Asymptotic Fredholm modules and a non-vanishing theorem for K-theoretic indices over non-compact spaces

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Abstract: We introduce a concept of asymptotic Fredholm module to prove a non-vanishing theorem for K-theoretic indices of elliptic operators over non-compact spaces. The non-vanishing theorem is applied to study positive scalar curvature and spectrum of Laplacian on non-compact spaces.

1 Introduction

In this paper we shall introduce a concept of asymptotic Fredholm module over a C^*-algebra to study non-vanishing of K-theory information. In the case of non-compact spaces we shall use vector bundles of small variation to construct asymptotic Fredholm modules and compute their pairings with K-theoretic indices of elliptic operators. The computation of K-theoretic indices is used to study non-existence of metrics with positive scalar curvature and the spectrum of the Laplacian operator acting on the space of L^2-forms.

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2 Asymptotic Fredholm modules

In this section we introduce the concept of asymptotic Fredholm module and define its pairing with K-theory.

Let A be a C^*-algebra. We consider sequence of triples (E_n, ϕ_n, F_n) where $n \in \mathbb{Z}_+$, the set of positive integers, E_n is a separable graded Hilbert space, ϕ_n is a graded map from A to $B(E_n)$, the algebra of all bounded operators acting on E_n, and F_n is a linear bounded operator in $B(E_n)$ with degree 1.

Definition 2.1 *A sequence of triples (E_n, ϕ_n, F_n) is called an asymptotic Fredholm module over A if*
(1) $[F_n, \phi_n(a)]$, $(F_n^2 - I)\phi_n(a)$, and $(F_n - F_n^*)\phi_n(a)$ are compact for all $a \in A$;
(2) for all $a, b \in A, \lambda \in \mathbb{C}$, the following norm limits vanish:

$$\lim_{n\to\infty}(\phi_n(a + \lambda b) - \phi_n(a) - \lambda\phi_n(b)) = 0$$

$$\lim_{n\to\infty}(\phi_n(ab) - \phi_n(a)\phi_n(b)) = 0$$

$$\lim_{n\to\infty}(\phi_n(a^*) - \phi_n(a)^*) = 0.$$

Notice that the concept of asymptotic Fredholm module is an asymptotic version of the usual concept of Fredholm module [17] [16] and is closely related to the concept of asymptotic morphism introduced by Connes and Higson [7].

An asymptotic Fredholm module (E_n, ϕ_n, F_n) is said to be degenerate if $[F_n, \phi_n(a)]$, $(F_n^2 - I)\phi_n(a)$, and $(F_n - F_n^*)\phi_n(a)$ are 0 for all $a \in A$.

A particularly useful class of asymptotic Fredholm modules are asymptotic quasihomomorphisms, an asymptotic version of Cuntz's quasihomomorphisms [9].

For each $n \in \mathbb{Z}_+$, let H_n be a Hilbert space, let $\phi_n^{(0)}$ and $\phi_n^{(1)}$ be maps from A to $B(H_n)$, the algebra of all bounded operators acting on H_n.

Definition 2.2 *A sequence of pairs $(\phi_n^{(0)}, \phi_n^{(1)})$ is said to be an asymptotic quasihomomorphism over A if*
(1) $\phi_n^{(0)}(a) - \phi_n^{(1)}(a)$ are compact for all $a \in A$;
(2) for all $a, b \in A, \lambda \in \mathbb{C}, i = 0, 1$, the following norm limits vanish:

$$\lim_{n\to\infty}(\phi_n^{(i)}(a + \lambda b) - \phi_n^{(i)}(a) - \lambda\phi_n^{(i)}(b)) = 0$$

$$\lim_{n\to\infty}(\phi_n^{(i)}(ab) - \phi_n^{(i)}(a)\phi_n^{(i)}(b)) = 0$$
$$\lim_{n\to\infty}(\phi_n^{(i)}(a^*) - \phi_n^{(i)}(a)^*) = 0.$$

Notice that an asymptotic morphism ϕ_t (in the sense of Connes and Higson) from A to K, the algebra of all compact operators, naturally gives rise to an asymptotic quasihomomorphism $(\phi_n, 0)$.

An asymptotic quasihomomorphism $(\phi_n^{(0)}, \phi_n^{(1)})$ gives rise to an asymptotic Fredholm module (E_n, ϕ_n, F_n) defined by: $E_n = H_n \oplus H_n$ with the grading operator $1 \oplus -1$, $\phi_n = \phi_n^{(0)} \oplus \phi_n^{(1)}$, and

$$F_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We shall show that every asymptotic Fredholm module is equivalent to an asymptotic homomorphism.

An asymptotic Fredholm module (E_n, ϕ_n, F_n) is said to be unitarily equivalent to another asymptotic Fredholm module (E_n', ϕ_n', F_n') if there exists a unitary in $B(E_n, E_n')$ interwining ϕ_n with ϕ_n', and F_n with F_n'.

