HOMOLOGY OF ALGEBRAS OF FAMILIES OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We compute the Hochschild, cyclic, and periodic cyclic homology groups of algebras of families of Laurent complete symbols on manifolds with corners. We show in particular that the spectral sequence associated with Hochschild homology degenerates at E^2 and converges to Hochschild homology. As a byproduct, we deduce an identification of the space of residue traces on fibrations by manifolds with corners. In the process, we prove several general results about algebras of complete symbols on manifolds with corners.

CONTENTS

INTRODUCTION

Some of the main tools in the applications of Non-commutative geometry to index theory and other areas of mathematics are the Hochschild and periodic cyclic homology groups. Hochschild homology, for example, can be used to understand the residue trace introduced by Guillemin and Wodzicki [16, 50]. Other higher residue cocycles appear when studying more complicated singular spaces.

In this paper, we study the Hochschild homology of certain algebras of complete symbols. Recall that an algebra of complete symbols is the quotient of the algebra of all pseudodifferential operators by the ideal of regularizing (or order $-\infty$) operators. Previously, results in this direction were obtained in [4, 10, 22, 32, 27, 37], and by Wodzicki [51] (unpublished). See also [38].

Our algebras of complete symbols can be obtained as algebras of complete symbols on differentiable groupoids [23, 34, 41]. For this large class of examples, it

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has been shown in [4] that the periodic cyclic homology can be computed, without any further assumption on the groupoid under consideration, in terms of the Laurent cohomology spaces of the cosphere bundle of the associated Lie algebroid. The Hochschild homology groups of algebras of complete symbols on differentiable groupoids, however, cannot be described in general in a simple, uniform way for all differentiable groupoids. Finding the right language in which to express these Hochschild homology groups seems to be a problem in itself–clearly an interesting one.

In the present paper, we are concerned with a class of algebras of complete symbols that consist of families of pseudodifferential operators on a fibration by manifolds with corners and with Laurent singularities at the hyperfaces (see Section 3 for precise definitions). Let us consider then a fibration $\pi : M \to B$ of a manifold with corners over a smooth manifold B. Let $\mathcal{A}_{\mathcal{L}}(M|B;E)$ be the algebra of complete symbols associated to the algebra of smooth families of pseudodifferential operators on the fibers of π and with Laurent singularities at the hyperfaces. (The precise construction of this algebra is done using groupoids, see Section 3.) To our surprise, the resulting algebra turns out not to depend on the groupoid $\mathcal{G}$ used to define it, but only on $\pi : M \to B$, as long as the groupoid $\mathcal{G}$ satisfies the assumptions (13) and (14) of Section 3.

Let $S^*_{vert}(M) := (T^*_{vert}M \smallsetminus 0)/\mathbb{R}^*_+$ denote the sphere bundle of the vertical cotangent bundle to the fibration $\pi : M \to B$. The periodic cyclic homology of the algebra $\mathcal{A}_{\mathcal{L}}(M|B;E)$ of Laurent vertical complete symbols with coefficients in a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E, is given by Theorem 3:

$$\mathrm{HP}_j(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \bigoplus_{k\in\mathbb{Z}} \mathrm{H}^{j+2k}_{c,\mathcal{L}}(S^*_{vert}(M)\times S^1), \quad j=0,1, \tag{1}$$

where $\mathrm{H}_{c,\mathcal{L}}$ denotes compactly supported Laurent cohomology (it is the compactly supported cohomology of a certain space functorially associated to $S^*_{vert}(M)$).

We also compute the Hochschild homology groups of the algebras $\mathcal{A}_{\mathcal{L}}(M|B;E)$. For B reduced to a point, this was done in [4]. For families, the computation requires new ingredients, and, in particular, we may obtain infinite dimensional groups. The result is then:

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \bigoplus_{k+h=m} \Omega^h_c(B, \mathcal{F}^{2p-k}), \tag{2}$$

where $\mathcal{F}^*$ is the local coefficient system over B given by the Laurent cohomology of the fibers of $S^*_{vert}(M)\times S^1$ and p is the dimension of the fibers of $\pi : M \to B$ (Theorem 6). This result leads, in particular, to an explicit description of the space of residue traces on the algebras of families of pseudodifferential operators that we consider. For example, when M is smooth (no corners) and π has connected fibers, we obtain that

$$\mathrm{HH}^0(\mathcal{A}(M|B;E)) \simeq \mathcal{C}^{-\infty}(B) := \mathcal{C}^\infty_c(B)', \tag{3}$$

that is, that the space of (super)traces on $\mathcal{A}(M|B;E)$ identifies with the space of distributions on B. (Note that in this case $\mathcal{A}_{\mathcal{L}}(M|B;E) = \mathcal{A}(M|B;E)$.) The space of traces in the general case of fibrations $\pi : M \to B$ when M has corners is also explicitly described in terms of the minimal faces of the fibers of π (Theorem 9).

Let us now briefly describe the contents of each section. In Section 1, we briefly recall the basic definitions of $\mathbb{Z}/2\mathbb{Z}$-graded homologies for topologically filtered algebras and give an appropriate criterion for the convergence of the associated spectral sequences. Section 2 is devoted to the description of the algebras of complete symbols that we are interested in. In Section 3, we fix the assumptions on our groupoids and also prove that these algebras depend only on $\pi : M \to B$, as long as Assumptions (13) and (14) are satisfied. Section 4 is devoted to the computation of the Hochschild homology of our algebras of complete symbols. Along the way, we compute several other homology groups associated to certain Poissson manifolds. In Section 5, we extend the main results of the previous sections to the relative case. The last section, Section 6, treats in detail a few examples. In particular, we obtain a description of the space of traces on our algebras of complete symbols. Note that in this paper almost all results are formulated in the $\mathbb{Z}/2\mathbb{Z}$-graded case, in view of some possible applications.

We hope that the results of this paper will find applications to the index theorem for families [1, 7] or to its generalization to families of fibrations by manifolds with boundary [8].

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1. Topological filtered algebras

Topologically filtered algebras were introduced in [4] to provide a natural framework for the algebras of complete symbols associated to algebras of pseudodifferential operators. In this section we review the definition of topologically filtered algebras and a few other relevant facts. The complexes computing the various homologies of these algebras have to be defined appropriately. In view of the applications that we have in mind, we have found it necessary to extend our setting to include that of $\mathbb{Z}/2\mathbb{Z}$-graded algebras. For basic facts about pseudodifferential operators, see one of the many nice monographs available [36], [43], or [45].

We begin by recalling the definitions of Hochschild and cyclic homology groups of a topological algebra $\mathcal{A}$. A good reference is Connes' book [12]. See also [20, 26]. See [18] for the homology of $\mathbb{Z}/2\mathbb{Z}$-graded algebras. These definitions have to be (slightly) modified when the multiplication of our algebra is only separately continuous. We thus discuss also the changes necessary to handle the class of algebras that we are interested in, that of "topologically filtered algebras" (Definition 1), and then we prove some results on the homology of these algebras.

First we consider the case of a topological algebra $\mathcal{A}$. Here "topological algebra" has the usual meaning, that is, $\mathcal{A}$ is a real or complex algebra, which is at the same time a locally convex space such that the multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous when $\mathcal{A} \times \mathcal{A}$ is endowed with the product topology. Denote by $\widehat{\otimes}$ the projective tensor product and $\mathcal{H}_n(\mathcal{A}) := \mathcal{A}^{\widehat{\otimes} n+1}$, the completion of $\mathcal{A}^{\otimes n+1}$ in the topology of the projective tensor product. Also, we denote as usual by $\partial a \in \mathbb{Z}/2\mathbb{Z}$ the degree of an element in the $\mathbb{Z}/2\mathbb{Z}$-graded algebra and by b' and b the Hochschild differentials:

$$\begin{aligned} b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n, \\ b(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) + (-1)^{n+\mu} a_n a_0 \otimes \ldots \otimes a_{n-1}, \end{aligned} \tag{4}$$

where $\mu = \partial a_n(\partial a_0 + \ldots + \partial a_{n-1})$.

The *Hochschild homology groups* of the algebra $\mathcal{A}$, denoted $\mathrm{HH}_*(\mathcal{A})$, are then the homology groups of the complex $(\mathcal{H}_n(\mathcal{A}), b)$. By contrast, the complex $(\mathcal{H}_n(\mathcal{A}), b')$ is often acyclic, for example when $\mathcal{A}$ has a unit. A topological algebra $\mathcal{A}$ for which $(\mathcal{H}_n(\mathcal{A}), b')$ is acyclic is called *H-unital* (or, better, *topological H-unital*), following Wodzicki [50].

We now define cyclic homology. Assume first that $\mathcal{A}$ is unital. We shall use the notation of [11]. See also [20].

$$\begin{aligned} s(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n, \\ t(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= (-1)^{n+\mu} a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}, \\ B_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) &= s \sum_{k=0}^{n} t^k (a_0 \otimes a_1 \otimes \ldots \otimes a_n), \quad \text{and} \quad B = (1-t)B_0, \end{aligned} \tag{5}$$

where $\mu = \partial a_n(\partial a_0 + \ldots + \partial a_{n-1})$, as above. Then $[b, B]_+ := bB + Bb = B^2 = b^2 = 0$, and hence, if we define

$$\mathcal{C}(\mathcal{A})_n = \bigoplus_{k \geq 0} \mathcal{H}_{n-2k}(\mathcal{A}), \tag{6}$$

$(\mathcal{C}(\mathcal{A}), b + B)$, is a complex, called *the cyclic complex* of $\mathcal{A}$, whose homology is by definition the *cyclic homology* of $\mathcal{A}$, as introduced in [11] and [46]. The cyclic homology groups of the algebra $\mathcal{A}$ are denoted $\mathrm{HC}_n(\mathcal{A})$. For algebras without unit $\mathcal{A}$ one considers the algebra with adjoint unit $\mathcal{A}^+$ and then the cyclic homology of $\mathcal{A}$ is the kernel of the map $\mathrm{HC}_n(\mathcal{A}^+) \to \mathrm{HC}_n(\mathbb{C})$ induced by the augmentation morphism $\mathcal{A}^+ \to \mathbb{C}$.

Consideration of the natural periodicity morphism $\mathcal{C}_n(\mathcal{A}) \to \mathcal{C}_{n-2}(\mathcal{A})$ easily shows that cyclic and Hochschild homology are related by a long exact sequence

$$\ldots \to \mathrm{HH}_n(\mathcal{A}) \xrightarrow{I} \mathrm{HC}_n(\mathcal{A}) \xrightarrow{S} \mathrm{HC}_{n-2}(\mathcal{A}) \xrightarrow{B} \mathrm{HH}_{n-1}(\mathcal{A}) \xrightarrow{I} \ldots, \tag{7}$$

with the maps I, B, and S explicitly determined. The map S is also called the *periodicity* operator. See [11, 28]. This exact sequence exists whether or not $\mathcal{A}$ is endowed with a topology.

For the algebras that we are interested in, however, the multiplication is usually only separately continuous, but there will usually exist an increasing multi-filtration $F^m_{n,l}\mathcal{A} \subset \mathcal{A}$ of $\mathcal{A}$,

$$F^m_{n,l}\mathcal{A} \subset F^{m'}_{n',l'}\mathcal{A}, \quad \text{if } n \leq n',\ l \leq l', \text{ and } m \leq m',$$

by closed, complemented subspaces, invariant under the $\mathbb{Z}_2$-grading, and satisfying the following properties:

1. $\mathcal{A} = \cup_{n,l,m} F^m_{n,l}\mathcal{A}$;
2. The union $\mathcal{A}_n := \cup_{m,l} F^m_{n,l}\mathcal{A}$ is a closed subspace such that
$$F^m_{n,l}\mathcal{A} = \mathcal{A}_n \cap \big(\cup_j F^m_{j,l}\mathcal{A} \big);$$

3. Multiplication maps $F^m_{n,l}\mathcal{A} \otimes F^m_{n',l'}\mathcal{A}$ to $F^m_{n+n',l+l'}\mathcal{A}$;
4. The maps
$$F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A} \otimes F^m_{n',l'}\mathcal{A}/F^m_{n'-1,l'}\mathcal{A} \to F^m_{n+n',l+l'}\mathcal{A}/F^m_{n+n'-1,l+l'}\mathcal{A}$$
induced by multiplication are continuous;
5. The quotient $F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A}$ is a nuclear Frechet space in the induced topology;
6. The natural map
$$F^m_{n,l}\mathcal{A} \to \lim_{\leftarrow} F^m_{n,l}/F^m_{n-j,l}\mathcal{A}, \quad j \to \infty$$
is a homeomorphism; and
7. The topology on $\mathcal{A}$ is the strict inductive limit of the subspaces $F^n_{n,n}\mathcal{A}$, as $n \to \infty$ (recall that $F^n_{n,n}\mathcal{A}$ is assumed to be closed in $F^{n+1}_{n+1,n+1}\mathcal{A}$).

Definition 1. An algebra $\mathcal{A}$ satisfying the conditions 1–7, above, will be called a *topologically filtered* algebra.

It follows from the definition that if $\mathcal{A}^{[m]} := \cup_{n,l} F^m_{n,l}\mathcal{A}$, then $\mathcal{A}^{[m]}$ is actually a subalgebra of $\mathcal{A}$ which is topologically filtered in its own, but with multi-filtration independent of m.

