L-FUNCTIONS OF SIEGEL MODULAR FORMS AND TWIST OPERATORS

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Abstract. It is a survey of linear twists of Siegel modular forms with Dirichlet characters and new applications to L-functions of modular forms. Twist groups consistent with twist operators are considered. Spinor L-functions of modular forms for twist groups are interpreted as zeta-functions of twisted forms by using commutation relations between twist operators and Hecke operators. Applications of the twist technique to investigation of analytic properties of L-functions of modular forms include functional equation in the case of cusp forms of genus 1 and a detailed sketch of proof of analytic continuation for L-functions of cusp forms of genus 2 .

Introduction

Linear twist operators on Siegel modular forms were introduced in paper [5]. If a Siegel modular form F given on the *upper half-plane of genus* $n \geq 1$,

$$\mathbb{H}^n = \left\{ Z = X + iY \in \mathbb{C}_n^n \;\middle|\; {}^tZ = Z,\ Y > 0 \qquad (i = \sqrt{-1}) \right\},$$

by a Fourier series

$$F(Z) = \sum_{N \in \mathfrak{N}^n, N \geq 0} f(N)\, e^{2\pi i Tr(NZ)} \tag{1}$$

with constant Fourier coefficients $f(A)$ numerated by positive semi-definite matrices A of the set

$$\mathfrak{N}^n = \left\{ N = (n_{\alpha\beta}) \in \frac{1}{2}\mathbb{Z}_n^n \;\middle|\; {}^tN = N, \quad n_{11}, n_{22}, \dots, n_{nn} \in \mathbb{Z} \right\}$$

of all symmetric half-integer matrices of order n with integral entries on principal diagonal, then the *twist of F with a Dirichlet character χ and a $p\,(arameter)-$matrix* L of the form

$$L = {}^tL \in \mathbb{Z}_n^n \tag{2}$$

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is defined by the series

$$(F|\mathcal{T}(\chi, L))(Z) = \sum_{N\in\mathfrak{N}^n,\, N\geq 0} \chi(Tr(LN))f(N)\, e^{2\pi i Tr(NZ)} \qquad (Z \in \mathbb{H}^n). \tag{3}$$

The operator

$$|\mathcal{T}(\chi, L) : F \mapsto F|\mathcal{T}(\chi, L) \tag{4}$$

is called the *twist operator with character* χ *and p-matrix* L. It was shown in [5] that under some assumptions twist operators transform modular forms into modular forms (for another group) and commute in certain sense with Hecke operators. It was also proved that the spinor zeta-function of a twisted modular form can be interpreted as the L-function of the initial modular form with the character of the twist. In the present paper we generalize, refine, extend, and apply these results.

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Contents. In section 1 we, on one hand, briefly recall some basic constructions of the paper [5] and, on the other hand, generalize and simplify some results of that paper. In particular, we consider natural congruence subgroups (*twist groups*) for which spaces of modular forms are invariant with respect to the twist operators and related operators and consider their commutation relations. In section 2, after short reminder of definitions of Hecke–Shimura rings and Hecke operators, we establish relations between the rings and the operators for the twist groups and related groups and consider commutation relations between regular Hecke operators and twist operators for twist groups. In section 3 it is proved, in particular, that spinor zeta-function of an twisted eigenfunction for all regular Hecke operators with respect to a twist group is equal to L-function of the initial form with the character of twist (Theorem 3.2). In section 4 we use the reduction theorem 3.2 to prove an implication of Atkin-Lehner theory [7] of "new" forms for the twist groups of genus $n = 1$. In section 5 we apply the twist technique to preliminary investigation of analytic continuation of L-functions of cusp forms of genus 2 for twist groups.

Notation. The letters $\mathbb{P}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are reserved for the set of positive rational prime numbers, the set of positive rational integers, the ring of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

$\mathbb{X}^m_n$ is the set of all $m \times n$-matrices with entries in a set $\mathbb{X}$. If M is a matrix, ${}^tM$ always denotes the transpose of M, $Tr(M)$ for a square M is the sum of diagonal elements of M. If Y is a real symmetric matrix, then $Y > 0$ (respectively, $Y \geq 0$) means that Y is positive definite (respectively, positive semi-definite). $\mathbf{1}$ and $\mathbf{0}$ is the unit matrix and the zero matrix of dimension clear from content, otherwise we write 1_r and 0_r^n for the unit matrix of order r and the zero matrix of dimension $n \times r$, respectively. The letter i usually denotes $\sqrt{-1} \in \mathbb{H}^1$. The bar over a complex number or character means the complex conjugation. We often use the notation

$$A[B] = {}^tBAB$$

for two matrices of suitable dimension.

§1. Twist operators

Petersson operators. Let us consider the connected component of the unit in the general real symplectic group of genus n, i.e., the group

$$\mathbb{G}^n = \left\{ M \in \mathbb{R}^{2n}_{2n} \mid {}^tMJ_nM = \mu(M)J_n, \quad \mu(M) > 0 \right\} \qquad \left(J_n = \left(\begin{smallmatrix} 0_n & 1_n \\ -1_n & 0_n \end{smallmatrix} \right) \right).$$

The positive scalar factor $\mu(M)$ is called the *multiplier of* M. The group $\mathbb{G}^n$ acts as a group of analytic automorphisms on the upper half-plane $\mathbb{H}^n$ by the rule

$$\mathbb{G}^n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto M\langle Z \rangle = (AZ+B)(CZ+D)^{-1} \qquad (Z \in \mathbb{H}^n) \quad (0.1)$$

and operates on functions $F : \mathbb{H}^n \mapsto \mathbb{C}$ by (*normalized*) *Petersson operators of an integral weight* k,

$$\mathbb{G}^n \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|_k M = \mu(M)^{nk-\langle n \rangle} \det(CZ+D)^{-k} F(M\langle Z \rangle), \quad (1.1)$$

where $\langle n \rangle = n(n+1)/2$. The normalizing factor $\mu(M)^{nk-\langle n \rangle}$, for $n = 1$ going back to Hecke, is not significant by itself, but it simplifies a number of formulas related to Hecke operators. The Petersson operators transform isomorphically the space functions on $\mathbb{H}^n$ into itself, map holomorphic functions to holomorphic functions, and satisfy the rule

$$F|_k MM' = F|_k M|_k M' \text{ for all } M, M' \in \mathbb{G}^n \quad (1.2)$$

(see, e.g., [4; Lemmas 1.4.1, 1.4.2]).

Modular forms. We recall that a (*holomorphic*) *modular form of genus* $n \geq 1$ *and integral weight* k for a subgroup Λ of finite index in the modular group of genus n,

$$\Gamma^n = Sp_n(\mathbb{Z}) = \left\{ M \in \mathbb{G}^n \cap \mathbb{Z}^{2n}_{2n} \mid \mu(M) = 1 \right\}, \quad (1.3)$$

is defined as a holomorphic on $\mathbb{H}^n$ function F satisfying

$$F|_k M = F \qquad \text{for each} \quad M \in \Lambda, \quad (1.4)$$

and, if $n = 1$, regular at all cusps of Λ. All such functions form a finite-dimensional over $\mathbb{C}$ linear space $\mathfrak{M}_k(\Lambda)$. The forms equal to zero at all cusps are called *cusp forms* and form the subspace of *cusp forms* $\mathfrak{N}_k(\Lambda)$.

We shall be mainly interested in spaces of all modular forms $\mathfrak{M}_k(\widehat{\Gamma}^n(m))$ and cusp forms $\mathfrak{N}_k(\widehat{\Gamma}^n(m))$ of fixed genus n, integral weight k, and level m for the *twist congruence subgroups* of the form

$$\widehat{\Gamma}^n(m) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \mid C \equiv \mathbf{0} \pmod{m^2},\, A \equiv D \equiv \mathbf{1} \pmod{m} \right\} \tag{1.5}$$

and some subspaces of these spaces. The group $\widehat{\Gamma}^n(m)$ is conjugated to the *principal congruence subgroup of level* m,

$$\Gamma^n(m) = \{ M \in \Gamma^n \mid M \equiv 1_{2n} \pmod{m} \} :$$

$$\widehat{\Gamma}^n(m) = V(m)\Gamma^n(m)V(m)^{-1}, \quad \text{where} \quad V(m) = V^n(m) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & m\mathbf{1} \end{pmatrix}.$$

Each modular form of integral weight for the group $\widehat{\Gamma}(m) = \widehat{\Gamma}^n(m)$ has a Fourier expansion of the form (1) absolutely and uniformly convergent on subsets of the shape

$$\mathbb{H}^n_\varepsilon = \{ Z = X + iY \in \mathbb{H}^n \mid Y \geq \varepsilon 1_n \} \quad \text{with} \quad \varepsilon > 0 \tag{1.6}$$

The groups of the form (1.5) will be called *twist groups* (*of genus n and level m*).

Twist operators and Petersson operators. Returning to twist operator, the following proposition allows one to reduce twist operators (4) to linear combinations of Petersson operators (1.1).

Proposition 1.1. *Let F be a function on $\mathbb{H}^n$ with Fourier expansion* (1) *absolutely and uniformly convergent on compacts, χ a Dirichlet character modulo $m \in \mathbb{N}$, and L a matrix satisfying* (2)*, then the twist* (3) *of the function F with character χ and p-matrix L can be written with the help of Petersson operators* (1.1) *of an arbitrary integral weight k in the form*

$$F|\mathcal{T}(\chi, L) = \frac{1}{m} \sum_{r,l \bmod m} \chi(r) e^{-\frac{2\pi i r l}{m}} F|_k U(m^{-1} l L), \tag{1.7}$$

where, for a real symmetric matrix B of order n, we use the notation

$$U(B) = \begin{pmatrix} \mathbf{1} & B \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \in \mathbb{G}^n. \tag{1.8}$$

If, in addition, the character χ is primitive modulo m, then the formula (1.7) *can be written in the form*

$$F|\mathcal{T}(\chi, L) = \frac{g(\chi)}{m} \sum_{l \bmod m} \overline{\chi}(-l) F|_k U(m^{-1} l L), \tag{1.9}$$

where, for a Dirichlet character χ modulo $m \in \mathbb{N}$,

$$g(\chi) = g(\chi,\, 1) \tag{1.10}$$

is a particular value of the Gaussian sum

$$g(\chi,\, l) = \sum_{r \bmod m} \chi(r)\, e^{\frac{2\pi i\, l r}{m}} \qquad (l \in \mathbb{Z}). \tag{1.11}$$

Proof. As it easily follows from definitions and Fourier expansion (1), we get

$$(F|_k U(m^{-1}lL))(Z) = F(Z + m^{-1}lL) = \sum_{N \in \mathfrak{N}^n,\, N \geq 0} f(N) e^{\frac{2\pi i\, l}{m} Tr(LN)} e^{2\pi i Tr(NZ)}.$$

Hence we have

$$\frac{1}{m} \sum_{r, l \bmod m} \chi(r) e^{-\frac{2\pi i r l}{m}} F|_k U(m^{-1}lL)$$

$$\begin{aligned}
&= \frac{1}{m} \sum_{r,\, l \bmod m} \chi(r) e^{-\frac{2\pi\, i r l}{m}} \sum_{N \in \mathfrak{N}^n,\, N \geq 0} f(N) e^{\frac{2\pi i l}{m} Tr(LN)} e^{2\pi i Tr(NZ)} \\
&= \sum_{r \bmod m} \chi(r) \sum_{N \in \mathfrak{N}^n,\, N \geq 0} \frac{1}{m} \sum_{l \bmod m} e^{\frac{2\pi i l}{m}(-r + Tr(LN))} f(N) e^{2\pi i Tr(NZ)} \\
&= \sum_{N \in \mathfrak{N}^n,\, N \geq 0} \chi(Tr(LN)) f(N) e^{2\pi i Tr(NZ)} = (F|\mathcal{T}(\chi,\, L))(Z),
\end{aligned}$$

which proves the formula (1.7).

We recall that a Dirichlet character χ is called *primitive character modulo m* if it is a character modulo m and not a character modulo any proper divisor of m. If the character χ is primitive modulo m, then the Gaussian sums (1.10)–(1.11) satisfy the relations

$$|g(\chi)| = \sqrt{m} \quad \text{and} \quad g(\chi,\, l) = \overline{\chi}(l) g(\chi), \tag{1.12}$$

where $\overline{\chi}$ is the complex conjugate character (see, e.g. [11, Proposition 21]).

If χ is primitive, then by (1.7) and (1.12) we obtain

$$\begin{aligned}
F|\mathcal{T}(\chi,\, L) &= \frac{1}{m} \sum_{l \bmod \mathrm{m}} \left(\sum_{r \bmod m} \chi(r) e^{-\frac{2\pi i r l}{m}} \right) F|_k U(m^{-1}lL) \\
&= \frac{1}{m} \sum_{l \bmod \mathrm{m}} \overline{\chi}(-l) g(\chi) F|_k U(m^{-1}lL). \qquad \square
\end{aligned}$$

By direct multiplication of block-matrices, we obtain that, for every matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of order $2n$ with blocks of order n, matrices of the form (1.8) satisfy the relations

$$U(B') \begin{pmatrix} A & B \\ C & D \end{pmatrix} U(B'')^{-1} = \begin{pmatrix} A + B'C & B'D - AB'' + B - B'CB'' \\ C & -CB'' + D \end{pmatrix}. \tag{1.13}$$

Lemma 1.2. *Let $U(m^{-1}L)$ be a matrix of the form* (1.8) *with L satisfying* (2) *and $m \in \mathbb{N}$, then*

(1) *the twist group $\widehat{\Gamma}(m) = \widehat{\Gamma}^n(m)$ satisfies the relation*

$$U(m^{-1}L)\widehat{\Gamma}(m)U(m^{-1}L)^{-1} = \widehat{\Gamma}(m); \tag{1.14}$$

(2) *the operator $|_k U(m^{-1}L)$ map the spaces $\mathfrak{M}_k(\widehat{\Gamma}(m))$ and $\mathfrak{N}_k(\widehat{\Gamma}(m))$ onto themselves.*

Proof. By formula (1.13), for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widehat{\Gamma}(m)$, we obtain the matrix

$$U(m^{-1}L)\begin{pmatrix} A & B \\ C & D \end{pmatrix}U(m^{-1}L)^{-1}$$

$$= \begin{pmatrix} A + m^{-1}LC & m^{-1}(LD - AL) + B - m^{-2}LCL \\ C & -m^{-1}CL + D \end{pmatrix},$$

which obviously again belongs to $\widehat{\Gamma}(m)$. It proves that the left side of (1.14) is contained in the right side. The inverse inclusion follows similarly, since clearly $U(B)^{-1} = U(-B)$.

