On the Index of Elliptic Operators on a Wedge

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0 Introduction

In this paper we derive an index formula for an elliptic operator in the so-called *wedge algebra* introduced and developed by the second author and his school [4, 5, 6]. We confine ourselves to the most simplest case both from analytical and topological point of view. The simplest wedge is a direct product $W = \mathbb{R}^q \times X^\wedge$ where $\mathbb{R}^q$ is called the edge and $X^\wedge = X \times \mathbb{R}_+$ is a stretched cone over a smooth compact manifold without boundary X. Sometimes we will call $\mathbb{R}^q$ the base and $X^\wedge$ the fiber of the wedge.

The analysis of pseudo-differential operators on such a manifold is naturally performed in the framework of operator-valued symbols: we consider pseudo-differential operators on the base $\mathbb{R}^q$ with symbols $a(y,\eta)$ taking values in the so-called *cone algebra with asymptotics* $C(X^\wedge)$ on fibres. In general, the cone algebra with asymptotics deals with meromorphic Mellin symbols. For the index theory a minimal asymptotic information is sufficient: we consider Mellin symbols holomorphic in a narrow strip $S = \{\Re z-(n+1)/2+\gamma_0 \in (-\varepsilon,\varepsilon)\}$ around a fixed weight line $\Gamma_{(n+1)/2-\gamma_0}$. In other words, we deal with empty asymptotic data in the weight strip. The cone and the wedge theories under these assumptions are much simpler than in the general case. For the

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reader's convenience we give a brief description of the cone and wedge algebras in our particular case. The proofs are mostly omitted or sketched, for more details the reader is referred e.g. to [6].

A general concept of pseudo-differential operators with operator-valued symbols acting in Hilbert spaces was studied in the work [3]. Index theory for such operators includes the following problems:

1. elaborate a notion of topological index (t-index) and the corresponding concept of ellipticity (t-ellipticity),

2. the same problem for analytical index, in other words, one has to elaborate the notion of ellipticity (a-ellipticity) implying the Fredholm property of the operator,

3. prove the index theorem, that is show that both indices coincide if the symbol is both a-elliptic and t-elliptic.

In the case of the wedge algebra the operator-valued symbols $a(y,\eta)$ in question are operator families on $T^*(\mathbb{R}^q) = \mathbb{R}^{2q}$. The minimal assumptions for item 1 is that for any point (y,η) the symbol $a(y,\eta)$ is a Fredholm operator invertible outside a compact set. Under these assumptions the so-called *index bundle* $\operatorname{ind} a(y,\eta)$ is defined as an element of $K_c(\mathbb{R}^{2q}) \cong \mathbb{Z}$ (see [3]). The topological index of the operator $A = \mathrm{Op}$ is defined then with the help of the Chern character

$$\operatorname{ind}_t A = \int_{\mathbb{R}^{2q}} \operatorname{ch}\{\operatorname{ind} a(y,\eta)\} \in \mathbb{Z}.$$

The second item is more subtle. Of course, the symbol classes $S^m(\mathbb{R}^q \times \mathbb{R}^q)$ similar to the scalar-valued case may be introduced by means of the estimates

$$\|\partial_y^\alpha \partial_\eta^\beta a(y,\eta)\| \le C\langle\eta\rangle^{m-|\beta|}. \tag{0.1}$$

It turns out, however, that these estimates together with t-ellipticity *do not imply* a-ellipticity (see [3]). This is because they do not control fibrewise properties of $\partial_y^\alpha \partial_\eta^\beta a(y,\eta)$ such as compactness for $|\beta| > m$ or the Hilbert-Schmidt property giving only norm decay for large η. In [3] the symbols of order 0 with compact fibre variation were introduced for which t-ellipticity and estimates (0.1) do imply a-ellipticity in appropriate Sobolev spaces, and the index theorem was proved.

For the wedge algebra we can not apply directly the methods of [3] since a non-trivial twisted group action on fibres is involved in this case. Instead we use the notion of ellipticity for wedge operators in terms of their interior and edge principal symbols (see e.g [6]). It implies the Fredholm property in the spaces $W^{s,\gamma}$ via parametrix construction. At the same time this construction provides fibrewise parametrices which are precisely inverse operators for large y, η. Thus, t-ellipticity also holds, and we prove the index theorem in the form

$$\operatorname{ind} A = \frac{1}{(2\pi i)^q q!} \int_{\mathbf{R}^{2q}} \operatorname{tr}(dr \wedge da + (rda)^2)^q. \qquad (0.2)$$

Here $A = \operatorname{Op}(a(y,\eta))$ is an elliptic wedge operator, $r(y,\eta)$ its fibrewise parametrix up to trace class operators. Observe that the integrand in (0.2) is a differential form with compact support representing the Chern character $\operatorname{ch}\{\operatorname{ind} a(y,\eta)\}$.

The proof is based on an analytical approach suggested in [1] and algebraic machinery developed in [2]. We introduce the so-called *algebraic index* in the algebra of formal operator-valued symbols as an intermediate step between analytical and topological indices. The analytical part of the paper consists in the theorem on the regularized trace of a product. In section 1 we briefly discuss the cone and the wedge Sobolev spaces $K^{s,\gamma}$ and $W^{s,\gamma}$, especially trace class operators in these spaces. In section 2 the needed special cases of the cone and wedge algebras are discussed. Here the fibrewise properties of the derivatives in (0.1) are controlled by a special grading of $W^{s,\gamma}$. The next section deals with the parametrix construction. Our treatment is rather brief, more details may be found in [6] and references therein. In section 4 the theorem on a regularized trace of a product is proved. We have made some improvements comparing with the original proof in [1] allowing to avoid an analytical continuation in orders for operator-valued symbols. The proof of the index formula is briefly discussed in section 5 following [1, 2].

1 Cone and Wedge Sobolev Spaces

1.1 Cone Sobolev Spaces

By *a stretched cone* $X^\wedge$ with a base manifold X we mean a cartesian product $X \times \overline{\mathbb{R}}_+$ with the action of the group $\mathbb{R}_+$

$$\lambda(x,t) = (x, \lambda t) \tag{1.1}$$

$\lambda \in \mathbb{R}_+$, $(x,t) \in X \times \mathbb{R}_+$. The base X is supposed to be a smooth compact n-dimensional manifold without boundary. For a coordinate neighborhood $U \subset X$ we denote by $U^\wedge = U \times \mathbb{R}_+$ the stretched conical neighborhood in $X^\wedge$. We use a notation $V^\Delta \subset \mathbb{R}^{n+1}$ for a geometrical conical neighborhood corresponding to a coordinate neighborhood $V \subset S^n$ on the unit sphere in $\mathbb{R}^{n+1}$. The group $\mathbb{R}_+$ acts on V^Δ by homotheties. By a conical coordinate diffeomorphism $\chi : U^\wedge \to V^\Delta$ we mean a diffeomorphism which commutes with the action of $\mathbb{R}_+$. The inverse diffeomorphism

$$\chi^{-1} : V^\Delta \ni \tilde{x} \mapsto (x,t) \in U^\wedge \tag{1.2}$$

may be thought of as a passage to polar coordinates.

There are several modifications of the Sobolev spaces adopted to the conical structure.

1. The spaces $H^s(X^\wedge)$. For a function $u \in C_0^\infty(X^\wedge)$ with support in a stretched conical neighborhood $U^\wedge$ we take its push-forward

$$(\chi_* u)(\tilde{x}) = u(\chi^{-1}(\tilde{x}))$$

under conical coordinate diffeomorphism (1.2) and define

$$\|u\|_{H^s(X^\wedge)} = \|\chi_* u\|_{H^s(\mathbb{R}^{n+1})}.$$

The general case may be reduced to the above special one by taking a finite coordinate covering U_i of X and the corresponding partition of unity

$\rho_i(x)$. For a stretched conical neighborhood $U_i^\wedge$ we take a coordinate diffeomorphism $\chi_i : U_i^\wedge \to V_i^\Delta$ and set

$$\|u\|^2_{H^s(X^\wedge)} = \sum_i \|\chi_{i*}\rho_i u\|^2_{H^s(\mathbb{R}^{n+1})}. \tag{1.3}$$

It may be easily verified that the norm (1.3) does not depend on the covering, partition of unity and coordinate diffeomorphisms up to equivalence.

2. The weighted Sobolev spaces $H^{s,\gamma}(X^\wedge)$. Again we may consider a special case of $u(x,t) \in C_0^\infty(X^\wedge)$ with support in a conical coordinate neighborhood $U^\wedge$. Let $\hat{u}(\xi, z)$ denote the Mellin transform of $u(x,t)$ with respect to t, that is

$$\hat{u}(z) = \int_0^\infty t^{z-1} u(t)dt,$$

and the Fourier transform with respect to x. Then

$$\|u\|^2_{H^{s,\gamma}(X^\wedge)} = \int_{\mathbb{R}^n} \int_{\Gamma_{(n+1)/2-\gamma}} (1 + |\xi|^2 + |z|^2)^s |\hat{u}(\xi, z)|^2 dz d\xi \tag{1.4}$$

where $\Gamma_{(n+1)/2-\gamma} = \{z : \Re z = (n+1)/2 - \gamma\}$ is a so-called weight line on the complex plane.

Like before we can reduce the general case to this special one.

3. The cone spaces $K^{s,\gamma}(X^\wedge)$. Take a cut-off function $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ which is equal to 1 near $t = 0$ and set

$$\|u\|_{K^{s,\gamma}(X^\wedge)} = \|\omega(t)u\|_{H^{s,\gamma}(X^\wedge)} + \|(1 - \omega(t))u\|_{H^s(X^\wedge)}. \tag{1.5}$$

So, the space $K^{s,\gamma}$ is a "mixture" of the weighted space $H^{s,\gamma}$ near $t = 0$ and the usual Sobolev space $H^s(\mathbb{R}^{n+1})$ near $t = \infty$. The choice of $\omega(t)$ does not affect the norm (1.5) up to equivalence.

For any fixed cut-off function $\omega_0(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ equal to 1 near $t = 0$ we have four bounded multiplication operators

$$K^{s,\gamma} \xrightarrow{\omega_0} H^{s,\gamma} \tag{1.6}$$

$$H^{s,\gamma} \xrightarrow{\omega_0} K^{s,\gamma} \tag{1.7}$$

$$K^{s,\gamma} \xrightarrow{1-\omega_0} H^s \tag{1.8}$$

$$H^s \xrightarrow{1-\omega_0} K^{s,\gamma}. \tag{1.9}$$

The group $\mathbb{R}_+$ acts on any of these functional spaces since it acts on cones. It is convenient to modify this action by a factor, namely

$$(\kappa_\lambda u)(t,x) = \lambda^{\frac{n+1}{2}} u(\lambda t, x). \tag{1.10}$$

A general result concerning strongly continuous actions of $\mathbb{R}_+$ on Banach spaces consists in the following estimate for the norm of κ_λ

$$\|\kappa_\lambda\| \le C \left(\max\left(\lambda, \frac{1}{\lambda}\right)\right)^M \tag{1.11}$$

for some $C > O$ and $M \ge 1$ (see [8]). For our spaces $H^s(\mathbb{R}^{n+1})$, $H^{s,\gamma}(X^\wedge)$, $K^{s,\gamma}(X^\wedge)$ the estimate (1.11) for the action (1.10) may be proved directly.

There are continuous embeddings

$$H^{s_1} \hookrightarrow H^{s_2},\ H^{s_1,\gamma_1} \hookrightarrow H^{s_2,\gamma_2},\ K^{s_1,\gamma_1} \hookrightarrow K^{s_2,\gamma_2}$$

for $s_1 \ge s_2$, $\gamma_1 \ge \gamma_2$.

Lemma 1.1 *For $\psi(x) \in C_0^\infty(\mathbb{R}^{n+1})$ denote by $M(\lambda)$, $\lambda \ge 1$ a multiplication operator*

$$M(\lambda) : H^s(\mathbb{R}^{n+1}) \overset{\psi(\frac{x}{\lambda})}{\longrightarrow} H^{s-N}(\mathbb{R}^{n+1}), \tag{1.12}$$

$N \ge 0$. If $N > (n+1)/2$, then $M(\lambda)$ is a Hilbert-Schmidt operator and the following estimate holds for its Hilbert-Schmidt norm

$$\|M(\lambda)\|_{HS} \le C\lambda^{\frac{n+1}{2}} \tag{1.13}$$

Proof. Applying Fourier transform to both sides of the equation

$$v(x) = \psi\left(\frac{x}{\lambda}\right) u(x),$$

we obtain

$$\widehat{v}(\xi_1) = \int \lambda^{n+1} \widehat{\psi}(\lambda(\xi_1 - \xi)) \widehat{u}(\xi) d\xi.$$

To represent this operator as an integral operator between L^2-spaces, introduce the following functions

$$\widehat{v}_1(\xi_1) = \langle \xi_1 \rangle^{s-N} \widehat{v}(\xi_1)$$

$$\widehat{u}_1(\xi) = \langle\xi\rangle^s \widehat{u}(\xi).$$

The operator (1.12) may be rewritten as

$$\widehat{v}_1(\xi_1) = \int K(\xi_1,\xi)\widehat{u}_1(\xi)d\xi$$

where $\widehat{u}_1, \widehat{v}_1 \in L^2(\mathbb{R}^{n+1})$ and

$$K(\xi_1,\xi) = \lambda^{n+1}\frac{\langle\xi_1\rangle^{s-N}}{\langle\xi\rangle^s}\widehat{\psi}(\lambda(\xi_1-\xi)).$$

The Hilbert- Schmidt norm of the operator $M(\lambda)$ is equal to the L^2-norm of its kernel $K(\xi_1,\xi)$, so that

$$\begin{aligned}\|M(\lambda)\|_{HS}^2 &= \int |K(\xi_1,\xi)|^2 d\xi_1 d\xi \\ &= \int \frac{\langle\xi_1/\lambda\rangle^{2(s-N)}}{\langle\xi/\lambda\rangle^{2s}}|\widehat{\psi}(\xi_1-\xi)|^2 d\xi_1 d\xi.\end{aligned}$$

By Peetre's inequality

$$\frac{\langle\xi_1/\lambda\rangle^{2(s-N)}}{\langle\xi/\lambda\rangle^{2(s-N)}} \le C\langle(\xi_1-\xi)/\lambda\rangle^{2|s-N|}.$$

Since $\lambda \ge 1$, the right-hand side does not exceed $C\langle\xi_1-\xi\rangle^{2|s-N|}$, whence

$$\begin{aligned}&\|M(\lambda)\|_{HS}^2 \le C\int\langle\xi_1-\xi\rangle^{2|s-N|}|\widehat{\psi}(\xi_1-\xi)|^2 d\xi_1 \int\langle\xi/\lambda\rangle^{-2N}d\xi \\ &\le C_1\lambda^{n+1}\int\langle\xi\rangle^{-2N}d\xi\end{aligned}$$

proving the lemma.

