Wild ramification and a vanishing cycles formula

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Abstract

In this paper we prove an explicit formula which compares the dimensions of the spaces of vanishing cycles in a Galois cover of degree p between formal germs of curves over a complete discrete valuation ring of inequal characteristics $(0,p)$. This formula can be easily generalised to the case of a Galois cover with group which is nilpotent or which has a normal p-sylow subgroup. The results of this paper play a key role in [Sa-1] where is studied the semi-stable reduction of Galois covers of degree p above semi-stable curves over a complete discrete valuation ring of inequal characteristics $(0,p)$, as well as the Galois action on these covers.

0. Introduction. Let R be a complete discrete valuation ring of inequal characteristics, with uniformiser π, fraction field K, and algebraically closed residue field k of characteristic p. In this paper we investigate Galois covers of degree p between formal germs of R-curves at closed points. Our main result is the following formula which compares the dimensions of the spaces of vanishing cycles at the corresponding closed points. More precisely we have the following:

Theorem (2.4). *Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, with $\mathcal{X}_k$ reduced. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ normal and local. Assume that the special fibre $\mathcal{Y}_k := \mathcal{Y} \times_R k$ of $\mathcal{Y}$ is reduced. Let $\{\wp_i\}_{i\in I}$ be the minimal prime ideals of $\hat{\mathcal{O}}_x$ which contain π and which correspond to the branches $(\eta_i)_{i\in I}$ of the special fibre $\mathcal{X}_k := \mathcal{X} \times_R k$ of $\mathcal{X}$ at x, and let $\mathcal{X}_i := \operatorname{Spf} \hat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. For each $i \in I$, the above cover f induces a torsor $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ under a finite and flat R-group scheme G_i of rank p, above the boundary $\mathcal{X}_i$. For each $i \in I$, let $(G_{k,i}, m_i, h_i)$ be the reduction type of f_i as defined in [Sa] 3.2. Let y be the closed point of $\mathcal{Y}$. Then one has the following* **"local Riemman-Hurwitz formula"***:*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

Where g_y (resp. g_x) denotes the genus of the singularity at y (resp. x), d_η is the degree *of the divisor of ramification in the morphism $f_\eta : \mathcal{Y}_\eta \to \mathcal{X}_\eta$ induced by f on the generic fibre, and $d_s := \sum_{i\in I^{\mathrm{rad}}}(m_i-1)(p-1)+\sum_{i\in I^{\mathrm{et}}}(m_i-1)(p-1)$, where I^{rad} is the subset of I consisting of those i for which $G_{k,i}$ is radicial, and I^{et} is the subset of I consisting of those i for which $G_{k,i}$ is étale and $m_i \neq 0$. Hier $G_{i,k}$ denotes the special fibre of the group scheme G_i.*

In particular the genus g_y of y depends only on the genus g_x of x, the ramification datas on the generic fibre in the above morphism $f : \mathcal{Y} \to \mathcal{X}$, and its degeneration type on the boundaries of the formal fibre $\mathcal{X}$. The above formula can be easily extended to the case of a Galois cover with group G which is nilpotent, or a group which has a normal p-Sylow subgroup. Our method to prove such a formula is to construct, using formal patching techniques à la Harbater, a compactification $\tilde{f} : Y \to X$ of the above cover $f : \mathcal{Y} \to \mathcal{X}$ (cf. 2.3.2). The formula follows then by comparing the genus of the special and the generic fibres of Y in this compactification.

As an application, and using the above formula one can obtain interesting results in the case where $\mathcal{X}$ is the formal germ of a semi-stable R-curve (cf. 3.1 and 3.2), in particular one can predict in this case if $\mathcal{Y}$ is semi-stable or not. We give several examples which illustrate this situation namely the case of Galois covers of degree p between formal germs of semi-stable R-curves (cf. 3.1.3, 3.1.4 and 3.2.4). In particular one can classify étale Galois covers of degree p between annuli (cf. 3.2.5). The above results play a key role in [Sa-1] in order to exhibit and realise the "degeneration datas" associated to Galois covers of degree p above a proper semi-stable R-curve.

I. Formal and rigid patching.

In what follows we explain the procedure which allows to construct (Galois)-covers of curves in the setting of formal or rigid geometry by glueing together covers of formal affine or affinoid rigid curves with covers of formal fibres at closed points of the special fibre. We refer the reader to the exposition in [Pr] for a discussion of patching results and for detailled references on the subject.

Let R be a complete discrete valuation ring with fractions field K, residue field k, and uniformiser π. Let X be an admissible formal R-scheme which is an R-**curve**, by which we mean that the special fibre $X_k := X\times_R k$ is a reduced one dimensional k-scheme of finite type. Let Z be a finite set of closed points of X_k. For a point $x \in Z$, let $X_x := \operatorname{Spf}\hat{O}_{X,x}$ be the formal completion of X at x, which is the **formal fibre** at the point x. Also let X' be a formal open subset of X whose special fibre is $X_k - Z$. For each point $x \in Z$, let $\{\wp_i\}_{i=1}^n$ be the set of minimal prime ideals of $\hat{\mathcal{O}}_{X,x}$ which contain π, they correspond to the *branches* $(\eta_i)_{i=1}^n$ of the completion of X_k at x, and let $X_{x,i} := \operatorname{Spf}\hat{\mathcal{O}}_{x,\wp_i}$ be the formal completion of the localisation of X_x at $\wp_i$. The ring $\hat{\mathcal{O}}_{x,\wp_i}$ is a complete discrete valuation ring. The set $\{X_{x,i}\}_{i=1}^n$ is the set of **boundaries** of the formal fibre X_x. For each $i \in \{1, n\}$ we have a canonical morphism $X_{x,i} \to X_x$.

1.1. Definition. With the same notations as above a *(G)-cover patching data* for the pair (X, Z) consists of the following:

a) A finite (Galois) cover $Y' \to X'$ (with group G).

b) For each point $x \in Z$, a finite (Galois) cover $Y_x \to X_x$ (with group G).

The above datas a) and b) must satisfy to the following condition:

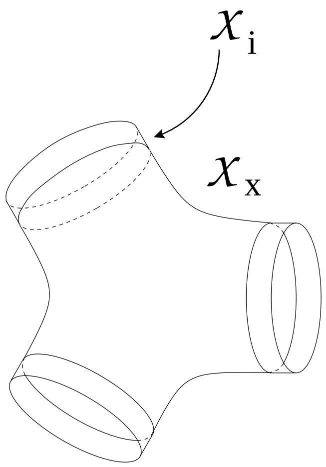
c) If $\{X_{x,i}\}_{i=1}^n$ are the boundaries of the formal fibre at the point x, then for each $i \in \{1, n\}$ is given a (G-equivariant) isomorphism $\sigma_i : Y_x \times_{X_x} X_{x,i} \simeq Y' \times_{X'} X_{x,i}$.

1.2. Proposition / Formal patching. *Given a (G)-cover patching data as in 1.1 there exists a unique, up to isomorphism, (Galois) cover $Y \to X$ (with group G) which induces the above (G)-covers in a) (resp. in b) when restricted to X' (resp. when pulled back to X_x for each point $x \in Z$).*

The proof of 1.2 is an easy consequence of theorem 3.4. in [Pr], and which is due to Ferrand and Raynaud.

1.3. Remark. With the same notations as above let $x \in Z$ and let $\tilde{X}_k$ be the normalisation of X_k. There is a one to one correspondance between the set of points of $\tilde{X}_k$ above x and the set of boundaries of the formal fibre at the point x. Let x_i be the point of $\tilde{X}_k$ above x which corresponds to the boundary $X_{x,i}$, for $i \in \{1, n\}$. Assume that the point $x \in X_k(k)$ is rational. Then the completion of $\tilde{X}_k$ at x_i is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ in one variable over k, where t_i is a local parameter at x_i. The complete local ring $\hat{\mathcal{O}}_{x,\wp_i}$ is a *two dimensional discrete valuation ring* whose residue field is isomorphic to $k((t_i))$. Let T_i be an element of $\hat{\mathcal{O}}_{x,\wp_i}$ which lifts t_i, such an element is called a *parameter* of $\hat{\mathcal{O}}_{x,\wp_i}$. Then it follows from [Bo] that there exists an isomorphism $\hat{\mathcal{O}}_{x,\wp_i} \simeq R[[T_i]]\{T_i^{-1}\}$, where $R[[T_i]]\{T_i^{-1}\}$ is the ring of formal power series $\sum_{i \in \mathbb{Z}} a_i T^i$ with $\lim_{i \to -\infty} |a_i| = 0$, and where $|\ |$ is an absolute value of K associated to its valuation.