To prove the part (2) it actually suffices to check, for example, that the function $F' = F|_k U(m^{-1}L)$ for $F \in \mathfrak{M}_k(\widehat{\Gamma}(m))$ satisfies $F'|_k M = F'$ for all $M \in \widehat{\Gamma}(m)$. By (1.2) we have

$$F'|_k M = F|_k U(m^{-1}L)M = F|_k U(m^{-1}L)MU(m^{-1}L)^{-1}|_k U(m^{-1}L) = F',$$

since, by (1.14), $U(m^{-1}L)MU(m^{-1}L)^{-1} \in \widehat{\Gamma}(m)$. □

The following theorem is a direct consequence of Proposition 1.1 and Lemma 1.2.

Theorem 1.3. *The twist operator $|\mathcal{T}(\chi, L)$ with every Dirichlet character χ modulo m and every p-matrix L satisfying* (2) *maps the spaces $\mathfrak{M}_k(\widehat{\Gamma}(m))$ and $\mathfrak{N}_k(\widehat{\Gamma}(m))$ of modular forms and cusp forms of weight k for the twist group $\widehat{\Gamma}(m)$ into themselves.*

This theorem makes it natural to restrict consideration of twist operators to spaces of modular forms for the groups of the form $\widehat{\Gamma}(m)$ and justifies the term "twist groups".

Twist operators and congruence operators. Let us consider now the action of Petersson operators corresponding to elements of the group

$$\Gamma_0(m^2) = \Gamma_0^n(m^2) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^n \mid C \equiv \mathbf{0} \pmod{m^2} \right\} \tag{1.15}$$

on spaces of modular forms for the groups $\widehat{\Gamma}(m)$ and their relation to twist operators. It follows from [4; Lemma 3.3.2(1)] that the natural mapping of the group $\Gamma_0(m^2)$

to the group $GL_n(\mathbb{Z}/m\mathbb{Z})$ of nonsingular matrices of order n over the residue ring $\mathbb{Z}/m\mathbb{Z}$ given by

$$\Gamma_0(m^2) \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto D \mod m \in GL_n(\mathbb{Z}/m\mathbb{Z}) \tag{1.16}$$

is an epimorphic homomorphism with the kernel $\widehat{\Gamma}(m)$, which is therefore a normal subgroup of $\Gamma_0(m^2)$. For a matrix D of the group $GL_n(\mathbb{Z}/m\mathbb{Z})$ we shall denote by

$$\varrho(D) \in \Gamma_0(m^2) \tag{1.17}$$

an inverse image of D under this mapping. Since $\widehat{\Gamma}(m)$ is a normal subgroup of $\Gamma_0(m^2)$, we conclude that cosets

$$\widehat{\Gamma}(m)\varrho(D) = \varrho(D)\widehat{\Gamma}(m) = \widehat{\Gamma}(m)\varrho(D)\widehat{\Gamma}(m) \tag{1.18}$$

are independent of the choice of $D \in GL_n(\mathbb{Z}/m\mathbb{Z})$. Hence each operator

$$\begin{aligned} |_k\varrho(D) : F \mapsto F|_k\varrho(D) \\ = F|_k \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(m^2),\ F \in \mathfrak{M}_k(\widehat{\Gamma}(m)) \right) \end{aligned} \tag{1.19}$$

depends only on matrix D modulo m and maps the space $\mathfrak{M}_k(\widehat{\Gamma}(m))$ onto itself. Besides, the operators satisfy

$$|_k\varrho(D)|_k\varrho(D') = |_k\varrho(DD') \qquad (D,\, D' \in GL_n(\mathbb{Z}/m\mathbb{Z})). \tag{1.20}$$

The following proposition generalizes Propositions 1.5 and 1.6 of the paper [5].

Proposition 1.4. *Let $F \in \mathfrak{M}_k(\widehat{\Gamma}(m))$, then in the notation and under the assumptions of Theorem* 1.3 *and notation* (1.19)*, for each $D \in GL_n(\mathbb{Z}/m\mathbb{Z})$ the identity is valid*

$$(F|\mathcal{T}(\chi,\, L))|_k\varrho(D) = (F|_k\varrho(D))|\mathcal{T}(\chi,\, L[D]), \tag{1.21}$$

where $L[D] = {}^tLDL$; if, moreover, the matrix D satisfies the congruence

$$L[D] \equiv \nu(D)L \pmod m \tag{1.22}$$

with a scalar $\nu(D)$ invertible modulo m, then the formula (1.21) *turns into*

$$(F|\mathcal{T}(\chi,\, L))|_k\varrho(D) = \chi(\nu(D))(F|_k\varrho(D))|\mathcal{T}(\chi,\, L); \tag{1.23}$$

in particular, for all $d \in \mathbb{N}$ prime to m the relation

$$(F|\mathcal{T}(\chi,\, L))|_k\tau(d) = \chi(d^2)(F|_k\tau(d))|\mathcal{T}(\chi,\, L) \tag{1.24}$$

is valid, where

$$\tau(d) = \tau^n(d) = \varrho(d \cdot 1_n) \tag{1.25}$$

Proof. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(m^2)$, then by (1.7) and (1.2) we can write

$$\begin{aligned}
&(F|\mathcal{T}(\chi,\, L))|_k\varrho(D)\\
&= (F|\mathcal{T}(\chi,\, L))|_k M = \frac{1}{m}\sum_{r,l \bmod m} \chi(r)e^{-\frac{2\pi irl}{m}} F|_k U(m^{-1}lL)|_k M\\
&= \frac{1}{m}\sum_{r,l \bmod m} \chi(r)e^{-\frac{2\pi irl}{m}} F|_k U(m^{-1}lL)MU(m^{-1}lL[D])^{-1}U(m^{-1}lL[D])\\
&= \frac{1}{m}\sum_{r,l \bmod m} \chi(r)e^{-\frac{2\pi irl}{m}} F|_k M_l'|_k U(m^{-1}lL[D]),
\end{aligned}$$

where $M_l' = U(m^{-1}lL)MU(m^{-1}l\,{}^tDLD)^{-1}$. By formula (1.13) with $B' = m^{-1}lL$ and $B'' = m^{-1}lL[D]$, we obtain

$$\begin{aligned}
M_l' &= \begin{pmatrix} A + m^{-1}lLC & m^{-1}l(LD - AL[D] + B - m^{-2}l^2LC\text{Ł}[D] \\ C & -m^{-1}lCL[D] + D \end{pmatrix}\\
&= \begin{pmatrix} A + m^{-1}lLC & -m^{-1}l(-A\,{}^tD + 1_n)LD + B - m^{-2}l^2LC\,{}^tDLD \\ C & -m^{-1}lC\,{}^tDLD + D \end{pmatrix}.
\end{aligned}$$

This matrix is clearly integral. Besides, it belongs to the group $\mathbb{G}^n$ and satisfies $\mu(M_l') = \mu(M) = 1$. Since $C \equiv \mathbf{0} \pmod{m^2}$, it follows that $M_l' \in \Gamma_0(m^2)$. Since the $D-$block of this matrix is $-m^{-1}lC\,{}^tDLD + D \equiv D \pmod{m}$, it follows from the above that $F|_k M_l' = F|_k\varrho(D)$. Again, by formula (1.7), we obtain

$$\begin{aligned}
(F|\mathcal{T}(\chi,\, L))|_k\varrho(D) &= \frac{1}{m}\sum_{r,l \bmod m} \chi(r)e^{-\frac{2\pi irl}{m}} (F|_k\varrho(D))|_k U(m^{-1}lL[D])\\
&= (F|_k\varrho(D))|\mathcal{T}(\chi,\, L[D]).
\end{aligned}$$

The formula (1.23) follows from (1.21) and the definition of twist operators. □

Speaking on the operators corresponding to elements (1.25), it will be convenient to split the space $\mathfrak{M}_k(\widehat{\Gamma}(m))$ into invariant subspaces of operators $|_k\tau(d) = |_k\tau^n(d) = |_k\varrho(d\mathbf{1})$ with all d prime to m. The mapping $d \mapsto |_k\tau(d)$ defines a representation of the multiplicative Abelian group $GL_1(\mathbb{Z}/m\mathbb{Z})$ on the space $\mathfrak{M}_k(\widehat{\Gamma}(m))$. Thus, this space is a direct sum of one-dimensional invariant subspaces. If $F|_k\tau(d) = \psi(d)F$ with all d prime to m, then ψ is a character of $GL_1(\mathbb{Z}/m\mathbb{Z})$, which can be considered as a Dirichlet character modulo m. Then the direct sum decomposition holds:

$$\mathfrak{M}_k(\widehat{\Gamma}(m)) = \bigoplus_{\psi \in \mathrm{Char}(GL_1(\mathbb{Z}/m\mathbb{Z}))} \mathfrak{M}_k(\widehat{\Gamma}(m),\, \psi), \tag{1.26}$$

where

$$\mathfrak{M}_k(\widehat{\Gamma}(m),\, \psi) = \left\{ F \in \mathfrak{M}_k(\widehat{\Gamma}(m)) \;\middle|\; F|_k\tau(d) = \psi(d)F,\ \gcd(d,\, m) = 1 \right\}. \tag{1.27}$$

In this notation, formulas (1.24) can be written in the form

$$(F|\mathcal{T}(\chi,\, L))|_k\tau(d) = \chi(d^2)\psi(d)F|\mathcal{T}(\chi,\, L) \qquad (F \in \mathfrak{M}_k(\widehat{\Gamma}(m),\, \psi)) \tag{1.28}$$

valid for all $d \in \mathbb{N}$ prime to m.

Twist operators and star operator. Along with operators (1.19) the Petersson operators corresponding to matrices of order $2n$ of the form

$$\Omega = \Omega^n(m) = \begin{pmatrix} \mathbf{0} & m^{-1}\mathbf{1} \\ -m\mathbf{1} & \mathbf{0} \end{pmatrix} \tag{1.29}$$

will be also useful. It is easy to see that $\Omega \in \mathbb{G}^n$ with $\mu(\Omega) = 1$,

$$\Omega^{-1} = -\Omega, \quad \text{and} \quad \Omega^2 = -1_{2n}. \tag{1.30}$$

It follows from obvious relations

$$\Omega \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Omega^{-1} = \Omega^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Omega = \begin{pmatrix} D & -C/m^2 \\ -m^2 B & A \end{pmatrix}, \tag{1.31}$$

valued for every $2n$-matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ with n-blocks A, B, C, D, and from definitions of the group $\widehat{\Gamma}(m)$ that

$$\Omega\widehat{\Gamma}(m)\Omega^{-1} = \Omega^{-1}\widehat{\Gamma}(m)\Omega = \widehat{\Gamma}(m). \tag{1.32}$$

Coming back to modular forms we obtain the following assertions.

Proposition 1.5. (1) *In the above notation, the linear operator*

$$|_k\Omega : F \mapsto F^* = F|_k\Omega \tag{1.33}$$

maps the spaces $\mathfrak{M}_k(\widehat{\Gamma}(m)))$ and $\mathfrak{N}_k(\widehat{\Gamma}(m))$, respectively, onto themselves and satisfies the relation

$$(F^*)^* = (-1)^k F; \tag{1.34}$$

(2) *the operators* (1.33) *and* (1.19) *on the spaces $\mathfrak{M}_k(\widehat{\Gamma}(m))$ and $\mathfrak{N}_k(\widehat{\Gamma}(m))$ satisfy the relation*

$$|_k\Omega|_k\varrho(D) = |_k\varrho({}^tD^{-1})|_k\Omega, \tag{1.35}$$

where D^{-1} is the inverse of D modulo m;

(3) *in the notation of Theorem* 1.3 *and Proposition* 1.4, *the formula holds*

$$\big(F|\mathcal{T}(\chi, L)\big)^*|_k\varrho(D) = \big((F|_k\varrho({}^tD^{-1}))|\mathcal{T}(\chi, L[{}^tD^{-1}])\big)^* \tag{1.36}$$

if, moreover, the matrix ${}^tD$ satisfies the congruence (1.22), *then this formula turns into*

$$(F|\mathcal{T}(\chi, L))^*|_k\varrho(D) = \overline{\chi}(\nu({}^tD))\big((F|_k\varrho({}^tD^{-1}))|\mathcal{T}(\chi, L)\big)^*, \tag{1.37}$$

in particular,

$$(F|\mathcal{T}(\chi, L))^*|_k\tau(d) = \overline{\chi}(d^2)\overline{\psi}(d)\big(F|\mathcal{T}(\chi, L)\big)^* \qquad (F \in \mathfrak{M}_k(\widehat{\Gamma}(m), \psi)) \tag{1.38}$$

for all $d \in \mathbb{N}$ *prime to* m*, where* $\tau(d)$ *are matrices* (1.25) *and* $\overline{\chi}$*,* $\overline{\psi}$ *are conjugate characters.*

Proof. From (1.33), (1.2), and (1.32), for $M \in \widehat{\Gamma}(m)$ and $F \in \mathfrak{M}_k(\widehat{\Gamma}(m))$ or $\mathfrak{N}_k(\widehat{\Gamma}(m))$ we get

$$F^*|_k M = F|_k \Omega M = F|_k \Omega M \Omega^{-1}\Omega = F|_k \Omega M \Omega^{-1}|_k \Omega = F|_k \Omega = F^*.$$

The relation (1.34) follows from (1.30). The rest of the part (1) is clear.