□

Lemma 1.2 *Let* $\omega(t) \in C_0^\infty(\overline{\mathbb{R}_+})$, $\omega = 1$ *near* $t = 0$. *For* $\lambda \ge 1$ *let* $M(\lambda)$ *be a multiplication operator*

$$M(\lambda) : K^{s,\gamma}(X^\wedge) \xrightarrow{\omega(\frac{t}{\lambda})} K^{s-N,\gamma-\delta}(X^\wedge) \tag{1.14}$$

where $N \ge 0$, $\delta \ge 0$. *If* $N > (n+1)/2$ *and* $\delta > 0$, *then* $M(\lambda)$ *is a Hilbert-Schmidt operator and the estimate* (1.13) *holds.*

Proof. Take cut-off functions $\omega_0(t), \omega_1(t), \omega_2(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ equal to 1 near $t = 0$ and such that

$$\omega\omega_0 = \omega_0; \quad \omega_0\omega_1 = \omega_1, \quad \omega_2\omega_0 = \omega_0.$$

Then for any $\lambda \geq 1$ we have

$$\omega\left(\frac{t}{\lambda}\right) = \omega_0(t) + (1 - \omega_0(t))\omega\left(\frac{t}{\lambda}\right)$$

which gives a decomposition of $M(\lambda)$ into a sum of two multiplication operators

$$M(\lambda) = M_1 + M_2(\lambda)$$

where

$$M_1 : K^{s,\gamma}(X^\wedge) \xrightarrow{\omega_2(t)} H^{s,\gamma}(X^\wedge) \xrightarrow{\omega_0(t)} H^{s-N,\gamma-\delta}(X^\wedge) \xrightarrow{\omega_2(t)} K^{s-N,\gamma-\delta}(X^\wedge)$$

$$M_2(\lambda) : K^{s,\gamma}(X^\wedge) \xrightarrow{1-\omega_0(t)} H^{s}(X^\wedge) \xrightarrow{\omega(\frac{t}{\lambda})} H^{s-N}(X^\wedge) \xrightarrow{1-\omega_1(t)} K^{s-N,\gamma-\delta}(X^\wedge).$$

By (1.6)–(1.9) the extreme left and right operators in these sequences are bounded, so it is sufficient to estimate Hilbert-Schmidt norms of multiplication operators in the middle. For the operator

$$H^s(X^\wedge) \xrightarrow{\omega(\frac{t}{\lambda})} H^{s-N}(X^\wedge)$$

the desired estimate follows directly from Lemma 1.1. Since M_1 does not depend on λ, it is sufficient to prove that

$$H^{s,\gamma}(X^\wedge) \xrightarrow{\omega_0(t)} H^{s-N,\gamma-\delta}(X^\wedge)$$

is a Hilbert-Schmidt operator for $N > (n+1)/2, \quad \delta > 0$. Using partition of unity, we come to the multiplication operator $\omega_0(t)\rho(x)$ where $\rho(x)$ is supported in a coordinate neighborhood $U \subset X$. Using Mellin transform with respect to t and Fourier transform with respect to x, we get

$$\widehat{v}(z_1, \xi_1) = \int_{\mathbb{R}^n}\int_{\Gamma_{(n+1)/2-\gamma}} \widehat{\omega}_0(z_1 - z)\widehat{\rho}(\xi_1 - \xi)\widehat{u}(z, \xi)dzd\xi$$

for $v = \omega_0\rho u$. The function

$$\widehat{\omega}_0(z) = \int_0^\infty t^{z-1}\omega_0(t)dt = -\frac{1}{z}\int_0^\infty t^z \frac{d}{dt}\omega_0(t)dt$$

is holomorphic in the whole plane $\mathbb{C}$ except the first order pole at $z = 0$ and decreases rapidly along vertical lines Γ_β. In particular, if $z \in \Gamma_{(n+1)/2-\gamma}$ and $z_1 \in \Gamma_{(n+1)/2-\gamma+\delta}$ with $\delta > 0$, then

$$|\widehat{\omega}(z_1 - z)| \leq \frac{C}{\delta}\langle \Im(z_1 - z)\rangle^{-M}$$

for any $N > 0$. We proceed further similarly to Lemma 1.2. Introducing

$$\widehat{v}_1(z_1, \xi_1) = (1 + |\xi_1|^2 + |z_1|^2)^{(s-N)/2}\widehat{v}(z_1, \xi)$$

$$\widehat{u}_1(t, \xi) = (1 + |\xi|^2 + |z|^2)^{s/2}\widehat{u}(z, \xi),$$

we come to an integral operator

$$\widehat{v}_1(z_1, \xi) = \int K(z_1, \xi_1, z, \xi)\widehat{u}_1(z, \xi)dzd\xi$$

between L^2-spaces whose kernel is

$$K(z_1, \xi_1, z, \xi) = \frac{(1 + |\xi_1|^2 + |z_1|^2)^{(s-N)/2}}{(1 + |\xi|^2 + |z|^2)^{s/2}}\widehat{\omega}(z_1 - z)\widehat{\rho}(\xi_1 - \xi)$$

$$= (1 + |\xi|^2 + |z|^2)^{-N/2}\frac{(1 + |\xi_1|^2 + |z_1|^2)^{(s-N)/2}}{(1 + |\xi|^2 + |z|^2)^{(s-N)/2}}\widehat{\omega}(z_1 - z)\widehat{\rho}(\xi_1 - \xi)$$

Thus, for the Hilbert-Schmidt norm of M_1 we obtain

$$\|M_1\|^2_{HS} \leq \int_{\Gamma_{(n+1)/2-\gamma}} |dz| \int_{\Gamma_{(n+1)/2-\gamma+\delta}} |dz_1| \int_{\mathbf{R}^{2n}} |K(z_1, \xi_1, z, \xi)|^2 d\xi_1 d\xi.$$

By Peetre's inequality

$$\frac{(1 + |\xi_1|^2 + |z_1|^2)^{s-N}}{(1 + |\xi|^2 + |z|^2)^{s-N}} \leq C(1 + |\xi_1 - \xi|^2 + |z_1 - z|^2)^{|s-N|}$$

The integral over ξ_1, z_1 converges since $\widehat{\omega}$ and $\widehat{\rho}$ are rapidly decreasing functions while the integral over ξ, z converges, provided $N > (n+1)/2$.

□

1.2 Wedge Sobolev Spaces

The simplest model of *a stretched wedge* is a cartesian product

$$W = \mathbb{R}^q \times X^\wedge = \mathbb{R}^q \times \mathbb{R}_+ \times X \tag{1.15}$$

where $X^\wedge = \mathbb{R}_+ \times X$ is a stretched cone with a base manifold X of dimension n. In local charts on X the points of W are represented by triples

$$(y, t, x) \in \mathbb{R}^q \times \mathbb{R}_+ \times \mathbb{R}^n.$$

The wedge Sobolev spaces $W^{s,\gamma}$ are adopted to the fibering structure (1.15). The function $u = u(y, t, x) \in C_0^\infty(W)$ is considered as a function on $\mathbb{R}^q$ with values in the cone Sobolev space $K^{s,\gamma}(X^\wedge)$. Introduce the so-called *smooth norm function* $[\eta]$ for $\eta \in \mathbb{R}^q$ which is greater than or equal to 1 everywhere and $[\eta] = |\xi|$ for sufficiently large $|\xi| \geq C$. Obviously we have

$$[\eta] \sim \langle\eta\rangle = (1 + |\eta|^2)^{1/2}$$

where $\sim$ means that two-sided estimates hold

$$0 < C_1 \leq \frac{[\eta]}{\langle\eta\rangle} \leq C_2.$$

We also introduce a notation

$$\kappa(\eta) = \kappa_{[\eta]}$$

for the action (1.10) on $K^{s,\gamma}(X^\wedge)$. Then the norm in $W^{s,\gamma}$ is given by

$$\|u\|^2_{W^{s,\gamma}} = \int_{\mathbb{R}^q} [\eta]^{2s} \|\kappa^{-1}(\eta)\hat{u}(\eta)\|^2_{K^{s,\gamma}} d\eta \tag{1.16}$$

where

$$\hat{u}(\eta) = F_{y \to \eta}[u(y, t, x)]$$

is the Fourier transform with respect to y. A detailed exposition of the wedge Sobolev spaces (as well as the cone Sobolev spaces $K^{s,\gamma}$) may be found in [4, 8]. Here we would like to emphasize one interesting and important property of these spaces. Although the fibering structure of the wedge is involved in the definition (1.16) it turns out that for functions with supports away from the edge the norm (1.16) is equivalent to the usual Sobolev norm in $H^s(\mathbb{R}^{q+n+1})$ [6].

For any $N, \delta \geq 0$ there is a continuous embedding $W^{s,\gamma} \hookrightarrow W^{s-N,\gamma-\delta}$. We are interested in the Hilbert-Schmidt properties of this embedding.

Theorem 1.3 *Let* $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega(t) = 1$ *near* $t = 0$, $\varphi(y) \in C_0^\infty(\mathbb{R}^q)$ *and let*

$$M : W^{s,\gamma} \stackrel{\omega(t)\varphi(y)}{\longrightarrow} W^{s-N,\gamma-\delta} \tag{1.17}$$

be a multiplication operator followed by embedding. Then M *is a Hilbert-Schmidt operator for* $\delta > 0$ *and* $N > (q+n+1)/2$.

Proof. We proceed similarly to Lemma 1.1. Suppressing dependence on $x \in X$ denote

$$v(y,t) = \varphi(y)\omega(t)u(y,t)$$

or applying the Fourier transform with respect to $y \in \mathbb{R}^q$

$$\widehat{v}(\eta_1,t) = \int \widehat{\varphi}(\eta_1 - \eta)\omega(t)\widehat{u}(\eta,t)d\eta. \tag{1.18}$$

Introducing the functions

$$\widehat{u}_1(\eta,t) = [\eta]^s\kappa^{-1}(\eta)\widehat{u}(\eta,t)$$

$$\widehat{v}_1(\eta_1,t) = [\eta_1]^{s-N}\kappa^{-1}(\eta_1)\widehat{v}(\eta_1,t)$$

we represent the operator (1.18) as an integral operator

$$\widehat{v}_1(\eta_1,t) = \int \frac{[\eta_1]^{s-N}}{[\eta]^s}\widehat{\varphi}(\eta_1-\eta)\kappa^{-1}(\eta_1)\omega(t)\kappa(\eta)\widehat{u}_1(\eta,t)d\eta$$

between L^2-spaces $L^2(\mathbb{R}^q, K^{s,\gamma}(X^\wedge)) \to L^2(\mathbb{R}^q, K^{s-N,\gamma-\delta}(X^\wedge))$. Its kernel

$$K(\eta_1,\eta) = \frac{[\eta_1]^{s-N}}{[\eta]^{s-N}}\widehat{\varphi}(\eta_1-\eta)\frac{1}{[\eta]^N}\kappa^{-1}(\eta_1)\omega(t)\kappa(\eta)$$

is a function in $(\eta_1,\eta) \in \mathbb{R}^{2q}$ with values in the space $\mathcal{L}(K^{s,\gamma}, K^{s-N,\gamma-\delta})$. Using Lemma 1.2, we will show that $K(\eta_1,\eta)$ is a Hilbert-Schmidt operator and estimate its Hilbert-Schmidt norm. Apart from a constant factor our operator is

$$\kappa^{-1}(\eta_1)\omega(t)\kappa(\eta) = \kappa^{-1}(\eta_1)\kappa(\eta)\omega\left(\frac{t}{[\eta]}\right) = \kappa_{[\eta]/[\eta_1]}\omega\left(\frac{t}{[\eta]}\right).$$

Applying (1.11) and then using Peetre's inequality, we get

$$\|\kappa_{[\eta]/[\eta_1]}\| \leq C\left(\max\left(\frac{[\eta]}{[\eta_1]}, \frac{[\eta_1]}{[\eta]}\right)\right)^M \leq C_1[\eta_1 - \eta]^M$$

where $\| \ \|$ means the operator norm in $K^{s-N,\gamma-\delta}$. Next, by Lemma 1.2 the Hilbert-Schmidt norm of the multiplication operator by $\omega\left(\frac{t}{[\eta]}\right)$ acting from $K^{s,\gamma}$ to $K^{s-N,\gamma-\delta}$ may be estimated as $C[\eta]^{(n+1)/2}$. Since

$$\frac{[\eta_1]^{s-N}}{[\eta]^{s-N}} \leq C[\eta_1 - \eta]^{|s-N|},$$

we come to the following estimate

$$\|K(\eta_1, \eta)\|_{HS} \leq C[\eta_1 - \eta]^{|s-N|+M}|\hat{\varphi}(\eta_1 - \eta)|[\eta]^{-N+(n+1)/2}.$$

Thus,

$$\|M\|_{HS}^2 = \int_{\mathbf{R}^{2q}} \|K(\eta_1, \eta)\|_{HS}^2 d\eta d\eta_1$$
$$\leq C\int_{\mathbf{R}^q} |[\eta_1 - \eta]^{|s-N|+M}\hat{\varphi}(\eta_1 - \eta)|^2 d\eta_1 \int_{\mathbf{R}^q} [\eta]^{-2N+n+1} d\eta.$$

The first integral converges since $\hat{\varphi}$ is a rapidly decreasing function while the second one converges in virtue of the inequality $2N > n + 1 + q$. □

Corollary 1.4 *For* $\delta > 0$, $N > n+q+1$ *the operator* M *given by* (1.17) *belongs to the trace class.*

Proof. Let $\omega_1(t)$, $\varphi(y)$ be other functions with compact supports and such that $\omega(t)\omega_1(t) = \omega(t)$, $\varphi(y)\varphi_1(y) = \varphi(y)$. The operator M may be factorized by

$$M : W^{s,\gamma} \overset{\varphi(y)\omega(t)}{\longrightarrow} W^{s-N/2,\gamma-\delta/2} \overset{\varphi_1(y)\omega_1(t)}{\longrightarrow} W^{s-N,\gamma-\delta},$$

both factors being the Hilbert-Schmidt operators, whence the assertion follows. □

2 The Cone and the Wedge Algebras

2.1 The Cone Algebra

We will need the very special case of *a cone algebra with asymptotics.* For reader's convenience we give here its brief description, the wedge algebra will be considered in the next subsection. As a rule, the proofs are omitted, more detailed exposition may be found in [6].

