Critical values for L-functions on $GSp_2 \times Gl_2$

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Abstract

This paper deals with Deligne's conjecture on the critical values of L-functions. Let $Z_{G\otimes h}(s)$ denote the tensor product L-function attached to a Siegel modular form G of weight k and an elliptic cusp form h of weight l. Let the first Fourier-Jacobi coefficient of G be not identically zero. Then Deligne's conjecture is fully proven for $Z_{G\otimes h}(s)$, when $l \leq 2k-2$ and partly for the remaining case.

Introduction

The purpose of this paper is to prove parts of Deligne's conjectures [De79] on the critical values for L-functions attached to Siegel modular forms and the standard representation of the L-group of $GSp_2 \times Gl_2$.
Let S_n^k denote the space of cuspidal Siegel modular forms of genus n and weight k. Let $h \in S_1^{2\tilde{k}_h}, G \in S_2^{2\tilde{k}_G}$ be eigenforms, whose Fourier coefficients lie in the field generated by the Hecke eigenvalues. Let $Z_{G\otimes h}(s)$ be the twisted spinor L-function, the convolution of the spinor L-function of G and Hecke's L-function of h. We normalize the Petersson scalar product $\langle\, ,\, \rangle$, such that

$$\langle F\, ,\, F\rangle = \int_{Sp_n(\mathbb{Z})\setminus\mathbb{H}_n} \left|f(Z)\right|^2 \ \det(Y)^{k-n-1} dX\ dY,$$

where $F \in S_n^k$ and $Z = X + iY$.
Deligne's conjectures (worked out by H. Yoshida [Yo99]) on the critical values of $Z_{G\otimes h}(s)$, by assuming the existence of motives $M(h)$ and $M(G)$ attached to the modular forms, can be stated as follows

Conjecture A Let $2\tilde{k}_h > 4\tilde{k}_G - 2$. Then the critical values of $Z_{G\otimes h}(s)$ are given by $C_I = [4\tilde{k}_G - 2, 2\tilde{k}_h - 1] \cap \mathbb{Z}$. Let $m \in C_I$, then

$$Z^*_{G\otimes h}(m) = \frac{Z_{G\otimes h}(m)}{\pi^{4m-8\tilde{k}_G+6}\langle h\, ,\, h\rangle^2} \in \overline{\mathbb{Q}} \qquad \text{and} \tag{1}$$

$$\Big(Z^*_{G\otimes h}(m)\Big)^\sigma = Z^*_{G^\sigma\otimes h^\sigma}(m) \qquad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}). \tag{2}$$

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Conjecture B Let $2\tilde{k}_h < 4\tilde{k}_G - 2$. Then the critical values of $Z_{G\otimes h}(s)$ are given by $C^a_{II} = [2\tilde{k}_G, 2\tilde{k}_G + 2\tilde{k}_h - 3] \cap \mathbb{Z}$ if $2\tilde{k}_G \geq 2\tilde{k}_h$ and $C^b_{II} = [2\tilde{k}_h, 4\tilde{k}_G - 3] \cap \mathbb{Z}$ if $2\tilde{k}_G < 2\tilde{k}_h$. Let m be a critical value, then

$$Z^*_{G\otimes h}(m) = \frac{Z_{G\otimes h}(m)}{\pi^{4m-4\tilde{k}_G-2\tilde{k}_h+4}\langle h\,,\,h\rangle\langle G\,,G\rangle} \in \overline{\mathbb{Q}} \qquad \text{and} \tag{3}$$

$$\Big(Z^*_{G\otimes h}(m)\Big)^\sigma = Z^*_{G^\sigma\otimes h^\sigma}(m) \qquad \text{for all } \sigma \in \mathrm{Aut}(\mathbb{C}). \tag{4}$$

Supplement: In the case $2\tilde{k}_h = 4\tilde{k}_G - 2$ there are no critical values.

For a description of this (conjectural) set of critical values in terms of the Γ-factor of the functional equation, see section 5.

Up to now only the following results had been known: Let the weights of G and h be equal. Furusawa [Fu93] proved $Z^*_{G\otimes h}(4\tilde{k}_h - 3)$ is algebraic, provided that there exists a non-trivial Fourier coefficient of G paramatrized by a half-integral positive definit matrix T, such that $-4\det T$ is a fundamental discriminant. Some years later (under similiar assumptions, but using a different integral representation) we [He99] described how the absolute Galois group acts on this critical value.
We will always assume that the first Fourier-Jacobi coefficient of G is not identically zero. (Numerical computations of Skoruppa [Sk92] suggest that this assumption may be always satisfied - not just only for Saito-Kurokawa lifts, where it is obvious). In this paper we prove, that then Deligne's conjecture B is true. Moreover we present some evidence for conjecture A by confirming it for all G, which are Saito-Kurokawa lifts. We mention that conjecture A is similar in nature to the "unbalanced case" for triple-L-functions, which is still open in general, see [Or87], [G-H93], [H-K91].
The main ingredients in our proof are the integral representation of $Z_{G\otimes h}(s)$ found in [He99] (and technical modifications given in [Bo-He99]), Shimura's concept of nearly holomorphic modular forms [Sh87], and some more or less well known facts on Siegel-type Eisenstein series, special values of Hecke- and Rankin L-series and quotients of scalar products.

Our paper is organized as follows: In the first three sections we collect facts about the L-functions in question, about differential operators and about Eisenstein series. Most of the material presented here is more or less well-known (at least in principle), but needs to be stated in the form we want to use it. Section 4 contains computations mainly concerned with the role of the auxiliary modular form F from the Maaß space, which occurs in our integral representation of $Z_{G\otimes h}$. In the final section 5 we present and prove our main results (theorems 5.1 and 5.2).

Notation: For matrices M and N we denote by M^t, $\det(M)$, and $\mathrm{tr}(M)$ its transpose, determinant, and trace and we put $M[N] = N^t MN$ (whenever it makes sense). For a square matrix Z we put $e\{Z\} = e^{2\pi i tr(Z)}$. The symbol Γ_n occurs with two meanings, on one hand as the Siegel modular group and on the other hand as a generalized Γ-function.

1 Siegel modular forms and their automorphic L-functions

The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ acts on Siegel's upper-half-space $\mathbb{H}_n$ of genus n as a group of biholomorphic automorphism by

$$(M, Z) \longmapsto M(Z) = (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{R})$ with $A, B, C, D \in \mathbb{R}^{n,n}$; we denote by $j(M, Z) = det(CZ + D)$ the standard automorphy factor. For $\alpha, \beta \in \mathbb{C}$ we define the slash-operator $|_{\alpha,\beta}$ for functions f on $\mathbb{H}_n$ and $M \in Sp_n(\mathbb{R})$ by

$$(f \mid_{\alpha,\beta} M)(Z) = j(M, Z)^{-\alpha} \overline{j(M, Z)}^{-\beta} f(M(Z)).$$

and we write $f \mid_k$ instead of $f \mid_{k,0}$. We denote by M_n^k (S_n^k) the space of (cuspidal) Siegel modular forms of degree n and weight k for the Siegel modular group $\Gamma_n := Sp_n(\mathbb{Z})$; every $F \in S_n^k$ satisfies the transformation law

$$F(Z) = F(M(Z))\ j(M, Z)^{-k} \text{ for all } M \in \Gamma_n,\ Z \in \mathbb{H}_n. \tag{5}$$

Sometimes we prefer to write $2\tilde{k}$ instead of k for the weight; the reason for this is that the differential operators $\partial_{n,k}^{(\mu)}$ (to be explained below) change the weight by 2μ.