If $\varrho(D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then it follows from (1.31) and definitions that

$$\Omega \varrho(D) = \Omega \varrho(D) \Omega^{-1} \Omega = \varrho(A)\Omega = \varrho({}^tD^{-1})\Omega,$$

since ${}^tAD \equiv 1_n \pmod{m}$. The rest of part (2) follows from (1.2).

The identities (1.36)–(1.38) follow from (1.35), (1.21), (1.24) and definitions. □

The operator (1.33) will be called the *star operator*.

§2. Twist operators and Hecke operators

Here we shall consider relations of Hecke operators with the twist operators $|\mathcal{T}(\chi, L)$, where χ is a Dirichlet character modulo m and L is a parameter matrix (2). But first we shall briefly recall definitions of Hecke–Shimura rings and Hecke operators. For details see [4, Chapters 3 and 4].

Hecke–Shimura rings and Hecke operators. Let Δ be a multiplicative semigroup and Λ a subgroup of Δ such that every double coset $\Lambda M \Lambda$ of Δ modulo Λ is a finite union of left cosets $\Lambda M'$. Let us consider the vector space over a field, say, the field $\mathbb{C}$ of complex numbers, consisting of all formal finite linear combinations with coefficients in $\mathbb{C}$ of symbols (ΛM) with $M \in \Delta$ being in one-to-one correspondence with left cosets ΛM of the set Δ modulo Λ. The group Λ naturally acts on this space by right multiplication defined on the symbols (ΛM) by $(\Lambda M)\lambda = (\Lambda M \lambda)$ with $M \in \Delta$ and $\lambda \in \Lambda$. We denote by

$$\mathcal{H}(\Lambda, \Delta) = \mathcal{H}_{\mathbb{C}}(\Lambda, \Delta)$$

the subspace of all Λ–invariant elements. The multiplication of elements of $\mathcal{H}(\Lambda, \Delta)$ given by the formula

$$(\sum_\alpha a_\alpha(\Lambda M_\alpha))(\sum_\beta b_\beta(\Lambda M'_\beta)) = \sum_{\alpha,\beta} a_\alpha b_\beta (\Lambda M_\alpha M'_\beta)$$

does not depend on the choice of representatives M_α and N_β in the corresponding left cosets, and turns the linear space $\mathcal{H}(\Lambda, \Delta)$ into an associative algebra over $\mathbb{C}$ with the unity element $(\Lambda 1_\Lambda)$, called *the Hecke–Shimura ring of* Δ *relative to* Λ (*over* $\mathbb{C}$). Elements

$$T(M) = T(M)_\Lambda = (\Lambda M \Lambda) = \sum_{M' \in \Lambda \backslash \Lambda M \Lambda} (\Lambda M') \qquad (M \in \Delta) \tag{2.1}$$

being in one-to-one correspondence with double cosets of $\varDelta$ modulo $\varLambda$ belong to $\mathcal{H}(\varLambda,\,\varDelta)$ and form a free basis of the ring over $\mathbb{C}$. For brevity, the symbols $(\varLambda M)$ and $T(M)$ will be referred as *left* and *double classes* (*of* $\varDelta$ *modulo* $\varLambda$), respectively.

Let us suppose now that the semigroup

$$\varDelta = \Delta^n = \mathbb{Z}^{2n}_{2n} \bigcap \mathbb{G}^n \tag{2.2}$$

consists of all integral matrices contained in the group $\mathbb{G}^n$, and the group $\varLambda$ is a subgroup of finite index in the modular group Γ^n. Then the conditions of the definition are fulfilled, and we can define the Hecke–Shimura ring

$$\mathcal{H}(\varLambda) = \mathcal{H}(\varLambda,\,\Delta^n). \tag{2.3}$$

Next, we shall define a linear representation of this ring on the space $\mathfrak{M}_k(\varLambda)$ of modular forms of weight k for the group Λ by *Hecke operators*:

$$\mathcal{H}(\varLambda) \ni T = \sum_{\alpha} a_\alpha(\varLambda M_\alpha) : F \mapsto F|T = F|_k T = \sum_{\alpha} a_\alpha F|_k M_\alpha, \tag{2.4}$$

where $|_k M_\alpha$ are the Petersson operators (1.1). The Hecke operators are independent of the choice of representatives in corresponding left cosets and map the spaces $\mathfrak{M}_k(\varLambda)$ and $\mathfrak{N}_k(\varLambda)$ into themselves.

Regular Hecke–Shimura rings and Hecke operators for $\widehat{\Gamma}^n(m)$. We shall mainly be interested not in the entire Hecke–Shimura ring (2.3), but rather in certain subrings, called *m-regular subrings* of a fixed genus $m \in \mathbb{N}$, which are defined for the groups $\varLambda$ of the shape $\Gamma = \Gamma^n$ and $\widehat{\Gamma}(m) = \widetilde{\Gamma}^n(m)$ with a fixed $n \in \mathbb{N}$ as Hecke–Shimura rings

$$\mathcal{H}_{(m)}(\varLambda) = \mathcal{H}(\varLambda,\,\Delta_{(m)}(\varLambda)), \tag{2.5}$$

of the group $\varLambda$ and m-regular semigroups $\Delta_{(m)}(\varLambda)$ given, respectively, by the conditions

$$\Delta_{(m)} = \Delta_{(m)}(\Gamma) = \{M \in \Delta^n \mid \gcd(m, \mu(M)) = 1\},$$

and

$$\Delta_{(m)}(\widehat{\Gamma}(m)) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_{(m)} \mid A \equiv \mathbf{1} \pmod{m},\, C \equiv \mathbf{0} \pmod{m^2} \right\}.$$

The corresponding Hecke operators are called *$m-$regular Hecke operators*. It turns out that the rings (2.5) for the groups $\Gamma = \Gamma^n$ and $\widehat{\Gamma}(m) = \widetilde{\Gamma}^n(m)$ are naturally isomorphic. More exactly, by an easy modification of the proof of Theorem 3.3.3 of the book [4], we obtain the following proposition.

Proposition 2.1. (1) *The linear map of the m-regular subrings* (2.5)*, defined on double classes* (2.1) *by conditions*

$$\mathcal{H}_{(m)}(\widehat{\Gamma}(m)) \ni T_{\widehat{\Gamma}(m)}(M) \mapsto T_{\Gamma}(M) \in \mathcal{H}_{(m)}(\Gamma) \qquad (M \in \Delta_{(m)}(\widehat{\Gamma}(m))), \tag{2.6}$$

is a ring isomorphism of the corresponding Hecke–Shimura rings. Moreover, the decompositions of related double classes into disjoint left classes are naturally correspond to each other:

$$T_{\widehat{\Gamma}(m)}(M) = \sum_{\alpha}(\widehat{\Gamma}(m)M_{\alpha}) \Longrightarrow T_{\Gamma}(M) = \sum_{\alpha}(\Gamma M_{\alpha}) \qquad (M \in \Delta_{(m)}(\widehat{\Gamma}(m))),$$

(2) *The regular Hecke operators are compatible with the mapping defined by* (2.6) *and natural embedding of the corresponding spaces of modular forms of weight k: if $F \in \mathfrak{M}_k(\Gamma) \subset \mathfrak{M}_k(\widehat{\Gamma}(m))$ and $T_{\widehat{\Gamma}(m)} \mapsto T_{\Gamma}$, then $F|T_{\widehat{\Gamma}(m)} = F|T_{\Gamma}$.*

The following proposition describes structure of the $m-$regular rings.

Proposition 2.2. *Each of the regular rings* (2.5) *for the groups Λ of the form $\Gamma = \Gamma^n$ or $\widehat{\Gamma}(m) = \widetilde{\Gamma}^n(m)$ is a commutative integral domain generated over $\mathbb{C}$ by algebraically independent elements*

$$\begin{cases} T_{\Lambda}(p) = T_{\Lambda}\big(\mathrm{diag}(\underbrace{1,\dots,1}_{n},\underbrace{p,\dots,p}_{n})\big), \\ T^j_{\Lambda}(p^2) = T_{\Lambda}\big(\varrho_j(p)\mathrm{diag}(\underbrace{1,\dots,1}_{n-j},\underbrace{p,\dots,p}_{j},\underbrace{p^2,\dots,p^2}_{n-j},\underbrace{p,\dots,p}_{j})\big) \quad (1 \le j \le n), \end{cases}$$

where p runs over all prime numbers not dividing m, and where

$$\varrho_j(p) = \varrho^n_j(p) = \varrho\big(\mathrm{diag}(\underbrace{1,\dots,1}_{n-j},\underbrace{p,\dots,p}_{j}\big) \quad (1 \le j \le n)$$

are matrices of the form (1.17)*.*

The elements $T_{\Lambda}(p), T^1_{\Lambda}(p^2), \dots, T^n_{\Lambda}(p^2)$ with prime $p \nmid m$ generate the $p-$local subring

$$\mathcal{H}_{(m)}(\Lambda)_p = \mathbb{C}\left[T_{\Lambda}(p), T^1_{\Lambda}(p^2), \dots, T^n_{\Lambda}(p^2)\right] \subset \mathcal{H}_{(m)}(\Lambda), \tag{2.7}$$

consisting of all finite linear combinations of elements $T_{\Lambda}(M)$ with $\mu(M) \mid p^{\infty}$.

Proof. The assertion for the case $\Lambda = \Gamma^n$ was proved in [4, Theorem 3.3.23(1)]. The other case follows from this case by Proposition 2.1. □

Regular Hecke operators and twist operators. We turn now to relation of regular Hecke operators and twist operators.

Theorem 2.3. *For the group $\Lambda = \widehat{\Gamma}(m)$, let $M \in \Delta_{(m)}(\Lambda)$, and let $F \in \mathfrak{M}_k(\Lambda)$ be a modular form of weight k for Λ. Let χ be a primitive Dirichlet character modulo m and L be a matrix of the form (2). Then the following commutation relation holds for the action of Hecke operator $T_\Lambda(M)$ and the twist operator $\mathcal{T}(\chi, L)$ on the form F:*

$$(F|\mathcal{T}(\chi, L))|T_\Lambda(M) = \chi(\mu(M))(F|T_\Lambda(M))|\mathcal{T}(\chi, L). \tag{2.8}$$

Proof. By the formulas (1.9), (2.1), (2.4), and (1.2), we obtain

$$(\mathcal{T}(\chi, L)F)|T_\Lambda(M) = \frac{g(\chi)}{m} \sum_{l \bmod m} \sum_{M' \in \Lambda\backslash\Lambda M\Lambda} \overline{\chi}(-l)F|_k U(m^{-1}lL)M' \tag{2.9}$$

$$= \frac{g(\chi)}{m} \sum_{l=0}^{m-1} \sum_{M' \in \Lambda\backslash\Lambda M\Lambda} \chi(\mu(M))\overline{\chi}(-l\mu(M))$$
$$\times (F|_k U(m^{-1}lL)M'U(-m^{-1}l\mu(M)L))|_k U(m^{-1}l\mu(M)L)$$

$$= \chi(\mu(M))\frac{g(\chi)}{m} \sum_{l=0}^{m-1} \overline{\chi}(-l')$$
$$\times \left(\sum_{M' \in \Lambda\backslash\Lambda M\Lambda} F|_k U(m^{-1}lL)M'U(-m^{-1}l'L)) \right) |_k U(m^{-1}l'L),$$

where $l' = l\mu(M)$. For every matrix

$$M' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Lambda M \Lambda \subset \Delta_{(m)}(\Lambda)$$

and $l = 0, 1, \ldots, m-1$, by formula (1.13) we can write

$$M'_l = U(m^{-1}lL)M'U(-m^{-1}l'L) \tag{2.10}$$

$$= \begin{pmatrix} A' + m^{-1}lLC' & -m^{-1}(lLD' - l'A'L) + B' - m^{-2}ll'LC'L \\ C' & -m^{-1}l'C'L + D' \end{pmatrix} = \begin{pmatrix} A'_l & B'_l \\ C'_l & D'_l \end{pmatrix},$$

Each such matrix belongs to $\Delta_{(m)}(\Lambda)$, because the matrices A'_l, C'_l, and D'_l are clearly integral matrices satisfying congruences $A'_l \equiv A' \equiv \mathbf{1} \pmod{m}$, $C'_l = C' \equiv \mathbf{0} \pmod{m^2}$, $D'_l \equiv D' \pmod{m}$, and the matrix B'_l is integral, since $lLD' - l'A'L \equiv l\mu(M)L - l'L = l'(L - L) \equiv 0 \pmod{m}$. By Lemma 1.2(1), we have the equalities

$$U(m^{-1}lL)\Lambda U(-m^{-1}lL) = \Lambda \text{ and } U(m^{-1}l'L)\Lambda U(-m^{-1}l'L) = \Lambda.$$