The operators of the cone algebra are defined first on functions

$$u(t) = u(t,x) \in C_0^\infty(X^\wedge) = C_0^\infty(\mathbb{R}_+, C^\infty(X))$$

and then are extended by continuity to the cone Sobolev spaces

$$A : K^{s_1,\gamma_1} \to K^{s_2,\gamma_2} \tag{2.1}$$

with some $s_1, s_2, \gamma_1, \gamma_2 \in \mathbb{R}^n$. One of the indices s_1, s_2 may be chosen arbitrarily. If for any $s_1 \in \mathbb{R}^n$ we may take $s_2 = s_1 - m$ the operator is said to be of order m (more precisely, not greater then m). If (2.1) holds for any $s_1, s_2 \in \mathbb{R}^n$, the operator is called *smoothing.*

The indices γ_1, γ_2 are more rigid, neither of them, in general, may be chosen arbitrarily. We suppose that there are two fixed weights α_1, α_2 such that for sufficiently small $\varepsilon > 0$ (2.1) holds for any $\gamma_1 \in (\alpha_1 - \varepsilon, \alpha_1 + \varepsilon)$ and

$$\gamma_2 = \alpha_2 + \gamma_1 - \alpha_1 \in (\alpha_2 - \varepsilon, \alpha_2 + \varepsilon).$$

These two weights α_1, α_2 will be called *weight data.* If γ_1, γ_2 may be chosen independently within the *weight intervals*

$$\gamma_1 \in (\alpha_1 - \varepsilon, \alpha_1 + \varepsilon), \quad \gamma_2 \in (\alpha_2 - \varepsilon, \alpha_2 + \varepsilon), \tag{2.2}$$

the operator is called *flattening* (with respect to the given weights α_1, α_2). If the operator is both smoothing and flattening, it is called a *Green operator* (with respect to the given weights α_1, α_2). The set of Green operators is denoted by $C_G = C_G(\alpha_1, \alpha_2)$.

The operators (2.1) of finite order form an algebra (provided the indices match) and Green operators form an ideal. Since $K^{s,\gamma}$ are dual spaces, the adjoint operator A^* belongs to the algebra if A does, moreover, $A^* \in C_G$ if $A \in C_G$.

Now we define the cone algebra $C = C(X^\wedge)$. An operator A of order $m \in \mathbb{R}$ (notation $A \in C^m$) consists of three summands

$$\begin{aligned} A &= A_F + A_M + A_G = \\ &= \omega_\infty(t)t^{-m}\mathrm{Op}_F(a_i(t,t\tau))\tilde{\omega}_\infty(t) + \\ &+ \omega_0(t)t^{-m}\mathrm{Op}_M(a_v(t,z))\tilde{\omega}_0(t) + A_G \end{aligned} \tag{2.3}$$

called *Fourier, Mellin and Green operators* respectively. Here $\omega_0(t), \omega_\infty(t)$ form a partition of unity on $\overline{\mathbb{R}}_+, \omega_0(t) \equiv 1$ near the vertex $t = 0$ and has compact support, $\omega_\infty(t) = 1 - \omega_0(t)$, the functions $\tilde{\omega}_0, \tilde{\omega}_\infty$ are equal to 1 on the supports of ω_0, ω_∞ respectively and $\tilde{\omega}_0, 1 - \tilde{\omega}_\infty \in C_0^\infty(\overline{\mathbb{R}}_+)$.

The first item in (2.3) is a usual *Fourier pseudo-differential operator*

$$\mathrm{Op}_F(a_i)u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\tau(t-t_1)}a_i(t,t\tau)u(t_1)dt_1 \tag{2.4}$$

with a Fuchs-type symbol $a_i(t, t\tau)$ (the subscript i stands for *interior*). The values of the symbol are pseudo-differential operators of order m on X, so in a local coordinate chart on X we have

$$a_i(t, x, \tilde{\tau}, \xi) \in S_{cl}^m(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{n+1}). \tag{2.5}$$

The second item in (2.1) is a *Mellin pseudo-differential operator*

$$\mathrm{Op}_M(a_v(t,z))u(t) = \frac{1}{2\pi i}\int_{\Gamma_{(n+1)/2-\gamma}} dz \int_0^\infty \left(\frac{t_1}{t}\right)^z a_v(t,z)u(t_1)\frac{dt_1}{t_1}. \tag{2.6}$$

Here $a_v(t,z)$ (v stands for *vertex*) is a holomorphic function in a *weight-strip*

$$S = \{|Rez - (n+1)/2 - \gamma_0| < \varepsilon\}$$

for some fixed γ_0 and $\varepsilon > 0$ with values in pseudo-differential operators on X, so that in a coordinate chart on X we have

$$a_v(t, x, z, \xi) \in S_{cl}^m(\overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \Gamma_\beta \times \mathbb{R}^n)$$

on any line $\Gamma_\beta = \{\Re z = \beta\} \subset S$ uniformly in $|\beta - (n+1)/2 + \gamma_0| \le \varepsilon_0 < \varepsilon$.

The operators (2.4), (2.6) under these assumptions have bounded extension

$$A_F + A_M : K^{s,\gamma} \to K^{s-m,\gamma-m} \tag{2.7}$$

for any $s \in \mathbb{R}^n$ and $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$. Without loss of generality we may take $\gamma_0 = 0$ in the sequel.

The operator (2.4) and (2.6) must be *compatible* in the sense that for any functions $\varphi, \psi \in C_0^\infty(X^\wedge)$ the inclusion

$$\varphi(\mathrm{Op}_F(a_i) - \mathrm{Op}_M(a_v))\psi \in L^{-\infty}(X^\wedge) \tag{2.8}$$

holds (see [6, 8.1.3, Theorem 2]). The notation $L^{-\infty}$ means smoothing operators. Observe that (2.8) is also an infinitely flattening operator since $\varphi, \psi \equiv 0$ in the neighborhood of $t = 0$.

The last item in (2.3) is a Green operator usually with respect to the weights $\gamma_0, \gamma_0 - m$. In this case the whole operator $A = A_F + A_M + A_G$ acts in the spaces (2.7). Sometimes (these cases will be specified) we will need Green operator with respect to the weights $\gamma_0, \gamma_0 - l$ with $l \geq m$. In these cases (2.7) implies

$$A = A_F + A_M + A_G : K^{s,\gamma} \to K^{s-m,\gamma-l} \tag{2.9}$$

because of the embedding

$$K^{s-m,\gamma-m} \hookrightarrow K^{s-m,\gamma-l}.$$

We also will use an obvious modification of operators (2.3)

$$A : \ K^{s,\gamma} \oplus \mathbb{C}^{N_-} \to K^{s-m,\gamma-m} \oplus \mathbb{C}^{N_+}.$$

According to these decompositions the operators are represented by 2×2 matrices. The first two items in (2.3) act in direct summands $K^{s,\gamma}$, so, more accurately, they should be represented by a matrix

$$\begin{pmatrix} A_F + A_M & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.10}$$

However, we will not use these pedantic notations keeping in mind that A_F and A_M enter only the left upper corner of the full matrix. As for the Green item, it is represented by a full matrix which admits an extension

$$A_G = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} : \begin{array}{c} K^{s_1,\gamma_1} \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \longrightarrow \begin{array}{c} K^{s_2,\gamma_2-l} \\ \oplus \\ \mathbb{C}^{N_+} \end{array} . \tag{2.11}$$

for any $s_1, s_2 \in \mathbb{R}^n, \quad \gamma_1, \gamma_2 \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$.

Let us list some important properties of the cone operators (2.3).

1. If $A \in C^{m_1}$ and $B \in C^{m_2}$ then $AB \in C^{m_1+m_2}$, so C is an algebra called the *cone algebra*. For the interior and vertex symbols of the product we have the following asymptotic sums

$$a_i \circ b_i \sim \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \partial_\xi^k a_i \partial_t^k b_i \tag{2.12}$$

$$a_v \circ b_v \sim \sum_{k=0}^{\infty} \frac{1}{k!} \partial_z^k a_v (t\partial_t)^k b_v. \tag{2.13}$$

   These expansions are called *Leibnitz products* for interior (2.12) and vertex (2.13) symbols. If the symbols a_i, a_v and b_i, b_v satisfy compatibility conditions (2.8), then (2.12) and (2.13) are automatically compatible.

2. Green operators form an ideal C_G in the cone algebra. The whole calculus is performed modulo Green remainders. In particular, the choice of cut-off functions $\tilde{\omega}_0, \tilde{\omega}_\infty$ as well as the partition of unity ω_0, ω_∞ is unessential modulo C_G.

There is another ideal denoted by C_{M+G} which consists of smoothing operators (2.3). Since the Fourier item A_F is necessarily flattening (since $\omega_\infty = \tilde{\omega}_\infty = 0$ near $t = 0$), it means that smoothing A_F are actually Green operators. So, we may assume that $a_i \equiv 0$ including the item A_F into A_G. The operators from C_{M+G} are called *smoothing Mellin operators.* It is especially important to specify the weight data since the information concerning orders is "forgotten" for smoothing operators. For the Mellin operator

$$A_M = \omega_0(t) t^{-l} \mathrm{Op}_M(a_v(t,z)) \tilde{\omega}_0(t) \tag{2.14}$$

with $a_v \in S^{-\infty}$ natural weight data are $\gamma_0, \gamma_0 - l$ (l has nothing to do with the order which is $-\infty$).

Remark 2.1 The smoothing operator (2.14) is in fact a Green operator with respect to the weight data $\gamma_0, \gamma_0 - l$ if $a_v(0,z) = 0$. Indeed, it is flattening since a_v may be written in the form $a_v(t,z) = tb(t,z)$ with some smoothing symbol $b(t,z)$ and the factor t gives a gain of weight at least by 1. Thus, any smoothing Mellin symbol may be decomposed

$$a_v(t,z) = a_v(0,z) + (a_v(t,z) - a_0(0,z))$$

into a Mellin symbol not depending on t and a symbol vanishing at $t = 0$. It means that the corresponding operator (2.14) modulo Green operators may be represented by a symbol independent of t (of course, all the Mellin symbols in question are holomorphic in the weight strip S).

Remark 2.2 Any cone operator (2.3) is uniquely defined modulo C_{M+G} by its interior symbol $a_i(t, t\tau)$. Indeed, there is a standard way to construct a Mellin symbol $a_v(t,z)$ holomorphic in the whole plane $z \in \mathbb{C}$ and satisfying the compatibility condition (2.8) [6, 8.13, Theorem 2]. This procedure will be referred to as "Mellinization". Any other choice of the compatible Mellin symbol differs by a cone operator with smoothing interior symbol, so the difference belongs to the ideal C_{M+G}.

We finish our description of the cone algebra by two important additional properties of Green operators.

Lemma 2.3 *For any Green operator $G \in C_G$ with respect to the weight data γ, γ and any cut-off function $\omega(t) \in C_0^\infty(\overline{\mathbb{R}}_+)$ the operator*

$$\omega G : K^{s,\gamma} \to K^{s,\gamma}$$

belongs to the trace class.

Proof. Indeed, the operator may be factorized

$$K^{s,\gamma} \to K^{s+N,\gamma+\delta} \stackrel{\omega(t)}{\to} K^{s,\gamma}$$

with $N > n + 1$ and $\delta > 0$. The second operator belongs to the trace class (cf. Corollary 1.4). □

Lemma 2.4 *In the assumption of Lemma 2.3 let*

$$1 + G : K^{s,\gamma} \to K^{s,\gamma}$$

be invertible for at least one $s = s_0$. Then it is invertible for any $s \in \mathbb{R}^n$ and $(1+G)^{-1} - 1$ is a Green operator.

Proof. If u belongs to the kernel of $1+G$, then $u = -Gu \in K^{s,\gamma}$ for any s. So, the kernel does not depend on s. The same is true for the cokernel.

Next, denoting

$$G_1 = (1+G)^{-1} - 1,$$

we have

$$G_1 = -G(1+G_1).$$

Since $1 + G_1$ is bounded in the space $K^{s,\gamma}$ and G belongs to the ideal C_G, it follows that G_1 also belongs to C_G.