An asymptotic Fredholm module (E_n, ϕ_n, F_n) is said to be a compact perturbation of another asymptotic Fredholm module (E_n, ϕ_n, F_n') if $(F_n - F_n')\phi_n(a)$ is compact for all $a \in A$.

We define an equivalence relation on the set of all asymptotic Fredholm modules over A to be the equivalence relation generated by unitary equivalence, compact perturbation and addition of degenerate asymptotic Fredholm modules.

The following useful lemma is an asymptotic version of a result of Cuntz [9].

Lemma 2.3 *Every asymptotic Fredholm module is equivalent to an asymptotic quasihomomorphism.*

Proof: Let (E_n, ϕ_n, F_n) be an asymptotic Fredholm module over A. We can assume $F_n^* = F_n$ by replacing F_n with $(F_n + F_n^*)/2$ (this is a compact perturbation). We can further assume $||F_n|| \leq 1$ by replacing F_n with its compact perturbation $g(F_n)$, where g is the continuous function on $\mathbb{R}$ such that $g(x) = -1$

for $x \leq -1$, $g(x) = x$ for $-1 \leq x \leq 1$, and $g(x) = 1$ for $x \geq 1$. (E_n, ϕ_n, F_n) is equivalent to $(E_n \oplus E_n, \phi_n \oplus 0, G_n)$, where the grading on $E_n \oplus E_n$ is given by the grading operator $\epsilon' = \epsilon \oplus -\epsilon$ (ϵ is the grading operator on E_n), and

$$G_n = \begin{pmatrix} F_n & \epsilon\sqrt{I - F_n^2} \\ \epsilon\sqrt{I - F_n^2} & F_n \end{pmatrix}.$$

Notice that $G_n = G_n^* = G_n^{-1}$. Let $E'_n = E_n \oplus E_n$, $\phi'_n = \phi_n \oplus 0$. $(E_n \oplus E_n, \phi_n \oplus 0, G_n)$ is equivalent to $(E'_n \oplus E'_n, \phi'_n \oplus 0, G_n \oplus -G_n)$, where the grading operator on $E'_n \oplus E'_n$ is defined to be $\epsilon' \oplus -\epsilon'$. But this is unitarily equivalent to $(E'_n \oplus E'_n, (ad\ U)(\phi'_n \oplus 0), F)$, where

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives rise to an asymptotic homomorphism. ■

With the help of Lemma 2.3 we can now define the pairing between asymptotic Fredholm modules and K-theory.

We shall first assume that our asymptotic Fredholm module is an asymptotic quasihomomorphism $(\phi_n^{(0)}, \phi_n^{(1)})$. Let p be a projection in $M_k(A)^+$ representing an element in $K_0(A)$. $\phi_n^{(0)}$ and $\phi_n^{(1)}$ can be naturally extended to maps from A^+ to $B(H_n)$. Let

$$p_n^{(0)} = f((\phi_n^{(0)}(p) + \phi_n^{(0)}(p)^*)/2)$$

$$p_n^{(1)} = f((\phi_n^{(1)}(p) + \phi_n^{(1)}(p)^*)/2),$$

where f is a continuous function on $\mathbb{R}$ such that $f(x) = 1$ all $x \in [2/3, 4/3]$ and $f(x) = 0$ for all $x \in [-1/3, 1/3]$. By the properties of asymptotic quasihomomorphism we know that there exists $N > 0$ such that $(p_n^{(0)}, p_n^{(1)})$ is a pair of projections in $B(H_n)$ such that $p_n^{(0)} - p_n^{(1)}$ is compact for all $n > N$. This implies that $p_n^{(1)} p_n^{(0)}$ is a Fredholm operator from $p_n^{(0)} H_n$ to $p_n^{(1)} H_n$ for all $n > N$. We define the pairing between the asymptotic quasihomomorphism $(\phi_n^{(0)}, \phi_n^{(1)})$ and the K-theory element $[p]$ by:

$$< (\phi_n^{(0)}, \phi_n^{(1)}), [p] >= index(p_n^{(1)} p_n^{(0)}),$$

Where $n > N$ and $index(p_n^{(1)}p_n^{(0)})$ is the Fredholm index of the operator $p_n^{(1)}p_n^{(0)}$ from $p_n^{(0)}H_n$ to $p_n^{(1)}H_n$.

The pairing of a general asymptotic Fredholm module with the K-theory can be defined by using the asymptotic quasihomomorphism equivalent to the given asymptotic Fredholm module in the proof of Lemma 2.3.

3 Vector bundles with small variation

In this section we introduce the concept of vector bundles with small variation over a non-compact proper metric space X and show how to construct asymptotic Fredholm modules using vector bundles with small variation.

Let X be a proper metric space. Recall that the properness of X means that every closed ball in X is compact.