1.4. Rigid patching. The analog of the above result is well known in rigid geometry, which is not surprising because of the link between rigid and formal geometry (cf. [Ra], [Bo-Lu]). We explain briefly the patching procedure in this context locally. Let $\mathcal{X} := \operatorname{Spm} A$ be an affinoid reduced curve, and let X be a formal model of $\mathcal{X}$ with special fibre X_k. Let x be a closed point of X_k, and let $\mathcal{X}_x$ be the *formal fibre* of $\mathcal{X}$ at x, which is a non quasi-compact rigid space and which consists of the set of points of $\mathcal{X}$ which reduce to the point x. The structure of the *boundary* of $\mathcal{X}_x$ is well known and depends functorially on the normalisation of the complete local ring $\hat{\mathcal{O}}_{X_k,x}$ (cf. [Bo-Lu-1]). Namely this boundary decomposes into a disjoint union of *semi-open annuli* $\mathcal{X}_i := \mathcal{X}_{x,i}$ one corresponding to each minimal prime ideal η_i of $\hat{\mathcal{O}}_{X_k,x}$. Let $\mathcal{X}' := \mathcal{X} - \mathcal{X}_x$ which is a quasi-compact rigid space. Let $f' : \mathcal{Y}' \to \mathcal{X}'$ and $f_x : \mathcal{Y}_x \to \mathcal{X}_x$ be (Galois)-covers (with group G). The (G)-cover $f' : \mathcal{Y}' \to \mathcal{X}'$ extends to (G)-covers $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ of each of the components of the boundary of $\mathcal{X}_x$, and the germ of such an extension is unique (cf. [Ra]). A (G)-patching data in this context are (G-equivariant) isomorphisms between the germs of $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ and the restriction of the initial (G)-cover $f_x : \mathcal{Y}_x \to \mathcal{X}_x$ to $\mathcal{X}_i$. The rigid patching result is that given a (G)-patching data as above then there exists a unique, up to isomorphism, (G)-cover $f : \mathcal{Y} \to \mathcal{X}$ which induces the above covers above $\mathcal{X}'$ and $\mathcal{X}_x$ when restricted to this analytic subspaces.



1.5. Local-global principle. As a direct consequence of the above patching results one obtains a *local-global principle*, which is certainly well known to the experts, for lifting of covers of curves. More precisely we have the following:

1.6. Proposition. *let X be a proper and flat algebraic (or formal) R-curve and let $Z := \{x_i\}_{i=1}^n$ be a finite set of closed points of X. Let $f_k : Y_k \to X_k$ be a finite generically separable (Galois)-cover (of group G) whose branch locus is contained in Z. Assume that for each $i \in \{1, n\}$ there exists a (Galois)-cover $f_i : Y_i \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x_i}$ (of group G) which lifts the cover $\hat{Y}_{k,i} \to \operatorname{Spec} \hat{\mathcal{O}}_{X_k,x_i}$ induced by f_k, where $\hat{Y}_{k,i}$ denotes the completion of Y_k above x_i. Then there exists a unique, up to isomorphism, (Galois)-cover $f : Y \to X$ (of group G) which lifts the cover f_k, and which is isomorphic to the cover f_i when pulled back to $\operatorname{Spf} \hat{\mathcal{O}}_{X,x_i}$, for each $i \in \{1, n\}$.*

Proof. After passing to the formal completion of X along its special fibre we may reduced to the case where $\mathcal{X}$ is a formal R-curve. We treat the case where $Z = \{x\}$ consists of one point (the general case is similar). Let $U_k := \mathcal{X}_k - \{x\}$, and let $\mathcal{U}$ be a formal open in $\mathcal{X}$ whose special fibre equals U_k. The étale cover $f'_k : V_k \to U_k$ induced by f_k above

U_k can be lifted, uniquely, by the theorems of lifting of étale covers (cf. [Gr]) to an étale formal cover $f' : \mathcal{V} \to \mathcal{U}$. Let $\{\wp_i\}_{i=1}^n$ be the minimal prime ideals of $\hat{\mathcal{O}}_{X,x}$ which contain π, and let $\mathcal{X}_i := \mathcal{X}_{x,i} := \operatorname{Spf} \hat{\mathcal{O}}_{x,\wp_i}$ be the formal completion of the localisation of $\hat{\mathcal{O}}_{X,x}$ at $\wp_i$. We have canonical morphisms $\mathcal{X}_i \to \mathcal{X}$, and $\mathcal{X}_i \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$. The cover f' (resp the given cover $f_i : Y_i \to \operatorname{Spec} \hat{\mathcal{O}}_{X,x_i}$) induces (by pull back) a cover $f'_i : \mathcal{Y}_i \to \mathcal{X}_i$ (resp. a cover $f_i : \mathcal{Y}_i \to \mathcal{X}_i$). For each $i \in \{1, n\}$, the cover f_i and f'_i by construction are isomorphic when restricted to the special fibre $\operatorname{Spec} k((t_i))$ of $\mathcal{X}_i$. Since both f_i and f'_i are étale and $\mathcal{X}_i$ is local and complete we deduce that they are isomorphic. Hence we obtain a patching data which allows us to patch the covers $f' : \mathcal{V} \to \mathcal{U}$ and $f_i : Y_i \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x_i}$ in order to obtain a cover $f : \mathcal{Y} \to \mathcal{X}$ with the required properties. Now thanks to the formal GAGA theorem this cover is algebraic $f : Y \to X$ and has the desired properties. Moreover if the starting datas are Galois then the constructed cover is also Galois with the same Galois group.

1.7. Remark. Although the formal patching result 1.2 and the rigid patching result 1.4 are equivalent, we opted in this paper to use the formal patching result and the framework of formal geometry since this seems to be more convenient for most readers. However it should be clear that one could also use the framework of rigid geometry and adapt the content of this paper to this setting.

II. Computation of vanishing cycles.

The main result of this section is 2.4 which gives a **formula** which compares the dimensions of the space of vanishing cycles in a Galois cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ of group $\mathbb{Z}/p\mathbb{Z}$ between formal germs of R-curves, where R is a complete discrete valuation ring of inequal characteristic which contains a primitive p-th root of unity, where p is the residue characteristic, in terms of the degeneration type of $\tilde{f}$ above the boundaries of $\mathcal{X}$.

2.1. In this section we consider a complete discrete valuation ring R of inequal characteristic, with residue characteristic $p > 0$, and which contains a primitive p-th root of unity ζ. We denote by K the fraction field of R, π a uniformising parameter of R, k the residue field of R, and $\lambda := \zeta - 1$. We also denote by v_K the valuation of K which is normalised by $v_K(\pi) = 1$. **We assume that the residue field k is algebraically closed**. By a (formal) R-curve we mean a (formal) R-scheme of finite type which is normal, flat, and whose fibres have dimension 1. For an R-scheme X, we denote by $X_K := X \times_{\operatorname{Spec} R} \operatorname{Spec} K$ the *generic* fibre of X, and $X_k := X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ its *special* fibre. In what follows by a *(formal) germ* $\mathcal{X}$ of an R-curve we mean that $\mathcal{X} := \operatorname{Spec} \mathcal{O}_{X,x}$ is the (resp. $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ is the formal completion of the) spectrum of the local ring of an R-curve X at a closed point x. Let $\mathcal{O}_x$ be the local ring of $\mathcal{X}_k$ at x. Let $\delta_x := \dim_k \tilde{\mathcal{O}}_x/\mathcal{O}_x$ where $\tilde{\mathcal{O}}_x$ is the normalisation of $\mathcal{O}_x$ in its total ring of fractions, and let r_x be the number of maximal ideals in $\tilde{\mathcal{O}}_x$. The **contribution to the arithmetic genus** of the point x is by definition $g_x := \delta_x - r_x + 1$. We will call the integer g_x the **genus** of the point x. The following lemma is easy to prove (cf. for example [Bo-Lu-1]).