□

2.2 The Wedge Algebra

We consider now our main object *the wedge algebra* $\mathcal{Y}$ for a model wedge $W = \mathbb{R}^q \times X^\wedge = \mathbb{R}^q \times \mathbb{R}_+ \times X$. Let $m \in \mathbb{R}$ and $l \geq m$. We define a wedge operator of order m with respect to the weight data $\gamma_0, \gamma_0 - l$ as a pseudodifferential operator on $\mathbb{R}^q$ with a symbol $a(y,\eta)$ whose values are cone operators from C^m satisfying the conditions below. We use notation

$$A = \mathrm{Op}_F(a(y,\eta)) \in \mathcal{Y}^m.$$

According to (2.3)

$$a(y,\eta) = a_F(y,\eta) + a_M(y,\eta) + a_G(y,\eta). \tag{2.15}$$

This operator depending on parameters $(y,\eta) \in \mathbb{R}^{2q}$ is considered in the spaces

$$a(y,\eta) : E \to \tilde{E}$$

with

$$E = K^{s,\gamma} \oplus \mathbb{C}^{N_-}, \tilde{E} = K^{s-m,\gamma-l} \oplus \mathbb{C}^{N_+}$$

for $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$.

First, we suppose that the estimates hold

$$||\kappa^{-1}(\eta)\partial_y^\alpha \partial_\eta^\beta a(y,\eta)\kappa(\eta)||_{L(E,\widetilde{E})} \leq C[\eta]^{m-|\beta|}. \tag{2.16}$$

Here $\kappa(\eta) = \kappa[\eta]$ means the action (1.10) in the spaces $K^{s,\gamma}$ and a trivial action in $\mathbb{C}^{N_\pm}$. The symbol spaces defined by estimates (2.16) will be denoted by $S^m(\mathbb{R}^q_y \times \mathbb{R}^q_\eta, E, \widetilde{E})$ or simply $S^m(\mathbb{R}^{2q})$. We will also use the spaces $S^m_{cl}(\mathbb{R}^q_y \times \mathbb{R}^q_\eta, E, \widetilde{E})$ for which

$$a(y,\eta) \sim \sum_{j=0}^{\infty} a_{m-j}(y,\eta)$$

with $a_{m-j} \in S^{m-j}(\mathbb{R}^{2q})$ being homogeneous of order $m-j$ for large η:

$$a(y,\lambda\eta) = \lambda^{m-j}\kappa(\lambda)a_{m-j}(y,\eta)\kappa^{-1}(\lambda) \tag{2.17}$$

for $\lambda \geq 1$, $|\eta| \geq C > 0$. Any such function defines uniquely a purely homogeneous function $\widetilde{a}_{m-j}(y,\eta)$ for $\eta \neq 0$ satisfying (2.17) everywhere with any $\lambda > 0$. Conversely, having a homogeneous function $\widetilde{a}_m(y,\eta)$ of order m we define a symbol $a_m(y,\eta) = \chi(\eta)\widetilde{a}_m(y,\eta)$ by means of a cut-off function $\chi(\eta)$ which is equal to 0 for $|\eta| \leq c/2$ and to 1 for $|\eta| \geq c$. Clearly, this symbol satisfies (2.16).

Next, we would like all the three items in (2.15) to satisfy estimates (2.16) separately. To this end we take $a_F(y,\eta)$ and $a_M(y,\eta)$ in the form

$$a_F(y,\eta) = \omega_\infty(t[\eta])t^{-m}\mathrm{Op}_F(a_i(y,t,t\eta,t\tau))\widetilde{\omega}_\infty(t[\eta]) \tag{2.18}$$

$$a_M(y,\eta) = \omega_0(t[\eta])t^{-m}\mathrm{Op}_M(a_v(y,t,t\eta,z))\widetilde{\omega}_0(t[\eta]). \tag{2.19}$$

Here $\mathrm{Op}_F, \mathrm{Op}_M$ mean Fourier and Mellin pseudo-differential operators with respect to the variable t (cf. (2.4), (2.6)). The interior and the vertex symbols a_i and a_v satisfy the same conditions as for the cone algebra with regard to the specific dependence on η:

1. $a_i(y,t,t\eta,t\tau)$ is a pseudo-differential operator on X with

$$a_i(y,t,\widetilde{\eta},\widetilde{\tau}) \in C^\infty(\mathbb{R}^q_y \times \overline{\mathbb{R}}_+, L^m_{cl}(X;\mathbb{R}^{q+1}_{\widetilde{\eta},\widetilde{\tau}})), \tag{2.20}$$

2. $a_v(y,t,t\eta,z)$ is holomorphic in the strip

$$z \in S = \{\Re z \in ((n+1)/2 - \gamma_0 - \varepsilon, (n+1)/2 - \gamma_0 + \varepsilon)\}$$

for some $\varepsilon > 0$ and on any weight line Γ_β

$$a_v(y,t,\tilde{\eta},z) \in C^\infty(\mathbb{R}^q_y \times \overline{\mathbb{R}}_+, L^m_{cl}(X; \mathbb{R}^{q+1}_{\tilde{\eta},\tilde{\tau}} \times \Gamma_\beta)) \tag{2.21}$$

uniformly in $\beta = (n+1)/2 - \gamma_0 - \gamma$ with $|\gamma| \le \varepsilon_0 < \varepsilon$,

3. the compatibility condition holds

$$\mathrm{Op}_F(a_i(y,t,t\eta,t\tau)) - \mathrm{Op}_M(a_v(y,t,t\eta,z)) \in C^\infty(\mathbb{R}^q_y, L^{-\infty}(X^\wedge; \mathbb{R}^q_\eta)) \tag{2.22}$$

(see [6, 9.2.3, Theorem 1]).

As for the Green item $a_G(y,\eta)$ we assume

$$a_G(y,\eta) \in S^m_{cl}(\mathbb{R}^n_y \times \mathbb{R}^n_\eta, K^{s_1,\gamma_1} \otimes \mathbb{C}^{N_-}, K^{s_2,\gamma_2} \otimes \mathbb{C}^{N_+}) \tag{2.23}$$

for any $s_1, s_2 \in \mathbb{R}^n$, $|\gamma_1 - \gamma_0| < \varepsilon$, $|\gamma_2 - \gamma_0 + l| < \varepsilon$.

Similarly to the cone algebra the Green symbol is given by 2×2 matrix while a_F and a_M enter only the left upper corner of the matrix.

Instead of (2.20) we may also assume that in a local chart $U \subset X$ the operator $a_i(y,t,t\eta,t\tau)$ is given by a complete symbol $a_i(y,t,x,t\eta,t\tau,\xi)$ with

$$a_i(y,t,x,\tilde{\eta},\tilde{\tau},\xi) \in S^m_{cl}(\mathbb{R}^q \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{q+1+n}). \tag{2.24}$$

Following [6], we call such symbols a_i *edge-degenerate.* The space of the operator-valued symbols $a(y,\eta)$ with the above properties (2.20)–(2.23) will be denoted by $\mathcal{R}^m$ (the weight data are tacitly meant). So, $\mathcal{Y}^m$ is the space of operators $\mathrm{Op}_F(a(y,\eta))$ with $a(y,\eta) \in \mathcal{R}^m$. The properties of the wedge operator are mostly similar to those of the cone operators.

1. Any $A \in \mathcal{Y}^m$ acts continuously in the wedge Sobolev spaces

$$A : W^{s,\gamma} \to W^{s-m,\gamma-l}$$

for any $s \in \mathbb{R}^n$ and $\gamma \in (\gamma_0 - \varepsilon, \gamma_0 + \varepsilon)$.

2. The wedge operators form an algebra, so that for $A \in \mathcal{Y}^{m_1}, B \in \mathcal{Y}^{m_2}$ $AB \in \mathcal{Y}^{m_1+m_2}$.

3. The ideals C_G, C_{M+G} in the cone algebra generate corresponding ideals $\mathcal{R}_G, \mathcal{R}_{M+G}$ and $\mathcal{Y}_G, \mathcal{Y}_{M+G}$ by the requirement that the symbols $a(y,\eta)$ take values in C_G or C_{M+G}.

These ideals have an additional filtration $\mathcal{R}_G^m, \mathcal{R}_{M+G}^m, \mathcal{Y}_G^m, \mathcal{Y}_{M+G}^m$ defined by the order m of the operator-valued symbol $a(y,\eta)$. In contrast to the cone algebra this order is not forgotten, it enters the relations (2.23), (2.16). All the properties of these ideals such as Lemmas 2.3, 2.4 and Remarks 2.1, 2.2 remain valid *fibrewise*, that is for $a(y,\eta)$ considered at any fixed value of y,η as an operator in the cone algebra. Because of the edge-degeneracy Remark 2.1 takes a more sharpened form:

$$a_v(y,t,t\eta,z) \equiv a_v(y,0,0,z) \mod \mathcal{Y}_G^m \tag{2.25}$$

for a smoothing Mellin symbol $a_v(y,t,t\eta,z)$.

To get global properties of $A = \mathrm{Op}(a(y,\eta)) \in \mathcal{Y}$ from the fibrewise properties of the symbol $a(y,\eta) \in \mathcal{R}$ we need some *stabilization conditions* for large t and y.

Definition 2.5 *We say that $A \in \mathcal{Y}$ stabilizes to 0 if there is a constant $C > 0$ such that for any function $u(t,y) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^q, C_0^\infty(X))$ supported outside a ball $t^2 + |y|^2 \geq C^2$ we have $Au = 0$. If $A = 1 + B$ where 1 means an identity operator and B stabilizes to 0, we say that A stabilizes to 1.*

For example, if the Green item $a_G(y,\eta)$ in (2.15) is absent, this definition means that the symbols $a_i(y,t,t\eta,t\eta)$ and $a_v(y,t,t\eta,z)$ vanish for $|y|^2 + t^2 \geq C^2$. The case when A stabilizes to 1 gives analogous conditions: $a_i \equiv 1$ for $|y|^2 + t^2 \geq C^2$.

The following lemma is crucial for the index theory. Roughly speaking, it says that derivation along the base covariable η not only gives better norm decay for large η, but improves fibrewise properties of the symbol.

Lemma 2.6 *Let $A \in \mathcal{R}^m$ with respect to the weight data $\gamma, \gamma - l$ $(l \geq m)$ and let $a(y,\eta)$ stabilize to 0. Then for $|\beta| > 0$ and some $\delta > 0$*

$$\partial_y^\alpha \partial_\eta^\beta a(y,\eta) \in S^{m-|\beta|}(\mathbb{R}^{2q}, K^{s,\gamma} \oplus C^{N-}, K^{s-m+|\beta|,\gamma-l+\delta} \oplus C^{N+}). \tag{2.26}$$

Observe that (2.26) is a sharpening of (2.16) since the space $\widetilde{E} = K^{s-m,\gamma-l}$ in (2.16) is replaced by $K^{s-m+|\beta|,\gamma-l+\delta}$ which has better smoothness and weight.

Proof. Let us consider first a Green symbol

$$a_G(y,\eta) \in S^m_{cl}(\mathbb{R}^{2q}, K^{s_1,\gamma_1} \oplus C^{N-}, K^{s_2,\gamma_2} \oplus C^{N+}). \tag{2.27}$$

From (2.17) it follows that for a homogeneous function a_{m-j} its derivatives $\partial_y^\alpha a_{m-j}$ have the same degree of homogeneity as a_{m-j} itself, while each derivation with respect to η diminishes this degree by 1. Hence,

$$\partial_y^\alpha \partial_\eta^\beta a_G(y,\eta) \in S^{m-|\beta|}_{cl}(\mathbb{R}^{2q}, K^{s_1,\gamma_1} \oplus C^{N-}, K^{s_2,\gamma_2} \oplus C^{N+})$$

that is $\partial_y^\alpha \partial_\eta^\beta a_G$ is again a Green symbol of the order $m - |\beta|$ with respect to the same weight data $\gamma, \gamma - l$. But for Green symbols we may restrict weight intervals $\gamma_1 \in (\gamma - \delta, \gamma + \delta)$, $\gamma_2 \in (\gamma - l - \delta, \gamma - l + \delta)$ to embedded ones $\gamma_1 \in (\gamma - \delta/2, \gamma + \delta/2)$, $\gamma_2 \in (\gamma - l, \gamma - l + \delta)$. It means that the Green operator with respect weight data $\gamma, \gamma - l$ is also a Green operator with respect to $\gamma, \gamma - l + \delta/2$. It remains to consider the operator-valued symbols of the form

$$\begin{aligned} a(y,\eta) &= a_F(y,\eta) + a_M(y,\eta) \\ &= \omega_\infty(t[\eta])t^{-m}\mathrm{Op}_F(a_i(y,t,t\eta,t\tau))\widetilde{\omega}_\infty(t[\eta]) \\ &\quad + \omega_0(t[\eta])t^{-m}\mathrm{Op}_M(a_v(y,t,t\eta,z))\widetilde{\omega}_0(t[\eta]) \end{aligned} \tag{2.28}$$

omitting the Green items. Conjugation by $\kappa(\eta) = \kappa_{[\eta]}$ yields

$$\begin{aligned} &\kappa^{-1}(\eta)a(y,\eta)\kappa(\eta) = \\ &= [\eta]^m t^{-m}\omega_\infty(t)\mathrm{Op}_F(a_i(y,\frac{t}{[\eta]},t\frac{\eta}{[\eta]},t\tau))\widetilde{\omega}_\infty(t) \\ &\quad + [\eta]^m t^{-m}\omega_0(t)\mathrm{Op}_M(a_v(y,\frac{t}{[\eta]},t\frac{\eta}{[\eta]},z))\widetilde{\omega}_0(t). \end{aligned} \tag{2.29}$$

The operator (2.29) is bounded in the norm of $\mathcal{L}(K^{s,\gamma}, K^{s-m,\gamma-m})$ for any y, η, we need to show that apart from the factor $[\eta]^m$ it is uniformly bounded with respect to y, η. It is really the case since the variables $y, t\eta/[\eta], t/[\eta]$ vary over compact sets (details may be found in [6, Chapter 9]).