For topologically filtered algebras, the multiplication is not necessarily continuous, and the definition of the Hochschild and cyclic homologies using the projective tensor product of the algebra $\mathcal{A}$ with itself, as above, is not very useful. For this reason, we change the definition of the space $\mathcal{H}_m(\mathcal{A})$ to be an inductive limit:
$$\mathcal{H}_q(\mathcal{A}) \simeq \lim_{\rightarrow} (F^n_{n,n}\mathcal{A})^{\widehat{\otimes} q+1},$$
the tensor product being the (complete) projective tensor product. The Hochschild homology of $\mathcal{A}$ is then still the homology of the complex $(\mathcal{H}(\mathcal{A}), b)$. Since the projective tensor product is compatible with the projective limits, we also have
$$\mathcal{H}_q(\mathcal{A}) = \lim_{\rightarrow} \left(\lim_{\leftarrow} (F^n_{n,n}\mathcal{A}/F^n_{k,n}\mathcal{A})^{\widehat{\otimes} q+1} \right), \tag{8}$$
with the induced topology, where first $k \to -\infty$ (in the projective limit) and then $n \to \infty$ (in the inductive limit). The operator B extends to a well defined map $B : \mathcal{H}_q(\mathcal{A}) \to \mathcal{H}_{q+1}(\mathcal{A})$, which allows us to define the cyclic complex and the cyclic homology of the algebra $\mathcal{A}$ as the homology of the complex $(\mathcal{C}_*(\mathcal{A}), b + B)$, with $\mathcal{C}_q(\mathcal{A}) := \oplus \mathcal{H}_{q-2k}(\mathcal{A})$, as for topological algebras.

We also observe that both the Hochschild and cyclic complexes have natural filtrations given by
$$F_p\mathcal{H}_q(\mathcal{A}) := \lim_{\rightarrow} \left(\lim_{\leftarrow} \widehat{\otimes}^q_{j=0} \left(F^m_{k_j,m}\mathcal{A}/F^m_{k,m}\mathcal{A} \right) \right), \tag{9}$$
where $k_0 + \ldots + k_n \leq p$ defines the filtration. The projective and inductive limits are such that first $k, l \to -\infty$ (in the projective limit) and then $m \to \infty$ (in the inductive limit). It follows from the definitions that the operators S, B and I preserve these filtrations.

For any topologically filtered algebra, we denote
$$Gr(\mathcal{A}) := \oplus_n \mathcal{A}_n/\mathcal{A}_{n-1}$$

the *graded algebra* associated to $\mathcal{A}$, where $\mathcal{A}_n$ is the union $\cup_{l,m} F^m_{n,l}\mathcal{A}$, as before. Its topology is that of an inductive limit of Frechet spaces:

$$Gr(\mathcal{A}) \simeq \lim_{N,m,l\to\infty} \oplus_{n=-N}^{N} F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A}.$$

For the algebras like $Gr(\mathcal{A})$, we need yet a third way of topologizing its iterated tensor products. The correct definition is then

$$\mathcal{H}_q(Gr(\mathcal{A})) \simeq \lim_{N,m,l\to\infty} (\oplus_{n=-N}^{N} F^m_{n,l}\mathcal{A}/F^m_{n-1,l}\mathcal{A})^{\widehat{\otimes} q+1}.$$

The Hochschild homology of $Gr(\mathcal{A})$ is the homology of the complex $(\mathcal{H}_*(Gr(\mathcal{A})),b)$. The operator B again extends to a map $B : \mathcal{H}_q(Gr(\mathcal{A})) \to \mathcal{H}_{q+1}(Gr(\mathcal{A}))$ and we can define the cyclic homology of $Gr(\mathcal{A})$ as above. The operators S, B and I associated to $\mathcal{H}_q(Gr(\mathcal{A}))$ are the graded operators associated with the corresponding operators (also denoted S, B and I) on $\mathcal{H}_q(\mathcal{A})$.

The Hochschild and cyclic complexes of the algebra $Gr(\mathcal{A})$ decompose naturally as direct sums of complexes indexed by $p \in \mathbb{Z}$. For example, $\mathcal{H}_q(Gr(\mathcal{A}))$ is the direct sum of the subspaces $\mathcal{H}_q(Gr(\mathcal{A}))_p$, where

$$\mathcal{H}_q(Gr(\mathcal{A}))_p = \lim_{m,N,l\to\infty} \bigoplus_{k_j} \left(\widehat{\otimes}_{j=0}^{n} F^m_{k_j,l}\mathcal{A}/F^m_{k_j-1,l}\mathcal{A}\right),$$
$$\text{where } \ k_0 + k_1 + \ldots + k_n = p \ \text{ and } \ -N \leq k_j \leq N,$$

with the inductive limit topology. The corresponding subcomplexes of the cyclic complex are defined similarly. We denote by $\mathrm{HH}_*(Gr(\mathcal{A}))_p$ and $\mathrm{HC}_*(Gr(\mathcal{A}))_p$ the homologies of the corresponding complexes (Hochschild and, respectively, cyclic).

The following two results are well known consequences of standard results in homological algebra (for topologically filtered algebras they were proved in [4]).

Lemma 1. *Let $\mathcal{A}$ be a topologically filtered algebra. Then the natural filtrations on the Hochschild and cyclic complexes of $\mathcal{A}$ define spectral sequences $\mathrm{EH}^r_{k,h}$ and $\mathrm{EC}^r_{k,h}$ such that*

$$\mathrm{EH}^1_{k,h} \simeq \mathrm{HH}_{k+h}(Gr(\mathcal{A}))_k \ \ \textit{and} \ \ \mathrm{EC}^1_{k,h} \simeq \mathrm{HC}_{k+h}(Gr(\mathcal{A}))_k.$$

Moreover, the periodicity morphism S induces a morphism $S' : \mathrm{EC}^r_{k,h} \to \mathrm{EC}^r_{k,h-2}$ of spectral sequences. For $r = 1$, the morphism S' is the graded map associated to the periodicity operator $S : \mathrm{HC}_n(Gr(\mathcal{A})) \to \mathrm{HC}_{n-2}(Gr(\mathcal{A}))$ and the natural filtration of the groups $\mathrm{HC}_n(Gr(\mathcal{A}))$.

Proof. The filtration $F_p\mathcal{H}_q(\mathcal{A})$ of the complex computing the Hochschild homology of $\mathcal{A}$ gives rise to a spectral sequence $(E^r)_{r\geq 1}$ with

$$E^1_{k,h} = \mathrm{H}_{k+h}(F_k\mathcal{H}(\mathcal{A})/F_{k-1}\mathcal{H}(\mathcal{A})),$$

by standard homological algebra. By the definition of the Hochschild complex of $Gr(\mathcal{A})$, we have:

$$\mathrm{H}_{k+h}(F_k\mathcal{H}(\mathcal{A})/F_{k-1}\mathcal{H}(\mathcal{A})) \simeq \mathrm{HH}_{k+h}(Gr(\mathcal{A}))_k.$$

This completes the proof for Hochschild homology. The proof for cyclic homology is similar. ☐

In our considerations below, we shall need the following classical result, which was proved for topologically filtered algebras in [4]. Due to the importance of this result in what follows and for the convenience of the reader, we include a proof of it.

Theorem 1. *Fix an integer N and $a \geq 1$. Let $\mathcal{A}$ be a topologically filtered algebra and let $\mathcal{A}^{[m]} := \cup_{n,l} F^m_{n,l}\mathcal{A}$ be such that $\mathrm{EH}^a_{k,h}(\mathcal{A}^{[m]}) = 0$, for all $k < N$. Then the spectral sequence $\mathrm{EH}^r_{k,h} = \mathrm{EH}^r_{k,h}(\mathcal{A})$ defined in Lemma 1 converges to $\mathrm{HH}_{k+h}(\mathcal{A})$. More precisely, we have*

$$\mathrm{HH}_j(\mathcal{A}) \simeq \oplus_{k=N}^{\infty} \mathrm{EH}^{\infty}_{k,j-k}\,.$$

A similar result holds for the cyclic homology spectral sequence.

Proof. Assume first that the filtration of $\mathcal{A}$ is independent of m. We have

$$\mathcal{H}(\mathcal{A}) = \lim_{\leftarrow} \mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A}).$$

This enables to write, for every fixed q, the well known associated $\lim^1$ exact sequence (see [4][Lemma 6], for example)

$$0 \to \lim_{\leftarrow}{}^1\, \mathrm{H}_{q+1}(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})) \to \mathrm{H}_q(\mathcal{H}(\mathcal{A})) \to \lim_{\leftarrow} \mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})) \to 0.$$

Now the spectral sequence $E^r_{k,h}(p)$ associated with the Hochschild homology of the filtered complex $\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})$ converges because it is the shift of a first quadrant spectral sequence and we have:

$$\mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A})) \cong \oplus_{t+s=q} E^{\infty}_{t,s}(p).$$

Therefore, the homology groups $\mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_p\mathcal{H}(\mathcal{A}))$ are endowed with a filtration $\mathcal{F}_t(p)$ so that

$$\mathcal{F}_t(p)/\mathcal{F}_{t-1}(p) \cong E^{\infty}_{t,q-t}(p). \tag{10}$$

Furthermore, the spectral sequence $E^r_{k,h}(p)$ satisfies:

$$E^r_{k,h}(p) = \begin{cases} 0 & \text{if } k \leq p \text{ and} \\ \mathrm{EH}^r_{k,h} & \text{if } k > p + r \end{cases} \tag{11}$$

Let us choose a particular projective system by setting

$$A_n := \mathrm{H}_q(\mathcal{H}(\mathcal{A})/F_{N-na}\mathcal{H}(\mathcal{A})),\ B_n := \mathcal{F}_{N-na+a}(N-na)\,,\ \text{and}\ \ C_n := A_n/B_n.$$

Then the ker-coker lemma [2] for the short exact sequence

$$0 \to \Pi B_n \hookrightarrow \Pi A_n \to \Pi C_n \to 0$$

gives rise to the following exact sequence [4][Lemma 7]:

$$0 \to \lim_{\leftarrow} B_n \to \lim_{\leftarrow} A_n \to \lim_{\leftarrow} C_n \to \lim_{\leftarrow}{}^1 B_n \to \lim_{\leftarrow}{}^1 A_n \to \lim_{\leftarrow}{}^1 C_n \to 0.$$

By conditions (10) and (11), the natural map $A_{n+1} \to A_n$ induces the zero map $B_{n+1} \to B_n$ and an isomorphism $C_{n+1} \to C_n$, for $n \geq 2$. Therefore we get:

$$\lim_{\leftarrow}{}^1 A_n = 0 \text{ and } \lim_{\leftarrow} A_n = C_{n_0}, \quad \forall n_0 \geq 2.$$

And hence, finally,

$$\mathrm{HH}_q(\mathcal{A}) \cong C_{n_0} = \oplus_{l \geq N} E^{\infty}_{l,q-l} \cong \oplus_{l \in \mathbb{Z}} \mathrm{EH}^{\infty}_{l,q-l}\,.$$

The proof in the general case when the filtration $F^m_{n,l}\mathcal{A}$ of $\mathcal{A}$ depends on m is deduced using that homology is compatible with taking direct limits. $\square$

The above isomorphism is not natural, in general, but comes from a filtration $F_k \operatorname{HH}_q(\mathcal{A})$ of $\operatorname{HH}_q(\mathcal{A})$ whose subquotients $F_k \operatorname{HH}_q(\mathcal{A})/F_{k-1} \operatorname{HH}_q(\mathcal{A})$ identify naturally with $\operatorname{EH}^{\infty}_{k,q-k}$, see [30, 31].

It is useful to mention here that the composite map

$$\operatorname{HH}_q(\mathcal{A}) \xrightarrow{\mathrm{I}} \operatorname{HC}_q(\mathcal{A}) \xrightarrow{\mathrm{B}} \operatorname{HH}_{q+1}(\mathcal{A})$$

preserves the filtrations and hence it induces natural maps

$$(\mathrm{B} \circ \mathrm{I})^{(r)} : \operatorname{EH}^r_{k,h}(\mathcal{A}) \to \operatorname{EH}^r_{k,h+1}(\mathcal{A}).$$

For $r = 1$, this map is the composition of the corresponding morphisms

$$\operatorname{HH}_q(Gr(\mathcal{A})) \to \operatorname{HC}_q(Gr(\mathcal{A})) \to \operatorname{HH}_{q+1}(Gr(\mathcal{A}))$$

for the graded algebra of $\mathcal{A}$.

2. Algebras of complete symbols

We now introduce the algebras of complete symbols that we study in this paper.

We shall follow the standard notation for groupoids and Lie algebroids, using the conventions of [23]. In particular, if $\mathcal{G}$ is a differentiable groupoid with space of units M, then $d, r : \mathcal{G} \to M$ denote the domain and range maps, respectively, so that the composition gg' of two elements $g, g' \in \mathcal{G}$ is defined if, and only if, $d(g) = r(g')$.

We shall also follow [4] for some specific constructions involving manifolds with corners, some of which are recalled below. As in that paper, we are interested in certain specific groupoid algebras associated to manifolds with corners. If $\mathcal{G}$ is a differentiable groupoid with space of units M and $E \to M$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle, then we shall denote by

$$\Psi^{\infty}(\mathcal{G}; E) = \cup_{m\in\mathbb{Z}} \Psi^m(\mathcal{G}; E)$$

the algebra of pseudodifferential operators on $\mathcal{G}$ acting on sections of the vector bundle r^*E. We also define

$$\Psi^{-\infty}(\mathcal{G}; E) := \cap_{m\in\mathbb{Z}} \Psi^m(\mathcal{G}; E).$$

(see [23] or [41] for definitions). These two algebras are naturally $\mathbb{Z}/2\mathbb{Z}$-graded.