2.1.1. Lemma. *Let X_k be a proper reduced algebraic curve over k. Let $\tilde{X}_k \to X_k$ be the normalisation of X_k, and let $\{X_i\}_{i\in I}$ be the irreducible components of $\tilde{X}_k$. Let $\{x_j\}_{j\in J}$ be the singular point of X_k, which we assume to be rational. Let $g(X_k)$ (resp. $g(X_i)$) be the arithmetic genus of X_k (resp. the arithmetic genus of X_i). Then $g(X_k) = \sum_{i\in I} g(X_i) + \sum_{j\in J} g_{x_j}$.*

2.2. Let $f : Y \to X$ be a finite cover between R-curves. Assume that the special fibres X_k and Y_k are reduced. Let y be a closed point of Y and let x be its image in X, which we assume to be a rational point. Let $(x_j)_{j\in J}$ be the points of the normalisation $\tilde{X}_k$ of X_k above x, and for a fixed j let $(y_{i,j})_{i\in I_j}$ be the points of the normalisation $\tilde{Y}_k$ of Y_k which are above x_j. Assume that the morphism $f_k : Y_k \to X_k$ is **generically étale**. Under this assumption we have the following *local Riemann-Hurwitz formula* which is due to Kato (cf. [Ka], and [Ma-Yo]):

$$(1)\ (g_y + \delta_y - 1) = n(g_x + \delta_x - 1) + d_K - d_k^w$$

Where n is the local degree at y, which is the degree of the morphism $\operatorname{Spec} \hat{\mathcal{O}}_{Y,y} \to \operatorname{Spec} \hat{\mathcal{O}}_{X,x}$ between the completion of the local rings of Y (resp X) at the point y (resp. x), d_K is the degree of the divisor of ramification in the morphism $\operatorname{Spec}(\hat{\mathcal{O}}_{Y,y} \otimes_R K) \to \operatorname{Spec}(\hat{\mathcal{O}}_{X,x} \otimes_R K)$. Let $d_{i,j}^w := v_{x_j}(\delta_{y_{i,j},x_j}) - e_{i,j} + 1$, where $\delta_{y_{i,j},x_j}$ is the discriminant ideal of the extension $\hat{\mathcal{O}}_{\tilde{X}_k,x_j} \to \hat{\mathcal{O}}_{\tilde{Y}_k,y_{i,j}}$ of complete discrete valuation ring, and $e_{i,j}$ its ramification index. The integer d_k^w is equal to the sum $\sum_{i,j} d_{i,j}^w$. In 2.4 we will obtain a formula similair to (1) in the case where f is Galois of group $\mathbb{Z}/p\mathbb{Z}$ and which includes the case where f_k is generically radicial.

2.3. Compactification process. Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve at a closed point x. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ local. We assume that the special fibre of $\mathcal{Y}_k$ is reduced (this can alaways be achieved after a finite extension of R). We will construct a compactification of the above cover $\tilde{f}$ which will allows us to compute the arithmetic genus of the closed point of $\mathcal{Y}$. More precisely we will construct a Galois cover $f : Y \to X$ of degree p, between proper algebraic R-curves, a closed point $y \in Y$ and its image $x = f(y)$, such that the formal germ of X (resp. Y) at x (resp. at y) equals $\mathcal{X}$ (resp. $\mathcal{Y}$), and such that the Galois cover $f_x : \operatorname{Spf} \hat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the above given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The construction of such a compactification is well known in the case where $\tilde{f}$ has étale reduction type on the boundaries (cf [Ma-Yo] and [Ra-2]). In the case of radicial reduction type of degree p on the boundaries one is able to carry out such a construction using the formal patching result in 1.2 and the result 3.1 in [Sa]. In fact it suffices to be able to treat the case of one boundary, which is easily done using the next proposition and the formal patching result.

2.3.1. Proposition. *Let $D := \operatorname{Spf} R < 1/T >$ be the formal closed disc centered at ∞. Let $\mathcal{D} := \operatorname{Spf} R[[T]]\{T^{-1}\}$, and let $\mathcal{D} \to D$ be the canonical morphism. Let $\tilde{f} : \mathcal{Y} \to \mathcal{D}$ be a non trivial torsor under a finite and flat R-group scheme of rank p, such that the special fibre of $\mathcal{Y}$ is reduced. Then there exists a Galois cover $f : Y \to D$ with group $\mathbb{Z}/p\mathbb{Z}$ whose pull back to $\mathcal{D}$ is isomorphic to the above given torsor $\tilde{f}$. More precisely, with the same notations as in [Sa] 3.1 we have the following depending on the several cases that occur there:*

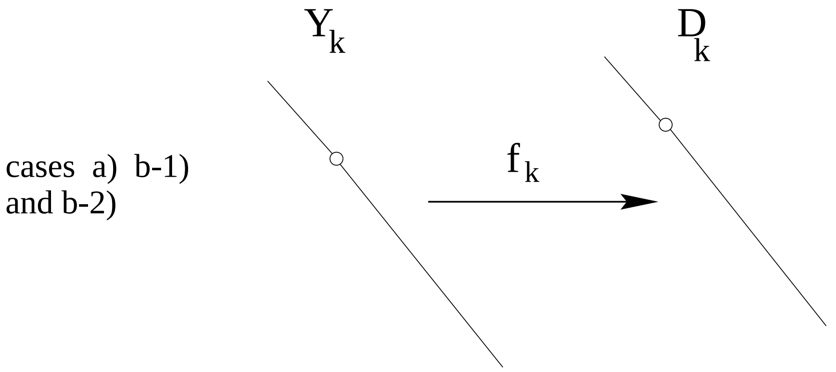
case a) consider the cover $f : Y \to D$ given generically by an equation $Z^p = \lambda^p T^m + 1$. This cover is an étale torsor above D under the group scheme $\mathcal{H}_{v_K(\lambda)}$ and induces an étale torsor $f_k : Y_k \to D_k$ in reduction. Moreover the genus of the smooth compactification of Y_k equals $(m-1)(p-1)/2$.

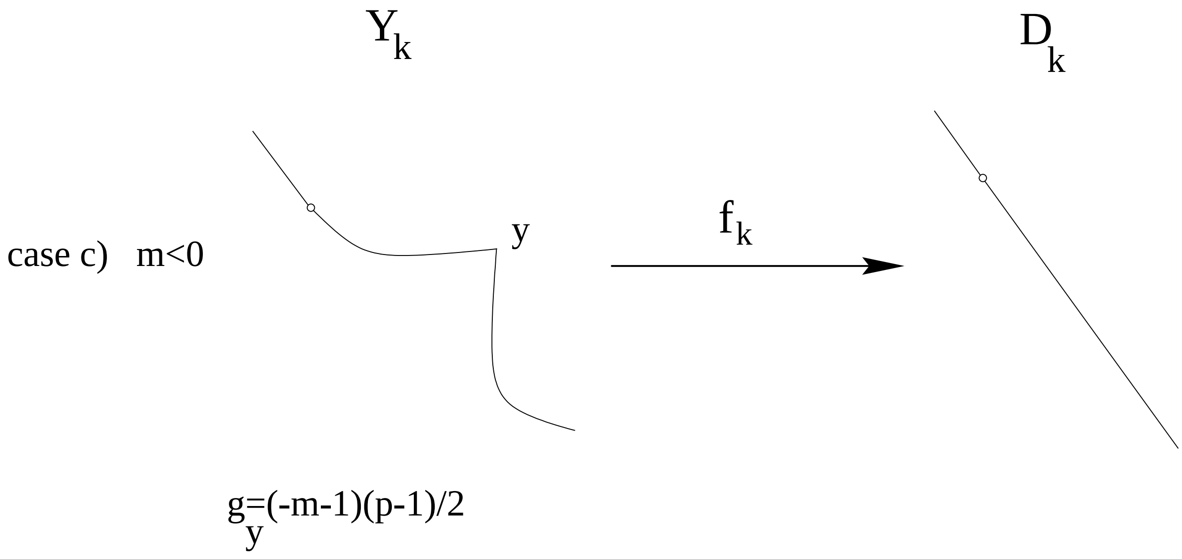
case b-1) Consider the cover $f : Y \to D$ given generically by an equation $Z^p = T^{-r}$ where $r := p - h$. This cover is ramified at the generic fibre only above the point at infinity, the finite morphism $f_k : Y_k \to X_k$ is a μ_p-torsor outside infinity, and Y_k is smooth at infinity. Moreover the genus of the smooth compactification of Y_k equals 0.