Let us apply $\partial/\partial\eta_k$ to (2.28). Denoting for brevity

$$a_i'(y,t,\tilde{\eta},\tilde{\tau}) = \frac{\partial}{\partial\tilde{\eta}_k} a_i(y,t,\tilde{\eta},\tilde{\tau})$$
$$a_v'(y,t,\tilde{\eta},z) = \frac{\partial}{\partial\tilde{\eta}_k} a_v(y,t,\tilde{\eta},z)$$

we get

$$\begin{aligned}
&\kappa^{-1}(\eta)\frac{\partial}{\partial\eta_k}a(y,\eta)\kappa(\eta) = \\
&= [\eta]^{m-1}t^{-m+1}\omega_\infty(t)\mathrm{Op}_F(a_i'(y,\frac{t}{[\eta]},t\frac{\eta}{[\eta]},t\tau))\tilde{\omega}_\infty(t) \\
&+[\eta]^{m-1}t^{-m+1}\omega_0(t)\mathrm{Op}_M(a_v'(y,\frac{t}{[\eta]},t\frac{\eta}{[\eta]},z))\tilde{\omega}_0(t) \\
&+[\eta]^{m-1}t^{-m+1}\omega_\infty(t)\mathrm{Op}_F\tilde{\omega}_\infty'(t)\frac{\partial[\eta]}{\partial\eta_k} \\
&+[\eta]^{m-1}t^{-m+1}\omega_0(t)\mathrm{Op}_M\tilde{\omega}_0'(t)\frac{\partial[\eta]}{\partial\eta_k} \\
&+[\eta]^{m-1}t^{-m+1}\omega_\infty'(t)\mathrm{Op}_F\tilde{\omega}_\infty(1-\tilde{\omega}_0)\frac{\partial[\eta]}{\partial\eta_k} \\
&+[\eta]^{m-1}t^{-m+1}\omega_0'(t)\mathrm{Op}_M\tilde{\omega}_0(1-\tilde{\omega}_\infty)\frac{\partial[\eta]}{\partial\eta_k} \\
&+[\eta]^{m-1}t^{-m+1}\omega_0'(t)(\mathrm{Op}_M-\mathrm{Op}_F)\tilde{\omega}_0\tilde{\omega}_\infty\frac{\partial[\eta]}{\partial\eta_k}
\end{aligned} \tag{2.30}$$

where

$$\mathrm{Op}_F = \mathrm{Op}_F(a_i(y,\frac{t}{[\eta]},t\frac{\eta}{[\eta]},t\tau))$$
$$\mathrm{Op}_M = \mathrm{Op}_M(a_v(y,\frac{t}{[\eta]},t\frac{\eta}{[\eta]},z)).$$

The first two summands in (2.30) have the same form as (2.29) with m replaced by $m-1$, hence they are uniformly bounded in $\mathcal{L}(K^{s,\gamma},K^{s-m+1,\gamma-m+1})$ apart from the factor $[\eta]^{m-1}$. The rest of the summands are infinitely smoothing and infinitely flattening for fixed y,η. For example, the third summand is smoothing because the supports of ω_∞ and $\tilde{\omega}'_\infty$ do not intersect and is infinitely flattening because ω_∞ vanishes near $t = 0$. The last summand is smoothing due to the compatibility condition (2.22) and flattening because $\omega'_0 = 0$ near $t = 0$. Taking into account the factor $[\eta]^{m-1}$ we see that all the terms but two first ones in (2.30) are Green symbols of order $m - 1$ with respect to any weight data. Homogeneous components may be obtained by the Taylor expansion of symbols a_i, a_v in t :

$$a_i(y, t, t\eta, t\tau) \sim \sum \partial_t^\alpha a_i(y, 0, t\eta, t\tau) \frac{t^\alpha}{\alpha!}$$

and similar expansion for a_v. In subsequent considerations such Green symbols may be thrown off. Finally, $\partial/\partial y_k$ applied to (2.28) do not affect orders and weights, so $\partial a/\partial y_k$ is an operator-valued symbol of the same type as $a(y, \eta)$.

Thus, using induction, we obtain

$$\partial_y^\alpha \partial_\eta^\beta a(y, \eta) \in S^{m-|\beta|}(\mathbb{R}^{2q}, K^{s,\gamma}, K^{s-m+|\beta|,\gamma-m+|\beta|})$$

for $a(y, \eta)$ given by (2.28).

It remains to observe that for $l \geq m$ and $|\beta| > 0$ we have an embedding

$$K^{s-m+|\beta|,\gamma-m+|\beta|} \hookrightarrow K^{s-m+|\beta|,\gamma-m+\delta}$$

for a $\delta > 0$ sufficiently small. □

3 Ellipticity and Parametrix Construction

In this section we consider elliptic operators of zero order in the wedge algebra. We also take $\gamma_0 = 0$ considering operator-valued symbols

$$a(y, \eta) : K^{s,\gamma} \oplus \mathbb{C}^{N_-} \rightarrow K^{s,\gamma} \oplus \mathbb{C}^{N_+}$$

for any $s \in \mathbb{R}$ and $\gamma \in (-\varepsilon, \varepsilon)$. In other words, $a(y, \eta) \in \mathcal{R}^0$ with respect to the weight data γ, γ and $A = \mathrm{Op}(a(y, \eta)) \in \mathcal{Y}^0$. Throughout this section we assume also that all the operators (and symbols) stabilize for $|y|^2 + t^2 \geq C^2$ either to 0 or to identity. Now, we define two *principal symbols* for the cone and wedge operators. For a cone operator A of the form (2.3) (with $m = 0$) its *interior principal symbol* is defined by

$$\sigma_\psi(A) = a_{i(0)}(t, x, t\tau, \xi) \tag{3.1}$$

for $t > 0$, $(\tau, \eta) \neq 0$. Here $a_{i(0)}(t, x, \tilde{\tau}, \xi)$ is the homogeneous part in $(\tilde{\tau}, \xi)$ of the highest degree 0 of the function (2.5) (with $m = 0$) written in the local coordinates on X. Similarly to the smooth case, the choice of local coordinates is unessential, so that (3.1) is a function on $T^*(intX^\wedge) \setminus 0$.

Next, the *conormal symbol* of A is defined by

$$\sigma_M(A) = a_v(0, z) : H^s(X) \to H^s(X). \tag{3.2}$$

It is an operator acting in Sobolev spaces on X and depending on a parameter $z \in \Gamma_{(n+1)/2-\gamma}$.

For the wedge operator of order 0 its interior principal symbol is defined similarly to (3.1)

$$\sigma_\psi(A) = a_{i(0)}(y, t, x, t\eta, t\tau, \xi) \tag{3.3}$$

where $a_{i(0)}$ means the homogeneous component of the highest degree zero of the function (2.24).

Finally, the *principal edge symbol* of $A \in \mathcal{Y}^0$ is an operator-valued function

$$\sigma_\wedge(A)(y, \eta) : \begin{matrix} K^{s,\gamma} \\ \oplus \\ \mathbb{C}^{N_-} \end{matrix} \longrightarrow \begin{matrix} K^{s,\gamma} \\ \oplus \\ \mathbb{C}^{N_+} \end{matrix} \tag{3.4}$$

defined for $\eta \neq 0$, homogeneous in η of the highest degree 0:

$$\sigma_\wedge(A)(y, \lambda\eta) = \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & 1 \end{pmatrix} \sigma_\wedge(A)(y, \eta) \begin{pmatrix} \kappa_\lambda^{-1} & 0 \\ 0 & 1 \end{pmatrix} \tag{3.5}$$

For the operator $A = \mathrm{Op}(a(y, \eta))$ with $a(y, \eta)$ given by (2.15), (2.18), (2.19), (2.23) the operator (3.4) is equal to

$$\begin{aligned} \sigma_\wedge(A) = {} & \omega_0(t|\eta|)\mathrm{Op}_M(a_v(y, 0, t\eta, z))\tilde{\omega}_0(t|\eta|) + \\ & + \omega_\infty(t|\eta|)\mathrm{Op}_F(a_i(y, 0, t\eta, t\tau))\tilde{\omega}_\infty(t|\eta|) + a_{G(0)}(y, \eta) \end{aligned} \tag{3.6}$$

where $a_{G(0)}(y, \eta)$ means the homogeneous component of the highest degree 0 of the classical operator-valued symbol a_G. Of course, homogeneity is understood in the sense of (3.5). The expression (3.6) may be rewritten (at least formally) as a limit

$$\lim_{\lambda\to+\infty}\begin{pmatrix}\kappa_\lambda^{-1} & 0\\ 0 & 1\end{pmatrix} a(y,\eta)\begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix} \tag{3.7}$$

where $a(y,\eta)\in\mathcal{R}^0$ is the operator-valued symbol (2.15). Although the symbol $a(y,\eta)$ is by no means classical, its principal edge symbol $\sigma_\wedge$ may be thought of as a homogeneous component of $a(y,\eta)$ of the highest degree 0 as (3.7) shows. In fact, the homogeneity of (3.6) in the sense of (3.5) is easily seen.

The principal symbols may be defined in the case $m\neq 0$ with obvious modifications, for example, $\sigma_\psi(A)=t^{-m}a_{i(m)}$. We confine ourselves to the case $m=0$ only.

All the introduced symbol maps are homomorphisms: for A_1, A_2 belonging to C^0 or to $\mathcal{Y}^0$ we have obviously

$$\sigma_\psi(A_1A_2)=\sigma_\psi(A_1)\sigma_\psi(A_2), \tag{3.8}$$

$$\sigma_M(A_1A_2)=\sigma_M(A_1)\sigma_M(A_2), \tag{3.9}$$

$$\sigma_\wedge(A_1A_2)=\sigma_\wedge(A_1)\sigma_\wedge(A_2). \tag{3.10}$$

Since $\sigma_\wedge(A)$ is a cone operator for any $y,\eta\neq 0$ it has a conormal symbol which may be found using (3.6) by throwing off the Green part and putting $t=0$. This yields

$$\sigma_M\sigma_\wedge(A)=a_v(y,0,0,z) \tag{3.11}$$

independent of η. Of course, $\sigma_M\sigma_\wedge$ is also a homomorphism.

Along with the notations $\sigma_\psi(A)$, $\sigma_\wedge(A)$, $\sigma_M\sigma_\wedge(A)$ we will use analogous notations $\sigma_\psi(a(y,\eta))$, $\sigma_\wedge(a(y,\eta))$, $\sigma_M\sigma_\wedge(a(y,\eta))$ for the operator-valued symbol $a(y,\eta)\in\mathcal{R}^0$.

Next, we define *elliptic operators* in the cone and wedge algebras C^0 or $\mathcal{Y}^0$ postulating that both principal symbols are invertible. So, an operator $A\in C^0$ is elliptic if the following two conditions hold.

1. $\sigma_\psi(A)$ is invertible, more precisely, $a_{i(0)}(t,\tilde\tau,x,\xi)$ is invertible for all $t\geq 0, x\in X$ and $(\tilde\tau,\xi)\neq 0$. This condition is usually called *interior ellipticity.*

2. $\sigma_M(A)(z) : H^s(X) \to H^s(X)$ is an invertible operator (at least for one $s = s_0$) for all $z \in \Gamma_{(n+1)/2-\gamma}$ (*conormal ellipticity*).

Similarly, an operator $A \in \mathcal{Y}^0$ or an operator-valued symbol $a(y,\eta) \in \mathcal{R}^0$ is called elliptic if interior ellipticity holds as in item 1 above and besides

2. $\sigma_\wedge(A)(y,\eta)$ is an invertible operator (3.5) for all $\eta \neq 0$ for at least one $s = s_0$ (*edge ellipticity*).

In the rest of this section we are dealing with parametrix constructions allowing one to obtain both t-ellipticity and a-ellipticity (see Introduction). Our main theorem reads as follows.

Theorem 3.1 *Let $a(y,\eta) \in \mathcal{R}^0$ be elliptic and stabilize to identity. Then there exists an operator-valued symbol $r(y,\eta) \in \mathcal{R}^0$ such that*

$$1 - r(y,\eta)a(y,\eta), \quad 1 - a(y,\eta)r(y,\eta) \tag{3.12}$$

are Green symbols vanishing outside a compact set in $(y,\eta) \in \mathbb{R}^{2n}$.

Having constructed the symbol $r(y,\eta)$ we use it as an initial step to construct successively more and more precise parametrices $R_N = \mathrm{Op}(r_N(y,\eta))$ in the algebra $\mathcal{Y}^0$.

Theorem 3.2 *Let $A \in \mathcal{R}^0$ be elliptic and stabilize to identity. Then for any integer $N > 0$ there exists an operator $\mathcal{R}_N \in \mathcal{Y}^0$ such that*

$$1 - R_N A, \quad 1 - A R_N$$

are Green operators of order $-N$ (that is belong to $\mathcal{Y}_G^{-N}$).

A similar theorem holds for the cone algebra. For further references let us formulate it separately.

Theorem 3.3 *Let $A \in C^0$ be elliptic and stabilize to identity. Then there exists an operator $R \in C^0$ such that*

$$1 - RA, \ \ 1 - AR \in C_G.$$

The last theorem is, of course, a parameter-independent particular case of Theorem 3.1. However, we need it in the proof of Theorem 3.1, so Theorem 3.3 must be proved independently. We obtain such a proof as a result of two initial steps in the proof of Theorem 3.1

Since Green operators in the cone algebra are compact (they even belong to the trace class by Lemma 2.3) Theorem 3.3 means that the operator A with invertible interior and conormal symbols is a Fredholm operator in the space $K^{s,\gamma}$. We need also an inverse statement: if $A \in C^0$ is a Fredholm operator in $K^{s,\gamma}$ for at least one $s = s_0$ then both its principal symbols (interior and conormal) are invertible. For the interior principal symbol this statement is a usual consequence of elliptic theory in the smooth case. For the conormal symbol it may be proved using the same ideas, see e.g. [4].

The above theorems assert not only the existence of the parametrix but also its belonging to a certain class. Lemma 2.4 gives a simple example of such a statement. The proofs of Theorems 3.1, 3.2, 3.3 are actually reductions of more complicated cases to this simplest one.

Proof of Theorem 3.1. We divide the proof into several steps.

