpha^1_0 = \underline{0}$. In particular, we have proved that if $\mathrm{rank}\mu^1_0 = m-2$ then $A^2_{r+1,\varphi_1} \equiv 0$. We treat these possibilities separately.

(1) *The case of* $A^2_{r+1,\varphi_1} \equiv 0$. Since $\mu^1_{r+1} \subset \mathrm{Ker} A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$, set

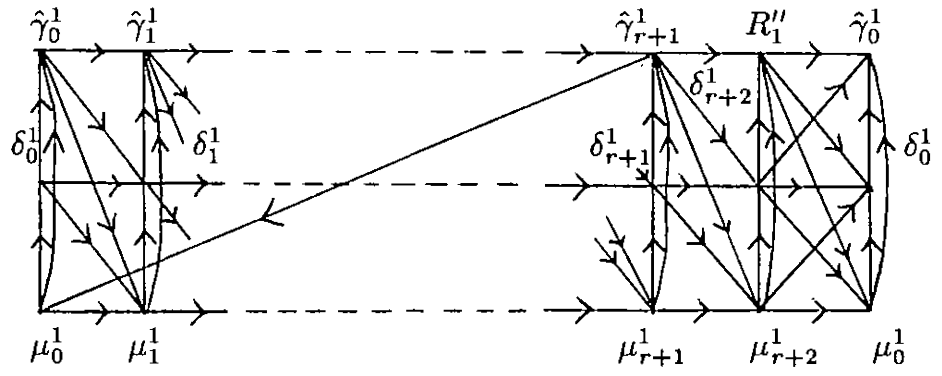
$$\begin{aligned} &\mu^1_{r+2} = \underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi_1)}_{(1,0)} \mid_{\mu^1_{r+1}}) \subset (R'_0 \oplus \alpha^0_0) \ominus G^{(r+1)}(\varphi_1)\ , \\ \text{and} \qquad &R'_1 = ((R'_0 \oplus \alpha^0_0) \ominus G^{(r+1)}(\varphi_1)) \ominus \mu^1_{r+2}\ . \end{aligned}$$

Then, we have the diagram (5.10), where $\mathrm{rank}\gamma^1_0 = 1, 2$. First, assume that $\mathrm{rank}\gamma^1_0 = 1$. This is just the same situation as (II-1), (1). Therefore, set $\underline{\varphi}_2 = (\underline{\varphi}_1 \ominus \mu^1_0) \oplus \mu^1_1$, then φ_2 has ∂'-isotropy order $\geq r+2$. Next, assume that $\mathrm{rank}\gamma^1_0 = 2$. Recall that $P^0_1 : \mu^1_0 \longrightarrow \alpha^0_1$ is an isomorphism. Set $\delta^1_0 = \underline{\mathrm{Im}} A^{\mu^1_{r+2},\gamma^1_0}_{(1,0)}$ then $\mathrm{rank}\delta^1_0 \leq 1$. Set $\delta^1_i = G^{(i)}_{\varphi_1}(\delta^1_0) \cap \gamma^1_i$ for $i = 1, \cdots, r+1$. If $\delta^1_0 = \underline{0}$, we only set $\underline{\varphi}_2 = (\underline{\varphi}_1 \ominus \mu^1_0) \oplus \mu^1_1$, thus we may assume that $\mathrm{rank}\delta^1_0 = 1$. We verify that

$$\delta^1_{r+1} \subset G^{(r+1)}(\varphi_1) \subset R'_0 \oplus \alpha^0_0\ .$$

We may use (4.4). Suppose that $\widetilde{P}^1\mid_{\delta^1_{r+1}} : \delta^1_{r+1} \longrightarrow \alpha^0_0$ is surjective. By (4.4), we see that $\underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} \mid_{\delta^1_{r+1}}) = \mu^1_0$, which is a contradiction because $A^{\delta^1_{r+1},\mu^1_0}_{(1,0)}$ can not be surjective by (5.10). Therefore, we have proved that $\widetilde{P}^1\mid_{\delta^1_{r+1}} \equiv 0$ and $\delta^1_{r+1} \subset \mathrm{Ker} A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$. Set $\delta^1_{r+2} = \underline{\mathrm{Im}}(A^{G^{(r+1)}(\varphi_1)}_{(1,0)} \mid_{\mu^1_{r+1} \oplus \delta^1_{r+1}}) \ominus \mu^1_{r+2} \subset R'_1$ and $R''_1 = R'_1 \ominus \delta^1_{r+2}$. Then, we have a diagram

(5.15)



where $\hat{\gamma}_i^1 = \gamma_i^1 \ominus \delta_i^1$ $(0 \leq i \leq r+1)$. Since rank$\hat{\gamma}_0^1 = 1$, we obtain $A_{(1,0)}^{\delta_{r+2}^1, \hat{\gamma}_0^1} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1 \ominus (\delta_0^1 \oplus \mu_0^1)) \oplus (\delta_1^1 \oplus \mu_1^1)$ then φ_2 has ∂'-isotropy order $\geq r+2$. We may regard the case of $\delta_0^1 = \underline{0}$ as a special case of this procedure. Note that rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}(\varphi_2),\varphi_2} = \mathrm{rank}\underline{\varphi}_2 - m$, where $m = 1, 2$.

(2) *The case of* $A_{r+1,\varphi_1}^2 \not\equiv 0$. Since $\alpha_{r+1}^1 \subset \mathrm{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, set

$$\alpha_{r+2}^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1)}\,|_{\alpha_{r+1}^1}) \subset (R_0' \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1)$$
$$\text{and} \qquad R_1' = ((R_0' \oplus \alpha_0^0) \ominus G^{(r+1)}(\varphi_1)) \ominus \alpha_{r+2}^1\ .$$

Moreover, set $\gamma_i^1 = G^{(i)}(\varphi_1) \ominus \mu_i^1$ and $\beta_i^1 = \mu_i^1 \ominus \alpha_i^1$ for $i = 0, 1, \cdots, r+1$. Then, we have the diagram $(5.1)_1$. We already know that rank$\gamma_0^1 = 1$, and that either rank$\beta_0^1 = 1$ or rank$\alpha_0^1 = 1$ holds according as $\widetilde{P}^1\,|_{\mu_{r+1}^1} \longrightarrow \alpha_0^0$ is surjective or not. It follows from $(5.1)_1$ and the fact rank$\gamma_0^1 = 1$ that $A_{(1,0)}^{\alpha_{r+2}^1,\gamma_0^1} \equiv 0$. Set $\underline{\varphi}_1^1 = (\underline{\varphi}_1 \ominus \alpha_0^1) \oplus \alpha_1^1$ then

$$\underline{\varphi}_1^1 = \gamma_0^1 \oplus \beta_0^1 \oplus \alpha_1^1\ , \qquad G^{(i)}(\varphi_1^1) = \gamma_i^1 \oplus \beta_i^1 \oplus \alpha_{i+1}^1\ (1 \leq i \leq r+1),$$
$$G^{(r+2)}(\varphi_1^1) \subset R_1' \oplus \beta_0^1 \oplus \alpha_0^1\ ,$$

hence $\beta_0^1 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\varphi_1^1),\varphi_1^1}$, $\beta_i^1 = G_{\varphi_1^1}^{(i)}(\beta_0^1)$ $(1 \leq i \leq r+1)$, and set $\beta_{r+2}^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1^1)}\,|_{\beta_{r+1}^1}) \subset R_1' \oplus \alpha_0^1$, $R_{11} = (R_1' \oplus \alpha_0^1) \ominus \beta_{r+2}^1$. We have the diagram $(5.3)_1$. Set $\delta_0^1 = \underline{\mathrm{Im}}A_{(1,0)}^{\beta_{r+2}^1,\hat{\alpha}_0^1}$, $\delta_i^1 = G_{\varphi_1^1}^{(i)}(\delta_0^1) \cap \hat{\alpha}_i^1$ for $i = 1, \cdots, r+1$. Observe that

$$\mathrm{rank}\delta_0^1 = \mathrm{rank}\alpha_1^1 = \mathrm{rank}\hat{\alpha}_0^1 - 1$$
$$\text{and} \qquad \delta_{r+1}^1 \subset \gamma_{r+1}^1 \oplus \alpha_{r+2}^1 \subset R_0' \oplus \alpha_0^0\ .$$

Denote by $\widetilde{P}_1^1 : \delta_{r+1}^1 \longrightarrow \alpha_0^0$ the Hermitian orthogonal projection. We show that $\delta_{r+1}^1 \subset \mathrm{Ker}A_{(1,0)}^{G^{(r+1)}(\varphi_1^1),\varphi_1^1}$. First, assume that $\widetilde{P}^1\,|_{\mu_{r+1}^1}$ is surjective, so that rank$\beta_0^1 = 1$. In this case, obviously, $A_{(1,0)}^{\delta_{r+1}^1,\beta_0^1} \equiv 0$. Next, assume that $\widetilde{P}^1\,|_{\mu_{r+1}^1}$ is not surjective, so that $\widetilde{P}^1\,|_{\mu_{r+1}^1} \equiv 0$, $P_1^0\,|_{\alpha_0^1} \equiv 0$, $\alpha_0^1 \subset \gamma_0^0$ and rank$\alpha_0^1 = 1$. Hence, $P_1^0\,|_{\beta_0^1} \colon \beta_0^1 \longrightarrow \alpha_1^0$ is an isomorphism. Suppose that $\widetilde{P}_1^1$ is surjective. We have

$$P_1^0 \circ A_{(1,0)}^{\delta_{r+1}^1,\beta_0^1}(v) = A_{(1,0)}^{\alpha_0^0,\alpha_1^0} \circ \widetilde{P}_1^1(v)\,, \qquad v \in C^\infty(\delta_{r+1}^1)\,, \tag{5.16}$$

which implies that $A_{(1,0)}^{\delta_{r+1}^1,\beta_0^1} : \delta_{r+1}^1 \longrightarrow \beta_0^1$ is surjective. However, this is a contradiction by $(5.3)_1$, hence, we see that $\widetilde{P}_1^1 \equiv 0$, and $A_{(1,0)}^{\delta_{r+1}^1,\beta_0^1} \equiv 0$ by (5.16). Therefore, $\delta_{r+1}^1 \subset \mathrm{Ker} A_{(1,0)}^{G^{(r+1)}(\varphi_1^1),\varphi_1^1}$. Set $\delta_{r+2}^1 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r+1)}(\varphi_1^1)}\,|_{\beta_{r+1}^1 \oplus \delta_{r+1}^1}) \ominus \beta_{r+2}^1 \subset R_{11}$ and $R'_{11} = R_{11} \ominus \delta_{r+2}^1$. Then, we have the diagram $(5.5)_1$, where $\mathrm{rank}\hat{\gamma}_0^1 = 1$. By $(5.5)_1$, we obtain $A_{(1,0)}^{\delta_{r+2}^1,\hat{\gamma}_0^1} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1^1 \ominus (\delta_0^1 \oplus \beta_0^1)) \oplus (\delta_1^1 \oplus \beta_1^1)$ then φ_2 has ∂'-isotropy order $\geq r+2$. Note that $\mathrm{rank}\underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+2)}(\varphi_2),\varphi_2} = \mathrm{rank}\underline{\varphi}_2 - 1$.

Finally, we treat (II-3).

(II-3) This case has the same type as that of φ_0^1 in (I). To compare this case with φ_0^1 in (I), reset α_i^0 by β_i^0 for $i = 0, 1, \cdots, r+1$. Set $\delta_0^0 = \underline{\mathrm{Im}} A_{(1,0)}^{\beta_{r+1}^0,\gamma_0^0}$, $\delta_i^0 = G_\varphi^{(i)}(\delta_0^0) \cap \gamma_i^0$ for $i = 1, \cdots, r$, and set $\hat{\gamma}_i^0 = \gamma_i^0 \ominus \delta_i^0$ for $i = 0, 1, \cdots, r+1$. Now, we have

$$\mathrm{rank}\beta_0^0 = \mathrm{rank}\delta_0^0 = \mathrm{rank}\hat{\gamma}_0^0 = 1\,.$$

Hence, $A_{(1,0)}^{\delta_r^0,\beta_0^0} \equiv 0$, and hence set $\delta_{r+1}^0 = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi)}\,|_{\beta_r^0 \oplus \delta_r^0}) \ominus \beta_{r+1}^0 \subset R'_0$ and $R'_{01} = R'_0 \ominus \delta_{r+1}^0$. Then, we have the diagram (5.5). One will find that the treatment of this case is rather easier than those of φ_0^1 in (I). We state only the essential parts. We use the same notation as in (I). Since $\mathrm{rank}\hat{\gamma}_0^0 = 1$, we obtain $A_{(1,0)}^{\delta_{r+1}^0,\hat{\gamma}_0^0} \equiv 0$. Set $\underline{\varphi}_1 = (\underline{\varphi} \ominus (\delta_0^0 \oplus \beta_0^0)) \oplus (\delta_1^0 \oplus \beta_1^0)$ then φ_1 has ∂'-isotropy order $\geq r+1$. It follows that $\mathrm{rank}\hat{\mu}_{r+1} = 1$, so that $\widetilde{P}^1(\alpha_{r+1}^1) \equiv 0$. (5.7) and the isomorphicity of P_1^0 imply that $\alpha_{r+1}^1 \subset \mathrm{Ker} A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$, so that $A_{r+1,\varphi_1}^3 \equiv 0$.