We shall denote by $\mathcal{O}(M)$ the space of smooth functions on the interior of M that have only Laurent singularities at the boundary faces. If every hyperface H of M has a defining function x_H, then $\mathcal{O}(M)$ is the ring generated by $\mathcal{C}^\infty(M)$ and x_H^{-1}. Let then

$$\mathcal{A}(\mathcal{G}; E) := \Psi^{\infty}(\mathcal{G}; E)/\Psi^{-\infty}(\mathcal{G}; E) \quad \text{and} \quad \mathcal{A}_{\mathcal{L}}(\mathcal{G}; E) = \mathcal{O}(M)\mathcal{A}(\mathcal{G}; E).$$

The $\mathbb{Z}/2\mathbb{Z}$-grading on E then provides us with a natural $\mathbb{Z}/2\mathbb{Z}$-grading on the algebras $\mathcal{A}(\mathcal{G}; E)$ and $\mathcal{A}_{\mathcal{L}}(\mathcal{G}; E)$ too.

Proposition 1. *Assume that $\mathcal{G}$ and M are as above and that M is σ-compact. Then the quotients $\mathcal{A}(\mathcal{G}; E)$ and $\mathcal{A}_{\mathcal{L}}(\mathcal{G}; E)$ are topologically filtered algebras.*

Proof. The proof is exactly the same as the one given for $E = \mathbb{C}$ in [4]. □

Let $A(\mathcal{G})$ be the Lie algebroid of $\mathcal{G}$ (see [23]) and let $S^*(\mathcal{G})$ be the sphere bundle of $A^*(\mathcal{G})$, that is, the set of unit vectors in the dual of the Lie algebroid of $\mathcal{G}$, and denote $\mathrm{H}_c^{[q]} = \oplus_{k\in\mathbb{Z}} \mathrm{H}_c^{q+2k}$.

Theorem 2. *Assume that the base M is σ-compact, then the periodic cyclic homology of the algebra $\mathcal{A}(\mathcal{G};E)$ is given by*

$$\mathrm{HP}_m(\mathcal{A}(\mathcal{G};E)) \simeq \mathrm{H}_c^{[m]}(S^*(\mathcal{G}) \times S^1). \tag{12}$$

Proof. An argument similar to that of Lemma 5 in Section 4 shows that the Hochschild homology is unchanged by introducing the extra vector bundle E and the $\mathbb{Z}/2\mathbb{Z}$-grading. Standard homological algebra arguments then show that the same is true for cyclic and periodic cyclic homology. The result follows then from the case $E = \mathbb{C}$ that was proved in [4]. □

To state the result for the algebra $\mathcal{A}_{\mathcal{L}}(\mathcal{G};E)$, we need first to recall a construction from [4] that will be used several times in what follows.

Let P be a manifold with corners. Then $\mathcal{L}(P)$ is a space naturally associated to P and defined as follows. Consider for each face F of P the space $F \times (S^1)^k$, where k is the codimension of the face. We establish a one-to-one correspondence between the canonical k copies of the unit circle in $(S^1)^k$ and the faces F' of P containing F, of dimension one higher than that of F. We then identify the points of the disjoint union $\cup F \times (S^1)^k$ as follows. If $F \subset F'$ and F' corresponds to the variable $\theta_i \in S^1$ we identify $(x, \theta_1, \dots, \theta_{i-1}, 1, \theta_{i+1}, \dots, \theta_k) \in F \times (S^1)^k$ to the point $(x, \theta_1, \dots, \theta_{i-1}, \theta_{i+1}, \dots, \theta_k) \in F' \times (S^1)^{k-1}$ (same x). The resulting quotient space is by definition $\mathcal{L}(P)$.

By construction, there exists a continuous map $p_{\mathcal{L}} : \mathcal{L}(P) \to P$. Let $J_\epsilon = S^1 \cup [1, 1+\epsilon) \subset \mathbb{C}$, for some $\epsilon > 0$, with S^1 identified with a subset of the complex plane. Then the space $\mathcal{L}(P)$ is locally modeled by $J_\epsilon^k \times \mathbb{R}^{n-k}$, above each point of P belonging to an open face of codimension k.

Suppose now that $P \to B$ is a fibration by manifolds with corners with B smooth. In particular, B is a smooth manifold. Let Q be the typical fiber of the fibration $P \to B$. Then we obtain by the above construction, a fibration $\mathcal{L}(P) \to B$, with typical fiber the locally compact space $\mathcal{L}(Q)$.

In [4], the periodic cyclic homology of several algebras of complete symbols was computed. These results include our algebras $\mathcal{A}_{\mathcal{L}}(\mathcal{G};E)$, when E is trivial. The result is the same in general.

Theorem 3. *For $j = 0, 1$, we have*

$$\mathrm{HP}_j(\mathcal{A}_{\mathcal{L}}(\mathcal{G};E)) \simeq \mathrm{H}_c^{[j]}(\mathcal{L}(S^*(\mathcal{G})) \times S^1).$$

Proof. When E is trivial one-dimensional, this result was proved in [4]. The general case is proved in the same way, using the same argument as in the proof of Lemma 5, which shows that the Hochschild homology of the algebras $\mathcal{A}_{\mathcal{L}}(\mathcal{G};E)$ does not depend on the bundle E, thanks to the Morita invariance of Hochschild homology. □

In particular, when M has no corners and $\mathcal{G} = M \times_B M$, the algebra $\Psi^\infty(\mathcal{G};E)$ is the algebra of smooth families of pseudodifferential operators along the fibers of $\pi : M \to B$ that have compactly supported Schwartz kernel. Let us denote by $\mathcal{A}(M|B;E) = \mathcal{A}(\mathcal{G};E)$ the algebra associated to this groupoid .

Corollary 1. *For any fibration $\pi : M \to B$ of smooth manifolds (without boundary), we obtain*

$$\mathrm{HP}_j(\mathcal{A}(M|B;E)) \simeq \mathrm{H}_c^{[j]}(S^*_{vert}(M) \times S^1), \quad j = 0, 1.$$

This leads to a complete determination of the periodic cyclic homology of the algebras $\mathcal{A}(\mathcal{G};E) := \Psi^\infty(\mathcal{G};E)/\Psi^\infty(\mathcal{G};E)$ and $\mathcal{A}_\mathcal{L}(\mathcal{G};E) := \mathcal{O}(M)\mathcal{A}(\mathcal{G};E)$. The result is moreover easily expressed in a uniform manner for all differentiable groupoids $\mathcal{G}$. The Hochschild homology of these algebras seems to be more difficult to compute, in general. Finding the groups $\mathrm{HH}_*(\mathcal{A}_\mathcal{L}(\mathcal{G};E))$ in general seems to depend on finding the right language in which to express this result. Needless to say, finding the right language to express the groups $\mathrm{HH}_*(\mathcal{A}_\mathcal{L}(\mathcal{G};E))$ and then determining them in general is a worthy problem.

We shall determine the groups $\mathrm{HH}_*(\mathcal{A}_\mathcal{L}(\mathcal{G};E))$ for a class of groupoids that, roughly speaking, consists of families of groupoids of the kind considered in [4]. We now proceed to describe this class in detail.

3. Families of manifolds with corners

To describe the class of differentiable groupoids $\mathcal{G}$ for which we shall determine the groups $\mathrm{HH}_*(\mathcal{A}_\mathcal{L}(\mathcal{G};E))$, we first describe the assumptions on the space of units of $\mathcal{G}$. We shall denote the space of units of $\mathcal{G}$ by M, where M is a differentiable manifold, possibly with corners, as before. We shall assume that there exists a smooth manifold (without corners) B and a map $\pi : M \to B$ that makes M a differentiable fiber bundle over B with fiber F. We regard this fiber bundle as a being the fiber bundle associated to a principal bundle with structure group $\mathrm{Diffeo}(F)$, the group of diffeomorphisms of F that map faces to faces. From now on and throughout the paper, we shall denote $n = \dim(M)$, $q = \dim(B)$. Also, we shall denote by p the dimension of the fibers of $\pi : M \to B$, so, in particular, $n = p + q$.

Fix M as above. We now describe our three assumptions on the groupoid $\mathcal{G}$. Our *first assumption* on $\mathcal{G}$ is that for any arrow $g \in \mathcal{G}$, the domain and range of g are in the same fiber of $\pi : M \to B$, that is,

$$\pi(d(g)) = \pi(r(g)), \quad \forall g \in \mathcal{G}. \tag{13}$$

The intuitive meaning of this condition is that the natural action of $\Psi^\infty(\mathcal{G};E)$ on $\mathcal{C}_c^\infty(M)$ via the vector representation [21, 41] is given by families of operators acting on the fibers of π.

Let $T_{vert}M$ be the vertical tangent bundle to the fibration $\pi : M \to B$. Denote as above by $\mathcal{O}(M)$ the space of smooth functions on the interior M_0 of M that have only Laurent (or rational) type singularities at the faces of M. Let us denote by $\varrho : A(\mathcal{G}) \to TM$ the anchor map of the Lie algebroid of $\mathcal{G}$. Our *second assumption* on $\mathcal{G}$ is that the map $\varrho_\Gamma : \Gamma(A(\mathcal{G})) \to \Gamma(T_{vert}M)$ defined by ϱ induces an isomorphism

$$\mathcal{O}(M) \otimes_{C^\infty(M)} \Gamma(A(\mathcal{G})) \simeq \mathcal{O}(M) \otimes_{C^\infty(M)} \Gamma(T_{vert}M), \tag{14}$$

of vector spaces. Clearly the above map preserves the Lie bracket, so we get an isomorphism of Lie algebras also.

Our *third and last assumption* on $\mathcal{G}$ is a local triviality condition on the algebra $\mathcal{A}_\mathcal{L}(\mathcal{G};E)$. To state this assumption, we need to fix the notation. For any open set $V \subset B$, we denote by $\mathcal{G}_V$ the reduction of $\mathcal{G}$ to $\pi^{-1}(V)$. Our previous assumptions on $\mathcal{G}$ give that $\mathcal{G}_V = (\pi \circ d)^{-1}(V)$. Similarly, for every point $b \in B$, we denote by $\mathcal{G}_b$ the reduction of $\mathcal{G}$ to $\pi^{-1}(b)$. Again, our assumptions give us that $\mathcal{G}_b = (\pi \circ d)^{-1}(b) = (\pi \circ r)^{-1}(b)$. Let us observe that

$$\mathcal{O}(\pi^{-1}(V)) \otimes \big(\Psi^\infty(V \times \mathcal{G}_b;E)/\Psi^{-\infty}(V \times \mathcal{G}_b;E)\big)$$

has a natural filtration and a natural completion to a topologically filtered algebra, denoted

$$\mathcal{O}(\pi^{-1}(V)) \otimes_{tf} \left(\Psi^{\infty}(V \times \mathcal{G}_b; E)/\Psi^{-\infty}(V \times \mathcal{G}_b; E)\right).$$

Then, for any $b \in B$, we assume the existence of an open neighborhood $V \subset B$ of b and a $\mathcal{C}^{\infty}(B)$-linear isomorphism

$$\begin{aligned}(15)\quad \mathcal{A}_{\mathcal{L}}(\pi^{-1}(V)|V; E_{|V}) &:= \mathcal{O}(\pi^{-1}(V)) \left(\Psi^{\infty}(\mathcal{G}_V; E)/\Psi^{-\infty}(\mathcal{G}_V; E)\right) \\ &\simeq \mathcal{O}(\pi^{-1}(V)) \otimes_{tf} \left(\Psi^{\infty}(V \times \mathcal{G}_b; E)/\Psi^{-\infty}(V \times \mathcal{G}_b; E)\right)\end{aligned}$$

of topologically filtered algebras, where $V \times \mathcal{G}_b$ is the product groupoid, with V consisting of just units and the operations being defined pointwise.

The three assumptions above, Equations (13), (14), and (15) are not independent, as we shall see shortly. See [52] for some basic facts about Poisson manifolds.

Lemma 2. *Assume that* (14) *is satisfied. Then the morphism ϱ above induces an isomorphism*

$$\mathcal{O}(M) \otimes_{C^{\infty}(M)} C^{\infty}(A^*(\mathcal{G})) \simeq \mathcal{O}(M) \otimes_{C^{\infty}(M)} C^{\infty}(T^*_{vert}M)$$

of Poisson algebras.

Proof. Let X, Y, and $Z \in \Gamma(A(\mathcal{G}))$. Then X, Y, and Z define functions (denoted by the same letter) $X, Y, Z : A^*(\mathcal{G}) \to \mathbb{R}$. Assume $Z = [X, Y]$. Then the Poisson bracket on $C^{\infty}(A^*(\mathcal{G}))$ is uniquely determined by $\{X, Y\} = Z$. The equation $\varrho([X, Y]) = [\varrho(X), \varrho(Y)]$ shows that the natural map

$$C^{\infty}(A^*(\mathcal{G})) \to C^{\infty}(T^*_{vert}M)$$

is a Poisson map. The proof is completed by including $\mathcal{O}(M)$-coefficients. $\square$

Let $M_0 := M \smallsetminus \partial M$ be as above the interior of M and let $T_{vert}M_0$ be the vertical tangent bundle to the smooth fibration $M_0 \to B$. Our second assumption implies, in particular, that the anchor map ϱ restricts to an isomorphism

$$A(\mathcal{G})|_{M_0} \simeq T_{vert}M_0,$$

of vector bundles.

We now discuss the relation between our three assumptions on $\mathcal{G}$. It turns out that these assumptions do not play equal roles. In fact, the second assumption implies the third one, and, under some weak assumptions on $\mathcal{G}$ (d-connectivity) it also implies the first assumption. The following considerations are however somewhat independent from the rest of the paper, and, for the purpose of Hochschild homology computations, the reader can just ignore some of the results below, but instead consider all three assumptions on $\mathcal{G}$.