1. For an operator-valued symbol $a(y,\eta) \in \mathcal{R}^0$ consider the corresponding complete interior symbol $a_i(y,t,x,t\eta,t\tau,\xi)$ in local coordinates on X with

$$a_i(y,t,x,\tilde{\eta},\tilde{\tau},\xi) \in S^0_{cl}(\mathbb{R}^q \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{q+1+n}). \tag{3.13}$$

Although the symbol classes S^m_{cl} in (3.13) are defined with respect to all the variables y,t,x and corresponding covariables η,τ,ξ, we treat y,η as parameters which are not involved into the Leibnitz product. In particular, η is a large parameter of parameter-dependent elliptic theory. Because of stabilization conditions we may assume that y,t vary on a compact set.

Now interior ellipticity implies that the symbol $a_i(y,t,x,t\eta,t\tau,\xi)$ is elliptic in the usual sense of parameter-dependent elliptic theory. Thus, we may construct the Leibnitz inverse for a_i, that is the symbol $r_i^1(y,t,x,t\eta,t\tau,\xi)$ with

$$r_i^1(y,t,x,\tilde{\eta},\tilde{\tau},\xi) \in S^0_{cl}(\mathbb{R}^q \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \times \mathbb{R}^{q+1+n}) \tag{3.14}$$

such that

$$1 - \sum_{|\alpha|=0}^{\infty} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi,\tau} r_i^1 \partial^\alpha_{x,t} a_i \in S^{-\infty}(\mathbb{R}^q \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{q+1+n}) \tag{3.15}$$

(y,η are parameters). These local complete symbols define a family of pseudo-differential operators on X

$$r_i^1(y,t,t\eta,t\tau) : H^s(X) \to H^s(X) \tag{3.16}$$

such that the operator-valued function $r_i^1(y,t,\tilde{\eta},\tilde{\tau})$ is smooth up to $t=0$. Such a family allows "Mellinization", that is one can construct another family $r_v^1(y,t,\tilde{\eta},z)$ smooth up to $t=0$ and holomorphic in $z\in\mathbb{C}$ such that the operators $\mathrm{Op}_F(r_i^1(y,t,t\eta,t\tau)$ and $\mathrm{Op}_M(r_v^1(y,t,t\eta,z))$ are compatible in the sense of (2.22).

Finally we define an operator-valued symbol

$$
\begin{aligned}
&r^1(y,\eta)=\\
&=\omega_\infty(t[\eta])\mathrm{Op}_F(r_i^1(y,t,t\eta,t\tau))\tilde{\omega}_\infty(t[\eta])\\
&+\omega_0(t[\eta])\mathrm{Op}_M(r_v^1(y,t,t\eta,z))\tilde{\omega}_0(t[\eta])
\end{aligned}
$$

belonging to $\mathcal{R}^0(\mathbb{R}^q\times\mathbb{R}^q,K^{s,\gamma},K^{s,\gamma})$. If necessary we may border this symbol by zeros to obtain a 2×2 matrix with the given left upper corner $r^1(y,\eta)$.

Thus, at the first step we have constructed an operator-valued symbol $r^1(y,\eta)\in\mathcal{R}^0$ satisfying the relation

$$
r^1(y,\eta)a(y,\eta)=1-b^1(y,\eta) \tag{3.17}
$$

in the cone algebra with a smoothing $b^1(y,\eta)$. Indeed, the interior symbol of (3.17) differs from 1 by a smoothing symbol in virtue of (3.15). In other words, we have $b^1\in\mathcal{R}^0_{M+G}$.

Remark 3.4 If a stabilizes to 1 then necessarily $N_+=N_-$, so that $a(y,\eta)$ acts in the space $K^{s,\gamma}\oplus\mathbb{C}^N$. By construction our symbol $r^1(y,\eta)$ also satisfies stabilization conditions, but it stabilizes to the matrix

$$
\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix} \tag{3.18}
$$

2. At the second step we correct $r^1(y,\eta)$ to obtain a symbol $r^2(y,\eta)\in\mathcal{R}^0$ which is a more accurate parametrix of $a(y,\eta)$, namely,

$$
1-r^2(y,\eta)a(y,\eta)=b^2(y,\eta)\in\mathcal{R}^0_G. \tag{3.19}
$$

Consider the edge symbol $\sigma_\wedge(a)$ given by (3.6). By the edge ellipticity it is an invertible (and hence Fredholm) operator in the space $K^{s,\gamma}\oplus\mathbb{C}^N$ for $\eta\neq 0$. In its turn, this property necessarily implies conormal ellipticity of the Fredholm operator $\sigma_\wedge(a)$. Thus,

$$
\sigma_M\sigma_\wedge(a)=a_v(y,0,0,z):H^s(X)\to H^s(X)
$$

has an inverse

$$a_v^{-1}(y,0,0,z) : H^s(X) \to H^s(X) \tag{3.20}$$

independently of η. Now, taking conormal symbols $\sigma_M\sigma_\wedge$ in (3.17), we get

$$r_v^1(y,0,0,z)a_v(y,0,0,z) = 1 - b_v^1(y,0,0,z)$$

with b_v^1 rapidly decreasing in the strip $z \in S$. By (3.20) we may rewrite the last equality in the form

$$r_v^1(y,0,0,z) = a_v^{-1}(y,0,0,z) + c(y,\eta), \tag{3.21}$$

with $c(y,z) = a_v^{-1}(y,0,0,z)b_v^1(y,0,0,z)$ which also decreases rapidly in the strip $z \in S$. Thus, the operator-valued symbol

$$\Delta r(y,\eta) = \omega_0(t[\eta])\mathrm{Op}_M(a_v^{-1}(y,0,0,z) - r_v^1(y,0,0,z))\tilde{\omega}_0(t[\eta])$$

is a smoothing Mellin one, and we set

$$r^2(y,\eta) = r^1(y,\eta) + \Delta r(y,\eta).$$

Now, by construction

$$\sigma_M\sigma_\wedge(r^2(y,\eta)) = a_v^{-1}(y,0,0,z) = (\sigma_M\sigma_\wedge(a))^{-1},$$

so that

$$\sigma_M\sigma_\wedge(1 - r^2(y,\eta)a(y,\eta)) = 0.$$

But the smoothing Mellin symbol vanishing at $t = 0$ is a Green symbol (see Remark 2.1), implying (3.19).

Remark 3.5 For the parameter-independent case with $N = 0$ these two steps prove Theorem 3.3.

3. At this step we construct the further correction $r^3 = r^2 + \Delta r$ with $\Delta r \in \mathcal{R}_G^0$, such that $\sigma_\wedge(r^3)$ is invertible for $\eta \neq 0$. So far we may affirm only that $\sigma_\wedge(r^2)$ is a Fredholm operator because of conormal ellipticity: $\sigma_M\sigma_\wedge(r^2) = (\sigma_M\sigma_\wedge(a))^{-1}$ is invertible. Its index does not depend on y, η for $\eta \neq 0$ and is equal to 0 since, for large y, r^2 stabilizes to the matrix (3.18) in $K^{s,\gamma} \oplus \mathbb{C}^N$, so that both kernel and cokernel coincide with $\mathbb{C}^N$.

There is a standard way to obtain an invertible operator starting with a Fredholm one by bordering the latter with finite-dimensional operators. This procedure looks as follows.

Denoting the principal homogeneous edge symbol $\sigma_\wedge(r^2)$ by $d(y,\eta)$ we may restrict ourselves to the compact set of parameters y,η with $|y| \leq C$ (because of stabilization condition) and $|\eta| = 1$ because of the homogeneity

$$d(y,\lambda\eta) = \kappa(\lambda)d(y,\lambda)\kappa^{-1}(\lambda).$$

For a Fredholm family $d(y,\eta) : K^{s,\gamma} \to K^{s,\gamma}$ on a compact parameter space one can find a map $k : \mathbb{C}^M \to K^{s,\gamma}$ independent of y,η with some $M \in \mathbb{N}$ such that the operator

$$(d(y,\eta),k) : \begin{array}{c} K^{s,\gamma} \\ \oplus \\ \mathbb{C}^M \end{array} \longrightarrow K^{s,\gamma} \tag{3.22}$$

is surjective. It may be done at any point y_0,η_0 using Fredholm property of $d(y_0,\eta_0)$. Since surjective operators form an open set in the space of all bounded operators, the surjectivity of (3.22) holds in some neighborhood of y_0,η_0. By compactness arguments we find a finite covering U_i, $i = 1,2,\ldots,m$ and corresponding maps $k_i : \mathbb{C}^{M_i} \to K^{s,\gamma}$ such that $(d(y,\eta),k_i)$ are surjective operators in U_i. Then $(d(y,\eta),k) = (d(y,\eta),k_1,k_2,\ldots,k_m)$ gives the desired operator with $M = M_1 + M_2 + \ldots + M_m$, surjective for all y,η. Its kernel

$$J_{y,\eta} = \mathrm{Ker}(d(y,\eta),k) \subset K^{s,\gamma} \oplus \mathbb{C}^M$$

is a subspace in $K^{s,\gamma} \oplus \mathbb{C}^M$ of dimension M since by surjectivity of $(d(y,\eta),k)$

$$\dim J_{y,\eta} = \mathrm{ind}(d(y,\eta),k) = \mathrm{ind}\, d(y,\eta) + M = M$$

because $\mathrm{ind}\, d(y,\eta) = 0$. Being of constant dimension this subspace depends smoothly on parameters y,η defining a vector bundle over $\mathbb{R}^q \times S^{q-1}$. Let

$$(p(y,\eta),q(y,\eta)) : J_{y,\eta} \to \mathbb{C}^M \tag{3.23}$$

be an isomorphism defined at least locally. Assuming for a moment that this isomorphism may be defined globally on $\mathbb{R}^q \times S^{q-1}$, we obtain an invertible matrix

$$\begin{pmatrix} d(y,\eta) & k \\ p(y,\eta) & q(y,\eta) \end{pmatrix} : \begin{array}{c} K^{s,\gamma} \\ \oplus \\ \mathbb{C}^M \end{array} \longrightarrow \begin{array}{c} K^{s,\gamma} \\ \oplus \\ \mathbb{C}^M \end{array}$$

which may be extended by homogeneity for $|y| \le C, \quad \eta \ne 0$ by

$$\begin{pmatrix} \kappa_{|\eta|} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d(y, \frac{\eta}{|\eta|}) & k \\ p(y, \frac{\eta}{|\eta|}) & q(y, \frac{\eta}{|\eta|}) \end{pmatrix} \begin{pmatrix} \kappa_{|\eta|} & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

and then define the needed operator-valued symbol $r^3(y,\eta)$ as a matrix

$$\begin{pmatrix} r^2(y,\eta) & \varphi(\eta)k(\eta) \\ \varphi(\eta)p(y,\eta) & \varphi(\eta)q(y,\eta) \end{pmatrix} \tag{3.24}$$

where $\varphi(\eta)$ is an excision function equal to 1 for $|\eta| \ge c > 0$. Clearly, $\Delta r = r^3 - r^2$ is a Green symbol since its left upper corner is identically zero. We may force the operators $a(y,\eta)$ and $r^3(y,\eta)$ to act in the same space $K^{s,\gamma} \oplus \mathbb{C}^{M+N}$ with the help of extra borderings obtaining new symbols of the form

$$a = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r^3 = \begin{pmatrix} * & 0 & * \\ 0 & 1 & 0 \\ * & 0 & * \end{pmatrix}.$$

It remains to show that the needed global bordering (3.24) actually exists.

Lemma 3.6 *Let*

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} : \begin{matrix} H \\ \oplus \\ \mathbb{C}^N \end{matrix} \longrightarrow \begin{matrix} H \\ \oplus \\ \mathbb{C}^N \end{matrix} \tag{3.25}$$

be an invertible family on a compact manifold V. Then $a_{11}(x)$ is a Fredholm operator in H and for any family of its parametrices $d(x)$ there exists a global invertible bordering

$$\begin{pmatrix} d(x) & k(x) \\ p(x) & q(x) \end{pmatrix} : \begin{matrix} H \\ \oplus \\ \mathbb{C}^M \end{matrix} \longrightarrow \begin{matrix} H \\ \oplus \\ \mathbb{C}^M \end{matrix}.$$

Proof. Let

$$\begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix} : \begin{matrix} H \\ \oplus \\ \mathbb{C}^N \end{matrix} \longrightarrow \begin{matrix} H \\ \oplus \\ \mathbb{C}^N \end{matrix}$$

be inverse matrix for (3.24). Then

$$1 - b_{11}a_{11} = b_{12}a_{21}; \quad 1 - a_{11}b_{11} = a_{12}b_{21}.$$

The operators on the right-hand sides are finite-dimensional, whence $b_{11}(x)$ is a family of parametrices for $a_{11}(x)$. Consider a homotopy of parametrices

$$d(x,t) = (1-t)b_{11}(x) + td(x)$$

$t \in [0,1]$. Reasoning as before for (3.22) and using compactness of $V \times [0,1]$, we find a map $k(x,t) : \mathbb{C}^M \to H$ such that the operator

$$(d(x,t), k(x,t)) : \begin{array}{c} H \\ \oplus \\ \mathbb{C}^M \end{array} \longrightarrow H$$

is surjective. Moreover, we may take $k(x,0) = b_{12}(x)$. Then

$$J = J_{x,t} = \operatorname{Ker}(d(x,t), k(x,t))$$

is a vector bundle over $V \times [0,1]$. Show that this bundle is trivial, that is there exist M linearly independent sections of J. For $t = 0$ such a basis is given by the column matrix

$$\begin{pmatrix} a_{12}(x) & 0 \\ a_{22}(x) & 0 \\ 0 & 1 \end{pmatrix}$$

where 1 means $(M-N) \times (M-N)$ identity matrix (we assume $M \geq N$). For $t \in [0,1]$ the existence of the basis follows from the covering homotopy theorem for the principal frame bundle $GL(J)$ [9].