case b-2) Consider the cover $f : Y \to D$ given generically by an equation $Z^p = T^{-\alpha}(T^{-m} + 1)$, where α is an integer such that $m + \alpha \equiv 0 \bmod (p)$. This cover is ramified at the generic fibre above infinity and above the distincts m-th root of unity. The finite morphism $f_k : Y_k \to D_k$ is a μ_p-torsor outside infinity and the distincts m-th root of unity, and Y_k is smooth. Moreover the genus of the smooth compactification of Y_k equals 0.

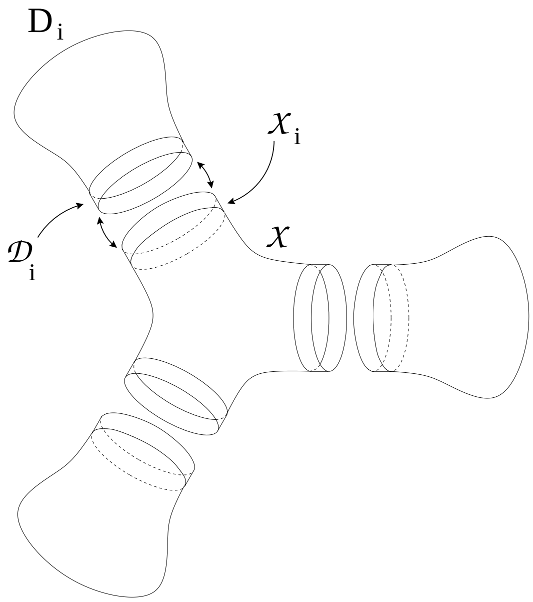
case c) First if $m \leq 0$ consider the cover $f : Y \to D$ given generically by an equation $Z^p = 1 + \pi^{np} T^m$. This cover is a torsor under the group scheme $\mathcal{H}_n$, and its special fibre $f_k : Y_k \to X_k$ is a torsor under α_p. The affine curve Y_k has a unique singular point y which is the point above infinity and $g_y = (-m-1)(p-1)/2$. Secondly if $m \geq 0$ consider the cover $f : Y \to D$ given generically by an equation $Z^p = T^{-\alpha}(T^{-m} + \pi^{pn})$ where α is as in b-2. This cover is ramified above infinity and the m distinct roots of π^{-np}. The finite morphism $f_k : Y_k \to X_k$ is an α_p-torsor outside infinity, and the special fibre Y_k of Y is smooth. Moreover in both cases the smooth compactification of the normalisation of Y_k has genus 0.

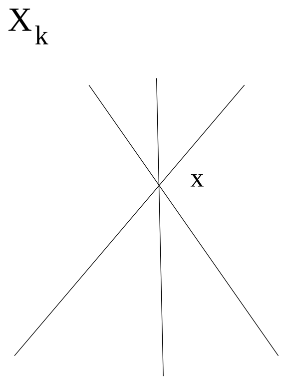
Proof. Case a is straitforward. In case b-1 one has only to justify that Y_k is smooth at infinity. the cover f is given by the equation $Z^p = T^{-r}$, and after using the Bezout identity one reduces to an equation $Z'^p = T^{-1}$ from which it follows directly that the complete local ring at the point above infinity is $B = R[[Z']]$ hence this point is smooth. In case b-2 one deduces in a similar way as above that Y_k is smooth. Case c: if m is negatif the α_p-torsor $f_k : Y_k \to X_k$ is given locally for the étale topology above the point at infinity by an equation $z^p = t^{-m}$ where $t := T \bmod (\pi)$. The computation of the arithmetic genus of the singularity above infinity follows then by a direct calculation (cf. example [Ra-1], or [Sai], 2.9). If m is positive and in order to see that Y_k is smooth one considers the Galois cover $f' : \mathcal{Y}' \to \mathcal{P}$ above the formal R-projective line $\mathcal{P}$, obtained by gluing D with the formal unit closed disc $D' := \operatorname{Spf} R < T >$ centered at 0, and given by the (same) equation $Z^p = 1 + \pi^{np} T^m$. The genus of the generic fibre of $\mathcal{Y}'$ is $(m-1)(p-1)/2$. The finite morphism $f'_k : \mathcal{Y}'_k \to \mathcal{P}_k$ is an α_p-torsor outside infinity, and outside 0 coincides with the morphism f_k. Above 0 this torsor is given locally for the étale topology by an equation $z^p = t^m$ where $t := T \bmod (\pi)$, hence the arithmetic genus above 0 equals $(m-1)(p-1)/2$ from which one deduces that Y_k is smooth.





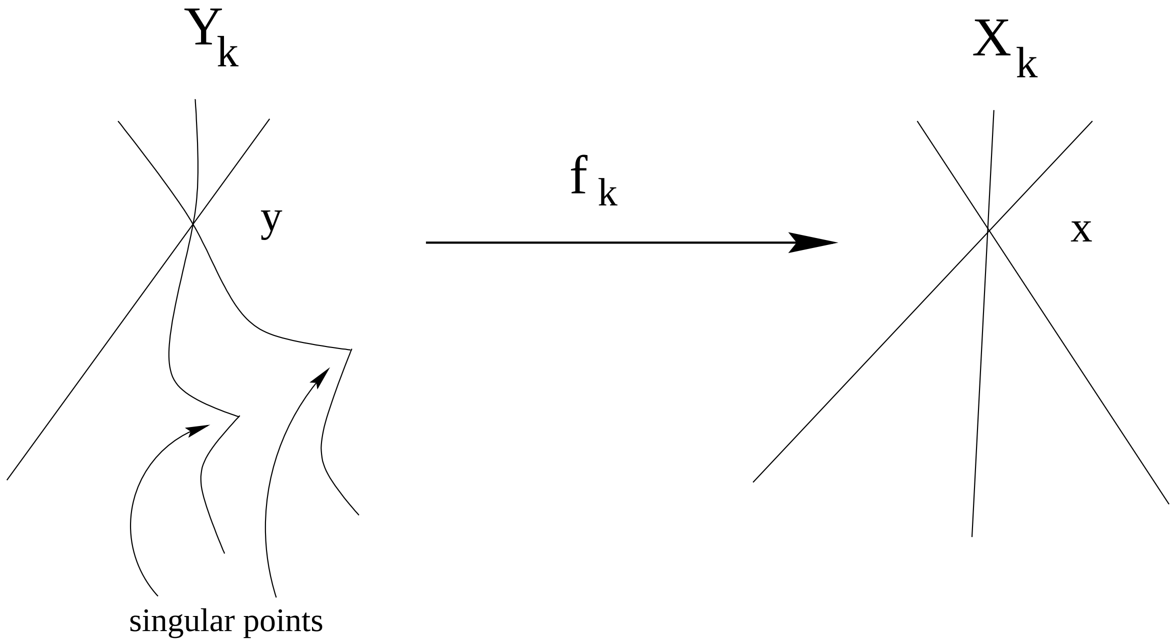
2.3.2. Proposition. *Let $\mathcal{X} := \operatorname{Spf} \hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed* *point x, and let $\{\mathcal{X}_i\}_i^n$ be the boundaries of $\mathcal{X}$. Let $\tilde{f}:\mathcal{Y}\to\mathcal{X}$ be a Galois cover of group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ local. Assume that $\mathcal{Y}_k$ and $\mathcal{X}_k$ are reduced. Then there exists a Galois cover $f:Y\to X$ of degree p, between proper algebraic R-curves, a closed point $y\in Y$ and its image $x=f(y)$, such that the formal germ of X (resp. Y) at x (resp. at y) equals $\mathcal{X}$ (resp. $\mathcal{Y}$), and such that the Galois cover $\operatorname{Spf}\hat{\mathcal{O}}_{Y,y}\to\operatorname{Spf}\hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the above given cover $\tilde{f}:\mathcal{Y}\to\mathcal{X}$. Moreover the formal completion of X along its special fibre has a covering which consists of n closed formal discs D_i which are patched with $\mathcal{X}$ along the boundaries $\mathcal{D}_i$, and the special fibre X_k of X consists of n smooth projective lines which intersect at the point x.*