First, let us notice that, in the same spirit as the above lemma, we get an isomorphism of the algebras of differential operators corresponding to $\mathcal{G}$ and to M. More precisely, let $\mathrm{Diff}(M, \mathcal{G})$ be the algebra of differential operators on M generated by $\mathcal{C}^{\infty}(M)$ and $\Gamma(A(\mathcal{G}))$. Similarly, let $\mathrm{Diff}(M)$ be the algebra of differential operators on M generated by $\mathcal{C}^{\infty}(M)$ and $\Gamma(TM)$. The anchor map ϱ then gives rise to an inclusion

$$(16)\qquad \varrho_{\mathrm{Diff}} : \mathrm{Diff}(M, \mathcal{G}) \to \mathrm{Diff}(M).$$

Proposition 2. *Suppose that the map $\varrho_\Gamma : \Gamma(A(\mathcal{G})) \to \Gamma(TM)$ defined by ϱ is injective. Then our second assumption on $\mathcal{G}$, Equation (14), is equivalent to the fact that $\varrho_{\mathrm{Diff}} : \mathrm{Diff}(M,\mathcal{G}) \to \mathrm{Diff}(M)$ induces an isomorphism*

$$\mathcal{O}(M)\,\mathrm{Diff}(M,\mathcal{G}) \to \mathcal{O}(M)\,\mathrm{Diff}(M).$$

Proof. The space of vector fields on a manifold coincides with the space of first order differential operators without constant term (i.e. that send the function constant equal to 1 to 0). Thus, the isomorphism $\mathcal{O}(M)\,\mathrm{Diff}(M,\mathcal{G}) \simeq \mathcal{O}(M)\,\mathrm{Diff}(M)$ is equivalent to the fact that $\mathcal{O}(M)\Gamma(A(\mathcal{G}))$ maps surjectively onto $\mathcal{O}(M)\Gamma(TM)$. Since this map is injective by assumption, the result follows. □

The algebras $\mathcal{A}_{\mathcal{L}}(\mathcal{G}; E)$ turn out to depend only on $\pi : M \to B$.

Theorem 4. *The algebras $\mathcal{A}_{\mathcal{L}}(\mathcal{G}; E)$ are independent of $\mathcal{G}$, as long as assumptions* (13) *and* (14) *are satisfied.*

Proof. Assume E is trivial, for simplicity. Let $\Psi^\infty_{prop}(M|B)$ be the algebra of smooth, properly supported families of operators acting on the fibers of

$$\pi_0 : M_0 := M \smallsetminus \partial M \to B.$$

Recall [21, 41] that the vector representation $\pi_v : \Psi^\infty(\mathcal{G}) \to \mathrm{End}(\mathcal{C}^\infty_c(M_0))$ is defined uniquely by

$$(\pi_v(P)f) \circ r = P(f \circ r),$$

see [41] for details. Assumption (13) shows that π_v factors through a morphism $\Psi^\infty(\mathcal{G}) \to \Psi^\infty_{prop}(M|B)$. Assumption (14) implies that $d^{-1}(x) \simeq \pi_0^{-1}(\pi_0(x))$ for all $x \in M_0$, so the action of some $P \in \Psi^\infty(\mathcal{G})$ on $\mathcal{G}$ is completely determined by the action of P on the fibers of π_0. This means that the vector representation π_v of $\Psi^\infty(\mathcal{G})$ is injective, so $\Psi^\infty(\mathcal{G})$ identifies with a subalgebra of $\Psi^\infty_{prop}(M|B)$. Consequently, $\mathcal{A}_{\mathcal{L}}(\mathcal{G})$ identifies with a subalgebra of $\mathcal{B} := \Psi^\infty_{prop}(M|B)/\Psi^{-\infty}_{prop}(M|B)$.

We now argue that Proposition 2 and asymptotic completeness implies that the image of $\Psi^\infty(\mathcal{G})$ in $\mathcal{B}$ is independent of $\mathcal{G}$. Indeed, it is enough to check that the image of $\Psi^m(\mathcal{G}) \to \mathcal{B}$ is independent of $\mathcal{G}$, for any m. Let $D \in \mathcal{O}(M)\,\mathrm{Diff}(M;\mathcal{G})$ be an elliptic differential operator in $\mathcal{O}(M)\Psi^k(\mathcal{G})$, for some fixed $k \geq 1$. Let Q be a parametrix of D. Then Proposition 2 implies that

$$\mathcal{O}(M)\,\mathrm{Diff}(M;\mathcal{G})[Q] = \mathcal{O}(M)\,\mathrm{Diff}(M)[Q]. \tag{17}$$

Let $\mathcal{G}_1$ be another differentiable groupoid satisfying Assumptions (13) and (14), then

$$\mathcal{O}(M)\,\mathrm{Diff}(M;\mathcal{G}_1)[Q] = \mathcal{O}(M)\,\mathrm{Diff}(M;\mathcal{G})[Q],$$

also. Because $\mathcal{O}(M)(\Psi^m(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})) \to \mathcal{B}$ is continuous and injective and the image of the space of operators of order at most m of $\mathcal{O}(M)\,\mathrm{Diff}(M;\mathcal{G})[Q]$ is dense in $\mathcal{O}(M)\Psi^m(\mathcal{G})/\Psi^{-\infty}(\mathcal{G}))$, we obtain that the *closure* of the range of $\Psi^m(\mathcal{G})$ in $\mathcal{B}$ does not depend on $\mathcal{G}$. By looking at the complete symbols of the images of $\mathcal{O}(M)(\Psi^m(\mathcal{G}))$ and $\Psi^m(\mathcal{G}_1)$ in $\mathcal{B}$ and using asymptotic completeness of the algebras of pseudodifferential operators $\mathcal{O}(M)(\Psi^\infty(\mathcal{G}))$ and $\Psi^\infty(\mathcal{G}_1)$, we obtain that the actual range of $\Psi^m(\mathcal{G})$ in $\mathcal{B}$ is independent of $\mathcal{G}$, as desired. □

Let us recall that $\mathcal{G}$ is *d-connected* if, and only if, all the sets $\mathcal{G}_x := d^{-1}(x)$ are connected.

Corollary 2. *Suppose $\mathcal{G}$ is a differentiable groupoid with units M. Then assumption* (14) *implies assumption* (15)*. If $\mathcal{G}$ is also d-connected, then* (14) *implies also assumption* (13)*.*

Proof. By Theorem 4, it is enough to check (15) for any fixed groupoid $\mathcal{G}$ satisfying (13). In particular, we can choose $\mathcal{G}$ to be locally a product, in which case (15) is trivially satisfied. (For example, we could take $\mathcal{G} = \mathcal{G}_{M,b}$, the b-groupoid defined in Section 6.)

Let $X_1, \ldots , X_m$ be sections of $A(\mathcal{G})$. We shall write $\varrho(X_j)$ for $\varrho_\Gamma(X_j)$. Then

$$\pi(\exp(\varrho(X_1)) \ldots \exp(\varrho(X_m))x) = \pi(x),$$

for all $x \in M$. The assumption that $\mathcal{G}$ be d-connected is equivalent to the assumption that, for any $g \in \mathcal{G}$, there exist $X_1, \ldots , X_m$ as above such that

$$r(g) = \exp(\varrho(X_1)) \ldots \exp(\varrho(X_m))d(g),$$

see [29]. □

It also follows from the above discussion that it is enough for our computations to consider a "typical" algebra for each fibration $\pi : M \to B$. There are several choices of "typical" algebras, in general. One possible choice, the b-calculus, as well as the result of our computations for these algebras, will be described in Section 6.

4. Hochschild homology of families

In this section, we compute the Hochschild homology of the algebras $\mathcal{A}_\mathcal{L}(M|B;E)$ introduced in the previous section. Recall that these algebras are algebras of complete symbols associated with a groupoid $\mathcal{G}$ with units M and a fibration $\pi : M \to B$ by manifolds with corners satisfying the assumptions of Equations (13) and (14). The results of this section are already interesting when the manifold M has no boundary. Recall that $n = \dim(M)$, $q = \dim(B)$, and $n = p + q$.

In addition to helping us eliminate our third assumption on $\mathcal{G}$, Equation (15), the introduction of the Laurent-type factors also simplifies the calculations, as in [32] and [4]. When B is reduced to a point $*$, this also ensures that the Hochschild homology of $\mathcal{A}_\mathcal{L}(M) := \mathcal{A}_\mathcal{L}(M|*)$ is finite dimensional. For example, the dimension of the space of traces on $\mathcal{A}_\mathcal{L}(M|*)$ is the number of minimal faces of M [4]. Moreover, the "cone algebras" described for example in [42] are more closely related to the algebras $\mathcal{A}_\mathcal{L}(M|B;E)$ than to the algebras $\mathcal{A}(M|B;E)$. See also [24, 37].

Our computations will use the Poisson structure of $A^*(\mathcal{G})$ and, more precisely, the "homogeneous Laurent-Poisson homology" of $A^*(\mathcal{G}) \smallsetminus 0$, where $A^*(\mathcal{G}) \smallsetminus 0$ is the dual of the Lie algebroid of $\mathcal{G}$, with the zero section removed. The homogeneous Laurent-Poisson homology of $A^*(\mathcal{G}) \smallsetminus 0$ is defined below and will be identified in terms of the "homogeneous, vertical Laurent-de Rham cohomology" of the fibration $A^*(\mathcal{G}) \smallsetminus 0 \to B$ (this cohomology is also defined below). The homogeneous Laurent-Poisson homology and the homogeneous vertical Laurent-de Rham cohomology are natural analogues of the Poisson and, respectively, de Rham cohomology, which are obtained, roughly speaking, by introducing Laurent type singularities at the corners of M and by considering homogeneous forms (on $A^*(\mathcal{G}) \smallsetminus 0$, for example). See [35, 53, 54] for more on Poisson cohomology.

We begin with the definition of the groups $\mathrm{H}^{i,j}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_l$, the homogeneous, vertical Laurent-de Rham cohomology of the fibration $A^*(\mathcal{G}) \smallsetminus 0 \to B$. Then we shall discuss Poisson homology and its variant, the homogeneous Laurent-Poisson homology.

Let X be a manifold with corners and let $\pi_0 : X \to B$ be a fiber bundle with B smooth. Let us call the sections of $T_{vert}X$ vertical vector fields, as it is customary. Then the sections of the dual $T^*_{vert}X$ are called vertical differential forms. There exists a natural (i.e. independent of any choices) differential operator

$$d_{vert} : \Gamma(\Lambda^k T^*_{vert}X) \to \Gamma(\Lambda^{k+1} T^*_{vert}X), \tag{18}$$

the *vertical de Rham differential*.

Every vertical vector field on X is also a vector field on X in the usual sense. On the other hand, a form on X restricts to a vertical form on X. Moreover, every vertical form on X is the restriction of a form on X, but we cannot choose that form in a canonical way. A convenient way to choose extensions of vertical forms is to consider a splitting of TX into vertical and horizontal parts. We shall hence fix from now an isomorphism (or splitting)

$$\Theta : TX \simeq T_{vert}X \oplus \pi_0^* TB. \tag{19}$$

The splitting Θ of Equation (19) gives rise to an embedding $\Theta_k : \Gamma(\Lambda^k T^*_{vert}X) \to \Omega^k(X)$. More generally, we get isomorphisms

$$\Lambda^k T^* X \simeq \bigoplus_{i+j=k} \Lambda^i T^*_{vert} X \otimes \pi_0^* \Lambda^j T^* B.$$

Let $\Omega^{i,j}(X) := \Gamma(X, \Lambda^i T^*_{vert}X \otimes \pi_0^* \Lambda^j T^* B)$. Then $\Omega^k(X) \simeq \bigoplus_{i+j=k} \Omega^{i,j}(X)$, and we also have isomorphisms

$$\Omega^{i,0}(X) \otimes_{\mathcal{C}^\infty(B)} \Omega^j(B) \ni \omega \otimes \eta \to \omega \wedge \pi_0^* \eta \in \Omega^{i,j}(X). \tag{20}$$

The embedding Θ_k can then be used to define a map $d_{vert} : \Omega^{i,0}(X) \to \Omega^{i+1,0}(X)$ (using the same notation is unlikely to cause any confusion in our case). We extend then d_{vert} to a map

$$d_{vert} : \Omega^{i,j}(X) \to \Omega^{i+1,j}(X),$$

by using the isomorphisms of the Equation (20) above and setting

$$d_{vert}(\omega \wedge \pi_0^* \eta) = d_{vert}(\omega) \wedge \pi_0^* \eta$$

if $\eta \in \Omega^{i,0}(X)$ and $\omega \in \Omega^j(B)$. Clearly $d_{vert}^2 = 0$. The extension d_{vert} that we obtain depends on the splitting Θ of Equation (19). The isomorphism class of the resulting complex, however, does not depend on Θ.

Let us denote by $\Omega^k_{\mathcal{L}}(X)$ the space of k-differential forms on the interior of X that have only rational (or Laurent) singularities near the corners. We shall sometimes call forms with these properties *Laurent*-differential forms. The above definitions and properties extend to $\Omega^k_{\mathcal{L}}(X)$ as follows. Let $\Omega^{i,j}_{\mathcal{L}}(X) := \mathcal{O}(X)\Gamma(X, \Lambda^i T^*_{vert} X \otimes \pi_0^* \Lambda^j T^* B)$. Then

$$\Omega^k_{\mathcal{L}}(X) \simeq \bigoplus_{i+j=k} \Omega^{i,j}_{\mathcal{L}}(X),$$

and, as before, we obtain a differential

$$d_{vert} : \Omega^{i,j}_{\mathcal{L}}(X) \to \Omega^{i+1,j}_{\mathcal{L}}(X).$$

We shall denote by $\mathrm{H}^{i,j}_{\mathcal{L}}(X) = \ker(d_{vert})/d_{vert}\Omega^{i-1,j}_{\mathcal{L}}(X)$ the homology of the above complex. Similarly, if compactly supported forms are considered, we obtain a complex whose homology we denote by $\mathrm{H}^{i,j}_{c,\mathcal{L}}(X)$.