(1) *The case of* $A_{r+1,\varphi_1}^2 \equiv 0$. If $\mu_0^1 = \underline{0}$, then φ_1 is a pluriharmonic map into $\mathbf{C}P^{n-1}$, hence we may assume that $\mu_0^1 \neq \underline{0}$. We see that $A_{(1,0)}^{\mu_{r+2}^1,\gamma_0^1} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}_1 \ominus \mu_0^1) \oplus \mu_1^1$ then φ_2 has ∂'-isotropy order $\geq r+2$. Note that $\mathrm{rank}\underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+1)}(\varphi_2),\varphi_2} = \mathrm{rank}\underline{\varphi}_2 - 1$.

(2) *The case of* $A_{r+1,\varphi_1}^2 \not\equiv 0$. Since $\mathrm{rank}\gamma_0^1 = 1$, we obtain $A_{(1,0)}^{\alpha_{r+2}^1,\gamma_0^1} \equiv 0$. Set $\underline{\varphi}_1^1 = (\underline{\varphi}_1 \ominus \alpha_0^1) \oplus \alpha_1^1$ then we have (5.11) and $(5.3)_1$. Since $P^1 : \delta_0^1 \longrightarrow \alpha_0^1$ is surjective, it follows from (5.12) that $P_1 : \delta_0^1 \longrightarrow \alpha_1^1$ is surjective, hence $\mathrm{rank}\delta_0^1 = \mathrm{rank}\hat{\alpha}_0^1 - 1$ and P_1 is an isomorphism. By (5.6), we see that $\delta_{r+1}^1 \subset \gamma_{r+1}^1 \oplus \alpha_{r+2}^1 \subset R'_{01} \oplus \delta_0^0 \oplus \beta_0^0$,

where $\alpha^1_{r+2} \subset R'_{01} \oplus \delta^0_0 \oplus \beta^0_0$ because $\alpha^1_{r+1} \subset \mathrm{Ker}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)}$. Set $\delta^0_0 \oplus \beta^0_0 = \hat{\mu}_{r+1} \oplus \hat{\nu}_{r+1}$, where $\mathrm{rank}\hat{\mu}_{r+1} = \mathrm{rank}\hat{\nu}_{r+1} = 1$. Suppose that $\widetilde{P}^1_{\hat{\nu}} : \delta^1_{r+1} \longrightarrow \hat{\nu}_{r+1}$ is surjective. Then, by (5.13) we see that $A^{\delta^1_{r+1},\beta^1_0}_{(1,0)} : \delta^1_{r+1} \longrightarrow \beta^1_0$ is surjective because $\hat{P}^0_1$ is an isomorphism. However, this contradicts the diagram $(5.3)_1$. Since $\mathrm{rank}\hat{\gamma}^1_0 = 1$, we obtain $A^{\delta^1_{r+2},\hat{\gamma}^1_0}_{(1,0)} \equiv 0$. Set $\underline{\varphi}_2 = (\underline{\varphi}^1_1 \ominus (\delta^1_0 \oplus \beta^1_0)) \oplus (\delta^1_1 \oplus \beta^1_1)$, then φ_2 has ∂'-isotropy order $\geq r+2$. Note that $\mathrm{rank}\underline{\mathrm{Im}}A^{G^{(r+2)}(\varphi_2),\varphi_2}_{(1,0)} = \mathrm{rank}\underline{\varphi}_2 - 1$.

In summary, we have

Proposition 5.1. *Let* $\varphi : M \setminus S_\varphi \longrightarrow G_3(\mathbf{C}^n)$ *be a pluriharmonic map. Assume that* φ *has* ∂'*-isotropy order* r. *Then,* $A^3_{r,\varphi} \equiv 0$.

(I) If $A^2_{r,\varphi} \not\equiv 0$, *set* $\alpha^0 = \underline{\mathrm{Im}}A^2_{r,\varphi}$. *Then,* $\alpha^0 \subset \mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^{\varphi}_{(1,0)})$, *and define* φ^1 *from* φ *by the forward replacement of* α^0. *Then,* φ^1 *has* ∂'*-isotropy order* r *and satisfies* $A^2_{r,\varphi^1} \equiv 0$. *Set* $\beta^0 = \underline{\mathrm{Im}}A_{r,\varphi^1}$ *and* $\delta^0 = \underline{\mathrm{Im}}A^{G^{(r+1)}_{\varphi^1}(\beta^0),\varphi^1\ominus\beta^0}_{(1,0)}$, *then* $\beta^0 \oplus \delta^0 \subset \mathrm{Ker}(A^{(\varphi^1)^\perp}_{(1,0)} \circ A^{\varphi^1}_{(1,0)})$. *Define* φ_1 *from* φ^1 *by the forward replacement of* $\beta^0 \oplus \delta^0$, *then* φ_1 *has* ∂'*-isotropy order* $\geq r+1$ *and satisfies* $A^3_{r+1,\varphi_1} \equiv 0$.

(II) If $A^2_{r,\varphi} \equiv 0$, *set* $\alpha^0 = \underline{\mathrm{Im}}A_{r,\varphi}$ *and* $\delta^0 = \underline{\mathrm{Im}}A^{G^{(r+1)}_{\varphi}(\alpha^0),\varphi\ominus\alpha^0}_{(1,0)}$. *Then,* α^0, $\delta^0 \subset \mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^{\varphi}_{(1,0)})$, $\mathrm{rank}\alpha^0 = 1, 2$, *and* $\mathrm{rank}\delta^0 = 0, 1$.

(II-1) If $\mathrm{rank}\alpha^0 = 2$, *then* $\delta^0 = \underline{0}$ *and define* φ_1 *from* φ *by the forward replacement of* α^0,

(II-2) if $\mathrm{rank}\alpha^0 = 1$ *and* $\delta^0 = \underline{0}$, *then define also* φ_1 *from* φ *by the forward replacement of* α^0,

(II-3) if $\mathrm{rank}\alpha^0 = \mathrm{rank}\delta^0 = 1$, *then define* φ_1 *from* φ *by the forward replacement of* $\alpha^0 \oplus \delta^0$.

Then, φ_1 *has* ∂'*-isotropy order* $\geq r+1$ *and satisfies* $A^3_{r+1,\varphi_1} \equiv 0$.

Moreover, for each φ_1 *in (I), (II), the following are true :*

(0) If $A_{r+1,\varphi_1} \equiv 0$, φ_1 *is a pluriharmonic map into* $\mathbf{C}P^{n-1}$ *or* $G_2(\mathbf{C}^n)$ *(the latter case occurs only for (II-2)).*

(1) If $A^2_{r+1,\varphi_1} \equiv 0$ *and* $A_{r+1,\varphi_1} \not\equiv 0$, *set*

$$\mu^1 = \underline{\mathrm{Im}}A_{r+1,\varphi_1} \qquad \textit{and} \qquad \delta^1 = \underline{\mathrm{Im}}A^{G^{(r+2)}_{\varphi_1}(\mu^1),\varphi_1\ominus\mu^1}_{(1,0)} .$$

Then, μ^1, $\delta^1 \subset \mathrm{Ker}(A^{\varphi_1^\perp}_{(1,0)} \circ A^{\varphi_1}_{(1,0)})$ *and* $\mathrm{rank}\delta^1 = 0, 1$ *(the latter case occurs only for (II-2)). Define* φ_2 *from* φ_1 *by the forward replacement of* $\mu^1 \oplus \delta^1$, *then* φ_2 *has*

∂'-isotropy order $\geq r+2$ and satisfies rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}(\varphi_2),\varphi_2} = \mathrm{rank}\underline{\varphi}_2 - m$, *where $m = 1, 2$ (the latter case occurs only for (II-2)).*

(2) If $A^2_{r+1,\varphi_1} \not\equiv 0$, set $\alpha^1 = \underline{\mathrm{Im}}A^2_{r+1,\varphi_1}$. Then, $\alpha^1 \subset \mathrm{Ker}(A^{\varphi_1^\perp}_{(1,0)} \circ A^{\varphi_1}_{(1,0)})$, and define φ_1^1 from φ_1 by the forward replacement of α^1. Then, φ_1^1 has ∂'-isotropy order $r+1$ and satisfies $A^2_{r+1,\varphi_1^1} \equiv 0$. Set $\beta^1 = \underline{\mathrm{Im}}A_{r+1,\varphi_1^1}$ and $\delta^1 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}_{\varphi_1^1}(\beta^1),\varphi_1^1\ominus\beta^1}$, then β^1, $\delta^1 \subset \mathrm{Ker}(A^{(\varphi_1^1)^\perp}_{(1,0)} \circ A^{\varphi_1^1}_{(1,0)})$. Define φ_2 from φ_1^1 by the forward replacement of $\beta^1 \oplus \delta^1$, then φ_2 has ∂'-isotropy order $\geq r+2$ and satisfies rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}(\varphi_2),\varphi_2} =$ rank$\underline{\varphi}_2 - 1$.

Using Proposition 5.1, we may prove the following

Theorem 5.1. *Let $\varphi : M \setminus S_\varphi \longrightarrow G_3(\mathbf{C}^n)$ be a pluriharmonic map. Assume that φ has finite ∂'-isotropy order and $n \leq 15$. Then, there is a sequence $\{\varphi_i\}_{i=0}^N$ of pluriharmonic maps such that*
(1) $\varphi_0 = \varphi$,   (2) $\varphi_N : M \setminus S_{\varphi_N} \longrightarrow \mathbf{C}P^{n-1}$ or $G_2(\mathbf{C}^n)$,
(3) for $i = 0, 1, \cdots, N-1$, each φ_i has finite ∂'-isotropy order, and φ_{i+1} is obtained from φ_i by the forward replacement of α^i, where α^i is a holomorphic subbundle of $\underline{\varphi}_i$ contained in $\mathrm{Ker}(A^{\varphi_i^\perp}_{(1,0)} \circ A^{\varphi_i}_{(1,0)})$.

Proof. Construct φ_2 from φ, using Proposition 5.1. Let r be the ∂'-isotropy order of φ_2. Then, $r \geq 3$. Set $\alpha_0 = \underline{\mathrm{Im}}A_{r,\varphi_2}$, $\alpha_i = G^{(i)}_{\varphi_2}(\alpha_0)$ for $i = 1, \cdots, r$ and $\gamma_0 = \underline{\varphi}_2 \ominus \alpha_0$, $\gamma_i = G^{(i)}(\varphi_2) \ominus \alpha_i$ for $i = 1, \cdots, r$. By Proposition 5.1, we have rank$\gamma_0 = m$, and rank$\alpha_0 =$ rank$\underline{\varphi}_2 - m$, where $m = 1, 2$. If $\alpha_0 = \underline{0}$, then φ_2 is a pluriharmonic map into $\mathbf{C}P^{n-1}$ or $G_2(\mathbf{C}^n)$, hence we may assume that $\alpha_0 \neq \underline{0}$. Set $R = \underline{\varphi}_2^\perp \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi_2))$. We have a diagram

(5.17)
$$\begin{tikzcd}
\gamma_0 \arrow[r] \arrow[ddr] & \gamma_1 \arrow[r, dashed] \arrow[ddr] & \cdots \arrow[r, dashed] \arrow[ddr] & \gamma_r \arrow[ddlll] \arrow[ddrr] & & \gamma_0 \\
 & & & & R & \\
\alpha_0 \arrow[uu] \arrow[r] & \alpha_1 \arrow[uu] \arrow[r, dashed] & \cdots \arrow[r, dashed] & \alpha_r \arrow[uu] \arrow[uurr] \arrow[lll, bend left] & & \alpha_0 \arrow[uu]
\end{tikzcd}$$

We have two possibilities : (1) $\alpha_i = \underline{0}$ for some $1 \leq i \leq r$, (2) any α_i $(1 \leq i \leq r)$ is non-zero.
(1) Set $\tilde{\varphi} = (\underline{\varphi}_2 \ominus \alpha_0) \oplus \alpha_1$. Then, by (5.17) we see that either, $\tilde{\varphi}$ is a pluriharmonic map into $\mathbf{C}P^{n-1}$ or $G_2(\mathbf{C}^n)$, or $\tilde{\varphi}$ has ∂'-isotropy order $r+1$ and rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\tilde{\varphi}),\tilde{\varphi}}$

$= \mathrm{rank}\underline{\widetilde{\varphi}} - m$, where $m = 1, 2$.
(2) Since $n \leq 15$, one of $\underline{\varphi}_2$, $G^{(i)}(\varphi_2)$ $(1 \leq i \leq r)$ has rank ≤ 3 and ∂'-isotropy order r. Hence, by Proposition 5.1, either, we have a pluriharmonic map into $\mathbf{C}P^{n-1}$ or $G_2(\mathbf{C}^n)$, or we have a pluriharmonic map $\widetilde{\varphi}$ which has ∂'-isotropy order $r+2$ and satisfies $\mathrm{rank}\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+2)}(\tilde{\varphi}),\tilde{\varphi}} = \mathrm{rank}\underline{\widetilde{\varphi}} - m$, where $m = 1, 2$.
Since the ∂'-isotropy order can not be so large, repeating this procedure we see that φ is reduced to a pluriharmonic map into $\mathbf{C}P^{n-1}$ or $G_2(\mathbf{C}^n)$, and each φ_i in the sequence has the desired properties by Proposition 2.3. q.e.d.