The $\mathbb{Q}$-subspaces of those (cuspidal) Siegel modular forms, which have rational Fourier coefficients, will be denoted by $M_n^k(\mathbb{Q})$ and $S_n^k(\mathbb{Q})$. It is well known that $M_n^k = M_n^k \otimes_{\mathbb{Q}} \mathbb{C}$, therefore we have a natural action $(\sigma, f) \mapsto f^\sigma$ of $Aut(\mathbb{C})$ on M_n^k (by acting on the Fourier coefficients).
For arbitrary complex valued functions F and G on $I\!H_n$, which satisfy (5), we define the Petersson integral, (if convergent) by

$$\langle F, G\rangle = \int_{\Gamma\backslash I\!H_n} F(Z)\,\overline{G(Z)}\,\delta(Z)^k d^*Z. \tag{6}$$

Here $Z = X + i\,Y$,$\delta(Z) = \det(Y)$ and $d^*Z = \det(Y)^{-(n+1)}dXdY$ denotes the symplectic volume element. Every $F \in S_n^k$ has a Fourier expansion of the form $F(Z) = \sum_T a_T^F\ e\{TZ\}$ and a Fourier-Jacobi expansion of the form

$$F\begin{pmatrix} \tau & z \\ z^t & \tau' \end{pmatrix} = \sum_{t=1}^{\infty} \Phi_t^F(\tau, z)\ e\{t\tau'\}, \qquad (\tau' \in I\!H) \tag{7}$$

with Fourier-Jacobi coefficients $\Phi_t^F(\tau, z)$. The $\sum_T$ runs over all half-integral positive definite matrices. These coefficients are Jacobi cusp forms of degree $(n-1, 1)$, index t and weight k. We denote by $\langle\,;\,\rangle_J$ the Petersson scalar product of Jacobi forms (cf. [He99], section 2.3 and 3.4). Further we denote by $S_2^{k,M}$ the Maaß space of weight k, which is a well known subspace of S_2^k (see [He99] Definition 3.14).
We will need some embeddings of domains and groups: Let $n = n_1 + \ldots + n_r$ be a partition, then we put

$$\mathbf{i}_{n_1,\ldots,n_r} : \begin{cases} I\!H_{n_1} \times \ldots \times I\!H_{n_r} & \longrightarrow & I\!H_n \\ (z_1, \ldots, n_r) & \longmapsto & diag(z_1, \ldots, z_r) \end{cases}$$

and we use the same symbol for the corresponding embedding of groups:

$$\mathbf{i}_{n_1,\ldots,n_r} : \begin{cases} Sp_{n_1}(\mathbb{R}) \times \ldots \times Sp_{n_r}(\mathbb{R}) & \longrightarrow & Sp_n(\mathbb{R}) \\ (g_1, \ldots, g_r) & \longmapsto & \begin{pmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_r & & & b_r \\ c_1 & & & c_1 & & \\ & \ddots & & & \ddots & \\ & & c_r & & & d_r \end{pmatrix} \end{cases}$$

with $g_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$.

Let $F \in S_n^k$ be a Hecke eigenform with Satake-parameters $(\alpha_{0,p}, \alpha_{1,p}, \ldots, \alpha_{n,p})$. Then the spinor zeta function $Z_F(s)$ of F is given by

$$\begin{aligned} Z_F^n(s) &= \prod_p \left\{Q_{p,F}^n(p^{-s})\right\}^{-1}, \text{ where} && (8) \\ Q_{p,F}^n(X) &= (1-\alpha_{0,p}X)\prod_{r=1}^n \prod_{1\le i_1<\cdots<i_r\le n}(1-\alpha_{0,p}\alpha_{i_1,p}\cdot\ldots\cdot\alpha_{i_r,p}X). \end{aligned}$$

We call a non-trivial Hecke eigenform $F \in S_n^k$ normalized, if the Fourier coefficients lie in the field generated by the eigenvalues of F; in the special case $n = 1$ we may (and always will) require in addition that $a_1^F = 1$. If $f \in S_1^k$ is a normalized Hecke eigenform, one usually puts $\alpha_p = \alpha_{0,p}$ and $\beta_p = \alpha_{0,p}\alpha_{1,p}$ and writes $L_f(s)$ for $Z_f^1(s)$ and $L_{p,f}(X)$ for $Q_{p,f}^1(X)$. Thus $a_p^f = \alpha_p + \beta_p$ and $L_{p,f}(X) = 1 - a_p^f X + p^{k-1}X^2$. Let $F \in S_2^{k,M}$ be a Hecke eigenform, then the spinor L-function of F splits, i.e.

$$Z_F^2(s) = \zeta(s-k+1)\ \zeta(s-k+2)L_f(s) \tag{9}$$

where $f \in S_1^{2k-2}$ corresponds to F by the Saito-Kurokawa lift (cf. [He99], Theorem 3.16). With this notation we introduce the L-function $Z_{G\otimes h}$ corresponding to two eigenforms $G \in S_2^k$ and $h \in S_1^l$ with parameters $\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}$ and α_p, β_p for every prime number p:

$$\begin{aligned} Z_{G\otimes h}(s) &= \prod_p \left\{Q_{p,G}^2(\alpha_p p^{-s})\ Q_{p,G}^2(\beta_p p^{-s})\right\}^{-1} \\ &= \prod_p \left\{Z_{p,G\otimes h}(p^{-s})\right\}^{-1}. && (10) \end{aligned}$$

In other words, this is the L-function attached to the standard representation of the L-group of $GSp_2 \times Gl_2$ and the automorphic representation $\pi_G \otimes \pi_h$ corresponding to G and h.

Theorem [Bo-He99] *Let k_h and k_G be even natural numbers. Let h be an elliptic cusp form of weight k_h and G a Siegel cusp form of degree 2 of weight k_G. We assume that h and G are Hecke eigenforms, where the first Fourier-Jacobi coefficient of G is not zero. Let $Z_{G\otimes h}(s)$ denote the L-function of G*

and h attached to the standard representation of the L-group of $GSp_2 \times Gl_2$ (cf. (10)). Then $Z_{G\otimes h}(s)$ has a meromorphic continuation to the whole complex plane and satisfies the functional equation

$$\widehat{Z_{G\otimes h}}(s) = \widehat{Z_{G\otimes h}}(k_h + 2k_G - 3 - s), \tag{11}$$

where

$$\begin{aligned} \widehat{Z_{G\otimes h}}(s) &= \gamma_\infty(s)\ Z_{G\otimes h}(s) \\ \gamma_\infty(s) &= (2\pi)^{-4s}\Gamma(s-k_G+2)\ \Gamma(s) \\ &\quad \times\Gamma(s-k_G+1)\ \Gamma(s-k_h+1). \end{aligned}$$