Define the horizontal differential $d_{hor} : \Omega^{i,j}_{\mathcal{L}}(X) \to \Omega^{i,j+1}_{\mathcal{L}}(X)$ as the component of bidegree $(0,1)$ of d. Then $\partial := d - d_{vert} - d_{hor}$ is known to be a differential and to have bidegree $(-1,2)$. See for instance [47].

The equality $d^2 = 0$ is equivalent to the following relations:

$$d_{vert}d_{hor} + d_{hor}d_{vert} = 0, \quad d_{hor}^2 + \partial d_{vert} + d_{vert}\partial = 0$$
$$d_{vert}^2 = 0, \quad \text{and} \ \partial d_{hor} + d_{hor}\partial = 0.$$

The vertical Laurent-de Rham cohomology can be computed in a fairly explicit manner. Indeed, let $\mathcal{F}^k$ be the local coefficient system determined by the Laurent cohomology groups of the fibers of $X \to B$. Thus $\mathcal{F}^k$ is a canonically flat vector bundle over B whose fiber at $b \in B$ is

$$\mathcal{F}^k(b) = \mathrm{H}^k_{c,\mathcal{L}}(\pi_0^{-1}(b)) := \mathrm{H}^k_{c,\mathcal{L}}(\pi_0^{-1}(b)|b). \tag{21}$$

Let $\Omega^k_c(B)$ be the space of compactly supported k-forms on B.

Proposition 3. *Using the above notation, we have that*

$$\mathrm{H}^{k,h}_{c,\mathcal{L}}(X|B) \simeq \Omega^h_c(B) \otimes_{\mathcal{C}^\infty(B)} \Gamma(\mathcal{F}^k) =: \Omega^h_c(B;\mathcal{F}^k), \tag{22}$$

the space of compactly supported h-forms on B with values in $\mathcal{F}^k$. In particular, the vertical Laurent-de Rham cohomology groups $\mathrm{H}^{k,h}_{c,\mathcal{L}}(X|B)$ are independent of the splitting $TX \simeq T_{vert}X \oplus \pi_0^*TB$ used to define them, see Equation (19).

In addition, the action induced by the horizontal de Rham differential d_{hor} on $\mathrm{H}^{k,h}_{c,\mathcal{L}}(X|B)$ is isomorphic under (22) *to the de Rham differential on B with coefficients in the locally constant sheaf $\mathcal{F}^k$.*

Proof. The above formula is checked right away when $\pi : X \to B$ is a trivial fiber bundle (i.e. $X = B \times F$) by using the Künneth formula for the tensor product of complexes of nuclear vector spaces, [19]. Moreover, an automorphism of the trivial fiber bundle $X = B \times F$ does not affect the isomorphism of the proposition. A partition of unity argument then completes the proof. □

Let $A^*(\mathcal{G}) \smallsetminus 0$ be obtained from the vector bundle $A^*(\mathcal{G})$, as before, by removing the zero section. We are interested in the above constructions when $X = A^*(\mathcal{G}) \smallsetminus 0$ and $\pi_0 : A^*(\mathcal{G}) \smallsetminus 0 \to B$ is obtained from the composition of the maps $A^*(\mathcal{G}) \to M$ and $\pi : M \to B$. More precisely, for us, the relevant cohomology groups are the cohomology groups obtained by considering homogeneous forms. Let then $\Omega^{i,j}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_l$ be the space of l-homogeneous forms in

$$\Omega^{i,j}_{\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0) = \mathcal{O}(M)\Gamma(A^*(\mathcal{G}) \smallsetminus 0, \Lambda^i T_{vert}(A^*(\mathcal{G}) \smallsetminus 0)^* \otimes \pi_0^*\Lambda^j T^*B)$$

whose support projects onto a compact subset of M. Here the homogeneity is considered with respect to the natural action of $\mathbb{R}^*_+$ on $A^*(\mathcal{G}) \smallsetminus 0$ by dilations. We denote then by

$$\mathrm{H}^{i,j}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_l$$

the homology of the complex $\Omega^{i,j}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_l$ with respect to the vertical de Rham differential d_{vert}. We shall call these groups the *homogeneous, vertical Laurent-de Rham cohomology groups of $A^*(\mathcal{G})$*.

Similar constructions and definitions are obtained with $T^*_{vert}M$ in place of $A^*(\mathcal{G})$. Our second assumption on the groupoid $\mathcal{G}$, Equation (14), gives that the two cohomologies are isomorphic.

Lemma 3. *The anchor map $\varrho : A(\mathcal{G}) \to T_{vert}M$ induces a natural isomorphism*

$$\mathrm{H}^{i,j}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_l \simeq \mathrm{H}^{i,j}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_l.$$

These groups vanish if $l \neq 0$ and, for $l = 0$, we have

$$\mathrm{H}^{i,j}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_0 \simeq \mathrm{H}^{i,j}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_0$$
$$\cong \mathrm{H}^{i,j}_{c,\mathcal{L}}(S^*_{vert}(M) \times S^1|B) \cong \mathrm{H}^{i,j}_{c,\mathcal{L}}(S^*(\mathcal{G}) \times S^1|B).$$

Proof. The map ϱ induces an isomorphism of the corresponding complexes, by Equation (14) and the definition of the spaces $\Omega^{i,j}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_l$. The vanishing of the groups $\mathrm{H}^{i,j}_{c,\mathcal{L}}(T^*_{vert} \smallsetminus 0|B)_l$ for $l \neq 0$ follows from the homotopy invariance of de Rham cohomology. The computation of the 0-homogeneous cohomology spaces is classical, see for instance [10, 32]. □

We can thus replace $A^*(\mathcal{G})$ with $T^*_{vert}M$ for the rest of our computations of the homogeneous, vertical Laurent-de Rham cohomology groups of $A^*(\mathcal{G})$.

The homogeneous, vertical Laurent-de Rham cohomology can be computed using a method similar to the one we used to determine the non-homogeneous homology. Indeed, let $\mathcal{F}^k$ be the local coefficient system determined by the Laurent cohomology groups of the fibers of $\pi_0 : S^*_{vert}(M) \times S^1 \to B$. Thus $\mathcal{F}^k$ is a canonically flat vector bundle over B whose fiber at $b \in B$ is

$$\mathcal{F}^k(b) = \mathrm{H}^k_{c,\mathcal{L}}(\pi_0^{-1}(b)) = \mathrm{H}^k_c(\mathcal{L}(\pi_0^{-1}(b))). \tag{23}$$

Proposition 4. *Using the above notation, we have that*

$$\mathrm{H}^{k,h}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_0 \simeq \Omega^h_c(B) \otimes_{\mathcal{C}^\infty(B)} \Gamma(\mathcal{F}^k) =: \Omega^h_c(B; \mathcal{F}^k). \tag{24}$$

For $l \neq 0$, the groups $\mathrm{H}^{k,h}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_l$ vanish. The horizontal de Rham differential d_{hor} induces a differential on $\mathrm{H}^{k,h}_{c,\mathcal{L}}(X|B)$ which is isomorphic under (24) to the de Rham differential on B with coefficients in the locally constant sheaf $\mathcal{F}^k$.

Proof. The proof is completely similar to that of Proposition 3. The vanishing of the groups $\mathrm{H}^{k,h}_{c,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0|B)_l$ for $l \neq 0$ follows again from the homotopy invariance of Laurent-de Rham cohomology. □

Let us now introduce the Poisson homology groups that we are interested in. The following considerations apply to any regular Poisson structure. Recall that the Poisson structure on $A^*(\mathcal{G})$ is defined by a two tensor

$$G \in \mathcal{C}^\infty(A^*(\mathcal{G}), \Lambda^2 T A^*(\mathcal{G}))$$

so that $\{f, g\} = i_G(df \wedge dg)$. Clearly, the tensor G must satisfy some non-trivial conditions for the map $\{\ ,\ \}$ to satisfy the Jacobi identity. These conditions turn out to be equivalent to $[G, G]_{SN} = 0$, where $[\ ,\ ]_{SN}$ is the Schouten-Nijenhuis bracket [48]. The formula for the Poisson bracket is determined in terms of the Lie algebra structure on the space of sections of $A(\mathcal{G})$. (This was recalled in Lemma 2.)

Let $i_G : \Omega^k(A^*(\mathcal{G})) \to \Omega^{k-2}(A^*(\mathcal{G}))$ be the contraction with the tensor G. Then we obtain as in [9] a differential

$$\delta := i_G \circ d - d \circ i_G : \Omega^k(A^*(\mathcal{G})) \to \Omega^{k-1}(A^*(\mathcal{G})). \tag{25}$$

Explicitly, for any $(f_0, \dots, f_k) \in \mathcal{C}^\infty(A^*(\mathcal{G}))^{k+1}$, the differential δ is given by the formula

$$\delta(f_0 df_1 df_2 \dots df_k) = \sum_{1\le j\le k} (-1)^{j+1}\{f_0, f_j\} df_1 \dots \widehat{df_j} \dots df_k + \sum_{1\le i<j\le k} (-1)^{i+j} f_0 d\{f_i, f_j\} df_1 \dots \widehat{df_i} \dots \widehat{df_j} \dots df_k. \tag{26}$$

When $\mathcal{G}$ is our groupoid and under the Assumptions (13) and (14), we can use the splitting Θ of Equation (19). One then shows that δ decomposes into the sum of two bihomogeneous differentials [48]

$$\delta = \delta_{vert} + \alpha,$$

where $\delta_{vert} := i_G \circ d_{vert} - d_{vert} \circ i_G$ is the vertical Poisson differential. The vertical Poisson differential has bidegree $(-1, 0)$. The extra term α has bidegree $(-2, +1)$ and is in fact given by [5]:

$$\alpha = i_G \circ d_{hor} - d_{hor} \circ i_G.$$

See [5]. In particular, the commutator $[i_G, \partial]$ is trivial. It is straightforward to see that the restriction of δ_{vert} to vertical differential forms is given by the following local expression

$$\begin{aligned} \delta_{vert}(f_0 d_{vert} f_1 d_{vert} f_2 \dots d_{vert} f_k) &= \sum_{1\le j\le k} (-1)^{j+1}\{f_0, f_j\} d_{vert} f_1 \dots \widehat{d_{vert} f_j} \dots d_{vert} f_k \\ &+ \sum_{1\le i<j\le k} (-1)^{i+j} f_0 d_{vert}\{f_i, f_j\} d_{vert} f_1 \dots \widehat{d_{vert} f_i} \dots \widehat{d_{vert} f_j} \dots d_{vert} f_k. \end{aligned} \tag{27}$$

The above formula determines δ_{vert} on $\Omega^{j,0}(A^*(\mathcal{G}))$. To determine δ_{vert} in general, we can use the following lemma.

Lemma 4. *Let $\alpha \in \Omega^i(A^*(\mathcal{G}))$, let $\beta \in \Omega^j(B)$, and let $\pi_0 : A^*(\mathcal{G}) \to B$ be the composite projection. Then*

$$\delta(\alpha \wedge \pi_0^*(\beta)) = \delta(\alpha) \wedge \pi_0^*(\beta) \quad \textit{and} \quad \delta_{vert}(\alpha \wedge \pi_0^*(\beta)) = \delta_{vert}(\alpha) \wedge \pi_0^*(\beta).$$

Proof. It is enough to check the first equation when $\beta = g$ or $\beta = dg$, for some smooth function g on B. Our claim then follows from the fact that $\{f, g \circ \pi_0\} = 0$, for any smooth function f on M and from the explicit formula for δ, Equation (26).

The equation for $\delta_{vert} := i_G \circ d_{vert} - d_{vert} \circ i_G$ follows from the equation for δ by checking bidegrees. □

The formula of Equation (26) is valid also when M has corners and it is easy to check that the differential δ is homogeneous of degree -1 with respect to the action of $\mathbb{R}_+^*$ on $A^*(\mathcal{G}) \smallsetminus 0$. Let $\Omega^k(A^*(\mathcal{G}) \smallsetminus 0)_l$ be the space of k-forms on $A^*(\mathcal{G}) \smallsetminus 0$ that are homogeneous of order l. We hence obtain a differential

$$\delta : \Omega^k(A^*(\mathcal{G}) \smallsetminus 0)_l \to \Omega^{k-1}(A^*(\mathcal{G}) \smallsetminus 0)_{l-1}.$$

Let $\Omega^k_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_l$ be the subspace of $\Omega^k(A^*(\mathcal{G}) \smallsetminus 0)_l$ consisting of forms whose support projects onto a compact subset of M, as before. Because δ preserves the support, it maps the space $\Omega^k_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_l$ to $\Omega^{k-1}_{rc}(A^*(\mathcal{G}) \smallsetminus 0)_{l-1}$.

The same result holds with δ_{vert} and we have:

$$\delta_{vert} : \Omega^k_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_l \to \Omega^{k-1}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_{l-1}.$$

We obtain in this way a direct sum of complexes $(\mathcal{P}^k)_{k\in\mathbb{Z}}$

$$\mathcal{P}^k : \quad 0 \to \mathcal{P}^k_{2p+q} \xrightarrow{\delta} \mathcal{P}^k_{2p+q-1} \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{P}^k_{-k} \to 0, \tag{28}$$

where $\mathcal{P}^k_l = \Omega^{k+l}_{rc,\mathcal{L}}(A^*(\mathcal{G}) \smallsetminus 0)_l$.