6. Pluriharmonic maps into $G_4(\mathbf{C}^n)$.

Let $\varphi : M \setminus S_\varphi \longrightarrow G_4(\mathbf{C}^n)$ be a pluriharmonic map, where M is a compact complex manifold with $c_1(M) > 0$. We also assume that φ has finite ∂'-isotropy order, say r. In this section, we present a method for increasing the ∂'-isotropy order of φ by only one. However, the result of this section, together with the results in sections 3 $\sim$ 5, yields the explicit construction of any pluriharmonic map into $G_4(\mathbf{C}^n)$ under the restriction on n.
Define $A_{r,\varphi}$ as in section 5, then $A_{r,\varphi}$ is nilpotent. There are three possibilities :
(I) $A^4_{r,\varphi} \equiv 0$ and $A^3_{r,\varphi} \not\equiv 0$, (II) $A^3_{r,\varphi} \equiv 0$ and $A^2_{r,\varphi} \not\equiv 0$, (III) $A^2_{r,\varphi} \equiv 0$.
As in section 5, we treat these three cases separately.

Firat of all, we prepare a proposition, which is used to avoid the repetition of argument and also useful for the future investigation.

Proposition 6.1 ([O-U2]). *Let $\varphi : M \setminus S_\varphi \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map. Assume that φ has finite ∂'-isotropy order, say r, and satisfies $A^2_{r,\varphi} \equiv 0$,* $\mathrm{rank}\underline{\mathrm{Im}}A_{r,\varphi} = 1$. *Then, there is a holomorphic subbundle τ of $\underline{\varphi}$, which is contained in* $\mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^{\varphi}_{(1,0)})$, *such that $\widetilde{\varphi}$ defined from φ by the forward replacement of τ has ∂'-isotropy order $\geq r+1$.*

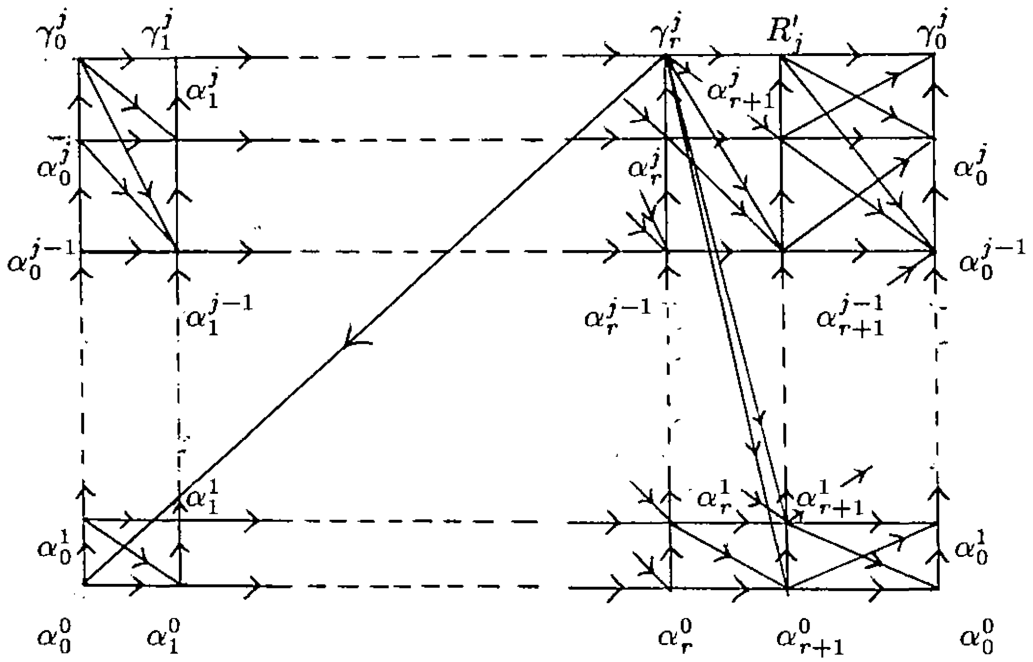
Proof. Set $\alpha^0_0 = \underline{\mathrm{Im}}A_{r,\varphi}$, $\alpha^0_i = G^{(i)}_\varphi(\alpha^0_0)$ for $i = 1, \cdots, r$. Since $\alpha^0_r \subset \mathrm{Ker}A^{G^{(r)}(\varphi),\varphi}_{(1,0)}$, set $\alpha^0_{r+1} = \underline{\mathrm{Im}}(A^{G^{(r)}(\varphi)}_{(1,0)}|_{\alpha^0_r}) \subset R$, where $R = \underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi))$. Set $\gamma^0_i = G^{(i)}(\varphi) \ominus \alpha^0_i$ for $i = 0, 1, \cdots, r$ and $R'_0 = R \ominus \alpha^0_{r+1}$. Then, we have the diagram (4.2). Thus, $A^{\alpha^0_{r+1},\gamma^0_0}_{(1,0)}$ is holomorphic, and hence set $\alpha^1_0 = \underline{\mathrm{Im}}A^{\alpha^0_{r+1},\gamma^0_0}_{(1,0)}$. If $\alpha^1_0 \neq \underline{0}$, set $\alpha^1_i = G^{(i)}_\varphi(\alpha^1_0) \cap \gamma^0_i$ for $i = 1, \cdots, r$. Then, we have $\alpha^1_r \subset \mathrm{Ker}A^{G^{(r)}(\varphi),\varphi}_{(1,0)}$ by $\mathrm{rank}\alpha^0_0 = 1$. Moreover, if we set $\alpha^1_{r+1} = \underline{\mathrm{Im}}(A^{G^{(r)}(\varphi)}_{(1,0)}|_{\alpha^0_r \oplus \alpha^1_r}) \ominus \alpha^0_{r+1} \subset R'_0$, $R'_1 = R'_0 \ominus \alpha^1_{r+1}$, and $\gamma^1_i = \gamma^0_i \ominus \alpha^1_i$ for $i = 0, 1, \cdots, r$, then we see that $A^{\alpha^1_i,\alpha^1_{i+1}}_{(1,0)}$ $(0 \leq i \leq r)$

and $A_{(1,0)}^{\alpha^1_{r+1},\gamma^1_0}$ are all holomorphic. We claim that

Claim. If $A_{(1,0)}^{\alpha^j_{r+1},\gamma^j_0}$ is holomorphic and $\underline{\mathrm{Im}}A_{(1,0)}^{\alpha^j_{r+1},\gamma^j_0} \neq \underline{0}$, then set $\alpha_0^{j+1} = \underline{\mathrm{Im}}A_{(1,0)}^{\alpha^j_{r+1},\gamma^j_0}$, $\alpha_i^{j+1} = G_\varphi^{(i)}(\alpha_0^{j+1}) \cap \gamma_i^j$ for $i = 1, \cdots, r$. Then, we have $\alpha_r^{j+1} \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi),\varphi}$. Moreover, if we set $\alpha_{r+1}^{j+1} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi)} |_{(\bigoplus_{k=0}^{j+1} \alpha_r^k)}) \ominus (\bigoplus_{k=0}^{j} \alpha_{r+1}^k) \subset R'_j$, $R'_{j+1} = R'_j \ominus \alpha_{r+1}^{j+1}$ and $\gamma_i^{j+1} = \gamma_i^j \ominus \alpha_i^{j+1}$ for $i = 0, 1, \cdots, r$, then we see that $A_{(1,0)}^{\alpha_i^{j+1},\alpha_{i+1}^{j+1}}$ $(0 \leq i \leq r)$ and $A_{(1,0)}^{\alpha^{j+1}_{r+1},\gamma^{j+1}_0}$ are all holomorphic.

This *Claim* follows from the induction on j and the following diagram

(6.1)



where we omit the non-essential arrays (see *Convention* below). By (6.1) and *Claim*, we see that, for any $j = 0, 1, \cdots$, $A_{(1,0)}^{\alpha^j_{r+1},\gamma^j_0}$ can not be surjective, hence there exists an nonnegative integer s such that $A_{(1,0)}^{\alpha^s_{r+1},\gamma^s_0} \equiv 0$. Set $\tau = \bigoplus_{j=0}^{s} \alpha_0^j$ and define $\widetilde{\varphi}$ by $\underline{\widetilde{\varphi}} = (\underline{\varphi} \ominus \tau) \oplus G'_\varphi(\tau)$. Then, it follows from (6.1) that

$$\underline{\widetilde{\varphi}} = \gamma_0^s \oplus (\bigoplus_{j=0}^{s} \alpha_1^j) \,, \quad G^{(i)}(\widetilde{\varphi}) = \gamma_i^s \oplus (\bigoplus_{j=0}^{s} \alpha_{i+1}^j) \; (1 \leq i \leq r) \,,$$

$$G^{(r+1)}(\widetilde{\varphi}) \subset R'_s \oplus (\bigoplus_{j=0}^{s} \alpha_0^j) \,,$$

hence $\widetilde{\varphi}$ has ∂'-isotropy order $\geq r+1$. q.e.d.

Hereafter, we use the following convention for simplicity

Convention. Given a diagram of type (6.1), we omit the non-essential arrays, where there are the arrays in the following cases :
(1) $\alpha_i^k \longrightarrow \alpha_i^l$ $(0 \leq i \leq r+1; 0 \leq k < l \leq j)$,
(2) $\alpha_i^m \longrightarrow \gamma_i^j$ $(0 \leq m \leq j; 0 \leq i \leq r+1)$, (3) $\alpha_{r+1}^m \longrightarrow R'_j$ $(0 \leq m \leq j)$,
(4) $\alpha_i^l \longrightarrow \alpha_{i+1}^k$ $(0 \leq i \leq r+1; 0 \leq k < l \leq j)$ with $\alpha_{r+2}^k = \alpha_0^k$,
(5) $\gamma_i^j \longrightarrow \alpha_{i+1}^m$ $(0 \leq i \leq r+1; 0 \leq m \leq j)$ with $\gamma_{r+1}^j = R'_j$, $\alpha_{r+2}^m = \alpha_0^m$.

Now, we start from the case (I).
(I) Set $\kappa_1 = \underline{\mathrm{Im}}A_{r,\varphi}$, $\kappa_2 = \underline{\mathrm{Im}}(A_{r,\varphi}\,|_{\kappa_1})$ and $\alpha_0 = \underline{\mathrm{Im}}(A_{r,\varphi}\,|_{\kappa_2})$. Then, $\alpha_0 \subset \mathrm{Ker}A_{r,\varphi}$. Set

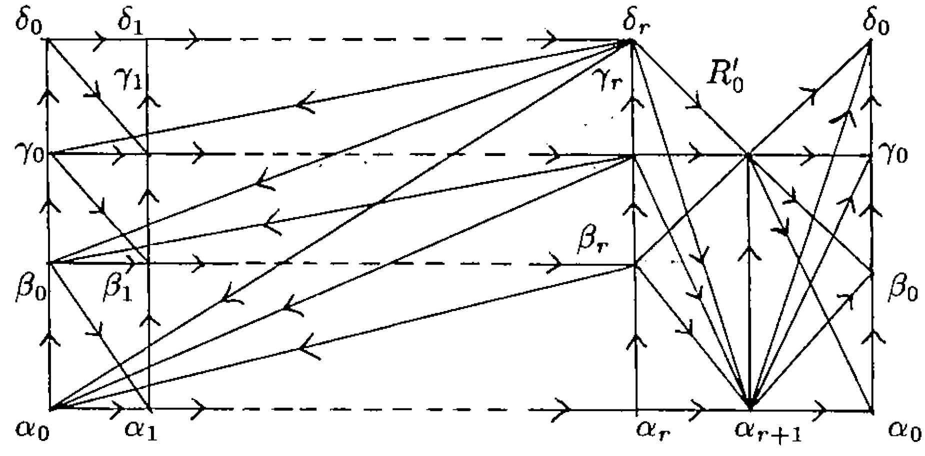
$$\alpha_i = G_\varphi^{(i)}(\alpha_0)\;(1 \leq i \leq r)\,,\quad \beta_0 = \kappa_2 \ominus \alpha_0\,,\quad \beta_i = G_\varphi^{(i)}(\kappa_2) \ominus \alpha_i\;(1 \leq i \leq r),$$

$$\gamma_0 = \kappa_1 \ominus \kappa_2\,,\quad \gamma_i = G_\varphi^{(i)}(\kappa_1) \ominus G_\varphi^{(i)}(\kappa_2)\;(1 \leq i \leq r)\,,$$

$$\delta_0 = \underline{\varphi} \ominus \kappa_1\,,\quad \delta_i = G^{(i)}(\varphi) \ominus G_\varphi^{(i)}(\kappa_1)\;(1 \leq i \leq r)\,,\quad R = \underline{\varphi}^{\perp} \ominus \Big(\bigoplus_{j=0}^{r} G^{(j)}(\varphi)\Big)\,.$$