Definition 3.1 *A sequence of compactly supported vector bundles V_n $(n \in \mathbb{Z}_+)$ over X is said to have small variation if each V_n can be represented by a projection P_n in $M_{k_n}(C_0(X))^+$ for some positive integer k_n such that for every $r > 0$*

$$\lim_{n\to\infty} sup_{(x,y)\in X\times X:d(x,y)\leq r}||p_n(x) - p_n(y)|| = 0,$$

where $C_0(X)$ is the algebra of all continuous functions vanishing at infinity over X, $M_{k_n}(C_0(X))$ is the the algebra of all $k_n \times k_n$ matrices over $C_0(X)$, and $M_{k_n}(C_0(X))^+$ is obtained from $M_{k_n}(C_0(X))$ by adjoining an identity.

A sequence of vector bundles with small variation asymptotically does not distinguish points within bounded distance. So it can be considered as a notion of "large scale" vector bundle. The concept of vector bundles with small variation is in spirit closely related to the concept of almost flat bundles [5] [13] although there does not seem to be a direct connection between the two concepts.

Next we shall use a sequence of vector bundles with small variation to construct an asymptotic Fredholm module over the C^*-algbera $C^*(X)$. $C^*(X)$ plays a key role in the index theory for non-compact spaces since its K-theory is the receptacle of K-theoretic indices of elliptic operators on X. For the convenience of the readers we shall briefly recall the definition of $C^*(X)$.

Throughout this paper an X-module is a separable Hilbert space equipped with a faithful and non-degenerate representation of $C_0(X)$ whose range contains no non-zero compact operator.

Definition 3.2 *Let H_X be an X-module. The support of a bounded operator $T : H_X \to H_X$ is defined to be the complement (in $X \times X$) of the set of all points $(x, y) \in X \times X$ for which there exist functions $f \in C_0(X), g \in C_0(X)$ such that $gTf = 0$, and $f(x) \neq 0$, $g(y) \neq 0$.*

Definition 3.3 *Let H_X be a X-module; let T be a bounded linear operator acting on H_X.*
(1) The propagation of T is defined to be: $sup\{d(x, y) : (x, y) \in Supp(T)\}$;
(2) T is said to be locally compact if fT and Tf are compact for all $f \in C_0(X)$.

Definition 3.4 *([19]) Let H_X be an X-module. $C^*(X, H_X)$ is defined to be the C^*-algebra generated by all locally compact operators acting on H_X with finite propagations.*

It is easy to show that $C^*(X, H_X)$ does not depend on the choice of H_X (up to isomorphism) [19]. For this reason $C^*(X, H_X)$ will sometimes be abbreviated as $C^*(X)$.

Let

$$E_{n,0} = \oplus_{i=1}^{l_n} H_X,$$

$$E_{n,1} = P_n(\oplus_{i=1}^{k_n} H_X),$$

where l_n is the rank of the vector bundle V_n in the Definition 3.1 and k_n is as in Definition 3.1. Define

$$E_n = E_{n,0} \oplus E_{n,1},$$

where the grading operator on E_n is defined to be $1 \oplus -1$. We define a graded map ϕ_n from $C^*(X)$ to $B(E_n)$ by:

$$\phi_n = \phi_{n,0} \oplus \phi_{n,1},$$

where $\phi_{n,0}$ is the $*$-homomorphism from $C^*(X)$ to $B(E_{n,0})$ defined by:

$$\phi_{n,0}(a) = \oplus_{i=1}^{l_n} a$$

for all $a \in C^*(X)$, and $\phi_{n,1}$ is the map from $C^*(X)$ to $B(E_{n,1})$ defined by:

$$\phi_{n,1}(a) = P_n(\oplus_{i=1}^{n} a)$$

for all $a \in C^*(X)$. Without loss of generality we can choose P_n such that for each n there exists a compact subset K_n of X for which

$$P_n(x) = \oplus_{i=1}^{l_n} I \oplus_{i=l_n+1}^{k_n} 0$$

for all $x \in X - K_n$. Let α_n be a unitary operator from $E_{n,1}$ to $E_{n,0}$ such that (1) $Supp(\alpha_n v) \subseteq K_n$ if $Supp(v) \subseteq K_n$, where $Supp(v)$ is the complement of the set of all points x in X such that there exists $f \in C_0(X)$, $f(x) \neq 0$, $fv = 0$; (2) $\alpha_n v = v$ if $Supp(v) \subseteq X - K_n$.

We define an operator F_n on E_n by:

$$F_n = \begin{pmatrix} 0 & \alpha_n \\ \alpha_n^* & 0 \end{pmatrix}.$$

A metric space X is said to have bounded geometry if there exists a subspace Γ such that
(1) there is $c > 0$ such that $d(x, \Gamma) \leq c$ for all $x \in X$;
(2) for each $r > 0$, there is $N(r)$ such that the number of elements in $B_\Gamma(\gamma, r)$ is at most $N(r)$ for all $\gamma \in \Gamma$, where $B_\Gamma(\gamma, r) = \{x \in \Gamma : d(x, \gamma) \leq r\}$.