n components

Proof. Let $\{\wp_i\}_{i=1}^n$ be the minimal prime ideals of $\hat{\mathcal{O}}_x$ which contain π and which correspond to the branches $(\eta_i)_{i=1}^n$ of $\mathcal{X}_k$ at x, and let $\mathcal{D}_i := \text{Spf}\, \hat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. If T_i is a lifting of a uniformising parameter of the branche (η_i) of $\mathcal{X}_k$ at x then $\hat{\mathcal{O}}_{\wp_i}$ is isomorphic to $R[[T_i]]\{T_i^{-1}\}$. For each $i \in \{1, n\}$ consider a formal closed disc $D_i := \text{Spf}\, R < 1/T_i >$ centered at infinity, and the canonical morphism $\mathcal{D}_i := \text{Spf}\, R[[T_i]]\{T_i^{-1}\} \to D_i$. As a consequence of the formal patching result, which is valid for coherent sheaves (cf. [Pr] theorem 3.4), one can patch $\mathcal{X} := \text{Spf}\, \hat{O}_x$ with the D_i, via the choice for each i of an automorphism of $\mathcal{D}_i$, in order to construct a proper formal R-curve X, and a closed point $x \in X$, such that the formal completion of X at x equals $\mathcal{X}$. The special fibre X_k of X is a union of n smooth k-projective lines which intersect at the point x. Now the given cover $\tilde{f}$ induces a torsor $f_i : \mathcal{Y}_i \to \mathcal{D}_i$, under a finite and flat R-group scheme of rank p for each i, and by the above lemma 3.3.1 one can find Galois covers $Y_i \to D_i$ of degree p which after pull back to $\mathcal{D}_i$ coincide with f_i, for each $i \in \{1, n\}$. The formal patching result again allows us then to patch these covers in order to construct a Galois cover $f : Y \to X$ of degree p with the desired properties. The formal R-curve is proper. By the formal GAGA theorems X is algebraic and the Galois cover $f : Y \to X$ is also algebraic.



The next result is the main one of this paper, it gives a **formula** which compares the dimensions of the space of vanishing cycles in a Galois cover of degree p between formal fibres.

2.4. Theorem. *Let $\mathcal{X} := \text{Spf}\, \hat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, with $\mathcal{X}_k$ reduced. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ local,* *and $\mathcal{Y}_k$ reduced. Let $\{\wp_i\}_{i\in I}$ be the minimal prime ideals of $\hat{\mathcal{O}}_x$ which contain π and which correspond to the branches $(\eta_i)_{i\in I}$ of $\mathcal{X}_k$ at x, and let $\mathcal{X}_i := \operatorname{Spf}\hat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. For each $i \in I$, the above cover $\tilde{f}$ induces a torsor $\tilde{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$ under a finite and flat R-group scheme of rank p, above the boundary $\mathcal{X}_i$. For each $i \in I$, let $(G_{k,i}, m_i, h_i)$ be the reduction type of $\tilde{f}_i$ (cf. [Sa] 3.2). Let y be the closed point of $\mathcal{Y}$. Then one has the following* **"local Riemman-Hurwitz formula"***:*

$$(2) \quad 2g_y - 2 = p(2g_x - 2) + d_\eta - d_s$$

Where d_η is the degree of the divisor of ramification in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathcal{X}_K$ induced by $\tilde{f}$, where $\mathcal{X}_K := \operatorname{Spec}(\hat{\mathcal{O}}_x \otimes_R K)$ and $\mathcal{Y}_K := \operatorname{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y} \otimes_R K)$, and $d_s := \sum_{i\in I^{\mathrm{rad}}}(m_i - 1)(p-1) + \sum_{i\in I^{\mathrm{et}}}(m_i - 1)(p-1)$, where I^{rad} is the subset of I consisting of those i for which $G_{k,i}$ is radicial, and I^{et} is the subset of I consisting of those i for which $G_{k,i}$ is étale and $m_i \neq 0$.

Proof. By 2.3.2 one can compactify the given morphism $\tilde{f}$. More precisely we constructed a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves, a closed point $y \in Y$ and its image $x = f(y)$, such that the formal germ of X (resp. Y) at x (resp. at y) equals $\mathcal{X}$ (resp. $\mathcal{Y}$), and such that the Galois cover $\operatorname{Spf}\hat{\mathcal{O}}_{Y,y} \to \operatorname{Spf}\hat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The special fibre of $\mathcal{X}$ consists (by construction) of n-ditincts smooth projective lines which meet at the closed point x. The formal completion of X along its special fibre has a covering which consists of $|I|$ formal closed unit discs which are patched with the formal fibre $\mathcal{X}$ along the boundaries $\mathcal{X}_i$. The above formula (2) follows then by comparing the arithmetic genus of the generic fibre Y_K of Y and the one of the special fibre Y_k. By the precise informations given in 3.3.1 one can easily deduce that $g(Y_K) = pg_x + (1-p) + d_\eta/2 + \sum_{i\in I_b}(-m_i + 1)(p-1)/2 + \sum_{i\in I_{c,>}}(-m_i + 1)(p-1)$ where I_b is the subset of I consisting of those i for which the degeneration data correspond to the one in case b of [Sa] 3.1, and $I_{c,>}$ is the subset of I consisting of those i for which the degeneration data correspond to the one in case c in [Sa] 3.1, and with m positif. On the other hand one has $g(Y_k) = g_y + \sum_{i\in I_{a,>}}(-m_i - 1)(p-1)/2 + \sum_{i\in I_{c,<}}(-m_i + 1)(p-1)$, where $I_{a,>}$ is the subset of I consisting of those i for which the degeneration data correspond to the one in case a of 2.4.1 with $m_i \neq 0$, and $I_{c,<}$ is the subset of I consisting of those i for which the degeneration data correspond to the one in case c in [Sa] 3.1, and with $m_i < 0$. Now since Y is flat $g(Y_K) = g(Y_k)$ from which directly follows the formula (2).