Then, $\alpha_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi),\varphi}$, and hence set $\alpha_{r+1} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi)}\,|_{\alpha_r}) \subset R$ and $R'_0 = R \ominus \alpha_{r+1}$. Then, we have a diagram

(6.2)



By (6.2), and keeping *Convention* in mind, we have a holomorphic circuit

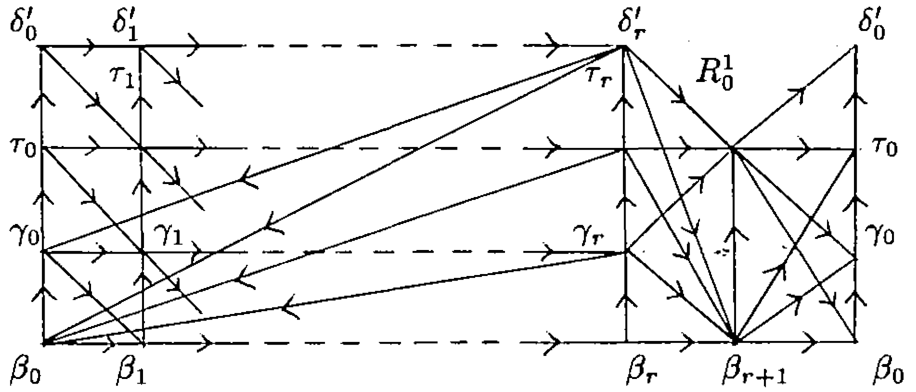
$$\{\alpha_0, \alpha_1, \cdots, \alpha_{r+1}, \delta_0, \delta_1, \cdots, \delta_r, \gamma_0, \gamma_1, \cdots, \gamma_r, \beta_0, \beta_1, \cdots, \beta_r, \alpha_0\}\,,$$

which must vanish, where we note that $\mathrm{rank}\alpha_0 = \mathrm{rank}\beta_0 = \mathrm{rank}\gamma_0 = \mathrm{rank}\delta_0 = 1$. Since each (1,0)-second fundamental form from δ_0 to α_0 is surjective, we obtain $A^{\alpha_{r+1},\delta_0}_{(1,0)} \equiv 0$. Set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \alpha_0) \oplus \alpha_1$ then

$$\underline{\varphi}^1 = \delta_0 \oplus \gamma_0 \oplus \beta_0 \oplus \alpha_1 \ , \quad G^{(i)}(\varphi^1) = \delta_i \oplus \gamma_i \oplus \beta_i \oplus \alpha_{i+1} \ (1 \le i \le r) \ ,$$
$$G^{(r+1)}(\varphi^1) \subset R'_0 \oplus \gamma_0 \oplus \beta_0 \oplus \alpha_0 \ ,$$

so that $\beta_0 \oplus \gamma_0 = \underline{\mathrm{Im}}A^{G^{(r)}(\varphi^1),\varphi^1}_{(1,0)}$, $\beta_i \oplus \gamma_i = G^{(i)}_{\varphi^1}(\beta_0 \oplus \gamma_0)$ $(1 \le i \le r)$, $\beta_0 = \underline{\mathrm{Im}}A^2_{r,\varphi^1}$, $\beta_i = G^{(i)}_{\varphi^1}(\beta_0)$ $(1 \le i \le r)$, $\beta_r \subset \mathrm{Ker}A^{G^{(r)}(\varphi^1),\varphi^1}_{(1,0)}$. Set $\beta_{r+1} = \underline{\mathrm{Im}}(A^{G^{(r)}(\varphi^1)}_{(1,0)} |_{\beta_r}) \subset R'_0 \oplus \alpha_0$, $R^1_0 = (R'_0 \oplus \alpha_0) \ominus \beta_{r+1}$ and $\hat{\delta}_i = \delta_i \oplus \alpha_{i+1}$ for $i = 0, 1, \cdots, r$. We have the same type diagram as (5.1), and we see that $A^{\beta_{r+1},\hat{\delta}_0}_{(1,0)}$ is holomorphic, hence can not be surjective. Set $\tau_0 = \underline{\mathrm{Im}}A^{\beta_{r+1},\hat{\delta}_0}_{(1,0)}$ then $\mathrm{rank}\tau_0 \le \mathrm{rank}\hat{\delta}_0 - 1$ and, in fact, $\mathrm{rank}\tau_0 = \mathrm{rank}\hat{\delta}_0 - 1$ by the surjectivity of the projection $\tau_0 \longrightarrow \alpha_1$ and the fact $\mathrm{rank}\delta_0 = 1$. Moreover, τ_0 is contained in the kernel of $A^{\hat{\delta}_r,\gamma_0}_{(1,0)} \circ A^{\hat{\delta}_{r-1},\hat{\delta}_r}_{(1,0)} \circ \cdots \circ A^{\hat{\delta}_0,\hat{\delta}_1}_{(1,0)}$. Set $\tau_i = G^{(i)}_{\varphi^1}(\tau_0) \cap \hat{\delta}_i$ $(1 \le i \le r)$, $\delta'_i = \hat{\delta}_i \ominus \tau_i$ $(0 \le i \le r)$. We have a diagram

(6.3)



Note that $\mathrm{rank}\delta'_0 = 1$, which is also true even if $\tau_0 = \underline{0}$. Set $\underline{\varphi}^2 = (\underline{\varphi}^1 \ominus \beta_0) \oplus \beta_1$ then

$$\underline{\varphi}^2 = \delta'_0 \oplus \tau_0 \oplus \gamma_0 \oplus \beta_1 \ , \quad G^{(i)}(\varphi^2) = \delta'_i \oplus \tau_i \oplus \gamma_i \oplus \beta_{i+1} \ (1 \le i \le r) \ ,$$
$$G^{(r+1)}(\varphi^2) \subset R^1_0 \oplus \tau_0 \oplus \gamma_0 \oplus \beta_0 \ ,$$

so that $\tau_0 \oplus \gamma_0 = \underline{\mathrm{Im}}A^{G^{(r)}(\varphi^2),\varphi^2}_{(1,0)}$, $\tau_i \oplus \gamma_i = G^{(i)}_{\varphi^2}(\tau_0 \oplus \gamma_0)$ $(1 \le i \le r)$ and $\tau_r \oplus \gamma_r \subset \mathrm{Ker}A^{G^{(r)}(\varphi^2),\varphi^2}_{(1,0)}$. Set $\mu_i = \tau_i \oplus \gamma_i$ $(0 \le i \le r)$, $\varepsilon_i = \delta'_i \oplus \beta_{i+1}$ $(0 \le i \le r)$ and set $\mu_{r+1} = \underline{\mathrm{Im}}(A^{G^{(r)}(\varphi^2)}_{(1,0)} |_{\mu_r}) \subset R^1_0 \oplus \beta_0$, $R^2_0 = (R^1_0 \oplus \beta_0) \ominus \mu_{r+1}$. We have a diagram

$$\tag{6.4}\begin{tikzcd}
\varepsilon_0 \arrow[r] \arrow[dr] & \varepsilon_1 \arrow[rr, dashed] \arrow[dr] & {} \arrow[dr] & \varepsilon_r \arrow[r] \arrow[dr] \arrow[dlll] & R_0^2 \arrow[r] \arrow[dr] & \varepsilon_0 \\
\mu_0 \arrow[u] \arrow[r] & \mu_1 \arrow[u] \arrow[rr, dashed] & {} & \mu_r \arrow[u] \arrow[r] & \mu_{r+1} \arrow[u] \arrow[r] \arrow[ur] & \mu_0 \arrow[u]
\end{tikzcd}$$

We need the following

Lemma 6.1. *Set* $A = A^{\varepsilon_r,\mu_0}_{(1,0)} \circ A^{\varepsilon_{r-1},\varepsilon_r}_{(1,0)} \circ \cdots \circ A^{\varepsilon_0,\varepsilon_1}_{(1,0)}$ *and* $B = A^{\mu_{r+1},\varepsilon_0}_{(1,0)} \circ A^{\mu_r,\mu_{r+1}}_{(1,0)} \circ \cdots \circ A^{\mu_0,\mu_1}_{(1,0)}$. *Set* $\eta = \underline{\mathrm{Im}}B$ *and* $\nu = \underline{\mathrm{Im}}(A \circ B)$. *Then,* $\mathrm{rank}\eta = \mathrm{rank}\varepsilon_0 - 1$, $\mathrm{rank}\nu = \mathrm{rank}\mu_0 - 1$ *and* $\nu \subset \mathrm{Ker}B$.

Proof. By (6.4), $A \circ B$ is holomorphic, hence nilpotent. Therefore, $A^{\mu_{r+1},\varepsilon_0}_{(1,0)}$ can not be surjective, hence $\mathrm{rank}\underline{\mathrm{Im}}B \leq \mathrm{rank}\varepsilon_0 - 1$. On the other hand, if we denote by $P_1 : \varepsilon_0 \longrightarrow \beta_1$ and $P^0 : \mu_{r+1} \longrightarrow \beta_0$ the Hermitian orthogonal projections, we have

$$P_1 \circ A^{\mu_{r+1},\varepsilon_0}_{(1,0)}(v) = A^{\beta_0,\beta_1}_{(1,0)} \circ P^0(v)\,, \qquad v \in C^\infty(\mu_{r+1})\,. \tag{6.5}$$

Since P^0 is surjective by (6.3) and $A^{\beta_0,\beta_1}_{(1,0)}$ is surjective, it follows from (6.5) that $P_1 |_{\underline{\mathrm{Im}}B}: \underline{\mathrm{Im}}B \longrightarrow \beta_1$ is surjective, hence $\mathrm{rank}\underline{\mathrm{Im}}B \geq \mathrm{rank}\beta_1 = \mathrm{rank}\varepsilon_0 - 1$. Therefore, we have proved that $\mathrm{rank}\underline{\mathrm{Im}}B = \mathrm{rank}\varepsilon_0 - 1$ and $P_1 |_{\underline{\mathrm{Im}}B}$ is an isomorphism. Thus, in case $\underline{\mathrm{Im}}B = \underline{0}$, we have $\beta_1 = \underline{0}$, which contradicts the diagram (6.2) and the assumption $A^3_{r,\varphi} \not\equiv 0$. Hence, we may assume that $\underline{\mathrm{Im}}B \neq \underline{0}$. Denote by $q_{r+1} : \varepsilon_r \longrightarrow \beta_{r+1}$ and $q^0 : \mu_0 \longrightarrow \tau_0$ the Hermitian orthogonal projections. Since $A^{\delta'_r,\tau_0}_{(1,0)} \equiv 0$ by (6.3), we have

$$q^0 \circ A^{\varepsilon_r,\mu_0}_{(1,0)}(w) = A^{\beta_{r+1},\tau_0}_{(1,0)} \circ q_{r+1}(w)\,, \qquad w \in C^\infty(\varepsilon_r)\,. \tag{6.6}$$

Set $\eta_r = G^{(r)}_{\varphi^2}(\underline{\mathrm{Im}}B) \cap \varepsilon_r$. By (6.3), we see that $q_{r+1} |_{\eta_r}: \eta_r \longrightarrow \beta_{r+1}$ is surjective, because $P_1 |_{\underline{\mathrm{Im}}B}$ is an isomorphism and all $A^{\beta_i,\beta_{i+1}}_{(1,0)}$ $(1 \leq i \leq r)$ are surjective. Since $A^{\beta_{r+1},\tau_0}_{(1,0)}$ is also surjective by the definition, it follows from (6.6) that $q^0 |_{\underline{\mathrm{Im}}(A\circ B)}$: $\underline{\mathrm{Im}}(A \circ B) \longrightarrow \tau_0$ is surjective, hence $\mathrm{rank}\nu \geq \mathrm{rank}\tau_0 = \mathrm{rank}\mu_0 - 1$, where we have put $\nu = \underline{\mathrm{Im}}(A \circ B)$. On the other hand, since $A \circ B$ is nilpotent, we must have $\mathrm{rank}\nu \leq \mathrm{rank}\mu_0 - 1$, thus we obtain $\mathrm{rank}\nu = \mathrm{rank}\mu_0 - 1$. We show that $\nu \subset \mathrm{Ker}B$. Set $\nu_{r+1} = G^{(r+1)}_{\varphi^2}(\nu) \subset \mu_{r+1}$. Suppose that $P^0 |_{\nu_{r+1}}: \nu_{r+1} \longrightarrow \beta_0$ is surjective. Then, by (6.5) we see that $P_1(\underline{\mathrm{Im}}(B |_\nu)) = P_1(\underline{\mathrm{Im}}B)$, which, together with the isomorphicity of P_1, implies that $\underline{\mathrm{Im}}(B |_\nu) = \underline{\mathrm{Im}}B$. However, this contradicts the nilpotency of $A \circ B$. Therefore, we have proved that $P^0 |_{\nu_{r+1}}$ is not surjective, hence

$P^0 \mid_{\nu_{r+1}} \equiv 0$ by rank$\beta_0 = 1$. Then, again, by (6.5) we see that $A_{(1,0)}^{\mu_{r+1},\varepsilon_0} \mid_{\nu_{r+1}} \equiv 0$, hence $\nu \subset \mathrm{Ker}B$. q.e.d.