Lemma 3.5 *If X is a proper metric space with bounded geometry, then the sequence of triples (E_n, ϕ_n, F_n) defined as above is an asymptotic Fredholm module over $C^*(X)$.*

Proof: All we need to do is to verify that

$$\lim_{n\to\infty}(\phi_{n,1}(a)\phi_{n,1}(b) - \phi_{n,1}(ab)) = 0$$

for all a and b. It is enough to show that for any operator a acting on H_X with finite propagation

$$(1) \qquad \lim_{n\to\infty}(P_n(\oplus_{i=1}^{k_n} a) - (\oplus_{i=1}^{k_n} a)P_n) = 0.$$

We assume that a has propagation r. The bounded geometry property of X implies that we can decompose $X = \cup_{i=1}^{m} X_i$ for some finite m such that $X_i \cap X_{i'} = \emptyset$ if $i \neq i'$, and each X_i is the disjoint union of a sequence of uniformly bounded Borel sets $\{U_{ij}\}_j$ such that $d(U_{ij}, U_{ij'}) > r$ if $j \neq j'$. Let χ_{ij} be the characteristic function of U_{ij}. The representation of $C_0(X)$ on H_X can be extended to that of the algebra of bounded Borel functions. The properties of the decomposition of X and the fact that a has propagation r imply that there is $C > 0$ such that

$$(2) \qquad ||P_n(\oplus_{i=1}^{k_n} a) - (\oplus_{i=1}^{k_n} a)P_n)||$$

$$\leq C sup_{i,j,i',j'} ||\chi_{ij}(P_n(\oplus_{i=1}^{k_n} a) - (\oplus_{i=1}^{k_n} a)P_n)\chi_{i'j'}||.$$

But

$$(3) \qquad ||\chi_{ij}(P_n(\oplus_{i=1}^{k_n} a) - (\oplus_{i=1}^{k_n} a)P_n)\chi_{i'j'}||$$

$$\leq ||\chi_{ij}(P_n - P(x_{ij}))(\oplus_{i=1}^{k_n} a)\chi_{i'j'}|| + ||\chi_{ij}(\oplus_{i=1}^{k_n} a)(P_n - P_n(x_{i'j'}))\chi_{i'j'}||$$

$$+ ||\chi_{i,j}(\oplus_{i=1}^{k_n} a)(P_n(x_{ij}) - P_n(x_{i'j'}))\chi_{i'j'}||,$$

where x_{ij} is a point in U_{ij} and $x_{i'j'}$ is a point in $U_{i'j'}$.

(3), together with (2) and the small variation property of P_n, implies the desired identity (1). ■

4 A non-vanishing theorem for K-theoretic indices over non-compact spaces

The main result of this section is the following:

Theorem 4.1 *Let X be a non-compact proper metric space with bounded geometry, and let $[D]$ be a K-homology class in $K_0(X) = KK(C_0(X), \mathbb{C})$. If V_n is a sequence of vector bundles V_n with small variation on X and (E_n, ϕ_n, F_n) is its associated asymptotic Fredholm module, then*

$$< (E_n, \phi_n, F_n), Index[D] > = < [D], [V_n] >$$

for all $n > N$, where N is some large integer, and $< [D], [V_n] >$ is the pairing between the K-homology class $[D]$ and the K-theory class $[V_n] - [X \times \mathbb{C}^{l_n}]$ (l_n is the rank of the vector bundle V_n).

Proof: Our theorem follows from Lemmas 4.3 and 4.4 in this section. ∎

We emphasize that $< [D], [V_n] >$ is computable. In the case that X is a complete Riemannian manifold, $< [D], [V_n] >$ can be computed by the Atiyah-Singer index formula. For example, if X is a complete Riemannian spin manifold and D is a Dirac operator, then $< [D], [V_n] >=< \hat{A}(X)ch(V_n), [M] >$.

Roe's index theorem [19] obtained using the Higson corona vX is a special case of Theorem 4.1 since every element in the range of the boundary map from $K^1(vX)$ to $K_0(C_0(X))$ has a sequence of representatives with small variation (see [26] for more details on the boundary map).

The following non-vanishing theorem is a consequence of Theorem 4.1.

Corollary 4.2 *Let X be a non-compact proper metric space with bounded geometry and let $[D]$ be a K-homology class in $K_0(X)$. If there exists a sequence of vector bundles V_n with small variation on X such that the pairing $< [D], [V_n] > \neq 0$ for infinitely many n, then $Index([D]) \neq 0$ in $K_0(C^*(X))$.*

For the convenience of readers we shall briefly recall the definition of the index map from $K_i(X)$ to $K_i(C^*(X))$. A K-homology class $[D]$ in $K_0(X)$ can be represented as a pair $(H_X \oplus H_X, D)$, where the grading on $H_X \oplus H_X$ is given by the grading operator $1 \oplus -1$, and D is a bounded operator of degree one on $H_X \oplus H_X$ such that $(D^* - D)f$, $(D^2 - I)f$ and $[D, f]$ are compact operators for all $f \in C_0(X)$. If X is a complete Riemannian manifold, then Dirac type operators on X naturally give rise to K-homology classes.