We shall denote the homology groups of the above complex by

$$\mathrm{H}^{\delta}_{\mathcal{L},k+l}(A^*(\mathcal{G}) \smallsetminus 0|B)_l := \frac{\ker(\delta : \mathcal{P}^k_l \to \mathcal{P}^k_{l-1})}{\delta(\mathcal{P}^k_{l+1})}.$$

In the same way we define the vertical *homogeneous Laurent-Poisson homology* groups using δ_{vert} instead of δ and denote them by

$$\mathrm{H}^{\delta_{vert}}_{\mathcal{L},k+l}(A^*(\mathcal{G}) \smallsetminus 0|B)_l := \frac{\ker(\delta_{vert} : \mathcal{P}^k_l \to \mathcal{P}^k_{l-1})}{\delta_{vert}(\mathcal{P}^k_{l+1})}.$$

Furthermore, we define for any (i,j)

$$\mathcal{P}^{i,j}_l := \Omega^{i+l,j}_{rc,\mathcal{L}}(T^*_{vert}M \smallsetminus 0)_l.$$

From the results of Section 3, we deduce that

$$\mathcal{P}^k_l \simeq \bigoplus_{i+j=k} \mathcal{P}^{i,j}_l.$$

Note that with respect to the splitting (19), we have:

$$\delta_{vert} : \mathcal{P}^{i,j}_l \to \mathcal{P}^{i,j}_{l-1},$$

and the vertical Laurent Poisson homology can be computed by fixing (i,j) and restricting to each $\mathcal{P}^{i,j} := \oplus_l \mathcal{P}^{i,j}_l$. However, the extra term α does not preserve $\mathcal{P}^{i,j}$, and sends $\mathcal{P}^{i,j}$ to $\mathcal{P}^{i-1,j+1}$.

The relevance for us of Poisson homology, in general, and of homogeneous Laurent-Poisson homology, in particular, is that they are related to the Hochschild homology groups of the algebras $\mathcal{A}_{\mathcal{L}}(M|B;E) := \mathcal{O}(M)(\Psi^\infty(\mathcal{G};E)/\Psi^{-\infty}(\mathcal{G};E))$ introduced in the previous section, where E is a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle. The following lemma makes this connection precise.

Lemma 5. *The algebra $\mathcal{A}_{\mathcal{L}}(M|B;E)$ is topologically H-unital. The* EH^2 *term of the spectral sequence associated to $\mathcal{A}_{\mathcal{L}}(M|B;E)$ by Lemma 1 is given by*

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{\delta}_{\mathcal{L},k+h}(A^*(\mathcal{G}) \smallsetminus 0|B)_k.$$

Proof. When E is a trivial one-dimensional vector bundle and $\mathcal{G}$ is an arbitrary groupoid, the above proposition was proved in [4, Proposition 7].

The extension to a non-trivial vector bundle and the $\mathbb{Z}/2\mathbb{Z}$-graded case is obtained as a consequence of Künneth formula as follows. Let us assume first that E is trivially graded. The graded algebra of $\mathcal{A}_{\mathcal{L}}(M|B;E)$ is the algebra generated by the homogeneous sections of the lift of E to $A^*(\mathcal{G}) \smallsetminus 0$. All these algebras are Morita equivalent, so their Hochschild homology is the same as that of the algebra corresponding to a trivial line bundle. Since the Hochschild homology of a commutative algebra can be represented as the homology of the space of global sections of a complex of sheaves, the statement about Morita equivalence is a local statement, which can then be proved locally using the Künneth formula. The spectral sequence of Lemma 1 then tells us that the Hochschild homology of the algebra $\mathcal{A}_{\mathcal{L}}(M|B;E)$ is independent of E.

In general, let $E = E_+ \oplus E_-$ be the decomposition of E into the direct sum of the $+1$ and, respectively, -1 eigenvalue of the grading automorphism. As above, we observe that the statement of the lemma is local, so we can assume that $E = E_+ \oplus E_-$ is such that both E_+ and E_- are trivial bundles. Denote by N the rank of E. Then $\mathcal{A}_{\mathcal{L}}(M|B;E) \simeq M_N(\mathcal{A}_{\mathcal{L}}(M|B))$.

Let A be a (topologically filtered) algebra A and N an integer. Assume that the grading automorphism of the algebra $M_N(A)$ is given by conjugation with a matrix in $M_N(\mathbb{C})$. Then $\mathrm{HH}_*(M_N(A)) \simeq \mathrm{HH}_*(A)$. This follows for example from the Künneth formula in ($\mathbb{Z}/2\mathbb{Z}$-graded) Hochschild homology (see [18]). □

As with the homogeneous, vertical de Rham homology, we can replace $A^*(\mathcal{G})$ in $\mathrm{H}^{\delta}_{\mathcal{L},k}(A^*(\mathcal{G}) \smallsetminus 0|B)_l$ with $T^*_{vert}M$, the vertical cotangent bundle to $\pi : M \to B$.

Lemma 6. *The anchor map $\varrho : A(\mathcal{G}) \to T_{vert}M$ induces an isomorphism*

$$\mathrm{H}^{\delta}_{\mathcal{L},k}(A^*(\mathcal{G}) \smallsetminus 0|B)_l \simeq \mathrm{H}^{\delta}_{\mathcal{L},k}(T^*_{vert}M \smallsetminus 0|B)_l.$$

Proof. This follows from Lemma 2 and the explicit formula for the Poisson bracket, Equation (26). □

We now proceed as usual and construct a chain map $*_{vert}$ from the complex that defines vertical Poisson homology to the complex that defines de Rham cohomology. The chain map $*_{vert}$ is, in a certain sense, a vertical symplectic $*$-operator. It corresponds to the canonical symplectic forms on the cotangent spaces of the fibers of $\pi : M \to B$. Denote by $M_b := \pi^{-1}(b)$, $b \in B$, and by ω_b the symplectic form on T^*M_b. There exists then a 2-form ω on $T^*_{vert}M := \cup_{b\in B} T^*M_b$ that restricts on each fiber T^*M_b to the form ω_b. This form is certainly not unique. There will be, however, a unique form $\omega \in \Omega^{2,0}_{\mathcal{L}}(T^*_{vert}M)$ with this property. We shall call this form ω the *vertical symplectic form* of $T^*_{vert}M$. It depends on the splitting of Equation (19).

The vertical symplectic volume form on $T^*_{vert}M$ is defined by analogy to be $vol_{vert}(M) := \omega^p/p!$. Next, we define $*_{vert} : \Omega^{k,0}_{\mathcal{L}}(T^*_{vert}M) \to \Omega^{2p-k,0}_{\mathcal{L}}(T^*_{vert}M)$ by the equation

$$\beta \wedge (*_{vert}\alpha) = (\beta, \alpha)_\omega \cdot vol_{vert}(M), \quad \forall\, \alpha, \beta \in \Omega^{k,0}_{\mathcal{L}}(T^*_{vert}M),$$

where $(\ ,\ )_\omega$ is the bilinear form induced by the symplectic form. Then we obtain that $*_{vert}(\Omega^{k,0}_{\mathcal{L}}(T^*_{vert}M)_l) = \Omega^{2p-k,0}_{\mathcal{L}}(T^*_{vert}M)_{l+p-k}$. Finally, to define

$$*_{vert} : \Omega^{i,j}_{\mathcal{L}}(T^*_{vert}M) \to \Omega^{2p-i,j}_{\mathcal{L}}(T^*_{vert}M)$$

in general, it is enough to define $*_{vert}\alpha$ when $\alpha = \eta \wedge \pi_0^*\beta$, with $\eta \in \Omega^{i,0}_{\mathcal{L}}(M)$ and $\beta \in \Omega^j(B)$, where $\pi_0 : T^*_{vert}M \to B$ is the induced projection. We set then

$$*_{vert}(\alpha) = *_{vert}(\eta \wedge \pi_0^*\beta) := *_{vert}(\eta) \wedge \pi_0^*\beta. \tag{29}$$

Similarly, we obtain again that $*_{vert}(\Omega^{i,j}_{\mathcal{L}}(T^*_{vert}M)_l) = \Omega^{2p-i,j}_{\mathcal{L}}(T^*_{vert}M)_{l+p-i}$.

The usual properties of the symplectic $*$-operator in relation to de Rham and Poisson homology extend to $*_{vert}$.

Proposition 5. *Let $*_{vert}$ be the operator defined above. Then*

(1) $*_{vert}^2 = id$;

(2) $(-1)^{i+1}d_{vert}\circ *_{vert} = *_{vert}\circ\delta_{vert}$ *on* $\Omega^{i,j}(T^*_{vert}M)$.
We can extend the range of both formulas to include homogeneous forms or forms with Laurent type singularities.

Proof. Both formulas are well known when B is reduced to a point [9]. Using a version with parameters of this particular case, we obtain that the two formulas are correct on $\Omega^{i,0}(T^*_{vert}M)$.

For the general case, let $\alpha\in\Omega^{i,0}(T^*_{vert}M)$ be of the form $\alpha=\eta\wedge\pi_0^*\beta$, where $\eta\in\Omega^{i,0}_{\mathcal{L}}(M)$ and $\beta\in\Omega^j_c(B)$, and $\pi_0 : T^*_{vert}M\to B$ is the induced projection. Then, using Equation (29), we obtain

$$*^2_{vert}(\alpha) = *^2_{vert}(\eta)\wedge\pi_0^*\beta = \alpha. \tag{30}$$

Similarly, using the definition of d_{vert}, Lemma 4, and Equation (29), we obtain

$$\begin{aligned}(-1)^{i+1}d_{vert}\circ *_{vert}(\alpha) &= (-1)^{i+1}d_{vert}\circ *_{vert}(\eta)\wedge\pi_0^*\beta\\ &= *_{vert}\circ\delta_{vert}(\eta)\wedge\pi_0^*\beta = *_{vert}\circ\delta_{vert}(\alpha).\end{aligned}$$

This is enough to complete the proof. □

We are ready now to determine the homogeneous, Laurent-Poisson homology groups of $A^*(\mathcal{G})$. Recall that p denotes the dimension of the fibers of $M\to B$ and q is the dimension of the manifold B. We set for any fixed $k\in\mathbb{Z}$,

$$K^{j,l} := \mathcal{P}^{k-j,j}_{l-j},$$

so that

$$\delta_{vert} : K^{j,l}\to K^{j,l-1} \text{ and } \alpha : K^{j,l}\to K^{j+1,l}.$$

To compute the homogeneous Laurent δ-homology of $A^*(\mathcal{G})\smallsetminus 0$, we use that the complex splits into subcomplexes $(\mathcal{P}^k,\delta)$. Thus we can fix the integer $k\in\mathbb{Z}$ and define a filtration of the above bicomplex $K^{j,l}$ by

$$F_h := \bigoplus_{l\in\mathbb{Z},j\le h} K^{j,l}.$$

Proposition 6. *The spaces* $\mathrm{H}^{\delta}_{\mathcal{L},k+l}(A^*(\mathcal{G})\smallsetminus 0|B)_l$ *and* $\mathrm{H}^{\delta_{vert}}_{\mathcal{L},k+l}(A^*(\mathcal{G})\smallsetminus 0|B)_l$ *are isomorphic, that is*

$$\mathrm{H}^{\delta}_{\mathcal{L},k+l}(A^*(\mathcal{G})\smallsetminus 0|B)_l \simeq \mathrm{H}^{\delta_{vert}}_{\mathcal{L},k+l}(A^*(\mathcal{G})\smallsetminus 0|B)_l.$$

Proof. Recall that we have

$$\delta=\delta_{vert}+\alpha \text{ and } \delta_{vert}\circ\alpha+\alpha\circ\delta_{vert}=0.$$

We then use for any fixed k the decomposition $\mathcal{P}^k\cong\bigoplus_{i+j=k}\mathcal{P}^{i,j}$ into a finite double complex and the decreasing filtration F_h defined above. This yields a spectral sequence $(E^r)_{r\ge 1}$ which converges to the δ-homology by classical homological arguments. The E^1 term of this spectral sequence is given by

$$E^1_{u,v} = \mathrm{H}^{2p-v-k+u,u}_{\mathcal{L}}(T^*_{vert}M\smallsetminus 0|B)_{p-k+u}.$$

But again by a homotopy argument, the homogeneous vertical de Rham cohomology space $\mathrm{H}^{2p-v-k+u,u}_{\mathcal{L}}(T^*_{vert}M\smallsetminus 0|B)_{p-k+u}$ is trivial unless $u=k-p$. Therefore, we get:

$$E^1_{u,v}=0 \text{ if } v\ne -k-p.$$

Hence for any $r\ge 1$, we see that $d^r=0$ and the spectral sequence collapses at E^1. The proof is thus complete. □

Theorem 5. *The homogeneous, Laurent-Poisson homology groups of $A^*(\mathcal{G})$ are given by*

$$\mathrm{H}^{\delta}_{\mathcal{L},k}(A^*(\mathcal{G}) \smallsetminus 0|B)_l \simeq \mathrm{H}^{p-l,k-l-p}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_0.$$

Proof. The vertical symplectic Hodge operator $*_{vert}$ yields isomorphisms

$$*_{vert} : \bigoplus_{i+j=k} \Omega^{i,j}_{\mathcal{L}}(T^*_{vert}M \smallsetminus 0)_l \to \bigoplus_{i+j=k} \Omega^{2p-i,j}_{\mathcal{L}}(T^*_{vert}M \smallsetminus 0)_{l+p-i}$$

which intertwine the δ_{vert} and d_{vert} differentials (up to a sign). Proposition 6 shows then that

$$\mathrm{H}^{\delta}_{\mathcal{L},k}(A^*(\mathcal{G}) \smallsetminus 0|B)_l \simeq \bigoplus_{i+j=k} \mathrm{H}^{2p-i,j}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_{l+p-i}.$$

But for $l+p-i \neq 0$, the cohomology spaces $\mathrm{H}^{2p-i,j}(T^*_{vert}M \smallsetminus 0|B)_{l+p-i}$ are trivial by the homotopy invariance of de Rham cohomology. Hence the only non trivial term is:

$$\mathrm{H}^{p-l,k-l-p}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_0,$$

and this completes the proof. □

We can now apply the results of Section 1 together with Theorem 5.