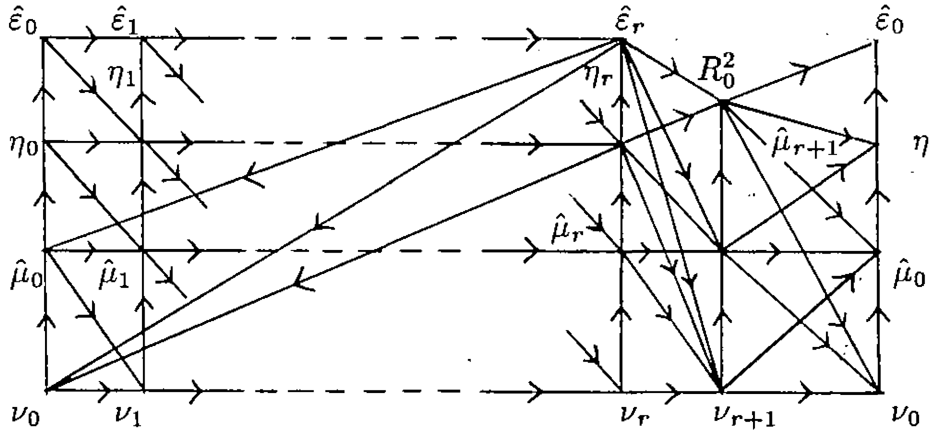
Set

$$\nu_0 = \nu\ ,\ \ \nu_i = G^{(i)}_{\varphi^2}(\nu_0)\ (1 \leq i \leq r+1)\ ,\ \ \hat{\mu}_i = \mu_i \ominus \nu_i\ (0 \leq i \leq r+1)\ ,$$

$$\eta_0 = \underline{\mathrm{Im}}B\ ,\ \ \eta_i = G^{(i)}_{\varphi^2}(\eta_0) \cap \varepsilon_i\ (1 \leq i \leq r)\ ,\ \ \hat{\varepsilon}_i = \varepsilon_i \ominus \eta_i\ (0 \leq i \leq r)\ .$$

Then, we have a diagram

(6.7)



Set $\underline{\varphi}^3 = (\underline{\varphi}^2 \ominus \nu) \oplus G'_{\varphi^2}(\nu)$, where ν is as in Lemme 6.1, then, by (6.7) we have

$$\underline{\varphi}^3 = \hat{\varepsilon}_0 \oplus \eta_0 \oplus \hat{\mu}_0 \oplus \nu_1\ , \qquad G^{(i)}(\varphi^3) = \hat{\varepsilon}_i \oplus \eta_i \oplus \hat{\mu}_i \oplus \nu_{i+1}\ (1 \leq i \leq r)\ ,$$
$$G^{(r+1)}(\varphi^3) \subset R_0^2 \oplus \hat{\mu}_{r+1} \oplus \hat{\mu}_0 \oplus \nu_0\ ,$$

so that $\hat{\mu}_0 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r)}(\varphi^3),\varphi^3}$, rank$\hat{\mu}_0 = 1$, $\hat{\mu}_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi^3),\varphi^3}$, hence φ^3 satisfies the conditions of Proposition 6.1. It follows from (6.1) and (6.7) that there is a holomorphic subbundle τ of $\underline{\varphi}^3$ with $\tau \subset \mathrm{Ker}(A_{(1,0)}^{(\varphi^3)^\perp} \circ A_{(1,0)}^{\varphi^3})$ and rankτ = rank$\underline{\varphi}^3-1$ such that φ_1 defined from φ^3 by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$ = rank$\underline{\varphi}_1 - 1$.

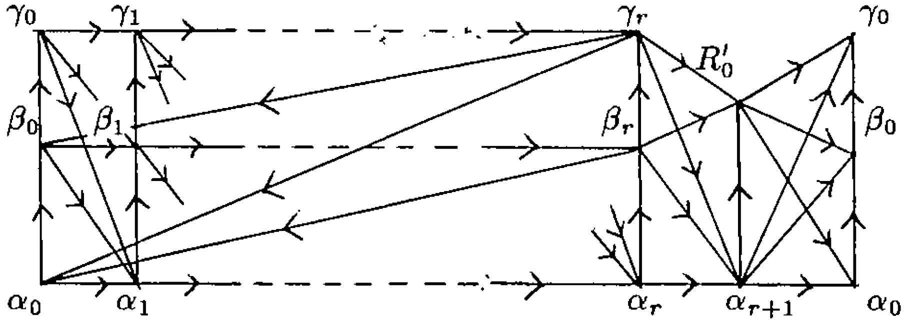
(II) Set $\kappa = \underline{\mathrm{Im}}A_{r,\varphi}$ and $\alpha_0 = \underline{\mathrm{Im}}(A_{r,\varphi} \mid_\kappa)$. Then, $\alpha_0 \subset \mathrm{Ker}A_{r,\varphi}$. Set

$$\alpha_i = G^{(i)}_\varphi(\alpha_0)\ (1 \leq i \leq r)\ ,\ \ \beta_0 = \kappa \ominus \alpha_0\ ,\ \ \beta_i = G^{(i)}_\varphi(\kappa) \ominus \alpha_i\ (1 \leq i \leq r)\ ,$$

$$\gamma_0 = \underline{\varphi} \ominus \kappa\ ,\ \ \gamma_i = G^{(i)}(\varphi) \ominus G^{(i)}_\varphi(\kappa)\ (1 \leq i \leq r)\ ,\ \ R = \underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^{r} G^{(j)}(\varphi))\ .$$

Then, $\alpha_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi),\varphi}$, and hence set $\alpha_{r+1} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi)} \mid_{\alpha_r}) \subset R$ and $R'_0 = R \ominus \alpha_{r+1}$. Then, we have a diagram

(6.8)



There are three possibilities :
(II-1) rankα_0 = rankβ_0 = 1, rankγ_0 = 2, (II-2) rankα_0 = rankγ_0 = 1, rankβ_0 = 2,
(II-3) rankβ_0 = rankγ_0 = 1, rankα_0 = 2.

(II-1) By (6.8), we see that rank$\underline{\mathrm{Im}}A_{(1,0)}^{\alpha_{r+1},\gamma_0} \leq 1$ and $\underline{\mathrm{Im}}A_{(1,0)}^{\alpha_{r+1},\gamma_0}$ is contained in the kernel of $A_{(1,0)}^{\gamma_r,\beta_0} \circ A_{(1,0)}^{\gamma_{r-1},\gamma_r} \circ \cdots \circ A_{(1,0)}^{\gamma_0,\gamma_1}$. First, assume that $A_{(1,0)}^{\alpha_{r+1},\gamma_0} \equiv 0$. Set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \alpha_0) \oplus \alpha_1$, then $\beta_0 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$, rank$\beta_0 = 1$ and $\beta_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$. Therefore, φ^1 satisfies the conditions of Proposition 6.1, and we see that there is a holomorphic subbundle τ of $\underline{\varphi}^1$ with $\tau \subset \mathrm{Ker}(A_{(1,0)}^{(\varphi^1)^\perp} \circ A_{(1,0)}^{\varphi^1})$ and rankτ = rank$\underline{\varphi}^1 - m$, where $m = 1, 2$, such that φ_1 defined from φ^1 by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies rank$\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1}$ = rank$\underline{\varphi}_1 - m$. Next, assume that rank$\underline{\mathrm{Im}}A_{(1,0)}^{\alpha_{r+1},\gamma_0} = 1$. Set $\delta_0 = \underline{\mathrm{Im}}A_{(1,0)}^{\alpha_{r+1},\gamma_0}$, $\delta_i = G_\varphi^{(i)}(\delta_0) \cap \gamma_i$ $(1 \leq i \leq r)$ and $\hat{\gamma}_i = \gamma_i \ominus \delta_i$ $(0 \leq i \leq r)$. We have the same type diagram as (6.3). Set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \alpha_0) \oplus \alpha_1$ then

$$\underline{\varphi}^1 = \hat{\gamma}_0 \oplus \delta_0 \oplus \beta_0 \oplus \alpha_1 \ , \qquad G^{(i)}(\varphi^1) = \hat{\gamma}_i \oplus \delta_i \oplus \beta_i \oplus \alpha_{i+1} \ (1 \leq i \leq r) \ ,$$
$$G^{(r+1)}(\varphi^1) \subset R_0' \oplus \delta_0 \oplus \beta_0 \oplus \alpha_0 \ ,$$

so that $\delta_0 \oplus \beta_0 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$, $\delta_i \oplus \beta_i = G_{\varphi^1}^{(i)}(\delta_0 \oplus \beta_0)$ $(1 \leq i \leq r)$ and $\delta_r \oplus \beta_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$. Set $\mu_i = \delta_i \oplus \beta_i$ $(0 \leq i \leq r)$, $\varepsilon_i = \hat{\gamma}_i \oplus \alpha_{i+1}$ $(0 \leq i \leq r)$, and set $\mu_{r+1} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi^1)}|_{\mu_r}) \subset R_0' \oplus \alpha_0$, $R_0^1 = (R_0' \oplus \alpha_0) \ominus \mu_{r+1}$. Then, we have the diagram (6.4), where we replace R_0^2 by R_0^1. Note that rank$\mu_0 = 2$. By the proof of Lemma 6.1, we obtain

Lemma 6.2. *Let A, B be as in Lemma 6.1. Set $\eta = \underline{\mathrm{Im}}B$ and $\nu = \underline{\mathrm{Im}}(A \circ B)$. Then,* rank$\eta$ = rank$\varepsilon_0 - 1$, rank$\nu = 1$ *and* $\nu \subset \mathrm{Ker}B$.

Set $\underline{\varphi}^2 = (\underline{\varphi}^1 \ominus \nu) \oplus G'_{\varphi^1}(\nu)$, where ν is as in Lemma 6.2, then by (6.7) we see that

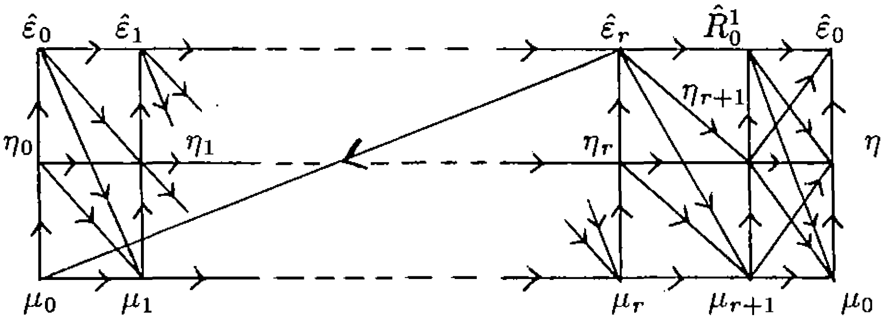
φ^2 satisfies the conditions of Proposition 6.1. It follows that there is a holomorphic subbundle τ of $\underline{\varphi}^2$ with $\tau \subset \mathrm{Ker}(A_{(1,0)}^{(\varphi^2)^\perp} \circ A_{(1,0)}^{\varphi^2})$ and $\mathrm{rank}\tau = \mathrm{rank}\underline{\varphi}^2 - 1$ such that φ_1 defined from φ^2 by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies $\mathrm{rank}\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} = \mathrm{rank}\underline{\varphi}_1 - 1$.

(II-2) By (6.8), we have $A_{(1,0)}^{\alpha_{r+1},\gamma_0} \equiv 0$. Set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \alpha_0) \oplus \alpha_1$ then $\beta_0 = \underline{\mathrm{Im}}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$, $\beta_i = G_{\varphi^1}^{(i)}(\beta_0)$ $(1 \leq i \leq r)$ and $\beta_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$. Reset $\mu_i = \beta_i$ $(0 \leq i \leq r)$, and set $\mu_{r+1} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi^1)}\,|_{\mu_r}) \subset R_0' \oplus \alpha_0$, $R_0^1 = (R_0' \oplus \alpha_0) \ominus \mu_{r+1}$ and $\varepsilon_i = \gamma_i \oplus \alpha_{i+1}$ $(0 \leq i \leq r)$. Then, we have the diagram (6.4), where we replace R_0^2 by R_0^1.