Thus, for each l, the matrix M'_l ranges the set

$$U(m^{-1}lL)\big(\varLambda\backslash\varLambda M\varLambda\big)U(-m^{-1}l'L) = \varLambda\backslash\varLambda(U(m^{-1}lL)MU(-m^{-1}l'L))\varLambda$$

It follows that for each l we can write

$$\sum_{M'\in\varLambda\backslash\varLambda M\varLambda} F|_k U(m^{-1}lL)M'U(-m^{-1}l'L) = F|T_\varLambda(U(m^{-1}lL)MU(-m^{-1}l'L)),$$

and in order to complete the proof it suffices to show that, for every l we have

$$\varLambda(U(m^{-1}lL)MU(-m^{-1}l'L))\varLambda = \varLambda M\varLambda \tag{2.11}$$

with $l' = l\mu(M)$, because in this case each inner sum in the big parenthesis on the right of (2.9) is equal to $F|T_\varLambda(M)$. The matrix $M_l = U(m^{-1}lL)MU(-m^{-1}l'L)$ and the matrix M, as we have seen above, both belong to the set $\Delta_{(m)}(\varLambda) \subset \Delta_{(m)}(\Gamma)$. Besides, $\mu(M_l) = \mu(M)$. It follows from Proposition 2.1 that the mapping (2.6) is one-to-one correspondence between sets of double cosets $\varLambda M\varLambda \subset \Delta_{(m)}(\varLambda)$ and $\Gamma M\Gamma \subset \Delta_{(m)}$. Therefore, in order to prove (2.11) it suffices to check that

$$\Gamma(U(m^{-1}lL)MU(-m^{-1}l'L))\Gamma = \Gamma M\Gamma \tag{2.12}$$

By [4, Lemma 3.3.6], in order to prove the equality (2.12) it is sufficient to prove that M_l and M have equal matrices of symplectic divisors, $\mathrm{sd}(M_l) = \mathrm{sd}(M)$. Let us set

$$mU(m^{-1}lL) = \begin{pmatrix} m\cdot\mathbf{1} & lL \\ \mathbf{0} & m\cdot\mathbf{1}\end{pmatrix} = N,\ mU(-m^{-1}l'L) = \begin{pmatrix} m\cdot\mathbf{1} & -l'L \\ \mathbf{0} & m\cdot\mathbf{1}\end{pmatrix} = N'.$$

Then we can write

$$U(m^{-1}lL) = \frac{1}{m}\gamma\,\mathrm{sd}(N)\gamma_1, \quad U(-m^{-1}l'L) = \frac{1}{m}\gamma_2\mathrm{sd}(N')\gamma_3,$$

where γ, γ_1, γ_2, $\gamma_3 \in \Gamma$. Hence, $M_l = \frac{1}{m^2}\gamma\,\mathrm{sd}(N)\gamma_1 M\gamma_2\mathrm{sd}(N')\gamma_3$. It follows that

$$m^2\mathrm{sd}(N)^{-1}\gamma^{-1}M_l\gamma_3^{-1} = \gamma_1 M\gamma_2\mathrm{sd}(N'). \tag{2.13}$$

Since $\mu(N) = \mu(N') = m^2$, it follows from definition of symplectic divisors that $\mathrm{sd}(m^2\mathrm{sd}(N)^{-1}) = \mathrm{sd}(N)$, $\mathrm{sd}(\gamma^{-1}M_l\gamma_3^{-1}) = \mathrm{sd}(M_l)$, $\mathrm{sd}(\gamma_1 M\gamma_2) = \mathrm{sd}(M)$, and $\mathrm{sd}(\mathrm{sd}(N')) = \mathrm{sd}(N')$. Since the number $\mu(M_l) = \mu(M)$ is coprime with $\mu(N) = \mu(N') = m^2$, the relation (2.13), by known properties of matrices of symplectic divisors, implies the relation

$$\mathrm{sd}(m^2\mathrm{sd}(N)^{-1})\mathrm{sd}(\gamma^{-1}M_l\gamma_3^{-1}) = \mathrm{sd}(\gamma_1 M\gamma_2)\mathrm{sd}(\mathrm{sd}(N')),$$

that is the relation

$$\mathrm{sd}(N)\mathrm{sd}(M_l) = \mathrm{sd}(M)\mathrm{sd}(N').$$

This equality of diagonal matrices obviously implies equalities $\mathrm{sd}(N) = \mathrm{sd}(N')$ and $\mathrm{sd}(M_l) = \mathrm{sd}(M)$, which proves the equality (2.12) and the theorem. $\square$

Corollary 2.4. *Under the assumptions of Theorem* 2.3*, if a modular form* $F \in \mathfrak{M}_k(\widehat{\Gamma}(m))$*, is an eigenfunction for the Hecke operator* $|T_{\widehat{\Gamma}(m)}(M)$*, where* $M \in \Delta_{(m)}(\widehat{\Gamma}(m))$*, with the eigenvalue* $\lambda(M)$*, then the function* $F|\mathcal{T}(\chi,\,L) \in \mathfrak{M}_k(\widehat{\Gamma}(m))$ *is an eigenfunction for the operator* $|T_{\widehat{\Gamma}(m)}(M)$ *with the eigenvalue* $\chi(\mu(M))\lambda(M)$*.*

Hecke operators and the star operator. Let us turn now to relations of regular Hecke operators and the star operator (1.33) defined with the help of matrix $\Omega = \Omega^n(m)$ of the form (1.29) on the space $\mathfrak{M}_k(\widehat{\Gamma}(m))$. For regular Hecke operators on the spaces of the form $\mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$ defined by (1.27) we now prove the following proposition.

Proposition 2.5. *The following assertions are valid for the group* $\Lambda = \widehat{\Gamma}(m)$*:*

(1) *Let* $F \in \mathfrak{M}_k(\Lambda,\,\psi)$ *and* $M \in \Delta_{(m)}(\Lambda)$*. Then the relation holds*

$$(F|T_\Lambda(M))^* = \psi(\mu(M))F^*|T_\Lambda(\tau(\mu(M))M^*), \tag{2.14}$$

where $\tau(\mu) = \tau^n(\mu)$ *is a matrix of the form* (1.25)*,* $G^* = G|_k\Omega$*, and*

$$M^* = \Omega^{-1}M\Omega \qquad (\Omega = \Omega^n(m)). \tag{2.15}$$

(2) *The matrix* $\tau(\mu(M))M^*$ *together with* M *belongs to semigroup* $\Delta_{(m)}(\Lambda)$*, and the mapping*

$$M \mapsto \check{M} = \tau(\mu(M))M^* \tag{2.16}$$

defines a bijection of the set $\Delta_{(m)}(\Lambda)$*, which is identical on the sets of double cosets modulo* Λ *contained in* $\Delta_{(m)}(\Lambda)$*, so that*

$$T_\Lambda(\check{M}) = T_\Lambda(M) \quad \text{for all} \quad M \in \Delta_{(m)}(\Lambda). \tag{2.17}$$

In particular, the relation (2.14) *can be rewritten in the form*

$$(F|T_\Lambda(M))^* = \psi(\mu(M))F^*|T_\Lambda(M) \qquad \big(F \in \mathfrak{M}_k(\Lambda,\,\psi),\, M \in \Delta_{(m)}(\Lambda)\big). \tag{2.18}$$

Proof. Using (1.33), (2.1), (2.4), (1.35), and (1.2), we have

$$(F|T_\Lambda(M))^* = F|T_\Lambda(M)|\Omega = \sum_{M' \in \Lambda\backslash\Lambda M\Lambda} F|_k M'\Omega$$

$$= \sum_{M' \in \Lambda\backslash\Lambda M\Lambda} F|_k\Omega\tau(\mu(M))^{-1}\Omega^{-1}\Omega\tau(\mu(M))\Omega^{-1}M'\Omega$$

$$= \sum_{M' \in \Lambda\backslash\Lambda M\Lambda} F|_k\tau(\mu(M))\Omega\tau(\mu(M))(M')^*$$

$$= \psi(\mu(M)) \sum_{M' \in \Lambda\backslash\Lambda M\Lambda} F^*|_k\tau(\mu(M))(M')^*$$

$$= \psi(\mu(M)) \sum_{M'' \in \Lambda\backslash\Lambda\tau(\mu(M))M^*\Lambda} F^*|_k M'' = \psi(\mu(M))F^*|T_\Lambda(\tau(M)M^*),$$

since $\Omega^{-1}\Lambda\Omega = \Lambda$, which proves the part (1).

Inclusion $\check{M} \in \Delta_{(m)}(\Lambda)$ follows from definitions. It follows from (1.30) and (1.35) that $(M^*)^* = M$ and

$$\check{\check{M}} = M \quad \text{for all} \quad M \in \Delta_{(m)}(\Lambda).$$

Thus, the mapping (2.16) is a bijection. According to Proposition 2.1 in order to prove the equality (2.17) it suffices to verify the inclusion $\check{M} \in \Gamma M \Gamma$, or that the matrices $\check{M}$ and M have equal matrices of symplectic divisors, $sd(\check{M}) = sd(M)$ (see [4, Lemma 3.3.6]). It follows from the definition that $\check{M}m\Omega^{-1} = \tau(\mu(M))m\Omega^{-1}M$. Since the multiplier $\mu(M) = \mu(\check{M})$ is coprime with the multiplier $\mu(m\Omega^{-1}) = m^2$, by known properties of symplectic divisors, we obtain

$$sd(\check{M})sd(m\Omega^{-1}) = sd(\check{M}m\Omega^{-1}) = sd(\tau(\mu(M))m\Omega^{-1}M) = sd(m\Omega^{-1})sd(M),$$

hence $sd(\check{M}) = sd(M)$. □

Note that the relation (1.35) imply the inclusion

$$F^* \in \mathfrak{M}_k(\widehat{\Gamma}(m), \overline{\psi}) \quad \text{if} \quad F \in \mathfrak{M}_k(\widehat{\Gamma}(m), \psi),$$

where $\overline{\psi}$ is the character conjugate to ψ.

Corollary 2.6. *A modular form $F \in \mathfrak{M}_k(\widehat{\Gamma}(m), \psi)$ is an eigenfunction for the Hecke operator $|T_\Lambda(M)$, where $\Lambda = \widehat{\Gamma}(m)$ and $M \in \Delta_{(m)}(\Lambda)$, with the eigenvalue $\lambda_F(M)$ if and only if the function $F^* \in \mathfrak{M}(\widehat{\Gamma}(m), \overline{\psi})$ is an eigenfunction for the operator $|T_\Lambda(M)$ with eigenvalue*

$$\lambda_{F^*}(M) = \overline{\psi}(\mu(M))\lambda_F(M). \tag{2.19}$$

Eigenfunctions of regular Hecke operators. As to existence of eigenfunctions for regular Hecke operators, the following proposition shows, in particular, that quite often spaces of modular forms are spanned by the common eigenfunctions.

Proposition 2.7. (1) *The operators corresponding to matrices $\tau(d)$ of the form* (1.25) *with d prime to m on spaces $\mathfrak{M}_k(\widehat{\Gamma}(m))$ commute with all regular Hecke operators. In particular, each of the subspaces $\mathfrak{M}_k(\widehat{\Gamma}(m), \psi)$ of the form* (1.27) *with $\psi \in \mathrm{Char}(\mathrm{GL}_1(\mathbb{Z}/m\mathbb{Z}))$ is invariant under each regular Hecke operator.*

(2) *Each of the subspaces cusp forms $\mathfrak{N}_k(\widehat{\Gamma}(m), \psi)$ has a basis consisting of common eigenfunctions for all regular Hecke operators; in particular, the whole space $\mathfrak{N}_k(\widehat{\Gamma}(m))$ has such a basis.*

Proof. The assertions are proved in [4, Lemma 4.1.5 and Theorem 4.1.8]. □

§3. Regular zeta functions of twisted forms and L-functions

Zeta functions and L-functions of modular forms. Let us consider sums of all different double cosets of fixed multipliers prime to m, which are contained in m-regular Hecke–Shimura rings $\mathcal{H}_{(m)}(\Lambda)$ of the form (2.5) for the groups Λ equal to $\Gamma = \Gamma^n$ and $\widehat{\Gamma}(m) = \widehat{\Gamma}^n(m)$, i.e., the elements of the form

$$T_\Lambda(a) = \sum_{M \in \Lambda\backslash\Delta_{(m)}(\Lambda)/\Lambda,\, \mu(M)=a} T_\Lambda(M), \quad \text{where } a \text{ is prime to } m. \tag{3.1}$$

Theorem 3.1. *The elements* (3.1) *for the groups* Λ *of the form* Γ *and* $\widehat{\Gamma}(m)$ *satisfy the following rules:*

$$T_\Lambda(a)T_\Lambda(a') = T_\Lambda(aa') \quad \textit{if } a \textit{ and } a' \textit{ are coprime}; \tag{3.2}$$

for each prime number p not dividing the number m, the formal power series over the ring $\mathcal{H}_{(m)}(\Lambda)$ *with coefficients* $T_\Lambda(1), T_\Lambda(p), T_\Lambda(p^2), \dots$ *is formally equal to a rational fraction with coefficients in* $\mathcal{H}_{(m)}(\Lambda)$*, whose denominator and numerator are polynomials of degree* 2^n *and at most* $2^n - 2$*, respectively,*

$$\sum_{\delta=0}^{\infty} T_\Lambda(p^\delta)t^\delta = Q_{p,\Lambda}(t)^{-1}R_{p,\Lambda}(t), \tag{3.3}$$

where

$$Q_{p,\Lambda}(t) = \sum_{i=0}^{2^n}(-1)^i\mathbf{q}^i_\Lambda(p)t^i, \qquad R_{p,\Lambda}(t) = \sum_{i=0}^{2^n-2}(-1)^i\mathbf{r}^i_\Lambda(p)t^i, \tag{3.4}$$

and coefficients $\mathbf{q}^i_\Lambda(p)$ *and* $\mathbf{r}^i_\Lambda(p)$ *belong to* $\mathcal{H}_{(m)}(\Lambda)$*. In addition, the coefficients of the denominator* $Q_{p,\Lambda}(t)$ *satisfy relations*

$$\mathbf{q}^0_\Lambda(p) = [\mathbf{1}]_\Lambda, \quad \mathbf{q}^1_\Lambda(p) = T_\Lambda(p), \quad \mathbf{q}^{2^n}_\Lambda(p) = \left(p^{n(n+1)/2}[\mathbf{p}]_\Lambda\right)^{2^{n-1}}, \tag{3.5}$$

and the symmetry relations

$$\mathbf{q}^{2^n-i}_\Lambda(p) = \left(p^{n(n+1)/2}[\mathbf{p}]_\Lambda\right)^{2^{n-1}-i}\mathbf{q}^i_\Lambda(p) \qquad (0 \le i \le 2^n), \tag{3.6}$$

where

$$[\mathbf{a}]_\Lambda = T_\Lambda\left(a\tau(a)\right) \tag{3.7}$$

and $\tau(a) = \tau^n(a)$ *has the form* (1.25).