2 Differential operators and Petersson scalar products

We denote by $\mathcal{N}_n^{(\mu)}$ the space of all polynomials of degree $\leq \mu$ in the entries of Y^{-1} with holomorphic functions on $\mathbb{H}_n$ as coefficients. Then $\mathcal{N}_n := \bigcup_\mu \mathcal{N}_n^\mu$ is the space of all nearly holomorphic functions in the sense of Shimura. Furthermore, for integral weight k

$$\mathcal{N}_n^{(k,\mu)} := \{f \in \mathcal{N}_n^{(\mu)} \mid f \mid_k \gamma = f \quad \text{for all} \quad \gamma \in \Gamma_n\}$$

is the (finite dimensional) space of nearly holomorphic modular forms (with the standard additional condition on the Fourier expansion when $n = 1$).
It is an important observation of Shimura, [Sh94, Thm.3.3] that (at least for $k > n + \mu$) an element of $\mathcal{N}_n^{k,\mu}$ can be written (uniquely) as a

$$f = h + \sum_s L_{k,s} t_s \tag{12}$$

where $h \in M_n^k$, the $L_{k,s}$ are certain non-holomorphic differential operators and the t_s are nearly holomorphic modular forms (the sum over s degenerates in our case to a single term, because we have only one archimedean place). By studying the adjoint of the operators $L_{k,s}$, Shimura shows that

$$\langle f, g\rangle = \langle h, g\rangle \qquad \text{for all} \quad g \in S_n^k \tag{13}$$

therefore $f \longmapsto h$ gives an essentially algebraic construction of a holomorphic projection. In the sequel we write $h = f_{hol}$ for this construction. At this point we mention that for C^∞ - modular forms satisfying a suitable growth condition there is also an analytic construction of a holomorphic projection, due to J.Sturm [St81] and further elaborated by Panciskin [Pa91]; this construction gives an explicit description of the Fourier coefficients of the holomorphic projection in terms of an integral involving the Fourier coefficient of the C^∞-form. We denote this holomorphic projection by f_{ana}. Both constructions coincide for nearly holomorphic cusp forms.

We will deal with nearly holomorphic modular forms arising from holomorphic modular forms by applying Maaß-type differential operators. Those operators (a slightly modified version of those introduced by Maaß [Ma53, Ma71]) are defined in the following way: For a variable $Z = (z_{ij})$ on $\mathbb{H}_n$, we put

$$\frac{d}{dZ} = \left(d_{ij} \frac{\partial}{\partial z_{ij}} \right)_{1 \le i,j \le n},$$

where $d_{jj} = 1$ and $d_{ij} = 1/2$ otherwise. Let α be a non-negative integer. We denote by

$$\partial_{n,\alpha} = (2\pi i)^{-n} \delta(Z)^{\frac{n-1}{2}-\alpha} \det\left(\frac{d}{dZ} \right) \delta(Z)^{\alpha - \frac{n-1}{2}},$$

the (modified) Maaß differential operator and for a nonnegative integer μ we put

$$\partial_{n,k}^{(\mu)} = \partial_{n,k+2\mu-2} \circ \cdots \circ \partial_{n,k}.$$

This operator satisfies

$$\partial_{n,k}^{(\mu)} (f \mid_\alpha g) = \left(\partial_{n,k}^{(\mu)} f \right) \mid_{\alpha+2\mu} g$$

for all $C^\infty-$functions f on $I\!H_n$ and all $g \in Sp_n(\mathbb{R})$ and it maps $\mathcal{N}_n^\nu$ to $\mathcal{N}_n^{\nu+n\mu}$; for the latter statement we refer to the works of Shimura (eg. [Sh94]: there (p.153) such a statement is proved for differential operators of type

$$D_\rho = \rho(Z - \bar{Z})^{-1} \frac{d}{dZ} \rho(Z - \bar{Z}),$$

where ρ is any rational representation of $Gl_n(\mathbb{C})$. In section 4 of loc.cit. it is then shown how to get our operators $\partial_{n,k}$ from operators of type D_ρ, see also [Sh81b]).

For $n=1$ we also need the adjoint of $\partial_{1,k}$ with respect to the Petersson scalar product. It is given explicitely by

$$\hat{\partial}_{1,k} := -\frac{1}{2\pi i} y^2 \frac{\partial}{\partial \bar{z}}$$

and we put

$$\hat{\partial}_{1,k}^{(\mu)} = \hat{\partial}_{1,k-2\mu+2} \circ \cdots \circ \hat{\partial}_{1,k}.$$

We will consider the following map, defined on C^∞- functions on $I\!\!H_5$:

$$\Lambda : F \longmapsto \partial_{5,k}^{(\nu)} F \longmapsto \left(\partial_{5,k}^{(\nu)} F\right) \circ \mathbf{i}_{1,2,2} \longmapsto \hat{\partial}_{1,k+2\nu}^{(\mu)}(\left(\partial_k^{(\nu)} F\right) \circ \mathbf{i}_{1,2,2}).$$

Only the symbol $\hat{\partial}_{1,k+2\nu}^{(\mu)}$ in the last step needs to be explained: It is an operator acting only on the first component z_1 of $(z_1, z_2, z_3) \in I\!\!H_1 \times I\!\!H_2 \times I\!\!H_2$. It is clear that $F \circ \mathbf{i}_{1,2,2}$ is nearly holomorphic with respect to all three variables $z_1 \in I\!\!H_1$, $z_2, z_3 \in I\!\!H_2$, if $F \in \mathcal{N}_5$; therefore Λ maps M_n^k to $\mathcal{N}_1^{(k+2\nu-2\mu,\nu-\mu)} \otimes \mathcal{N}_2^{(k+2\nu,2\nu)} \otimes \mathcal{N}_2^{(k+2\nu,2\nu)}$.
The theorem of Shimura mentioned above (see formula (12)) can now be applied to $\Lambda(F)$, $F \in M_5^k$ (separately for $z_1 \in I\!\!H_1$, $z_2 \in I\!\!H_2$, $z_3 \in I\!\!H_2$), at least as long as

$$k + 2\nu > 2 + 2\nu \qquad \text{and} \qquad k + 2\nu - 2\mu > 1 + \nu - \mu.$$

An inspection of the proof shows that the differential operators involved do not depend on F at all ! We can therefore describe the map Λ in the following way:

$$\Lambda = \mathcal{D}_{hol} + \sum (L_{i_1} \otimes L_{i_2} \otimes L_{i_3}) \circ \mathcal{D}_{i_1,i_2,i_3} \tag{14}$$

All the summands $\mathbf{S}$ map functions on $I\!\!H_5$ to functions on $I\!\!H_1 \times I\!\!H_2 \times I\!\!H_2$ and satisfy

$$\mathbf{S}\left(f \mid_k \mathbf{i}_{1,2,2}(g_1, g_2, g_3)\right) = (\mathbf{S} f) \mid_{k+2\nu-2\mu} g_1 \mid_{k+2\mu} g_2 \mid_{k+2\mu} g_3 \tag{15}$$

for all $(g_1, g_2, g_3) \in SL_2(\mathbb{R}) \times Sp_2(\mathbb{R}) \times Sp_2(\mathbb{R})$; it should be clear that on the right side of (15) g_i is considered as acting on z_i $(1 \leq i \leq 3)$. The L_{i_j} are differential operators of the kind occuring in (12) and acting on functions defined on $I\!H_1$ (for i_1) or $I\!H_2$ (for i_2 and i_3) and the $\mathcal{D}$'s are differential operators on $I\!H_5$, evaluated in $\mathbf{i}_{1,2,2}(I\!H_1 \times I\!H_2 \times I\!H_2)$, in particular, $\mathcal{D}_{hol}$ is a polynomial $\mathcal{P}$ in the entries of $\frac{d}{dZ}, Z \in I\!H_5$, evaluated in $\mathbf{i}_{1,2,2}(I\!H_1 \times I\!H_2 \times I\!H_2)$; the coefficients of this polynomial depend on k, μ and ν. We write $\Lambda = \Lambda(k,\nu,\mu)$, $\mathcal{D}_{hol} = \mathcal{D}_{hol}(k,\nu,\mu)$ and $\mathcal{P} = \mathcal{P}(k,\nu,\mu)$ if necessary. We summarize some properties of the polynomial $\mathcal{P}$ and the map Λ in