Without loss of generality we can assume $D^* = D$, $||D|| \leq 1$. Let $\{U_i\}_i$ be a locally finite and uniformly bounded open cover of X and $\{\phi_i\}_i$ be a continuous partition of unity subordinate to the open cover. Define

$$D' = \sum_i \phi_i^{\frac{1}{2}} D \phi_i^{\frac{1}{2}},$$

where the infinite sum converges in strong topology. It is not difficult to verify that $(H_X \oplus H_X, D')$ is equivalent to $(H_X \oplus H_X, D)$ in $K_0(X)$. Write

$$D' = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.$$

Let $S_0 = I - AA^*, S_1 = I - A^*A$. We define

$$Index[D] = \begin{pmatrix} I - S_1^2 & (S_1 + S_1^2)A^* \\ S_0 A & S_0^2 \end{pmatrix}.$$

Similarly we can define the index map from $K_1(X)$ to $K_1(C^*(X))$.

Let $H_{n,0}$ be the graded Hilbert space $\oplus_{i=1}^{l_n}(H_X \oplus H_X)$ with the grading operator $\epsilon_0 = \oplus_{i=1}^{l_n}(1 \oplus -1)$. Define $D_{X\times\mathbb{C}^{l_n}}$ to be the operator $\oplus_{i=1}^{l_n} D$ acting on $H_{n,0}$. Let $H_{n,1}$ be the graded Hilbert space $P_n(\oplus_{i=1}^{k_n}(H_X \oplus H_X))$ with the grading operator $\epsilon_1 = P_n(\oplus_{i=1}^{k_n}(1 \oplus -1))$, where P_n is as in Definition 3.1. Define D_{V_n} to be the operator $P_n(\oplus_{i=1}^{k_n} D)$ acting on $H_{n,1}$. Let α_n be as in section 3 and let β_n be the operator from $H_{n,1} = P_n(\oplus_{i=1}^{k_n} H_X) \oplus P_n(\oplus_{i=1}^{k_n} H_X)$ to $H_{n,0} = (\oplus_{i=1}^{l_n} H_X) \oplus (\oplus_{i=1}^{l_n} H_X)$ defined by: $\beta_n = \alpha_n \oplus \alpha_n$. Let $H_n = H_{n,0} \oplus H_{n,1}$ be the graded Hilbert space with grading given by $\epsilon = \epsilon_0 \oplus -\epsilon_1$. Define D_n to be the operator $D_{X\times\mathbb{C}^{l_n}}\epsilon_0 \oplus D_{V_n}\epsilon_1$ acting on the graded Hilbert space H_n. Let

$$C_n = \begin{pmatrix} 0 & \beta_n \\ \beta_n^* & 0 \end{pmatrix}.$$

Finally we define an operator G_n of degree one on the graded Hilbert space H_n by:

$$G_n = D_n + \sqrt{1 - D_n^2}\, C_n.$$

We can easily check that $G_n^2 = I \ mod\ K$. Hence we can define $index(G_n)$ to be the Fredholm index of the operator $G_n|_{H_{n,+}}$ from the positive eigenspace $H_{n,+}$ of ϵ to the negative eigenspace $H_{n,-}$ of ϵ.

Lemma 4.3

$$< [D], [V_n] >= index(G_n).$$

Proof: Let α be a continuous non-negative function on X such that $\alpha(x) \leq 1$ for all $x \in X$, $\alpha(x) = 0$ for all $x \in K_n$ and $\alpha(x) = 1$ for all x outside some compact subset of X, where K_n is as in Section 3. The K-theory element $[V_n] - [X \times \mathbb{C}^{l_n}]$ can be represented as a Kasparov module (E, ϕ, F) for $(\mathbb{C}, C_0(X))$, where E is the graded Hilbert module over $C_0(X)$ defined by $E = (\oplus_{i=1}^{l_n} C_0(X)) \oplus P_n(\oplus_{i=1}^{k_n} C_0(X))$ with the grading given $1 \oplus -1$, ϕ is the homomorphism from $\mathbb{C}$ to $B(E)$ satisfying $\phi(1) = I$, and

$$F = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}.$$

We need to compute the Kasparov product $(E, \phi, F)\hat{\otimes}_{C_0(X)}(D, H_X \oplus H_X)$. To do this we first identify $E\hat{\otimes}_{C_0(X)}(H_X \oplus H_X)$ with H_n by the following isomorphism h:

$$h(((\oplus_{i=1}^{l_n} x_i) \oplus P_n(\oplus_{i=1}^{k_n} x_{i+l_n}))\hat{\otimes}_{C_0(X)} y) = (\oplus_{i=1}^{l_n} x_i y) \oplus P_n(\oplus_{i=1}^{k_n} x_{i+l_n} y)$$

for all $((\oplus_{i=1}^{l_n} x_i) \oplus P_n(\oplus_{i=1}^{k_n} x_{i+l_n}))\hat{\otimes}_{C_0(X)} y \in E\hat{\otimes}_{C_0(X)}(H_X \oplus H_X)$.