Proof. According to the isomorphism of the rings $\mathcal{H}_{(m)}(\Gamma)$ and $\mathcal{H}_{(m)}(\Gamma)$ of Proposition 2.1, it suffices to prove the theorem only for the group $\Lambda = \Gamma = \Gamma^n$. But for the group Γ^n all of the assertions are well-known: the analog of relations (3.2) was proved in [12], the analog of summation formula (3.3) was established in [1], and relations analogous to (3.5)–(3.6) were checked in [4, §3.3.3]. □

It follows from (3.2) and (3.3) that the formal Dirichlet series with the coefficients $\eta(a)T_\Lambda(a)$ for a prime to m, where $\eta(a)$ is a completely multiplicative complex-valued function in a, can be expanded into formal Euler product:

$$D_\Lambda(s,\eta) = \sum_{a\in\mathbb{N},\,\gcd(a,m)=1} \frac{\eta(a)T_\Lambda(a)}{a^s} \tag{3.8}$$

$$= \prod_{p\in\mathbb{P},\,p\nmid m}\sum_{\delta=0}^{\infty}\eta(p)^\delta T_\Lambda(p^\delta)p^{-\delta s} = \prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\Lambda}(\eta(p)p^{-s})^{-1}R_{p,\Lambda}(\eta(p)p^{-s})$$

$$= \left(\prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\Lambda}(\eta(p)p^{-s})\right)^{-1}\left(\prod_{p\in\mathbb{P},\,p\nmid m} R_{p,\Lambda}(\eta(p)p^{-s})\right)$$

where the symbol a^s is considered just as a formal quasicharacter of the multiplicative semigroup $\mathbb{N}$ of positive integers. (We remind that, by Proposition 2.2, the rings $\mathcal{H}_{(m)}(\Lambda)$ are commutative.)

Let us turn now to regular Hecke operators on the spaces $\mathfrak{M}_k(\Lambda)$ of modular forms of weight k for these groups Λ. Suppose that we are given an eigenfunction $F\in\mathfrak{M}_k(\Lambda)$ for all Hecke operators $|T$ with $T\in\mathcal{H}_{(m)}(\Lambda)$,

$$F|T = \lambda_F(T)F \qquad (\forall\, T\in\mathcal{H}_{(m)}(\Lambda)). \tag{3.9}$$

We set $\lambda_F(T_\Lambda(a)) = \lambda_F(a)$ so that

$$F|T_\Lambda(a) = \lambda_F(a)F \qquad (\gcd(a,m)=1). \tag{3.10}$$

On replacing of all coefficients $T\in\mathcal{H}_{(m)}(\Lambda)$ in the formal identity (3.8) with the eigenvalues $\lambda_F(T)$ of the corresponding Hecke operators acting on the eigenfunction F, we obtain a formal Euler product expansion over $\mathbb{C}$ of the form

$$D_F(s,\eta) = \sum_{a\in\mathbb{N},\,\gcd(a,m)=1} \frac{\eta(a)\lambda_F(a)}{a^s} \tag{3.11}$$

$$= \left(\prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\,F}(\eta(p)p^{-s})\right)^{-1}\left(\prod_{p\in\mathbb{P},\,p\nmid m} R_{p,\,F}(\eta(p)p^{-s})\right),$$

where

$$Q_{p,\,F}(t) = \sum_{i=0}^{2^n}(-1)^i\lambda_F\big(\mathbf{q}^i_\Lambda(p)\big)t^i, \qquad R_{p,\,F}(t) = \sum_{i=0}^{2^n-2}(-1)^i\lambda_F\big(\mathbf{r}^i_\Lambda(p)\big)t^i. \tag{3.12}$$

Assuming now that the function $\eta(a)$ grows not faster than a constant power of a, and using known estimates of eigenvalues of Hecke operators, it is not hard to see that the infinite series and products occurring in the formal identity (3.11) converge absolutely and uniformly in a right half-plane Re $s > c$ of the variable s, depending on F and η, and so define there holomorphic functions in s. (For the case of the group $\Lambda = \Gamma^n$ see [3, §1.3]). We shall call the function

$$L_F(s,\,\eta) = \left(\prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\,F}(\eta(p)p^{-s})\right)^{-1} \tag{3.13}$$

the ($m-$*regular*) *L-function of the eigenfunction* F *with "character"* η and call the function

$$Z_F(s) = L_F(s,\,1) = \left(\prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\,F}(p^{-s})\right)^{-1} \tag{3.14}$$

the ($m-$*regular*) *zeta-function of the eigenfunction* F.

Zeta functions of twisted forms and L-functions.

Theorem 3.2. *For the group* $\Lambda = \widehat{\Gamma}(m) = \widehat{\Gamma}^n(m)$, *let* $F \in \mathfrak{M}_k(\Lambda)$ *be an eigenfunction for all regular Hecke operator* $|T_\Lambda(M)$ *with* $M \in \Delta_{(m)}(\Lambda)$. *Then, for every primitive Dirichlet character* χ *modulo* m *and each* p*-matrix* L, *the twisted form* $F|\mathcal{T}(\chi,\,L) \in \mathfrak{M}_k(\Lambda)$ *is an eigenfunction for all regular Hecke operator, and zeta-function of the twisted form in every domain of absolute convergence is equal to the L-function of the form* F *with character* χ*:*

$$Z_{F|\mathcal{T}(\chi,\,L)}(s) = L_F(s,\,\chi), \tag{3.15}$$

if the twisted form is not identically zero. Moreover, if a form $F \in \mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$, *where* ψ *is a Dirichlet character modulo* m, *then the form* $\big(F|\mathcal{T}(\chi,\,L)\big)^*$, *where the star is the mapping* (1.33), *contained in the space* $\mathfrak{M}_k(\widehat{\Gamma}(m),\,\overline{\chi^2\psi})$, *is also an eigenfunction for all regular Hecke operators for the group* $\widehat{\Gamma}(m)$, *and zeta-function of this form in every domain of absolute convergence is equal to the L-function of the form* F *with conjugated product of charactes* $\overline{\chi\psi}$*:*

$$Z_{(F|\mathcal{T}(\chi,\,L))^*}(s) = L_F(s,\,\overline{\chi\psi}). \tag{3.16}$$

Proof. By Corollary 2.4, the relation $F|T_\Lambda(a) = \lambda_F(a)F$ implies the relation

$$(F|\mathcal{T}(\chi,\,L))T_\Lambda(a) = \chi(a)\lambda_F(a)F$$

for each a prime to m. Hence, by (3.11) and (3.14), we obtain for each prime p not dividing m the relation

$$Q_{p,\,F|\mathcal{T}(\chi,\,L)}(t) = Q_{p,\,F}(\chi(p)t) \tag{3.17}$$

and the relation

$$Z_{F|T(\chi,\,L)}(s) = \left(\prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\,F|T(\chi,\,L)}(p^{-s})\right)^{-1}$$
$$= \left(\prod_{p\in\mathbb{P},\,p\nmid m} Q_{p,\,F}(\chi(p)p^{-s})\right)^{-1} = L_F(s,\,\chi).$$

In order to prove (3.16), we note that by Corollary 2.6 and relation (1.38), we get, for each a prime to m, the relation

$$\lambda_{(F|T(\chi,\,L))^*}(a) = \overline{\chi}^2(a)\overline{\psi}(a)\lambda_{F|T(\chi,\,L)}(a).$$

Hence and by (3.17), for each prime p not dividing m, follows the relation

$$Q_{p,\,(F|T(\chi,\,L))^*}(t) = Q_{p,\,F|T(\chi,\,L)}(\overline{\chi}^2(p)\overline{\psi}(p)t)$$
$$= Q_{p,\,F}(\chi(p)\overline{\chi}^2(p)\overline{\psi}(p))t) = Q_{p,\,F}(\overline{\chi}(p)\overline{\psi}(p)t)$$

and the relation (3.16). □

Note that, as follows from the definition of the subspaces $\mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$, every modular form $F \in \mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$ is an eigenfunction of each Hecke operator $|_k[\mathbf{a}]_\Lambda$ corresponding to element (3.7) for $\Lambda = \widehat{\Gamma}(m)$ with the eigenvalue $\psi(a)a^{nk-n(n+1)}$:

$$F|_k[\mathbf{a}]_\Lambda = \psi(a)a^{nk-n(n+1)}F \qquad (F \in \mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi), \quad \gcd(a,\,m) = 1). \tag{3.18}$$

According to (1.28) the twist with character χ of every modular form of the spaces or $\mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$ belongs to the space $\mathfrak{M}_k(\widehat{\Gamma}(m),\,\chi^2\psi)$. Hence, for twists of forms from $\mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$ we obtain relations

$$(F|\mathcal{T}(\chi,\,L))|_k[\mathbf{a}]_\Lambda = \chi(a^2)\psi(a)a^{nk-n(n+1)}(F|\mathcal{T}(\chi,\,L)) \quad (\gcd(a,\,m) = 1). \tag{3.19}$$

These relations and relations (3.5), (3.6) allows us to compute the constant and leading coefficients of the denominators of p-factors of zeta-functions of the twisted form $G = F|\mathcal{T}(\chi,\,L)$ of an eigenfunction $F \in \mathfrak{M}_k(\widehat{\Gamma}(m),\,\psi)$ in the form

$$\lambda_G\left(\mathbf{q}^0_\Lambda(p)\right) = 1, \qquad \lambda_G\left(\mathbf{q}^{2^n}_\Lambda(p)\right) = \left(p^{nk-\langle n\rangle}\chi(p^2)\psi(p)\right)^{2^{n-1}}, \tag{3.20}$$

and write the general symmetry relations for the coefficients of the denominators in the form

$$\lambda_G\left(\mathbf{q}^{2^n-i}_\Lambda(p)\right) = \left(p^{nk-\langle n\rangle}\chi(p^2)\psi(p)\right)^{2^{n-1}-i}\lambda_G\left(\mathbf{q}^i_\Lambda(p)\right) \quad (0 \le i \le 2^n). \tag{3.21}$$

In particular, if $n = 1$, then defined in (3.12) polynomials $Q_{p,\,F}(t)$ and $Q_{p,\,G}(t)$ have the form

$$Q_{p,\,F}(t) = 1 - \lambda_F(T_\Lambda(p))t + \psi(p)p^{k-1}t^2 \tag{3.22}$$

and

$$Q_{p,\,G}(t) = 1 - \lambda_G(T_\Lambda(p))t + \chi(p^2)\psi(p)p^{k-1}t^2 = Q_{p,\,F}(\chi(p)t). \tag{3.23}$$

§4. L-functions of cusp forms of genus 1

In this section we apply the reduction theorem 3.2 to the simplest case of modular forms in one variable and prove an implication of Atkin-Lehner theory [7] for "new" forms. Here we assume that the genus n is equal to 1.

By Proposition 2.7, when we speak of eigenfunctions of regular Hecke operators, we may restrict ourselves to consideration of modular forms contained in spaces of the form (1.27). Let $F \in \mathfrak{M}_k(\widehat{\Gamma}(m), \psi)$, where ψ is a multiplicative character modulo m, be a modular form of an integral weight k for the congruence subgroup $\widehat{\Gamma}(m) = \widetilde{\Gamma}^1(m)$ of the form (1.5) of the modular group $\Gamma = \Gamma^1$ with Fourier expansion

$$F(z) = f(0) + \sum_{a=1}^{\infty} f(a)e^{2\pi iaz} \qquad (z = x + iy \in \mathbb{H} = \mathbb{H}^1). \tag{4.1}$$

We shall assume that F is an eigenfunction for all regular Hecke operators $|T = |T_\Lambda$ with $T \in \mathcal{H}_{(m)}(\Lambda)$, where $\Lambda = \widehat{\Gamma}(m)$:

$$F|T = \lambda_F(T)F \qquad (\forall\ T = T_\Lambda \in \mathcal{H}_{(m)}(\Lambda)). \tag{4.2}$$

Further, we denote by

$$G = F|\mathcal{T}(\chi, 1) = F|\mathcal{T}(\chi)$$

the image of F under the twist operator (4) with a fixed primitive Dirichlet character χ modulo m and $p-$matrix $L = l = 1$. By the definition the function G has Fourier expansion

$$G(z) = \sum_{a=1}^{\infty} g(a)e^{2\pi iaz}, \text{ where } g(a) = \chi(a)f(a), \tag{4.3}$$

and, according to (1.28) belongs to the space $\in \mathfrak{M}_k(\widehat{\Gamma}(m), \chi^2\psi)$,

$$G \in \mathfrak{M}_k(\widehat{\Gamma}(m), \chi^2\psi). \tag{4.4}$$

Then, according to Corollary 2.4, the form G is an eigenfunction for all regular Hecke operators $|T_\Lambda$ with $\Lambda = \widehat{\Gamma}(m)$ and $T_\Lambda \in \mathcal{H}_{(m)}(\Lambda)$. Besides, the eigenvalues of Hecke operators corresponding to elements of the form (3.1) for $\Lambda = \widehat{\Gamma}(m)$ acting on G and F, respectively, satisfy the relation

$$\lambda_G(a) = \chi(a)\lambda_F(a) \qquad (a \in \mathbb{N}, \gcd(a, m) = 1). \tag{4.5}$$

Thus, we have two sequences of complex numbers associated with each of the eigenfunctions: in the first place, it is the sequence of Fourier coefficients, and in the second place, the sequence of eigenvalues of Hecke operators. The natural question is whether these sequences are related to each other. The question is interesting in two respects: first of all, because of multiplicative properties of Hecke operators and their eigenvalues, such relations could reveal multiplicative properties of Fourier coefficients, which often presents an arithmetical interest, besides, analytical properties of the modular forms considered as generating series for their Fourier coefficients may possibly be transferred to analytical properties of generating functions for the eigenvalues and corresponding Euler products.