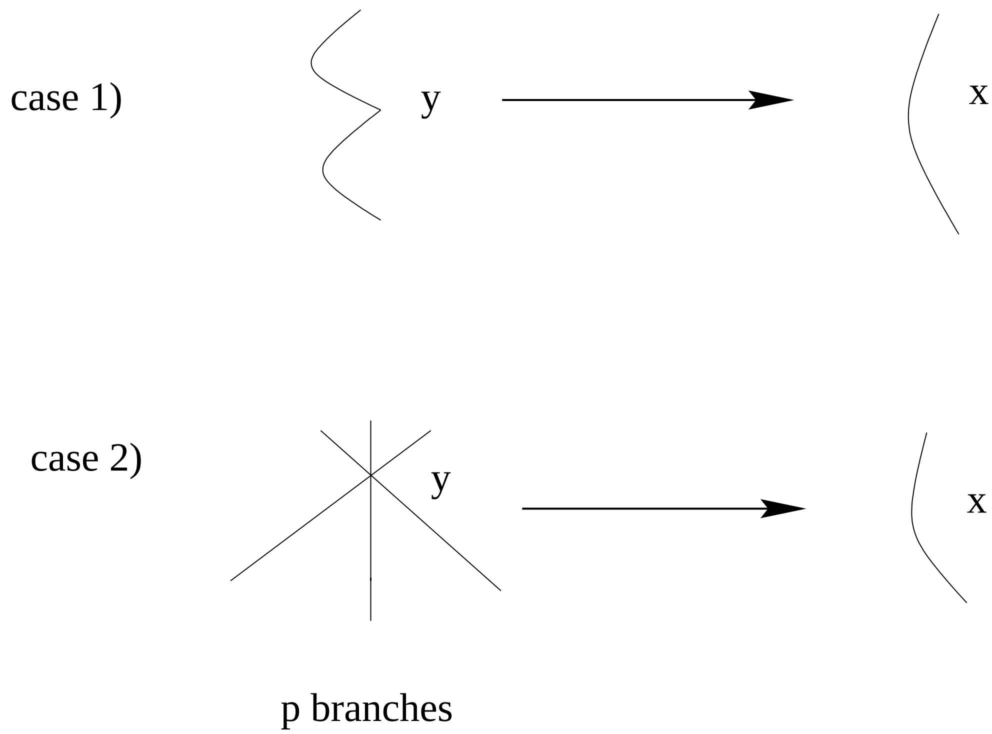
III. Galois covers of degree p above germs of semi-stable curves. In what follows we use the same notations as in 2.1. As a consequence of the above result 2.4 we will deduce some interesting results in this section in the case of a Galois cover $\mathcal{Y} \to \mathcal{X}$, where $\mathcal{X}$ is the formal germ of a semi-stable R-curve at a closed point. These results will play an important role in the paper [Sa-1] in order to exhibit, and realise, the degeneration datas which describe the semi-stable reduction of Galois covres of degree p.

3.1. We start with the case of a Galois cover of degree p above a germ of a **smooth** point.

3.1.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[T]]$ be the germ of a formal R-curve at a smooth point x. Let η be the generic point of the special fibre $\mathcal{X}_k$ of $\mathcal{X}$. The completion of the localisation of $\mathcal{X}$ at η is $\mathcal{X}_\eta := \operatorname{Spf} R[[T]]\{T^{-1}\}$, which is the boundary of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. Let y be the unique closed point of $\mathcal{Y}_k$. Let $\delta_K := r(p-1)$ be the degree of the divisor of ramification in the morphism $f : \mathcal{Y}_K \to \mathcal{X}_K$. We distinguish two cases:*

case 1) $\mathcal{Y}_k$ is **unibranche** *at y. Let (G_k, m, h) be the degeneration type of f above the boundary $\mathcal{X}_\eta$. Then necessarily $r - m - 1 \geq 0$, and $g_y = (r-m-1)(p-1)/2$.*

*case 2) $\mathcal{Y}_k$ has p-***branches** *at y. Then the cover f has an étale completely split reduction of type $(\mathbb{Z}/p\mathbb{Z}, 0, 0)$ on the boundary, i.e. the induced torsor above $\operatorname{Spf} R[[T]]\{T^{-1}\}$ is trivial, in which case $g_y = (r-2)(p-1)/2$.*



With the same notations as in 3.1.1, and as an immediate consequence, one can immediately see whether the point y is smooth or not. More precisely we have the following:

3.1.2. Corollary. *We use the same notations as in 3.1.1 Then y is a smooth point,* *which is equivalent to $g_y = 0$, if and only if $r = m+1$. This in the case of radicial reduction type on the boundary is equivalent to $r = -\operatorname{ord}(\omega)$, where ω is the associated differential form. In particular if the reduction is of multiplicative type on the boundary, i.e. $G_k = \mu_p$, then $g_y = 0$ only if $r = 1$ or $r = 0$, since $\operatorname{ord}(\omega) \geq -1$ in this case. Also if $r = 1$ and $g_y = 0$ then necessarily $G_k = \mu_p$.*

Next we give some examples of Galois covers of degree p above the formal germ of a smooth point which cover all the possibilities for the genus and the degeneration type on the boundary. Both in 3.1.3 and 3.1.4 we use the same notations as in 3.1.1. We first begin with examples with genus 0.

3.1.3. Examples. The following are examples given by explicit equations of the different cases, depending on the possible degeneration type, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$, and where $\mathbf{g_y} = \mathbf{0}$.

1) For $m > 0$ an integer prime to p, consider the cover given generically by the equation $X^p = 1 + \lambda^p T^{-m}$. Hier $r = m + 1$, and this cover has a reduction of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ on the boundary.

2) For $h \in \mathbb{F}_p^*$, consider the cover given generically by the equation $X^p = T^h$. Hier $r = 1$, and this cover has a reduction of type $(\mu_p, 0, h)$ on the boundary.

3) Consider the cover given generically by the equation $X^p = 1 + T$. Hier $r = 0$, and this cover has a reduction of type $(\mu_p, -1, 0)$ on the boudary.

4) For $n < v_K(\lambda)$, and $m < 0$, consider the cover given generically by the equation $X^p = 1 + \pi^{np} T^m$. Hier $r = -m + 1$, and this cover has a reduction of type $(\alpha_p, -m, 0)$ on the boudary.

5) For $n < v_K(\lambda)$, consider the cover given generically by the equation $X^p = 1 + \pi^{np} T$. Hier $r = 0$, and this cover has a reduction of type $(\alpha_p, -1, 0)$ on the boudary.

Next we give examples of Galois covers of degree p above formal germs of smooth points which lead to a singularity with positive genus.

3.1.4. Examples. The following are examples given by explicit equations of the different cases, depending on the possible reduction type, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$, and where $\mathbf{g_y} > \mathbf{0}$.

1) For $m > 0$ an integer prime to p, and $m' > m$, consider the cover given generically by the equation $X^p = 1 + \lambda^p (T^{-m} + \pi T^{-m'})$. Hier $r = m' + 1$, and this cover has a reduction of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ on the boundary. Moreover the genus g_y of the closed point y of $\mathcal{Y}$ equals $(m' - m)(p - 1)/2$.

2) For $h \in \mathbb{F}_p^*$, and $m > 0$ an integer prime to p, consider the cover given generically by the equation $X^p = T^{h'}(T^m + a)$, where h' is a positif integer such that $m + h' \equiv h \bmod (p)$, and $a \in \pi R$. Hier $r = m + 1$, and this cover has a reduction of type $(\mu_p, 0, h)$ on the

boundary. Moreover the genus g_y of the closed point y of $\mathcal{Y}$ equals $m(p-1)/2$.

3) For a positif integer m', an integer h such that $m'+h \equiv 0 \bmod (p)$, and $a \in \pi R$, consider the cover given generically by the equation $X^p = T^h(T^{m'}+a)(1+T^m)$. Hier $r = m'+1$, and this cover has a reduction of type $(\mu_p, -m, 0)$ on the boudary. Moreover the genus g_y of the closed point y of $\mathcal{Y}$ equals $(m'+m)(p-1)/2$.

4) For $n < v_K(\lambda)$, and integers $m > 0$ prime to p and $m < m'$, consider the cover given generically by the equation $X^p = 1+\pi^{np}(T^{-m}+\pi T^{-m'})$. Hier $r = m'+1$, and this cover has a reduction of type $(\alpha_p, -m, 0)$ on the boudary. Moreover the genus g_y of the closed point y of $\mathcal{Y}$ equals $(m'+m)(p-1)/2$.

Note that in cases 5, 6 of 3.1.3, and 4 of 3.1.4, and in order to realise these covers above $\mathcal{X}$, for a given n, one needs in general to perform a ramified extension of R.