Next we shall show that G_n is D_n-connection in the sense of Connes and Skandalis [8]. Given $x \in E$, let T_x be the operator from $H_X \oplus H_X$ to $E\hat{\otimes}_{C_0(X)}(H_X \oplus H_X)$ defined by: $T_x y = x\hat{\otimes}_{C_0(X)} y$ for all $y \in H_X \oplus H_X$. Write $x = (\oplus_{i=1}^{l_n} x_i) \oplus P_n(\oplus_{i=1}^{k_n} x_{i+l_n})$ according to the decomposition $E = (\oplus_{i=1}^{l_n} C_0(X)) \oplus P_n(\oplus_{i=1}^{k_n} C_0(X))$. Using the isomorphism h we can identify T_x with the operator from $H_X \oplus H_X$ to H_n defined by:

$$T_x y = (\oplus_{i=1}^{l_n} x_i y) \oplus P_n(\oplus_{i=1}^{k_n} x_{i+l_n} y)$$

for all $y \in H_X \oplus H_X$. Now we can easily check that

$$T_x D_n - (-1)^{deg(x)} G_n T_x \in K(H_X \oplus H_X, H_n)$$

for a homogeneous element x. Similarly we can verify that

$$D_n T_x^* - (-1)^{deg(x)} T_x^* F \in K(H_n, H_X \oplus H_X).$$

Hence by definition G_n is D_n-connection.

Using the isomorphism h we can identify the operator $F\hat{\otimes}I$ from $E\hat{\otimes}_{C_0(X)}(H_X \oplus H_X)$ to $E\hat{\otimes}_{C_0(X)}(H_X \oplus H_X)$ with the operator from H_n to H_n defined by:

$$F\hat{\otimes}I = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

with respect to the decomposition $H_n = H_{n,0} \oplus H_{n,1}$. We can verify that

$$[F\hat{\otimes}I, G_n] = \sqrt{1 - D_n^2}(F\hat{\otimes}I)^2 \ mod\ K.$$

Now by Definition 18.4.1 and Theorem 18.4.3 in [1] our lemma follows from the above identity and the fact that G_n is D_n-connection. ■

Lemma 4.4

$$< (E_n, \phi_n, F_n), Index[D] >= index(G_n)$$

for all $n > N$, where N is some large integer.

Proof: Let H_X and H'_X be two X-modules, $E = H_X \oplus H_X$ with the grading operator $1 \oplus -1$ and $E' = H'_X \oplus H'_X$ with the grading operator $1 \oplus -1$; let (T, T', U) be a triple where T is an operator of degree one acting on E satisfying $T^* = T, ||T|| \leq 1$, $T^2 - I \in C^*(X, H_X) \oplus C^*(X, H_X)$, T' is an operator of degree one acting on E' satisfying $(T')^* = T'$, $||T'|| \leq 1$, $(T')^2 - I \in C^*(X, H'_X) \oplus C^*(X, H'_X)$, and U is a unitary operator of degree 0 from E to E' such that $T^2 - U^*(T')^2U$ is compact. We shall first define the index of the triple (T, T', U). Let

$$T = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}.$$

Let $S_0 = I - AA^*, S_1 = I - A^*A$. We define

$$e(T) = \begin{pmatrix} I - S_1^2 & (S_1 + S_1^2)A^* \\ S_0 A & S_0^2 \end{pmatrix}.$$

$e(T)$ is similar to the following projection

$$p(T) = e(T)e^*(T)(I + (e(T) - e^*(T))(e^*(T) - e(T)))^{-1}.$$

Similarly we can define $p(T')$. We define $ind(T, T', U)$ to be the element in $\mathbb{Z} = KK(\mathbb{C}, \mathbb{C})$ represented by the quasihomomorphism (ϕ, ϕ') from $\mathbb{C}$ to $\mathbb{C}$, where ϕ is the homomorphism from $\mathbb{C}$ to $B(H_X \oplus H_X)$ satisfying $\phi(1) = p(T)$, and ϕ' is the homomorphism from $\mathbb{C}$ to $B(H_X \oplus H_X)$ satisfying $\phi'(1) = (U \oplus U)^*p(T)(U \oplus U)$.