Fourier coefficients of an eigenfunction and eigenvalues. The relation between Fourier coefficients of an eigenfunction and the corresponding eigenvalues for modular forms in one variable were discovered by Hecke and look very simple.

Lemma 4.1. *Suppose that a modular form* $F \in \mathfrak{M}_k(\widehat{\Gamma}(m), \psi)$ *with Fourier expansion* (4.1) *is an eigenfunction of the Hecke operator* $T(d) = T_{\widehat{\Gamma}(m)}(d)$ *of the form* (3.1) *for the group* $\widehat{\Gamma}(m)$ *with* d *prime to* m *and let* $\lambda_F(d)$ *be the corresponding eigenvalue. Then the relations hold*

$$\lambda_F(d)f(0) = f(0)\sum_{\alpha \mid d} \psi(\alpha)\alpha^{k-1} \tag{4.6}$$

and

$$\lambda_F(d)f(a) = \sum_{\alpha \mid a,\, d} \psi(\alpha)\alpha^{k-1} f\left(\frac{ad}{\alpha^2}\right) \qquad (a \geq 1), \tag{4.7}$$

where α *ranges over all positive divisors of* d *and common positive divisors of* a *and* d*, respectively. In particular,*

$$\lambda_F(d)f(1) = f(d). \tag{4.8}$$

Proof. It is well known that elements $T_\Gamma(d)$ for the group $\Gamma = \Gamma^1$ have decompositions into left cosets of the form

$$T_\Gamma(d) = \sum_{\substack{\alpha,\beta,\delta\in\mathbb{N},\\ \alpha\delta=d,\, 0\leq\beta\leq\delta}} \left(\Gamma\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\right).$$

Then it follows from Proposition 2.1 that the elements $T_{\widehat{\Gamma}(m)}(d)$ have decompositions of the form

$$T_{\widehat{\Gamma}(m)}(d) = \sum_{\substack{\alpha,\beta,\delta\in\mathbb{N},\\ \alpha\delta=d,\, 0\leq\beta<\delta}} \left(\widehat{\Gamma}(m)\tau(\alpha)\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\right).$$

By the definition of Hecke operators, we obtain

$$\begin{aligned}
F|_k T_{\widehat{\Gamma}(m)}(d) &= \sum_{\substack{\alpha,\beta,\delta\in\mathbb{N},\\ \alpha\delta=d,\, 0\le\beta<\delta}} F|_k \left(\tau(\alpha)\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\right)\\
&= \sum_{\substack{\alpha,\beta,\delta\in\mathbb{N},\\ \alpha\delta=d,\, 0\le\beta<\delta}} \psi(\alpha) d^{k-1}\delta^{-k} F\left(\frac{\alpha z+\beta}{\delta}\right)\\
&= \sum_{\substack{\alpha,\beta,\delta\in\mathbb{N},\\ \alpha\delta=d,\, 0\le\beta<\delta}} \psi(\alpha) d^{k-1}\delta^{-k} \left(f(0)+\sum_{a=1}^{\infty} f(a) e^{2\pi i a(\alpha z+\beta)\delta^{-1}}\right)\\
&= f(0)\sum_{\alpha|d}\psi(\alpha)\alpha^{k-1} + \sum_{\alpha|d}\psi(\alpha)\alpha^{k-1}\sum_{a'=1}^{\infty} f\left(\frac{a'd}{\alpha}\right) e^{2\pi i a'\alpha z}\\
&= f(0)\sum_{\alpha|d}\psi(\alpha)\alpha^{k-1} + \sum_{a=1}^{\infty}\left(\sum_{\alpha|a,\, d}\psi(\alpha)\alpha^{k-1} f\left(\frac{ad}{\alpha^2}\right)\right) e^{2\pi i a z}.
\end{aligned}$$

On the other hand we have

$$F|_k T_{\widehat{\Gamma}(m)}(d) = \lambda_F(d) F = \lambda_F(d) f(0) + \sum_{a=1}^{\infty} \lambda_F(d) f(a) e^{2\pi i a z}.$$

By comparing corresponding Fourier coefficients of the last two expansions we obtain the relations (4.6) and (4.7). The relation (4.8) follows from (4.7) for $a=1$. □

Multiplying both sides of (4.7) with a prime to m by $f(1)$ and using (4.8), we obtain multiplicative relations for the eigenvalues in the form

$$f(1)\lambda_F(d)\lambda_F(a) = f(1)\sum_{\alpha|\, a,\, d}\psi(\alpha)\alpha^{k-1}\lambda_F\left(\frac{ad}{\alpha^2}\right). \tag{4.9}$$

It is an easy consequence of these relations (see, e.g., [4, §4.3.1]) that the following formal Euler product factorizations holds for Dirichlet series with coefficients of the form $\eta(a)\lambda_F(a)$, where $\eta : \mathbb{N} \mapsto \mathbb{C}$ is a completely multiplicative function, say, a Dirichlet character,

$$\sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{\eta(a)\lambda_F(a)}{a^s} = \prod_{p\in\mathbb{P},\, p\nmid m}\left(1-\eta(p)\lambda_F(p)p^{-s}+\eta(p^2)\psi(p)p^{k-1-2s}\right)^{-1}. \tag{4.10}$$

On the other hand, by (4.8), for the same η as above, we get a formal identity of Dirichlet series

$$\sum_{a\in\mathbb{N},\, \gcd(a,m)=1}\frac{\eta(a)f(a)}{a^s} = f(1)\sum_{a\in\mathbb{N},\, \gcd(a,m)=1}\frac{\eta(a)\lambda_F(a)}{a^s}. \tag{4.11}$$

The formal relations (4.10)–(4.11) and known estimates of Fourier coefficients and eigenvalues imply identities between generating Dirichlet series for Fourier coefficients of eigenfunctions and corresponding zeta functions and $L-$functions in half-planes of absolute and uniform convergence.

Theorem 4.2. *Let a modular form $F \in \mathfrak{M}_k(\widehat{\Gamma}(m),\, \psi)$ with Fourier expansion* (4.1) *be an eigenfunction for all regular Hecke operators of the form $T(d) = T_{\widehat{\Gamma}(m)}(d)$ for d prime to m with the eigenvalues $\lambda_F(d)$, and let χ be a primitive Dirichlet character modulo $m \geq 1$. Then in the half-plane* $\operatorname{Re} s > k$, *if $f(0) \neq 0$, and in the half-plane* $\operatorname{Re} s > k/2 + 1$, *if F is a cusp form, the following identities are valid*

$$\sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{f(a)}{a^s} = f(1) \sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{\lambda_F(a)}{a^s} = f(1) Z_F(s), \tag{4.12}$$

and

$$\sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{\chi(a) f(a)}{a^s} = f(1) \sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{\chi(a)\lambda_F(a)}{a^s} = f(1) L_F(s,\, \chi), \tag{4.13}$$

where $Z_F(s)$ and $L_F(s,\, \chi)$ are the zeta function (3.14) *and the $L-$function* (3.13) *with character χ of the eigenfunction F, respectively.*

If $f(0) \neq 0$, then the zeta function and the $L-$function can be explicitly written in the form

$$Z_F(s) = \sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{1}{a^s} \sum_{a\in\mathbb{N},\, \gcd(a,m)=1} \frac{\psi(a)}{a^{s-k+1}} \tag{4.14}$$

and

$$L_F(s,\, \chi) = L(s,\, \chi) L(s-k+1,\, \chi\psi), \tag{4.15}$$

where

$$L(s,\, \eta) = \sum_{a=1}^{\infty} \frac{\eta(a)}{a^s} = \prod_{p\in\mathbb{P}} \left(1 - \frac{\eta(p)}{p^s}\right)^{-1} \qquad (\operatorname{Re} s > 1) \tag{4.16}$$

is the Dirichlet L-functions with character η.

Analytical properties of L-functions. The identity (4.13) allows one to investigate analytical properties of L-functions $L_F(s,\, \chi)$. According to Theorem 3.2, it suffices to consider zeta-functions of the twisted forms G. Analytical properties of Dirichlet $L-$functions are well known and we can limit ourselves to the consideration of $L-$functions of cusp forms.

Theorem 4.3. *Let a cusp form $F \in \mathfrak{N}_k(\widehat{\Gamma}(m),\, \psi)$ be an eigenfunction of all regular Hecke operators for the group $\widehat{\Gamma}(m)$, let $G = F|\mathcal{T}(\chi) \in \mathfrak{M}_k(\widehat{\Gamma}(m),\, \chi^2\psi)$ be the twist* (4.3) *of F with a primitive Dirichlet character χ modulo $m > 1$, and let $G^* \in \mathfrak{N}_k(\widehat{\Gamma}(m),\, \overline{\chi^2\psi})$ the image of G under the operator* (1.33). *Suppose that the first Fourier coefficient $f(1)$ of F is not zero. Then the following assertions hold:*

(1) *The zeta function* (3.14) *of the eigenfunction* G *in the half-plane* $\mathrm{Re}\, s > k/2+1$ *satisfies the identity*

$$\Psi_G(s) = m^{s-k/2}(2\pi)^{-s}\Gamma(s)Z_G(s) \tag{4.17}$$

$$= \frac{1}{g(1)}\left(m^{s-k/2}\int_{1/m}^{\infty} y^{s-1}G(iy)dy + (-i)^k m^{k/2-s}\int_{1/m}^{\infty} y^{k-1-s}G^*(iy)dy\right),$$

where $g(1)$ *is the first coefficient of the Fourier expansion* (4.3) *of* G, $\Gamma(s)$ *is the gamma-function, and* $G \mapsto G^*$ *is the mapping* (1.33).

(2) *The right hand part of the identity* (4.17) *is holomorphic for all* s. *Thus, the function* $\Psi_G(s)$ *has analytical continuation over whole* s*-plane as a holomorphic function.*

(3) *The function* $\Psi_G(s)$ *satisfies the functional equation*

$$\Psi_G(k-s) = (-i)^k\frac{g^*(1)}{g(1)}\Psi_{G^*}(s), \tag{4.18}$$

where, for the eigenfunction $G^* = (F|\mathcal{T}(\chi))^* \in \mathfrak{N}_k(\widehat{\Gamma}(m), \overline{\chi^2\psi})$ *of all regular Hecke operators, we set*

$$\Psi_{G^*}(s) = m^{s-k/2}(2\pi)^{-s}\Gamma(s)Z_{G^*}(s), \tag{4.19}$$

$Z_{G^*}(s)$ *is the zeta function of* G^*, *and* $g^*(1)$ *is the first Fourier coefficient of* G^*.

(4) *The function*

$$\Psi_F(s, \chi) = m^{s-k/2}(2\pi)^{-s}\Gamma(s)L_F(s, \chi), \tag{4.20}$$

where $L_F(s, \chi)$ *is the* $L-$*function* (3.13) *of* F *with character* χ, *has analytical continuation over whole* s*-plane as a holomorphic function and satisfies the functional equation*

$$\Psi_F(k-s, \chi) = (-i)^k\frac{g^*(1)}{f(1)}\Psi_F(s, \overline{\chi\psi}), \tag{4.21}$$

where $f(1)$ *and* $g^*(1)$ *are the first Fourier coefficients of* F *and* $(F|\mathcal{T}(\chi))^*$, *respectively.*

Proof. According to Theorem 3.2 and formula (4.3), we can write the identity (4.13) in the shape

$$R_G(s) = \sum_{a=1}^{\infty}\frac{g(a)}{a^s} = \sum_{a=1}^{\infty}\frac{\chi(a)f(a)}{a^s} = f(1)L_F(s, \chi) = f(1)Z_G(s). \tag{4.22}$$

Using the *Euler integral*

$$\int_0^{\infty} y^{s-1}e^{-\alpha y}dy = \Gamma(s)\alpha^{-s} \qquad (\alpha > 0,\ \mathrm{Re}\, s > 0), \tag{4.23}$$

where $\Gamma(s)$ is the gamma-function, we obtain

$$\int_0^\infty y^{s-1}G(iy)dy = \sum_{a=1}^\infty g(a)\int_0^\infty y^{s-1}e^{-2\pi ay}dy$$
$$= (2\pi)^{-s}\Gamma(s)\sum_{a=1}^\infty \frac{g(a)}{a^s} = (2\pi)^{-s}\Gamma(s)R_G(s) \qquad (\operatorname{Re} s > k/2+1).$$

Hence, by (4.22), we have

$$\Psi_G(s) = \frac{1}{g(1)}m^{s-k/2}(2\pi)^{-s}\Gamma(s)R_G(s) = \frac{1}{g(1)}m^{s-k/2}\int_0^\infty y^{s-1}G(iy)dy$$
$$= \frac{1}{g(1)}\left(m^{s-k/2}\int_{1/m}^\infty y^{s-1}G(iy)dy + m^{s-k/2}\int_0^{1/m} y^{s-1}G(iy)dy\right).$$

On replacing of y by $1/m^2y$, we can write

$$\int_0^{1/m} y^{s-1}G(iy)dy = \int_\infty^{1/m}(m^2y)^{1-s}G(i/m^2y)\frac{-dy}{m^2y^2}$$
$$= (-i)^k m^{k-2s}\int_{1/m}^\infty y^{k-1-s}(-imy)^{-k}G(i/m^2y)dy$$
$$= (-i)^k m^{k-2s}\int_{1/m}^\infty y^{k-1-s}G^*(iy)dy,$$

since, by definition, $G^*(z) = G|_k\begin{pmatrix} 0 & 1/m \\ -m & 0\end{pmatrix} = (-mz)^{-k}G(-1/m^2z)$. Substituting this expression on the right of (4.13), we came to the identity (4.17).