Proposition 7. *The* EH^2*-term of the spectral sequence associated in Lemma 1 to the Hochschild homology of the algebra* $\mathcal{A}_{\mathcal{L}}(M|B;E)$ *is given by:*

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{p-k,h-p}_{c,\mathcal{L}}(S^*_{vert}(M) \times S^1|B),$$

where $S^*_{vert}(M)$ *is the sphere bundle of the vertical cotangent bundle* $T^*_{vert}M$*.*

Proof. Denote by δ the Poisson differential on the vertically symplectic fibration $T^*_{vert}M \to B$. Lemma 5, Proposition 6 and Theorem 5 give by straightforward computation:

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{p-k,h-p}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_0.$$

Now a classical argument shows that [32]:

$$\mathrm{H}^{p-k,h-p}_{c,\mathcal{L}}(T^*_{vert}M \smallsetminus 0|B)_0 \simeq \mathrm{H}^{p-k,h-p}_{c,\mathcal{L}}(S^*_{vert}(M) \times S^1|B),$$

which completes the proof. □

In the following lemma, we shall denote by $\otimes_{tf}$ the completion of the tensor product of two algebras in the unique natural way that makes the completed tensor product a topologically filtered algebra. See also Equation (15) where $\otimes_{tf}$ was used before.

Lemma 7. *Assume that the fibration* $\pi : M \to B$ *and the bundle of algebras* $\mathcal{A}_{\mathcal{L}}(M|B;E)$ *are trivial; that is, assume that* $M = B \times F$ *and* $\mathcal{A}_{\mathcal{L}}(M|B;E) \simeq \mathcal{A}_{\mathcal{L}}(F;E) \otimes_{tf} \mathcal{C}^\infty_c(B)$ *as topologically filtered algebras. Then the spectral sequence associated in Lemma 1 to the Hochschild homology of the algebra* $\mathcal{A}_{\mathcal{L}}(M|B;E)$ *collapses at* EH^2*, converges, and we have*

$$\mathrm{HH}_k(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \oplus_{i+j=k} \mathrm{H}^{2p-i}_{c,\mathcal{L}}(S^*(F) \times S^1) \otimes \Omega^j_c(B).$$

Proof. We know from [4] that $\mathrm{HH}_j(\mathcal{A}_{\mathcal{L}}(F;E)) \simeq \mathrm{H}_c^{2p-j}(\mathcal{L}(S^*(F)) \times S^1)$. The usual Künneth formula for the Hochschild homology of algebras then gives that

$$
\begin{aligned}
(31)\quad \mathrm{HH}_k(\mathcal{C}_c^\infty(B) \otimes \mathcal{A}_{\mathcal{L}}(F;E)) &\simeq \oplus_{i+j=k} \mathrm{HH}_i(\mathcal{C}_c^\infty(B)) \otimes \mathrm{HH}_j(\mathcal{A}_{\mathcal{L}}(F;E)) \\
&\simeq \oplus_{i+j=k} \Omega_c^i(B) \otimes \mathrm{H}_{c,\mathcal{L}}^{2p-j}(S^*(F) \times S^1).
\end{aligned}
$$

The last equality is a consequence of the computation of $\mathrm{HH}_j(\mathcal{A}_{\mathcal{L}}(F;E))$ carried out in [4]. The inclusion $\mathcal{C}_c^\infty(B) \otimes \mathcal{A}_{\mathcal{L}}(F;E) \to \mathcal{C}_c^\infty(B) \otimes_{tf} \mathcal{A}_{\mathcal{L}}(F;E) = \mathcal{A}_{\mathcal{L}}(M|B;E)$ preserves the natural filtrations, and hence it induces a morphism of the corresponding spectral sequences. By Proposition 7, Equation (31), and naturality, this morphism is an isomorphism for the E^2-terms. We also obtain that all classes in the E^2 term of the spectral sequence associated to $\mathcal{C}_c^\infty(B) \otimes \mathcal{A}_{\mathcal{L}}(F;E)$ survive in the E^∞ term (they give rise to Hochschild homology classes), and hence the spectral sequence associated to $\mathcal{C}_c^\infty(B) \otimes \mathcal{A}_{\mathcal{L}}(F;E)$ degenerates at E^2. Consequently, the spectral sequence associated to $\mathcal{A}_{\mathcal{L}}(M|B;E)$ also degenerates at E^2.

Then Theorem 1 can be used to complete the proof. ☐

We now extend the above lemma to more general groupoids.

Lemma 8. *The spectral sequence* EH^r *associated to the Hochschild homology of* $\mathcal{A}_{\mathcal{L}}(M|B;E)$ *by Lemma 1 degenerates at* EH^2 *and converges to its Hochschild homology.*

Proof. Denote $\mathcal{A} = \mathcal{A}_{\mathcal{L}}(M|B;E)$ in this proof, for simplicity. The differential b of the Hochschild complex of $\mathcal{A}$ is $\mathcal{C}^\infty(B)$-linear, if $f \in \mathcal{C}^\infty(B)$ acts on $a_0 \otimes \ldots \otimes a_n$ by $f(a_0 \otimes \ldots \otimes a_n) := (fa_0) \otimes \ldots \otimes a_n$. The filtrations of the Hochschild complex are also preserved by the multiplication operators with functions $f \in \mathcal{C}^\infty(B)$. This shows that the spectral sequence associated to the Hochschild homology of $\mathcal{A}_{\mathcal{L}}(M|B;E)$ by Lemma 1 consists of $\mathcal{C}^\infty(B)$-modules.

Let $\mathcal{A}^K$ consist of the complete symbols with support in a fixed compact set $K \subset M$. Let $K_m \subset M$ be an exhausting sequence of M by compact subsets (that is, $M = \cup K_m$ and each K_m is contained in the interior of K_{m+1}). We can assume that $\mathcal{A}^{[m]} = \mathcal{A}^{K_m}$. (This sequence can be reduced to one set if M is compact.) To prove that $E^2_{k,h}(\mathcal{A}^{[m]}) = 0$ or that $d^r = 0$ for this spectral sequence, it is enough to check that $fE^2_{k,h}(\mathcal{A}^{[m]}) = 0$ (or $fd^r = 0$) for any f with support in K_m. But $fE^2_{k,h}(\mathcal{A}^{[m]}) = fE^2_{k,h}(\mathcal{A})$ if the support of f is inside K_m. This shows that $E^2_{k,h}(\mathcal{A}^{[m]}) = 0$ if $E^2_{k,h}(\mathcal{A}) = 0$.

We now reverse the argument. To prove that $E^2_{k,h}(\mathcal{A}) = 0$ or that $d^r = 0$ for this spectral sequence, it is enough to check that $fE^2_{k,h}(\mathcal{A}) = 0$ (or $fd^r = 0$) for any f in a trivializing open subset V of M (that is, a subset satisfying the conditions of our third assumption on the algebra $\mathcal{A} = \mathcal{A}_{\mathcal{L}}(M|B;E)$, Equation (15)), more precisely, satisfying that

$$
\begin{aligned}
(32)\quad \mathcal{A}_V := \mathcal{O}(\pi^{-1}(V)) &\left(\Psi^\infty(\mathcal{G}_V;E)/\Psi^{-\infty}(\mathcal{G}_V;E)\right) \\
&\simeq \mathcal{O}(\pi^{-1}(V)) \otimes_{tf} \left(\Psi^\infty(V \times \mathcal{G}_b;E)/\Psi^{-\infty}(V \times \mathcal{G}_b;E)\right)
\end{aligned}
$$

is a $\mathcal{C}^\infty(B)$-linear isomorphism. Then $fE^2_{k,h}(\mathcal{A}) = fE^2_{k,h}(\mathcal{A}_V)$. By Lemma 7, $E^r_{k,h} = 0$ if $k < -p$. Similarly, $fd^r = 0$ if $r \geq 2$.

This proves that the spectral sequence in Hochschild homology associated to the algebra $\mathcal{A}_{\mathcal{L}}(M|B;E)$ by Lemma 1 degenerates at EH^2. It also proves that the assumptions of Theorem 1 are satisfied, so the spectral sequence $\mathrm{EH}^r_{k,h}$ converges to Hochschild homology. □

Now we can state and prove the main theorem of this section. Let $\mathcal{G}$ be a differentiable groupoid whose space of units is a manifold with corners M which is the total space of a fibration $\pi : M \to B$, as before. Denote by $\mathcal{F}^j$ the local coefficient system defined by

$$\mathcal{F}^j(b) := \mathrm{HH}_{2p-j}(\mathcal{A}_{\mathcal{L}}(\pi^{-1}(b))) \simeq \mathrm{H}^j_{c,\mathcal{L}}(\pi_0^{-1}(b)) \simeq \mathrm{H}^j_c(\mathcal{L}(\pi_0^{-1}(b)))$$

with $\pi_0 : S^*_{vert}(M) \times S^1 \to B$ the natural projection.

Theorem 6. *Assume that B is smooth (without corners) and that $\mathcal{G}$ satisfies the assumptions* (13) *and* (14). *Let $\mathcal{A}_{\mathcal{L}}(M|B;E)$ be the algebra of Laurent type complete symbols on $\mathcal{G}$ with coefficients in the $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E. Then*

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \bigoplus_{k+h=m} \Omega^h_c(B, \mathcal{F}^{2p-k}). \tag{33}$$

Proof. We apply Lemma 8 and deduce that:

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \bigoplus_{i+j=m} \mathrm{EH}^2_{i,j}\,.$$

The computation of EH^2 was carried out in Proposition 7 and the result is:

$$\mathrm{EH}^2_{i,j} \simeq \mathrm{H}^{p-i,j-p}_{c,\mathcal{L}}(S^*_{vert}(M) \times S^1|B).$$

On the other hand, Proposition 3 applied to $X = S^*_{vert}(M) \times S^1$ gives:

$$\mathrm{H}^{p-i,j-p}_{c,\mathcal{L}}(S^*_{vert}(M) \times S^1|B) \simeq \Omega^{j-p}_c(B, \mathcal{F}^{p-i}).$$

Therefore we get

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \bigoplus_{i+j=m} \Omega^{j-p}_c(B, \mathcal{F}^{p-i}).$$

The conclusion follows by setting $i = k$ and $j = h$. □

Remark 1. The above proposition has to be modified only slightly if B is also a manifold with corners. For example, when B is compact, the result remains true if we replace $\Omega^j_c(B)$ with $\mathcal{O}(B)\Omega^j(B)$.

5. The relative case

Let again $\mathcal{G}$ be a groupoid with corners satisfying the assumptions (13) and (14) with respect to the fibration $\pi : M \to B$ of the manifold with corners M over the smooth manifold B. Let X be a union of faces of M. We shall denote by $\mathcal{I}_X$ the ideal of smooth functions on M that vanish to infinite order on X.

We shall consider in this section the algebra of Laurent complete symbols on $\mathcal{G}$ which vanish to infinite order over X and which represent pseudodifferential operators acting on sections of the $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E. This algebra is denoted by $\mathcal{A}_{\mathcal{L}}(M|B, X|B;E)$. Thus we have:

$$\mathcal{A}_{\mathcal{L}}(M|B, X|B;E) := \mathcal{O}(M)\mathcal{I}_X\mathcal{A}(M|B;E).$$

Note that if $X = \emptyset$, then we recover the algebras $\mathcal{A}_{\mathcal{L}}(M|B;E)$ studied in the previous sections. The proof of Proposition 3 in [4] extends to show that the algebras $\mathcal{A}_{\mathcal{L}}(M|B, X|B;E)$ are topologically filtered algebras.

For any fibrations $Y \to M \to B$, we denote by p_Y the projection $Y \to M$ and by π_Y the composite projection $Y \to B$. When Y is a manifold with corners, we shall denote by $p_{\mathcal{L}}$ the projection $\mathcal{L}(Y) \to M$ which is the composite map of $\mathcal{L}(Y) \to Y$ and $Y \to M$. This last notation is intended to simplify the statements of this section.