Lemma 6.3. *Let A, B be as in Lemma 6.1. Set $\eta_0 = \underline{\mathrm{Im}}B$. Then,* $\mathrm{rank}\eta_0 = \mathrm{rank}\varepsilon_0 - 1$ *and $(A \circ B)^2 \equiv 0$. Moreover, the following hold :*
(1) If $A \circ B \equiv 0$, set $\nu = \mu_0 \oplus \eta_0$. Then, $\nu \subset \mathrm{Ker}(A_{(1,0)}^{(\varphi^1)^\perp} \circ A_{(1,0)}^{\varphi^1})$, $\mathrm{rank}\nu = \mathrm{rank}\underline{\varphi}^1 - 1$, *and φ_1 defined from φ^1 by the forward replacement of ν has ∂'-isotropy order $\geq r+1$ and satisfies* $\mathrm{rank}\underline{\mathrm{Im}}A_{(1,0)}^{G^{(r+1)}(\varphi_1),\varphi_1} = \mathrm{rank}\underline{\varphi}_1 - 1$.
(2) If $A \circ B \not\equiv 0$, set $\nu = \underline{\mathrm{Im}}(A \circ B)$. Then, $\mathrm{rank}\nu = 1$ *and $\nu \subset \mathrm{Ker}B$.*

Proof. Since $\mathrm{rank}\gamma_0 = 1$, by the proof of Lemma 6.1 we see that $\mathrm{rank}\eta_0 = \mathrm{rank}\varepsilon_0 - 1$. Since $A \circ B$ is nilpotent and $\mathrm{rank}\mu_0 = 2$, we have $(A \circ B)^2 \equiv 0$.
(1) Set $\eta_i = G_{\varphi^1}^{(i)}(\eta_0) \cap \varepsilon_i$ $(1 \leq i \leq r)$. Since $\eta_0 \subset \mathrm{Ker}A$, we have $\eta_r \subset \mathrm{Ker}A_{(1,0)}^{G^{(r)}(\varphi^1),\varphi^1}$ and hence set $\eta_{r+1} = \underline{\mathrm{Im}}(A_{(1,0)}^{G^{(r)}(\varphi^1)}\,|_{\mu_r \oplus \eta_r}) \ominus \mu_{r+1} \subset R_0^1$, $\hat{R}_0^1 = R_0^1 \ominus \eta_{r+1}$ and $\hat{\varepsilon}_i = \varepsilon_i \ominus \eta_i$ $(0 \leq i \leq r)$. We have a diagram

(6.9)



Since $\mathrm{rank}\hat{\varepsilon}_0 = 1$, by (6.9) we see that $A_{(1,0)}^{\eta_{r+1},\hat{\varepsilon}_0} \equiv 0$. Set $\nu = \mu_0 \oplus \eta_0$ and $\underline{\varphi}_1 = (\underline{\varphi}^1 \ominus \nu) \oplus G'_{\varphi^1}(\nu)$. Then, ν and φ_1 have the desired properties.
(2) Since $A \circ B \not\equiv 0$ and $\mathrm{rank}\mu_0 = 2$, we obtain $\mathrm{rank}\nu = 1$, where $\nu = \underline{\mathrm{Im}}(A \circ B)$. Since $\mathrm{rank}\alpha_0 = \mathrm{rank}\gamma_0 = 1$, the proof for $\nu \subset \mathrm{Ker}B$ is quite similar to that of Lemma 6.1. q.e.d.

We may consider only the case (2) in Lemma 6.3. In this case, we also have the same type diagram as (6.7). Set $\underline{\varphi}^2 = (\underline{\varphi}^1 \ominus \nu) \oplus G'_{\varphi^1}(\nu)$, then we see that φ^2 satisfies the conditions of Proposition 6.1. It follows that there is a holomorphic subbundle τ of $\underline{\varphi}^2$ with $\tau \subset \mathrm{Ker}(A^{(\varphi^2)^\perp}_{(1,0)} \circ A^{\varphi^2}_{(1,0)})$ and $\mathrm{rank}\tau = \mathrm{rank}\underline{\varphi}^2 - 1$ such that φ_1 defined from φ^2 by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies $\mathrm{rank}\underline{\underline{\mathrm{Im}}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{rank}\underline{\varphi}_1 - 1$.

(II-3) By (6.8), we obtain $A^{\alpha_{r+1},\gamma_0}_{(1,0)} \equiv 0$. Set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \alpha_0) \oplus \alpha_1$ then $\beta_0 = \underline{\underline{\mathrm{Im}}}A^{G^{(r)}(\varphi^1),\varphi^1}_{(1,0)}$, $\mathrm{rank}\beta_0 = 1$ and $\beta_r = G^{(r)}_{\varphi^1}(\beta_0) \subset \mathrm{Ker}A^{G^{(r)}(\varphi^1),\varphi^1}_{(1,0)}$, hence φ^1 satisfies the conditions of Proposition 6.1. It follows that there is a holomorphic subbundle τ of $\underline{\varphi}^1$ with $\tau \subset \mathrm{Ker}(A^{(\varphi^1)^\perp}_{(1,0)} \circ A^{\varphi^1}_{(1,0)})$ and $\mathrm{rank}\tau = \mathrm{rank}\underline{\varphi}^1 - 1$ such that φ_1 defined from φ^1 by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies $\mathrm{rank}\underline{\underline{\mathrm{Im}}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{rank}\underline{\varphi}_1 - 1$.

(III) Set $\mu_0 = \underline{\underline{\mathrm{Im}}}A_{r,\varphi}$, then $\mu_0 \subset \mathrm{Ker}A_{r,\varphi}$. Set

$$\mu_i = G^{(i)}_\varphi(\mu_0)\ (1 \leq i \leq r)\ , \qquad \varepsilon_0 = \underline{\varphi} \ominus \mu_0\ ,$$

$$\varepsilon_i = G^{(i)}(\varphi) \ominus \mu_i\ (1 \leq i \leq r)\ , \qquad R = \underline{\varphi}^\perp \ominus (\bigoplus_{j=1}^{r} G^{(j)}(\varphi))\ .$$

Then, $\mu_r \subset \mathrm{Ker}A^{G^{(r)}(\varphi),\varphi}_{(1,0)}$, hence set $\mu_{r+1} = \underline{\underline{\mathrm{Im}}}(A^{G^{(r)}(\varphi)}_{(1,0)}\ |_{\mu_r}) \subset R$ and $R'_0 = R \ominus \mu_{r+1}$. We have a diagram

(6.10)
$$\begin{tikzcd}
\varepsilon_0 \arrow[r] \arrow[dr] & \varepsilon_1 \arrow[r, dashed] \arrow[dr] & \cdots \arrow[r, dashed] \arrow[dr] & \varepsilon_r \arrow[r] \arrow[dr] \arrow[dlll] & R'_0 \arrow[r] \arrow[dr] & \varepsilon_0 \\
\mu_0 \arrow[u] \arrow[r] & \mu_1 \arrow[u] \arrow[r, dashed] & \cdots \arrow[r, dashed] & \mu_r \arrow[u] \arrow[r] & \mu_{r+1} \arrow[u] \arrow[r] \arrow[ur] & \mu_0 \arrow[u]
\end{tikzcd}$$

There are three possibilities :
(III-1) $\mathrm{rank}\mu_0 = 1$, (III-2) $\mathrm{rank}\mu_0 = 2$, (III-3) $\mathrm{rank}\mu_0 = 3$.

(III-1) In this case, φ itself satisfies the conditions of Proposition 6.1. It follows that there is a holomorphic subbundle τ of $\underline{\varphi}$ with $\tau \subset \mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^{\varphi}_{(1,0)})$ and $\mathrm{rank}\tau = \mathrm{rank}\underline{\varphi} - m$, where $m = 1, 2, 3$, such that φ_1 defined from φ by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies $\mathrm{rank}\underline{\underline{\mathrm{Im}}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{rank}\underline{\varphi}_1 - m$.

(III-2) By (6.10), we have rank$\underline{\mathrm{Im}}A^{\mu_{r+1},\varepsilon_0}_{(1,0)} \leq 1$. First, assume that $A^{\mu_{r+1},\varepsilon_0}_{(1,0)} \equiv 0$. Set $\underline{\varphi}_1 = (\underline{\varphi} \ominus \mu_0) \oplus \mu_1$. Then, by (6.10) we see that φ_1 has ∂'-isotropy order $\geq r+1$ and satisfies rank$\underline{\mathrm{Im}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{rank}\underline{\varphi}_1 - m$, where $m = 1, 2$. Next, assume that rank$\underline{\mathrm{Im}}A^{\mu_{r+1},\varepsilon_0}_{(1,0)} = 1$. In the same way as Lemma 6.3, we have the following

Lemma 6.4. *Let A, B be as in Lemma 6.1. Set $\eta_0 = \underline{\mathrm{Im}}A^{\mu_{r+1},\varepsilon_0}_{(1,0)}$. Then,* rank$\eta_0 = 1$ *and $(A \circ B)^2 \equiv 0$. Moreover, the following hold :*
(1) If $A \circ B \equiv 0$, set $\nu = \mu_0 \oplus \eta_0$. Then, $\nu \subset \mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^{\varphi}_{(1,0)})$, rank$\nu = \mathrm{rank}\underline{\varphi} - 1$, *and φ_1 defined from φ by the forward replacement of ν has ∂'-isotropy order $\geq r+1$ and satisfies* rank$\underline{\mathrm{Im}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{rank}\underline{\varphi}_1 - 1$.
(2) If $A \circ B \not\equiv 0$, set $\nu = \underline{\mathrm{Im}}(A \circ B)$. Then, rank$\nu = 1$ *and* $\nu \subset \mathrm{Ker}B$.

We only consider the case (2) in Lemma 6.4. Set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \nu) \oplus G'_\varphi(\nu)$, then we see that φ^1 satisfies the conditions of Proposition 6.1 (cf. (6.7)). Moreover, it follows that there is a holomorphic subbundle τ of $\underline{\varphi}^1$ with $\tau \subset \mathrm{Ker}(A^{(\varphi^1)^\perp}_{(1,0)} \circ A^{\varphi^1}_{(1,0)})$ and rank$\tau = \mathrm{rank}\underline{\varphi}^1 - 1$ such that φ_1 defined from φ^1 by the forward replacement of τ has ∂'-isotropy order $\geq r+1$ and satisfies rank$\underline{\mathrm{Im}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{tank}\underline{\varphi}_1 - 1$.

Remark. If we simply set $\underline{\varphi}^1 = (\underline{\varphi} \ominus \mu_0) \oplus \mu_1$, then we see that φ^1 satisfies the conditions of Proposition 6.1. However, we can not get the information between rankτ and rank$\underline{\varphi}^1$.

(III-3) In this case, since rank$\varepsilon_0 = 1$, it follows from (6.10) that $A^{\mu_{r+1},\varepsilon_0}_{(1,0)} \equiv 0$. Set $\underline{\varphi}_1 = (\underline{\varphi} \ominus \mu_0) \oplus \mu_1$, then we see that φ_1 has ∂'-isotropy order $\geq r+1$ and satisfies rank$\underline{\mathrm{Im}}A^{G^{(r+1)}(\varphi_1),\varphi_1}_{(1,0)} = \mathrm{rank}\underline{\varphi}_1 - 1$.

In summary, we have the following

Proposition 6.2. *Let $\varphi : M \setminus S_\varphi \longrightarrow G_4(\mathbf{C}^n)$ be a pluriharmonic map. Assume that φ has ∂'-isotropy order r. Then, there is a sequence $\{\varphi^i\}_{i=0}^N$ of pluriharmonic maps such that*
(1) $\varphi^0 = \varphi$, (2) φ^N has ∂'-isotropy order $\geq r+1$ and satisfies rank$\underline{\mathrm{Im}}A^{G^{(r+1)}(\varphi^N),\varphi^N}_{(1,0)}$ $= \mathrm{rank}\underline{\varphi}^N - m$, *where $m = 1, 2, 3$,*
(3) for $i = 0, 1, \cdots, N-1$, each φ^i has ∂'-isotropy order r, and φ^{i+1} is obtained from φ^i by the forward replacement of α^i, where α^i is a holomorphic subbundle of $\underline{\varphi}^i$ contained in $\mathrm{Ker}(A^{(\varphi^i)^\perp}_{(1,0)} \circ A^{\varphi^i}_{(1,0)})$.