Since G is a cusp form, absolute values of both integrand on the right of (4.17) decrease exponentially as $y \to +\infty$. Hence, the both integral converge absolutely and uniformly for all $s \in \mathbb{C}$ and define everywhere holomorphic functions.

By formula (4.17) with G^* in place of G and $k-s$ in place of s we get

$$\Psi_{G^*}(k-s) = \frac{1}{g^*(1)}\Bigg(m^{k-s-k/2}\int_{1/m}^\infty y^{k-s-1}G^*(iy)dy$$
$$+ (-i)^k m^{k/2-(k-s)}\int_{1/m}^\infty y^{k-1-(k-s)}G^{**}(iy)dy\Bigg)$$

$$= i^k\frac{g(1)}{g^*(1)}\frac{1}{g(1)}\left((-i)^k\int_{1/m}^\infty y^{k-s-1}G^*(iy)dy + m^{s-k/2}\int_{1/m}^\infty y^{s-1}G(iy)dy\right)$$
$$= i^k\frac{g(1)}{g^*(1)}\Psi_G(s),$$

since, in view of (1.34), $(G^*)^* = (-1)^kG$. The functional equation (4.18) follows.

Finally, returning to the L-functions, by Theorem 3.2 we have the equality $\Psi_F(s,\chi) = \Psi_G(s)$, and the functional equation (4.21) follows from the functional equation (4.18), since $g(1) = \chi(1)f(1) = f(1)$ by (4.3), and $Z_{G^*}(s) = L(s,\overline{\chi\psi})$ by (3.16). $\square$

§5. L-functions of cusp forms of genus 2

L-functions of cusp forms for Γ^2. In this section we assume that the genus $n = 2$, and set

$$\Gamma = \Gamma^2, \quad \Upsilon = \Gamma_0^2(m^2) \quad \text{and } \Lambda = \widehat{\Gamma}^2(m) \tag{5.0}$$

with fixed integral $m > 1$. These and other notation and assumptions of this subsection will be preserved by the end of this section.

Let $F \in \mathfrak{N}_k(\Gamma)$ be a cusp form of integral weight k with the Fourier expansion

$$F(Z) = \sum_{N \in \mathfrak{N}^2,\, N>0} f(N)\, e^{2\pi i Tr(NZ)}. \tag{5.1}$$

The Fourier coefficients $f(N)$ satisfy the relations

$$f(N[V]) = (\det V)^k f(N) \qquad (N \in \mathfrak{N}^2,\, V \in GL_2(\mathbb{Z})) \tag{5.2}$$

and the estimate

$$|f(N)| \le c(\det N)^{k/2} \qquad (N \in \mathfrak{N}^2,\, N > 0) \tag{5.3}$$

with a constant $c = c_F$ depending only on F. Suppose that F is an eigenfunction for all $m-$regular Hecke operators for the group Γ. In particular, for each prime $p \nmid m$ we have

$$\begin{cases} F|_k T_\Gamma(p^\delta) = \lambda(p^\delta,\, F)F \quad (\delta = 0, 1, 2, \dots), \\ F|_k T_\Gamma^1(p^2) = \lambda_1(p^2,\, F)F, \\ F|_k T_\Gamma^2(p^2) = F|_k[\mathbf{p}]_\Gamma = p^{2k-6}F, \end{cases} \tag{5.4}$$

where elements $T_\Gamma(a)$ have the form (3.1) and elements $T_\Gamma^j(p^2)$ were defined in Proposition 2.2, and the last formula follows from (3.18). According to Theorem 3.1, the generating series for the eigenvalues $\lambda(p^\delta,\, F)$ with prime p not dividing m is a rational fraction of the form

$$\sum_{\delta=0}^{\infty} \lambda(p^\delta,\, F)t^\delta = R_{p,F}(t)\, Q_{p,F}(t)^{-1}$$

with the numerator $R_{p,F}$ of degree 2 and the denominator $Q_{p,F}(t)$ of degree 4. By [12, Theorem 2] and (5.4), the numerator and the denominator can be written in the form

$$\begin{aligned} R_{p,F}(t) &= 1 - p^{2k-4}t^2, \\ Q_{p,F}(t) &= 1 - \lambda(p,\, F)t + \big(p\lambda_1(p^2,\, F) + p^{2k-5}(p^2+1)\big)t^2 \\ &\quad - p^{2k-3}\lambda(p,\, F)t^3 + p^{4k-6}t^4. \end{aligned} \tag{5.5}$$

Then the ($m-$regular spinor) zeta function of the eigenfunction F has the form

$$Z_F(s) = \prod_{p \in \mathbb{P},\, p \nmid m} Q_{p,F}(p^{-s})^{-1}$$

and L-function of F with a Dirichlet character χ modulo m is defined by

$$L_F(s,\,\chi)=\prod_{p\in\mathbb{P},\,p\nmid m}Q_{p,F}(\chi(p)p^{-s})^{-1}. \tag{5.6}$$

It follows from estimates (5.3) and direct formulas for the action of Hecke operators on the Fourier coefficients that these Euler products and corresponding Dirichlet series converge absolutely and uniformly in a right half-plane of the complex variable s and therefore define there holomorphic functions in s.

On the other hand, let

$$G=F|\mathcal{T}(\chi,\,L)=\sum_{N\in\mathfrak{N}^2,\,N>0}g(N)\,e^{2\pi iTr(NZ)} \tag{5.7}$$

be a twist (3) of the form F with the character χ and a $p-$matrix $L={}^tL\in\mathbb{Z}_2^2$. Fourier coefficients of forms G and F are related by

$$g(N)=\chi(Tr(LN))\,f(N)\qquad(N\in\mathfrak{N}^2). \tag{5.8}$$

By Theorem 1.3, the form G is a cusp form of weight k for the group Λ. By Corollary 2.4, if the character χ is primitive modulo m, the function G is an eigenfunction for all $m-$regular Hecke operators for the group Λ. Assuming in addition that the form G is not identically zero, by Theorem 3.2, we conclude that L-function (5.6) of the form F is equal to the zeta function of G in every domain of absolute convergence of the functions,

$$L_F(s,\,\chi)=Z_G(s). \tag{5.9}$$

Thus, as in the case of modular forms of genus $n=1$, we have two sequences of complex numbers associated with the eigenfunctions: the sequence of Fourier coefficients and the sequence of eigenvalues of Hecke operators. The question on whether these sequences are related to each other is again of interest by similar reasons.

Unfortunately, relations between individual Fourier coefficients and eigenvalues look good only for modular form of genus $n=1$ (see, e.g., Lemma 4.1). For genera $n\geq 2$, one have to consider relations between appropriate generating functions such as zeta-functions for eigenvalues and appropriate Dirichlet series formed by Fourier coefficients. It turns out that, at least for $n=2$, as appropriate generating functions constructed by Fourier coefficients, one can take so-called "radial" Dirichlet series. For a half-integer positive definite matrix $N\in\mathfrak{N}^2$ and the cusp form G with Fourier expansion (5.7), the *radial Dirichlet series of* G *relative to the ray* $\{rN\mid r\in\mathbb{N}\}$ or just $N-$*ray series of* G is defined by

$$R_G(s,\,N)=\sum_{r=1}^{\infty}\frac{g(rN)}{r^s}. \tag{5.10}$$

It follows from estimations (5.3) that each radial series converges absolutely and uniformly in the right half-plane

$$\text{Re } s>k+1+\varepsilon\quad\text{with }\varepsilon>0 \tag{5.11}$$

and so defines there a holomorphic function in s. The idea to apply radial Dirichlet series to investigation of zeta functions of modular forms goes back to Hecke [8] in the case of one variable.

1−ray series of twisted forms and L-functions. As it was shown in [3, Chapter 2], each nonzero N−ray series of an eigenfunction for the group Γ has close relation with corresponding zeta function and can be used for its analytic investigation. The problem is that we do not know which one of N−ray series is nonzero and so are forced to consider all N−ray series, as this was done in [3] for $n = 1$. Consideration of different N−ray series although similar in general ideas can significantly differ in details depending on arithmetic of the binary quadratic form with matrix N and corresponding imaginary quadratic field.

Here, in order to illustrate the general ideas, we consider only the simplest case of radial series relative to the ray of the unit matrix $N = 1_2 = \mathbf{1}$, i.e., the $\mathbf{1}$−ray series

$$R_G(s) = R_G(s, \mathbf{1}) = \sum_{r=1}^{\infty} \frac{g(r\mathbf{1})}{r^s}. \tag{5.12}$$

Euler factorization of this series is closely related to arithmetic of the quadratic form $x_1^2 + x_2^2$ with matrix is 1_2, or, in other language, arithmetic of the ring of gaussian integers $\mathcal{O}(\mathfrak{G}) = \mathbb{Q}[\sqrt{-1}]$, i.e., the ring of integers of the imaginary quadratic field

$$\mathfrak{G} = \mathbb{Q}(\sqrt{-1}). \tag{5.13}$$

Each nonzero ideal of $\mathcal{O}(\mathfrak{G})$ is principal and uniquely up to an order decomposes into a product of prime ideals. L−function of the ring $\mathcal{O}(\mathfrak{G})$ with a character $\mathbf{x}$ of the multiplicative semigroup of nonzero integral ideals has the form

$$L_{\mathcal{O}(\mathfrak{G})}(s, \mathbf{x}) = \sum_{\mathfrak{a}} \frac{\mathbf{x}(\mathfrak{a})}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\mathbf{x}(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \qquad (\mathrm{Re}\ s > 1), \tag{5.14}$$

where $\mathfrak{a}$ and $\mathfrak{p}$ range over all nonzero integral ideals and prime ideals of the ring, respectively. For the arithmetic of quadratic fields and rings see, e.g., [4, Appendix 3].

The following theorem links the Dirichlet series (5.12) assigned to the twisted cusp form (5.7) with the L−function (5.6) of the initial form F.

Theorem 5.1. *Let $F \in \mathfrak{N}_k(\Gamma)$ be an eigenfunction for all m−regular Hecke operators for the group Γ, and χ – a primitive Dirichlet character modulo m. Then in every half-plane of absolute convergence L−function* (5.6) *of F with character χ satisfies the identity*

$$g\,(\mathbf{1})\, L_F(s, \chi) = L_{\mathcal{O}(\mathfrak{G})}(s - k + 2, \mathbf{x}) R_G(s), \tag{5.15}$$

where $R_G(s)$ is $\mathbf{1}$−ray series (5.12) *of the twist $G \in \mathfrak{N}_k(\Lambda)$ of F with the character χ and a p−matrix $L = {}^tL \in \mathbb{Z}_2^2$, $g(\mathbf{1})$ is the first coefficient of $R_G(s)$, and where $L_{\mathcal{O}(\mathfrak{G})}(s, \mathbf{x})$ is the L-series* (5.14) *of the ring of gaussian integers $\mathcal{O}(\mathfrak{G})$ with the norm extension*

$$\mathbf{x}(\mathfrak{a}) = \chi(N(\mathfrak{a})) = \chi(\mathfrak{a}\bar{\mathfrak{a}}) \tag{5.16}$$

of the character χ on ideals of the ring.