Proposition 2.1 *With the notations from above, we have*

a) $\mathcal{P}$ *is a polynomial of total degree* $5\nu - \mu$

b) The coefficients of $(2\pi i)^{5\nu+\mu}\mathcal{P}$ *are rational numbers*

c) $(2\pi i)^{2\mu}\mathcal{D}_{hol}$ *defines a map*

$$M_5^k(\mathbb{Q}) \longrightarrow M_1^{k+2\nu-2\mu}(\mathbb{Q}) \otimes M_2^{k+2\nu}(\mathbb{Q}) \otimes M_2^{k+2\nu}(\mathbb{Q})$$

d) For all $H \in M_5^k, h \in S_1^{k+2\nu-2\mu}, F, G \in S^{k+2\nu}$ *the following identity of threefold Petersson scalar products holds*

$$\langle\langle\langle\Lambda(H), h\rangle, F\rangle, G\rangle = \langle\langle\langle\mathcal{D}_{hol}(H), h, \rangle F\rangle, G\rangle$$

e) Assume in addition that h, F, G *from above are normalized eigenforms and* $H \in M_5^k(\mathbb{Q})$. *Then we have for all* $\sigma \in Aut(\mathbb{C})$:

$$\left(\frac{\langle\langle\langle(2\pi i)^{2\mu}\Lambda(H), f\rangle, F\rangle, G\rangle}{\langle f, f\rangle\langle F, F\rangle\langle G, G\rangle}\right)^\sigma = \frac{\langle\langle\langle(2\pi i)^{2\mu}\Lambda(H), h^\sigma\rangle, F^\sigma\rangle, G^\sigma\rangle}{\langle f^\sigma, f^\sigma\rangle\langle F^\sigma, F^\sigma\rangle\langle G^\sigma, G^\sigma\rangle}$$

in particular,

$$\frac{\langle\langle\langle(2\pi i)^{2\mu}\Lambda(H), f\rangle, F\rangle, G\rangle}{\langle f, f\rangle\langle F, F\rangle\langle G, G\rangle} \in K_f K_F K_G$$

where K_f, K_F, K_G *denote the fields generated by the Hecke eigenvalues of* f, F *and* G *respectively.*

Proof:
The statements a), b), c) are obvious, d) follows from (13) and (14); to prove e), one has to substitute $(2\pi i)^{2\mu}\Lambda$ by $(2\pi i)^{2\mu}\mathcal{D}_{hol}$ and to use standard properties of Petersson scalar products, see e.g. [He99, Theorem 1.9]

□

3 Eisenstein series at negative integers

We begin this section by recalling the definition of the Siegel type Eisenstein series $E_n^k(Z,s)$ on $\mathbb{H}_n \times \mathbb{C}$: Let $P_{n,0} = \{\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_n \mid C = 0\}$, then

$$E_n^k(Z,s) = \delta(Z)^s \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in P_{n,0}(\mathbb{Z}) \setminus Sp_n(\mathbb{Z})} \det(CZ+D)^{-k} \,|\det(CZ+D)|^{-2s} \tag{16}$$

is absolutely convergent for $k + 2\mathrm{Re}(s) > n+1$ and has a meromorphic continuation on the whole complex plane. It is well known that $E_n^k(Z,-\nu)$ has rational Fourier coefficients for $\nu = 0$ and $k > n+1$. But a more subtle statement is true :

Proposition 3.1 *Let $k > n+1$ be even and ν a non-negative integer smaller than $(k-n-1)/2$. Then $E_n^k(Z,-\nu)$ is a nearly holomorphic function in $\mathcal{N}_n^{k,n\nu}$, more precisely*

$$E_n^k(Z,-\nu) = \pi^{n\nu} \partial_{n,k-2\nu}^{(\nu)} F$$

where $F \in M_n^{k-2\nu}$ is a holomorphic modular form with rational Fourier coefficients.
Let k be not congruent to $\frac{n+2}{2}$ or $\frac{n+3}{2}$ modulo 4, then the statement above remains valid for $\nu \in [0, \frac{k}{2}-1) \cap \mathbb{Z}$.

Proof:
The conditions imposed on k and n guarantee that the Eisenstein series $E_n^{k-2\nu}(Z,s)$ evaluated at $s=0$ is a holomorphic modular form on $\mathbb{H}_n$. Moreover, the corresponding Fourier coefficients are rational numbers. This is a classical theorem of Siegel [Si39] if the weight is sufficiently large ($k - 2\nu > n+1$), in the case of smaller weights it follows from the work of Feit [Fe86], Shimura [Sh83] and Weissauer [We86] on Hecke summation, see also the paper of Haruki [Ha97] for a nice survey of this topic.
Now the Proposition can be proven in the following way: We apply the differential operator $\partial_{n,k}$ to the Eisenstein series (16); this gives, after interchanging the summation and the differentiating process (which is allowed), the formula

$$\partial_{n,k-2\nu}^{(\nu)} E_{n,0}^{k-2\nu}(Z,s) = (-4\pi)^{-n\nu} \prod_{j=1}^{\nu} \varepsilon_n(s+k-\nu-j) E_{n,0}^k(Z,s-\nu), \tag{17}$$

for k even and $k > 2\nu$, where $\varepsilon_n(s) = \prod_{j=0}^{n-1}(s - \frac{j}{2})$. To prove the proposition, we only have to observe that

$$(-4\pi)^{-n\nu} \prod_{j=1}^{\nu} \varepsilon_n(s + k - \nu - j) \Big|_{s=0} \neq 0.$$

$\square$

Combining Propositions 2.1 and 3.1 and slightly changing our notation we obtain

Corollary 3.2 *Let $h \in S_1^{2\tilde{k}_h}$ and $G, F \in S^{2\tilde{k}_G}$ be normalized Hecke eigenforms and $\tilde{\Delta} = \tilde{k}_G - \tilde{k}_h \geq 0$. Let*

$$\Phi(\partial_{2\tilde{k}_h}^{(\tilde{\Delta})} h, G, F; s) = \Big\langle \Big\langle \Big\langle E_5^{2\tilde{k}_G}(*, s) \circ \mathbf{i}_{1,2,2,}, \, \partial_{2\tilde{k}_h}^{(\tilde{\Delta})} h, G, F \Big\rangle \Big\rangle \Big\rangle \tag{18}$$

be the iterated Petersson scalar product of an Eisenstein series (restricted to $\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbb{H}_2$) against the automorphic forms $\partial_{2\tilde{k}_h}^{(\tilde{\Delta})} h, G$ and F. Let $\nu \in [0, \tilde{k}_G - 1) \cap \mathbb{Z}$. Then we have

$$\Phi^*(h, G, F; -\nu) = \frac{\Phi(\partial_{2\tilde{k}_h}^{(\tilde{\Delta})} h, G, F; s)}{\pi^{5\nu - 2\tilde{\Delta}} \langle h, h \rangle \langle G, G \rangle \langle F, F \rangle} \in K_h K_G K_F \tag{19}$$

and

$$\Big(\Phi^*(h, G, F; -\nu)\Big)^{\sigma} = \Phi^*(h^{\sigma}, G^{\sigma}, F^{\sigma}; \nu) \quad \textit{for all } \sigma \in Aut(\mathbb{C}). \tag{20}$$