3.2. Next we examine the case of Galois covers of degree p above formal germs at **double points**.

3.2.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[S,T]]/(ST-\pi^e)$ be the formal germ of an R-curve at an ordinary double point x of thikeness e. The special fibre of $\mathcal{X}$ consists of two irreducible components X_1 and X_2 with generic points η_1 and η_2, corresponding to the prime ideal (π,T) (resp. (π,S)) of $R[[S,T]]/(ST-\pi^e)$. The completion of the localisation of $\mathcal{X}$ at η_1 (resp. η_2) is isomorphic to $\mathcal{X}_1 := \operatorname{Spf} R[[S]]\{S^{-1}\}$ (resp. $\mathcal{X}_2 := \operatorname{Spf} R[[T]]\{T^{-1}\}$). These are the two boundaries of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. We assume that $\mathcal{Y}_k$ has* **two branches** *at the point y. Let $\delta_K := r(p-1)$ be the degree of the divisor of ramification in the morphism $f : Y_K \to X_K$. Let $(G_{k,i}, m_i, h_i)$ be the type of reduction on the two boundaries of $\mathcal{X}$, for $i = 1,2$. Then necessarily $r - m_1 - m_2 \geq 0$, and $g_y = (r-m_1-m_2)(p-1)/2$.*

3.2.2. Proposition. *We use the same notations as in 3.2.1. We consider the remaining cases:*
case 1) $\mathcal{Y}_k$ **has** $p+1$ **branches at** *y. We can assume that $\mathcal{Y}$ is completely split above η_1. Let $(G_{k,2}, m_2, h_2)$ be the reduction type on the second boundary of $\mathcal{X}$. Then necessarily $r - m_2 - 1 \geq 0$ and $g_y = (r-m_2-1)(p-1)/2$.*
case 2) $\mathcal{Y}_k$ **has** $2p$ **branches at** *y. In this case $\mathcal{Y}$ is completely split above η_1 and η_2 and*

$g_y = (r-2)(p-2)/2$.

With the same notations as in 3.2.1, and as an immediate consequence, one can recognise whether the point y is a double point or not. More precisely we have the following.

3.2.3. Corollary. *We use the same notations as in 3.2.1. Then y is an ordinary double point, which is equivalent to $g_y = 0$, if and only if x is an ordinary double point of thikeness divisible by p, and $r = m_1 + m_2$. Moreover if $g_y = 0$, if $r = 0$, and if $(G_{k,i}, m_i, h_i)$ is the reduction type on the boundary for $i = 1, 2$, then necessarily $h_1 + h_2 = 0$.*

Proof. We need only to justify the last assertion. If both h_1 and h_2 equals 0 there is nothing to prove. Otherwise assume $h_1 \neq 0$. Then $m_1 = m_2 = 0$ (because $g_y = 0 = m_1 + m_2$), and one sees easily that necessarily $G_{k,1} = G_{k,2} = \mu_p$. So the cover $f : \mathcal{Y} \to \mathcal{X}$ is in this case a μ_p-torsor and its reduction $f_k : \mathcal{Y}_k \to \mathcal{X}_k$ is also μ_p-torsor given by an equation $t^p = u$, and $\omega := du/u$ is the associated differential form. The restriction ω_i of ω to the i-th branch , for $i = 1, 2$, is the differential form associated to the μ_p-torsor $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ induced by f above the boundary $\mathcal{X}_i$ of $\mathcal{X}$. The equality $h_1 + h_2 = 0$ follows then from the fact that the sum of the residues of ω on each branch equals 0, which is a property of the regular differential form at a double point.

Next we give some examples of Galois covers of degree p, above the formal germ of a double point, which leads to singularities with genus 0, i.e. double points, and such that $r = 0$. These examples will be used in the paper [Sa-1] in order to realise the "degeneration datas".

3.2.4. Examples. The following are examples, given by explicit equations, of the different cases, depending on the possible type of reduction on the boundaries, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[S,T]]/(ST - \pi^e)$, with $\mathbf{r} = \mathbf{0}$, and where $\mathbf{g_y} = \mathbf{0}$, for a suitable choice of e and R. Note that $e = pt$ must be divisible by p. In all the following examples we have $r = 0$.

1) p-**Purity**: if f as above has an étale reduction type on the boundaries, and $r = 0$, then f is necessarily étale, and hence is completely split since $\mathcal{X}$ is strictly henselian.

2) Consider the cover given generically by an equation $X^p = T^h$, which leads to a reduction on the boundaries of type $(\mu_p, 0, h)$ and $(\mu_p, 0, -h)$.

3) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R choose t such that $tm < v_K(\lambda)$, and consider the cover given generically by an equation $X^p = 1 + S^m$, which leads to a reduction on the boundaries of type $(\mu_p, -m, 0)$ and $(\alpha_p, m, 0)$.

4) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R choose t such that $t = v(\lambda)/m$, and consider the cover given generically by an equation $X^p = \lambda^p/T^m + 1$, which leads to a reduction on the boundaries of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and

$(\mu_p, -m, 0)$.

5) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R, choose t such that $tm < v_K(\lambda)$, and consider the cover given generically by an equation $X^p = 1 + \lambda^p S^{-m}$, which leads to a reduction on the boundaries of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\alpha_p, -m, 0)$.

6) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R choose t and n such that $tm + n < v_K(\lambda)$, and consider the cover given generically by an equation $X^p = 1 + \pi^{np} S^m$, which leads to a reduction on the boundaries of type $(\alpha_p, -m, 0)$ and $(\alpha_p, m, 0)$.

In fact one can describe Galois covers of degree p above formal germs at double points, which are étale above the generic fibre, and with genus 0. Namely they are all of the form given in the examples 3.2.4. In particular these covers are uniquely determined, up to isomorphism, by their degeneration type on the boundaries. More precisely we have the following:

3.2.5. Proposition. *Let $\mathcal{X}$ be the formal germ of a semi-stable R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}_k$ reduced and local, and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$* **étale**. *Let $\mathcal{X}_i$, for $i = 1, 2$, be the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$, and let δ_i be the corresponding degree of the different (cf. 2.3.5). Let y be the closed point of $\mathcal{Y}$. Assume that $\mathbf{g_y} = \mathbf{0}$. Then there exists an isomorphism $\mathcal{X} \simeq \operatorname{Spf} R[[S, T]]/(ST - \pi^{tp})$ such that, if say $\mathcal{X}_2$ is the boundary corresponding to the prime ideal (π, S), the following holds:*

a) The cover f is generically given by an equation $X^p = T^h$, with $h \in \mathbb{F}_p^*$, which leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\mu_p, m = 0, h)$ and $(\mu_p, m = 0, -h)$. Hier $t > 0$ can be any integer. In this case $\delta_1 = \delta_2 = v_K(p)$.

b) The cover f is generically given by an equation $X^p = 1 + T^m$, for $m > 0$ an integer prime to p, such that $tm < v_K(\lambda)$. In particular in this case necessarily $t < v_K(\lambda)$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\alpha_p, m, 0)$ and $(\mu_p, -m, 0)$. In this case $\delta_2 = v_K(p) = \delta_1 + (p-1)tm$

c) The cover f is generically given by an equation $X^p = 1 + T^m$, with $m > 0$ an integer prime to p, such that $tm = v_K(\lambda)$. In particular in this case t divides $v_K(\lambda)$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\mu_p, -m, 0)$. In this case $\delta_2 = v_K(p) = \delta_1 + (p-1)tm$, and $\delta_1 = 0$.

d) The cover f is generically given by an equation $X^p = 1 + \lambda^p T^{-m}$, with $m > 0$ an integer prime to p such that $tm < v_K(\lambda)$, which leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\alpha_p, -m, 0)$. In particular we have necessarily $t < v_K(\lambda)$. In this case $\delta_1 = \delta_2 + (p-1)tm$, and $\delta_2 = 0$.

e) The cover f is generically given by an equation $X^p = 1 + \pi^{np} T^m$, for a positif *integer m prime to p, with $n < v_K(\lambda)$, and $n + tm < v_K(\lambda)$, which leads to a reduction on the boundaries of type $(\alpha_p, -m, 0)$ and $(\alpha_p, m, 0)$.* In particular we have necessarily $t < v_K(\lambda)$. In this case $\delta_2 = \delta_1 + (p-1)tm$, and $\delta_1 = (p-1)(v_K(\lambda) - (n+tm))$.