Without loss of generality we can assume that D has finite propagation. Using identity (1) in the proof of Lemma 3.5 it is not difficult to see that

$$(1) \qquad < (E_n, \phi_n, F_n), Index[D] >= ind(D_{X\times \mathbb{C}^{l_n}}, D_{V_n}, \beta_n^*)$$

for all $n > N$, where N is some large integer N.

Let $T = D_{X\times \mathbb{C}^{l_n}}, T' = D_{V_n}, U = \beta_n^*$. Let

$$B = \begin{pmatrix} T & \epsilon_0 U^* \sqrt{I-(T')^2}U \\ \epsilon_0 U^* \sqrt{I-(T')^2}U & U^*T'U \end{pmatrix},$$

$$B' = \begin{pmatrix} T' & \epsilon_1 \sqrt{I-(T')^2} \\ \epsilon_1 \sqrt{I-(T')^2} & T' \end{pmatrix}.$$

Notice that B is an operator of degree one acting on $H_{n,0} \oplus H_{n,0}$ with the grading operator $\epsilon_0 \oplus -\epsilon_0$, and B' is an operator of degree one acting on $H_{n,1} \oplus H_{n,1}$ with the grading operator $\epsilon_1 \oplus -\epsilon_1$. We can easily verify that $B^2 = I \ mod \ K$ and $(B')^2 = I$. Observe that we are using the same matrix trick here as in the proof of Lemma 2.3.

It is easy to see that

$$ind(U^*T'U, T', U) = 0.$$

Hence

$$(2) \qquad ind(T, T', U) = ind(T \oplus U^*T'U, T' \oplus T', U \oplus U)$$

$$= ind(B, B', U \oplus U),$$

where the last equality follows from the fact that $B - (T \oplus U^*T'U)$ is an element in $C^*(X, H_{n,0} \oplus H_{n,0})$, and $B' - (T' \oplus T')$ is an element in $C^*(X, H_{n,1} \oplus H_{n,1})$. The invertibility of B' implies that

$$ind(T, T', U) = index(B),$$

where $index(B)$ is the Fredholm index of $B|_{H_+}$ from the positive eigenspace H_+ of the grading operator $\epsilon_0 \oplus -\epsilon_0$ to the negative eigenspace H_- of $\epsilon_0 \oplus -\epsilon_0$.

An easy computation shows that

$$G_n = \begin{pmatrix} D_{X\times\mathbb{C}^{l_n}}\epsilon_0 & \sqrt{I - D^2_{X\times\mathbb{C}^{l_n}}}\,\beta_n \\ \sqrt{I - D^2_{V_n}}\,\beta_n^* & D_{V_n}\epsilon_1 \end{pmatrix}.$$

Let V be the unitary operator of degree 0 from $H_{n,0} \oplus H_{n,0}$ to $H_{n,0} \oplus H_{n,1}$ defined by $V = 1 \oplus U$. We have

$$VBV^* = \begin{pmatrix} D_{X\times\mathbb{C}^{l_n}} & \epsilon_0\beta_n\sqrt{I - D^2_{X\times\mathbb{C}^{l_n}}} \\ \epsilon_1\sqrt{I - D^2_{V_n}}\,\beta_n^* & D_{V_n} \end{pmatrix}.$$

This implies that

$$(\epsilon_0 \oplus \epsilon_1)G_n = VBV^* \ mod\ K.$$

Hence we have

$$index(G_n) = index(B).$$

Now our lemma follows from (1), (2) and the above identity. ■

5 The odd dimensional case

In this section we shall briefly discuss an odd dimensional analogue of Theorem 4.1.

Definition 5.1 *A sequence of elements in $K_1(C_0(X))$, the compact supported K_1-group of X, is said to have small variation if it can be represented by a sequence of unitaries u_n ($n \in \mathbb{Z}_+$) in $M_{k_n}(C_0(X))^+$ for some positive integer k_n such that for every $r > 0$*

$$\lim_n sup_{(x,y)\in X\times X: d(x,y)\le r} ||u_n(x) - u_n(y)|| = 0,$$

where $C_0(X)$ is the algebra of all continuous functions vanishing at infinity over X, $M_{k_n}(C_0(X))$ is the the algebra of all $k_n \times k_n$ matrices over $C_0(X)$, and $M_{k_n}(C_0(X))^+$ is obtained from $M_{k_n}(C_0(X))$ by adjoining an identity.