Using Proposition 6.2, we obtain the following

Theorem 6.1. *Let $\varphi : M \setminus S_\varphi \longrightarrow G_4(\mathbf{C}^n)$ be a pluriharmonic map. Assume that φ has finite ∂'-isotropy order and $n \leq 14$. Then, there is a sequence $\{\varphi_i\}_{i=0}^N$ of pluriharmonic maps such that*
(1) $\varphi_0 = \varphi$, (2) $\varphi_N : M \setminus S_{\varphi_N} \longrightarrow G_m(\mathbf{C}^n)$, $m = 1, 2, 3$,
(3) for $i = 0, 1, \cdots, N-1$, each φ_i has finite ∂'-isotropy order, and φ_{i+1} is obtained from φ_i by the forward replacement of α^i, where α^i is a holomorphic subbundle of $\underline{\varphi}_i$ contained in $\mathrm{Ker}(A_{(1,0)}^{\varphi_i^\perp} \circ A_{(1,0)}^{\varphi_i})$.

Proof. Construct φ_1 from φ using Proposition 6.2. Let r be the ∂'-isotropy order of φ_1. Then, we have $r \geq 2$. Set $\alpha_0 = \underline{\mathrm{Im}} A_{r,\varphi}$, $\alpha_i = G_{\varphi_1}^{(i)}(\alpha_0)$ for $i = 1, \cdots, r$ and $\gamma_0 = \underline{\varphi}_1 \ominus \alpha_0$, $\gamma_i = G^{(i)}(\varphi_1) \ominus \alpha_i$ for $i = 1, \cdots, r$. By Proposition 6.2, we have $\mathrm{rank}\gamma_0 = m$ and $\mathrm{rank}\alpha = \mathrm{rank}\underline{\varphi}_1 - m$, where $m = 1, 2, 3$. If $\alpha_0 = \underline{0}$, then φ_1 is a pluriharmonic map into $G_m(\mathbf{C}^n)$, where $m = 1, 2, 3$, hence we may assume that $\alpha_0 \neq \underline{0}$. Set $R = \underline{\varphi}_1^\perp \ominus (\bigoplus_{j=1}^r G^{(j)}(\varphi_1))$, then we have the diagram (5.17). In the same way as in the proof of Theorem 5.1, we have two possibilities : (1) $\alpha_i = \underline{0}$ for some $1 \leq i \leq r$, (2) any α_i $(1 \leq i \leq r)$ is non-zero.
(1) Set $\underline{\widetilde{\varphi}} = (\underline{\varphi}_1 \ominus \alpha_0) \oplus \alpha_1$. Then, by (5.17) we see that either, $\widetilde{\varphi}$ is a pluriharmonic map into $G_m(\mathbf{C}^n)$, where $m = 1, 2, 3$, or $\widetilde{\varphi}$ has ∂'-isotropy order $r+1$ and $\mathrm{rank}\underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+1)}(\tilde{\varphi}),\tilde{\varphi}} = \mathrm{rank}\underline{\widetilde{\varphi}} - m$, where $m = 1, 2, 3$.
(2) Since $n \leq 14$, one of $\underline{\varphi}_1$, $G^{(i)}(\varphi_1)$ $(1 \leq i \leq r)$ has rank ≤ 4 and ∂'-isotropy order r. Hence, by Proposition 6.2, either, we have a pluriharmonic map into $G_m(\mathbf{C}^n)$, where $m = 1, 2, 3$, or we have a pluriharmonic map $\widetilde{\varphi}$ which has ∂'-isotropy order $r+1$ and satisfies $\mathrm{rank}\underline{\mathrm{Im}} A_{(1,0)}^{G^{(r+1)}(\tilde{\varphi}),\tilde{\varphi}} = \mathrm{rank}\underline{\widetilde{\varphi}} - m$, where $m = 1, 2, 3$.
Repeating this procedure, we see that φ is reduced to a pluriharmonic map into $G_m(\mathbf{C}^n)$, where $m = 1, 2, 3$. q.e.d.

If we don't require the result about (2) in Theorem 6.1, we have

Theorem 6.2. *Let $\varphi : M \setminus S_\varphi \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map. Assume that $k = 3$ (resp. 4) and $n \leq 20$ (resp. 15). Then, by the successive procedures of the forward replacement, φ is reduced to an anti-holomorphic map $f : M \setminus S_f \longrightarrow G_t(\mathbf{C}^n)$ for some t.*

Proof. We show the case of $k = 3$. By Proposition 5.1, we can construct φ_2 which has ∂'-isotropy order ≥ 3. Since $n \leq 20$, either one of $\underline{\varphi}_2$, $G^{(i)}(\varphi_2)$ $(1 \leq i \leq 3)$ has rank ≤ 4 or any of $\underline{\varphi}_2$, $G^{(i)}(\varphi_2)$ $(1 \leq i \leq 3)$ has rank 5. The former case implies that we may construct $\widetilde{\varphi}$ from φ_2, which has ∂'-isotropy order ≥ 4 by Theorems 3.1, 4.1 and Propositions 5.1, 6.2. The latter case implies that $G^{(4)}(\varphi_2) \subset \varphi_2$, hence $\mathrm{rank}G^{(4)}(\varphi_2) \leq 4$. Hence, we can construct $\widetilde{\varphi}$ from $G^{(4)}(\varphi_2)$, which has ∂'-isotropy order ≥ 4 by Theorems 3.1, 4.1 and Propositions 5.1, 6.2. Repeating this procedure and noting that any pluriharmonic map with infinite ∂'-isotropy order is reduced to an anti-holomorphic map, we see that φ is reduced to an anti-holomorphic map by the successive procedures of the forward replacement. q.e.d.

7. A construction of pluriharmonic maps from rational maps.

In this section, we give the inverse of the procedures in Theorems 3.1, 4.1, 5.1, 6.1 and 6.2. For this purpose, we review the following propositions

Proposition 7.1 ([O-U2]). *Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map from a complex manifold. Let $\alpha \subset \mathrm{Ker}(A^{\varphi^\perp}_{(1,0)} \circ A^{\varphi}_{(1,0)})$ be a holomorphic subbundle of $\underline{\varphi}$ and let $\widetilde{\varphi}$ be defined from φ by the forward replacement of α. Then, $G'_\varphi(\alpha)$ is an anti-holomorphic subbundle of $\underline{\widetilde{\varphi}}$, $G'_\varphi(\alpha) \subset \mathrm{Ker}(A^{\tilde{\varphi}^\perp}_{(0,1)} \circ A^{\tilde{\varphi}}_{(0,1)})$ and, if $\underline{\mathrm{Ker}}A^{\varphi}_{(1,0)} = \underline{0}$, then φ is obtained from $\widetilde{\varphi}$ by the backward replacement of $G'_\varphi(\alpha)$.*

Proposition 7.2 ([O-U2]). *Let $\varphi : M \longrightarrow G_k(\mathbf{C}^n)$ be a pluriharmonic map from a complex manifold. Assume that $\underline{\mathrm{Ker}}A^{\varphi}_{(1,0)} \neq \underline{0}$. Then, there exists a pluriharmonic map $\psi : M \setminus S_\psi \longrightarrow G_t(\mathbf{C}^n)$ for some $0 \le t \le k-1$ and a non-zero anti-holomorphic subbundle β of $(\underline{\psi} \oplus G'(\psi))^\perp$ such that $\underline{\varphi} = \psi \oplus \beta$ over $M \setminus S_\psi$. Conversely, given $\psi : M \longrightarrow G_t(\mathbf{C}^n)$ a pluriharmonic map and a non-zero anti-holomorphic subbundle β of $(\psi \oplus G'(\psi))^\perp$ then φ defined by $\underline{\varphi} = \underline{\psi} \oplus \beta$ gives a pluriharmonic map $\varphi : M \setminus S_\varphi \longrightarrow G_k(\mathbf{C}^n)$ with $\underline{\mathrm{Ker}}A^{\varphi}_{(1,0)} \neq \underline{0}$, where $k = t + \mathrm{rank}\beta$.*

We remark that if we reverse the orientation of M we may use the concepts of ∂''-isotropy order and the backward replacement in place of those of ∂'-isotropy order and the forward replacement. First, we treat the case of infinite isotropy order.

Proposition 7.3. *Let $\varphi : M \setminus S_\varphi \longrightarrow G_k(\mathbf{C}^n)$ be any non-holomorphic pluriharmonic map with infinite ∂''-isotropy order, where M is a complex manifold. Then, there is a unique sequence $\{\varphi^i\}_{i=0}^N$ of pluriharmonic maps such that*
(1) $\varphi^N = \varphi$, (2) $\varphi^0 : M \setminus S_{\varphi^0} \longrightarrow G_t(\mathbf{C}^n)$ is a holomorphic map for some $t \in \mathbf{N}$, that is, a rational map $f : M \longrightarrow G_t(\mathbf{C}^n)$,
(3) for $i = 0, 1, \cdots, N-1$, $\underline{\mathrm{Ker}}A^{\varphi^i}_{(1,0)} = \underline{0}$, and each φ^{i+1} is obtained from φ^i by $\underline{\varphi}^{i+1} = G'(\varphi^i) \oplus \alpha^i$, where α^i is a holomorphic subbundle of $(G'(\varphi^i) \oplus \underline{\varphi}^i)^\perp$.

Proof. Since $G^{(-s)}(\varphi) = \underline{0}$ for some $s \in \mathbf{N}$, set $\underline{\varphi}^i = G^{(-s+1+i)}(\varphi)$ for $i = 0, 1, \cdots, s-1$. Since $G'(G^{(-s+1+i)}(\varphi)) \subset G^{(-s+2+i)}(\varphi)$, we have $G'(\varphi^i) \subset \underline{\varphi}^{i+1}$. Set $\alpha^i = \underline{\mathrm{Ker}}A^{\varphi^{i+1},\varphi^i}_{(0,1)}$, then by (1.2) and Proposition 7.2 we see that $\underline{\varphi}^{i+1} = G'(\varphi^i) \oplus \alpha^i$ and α^i is a holomorphic subbundle of $(G'(\varphi^i) \oplus G''(G'(\varphi^i)))^\perp$. Note that the condition $\underline{\mathrm{Ker}}A^{\varphi^i}_{(1,0)} = \underline{0}$ is equivalent to that $A^{\varphi^{i+1},\varphi^i}_{(0,1)} : \underline{\varphi}^{i+1} \longrightarrow \underline{\varphi}^i$ is surjective, which is satisfied by the definition (see (1.2)). Now, $N = s - 1$ and the existence is established. For the uniqueness, define the sequence $\{\varphi^i\}_{i=0}^N$ as in (3), where φ^0 is as in (2). We show that each α^i is uniquely determined by the condition (1). Suppose that $\underline{\varphi}^i \subsetneqq G^{(-N+i)}(\varphi)$ for some $1 \leq i \leq N-1$. Set $\beta^i = G^{(-N+i)}(\varphi) \ominus \underline{\varphi}^i$. Since, $\underline{\varphi}^{i+1}$ is a holomorphic subbundle of $(\underline{\varphi}^i)^\perp$, $A_{(0,1)}^{G^{(-N+i+1)}(\varphi),\beta^i}$ is surjective, and $\underline{\mathrm{Ker}}A_{(1,0)}^{\varphi^i} = \underline{0}$, it follows that $\underline{\varphi}^{i+1}$ can not have $G^{(-N+i+1)}(\varphi)$ as a direct sum factor and $\underline{\varphi}^{i+1} \subset G^{(-N+i+1)}(\varphi) \oplus \beta^i$. Thus, either $\underline{\varphi}^{i+1} \subsetneqq G^{(-N+i+1)}(\varphi)$ or $\underline{\varphi}^{i+1}$ has the non-trivial projection into β^i. The former case may be treated in the same way, and the latter one yields $\varphi^N \neq \varphi$ because $\underline{\mathrm{Ker}}A_{(1,0)}^{\varphi^j} = \underline{0}$ and $\underline{\mathrm{Ker}}A_{(1,0)}^{G^{(-j-1)}(\varphi)} = \underline{0}$ for any $0 \leq j \leq N-1$. Therefore, we have $\varphi^N \neq \varphi$. Next, suppose that $\underline{\varphi}^i \supsetneqq G^{(-N+i)}(\varphi)$ for some $1 \leq i \leq N-1$. If $\underline{\varphi}^i$ contains also $G^{(-N+i+1)}(\varphi)$, then $G^{(-N+i)}(\varphi) \subset \mathrm{Ker}A_{(1,0)}^{\varphi^i}$, which is a contradiction. Thus, $\underline{\varphi}^i$ has a proper holomorphic subbundle of $G^{(-N+i+1)}(\varphi)$ as a direct sum factor, hence, again, we have $\varphi^N \neq \varphi$. Finally, suppose that $G'(\varphi^{i-1}) \subsetneqq G^{(-N+i)}(\varphi)$, α^{i-1} has the non-trivial projection into the both of $G^{(-N+i+1)}(\varphi)$ and β^i, for some $1 \leq i \leq N-1$. This case also leads to the conclusion $\varphi^N \neq \varphi$. q.e.d.