Let again $S^*_{vert}(M)$ be the quotient bundle of $T^*_{vert}M \setminus 0$ by the radial action of $\mathbb{R}^*_+$. In [4], the computations of periodic cyclic homology recover the case of our algebra $\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)$. The result can be stated as follows:

Theorem 7. [4] *For $q = 0, 1$, we have:*

$$\mathrm{HP}_q(\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)) \simeq \mathrm{H}^{[q]}_c(\mathcal{L}(S^*_{vert}M) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X)).$$

Proof. Again by a Morita equivalence argument we can forget the bundle E. We apply Proposition 5 in [4] and obtain for our groupoid $\mathcal{G}$:

$$\mathrm{HP}_q(\mathcal{A}_{\mathcal{L}}(M|B, X|B)) \simeq \mathrm{H}^{[q]}_c(\mathcal{L}(S^*(\mathcal{G})) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X)),$$

But

$$\mathrm{H}^{[q]}_c(\mathcal{L}(S^*(\mathcal{G})) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X)) \simeq \mathrm{H}^{[q]}_{c,\mathcal{L}}((A^*(\mathcal{G}) \setminus 0) \setminus p^{-1}_{A^*(\mathcal{G})\setminus 0}(X))_0.$$

From Assumption (14), we thus deduce as in Lemma 6 that:

$$\mathrm{H}^{[q]}_{c,\mathcal{L}}((A^*(\mathcal{G}) \setminus 0) \setminus p^{-1}_{A^*\mathcal{G}\setminus 0}(X))_0 \simeq \mathrm{H}^{[q]}_{c,\mathcal{L}}((T^*_{vert}M \setminus 0) \setminus p^{-1}_{T^*_{vert}M\setminus 0}(X))_0.$$

The space $\mathrm{H}^{[q]}_{c,\mathcal{L}}((T^*_{vert}M \setminus 0) \setminus p^{-1}_{T^*_{vert}M\setminus 0}(X))_0$ is again isomorphic to the space $\mathrm{H}^{[q]}_c(\mathcal{L}(S^*_{vert}M) \times S^1 \setminus p_{\mathcal{L}}^{-1}(X))$, and this completes the proof. $\square$

Let us state now the corresponding results for Hochschild homology. Let $\mathcal{F}^l_X$ be the coefficient system over B given for any $b \in B$ by the relative cohomology space

$$\mathcal{F}^l_X(b) := \mathrm{H}^l_{\mathcal{L}}(S^*(\pi^{-1}(b)) \times S^1, p^{-1}_{S^*_{vert}M\times S^1}(X)).$$

Theorem 8. *The Hochschild homology spaces of the algebra $\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)$ are given by:*

$$\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)) \simeq \oplus_{k+h=m}\Omega^h_c(B, \mathcal{F}^{2p-k}_X).$$

Proof. We can again assume that the graded vector bundle E is trivial, one-dimensional. The proof of this theorem is similar to the proof of Theorem 6, replacing cohomology by relative cohomology and using excision. More precisely, the natural filtration of the topologically filtered algebra $\mathcal{A}_{\mathcal{L}}(M|B, X|B)$ by the order of the symbols gives rise to a spectral sequence for Hochschild homology given by Lemma 1. This spectral sequence was studied in [4] where the E^2-term was identified with the homogeneous Laurent Poisson relative homology of the Poisson manifold $A^*(\mathcal{G}) \setminus 0$. Indeed, we have [4][Proposition 7]:

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{\delta}_{\mathcal{L},k+h}((A^*(\mathcal{G}) \setminus 0) \setminus p^{-1}_{A^*(\mathcal{G})\setminus 0}(X))_k,$$

Now using the immediate extension of Lemma 6 to the relative case, we obtain:

$$\mathrm{H}^{\delta}_{\mathcal{L},k+h}((A^*(\mathcal{G}) \setminus 0) \setminus p^{-1}_{A^*(\mathcal{G})\setminus 0}(X))_k \simeq \mathrm{H}^{\delta}_{\mathcal{L},k+h}((T^*_{vert}M \setminus 0) \setminus p^{-1}_{T^*_{vert}M\setminus 0}(X))_k.$$

The vertical symplectic Hodge operator $*_{vert}$ preserves the forms vanishing above X, therefore we deduce using the proof of Proposition 7 that:

$$\mathrm{EH}^2_{k,h} \simeq \mathrm{H}^{p-k,h-p}_{c,\mathcal{L}}((S^*_{vert}(M) \times S^1)|B, p_{\mathcal{L}}^{-1}(X)|B).$$

The arguments ensuring the degeneracy of the spectral sequence at the second level in the proof of Lemma 8 again obviously extend to the relative case. In addition, we can apply Theorem 1 to deduce the convergence of the spectral sequence to Hochschild homology. Hence we finally obtain:

$$\begin{aligned}\mathrm{HH}_m(\mathcal{A}_{\mathcal{L}}(M|B, X|B; E)) &\simeq \oplus_{k+h=m} \mathrm{EH}^2_{k,h} \\ &\simeq \oplus_{k+h=m} \mathrm{H}^{p-k,h-p}_{c,\mathcal{L}}((S^*_{vert}(M) \times S^1)|B, p^{-1}_{\mathcal{L}}(X)|B).\end{aligned}$$

To end the proof we simply observe that Proposition 3 extends straightforward to the relative case. □

6. Examples and applications

We begin by providing a construction of a groupoid $\mathcal{G}$ satisfying the assumptions (13) and (14), for any fibration $\pi : M \to B$, where B is a smooth manifold (no corners) and M is a manifold, possibly with corners. For each fibration π as above we shall construct a canonical groupoid $\mathcal{G}_{M,b}$, the "B-groupoid," satisfying the assumptions (13) and (14).

Our examples are obtained by first defining the Lie algebroid of $\mathcal{G}$, and then by integrating it. Let $\mathcal{V}_b(M|B)$ be the space of vertical vector fields on M that are tangent to all faces of M. Then $\mathcal{V}_b(M|B)$ is a Lie algebra with respect to the Lie bracket of vector fields and is also a projective $\mathcal{C}^\infty(M)$-module. By the Serre-Swan theorem [20] there exists a vector bundle $A_b \to M$ such that $\mathcal{V}_b(M|B) \cong \Gamma(A)$, naturally. (See also Melrose and Piazza [33].)

The procedure in [39] then provides us with groupoids $\mathcal{G}_{M,b}$ whose Lie algebroid are isomorphic to A_b. The minimal groupoid with this property is obtained as follows. For each face F of M, consider the interior of that face, $F_0 := F \smallsetminus \partial F$. Assume F has codimension k. Define then $\mathcal{G}'_F$ to be the groupoid with units F_0 and with at most one arrow between any two units. (Thus $\mathcal{G}'_F$ is the groupoid associated to an equivalence relation: two units are connected by an arrow if, and only if, they are equivalent.) The equivalence relation that we consider is that $x, y \in F_0$ are equivalent if, and only if, they belong to the same connected component of a set of the form $\pi^{-1}(b) \cap F_0$. Let

$$\mathcal{G}_{M,b} = \cup \mathcal{G}'_F \times \mathbb{R}^k, \tag{34}$$

the union being a disjoint union, and with the induced groupoid structure. Then it can be checked directly that the charts provided in [39] define a smooth structure on $\mathcal{G}$ such that $A(\mathcal{G}_{M,b}) \cong A_b$. This structure must then be unique [39]. See also [14, 34].

Let $\mathcal{F}$ be the locally constant sheaf (or coefficient system) that associates to $b \in B$ the complex vector space with basis the minimal faces of $\pi^{-1}(b)$. (All faces of a manifold with corners are connected, by definition.)

Theorem 9. *Let $\pi : M \to B$ be as above. Then*

$$\mathrm{HH}_0(\mathcal{A}_{\mathcal{L}}(M|B; E)) \cong C^\infty_c(B, \mathcal{F}),$$

the space of compactly supported sections of the sheaf $\mathcal{F}$. The space of traces of $\mathcal{A}_{\mathcal{L}}(M|B; E)$ identifies with the dual of this space:

$$\mathrm{HH}^0(\mathcal{A}_{\mathcal{L}}(M|B; E)) \cong C^{-\infty}(B, \mathcal{F}) =: C^\infty_c(B, \mathcal{F})'.$$

Proof. This follows from Theorem 6. □

In the particular case when M is smooth, our construction simplifies and we obtain $\mathcal{G}_{M,b} = M \times_B M$. Then the algebra $\Psi^\infty(\mathcal{G})$ consists of differentiable families of pseudodifferential operators along the fibers of $M \to B$. Similarly, let us consider $\Psi^\infty(M|B;E)$, the algebra of *smooth* families of pseudodifferential operators along the fibers with coefficients in the $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle E, introduced in [1]. Let $\mathcal{A}(M|B;E) := \Psi^\infty(M|B;E)/\Psi^{-\infty}(M|B;E)$ be the algebra of vertical complete symbols. Then $\mathcal{A}(M|B;E)\mathcal{A}_{\mathcal{L}}(M|B;E)$, and we obtain

Corollary 3. *Let $\mathcal{F}^j$ be the locally constant sheaf given by the cohomology of the fibers of $S^*_{vert}(M) \times S^1 \to B$. Then we have:*

$$\mathrm{HH}_m(\mathcal{A}(M|B;E)) \simeq \oplus_{k+h=m}\Omega^h_c(B, \mathcal{F}^{2p-k}).$$

In particular, for $m = 0$ and provided that the fibers of $M \to B$ are connected, this isomorphism becomes

$$\mathrm{HH}_0(\mathcal{A}(M|B;E)) \simeq \mathcal{C}^\infty_c(B).$$

Therefore, the space of traces is given by

$$\mathrm{HH}^0(\mathcal{A}(M|B;E)) \simeq \mathcal{C}^{-\infty}(B) := \mathcal{C}^\infty_c(B)', \tag{35}$$

the space of distributions on the base manifold B.

Let ω be as before the vertical symplectic form on $T^*_{vert}M \to B$. The isomorphism of Equation (35) can be made more explicit as follows. Let $\mathcal{R}$ be the radial vector field on the fibers of $T^*_{vert}M \to B$ and $\alpha = i_{\mathcal{R}}(\omega^p/p!)$ the corresponding Liouville form on the fibers of $S^*_{vert}(M) \to B$. Let STr be the graded trace on the endomorphisms of the fibers of E, that is $\mathrm{STr}(A) = \mathrm{Tr}(\gamma \circ A)$, where γ is the involution defining the grading. Also, let π_* be the fiberwise integration on the fibers of $T^*_{vert}M$. Then, for any $a = \sum_{j\leq m} a_j$, we set

$$\tau_\mu(a) := \langle \mu, \pi_* \,\mathrm{STr}(a_{-p})\alpha\rangle. \tag{36}$$

This formula defines a super (or graded) trace on $\mathcal{A}(M|B;E)$ such that $\mu \to \tau_\mu$ is the isomorphism described in Corollary 3, for $m = 0$.

Fix a quantization function

$$q : \cup S^s(T^*_{vert}M; \mathrm{End}(E)) \to \cup \Psi^s(M|B;E), \tag{37}$$

where $S^s(T^*_{vert}M;\mathrm{End}(E))$ denotes classical vertical symbols of order s. The function q is thus assumed to be continuous and to satisfy $\sigma_s(q(a)) = a$ if a is a symbol of order s. Let ρ be a positive symbol on $T^*_{vert}M$, such that $\rho(\xi) = |\xi|$, for $|\xi| \geq 1$. We consider a family $D(z)$ such that

$$D(z) = q(\rho^z)B_1(z) + R(z), \tag{38}$$

where B_1 is a holomorphic function on $\mathbb{C}$ with values in $\Psi^0(M|B;E)$, $B_1(0) = 1$, $R(z)$ is a holomorphic function on $\mathbb{C}$ with values $\Psi^{-\infty}(M|B;E)$, and $R(0) = 0$. Then $D(0) = 1$.

Proposition 8. *Let μ be a distribution on B and $A \in \Psi^m(M|B;E)$ and $D(z)$ be as above. Then the function*

$$z \to F_A(z) := \langle \mu, \mathrm{STr}_b(AD(z))\rangle$$

is well defined for $Re(z) < -m - p$, where p is the dimension of the fibers of $\pi : M \to B$. The function F_A extends to a meromorphic function on $\mathbb{C}$, with at most simple poles at the integers. For $z = 0$, the residue of this function is up to constant $\tau_\mu(A)$.

Proof. This is proved as in the classical case when $D(z)$ is given by the complex powers of a positive elliptic operator (see [44, 17]). One can follow the approach from [40], for example. □

We remark that we used only a weak result from [39] (namely Theorem 2). Let us use this opportunity however to mention that there is a missing assumption in the gluing theorem of [39], Theorem 3. Here is the corrected version.

Theorem 10. *Let A be a Lie algebroid on a manifold with corners M. Suppose that M has an A-invariant stratification $M = \cup S$ such that, for each stratum S, the restriction A_S is integrabl and let $\mathcal{G}_S$ be d-simply connected differential groupoids such that $A(\mathcal{G}_S) \simeq A_S$. Then A is integrable if, and only if, the exponential map $\mathrm{Exp} : A \to \mathcal{G} = \cup \mathcal{G}_S$ is injective on an open neighborhood of the zero section of A for some (equivalently, for any) connection on A. Moreover, if these conditions are satisfied, then the disjoint union $\mathcal{G} = \cup \mathcal{G}_S$ is naturally a differentiable groupoid such that $A(\mathcal{G}) \simeq A$.*

The above condition on the map Exp is seen to be necessary from the work of Crainic and Fernandes [14], and also from some earlier results of Weinstein. The map Exp introduced in [39] seems to be essential for both the results of that paper and for the results of [14]. The second named author would like to thank M. Crainic for pointing out a possible problem with the original statement of the above theorem.

Let us mention for completeness the result for the computation of the cyclic homology of the algebras $\mathcal{A}_{\mathcal{L}}(M|B;E)$. Note first that

$$\mathrm{HC}_j(\mathcal{A}_{\mathcal{L}}(M|B;E)) \simeq \mathrm{HP}_j(\mathcal{A}_{\mathcal{L}}(M|B;E)) \quad \text{for } j \geq 2p+2,$$

from the SBI-exact sequence, because the Hochschild homology of $\mathcal{A}_{\mathcal{L}}(M|B;E)$ vanishes above this rank.

Proposition 9. *The spectral sequence $\mathrm{EC}^r_{k,h}$ associated to the cyclic homology complex of $\mathcal{A}_{\mathcal{L}}(M|B;E)$ has E^1-term given by $\mathrm{EC}^1_{k,h} = \Omega^{k+h}_{rc,\mathcal{L}}(S^*_{vert}M)$, if $k \neq 0$, and by*

$$\mathrm{EC}^1_{0,h} = \Omega^h(S^*_{vert}M \times S^1)/d\Omega^{h-1}(S^*_{vert}M \times S^1) \oplus \oplus_{j>0} \mathrm{H}^{h-2j}(S^*_{vert}M \times S^1).$$

The d_1 differential is induced by the Poisson différential and the spectral sequence converges to the cyclic homology of the algebra $\mathcal{A}_{\mathcal{L}}(M|B;E)$.

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