4 Explicit computations

Let $h \in S_1^{2\tilde{k}_h}$ and $G, F \in S_2^{2\tilde{k}_G}$ be normalized Hecke eigenforms. We assume F to be a Saito-Kurokawa lift of $f \in S_1^{2\tilde{k}_f}$, where $2\tilde{k}_f = 4\tilde{k}_F - 2$.
Let Φ_1^F, Φ_1^G be the first Fourier-Jacobi coefficients of F and G and let $\tilde{f}, \tilde{g}$ be the corresponding modular forms of half-integral weight. We denote by $\langle \, , \, \rangle_J$, $\langle \, , \, \rangle_\sim$ the Petersson scalar products on the space of Jacobi forms and modular forms of half-integral weight. Let K be a field. We put $a \sim_{K^\times} b$ for $a, b \in \mathbb{C}$, if there exists a $k \in K^\times$, such that $ak = b$. It is well known that $\langle \Phi_1^F, \Phi_1^G \rangle_J \sim_{\mathbb{Q}^\times} \langle \tilde{f}, \tilde{g} \rangle_\sim$. Let $F, \Phi_1^F, f, \tilde{f}$ be given. Then these liftings commute with $\sigma \in \mathrm{Aut}(\mathbb{C})$ by fixing $F \mapsto F^\sigma$.

Lemma 4.1 *Let $\tilde f, \tilde g$ as before. Then we have $\langle \tilde f, \tilde g\rangle_\sim \,/\, \langle \tilde f, \tilde f\rangle_\sim \in K_F K_G$ and*

$$\left(\frac{\langle \tilde f, \tilde g\rangle_\sim}{\langle \tilde f, \tilde f\rangle_\sim}\right)^\sigma = \frac{\langle \tilde f^\sigma, \tilde g^\sigma\rangle_\sim}{\langle \tilde f^\sigma, \tilde f^\sigma\rangle_\sim} \qquad \text{for all } \sigma \in Aut(\mathbb{C}). \tag{21}$$

This follows directly from [Sh81], Theorem 1.

Lemma 4.2 *Let F as before. Then*

$$\frac{L_f(2\tilde k_G)\langle \tilde f, \tilde f\rangle_\sim}{\pi^{2\tilde k_G}\langle F, F\rangle} = \left(\frac{L_f(2\tilde k_G)\langle \tilde f, \tilde f\rangle_\sim}{\pi^{2\tilde k_G}\langle F, F\rangle}\right)^\sigma = \frac{L_{f^\sigma}(2\tilde k_G)\langle \tilde f^\sigma, \tilde f^\sigma\rangle_\sim}{\pi^{2\tilde k_G}\langle F^\sigma, F^\sigma\rangle} \tag{22}$$

for all $\sigma \in Aut(\mathbb{C})$.

This follows from $L_f(2\tilde k_G) = a\, \pi^{2\tilde k_G}\langle F, F\rangle / \langle \tilde f, \tilde f\rangle$, where a is a positive rational number depending only on the weight of F (cf. [Ko84]).

Proposition 4.3 *Let F, G as before and $\nu \in [0, \tilde k_G - 1) \cap \mathbb{Z}$. Then we have*

$$A(F, G, \nu) \;=\; \frac{\langle \Phi_1^F, \Phi_1^G\rangle_J L_f(4\tilde k_G - 4 - 2\nu)}{\pi^{4\tilde k_G - 2\nu - 4}\langle F, F\rangle} \in K_F K_G \text{ and} \tag{23}$$

$$A(F, G, \nu)^\sigma \;=\; A(F^\sigma, G^\sigma, \nu) \qquad \text{for all } \sigma \in Aut(\mathbb{C}). \tag{24}$$

Proof:

The claim follows from well-known properties of critical values of Hecke L-functions, if we are allowed to replace $\langle F, F\rangle$ by $\langle f, f\rangle$. To justify this replacement we consider $B(F, f) := \langle F, F\rangle / \langle f, f\rangle$. Up to powers of π we can replace $\langle F, F\rangle$ by a critical value of the standard L-function $D(F, s)$ of F and $\langle f, f\rangle$ by the products of two critical values of the Hecke L function of f with opposite parity. Taking into account that $D(F, s) = \zeta(s)L(f, s+2\tilde k_G - 1)L(s+2\tilde k_G - 2)$ we easily obtain

$$B(F^\sigma, f^\sigma) = B(F, f)^\sigma \qquad \sigma \in Aut(\mathbb{C})$$

For details we refer to [He99, Prop.3.18].

□

5 On the proof of Deligne's conjecture

In this section we present the main results of our paper, proving conjecture B.
Before we start we want to make the folllowing comment on the "critical values":
Let γ_∞ be the Γ-factor occuring in the functional equation (11) of $Z_{G\otimes h}$. It is easy to see that (in all cases !) the set of $m \in \mathbb{Z}$ for which neither $\gamma_\infty(s)$ nor $\gamma_\infty(k_h + 2k_G - 3 - s)$ has a pole, coincides with the set described in conjectures A and B from the introduction (coming from Yoshida's analysis of Deligne's conjecture, assuming the existence of motives attached to G and h).

Theorem 5.1 *Let $\tilde{k}_h, \tilde{k}_G$ be natural numbers and $\tilde{k}_h \le \tilde{k}_G$. Let $h \in S_1^{2\tilde{k}_h}$, $G \in S_2^{2\tilde{k}_G}$ be normalized Hecke eigenforms and let us assume that the first Fourier-Jacobi coefficient of G is non-zero. Then the critical values of $Z_{G\otimes h}(s)$ are given by the integers $m \in [2\tilde{k}_G, 2\tilde{k}_G + 2\tilde{k}_h - 3]$. Moreover we have*

$$\begin{aligned} Z^*_{G\otimes h}(m) &= \frac{Z_{G\otimes h}(m)}{\pi^{4m-4\tilde{k}_G-2\tilde{k}_h+4}\langle h\,,\,h\rangle\langle G\,,G\rangle} \in \overline{\mathbb{Q}} \qquad \textit{and} && (25)\\ \Big(Z^*_{G\otimes h}(m)\Big)^\sigma &= Z^*_{G^\sigma\otimes h^\sigma}(m) \qquad \textit{for all } \sigma \in Aut(\mathbb{C}). && (26)\end{aligned}$$