Theorem 7.1. *Let $\varphi : M \setminus S_\varphi \longrightarrow \mathbf{C}P^{n-1}$ be any non ±-holomorphic pluriharmonic map, where M is a compact complex manifold with $c_1(M) > 0$. Then, there is a unique sequence $\{\varphi^i\}_{i=0}^N$ $(N \leq n-1)$ of pluriharmonic maps into $\mathbf{C}P^{n-1}$ such that*
(1) $\varphi^N = \varphi$, (2) $\varphi^0 : M \setminus S_{\varphi^0} \longrightarrow \mathbf{C}P^{n-1}$ is a holomorphic map, that is, a rational map $f : M \longrightarrow \mathbf{C}P^{n-1}$,
(3) for $i = 0, 1, \cdots, N-1$, each φ^{i+1} is obtained from φ^i by $\underline{\varphi}^{i+1} = G'(\varphi^i)$.

Proof. This follows from Theorem 3.1 and Proposition 7.3. q.e.d.

For the case of finite isotropy order, we have the following

Theorem 7.2. *Let $\varphi : M \setminus S_\varphi \longrightarrow G_2(\mathbf{C}^n)$ be any pluriharmonic map with finite ∂''-isotropy order, where M is a compact complex manifold with $c_1(M) > 0$. Then, there is a sequence $\{\varphi^i\}_{i=0}^N$ of pluriharmonic maps such that*
(1) $\varphi^N = \varphi$, (2) $\varphi^0 : M \setminus S_{\varphi^0} \longrightarrow \mathbf{C}P^{n-1}$, and φ^1 is obtained from φ^0 by $\underline{\varphi}^1 = \underline{\varphi}^0 \oplus \beta_0$, where β^0 is a holomorphic subbundle of $(\underline{\varphi}^0 \oplus G''(\varphi^0))^\perp$ so that $\underline{\mathrm{Im}}A_{(0,1)}^{G^{(-r)}(\varphi^1),\varphi^1} = \beta^0$ for some $r \in \mathbf{N}$, and if φ is non anti-holomorphic then $\mathrm{rank}\beta^0 = 1$,
(3) for $i = 1, \cdots, N-1$, each φ^{i+1} has ∂''-isotropy order $r - i$, and φ^{i+1} is obtained from φ^i by

$$\underline{\varphi}^{i+1} = \widetilde{\varphi}^i \oplus \beta^i \ , \ \ \underline{\widetilde{\varphi}}^i = (\underline{\varphi}^i \ominus \alpha^i) \oplus G'_{\varphi^i}(\alpha^i) \ ,$$

where α^i *is a holomorphic subbundle of* $\underline{\varphi}^i$ *so that* $\mathrm{rank}\alpha^i = \mathrm{rank}\underline{\varphi}^i - 1$ *and the Hermitian orthogonal projection* $P^i : \underline{\mathrm{Im}}A_{(0,1)}^{G^{(i-r-1)}(\varphi^i),\varphi^i} \longrightarrow \alpha^i$ *is an anti-holomorphic isomorphism, and* β^i *is a holomorphic subbundle of* $(\underline{\widetilde{\varphi}}^i \oplus G''(\widetilde{\varphi}^i))^\perp$ *so that* $\underline{\mathrm{Im}}A_{(0,1)}^{G^{(i-r)}(\varphi^{i+1}),\varphi^{i+1}} = G'_{\varphi^i}(\alpha^i) \oplus \beta^i$.

Proof. This follows from Theorem 4.1 and Propositions 7.1, 7.2. q.e.d.

The uniqueness for the choice of β^i may be expected if we assume that $\underline{\mathrm{Ker}}(A_{(1,0)}^{\varphi^i}\,|_{\alpha^i}) = \underline{0}$, however, in general, it seems to be difficult to determine α^i uniquely.

Theorem 7.3. *Let* $\varphi : M \setminus S_\varphi \longrightarrow G_k(\mathbf{C}^n)$ *be any pluriharmonic map with finite* ∂''*-isotropy order, where* M *is a compact complex manifold with* $c_1(M) > 0$. *Assume that* $k = 3$ *(resp. 4) and* $n \leq 15$ *(resp. 14). Then, there is a sequence* $\{\varphi^i\}_{i=0}^N$ *of pluriharmonic maps such that*
(1) $\varphi^N = \varphi$, *(2)* $\varphi^0 : M \setminus S_{\varphi^0} \longrightarrow G_t(\mathbf{C}^n)$, $1 \leq t \leq k-1$, *and* φ^1 *is obtained from* φ^0 *by* $\underline{\varphi}^1 = \underline{\varphi}^0 \oplus \beta^0$, *where* β^0 *is a holomorphic subbundle of* $(\underline{\varphi}^0 \oplus G''(\varphi^0))^\perp$, *and* φ^1 *has finite* ∂''*-isotropy order,*
(3) for $i = 1, \cdots, N-1$, *each* φ^{i+1} *has finite* ∂''*-isotropy order, and* φ^{i+1} *is obtained from* φ^i *by*

$$\underline{\varphi}^{i+1} = \underline{\widetilde{\varphi}}^i \oplus \beta^i\ ,\ \ \underline{\widetilde{\varphi}}^i = (\underline{\varphi}^i \ominus \alpha^i) \oplus G'_{\varphi^i}(\alpha^i)\ ,$$

where α^i *is a holomorphic subbundle of* $\underline{\varphi}^i$ *contained in* $\mathrm{Ker}(A_{(1,0)}^{(\varphi^i)^\perp} \circ A_{(1,0)}^{\varphi^i})$, *and* β^i *is a holomorphic subbundle of* $(\underline{\widetilde{\varphi}}^i \oplus G''(\widetilde{\varphi}^i))^\perp$.

Proof. This follows from Theorems 5.1, 6.1 and Propositions 7.1, 7.2. q.e.d.

Theorem 7.4. *Let* $\varphi : M \setminus S_\varphi \longrightarrow G_k(\mathbf{C}^n)$ *be any non-holomorphic pluriharmonic map. Assume that* $k = 3$ *(resp. 4) and* $n \leq 20$ *(resp. 15). Then, there is a sequence* $\{\varphi^i\}_{i=0}^N$ *of pluriharmonic maps such that*
(1) $\varphi^N = \varphi$, *(2)* $\varphi^0 : M \setminus S_{\varphi^0} \longrightarrow G_t(\mathbf{C}^n)$ *is a holomorphic map for some* $t \in \mathbf{N}$, *that is, a rational map* $f : M \longrightarrow G_t(\mathbf{C}^n)$,
(3) for $i = 0, 1, \cdots, N-1$, *each* φ^{i+1} *is obtained from* φ^i *by*

$$\underline{\varphi}^{i+1} = \underline{\widetilde{\varphi}}^i \oplus \beta^i\ ,\ \ \underline{\widetilde{\varphi}}^i = (\underline{\varphi}^i \ominus \alpha^i) \oplus G'_{\varphi^i}(\alpha^i)\ ,$$

where α^i *is a holomorphic subbundle of* $\underline{\varphi}^i$ *contained in* $\mathrm{Ker}(A_{(1,0)}^{(\varphi^i)^\perp} \circ A_{(1,0)}^{\varphi^i})$, *and* β^i *is a holomorphic subbundle of* $(\underline{\widetilde{\varphi}}^i \oplus G''(\widetilde{\varphi}^i))^\perp$.

Proof. This follows from Theorem 6.2 and Propositions 2.3, 7.1~7.3. q.e.d.

Remark. (1) Ohnita and Valli [O-V] proved the factorization theorem for the class of meromorphically pluriharmonic maps into the unitary group. We remark that, our class of pluriharmonic maps is wider than that of theirs, the method using the image or kernel of the second fundamental form, which is called the basic transform, is not established yet, and that even if it is established our results are not covered by it (cf. [Wd1]).
(2) Toledo suggested to the author that the analogy of their result [C-T] may hold, that is, any non-constant pluriharmonic map φ from compact complex manifold M into $\mathbf{C}P^{n-1}$ has a factorization of the form $\varphi = g \circ f$, where $f : M \longrightarrow S$ is a holomorphic map into a compact Riemann surface and $g : S \longrightarrow \mathbf{C}P^{n-1}$ is a harmonic map, if $\varphi(M)$ is not a geodesic arc in $\mathbf{C}P^{n-1}$. The exceptional case surely occurs when we set $M = T^m$, that is, m-dimensional complex torus, and consider the factorization, $f : T^m \longrightarrow S^1$ a totally geodesic map, $g : S^1 \longrightarrow \mathbf{C}P^{n-1}$ a totally geodesic immersion. Note that a totally geodesic map from a Kähler manifold is pluriharmonic (cf. [O-U1]). Thus, if we assume $c_1(M) > 0$, we may expect that S is a Riemann sphere and g is a branched minimal immersion. We will discuss it elsewhere.

Acknowledgements.

The author wishes to thank Professor Y. Ohnita for the useful communication of the topics in this paper. The author also wishes to thank Professors J. Eells and D. Toledo for kind suggestions. This work is done while the author stay at the Max-Planck-Institut für Mathematik in Bonn, he wishes to thank the Max-Planck-Institut for the hospitality.

References

[Bn] D. Burns, Harmonic maps from $\mathbf{C}P^1$ to $\mathbf{C}P^n$, in Harmonic maps (Proceedings, New Orleans 1980, 48–56), *Lecture Notes in Math.*, vol. **949**, Springer, Berlin, 1982.

[B-B-B-R] D. Burns, F. Burstall, P. de Bartolomeis and J. Rawnsley, Stability of harmonic maps of Kähler manifolds, *J. Differential Geom.* **30**(1989), 579–594.

[B-B] D. Burns and P. de Bartolomeis, Applications harmoniques stables dans $\mathbf{P}^n$, *Ann. Scient. Ec. Norm. Sup.* **21**(1988), 159–177.

[B-R] F. Burstall and J. Rawnsley, Spheres harmoniques dans les groupes de Lie compacts et curbes holomorphes dans les espaces homogenes, *C. R. Acad. Sc. Paris* **302**(1986), 709–712.

[B-S] F. Burstall and S. Salamon, Tournaments, flags and harmonic maps, *Math. Ann.* **277**(1987), 249–265.

[B-W] F. Burstall and J. C. Wood, The construction of harmonic maps into complex Grassmannians, *J. Differential Geom.* **23**(1986), 255–297.

[C-T] J. A. Carlson and D. Toledo, Harmonic mappings of Kähler manifolds to lacally symmetric spaces, *Publications Math. I.H.E.S.* **69**(1990), 173–201.

[C-W] S. S. Chern and J. G. Wolfson, Harmonic maps of the 2-sphere into a complex Grassmann manifold, II, *Ann. of Math.* **125**(1987), 301–335.

[D-Z] A. M. Din and W. J. Zakrzewski, General classical solutions in the $\mathbf{C}P^{n-1}$ model, *Nuclear Phys.* **B174**(1980), 397–406.

[E-L] J. Eells and L. Lemaire, Another report on harmonic maps, *Bull. London Math. Soc.* **20**(1988), 385–524.

[E-W] J. Eells and J. C. Wood, Harmonic maps from surfaces to complex projective spaces, *Adv. in Math.* **49**(1983), 217–263.

[G-S] V. Glaser and R. Stora, Regular solutions of the $\mathbf{C}P^n$ models and further generalizations, (1980), preprint.

[O-U1] Y. Ohnita and S. Udagawa, Stability, complex-analyticity and constancy of pluriharmonic maps from compact Kähler manifolds, *Math. Zeit.*, to appear.

[O-U2] Y. Ohnita and S. Udagawa, Complex-analyticity of pluriharmonic maps and their constructions, (Proceedings, Kyoto 1989), *Lecture Notes in Math.*, Springer, Berlin, to appear.

[O-V] Y. Ohnita and G. Valli, Pluriharmonic maps into compact Lie groups and factorization into unitons, *Proc. London Math. Soc.*, to appear.

[Rm] J. Ramanathan, Harmonic maps from S^2 to $G_{2,4}$, *J. Differential Geom.* **19** (1984),207–219.

[Ud] S. Udagawa, Holomorphicity of certain stable harmonic maps and minimal immersions, *Proc. London Math. Soc.* **57**(1988), 577–598.

[Uh] K. Uhlenbeck, Harmonic maps into Lie groups (Classical solutions of the chiral model), *J. Differential Geom.* **30**(1989), 1–50.

[V] G. Valli, On the energy spectrum of harmonic 2-spheres in unitary groups, *Topology* **27**(1988), 129–136.

[Wol] J. G. Wolfson, Harmonic sequences and harmonic maps of surfaces into complex Grassmann manifolds, *J. Differential Geom.* **27**(1988), 161–178.

[Wd1] J. C. Wood, The explicit construction and parametrization of all harmonic maps from the two-sphere to a complex Grassmannian, *J. Reine Angew. Math.* **386**(1988), 1–31.

[Wd2] J. C. Wood, Explicit construction and parametrization of harmonic two-spheres in the unitary group, *Proc. London Math. Soc.* **58**(1989), 608–624.

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
Federal Republic of Germany

and

Department of Mathematics
School of Medicine
Nihon University
Itabashi-Ku, Tokyo 173
Japan