Let the column matrix

$$e(x,t) = \begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} : \mathbb{C}^M \longrightarrow \begin{array}{c} H \\ \oplus \\ \mathbb{C}^M \end{array} \qquad (3.26)$$

of M linearly independent vectors of $H \oplus \mathbb{C}^M$ denote the global frame of J. Then the global trivialization of J is a row matrix $(p(x,t), q(x,t))$ which is a left inverse for (3.26), for example, we may take it in the form $(e^*e)^{-1}e^*$. □

4. Let us proceed with the construction of the parametrix. At the previous step we have constructed $r^3 = r^2 + \Delta r$ with a Green symbol Δr such that $\sigma_\wedge(r^3)$ is invertible for $\eta \neq 0$. Then

$$r^3 a = 1 - b^3$$

with a Green symbol b^3 and

$$\sigma_\wedge(r^3)\sigma_\wedge(a) = 1 - \sigma_\wedge(b^3).$$

By construction the left-hand side is invertible, so is the right-hand side. By Lemma (2.4)

$$(1 - \sigma_\wedge(b^3))^{-1} = 1 + w(y, \eta)$$

with a Green symbol $w(y,\eta)$ defined for $\eta \neq 0$. Multiplying it by an excision function $\varphi(\eta)$ equal to 1 for η large, we obtain a Green operator-valued symbol

$$w_1(y,\eta) = \varphi(\eta) w(y,\eta) \in \mathcal{R}^0_G.$$

Define $r^4 = (1 + w_1) r^3$. Then

$$r^4 a = (1 + w_1)(1 - b^3) = 1 - b^4$$

where $b_4 \in \mathcal{R}^0_G$ is a Green symbol such that $\sigma_\wedge(b^4) = 0$. By definition of Green symbols b^4 is a classical one and $\sigma_\wedge(b^4)$ is its homogeneous leading part of degree 0. Since $\sigma_\wedge(b^4)$ vanishes identically, the highest degree of homogeneity is equal to -1, thus $b^4 \in \mathcal{R}^{-1}_G$.

5. At the last step we construct the final correction r^5 such that

$$1 - r^5 a = b^5 \in \mathcal{R}_G$$

and $b^5 = 0$ outside a compact in $\mathbb{R}^{2q}$. To this end observe that $b^4 \in \mathcal{R}^{-1}_G$ satisfies the following estimate

$$\|\kappa^{-1}(\eta) b^4(y,\eta) \kappa(\eta)\| \leq C[\eta]^{-1}$$

where $\|\cdot\|$ means the operator norm in the space

$$\mathcal{L}(K^{s_1,\gamma_1} \oplus \mathbb{C}^{M+N}, K^{s_2,\gamma_2} \oplus \mathbb{C}^{M+N})$$

with any $s_1, s_2 \in \mathbb{R}$ and $\gamma_1, \gamma_2 \in (-\varepsilon, \varepsilon)$. Thus, the operator

$$\kappa^{-1}(\eta)(1 - b^4(y,\eta))\kappa(\eta)$$

is invertible for large η and so is $1 - b^4(y,\eta)$. Thus, using Lemma 2.4 once more, we may write for large η

$$(1 - b^4(y,\eta))^{-1} = 1 + w_2(y,\eta)$$

with a Green symbol $w_2(y,\eta)$ defined for $|\eta| \geq C > 0$. Multiplication by an excision function which is equal to 0 for $|\eta| < C$ and to 1 for $|\eta| > 2C$ yields a Green symbol

$$w_3(y,\eta) = \varphi(\eta)w_2(y,\eta) \in \mathcal{R}_G^{-1}.$$

We set $r^5 = (1 + w_3(y,\eta))r^4$.

Thus, we have constructed a left parametrix r^5 such that

$$1 - r^5 a \in \mathcal{R}_G^{-1}$$

and vanishes outside a compact in $\mathbb{R}^{2q}$ (for large y it is evident from stabilization conditions which are respected at each step of our construction).

The same reasoning gives a right parametrix with the same properties, so any of these parametrices is a two-sided parametrix. This completes the proof of Theorem 3.1. □

The proof of Theorem 3.2 will be given in the next section.

4 A regularized Trace of a Product

Consider two operators $A, B \in \mathcal{Y}^0$ with operator-valued symbols $a(y,\eta)$ and $b(y,\eta)$ of the form (2.9). In this section we study the operator

$$C_N = AB - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \mathrm{Op}(\partial_\eta^\alpha a(y,\eta)\partial_y^\alpha b(y,\eta)). \tag{4.1}$$

Using the notation

$$a \circ b|_N = \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\eta^\alpha a(y,\eta)\partial_y^\alpha b(y,\eta)$$

for the truncated Leibnitz product of symbols we may rewrite (4.1) in a more compact form

$$C_N = AB - \mathrm{Op}(a \circ b|_N).$$

The basic property of this operator is described in the following lemma.

Lemma 4.1 *For any fixed $\gamma \in (-\varepsilon, \varepsilon)$ there exists a $\delta > 0$ such that the operator*

$$C_N : W^{s,\gamma} \to W^{s+N,\gamma+\delta} \tag{4.2}$$

is bounded for any $s \in \mathbb{R}$.

The crucial point is that the regularization procedure (4.1) gives not only better smoothing properties but also better weights.

Proof. First introduce some notations. Let $a(y,\eta) \in \mathcal{R}^m$ be an operator-valued symbol stabilizing to 0 and acting between the spaces

$$K^{s,\gamma} \to K^{s-m,\gamma-l}, \ (l \geq m).$$

Denote by

$$\widehat{a}(\zeta,\xi) = \int e^{-ix\zeta} a(x,\xi)dx.$$

its Fourier transform with respect to y. Then for any $p > 0$ we have

$$\|\kappa^{-1}(\eta)\widehat{a}(\zeta,\eta)\kappa(\eta)\| \leq C_{pm}[\zeta]^{-p}[\eta]^m$$

where the norm means the operator norm in the above-mentioned spaces. This estimates are fulfilled by definition of the symbol classes

$$S^m(\mathbb{R}^{2q}, K^{s,\gamma}, K^{s-m,\gamma-l}).$$

We shall briefly write them in the form

$$\kappa^{-1}(\eta)\widehat{a}(\zeta,\eta)\kappa(\eta) = O([\zeta]^{-\infty}[\eta]^m).$$

Note that multiplication by $[\zeta]^q$ for any $q \in \mathbb{R}$ does not change this form.

Now, let us return to (4.1) with $A, B \in \mathcal{Y}^0$. In terms of the Fourier transform the operator B acts as

$$\widehat{Bu}(\eta) = \int \widehat{b}(\eta - \xi, \xi)\widehat{u}(\xi)d\xi.$$

Applying this formula once more, we obtain

$$\widehat{ABu}(\zeta) = \int\int \widehat{a}(\zeta - \eta, \eta)\widehat{b}(\eta - \xi, \xi)\widehat{u}(\xi)d\xi d\eta.$$

By Taylor's formula

$$\widehat{a}(\zeta - \eta, \eta) =$$
$$= \sum_{|\alpha|<N} \frac{1}{\alpha!}\partial_\xi^\alpha \widehat{a}(\zeta - \eta, \xi)(\eta - \xi)^\alpha + R_N(\zeta - \eta, \eta, \xi).$$

The sum here corresponds to the sum in (4.1), so for the remainder C_N we have

$$\widehat{C_N u}(\zeta) = \int\int R_N(\zeta - \eta, \eta, \xi)\widehat{b}(\eta - \xi, \xi)\widehat{u}(\xi)d\xi d\eta.$$

Using the integral form for the remainder in Taylor's formula

$$R_N(\xi - \eta, \eta, \xi) =$$
$$= N\int_0^1 (1-t)^{N-1} \sum_{|\alpha|=N} \frac{1}{\alpha!}\widehat{a}^{(\alpha)}(\xi - \eta, \xi + t(\eta - \xi))(\eta - \xi)^\alpha dt,$$

where $\widehat{a}^{(\alpha)}$ mean derivatives with respect to the second argument, we see that the Schwartz kernel $K_N(\zeta, \xi)$ of the operator C_N may be represented in the form

$$K_N(\zeta, \eta) =$$
$$= N\int_0^1 (1-t)^{N-1}dt \int \sum_{|\alpha|=N} \frac{1}{\alpha!}\widehat{a}^{(\alpha)}(\zeta - \xi - \theta, \xi + t\theta)\theta^\alpha \widehat{b}(\theta, \xi)d\theta. \quad (4.3)$$

Supposing that the symbols $a(y, \eta)$, $b(y, \eta)$ act between the spaces $K^{s,\gamma} \to K^{s,\gamma}$ and using Lemma 2.6, we see that $\widehat{a}^{(\alpha)}(\zeta - \xi - \theta, \xi + t\theta)$ acts between $K^{s,\gamma} \to K^{s+N,\gamma+\delta}$ with some positive δ. Since $\widehat{b}(\theta, \xi) : K^{s,\gamma} \to K^{s,\gamma}$ is bounded it implies the boundedness of

$$K_N(\zeta, \xi) : K^{s,\gamma} \to K^{s+N,\gamma+\delta}. \quad (4.4)$$

To estimate the norm in wedge Sobolev spaces we apply a standard trick and introduce a new kernel

$$\widetilde{K}_N(\zeta, \xi) = [\zeta]^{s+N}\kappa^{-1}(\zeta)K_N(\zeta, \xi)\kappa(\xi)[\xi]^{-s}. \quad (4.5)$$

Then the norm of the operator with the kernel (4.4) in the spaces

$$W^{s,\gamma} \to W^{s+N,\gamma+\delta}$$

is equal to the norm in L^2 of the operator with the kernel (4.5). Now,

$$\begin{aligned}
&\kappa^{-1}(\xi)\widehat{b}(\theta,\xi)\kappa(\xi) = O([\theta]^{-\infty}),\\
&\kappa^{-1}(\xi+t\theta)\widehat{a}^{(\alpha)}(\zeta-\xi-\theta,\xi+t\theta)\kappa(\xi+t\theta) =\\
&= O([\zeta-\xi-\theta]^{-\infty}[\xi+t\theta]^{-N})
\end{aligned}$$

and we obtain the following norm estimate for the operator with the kernel (4.5)

$$\begin{aligned}
&\|\widetilde{K}(\zeta,\xi)\| =\\
&= \int_0^1 (1-t)^{N-1}dt \int [\zeta]^{s+N}\|\kappa^{-1}(\xi+t\theta)\kappa(\xi)\|\\
&\times O([\zeta-\xi-\theta]^{-\infty}[\xi+t\theta]^{-N})\|\kappa^{-1}(\xi+t\theta)\kappa(\xi)\|O([\theta]^{-\infty})[\xi]^{-s}d\theta. \quad (4.6)
\end{aligned}$$

Next, we apply estimate (1.11) and Peetre's inequality to get

$$\begin{aligned}
&\|\kappa^{-1}(\zeta)\kappa(\xi+t\theta)\| \le C\max\left\{\frac{[\zeta]^M}{[\xi+t\theta]^M}, \frac{[\xi+t\theta]^M}{[\zeta]^M}\right\}\\
&\le C[\zeta-\xi t\theta]^M \le C[\zeta-\xi\theta]^M[(1-t)\theta]^M\\
&\le C[\zeta-\xi-\theta]^M[\theta]^M.
\end{aligned}$$

This term may be omitted since there are factors $[\zeta-\xi-\theta]^{-\infty}$ and $[\theta]^{-\infty}$ in (4.5). Similarly,

$$\|\kappa^{-1}(\xi+t\theta)\kappa(\xi)\| \le C\max\left\{\frac{[\xi]^M}{[\xi+t\theta]^M}, \frac{[\xi+t\theta]^M}{[\xi]^M}\right\} \le C[\theta]^M$$

and we may also omit this term. Finally, Peetre's inequality yields

$$[\xi+t\theta]^{-N} \le C[\theta]^N[\xi]^{-N}.$$

Substituting into (4.6) and using repeatedly Peetre's inequality we get

$$\begin{aligned}
&\|\widetilde{K}(\zeta,\xi)\| =\\
&= \int O([\zeta]^{s+N}[\xi]^{-s-N}[\zeta-\xi-\theta]^{-\infty}[\theta]^{-\infty}d\theta\\
&= \int O([\zeta-\xi-\theta]^{-\infty}[\theta]^{-\infty})d\theta.
\end{aligned}$$

Choosing M sufficiently large, we obtain

$$\int \|\widetilde{K}(\zeta,\xi)\| d\zeta \leq C \int\int \frac{d\zeta d\theta}{[\zeta-\xi-\theta]^M [\theta]^M} < \infty$$

and similarly

$$\int \|\widetilde{K}(\zeta,\xi)\| d\xi < \infty.$$

It means that the operator in L^2 with the kernel $\widetilde{K}(\zeta,\xi)$ is bounded proving the lemma. □

In particular, for $N=1$ we get that the operator

$$AB - \mathrm{Op}(a(y,\eta)b(y,\eta)) : W^{s,\gamma} \to W^{s+1,\gamma+\delta} \tag{4.7}$$

is bounded.

Proof of Theorem 3.2. Let $R_0 = \mathrm{Op}(r(y,\eta)) \in \mathcal{Y}^0$ be a pseudo-differential operator with the symbol $r(y,\eta)$ constructed in Theorem 3.1. We define

$$R_N = R_0 \sum_{k=0}^{N-1} (1-AR_0)^k = \sum_{k=0}^{N-1} (1-R_0A)^k R_0,$$

so that

$$1 - R_N A = (1-R_0A)^N, \;\; 1 - AR_N = (1-AR_0)^N.$$

Now,

$$1 - R_0 A = \mathrm{Op}(r(y,\eta)a(y,\eta)) - R_0A + \mathrm{Op}(1 - r(y,\eta)a(y,\eta)).$$

By (4.7) the first two terms give a bounded operator from $W^{s,\gamma}$ to $W^{s+1,\gamma+\delta}$. The third term belongs to $\mathcal{Y}_G^{-1}$ by Theorem 3.1, thus it is also bounded from $W^{s,\gamma}$ to $W^{s+1,\gamma+\delta}$. Both terms stabilize to 0. Thus, the operator

$$(1-R_0A)^N : W^{s,\gamma} \to W^{s+N,\gamma+\delta} \hookrightarrow W^{s,\gamma}$$

for $N > q+1+n$ belongs to the trace class by Corollary 1.4. The same reasoning may by applied to $(1-AR_0)^N$. □

Define the regularized trace of the product of A and B by

$$\mathrm{Tr}_N AB := \mathrm{Tr}\left\{ AB - \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \mathrm{Op}(\partial_\eta^\alpha a(y,\eta)\partial_y^\alpha b(y,\eta)) \right\}, \tag{4.8}$$

provided the trace on the right-hand side exists.