Proof:
Let $F \in S_2^{2\tilde{k}_G}$ be a normalized Hecke eigenform, assumed to be a Saito-Kurokawa lift of $f \in S_1^{2\tilde{k}_f}$. We choose F in such a way, that $\langle \Phi_1^F\,,\,\Phi_1^G\rangle_J \neq 0$. Then it follows from [Bo-He99]:

$$\begin{aligned}\Phi\Big(\partial^{(\tilde{\Delta})}_{2\tilde{k}_h} h, G, F; s\Big) &= a(s, 2\tilde{k}_G)\ \langle \Phi_1^F\,,\,\Phi_1^G\rangle_J \Gamma_\triangleright(s)\ \zeta_\triangleright(s) && (27)\\ &\quad \times L_f(2s + 4\tilde{k}_G - 4)\ Z_{G\otimes h}(s + 3\tilde{k}_G + \tilde{k}_h - 3),\end{aligned}$$

where

$$\begin{aligned} a(s, 2\tilde{k}_G) &= 2^{-6s-12\tilde{k}_G+13}\pi^{6-s-4\tilde{k}_G},\\ \zeta_\triangleright(s) &= \Big\{\zeta(2s+2\tilde{k}_G)\,\zeta(4s+4\tilde{k}_G-2)\zeta(4s+4\tilde{k}_G-4)\Big\}^{-1},\\ \Gamma_\triangleright(s) &= \frac{\Gamma(2\tilde{k}_G+s-\frac{3}{2})\Gamma(2\tilde{k}_G+s-2)}{\Gamma(2\tilde{k}_G+s)\Gamma(2\tilde{k}_G+s-\frac{1}{2})}\frac{\Gamma(s+\tilde{k}_G-\tilde{k}_h)}{\Gamma(s)}\\ &\quad \frac{\Gamma(s+\tilde{k}_G+\tilde{k}_h-1)\,\Gamma(s+3\tilde{k}_G+\tilde{k}_h-3)\,\Gamma(s+3\tilde{k}_G-\tilde{k}_h-2)}{\Gamma(2s+4\tilde{k}_G-3)}.\end{aligned}$$

Then $a(r,2\tilde{k}_G) \sim_{\mathbb{Q}} \pi^{6-4\tilde{k}_G-r}$ for $r\in\mathbb{Z}$ and

$$\begin{aligned}\zeta_{\triangleright}(-\nu) &\sim_{\mathbb{Q}^\times} \pi^{-(-10\nu+10\tilde{k}_G-6)}\\ \Gamma_{\triangleright}(-\nu) &\sim_{\mathbb{Q}^\times} \frac{\Gamma(-\nu+\tilde{k}_G-\tilde{k}_h)}{\Gamma(-\nu)} \qquad \text{for } \nu\in[0,\tilde{k}_G-1)\cap\mathbb{Z}.\end{aligned}$$

Hence $\Gamma_{\triangleright}(-\nu)\sim_{\mathbb{Q}} 1$ if $\nu\in[\tilde{k}_G-\tilde{k}_h,\tilde{k}_G-1)\cap\mathbb{Z}$. Moreover for such ν it is well known that $L_f(-2\nu+4\tilde{k}_G-4)$ does not vanish. Finally (27) gives a non-trivial relation between an Eisenstein series at negative integers and the twisted spinor L-function $Z_{G\otimes h}(m)$ at $m\in[2\tilde{k}_G+\tilde{k}_h-1,2\tilde{k}_G+2\tilde{k}_h-3]$. These are only half of the integers we are interested in, nevertheless nothing has been lost, the functional equation of $Z_{G\otimes h}(s)$ can be used to get the whole set of the critical values. Let $\nu\in[\tilde{k}_G-\tilde{k}_h,\tilde{k}_G-1)$ as before. Then

$$\begin{aligned}\Phi\left(\partial^{(\tilde{\Delta})}_{2\tilde{k}_h}h,G,F;-\nu\right) \;\sim_{\mathbb{Q}^\times}\; &\pi^{11\nu-14\tilde{k}_G+12}\langle\Phi_1^F\,,\,\Phi_1^G\rangle_J\;L_f(-2\nu+4\tilde{k}_G-4)\\ &Z_{G\otimes h}(-\nu+3\tilde{k}_G+\tilde{k}_h-3).\end{aligned}$$

We have already proven in Proposition 4.3 the relation

$$\langle\Phi_1^F\,,\,\Phi_1^G\rangle_J\;L_f(-2\nu+4\tilde{k}_G-4)\sim_{K_F^\times K_G^\times}\pi^{4\tilde{k}_G-2\nu-4}\langle F\,,\,F\rangle.$$

Putting everything together leads to

$$\Phi\left(\partial^{(\tilde{\Delta})}_{2\tilde{k}_h}h,G,F;-\nu\right)\sim_{\overline{\mathbb{Q}}^\times}\pi^{9\nu-10\tilde{k}_G+8}\langle F\,,\,F\rangle\;Z_{G\otimes h}(-\nu+3\tilde{k}_G+\tilde{k}_h-3). \tag{28}$$

Moreover we know how $\mathrm{Aut}(\mathbb{C})$ acts. Applying corollary 3.2 proves the theorem.

□

Theorem 5.2 *Let $\tilde{k}_h,\tilde{k}_G$ be natural numbers with $\tilde{k}_G<\tilde{k}_h<2\tilde{k}_G-1$. Let $h\in S_1^{2\tilde{k}_h}, G\in S_2^{2\tilde{k}_G}$ be normalized Hecke eigenforms and let us assume that the first Fourier-Jacobi coefficient of G is non-zero. Then the critical values of $Z_{G\otimes h}(s)$ are given by the integers $m\in[2\tilde{k}_h,4\tilde{k}_G-3]$. Moreover we have*

$$Z^*_{G\otimes h}(m) = \frac{Z_{G\otimes h}(m)}{\pi^{4m-4\tilde{k}_G-2\tilde{k}_h+4}\langle h\,,\,h\rangle\langle G\,,G\rangle}\in\overline{\mathbb{Q}} \qquad \textit{and} \tag{29}$$

$$\left(Z^*_{G\otimes h}(m)\right)^\sigma = Z^*_{G^\sigma\otimes h^\sigma}(m) \qquad \textit{for all } \sigma\in Aut(\mathbb{C}). \tag{30}$$

Proof:
For convenience we put $\mu = 2\tilde{k}_h - 2\tilde{k}_G > 0$. To study analytic properties of the L-function, in the case $2\tilde{k}_h > 2\tilde{k}_G$ we have introduced certain *anti-holomorphic* differential operators $\mathcal{A}^\circ\mathcal{D}^{\mu}_{n,s}$. We recall that those operators are defined on $I\!H_{2n}$, they are polynomials in the entries of $\frac{d}{dZ}$, restricted to $I\!H_n \times I\!H_n \hookrightarrow I\!H_{2n}$ via $\mathbf{i}_{n,n}$, therefore they map functions on $I\!H_{2n}$ to functions on $I\!H_n \times I\!H_n$. These operators satisfy

$$\mathcal{A}^\circ\mathcal{D}^{\mu}_{n,\beta}\left(f \mid_{\alpha,\beta} \mathbf{i}(g,g')\right) = \left(\mathcal{A}^\circ\mathcal{D}^{\mu}_{n,\beta} f\right) \mid^{(1)}_{\alpha,\beta+\mu} g \mid^{(2)}_{\alpha,\beta+\mu} g'$$

for all $g, g' \in Sp(n,\mathbb{R})$ and all C^∞-functions on $I\!H_{2n}$; the upper indices (1) and (2) attached to the slash-operator $\mid$ indicate to which of the two components of $I\!H_n \times I\!H_n$ the $\mid_{\alpha,\beta+\mu}$ is assigned to. For more details and properties see [Bo-He99, section 4]. We also recall briefly the relevant integral representation [Bo-He99], Theorem 5.3:

$$\begin{aligned}\Big\langle\Big\langle\Big\langle \delta(z_1)^s\delta(z_2)^s\delta(z_3)^{s+\mu}\mathcal{A}^\circ\mathcal{D}^{\mu}_{2,s}\big(G^{(5)}_{2\tilde{k}_h+s,s} \circ \mathbf{i}_{1,4}\big)\,(z_1 \times z_2 \times z_3); h(z_1)G(z_2)F(z_3)\Big\rangle\Big\rangle\Big\rangle \\ = \gamma^-_\infty(s)\ \langle \Phi^F_1;\Phi^G_1\rangle_J\ \zeta_\triangleleft(s)\ L_f(2s+2\tilde{k}_G+2\tilde{k}_h-4) \qquad (31) \\ Z_{G\otimes h}(s+2\tilde{k}_G+2\tilde{k}_h-3).\end{aligned}$$

Here

$$G^{(n)}_{k+s}(Z) = det(Y)^{-s}E^k_n(Z,s) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in P_{n,0}(\mathbb{Z})\setminus Sp_n(\mathbb{Z})} \det(CZ+D)^{-k}\ |\det(CZ+D)|^{-2s},$$

$$\zeta_\triangleleft(s) = \Big\{\zeta(2s+2\tilde{k}_h)\,\zeta(4s+4\tilde{k}_h-2)\zeta(4s+4\tilde{k}_h-4)\Big\}^{-1}$$

and $\gamma^-_\infty(s)$ is a certain Γ-factor.