In all cases we have $\delta_2 - \delta_1 = mt(p-1)$.

Proof. We explain briefly the proof. Since $g_y = 0$, the thikness $e = pt$ at the double point x of $\mathcal{X}$ is divisible by p. Say $\mathcal{X} \simeq \operatorname{Spf} A'$, where $A' = R[[T', S']]/(S'T' - \pi^{tp})$. The μ_p-torsor $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ is given by an equation $X^p = u_K$ where, u_K is a unit on $\mathcal{X}_K$. Such a unit can be uniquely written as $u_K := \pi^n T'^m u$, where n and m are integers, and $u \in R[[T', S']]/(S'T' - \pi^{tp})$ is a unit. Note first that necessarily $n \equiv 0 \bmod(p)$, since $\mathcal{Y}_k$ is reduced. Assume first that $\gcd(m, p) = 1$. Let $t' := T' \bmod(\pi)$, let $s' := S' \bmod(\pi)$, and let $\bar{u} := u \bmod(\pi)$ which is a unit of $k[[s', t']]/(s't')$. Let $T := T'u^{1/m}$. Then $\mathcal{X} \simeq \operatorname{Spf} A$, where $A = R[[S, T]]/(ST - \pi^{tp})$ for a suitable $S \in A$. Let $B := A[X, Y]/(X^p - T^m, Y^p - S^m, XY - \pi^t)$. Then B is a finite flat algebra over A which is integrally closed (because $B/\pi B$ is reduced), and $\mathcal{Y} = \operatorname{Spf} B$. The cover $f : \mathcal{Y} \to \mathcal{X}$ is thus generically given by an equation $X^p = T^m$, and we are in case a. Assume now that $m \equiv 0 \bmod(p)$. Hier we have two cases:

case 1) $\bar{u}$ is not a p-power. Then it is easy to see, that after changing the Kummer generator of the torsor f_K, one can assume that u is such that $\bar{u} = 1 + t'^m \bar{v}$, where m is a positif integer prime to p and $\bar{v}$ is a unit of $k[[s', t']]/(s't')$. In particular $\bar{u} = 1 + (t'\bar{v}^{1/m})^m = 1 + t^m$, where $t := t'\bar{v}^{1/m}$. Let $T := T'v^{1/m}$, where $v := (u-1)/T'^m$. We have $\mathcal{X} \simeq \operatorname{Spf} R[[S, T]]/(ST - \pi^{tp})$ for a suitable $S \in A$. The cover $f : \mathcal{Y} \to \mathcal{X}$ is thus generically given by an equation $X^p = 1 + T^m$, and we are in case b. Let $\mathcal{X}_1 := \operatorname{Spf} R[[S]]\{S^{-1}\}$ be the boundary of $\mathcal{X}$ corresponding to the ideal (π, T). The torsor $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induced by f above X_1 is generically given by an equation $X^p = 1 + \pi^{pt} S^{-m}$, which imply that $t \le v_K(\lambda)$, since f_1 is not completely split. Moreover we are in case *b* if $t < v_K(\lambda)$, and in case *c* if $t = v_K(\lambda)$.

case 2) $\bar{u}$ is a p-power. In this case, and after changing the Kummer generator of the torsor f_K, we can assume that this torsor is given by an equation $X^p = 1 + \pi^{n'} v$, where $v \in A'$ does not belong to the ideal $\pi A'$, and v is not a p-power mod π. Also one can check, after localisation and completion at the ideal (π, S') as above (namely by looking what happend above the boundary $\mathcal{X}_1$), that necessarily $n' \le p v_K(\lambda)$, and $n' = np$ is divisible by p (cf. for example [Gr-Ma] proof of 1.1, and [He] proof of 1.6, chap 5). Moreover after changing the Kummer generator of the torsor f_K, one can assume that $\bar{v} = t'^m \bar{v}'$, where m is an integer prime to p, and v' is a unit. Let $T := T'(v/T'^m)^{1/m}$. We have $\mathcal{X} \simeq \operatorname{Spf} R[[S, T]]/(ST - \pi^{tp})$ for a suitable $S \in A'$. The cover $f : \mathcal{Y} \to \mathcal{X}$ is thus generically given by an equation $X^p = 1 + \pi^{np} T^m$, and we are in case d or e. Let $\mathcal{X}_1$ be as above the boundary of $\mathcal{X}$ corresponding to the ideal (π, T). The torsor $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induced by f above X_1 is given by an equation $X^p = 1 + \pi^{p(n+t)} S^{-m}$, which imply that

$t+n \leq v_K(\lambda)$ since f_1 is not completely split. Moreover we are in case d if $t+n=v_K(\lambda)$ and in case e if $n+t<v_K(\lambda)$.

3.3. Variation of the different. (Compare with [He], 5.2) The following lemma, which is a direct consequence of 3.2.5, describes how does the degree of the different vary from one boundary to another in a cover $f:\mathcal{Y}\to\mathcal{X}$ as in 3.2.5.

3.3.1. Proposition. *Let $\mathcal{X}$ be the formal germ of a semi-stable R-curve at an ordinary double point x. Let $f:\mathcal{Y}\to\mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}_k$ reduced and local, and with $f_K:\mathcal{Y}_K\to\mathcal{X}_K$* **étale**. *Let y be the closed point of $\mathcal{Y}$. Assume that $\mathbf{g_y}=\mathbf{0}$, which imply necessarily that the thikness $e=pt$ of the double point x is divisible by p. For each integer $0<t'<t$, let $\mathcal{X}_{t'}\to\mathcal{X}$ be the blow-up of $\mathcal{X}$ at the ideal $(\pi^{pt'},T)$. The special fibre of $\mathcal{X}_{t'}$ consists of a projective line $P_{t'}$ which meets two germs of double points x and x'. Let η be the generic point of $P_{t'}$, and let v_η be the corresponding discrete valuation of the function field of $\mathcal{X}$. Let $f_{t'}:\mathcal{Y}_{t'}\to\mathcal{X}_{t'}$ be the pull back of f, which is a Galois cover of degree p, and let $\delta(t')$ be the degree of the different induced by this cover above v_η (cf. [Sa] 3.1). Also let $\mathcal{X}_i$, for $i=1,2$, be the boundaries of $\mathcal{X}$. Let $f_i:\mathcal{Y}_i\to\mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$, let $(G_{k,i},m_i,h_i)$ be their degeneration type, and let δ_i be the corresponding degree of the different (cf. [Sa] 3.1). Say $\delta_1=\delta(0)$, $\delta_2=\delta(t)$, and $\delta(0)\leq\delta(t)$. We have $m:=-m_1=m_2$ say is positif. Then the following holds:*

1) If $\delta(t')<v_K(p)$, for every $0\leq t'\leq t$. Then for $0\leq t_1\leq t_2\leq t$ we have $\delta(t_2)=\delta(t_1)+m(p-1)(t_2-t_1)$, and $\delta(t')$ is an increasing function of t'.

2) If there exists $0\leq t'\leq t$ such that $\delta(t')=v_K(p)$. Then there exists $0\leq t_1\leq t_2\leq t$ such that $\delta(t')=v_K(p)$ is constant for $t_1\leq t'\leq t_2$, $\delta(t')$ is increasing as t' increases from 0 to t_1, and $\delta(t')$ decreases as t' increases from t_2 to t.

3.3.2. Remark. In [Gr-Ma] and [He] where studied order p-automorphisms of open p-adic discs and p-adic annuli. In their aproach one writes such an automorphism as a formal series and one deduce some results, e.g. the only if part of 3.1.2 and 3.2.3, using the Weirstrass preparation theorem. The aproach adobted hier, and which consists on directly computing the vanishing cycles first, I believe provides another way to study such automorphisms. Namely these are those covers above formal fibres of semi-stable R-curves with genus 0, and one can easily writes down Kummer equations which lead to such covers as in 3.1.3, 3.2.4, and 3.2.5.

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