Theorem 4.2 *Let $A, B \in \mathcal{Y}^0$ and stabilize to 0. Then the regularized trace* $\mathrm{Tr}_N AB$ *exists for $N > q+1+n$ and does not depend on the order of factors, that is*

$$\mathrm{Tr}_N AB = \mathrm{Tr}_N BA. \tag{4.9}$$

Proof. The existence follows directly from Lemma 4.1 and Corollary 1.4. Integrating the kernel (4.3) over diagonal $\zeta = \xi$, we obtain

$$\mathrm{Tr}_N AB =$$
$$= N \int\int\int_0^1 (1-t)^{N-1} dt \,\mathrm{tr} \sum_{|\alpha|=N} \frac{1}{\alpha!} \partial_\xi^\alpha \widehat{a}(-\theta, \xi + t\theta)\theta^\alpha \widehat{b}(\theta, \xi) d\theta d\xi.$$

Next, we integrate by parts in ξ and change variables $\xi' = \xi + t\theta$, $\theta' = -\theta$. It yields

$$\mathrm{Tr}_N AB =$$
$$= N \int\int\int_0^1 (1-t)^{N-1} dt \,\mathrm{tr} \sum_{|\alpha|=N} \frac{1}{\alpha!} \widehat{a}(\theta', \xi')(\theta')^\alpha \partial_{\xi'}^\alpha \widehat{b}(-\theta', \xi' + t\theta') d\theta' d\xi'.$$

This expression coincides with the corresponding expression for $\mathrm{Tr}_N BA$ if one changes the order of factors under the trace sign. This proves the theorem. □

5 The index Formula

We are now in a position to derive an index formula for an elliptic wedge operator starting with the general formula

$$\mathrm{ind} A = \mathrm{Tr}(1 - R_N A) - \mathrm{Tr}(1 - A R_N) \tag{5.1}$$

for a Fredholm operator $A : H_0 \to H_1$. Here $R_N : H_1 \to H_0$ is a parametrix of A up to trace class operators. In our case $H_0 = H_1$ is the wedge Sobolev space $W^{s,\gamma}$ and $A \in \mathcal{Y}^0$ is an elliptic operator stabilizing to identity.

After analytical preparation of the preceding sections the scheme of [1, 2] goes almost without changes.

We begin with a definition of the algebra of formal symbols where the algebraic index lives, confining ourselves to a particular case of the general definition in [2]. Here we denote the formal parameter by h (instead of λ in [2]) as it is usual in deformation quantization.

The algebra $\mathcal{A}$ of *formal symbols* consists of formal power series

$$a(y,\eta,h) = \sum_{k=0}^{\infty} h^k a_k(y,\eta) \tag{5.2}$$

with

$$a_k(y,\eta) \in S^0(\mathbb{R}^{2q}, K^{s,\gamma} \oplus \mathbb{C}^M, K^{s,\gamma} \oplus \mathbb{C}^M).$$

We also suppose that a stabilizes to a constant (times identity operator) at large y. The formal *Leibnitz product* is defined by

$$a \circ b = \sum_{|\alpha|,p,q=0}^{\infty} h^{|\alpha|+p+q} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\eta^\alpha a_p \partial_y^\alpha b_q \tag{5.3}$$

A *trace ideal* $\mathcal{J} \in \mathcal{A}$ consists of formal symbols whose coefficients are Green symbols vanishing outside a compact in $\mathbb{R}^{2q}$ (that is not only for large y but for large η also). For $a \in \mathcal{J}$ define its *formal trace* by

$$\operatorname{Tr} a = \sum_{k=0}^{\infty} \frac{h^{k-q}}{(2\pi)^q} \int_{\mathbb{R}^{2q}} \operatorname{tr} a_k(y,\eta)\, dy d\eta \tag{5.4}$$

where tr means the trace of coefficients in the cone algebra. Being Green operators they are trace class operators in $K^{s,\gamma} \oplus \mathbb{C}^M$, the integral in (5.4) converges because a_k are compactly supported. So, $\operatorname{Tr} a$ is a formal Laurent series in h with negative exponents not exceeding q. Integration by parts shows that $\operatorname{Tr} a \circ b = \operatorname{Tr} b \circ a$ if one of the factors belongs to $\mathcal{J}$.

Next, define *elliptic* formal symbols and their indices similarly to (5.1). The symbol $a \in \mathcal{A}$ is called elliptic if there exists a symbol $r \in \mathcal{A}$ such that

$$1 - r \circ a, \quad 1 - a \circ r \in \mathcal{J}. \tag{5.5}$$

We define an *algebraic index* setting

$$\operatorname{ind} a = \operatorname{Tr}(1 - r \circ a) - \operatorname{Tr}(1 - a \circ r). \tag{5.6}$$

A basic property of the algebraic index is its stability under homotopies (see [2] or more recent paper [7]). As a consequence we obtain that the formal Laurent series (5.6) consists of the constant term only, since there is a homothety

$$H_\lambda : a(y, \eta, h) \mapsto a(y, \lambda\eta, \lambda h)$$

of the algebra $\mathcal{A}$ not affecting the index [2, 7]. Thus, the algebraic index may be treated as a number.

For an elliptic operator-valued symbol

$$a(y, \eta) \in \mathcal{R}^0(\mathbb{R}^{2q}, K^{s,\gamma} \oplus \mathbb{C}^N, K^{s,\gamma} \oplus \mathbb{C}^N)$$

satisfying stabilization conditions we may define its index in two different ways. First, the operator $A = \mathrm{Op}(a(y, \eta))$ is Fredholm in the space $W^{s,\gamma}$, and we have its analytical index defined by (5.1). On the other hand, the symbol $a(y, \eta)$ may be considered as a formal one consisting of the leading term only. Show that there exists a $r(y, \eta, h) \in \mathcal{A}$ satisfying (5.5). To this end let us denote by $r_0(y, \eta)$ the parametrix constructed in Theorem 3.1. Again $r_0(y, \eta)$ is treated as a formal symbol consisting of the leading term only. For such symbols we consider two products: the Leibnitz product $r_0 \circ a$ in the algebra $\mathcal{A}$ and the pointwise product $r_0(y, \eta)a(y, \eta)$ for operator-valued functions. We define

$$r(y, \eta, h) = r_0 \circ \sum_{k=0}^{\infty} (ar_0 - a \circ r_0)^{\circ k} \tag{5.7}$$

where the exponent $\circ k$ means the k-th power with respect to the product $\circ$. Clearly, (5.7) is meaningful as a formal symbol since $ar_0 - a \circ r_0$ has vanishing leading term.

Lemma 5.1 *The symbol r given by* (5.7) *is a parametrix of a in the algebra $\mathcal{A}$, that is satisfies* (5.5).

Proof. A direct check shows that (5.7) is a right parametrix. Indeed,

$$\begin{aligned} 1 - a \circ r &= 1 - (a \circ r_0 - ar_0 + ar_0) \circ \sum_{k=0}^{\infty} (ar_0 - a \circ r_0)^{\circ k} = \\ &= 1 + \sum_{k=0}^{\infty} (ar_0 - a \circ r_0)^{\circ(k+1)} + (1 - ar_0) \circ \sum_{k=0}^{\infty} (ar_0 - a \circ r_0)^{\circ k} \\ &- \sum_{k=0}^{\infty} (ar_0 - a \circ r_0)^{\circ k} = (1 - ar_0) \circ \sum_{k=0}^{\infty} (ar_0 - a \circ r_0)^{\circ k}. \end{aligned} \tag{5.8}$$

This expression belongs to $\mathcal{J}$ since $(1 - ar_0)$ does (it is a Green operator in $K^{s,\gamma} \oplus \mathbb{C}^M$ with compact support in y, η). Similarly, one can check that

$$r_l = \sum_{k=0}^{\infty} (r_0 a - r_0 \circ a)^{\circ k} \circ r_0$$

is a left parametrix, that is $1 - r_l \circ a \in \mathcal{J}$. In this case $r_l - r \in \mathcal{J}$ since we have

$$r_l \circ a \circ r \equiv r_l \equiv r \pmod{\mathcal{J}}.$$

Thus, r is a two-sided parametrix. $\square$

Now, the algebraic index of $a(y, \eta)$ is defined by (5.6) and the first part of the index theorem claims that *both indices, analytical and algebraic, coincide.* This statement is a simple consequence of Theorem 4.2. Indeed, for

$$r = \sum_{k=0}^{\infty} h^k r_k(y, \eta)$$

Lemma 2.6 implies that

$$r_k(y, \eta) \in \mathcal{R}^{-k}(\mathbb{R}^{2q}, K^{s,\gamma} \oplus \mathbb{C}^M, K^{s,\gamma} \oplus \mathbb{C}^M)$$

since r_k contains k derivatives with respect to ξ applied to $a(y, \eta)$ or $r_0(y, \eta)$. Introducing a notation

$$r|_N = \sum_{k=0}^{N-1} r_k(y, \eta)$$

for partial sums of formal series at $h = 1$, we may define a parametrix

$$R_N = \mathrm{Op}(r|_N)$$

slightly different from that constructed in Theorem 3.2. Then we have the following chain of equalities starting with the analytical index and finishing with the algebraic one:

$$\begin{aligned}
&\mathrm{Tr}(1 - \mathrm{Op}(r|_N)\mathrm{Op}(a)) - \mathrm{Tr}(1 - \mathrm{Op}(a)\mathrm{Op}(r|N)) = \\
&= \mathrm{Tr}(1 - \mathrm{Op}((r|_N \circ a)|_N)) - \mathrm{Tr}(1 - \mathrm{Op}((a \circ r|_N)|_N)) = \\
&= \mathrm{Tr}(1 - \mathrm{Op}((r \circ a)|_N)) - \mathrm{Tr}(1 - \mathrm{Op}((a \circ r)|_N)) = \\
&= \mathrm{Tr}(1 - r \circ a) - \mathrm{Tr}(1 - a \circ r).
\end{aligned} \tag{5.9}$$

The first equality is due to Theorem 4.2 since

$$\mathrm{Tr}\{\mathrm{Op}(r|_N)\mathrm{Op}(a) - \mathrm{Op}((r|_N \circ a)|_N)\}$$

is a regularized trace of $\mathrm{Op}(r|_N)$ and $\mathrm{Op}(a)$. The second equality follows because the difference

$$\mathrm{Op}((r|_N \circ a)|_N) - \mathrm{Op}((r \circ a)|_N)$$

may be written as a finite sum of the terms

$$\mathrm{Tr}\mathrm{Op}(\partial_\xi^\alpha r_k \partial_y^\alpha a)$$

with $|\alpha| < N, k < N, k + |\alpha| \geq N$. For N large enough this is equal to

$$\mathrm{Tr}\mathrm{Op}(\partial_\xi^\alpha a \partial_y^\alpha r_k)$$

as may be seen integrating by parts . Finally, the last equality in (5.9) follows since the algebraic index contains the constant term only (and thus does not depend on N) for sufficiently large N.

The second part of the index theorem which claims that the algebraic index is equal to the topological one given by (0.2) is a general fact for the algebra of formal symbols $\mathbb{R}^{2q}$, see [2]. Recall briefly how the proof runs.

Given $a, r \in \mathcal{A}$ satisfying (5.4) we construct matrices

$$\begin{aligned} P^0 &= E \circ G = \begin{pmatrix} 1 \\ a \end{pmatrix} \circ (1 - r \circ a, \quad r) = \\ &= \begin{pmatrix} 1 - r \circ a & r^* \\ a \circ (1 - r \circ a) & a \circ r \end{pmatrix}, \\ P^1 &= \begin{pmatrix} r \\ 1 \end{pmatrix} (0, \quad 1) = \begin{pmatrix} 0 & r \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Since $G \circ E = 1$, P^0 and P^1 are projectors:

$$P^0 \circ P^0 = P^0, \quad P^1 \circ P^1 = P^1.$$

Moreover, $P^0 - P^1 \in \mathcal{J}$ and

$$\mathrm{ind}\, a = \mathrm{Tr}(P^0 - P^1).$$

By periodicity theorem for the symbol algebra [2] this quantity may be expressed in terms of the leading symbols

$$p_0^0(y,\eta) = e(y,\eta)g(y,\eta) = \begin{pmatrix} 1 \\ a(y,\eta) \end{pmatrix} (1 - r_0 a, \quad r_0) =$$

$$= \begin{pmatrix} 1 - r_0 a & r_0 \\ a(1 - r_0 a) & a r_0 \end{pmatrix}$$

in the form

$$\operatorname{ind} a = \frac{1}{(2\pi i)^q q!} \int_{\mathbf{R}^{2q}} \operatorname{tr}(p_0^0 dp_0^0 \wedge dp_0^0)^q$$

and it remains to return to the symbols $a(y,\eta)$ and $r_0(y,\eta)$. Straightforward calculations show that

$$p_0^0 dp_0^0 \wedge dp_0^0 = e(dg \wedge de + gde \wedge gde)g = e(dr_0 \wedge da + r_0 da \wedge r_0 da)g,$$

so that

$$\operatorname{tr}(p_0^0 dp_0^0 \wedge dp_0^0)^q = \operatorname{tr}(dr_0 \wedge da + r_0 da \wedge r_0 da)^q.$$

This completes the proof of the index theorem.

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