If we assume $s = -\mu-\nu$ to be an integer, where $\nu \in [0, 2\tilde{k}_G - \tilde{k}_h - 2]$, then formula (31) can be described in a more explicit way:

$$\begin{aligned}\gamma^-_\infty(s) &\sim_{\mathbb{Q}^\times} \pi^{-(2\tilde{k}_G+2\tilde{k}_h+s-6)}\\ \zeta_\triangleleft(s) &\sim_{\mathbb{Q}^\times} \pi^{-(10s+10\tilde{k}_h-6)}\\ \langle\Phi^F_1;\Phi^G_1\rangle_J\ L_f(2s+2\tilde{k}_G+2\tilde{k}_h-4) &\sim_{K_f^\times} \pi^{2s+2\tilde{k}_G+2\tilde{k}_h-4}\langle F\,,\,F\rangle.\end{aligned}$$

Hence we obtained for the right side of (31):

$$\sim_{K_f^\times} \pi^{9(\mu+\nu)-10\tilde{k}_h+8} Z_{G\otimes h}(4\tilde{k}_G-3-\nu).$$

On the other hand, let us assume for the moment that

$$H_s(z_1,z_2,z_3) := \delta(z_1)^s\delta(z_2)^{s-\mu}\delta(z_3)^{s-\mu}\mathcal{A}^o\mathcal{D}^{\mu}_{2,s}(G^{(5)}_{2\tilde{k}_h+s,s})(z_1\times z_2\times z_3)$$

has at $s=-\mu-\nu$ a holomorphic projection (w.r.t. all three variables z_1,z_2,z_3); we call this holomorphic projection (in the analytic sense for z_2 and z_3) $H_{-\mu-\nu}(z_1,z_2,z_3)_{ana}$; it is an element of

$$\pi^{3\mu+5\nu}\times M_1^{k_h}(\mathbb{Q})\otimes S_2^{k_G}(\mathbb{Q})\otimes S_2^{k_G}(\mathbb{Q}).$$

Then we obtain for the left hand side of equation (31) the expression

$$a\,\pi^{5(\mu+\nu)-2\mu}\langle h\,,\,h\rangle\langle G\,,\,G\rangle\langle F\,,\,F\rangle, \tag{32}$$

where a is an algebraic number. Hence the quotient

$$\frac{\pi^{8+2\tilde{k}_h-12\tilde{k}_G+4\nu}Z_{G\otimes h}(4\tilde{k}_G-3-\nu)}{\langle h\,,\,h\rangle\langle G\,,\,G\rangle} \tag{33}$$

is an algebraic number and has the expected behaviour under the action of $\sigma\in\mathrm{Aut}(\mathbb{C})$. Again by the functional equation of the L-function, we obtain this for all critical values predicted by Deligne. To finish the proof of theorem5.2, we have to prove the assertion about the holomorphic projection: Using

$$E_n^k(Z,s)=\delta(Z)^sG^{(n)}_{k+s,s}$$

we get from Prop.3.1 that $G^{(5)}_{2k_h+s,s}$ is (at $s=-\mu-\nu$) of type

$$\delta(Z)^{\mu+\nu}\times\quad\text{nearly holomorphic function w.r.t. } Z$$

Then

$$G^{(5)}_{2\tilde{k}_h+s,s}(z_1\times Z)\,,\;(Z\in I\!H_4)$$

is a polynomial of degree $\leq\mu+\nu$ in the variable $y_1=\Im(z_1)$, and of degree $\leq 2\mu+2\nu$ as a polynomial in the entries of $y_2=\Im(z_2)$ and the same w.r.t. the entries of $y_3=\Im(z_3)$ and the entries of $y=\Im(z)$ with

$$z_1\times Z=\begin{pmatrix} z_1 & 0 & 0\\ 0 & z_2 & z\\ 0 & z^t & z_3\end{pmatrix},\qquad Z\in I\!H_4$$

and therefore

$$\mathcal{A}^o\mathcal{D}^{\mu}_{2,s}(G^{(5)}_{2\tilde{k}_h+s,s})(z_1 \times z_2 \times z_3)$$

is a polynomial of
degree $\leq \mu+\nu$ w.r.t. y_1,
degree $\leq 2\mu+2\nu$ w.r.t. the entries of y_2
degree $\leq 2\mu+2\nu$ w.r.t. the entries of y_3.
with holomorphic functions as coefficients; more precisely, this function has a Fourier expansion of type

$$\sum_T P(y_1, y_2, y_3, 2\pi T)a(T)e^{2\pi i(t_1 z_1 + tr(T_2 z_2) + tr(T_3 z_3))}$$

where

$$T = \begin{pmatrix} t_1 & * & * \\ * & T_2 & * \\ * & * & T_3 \end{pmatrix}$$

runs over all positive semidefinite half-integral matrices of size 5, $a(T)$ denotes the Fourier coefficient of the degree 5 Siegel Eisenstein series $G^{(5)}_{\tilde{k}_h-2\mu-2\nu,0}$ and P is a polynomial of the type described above (and also a polynomial in the entries of T); as a polynomial in all these variables it has coefficients in $\mathbb{Q}$. The polynomial P arises naturally in the following way: We recall first that (by a proof quite similar to the one in Prop.3.1) there is a differential operator $\mathcal{M}^{(\mu+\nu)}$ acting on $\mathbb{H}_5$ satisfying

$$\mathcal{M}^{(\mu+\nu)}G^{(5)}_{2\tilde{k}_h-2\mu-2\nu,0} = G^{(5)}_{2\tilde{k}_h-\mu-\nu,-\mu-\nu}.$$

Up to normalization this is the Maaß-operator introduced by H.Maaß himself [Ma53, Ma71]; the explicit form of this differential operator is not needed here. The polynomial P then comes up, when we apply the differential operators at hand to a function of type $f_T(Z) = e^{tr(TZ)}$ with $Z \in \mathbb{H}_5$:

$$\mathcal{A}^o\mathcal{D}^{\mu}_{2,-\mu-\nu}\mathcal{M}^{(\mu+\nu)}f_T = P(y_1, y_2, y_3, T)e^{t_1 z_1 + tr(T_2 z_2) + tr(T_3 z_3)}$$

As a simple consequence of the equivariance properties of the differential operators we obtain

$$P(T[A^t], y_1, y_2, y_3) = \det(A)^{\mu}P(T, y_1, y_2[a_2], y_3[a_3]) \tag{34}$$

for all

$$A = 1 \times a_2 \times a_3 \in GL(5,\mathbb{R})\,, a_2, a_3 \in GL(2,\mathbb{R})$$