We shall associate a sequence of elements with small variation in $K_1(C_0(X))$ to an asymptotic Fredholm module over $SC^*(X)$, the suspension C^*-algebra over $C^*(X)$. Recall that for any C^*-algebra A, SA is defined to be the C^*-algebra $\{f \in C([0,1],A) : f(0) = f(1) = 0\}$. For each n, define a projection P_n in $M_{2k_n}(SC_0(X))^+$ by: $P_n = w_t(u_n)(I \oplus 0)(w_t(u_n))^{-1}$, where $w_t(u) = (u \oplus I)v_t(u^{-1} \oplus I)v_t^{-1}$, and

$$v_t = \begin{pmatrix} cos(\pi t/2) & -sin(\pi t/2) \\ sin(\pi t/2) & cos(\pi t/2) \end{pmatrix}.$$

P_n is a sequence of projections with small variation. The method of Section 3 can be then used to construct an asymptotic Fredholm module (E_n, ϕ_n, F_n) (associated to P_n) over $SC^*(X)$. (E_n, ϕ_n, F_n) can be used to construct a pairing with $K_1(C^*(X))$ as follows. For any unitary u in $M_k(C_0(X))^+$ representing an element $[u] \in K_1(C^*(X))$, there is an associated projection $p(u)$ in $M_k(SC^*(X))^+$ defined by: $p(u) = w_t(u)(I \oplus 0)(w_t(u))^{-1}$. The pairing $< (E_n, \phi_n, F_n), [u] >$ is defined to be the pairing of the asymptotic Fredholm module (E_n, ϕ, F_n) with $p(u)$ for large n as in Section 2.

Theorem 5.2 *Let X be a non-compact proper metric space with bounded geometry and let $[D]$ be a K-homology class in $K_1(X) = KK^1(C_0(X), \mathbb{C})$. If u_n is a sequence of elements with small variation in $K_1(C_0(X))$ and (E_n, ϕ_n, F_n) is its associated asymptotic Fredholm module, then*

$$< (E_n, \phi_n, F_n), Index[D] > = < [D], [u_n] >$$

for all $n > N$, where N is some large integer, and $< [D], [u_n] >$ is the pairing between the K-homology class $[D]$ and the K-theory class $[u_n]$.

The proof of theorem 5.2 is similar to that of Theorem 4.1 and is therefore omitted.

6 Applications

In this section we shall apply our main results to study the positive scalar curvature problem and the spectrum of Laplacian. For simplicity we shall concentrate on the even dimensional case, i.e. the case of vector bundles with small variation. The odd dimensional analogues can also be proved using Theorem 5.2.

We shall first introduce a concept of large scale equivalence.

Definition 6.1 *([11]) Let d_1 and d_2 be two metrics on a space X. d_1 is said to be large scale equivalent to d_2 if for any $r > 0$ there exists $R > 0$ such that*
(1) if $d_1(x, y) \leq r$ for any pair of points x and y in X, then $d_2(x, y) \leq R$;
(2) if $d_2(x, y) \leq r$ for any pair of points x and y in X, then $d_1(x, y) \leq R$;

Notice that the concept of a sequence of vector bundles with small variation is invariant under large scale equivalence of metrics.

The following result follows from Corollary 4.2, the invariance of vector bundles with small variation under large scale equivalence and a standard Lichnerowicz type argument.

Theorem 6.2 *Let M be a spin complete Riemannian manifold with bounded geometry. If there is a sequence of vector bundles V_n on M with small variation such that $< \hat{A}(M)ch(V_n), [M] > \neq 0$ for all n, then there is no complete Riemannian metric on M which is large scale equivalent to the given metric and has uniform positive scalar curvature.*

The above result indicates that the existence of vectors bundles of small variation has stronger geometric implication than the existence of almost flat bundles since the concept of almost flat bundles does not seem to be invariant under large scale equivalence.

Theorem 6.3 *Let M be a complete oriented Riemannian manifold with bounded geometry. If there is a sequence of vector bundles V_n on M with small variation such that $< L(M)ch(V_n), [M] > \neq 0$ for all n, then 0 belongs to the spectrum of Laplacian acting on the space of L^2-forms.*

The above result follows from Theorem 4.1.

An idea of Gromov and Lawson in [13] can be used to construct vector bundles of small variation over a proper metric space X as follows. Let f_n be a sequence of continuous maps from X to a compact metric space Y such that

(1) for any $\epsilon > 0, r > 0$, there exists N such that $d(f_n(x), f_n(x')) < \epsilon$ for all $n > N$ and $d(x, x') < r$;

(2) f_n is constant outside a compact subset K_n of X.

If V is a vector bundle over Y, then $f_n^* V$ is a sequence of vector bundles of small variation over X. The most interesting case is perhaps when $Y = S^k$, the standard sphere of dimension k.

Recall that X is said to be uniformly contractible if for any $r > 0$, there exists $R > r$ such that every ball $B(x, r)$ can be contracted to a point within $B(x, R)$.

Question 6.4 *If X is a uniformly contractible Riemannian manifold with bounded geoemrty, do there exist a sequence of continuous maps f_n from X to $S^{dim(M)}$ satisfying the above conditions (1) and (2) such that each f_n has nonzero degree?*

A positive answer to the above question would imply Gromov and Lawson's conjecture that no compact $K(\pi, 1)$ manifold admits a metric with positive scalar curvature and Gromov's conjecture that the Laplacian acting on the space of L^2-forms on a uniformly contractible Riemannian manifold with bounded geometry contains 0 in its spectrum.

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