We claim that only the positive definite matrices T_2 and T_3 give nontrivial contributions to the series above:
This is (for T_3)a simple consequence of (34) applied to T of type

$$T = \begin{pmatrix} t_1 & * & * & 0 \\ * & T_2 & * & 0 \\ * & * & t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $a_2 = 1_2$, $a_3 = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$ with $\lambda \longrightarrow 0$. Exactly the same type of argument also works for T_2.
Therefore (by the same argument as in the case of holomorphic cusp forms) the function

$$H_{-\mu-\nu}(z_1, z_2, z_3) = \delta(z_1)^{-\mu-\nu}\delta(z_2)^{-\nu}\delta(z_3)^{-\nu}\sum_T P(y_1, y_2, y_3, 2\pi T)a(T)e^{2\pi i t_1 z_1 + tr(T_2 z_2) + tr(T_3 z_3)}$$

is, considered as a (nonholomorphic) modular form w.r.t. z_2 or z_3 of rapid decay, in particular the *analytic* procedure of holomorphic projection as described in [Pa91, St81] can be applied (the condition on the weight is automatically satisfied because we only have to consider weights $k_G \geq 10$). In this way we obtain a holomorphic cusp form with respect to z_2 and z_3, which as a function of z_1 is nearly holomorphic; its Fourier coefficients atttached to exponent matrices T_2 and T_3 are given by

$$\frac{\pi^{4k_G-6}}{\Gamma_2(k_G-\frac{3}{2})^2}\det(4T_2)^{k_G-\frac{3}{2}}\det(4T_3)^{k_G-\frac{3}{2}}\sum_T a(T)$$

$$\times \int_{y_2}\int_{y_3} y_1^{-\mu-\nu}P(y_1, y_2, y_3, 2\pi T)e^{-4\pi tr(T_2 y_2) - 4\pi tr(T_3 y_3}\det(y_2)^{k_G-\frac{3}{2}-\nu}\det(y_3)^{k_G-\frac{3}{2}-\nu}d^*y_2 dy_3^*$$

Here T runs over all positive semidefinite half-integral matrices with T_2 and T_3 as block matrices on the bottom of its diagonal; furthermore we use the standard notations

$$\Gamma_n(s) = \pi^{\frac{n(n-1)}{4}}\prod_{j=0}^{n-1}\Gamma(s-\frac{j}{2}) = \int \det(y)^s e^{-tr(y)}d^*y$$

$$d^*y = \det(y)^{-\frac{n+1}{2}}\prod_{i\leq j} dy_{ij}$$

Using

$$P(y_1, y_2, y_3, 2\pi T) = \pi^{2\mu} P(\pi y_1, \pi y_2, \pi y_3, 2T)$$

we obtain that

$$\frac{1}{\Gamma_2(k_G - \frac{3}{2})^2} \int_{y_2} \int_{y_3} P(y_1, y_2, y_3, 2\pi T) e^{-4\pi tr(T_2 y_2) - 4\pi tr(T_3 y_3)} \det(y_2)^{k_G - \frac{3}{2} - \nu} \det(y_3)^{k_G - \frac{3}{2} - \nu} d^* y_2 dy_3^*$$

is an element of

$$\pi^{2\mu + 4\nu + 6 - 4k_G} \times \mathbb{Q}[\pi y_1]$$

of degree $\leq \nu + \mu$ w.r.t. y_1.
Here we tacitly used that

$$\frac{1}{\Gamma_2(k_G - \frac{3}{2})^2} \int_{y_2} \int_{y_3} e^{-4tr(T_2 y_2) - 4tr(T_3 y_3)} \det(y_2)^{k_G - \frac{3}{2} - \nu} \det(y_3)^{k_G - \frac{3}{2} - \nu} d^* y_2 dy_3^* =$$

$$\left(\frac{\Gamma_2(k_G - \frac{3}{2} - \nu)}{\Gamma_2(k_G - \frac{3}{2})} \right)^2 \det(4y_2)^{k_G - \frac{3}{2} - \nu} \det(4y_3)^{k_G - \frac{3}{2} - \nu}$$

and that we may insert the polynomial P in the integrand essentially by differentiation with respect to the entries of T_2 and T_3.
Therefore we obtain that the (analytic) holomorphic projection of $H_{-\mu-\nu}$ with respect to z_2 and z_3 is a nearly holomorphic function of z_1 of the form

$$\pi^{3\mu + 5\nu} \times \sum_{j=0}^{\mu+\nu} \frac{1}{(\pi y_1)^j} h_j(z_1, z_2, z_3)$$

with h_j being holomorphic periodic functions of all variables with rational Fourier coefficients.
The assertion about $(H_{-\mu-\nu})_{ana}$ follows by considering the holomorphic part of the nearly holomorphic function from above (as described e.g. in [Sh76, lemma 7]) or by the formula (12).

□

At this point we should remark that we found it more convenient to use the analytic version of holomorphic projection for the "second case", i.e. $2\tilde{k}_G < 2\tilde{k}_h < 4\tilde{k}_G - 2$. It should also be possible to do it within the framework used for the "first case". This however would have required some extra considerations about the differential operators $\mathcal{A}^\circ \mathcal{D}^\mu_{n,s}$ which we prefered to avoid.

Let $2\tilde{k}_h > 4\tilde{k}_G - 2$. Then $\gamma^-_\infty(s_0)\ \zeta_\triangleleft(s_0)$, for s_0 chosen to study the critical values $m \in C_I$, is zero. This shows that that the method of the present paper cannot be used to study these critical values (we would get the "wrong" Petersson scalar products anyway !). Nevertheless let $G \in S_2^{k,M}$ be a normalized eigenform in the Maaß space . Then we are able to prove this conjecture by employing the splitting of $Z_G(s)$ given in (9), which relates the critical values of $Z_{G\otimes h}(s)$ to critical values of Hecke- and Rankin L-functions attached to elliptic cusp forms. It is then an easy exercise to prove Conjecture A and B directly using only the explicit computations of section 4.

Remark 5.3 *We have discovered in [He99] some surprising similiarities between analytic properties of the twisted spinor L-function $Z_{G\otimes h}(s)$ and the Rankin triple L-function $L_{f_1\otimes f_2\otimes f_3}(s)$ attached to three elliptic cuspidal eigenforms. This similarity also appears in the critical values, more precisely, we claim that Conjectures A and B are of exactly the same type as the one stated by Deligne and worked out by Blasius (see [Or87]) for the Rankin triple L-function, where a certain triangle condition related to the weights played a fundamental role.*
By introducing a virtual weight in the context of the twisted spinor L-function, the distinction between conjecture A and B and the relation to the triangle condition can be interpreted in the following way. Let $k_1 = 2\tilde{k}_h$ and $k_2 = k_3 = 2\tilde{k}_G$, where $2\tilde{k}_h \leq 2\tilde{k}_G$ and $k_1 = k_2 = 2\tilde{k}_G$ and $k_3 = 2\tilde{k}_h$ otherwise. Then $k_1 \leq k_2 \leq k_3$ and the hidden triangle condition is given by

$$\begin{aligned} k_3 &\geq k_1 + k_2 \qquad (\textit{Conjecture A}) \\ k_3 &< k_1 + k_2 \qquad (\textit{Conjecture B}). \end{aligned}$$

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