Proof. For the functions F and $G = F|\mathcal{T}(\chi, L)$, a prime number p not dividing m, and $\delta = 0, 1, 2, \dots$, we denote by $f(p^\delta; A)$ and $g(p^\delta; A)$ the Fourier coefficients of the functions $F|T_\Gamma(p^\delta)$ and $G|T_\Lambda(p^\delta)$, respectively, so that

$$\begin{aligned} F|T_\Gamma(p^\delta) &= \sum_{N\in\mathfrak{N}^2, N>0} f(p^\delta;\, N)e^{2\pi i Tr(NZ)} \\ G|T_\Lambda(p^\delta) &= \sum_{N\in\mathfrak{N}^2, N>0} g(p^\delta;\, N)e^{2\pi i Tr(NZ)}. \end{aligned} \tag{5.17}$$

By Theorems 2.3 and 2.1, we have

$$\begin{aligned} G|T_\Lambda(p^\delta) &= \big(F|\mathcal{T}(\chi, L)\big)|T_\Lambda(p^\delta) = \chi(p^\delta)\big(F|T_\Gamma(p^\delta)\big)|\mathcal{T}(\chi, L) \\ &= \sum_{N\in\mathfrak{N}^2, N>0} \chi(p^\delta)\chi(Tr(LN))f(p^\delta;\, N)e^{2\pi i Tr(NZ)}. \end{aligned}$$

Hence,

$$g(p^\delta;\, N) = \chi(p^\delta)\chi(Tr(LN))f(p^\delta;\, N). \tag{5.18}$$

According to [2, Theorem 1], for the matrix $M = \mathbf{1}$ and the discriminant $d = -4$ of the field $\mathfrak{G} = \mathbb{Q}(\sqrt{-1})$, using the laws of decomposition of prime numbers in prime ideals of the ring of integers of $\mathfrak{G}$, we obtain that for every positive integer r and prime number p not dividing rm the following formulas hold:

$$f(p^\delta;\, r\mathbf{1}) = \begin{cases} f(p^\delta r\mathbf{1}) + 2\sum_{\beta=1}^{\delta}(p^{k-2})^\beta f(p^{\delta-\beta}r\mathbf{1}), & \text{if } p \equiv 1 \pmod 4, \\ f(p^\delta r\mathbf{1}) + p^{k-2}f(p^{\delta-1}r\mathbf{1}), & \text{if } p = 2, \\ f(p^\delta r\mathbf{1}), & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Multiplying both sides of these relations for by $\chi(Tr(L(p^\delta r\mathbf{1})) = \chi(p^\delta)\chi(Tr(L(r\mathbf{1})))$ by formulas (5.18) we obtain the corresponding formulas for the function g:

$$g(p^\delta;\, r\mathbf{1}) = \begin{cases} g(p^\delta r\mathbf{1}) + 2\sum_{\beta=1}^{\delta}(\chi(p)p^{k-2})^\beta g(p^{\delta-\beta}r\mathbf{1}), & \text{if } p \equiv 1 \pmod 4, \\ g(p^\delta r\mathbf{1}) + \chi(p)p^{k-2}g(p^{\delta-1}r\mathbf{1}), & \text{if } p = 2, \\ g(p^\delta r\mathbf{1}), & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

We remind that the decomposition of a prime number p in products of prime ideals of the ring $\mathcal{O} = \mathcal{O}(\mathfrak{G}) = \mathbb{Q}[\sqrt{-1}]$ has the form $p = \mathfrak{p}\bar{\mathfrak{p}}$ with $\mathfrak{p} \neq \bar{\mathfrak{p}}$ in the first of listed cases, $p = \mathfrak{p}^2$ in the second case, and $p = \mathfrak{p}$ in the third case. Since the function G is an eigenfunction for Hecke operator $T_\Lambda(p^\delta)$ with the eigenvalue $\lambda_G(p^\delta)$, it follows that

$$\lambda_G(p^\delta)g(r\mathbf{1}) = g(p^\delta;\, r\mathbf{1}).$$

Multiplying both sides of the relation for the case $p \equiv 1 \pmod 4$ by t^δ and summing up on $\delta = 0, 1, \dots$, we obtain the formal identity in the variable t

$$
\begin{aligned}
g(r\mathbf{1}) \sum_{\delta=0}^{\infty} \lambda_G(p^\delta) t^\delta &= \sum_{\delta=0}^{\infty} g(p^\delta;\, r\mathbf{1}) t^\delta \qquad (5.19)\\
&= \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta + 2 \sum_{\delta=1}^{\infty} \sum_{\beta=1}^{\delta} (\chi(p)p^{k-2}t)^\beta g(p^{\delta-\beta} r\mathbf{1}) t^{\delta-\beta}\\
&= \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta + 2 \sum_{\delta=0}^{\infty} \sum_{\alpha,\beta\geq 0,\, \alpha+\beta=\delta} (\chi(p)p^{k-2}t)^\beta g(p^\alpha r\mathbf{1}) t^\alpha - 2 \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta\\
&= -\sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta + 2 \sum_{\beta=0}^{\infty} (\chi(p)p^{k-2}t)^\beta \sum_{\alpha=0}^{\infty} g(p^\alpha r\mathbf{1}) t^\alpha\\
&= \big(-1 + 2 \sum_{\beta=0}^{\infty} (\chi(p)p^{k-2}t)^\beta\big) \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta\\
&= \frac{1+\chi(p)p^{k-2}t}{1-\chi(p)p^{k-2}t} \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta = \frac{1-(\chi(p)p^{k-2}t)^2}{(1-\chi(p)p^{k-2}t)^2} \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1}) t^\delta.
\end{aligned}
$$

By formulas (5.5) and (5.6), we have the identity

$$
\sum_{\delta=0}^{\infty} \lambda_G(p^\delta) t^\delta = \sum_{\delta=0}^{\infty} \lambda_F(p^\delta)(\chi(p)t)^\delta = Q_{p,F}(\chi(p)t)^{-1}\big(1 - \chi(p^2)p^{2k-4}t^2\big). \qquad (5.20)
$$

In view of this identity, we can rewrite the identity (5.19) in the form

$$
g(r\mathbf{1}) Q_{p,F}(\chi(p)t)^{-1} = g(r\mathbf{1})(1-(\chi(p)p^{k-2}t)^2)^{-1} \sum_{\delta=0}^{\infty} \lambda_G(p^\delta) t^\delta
$$

$$
= (1-\chi(N(\mathfrak{p}))N(\mathfrak{p})^{k-2}t)^{-1}(1-\chi(N(\bar{\mathfrak{p}}))N(\mathfrak{p})^{k-2}t)^{-1} \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1})\, t^\delta,
$$

where $p = \mathfrak{p}\bar{\mathfrak{p}}$ and $N(\mathfrak{p}) = N(\bar{\mathfrak{p}}) = p$. Setting $t = p^{-s}$ and multiplying the both sides by r^{-s}, we can rewrite this identity in the form

$$
Q_{p,F}(\chi(p)p^{-s}) \sum_{\delta=0}^{\infty} g(p^\delta r\mathbf{1})(p^\delta r)^{-s} \qquad (5.21)
$$

$$
= g(r\mathbf{1}) r^{-s} (1-\chi(N(\mathfrak{p}))N(\mathfrak{p})^{k-2-s})(1-\chi(N(\bar{\mathfrak{p}}))N(\mathfrak{p})^{k-2-s}).
$$

Similarly, in the case $p = 2$ we get the formal identity in the variable t

$$
g(r\mathbf{1}) \sum_{\delta=0}^{\infty} \lambda_G(p^\delta) t^\delta = \sum_{\delta=0}^{\infty} g(p^\delta;\, r\mathbf{1}) t^\delta
$$

$$= \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta} + p^{k-2} \sum_{\delta=1}^{\infty} g(p^{\delta-1} r\mathbf{1}) t^{\delta}$$

$$= (1 + \chi(p) p^{k-2} t) \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta} = \frac{1 - (\chi(p) p^{k-2} t)^2}{1 - \chi(p) p^{k-2} t} \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta},$$

which, in view of (5.20), can be rewritten in the form

$$g(r\mathbf{1}) Q_{p,F}(\chi(p)t)^{-1} = g(r\mathbf{1})(1 - (\chi(p)p^{k-2})^2 t^2)^{-1} \sum_{\delta=0}^{\infty} \lambda_G(p^{\delta}) t^{\delta}$$

$$= (1 - \chi(N(\mathfrak{p})) N(\mathfrak{p})^{k-2} t)^{-1} \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta},$$

where $p = \mathfrak{p}^2$ and $N(\mathfrak{p}) = p$. Hence,

$$Q_{p,F}(\chi(p)p^{-s}) \sum_{\delta=0}^{\infty} g(r\mathbf{1})(p^{\delta} r)^{-s} \tag{5.22}$$

$$= g(r\mathbf{1}) r^{-s} (1 - \chi(N(\mathfrak{p})) N(\mathfrak{p})^{k-2-s}).$$

Finally, in the case $p \equiv 3 \pmod 4$ we have

$$g(r\mathbf{1}) \sum_{\delta=0}^{\infty} \lambda_G(p^{\delta}) t^{\delta} = \sum_{\delta=0}^{\infty} g(p^{\delta};\, r\mathbf{1}) t^{\delta}$$

$$= \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta} = \frac{1 - (\chi(p) p^{k-2} t)^2}{1 - (\chi(p) p^{k-2} t)^2} \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta},$$

hence

$$g(r\mathbf{1}) Q_{p,F}(\chi(p)t)^{-1} = g(r\mathbf{1})(1 - (\chi(p)p^{k-2})^2 t^2)^{-1} \sum_{\delta=0}^{\infty} \lambda_G(p^{\delta}) t^{\delta}$$

$$= (1 - \chi(N(\mathfrak{p})) N(\mathfrak{p})^{k-2} t^2)^{-1} \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1}) t^{\delta},$$

where $p = \mathfrak{p}$ and $N(\mathfrak{p}) = p^2$. Thus,

$$Q_{p,G}(\chi(p)p^{-s}) \sum_{\delta=0}^{\infty} g(p^{\delta} r\mathbf{1})(p^{\delta} r)^{-s} \tag{5.23}$$

$$= g(r\mathbf{1})\, r^{-s} (1 - \chi(N(\mathfrak{p})) N(\mathfrak{p})^{k-2-s}).$$

Applying the corresponding of the identities (5.21)–(5.23) in succession to all prime numbers p not dividing m, we obtain the formal identity

$$L_F(s,\,\chi)^{-1}\sum_{r=1}^{\infty}\frac{g(r\mathbf{1})}{r^s}=g(\mathbf{1})L_{\mathcal{O}}(s-k+2,\,\mathbf{x})^{-1}$$

(note that together with $\chi(r)$ coefficients $g(r\mathbf{1})=\chi(r)\chi(TrL)f(r\mathbf{1})$ are eual to zero, if r and m are not coprime). Hence, multiplying both sides by the product $L_{\mathcal{O}}(s-k+2,\,\mathbf{x})L_F(s,\,\chi)$ we obtain the identity (5.14) on formal level. Convergence of both side in a right half-plane of the variable s follows from the estimates of Fourier coefficients and similar known estimates for the eigenvalues. $\square$

Integral representations of 1−ray series. The identity (5.15) may as well reduce itself to the equality $0=0$, if the series $R_G(s)$ is identically equals to zero. This equality is equivalent to the condition $g(\mathbf{1})=0$. Otherwise, the identity reduces analytic properties of the L−function $L_F(s,\,\chi)$ to analytic properties of the ray series.

Let us set

$$\mathbf{X}=\left\{\begin{pmatrix}x & y\\ y & -x\end{pmatrix}\;\middle|\; x,\,y\in\mathbb{R}\right\},\tag{5.24}$$

and consider restriction of the function G on subset of $\mathbb{H}^2$ of the form

$$\mathbf{L}=\left\{X+it1_2\;\middle|\; X\in\mathbf{X},\,t>0\right\}\subset\mathbb{H}^2.\tag{5.25}$$

We shall refer to the real numbers x, y, and $t>0$ as *coordinates of the point* $\begin{pmatrix}x & y\\ y & -x\end{pmatrix}+it\mathbf{1}\in\mathbf{L}$. In the terms of the coordinates the restriction of G on $\mathbf{L}$ is given by the series

$$G\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix}+it\mathbf{1}\right)=\sum_{N=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}\in\mathfrak{N}^2,\,N>0} g(N)e^{2\pi i((a-c)x+by+i(a+c)t)}.$$

Hence we have

$$\int_{|x|\le 1/2,\,|y|\le 1/2} G\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix}+it\mathbf{1}\right)dx\,dy=\sum_{a=1}^{\infty}g\left(\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right)e^{-4\pi at},$$

and using the Euler formula (4.23), we came to the integral identity

$$\int_0^{\infty}t^{s-1}\left(\int_{|x|\le 1/2,\,|y|\le 1/2} G\left(\begin{pmatrix}x & y\\ y & -x\end{pmatrix}+it\mathbf{1}\right)dx\,dy\right)dt\tag{5.26}$$

$$=\Gamma(s)\sum_{a=1}^{\infty}(4\pi a)^{-s}g(a\mathbf{1})=(4\pi)^{-s}\Gamma(s)R_G(s)\qquad(\operatorname{Re}\,s>k+1).$$

Further transformation of this integral representation of the $\mathbf{1}$−ray series of the twist G is based on the remarkable circumstance that the real symplectic group $Sp_2(\mathbb{R})$ and its discrete subgroups (5.0) contain rather big subgroups operating as groups of analytical automorphisms on the subset $\mathbf{L} \subset \mathbb{H}^2$. Let us consider the subset $\mathbf{S}$ of the group $Sp_2(\mathbb{R})$ consisting of contained in $Sp_2(\mathbb{R})$ matrices of the form

$$M = \begin{pmatrix} a & a' & b & b' \\ -a' & a & b' & -b \\ c & c' & d & d' \\ c' & -c & -d' & d \end{pmatrix}. \tag{5.27}$$

It is easy to see that $\mathbf{S}$ is a group, and for each $M \in \mathbf{S}$ the restriction of the automorphism $Z \mapsto M\langle Z\rangle$ of the upper half-plane onto $\mathbf{L}$ defines a (real) analytical automorphism of the domain $\mathbf{L}$. Subgroups $\mathit{\Lambda}(\mathbf{S}) = \mathit{\Lambda} \cap \mathbf{S}$, where $\mathit{\Lambda}$ is one of the groups (5.0), are discrete subgroups of $\mathbf{S}$, as well as their common subgroup

$$\mathit{\Lambda}_\infty(\mathbf{S}) = \Gamma_\infty(\mathbf{S}) = \Upsilon_\infty(\mathbf{S}) = \Lambda_\infty(\mathbf{S}) = \left\{ \pm \begin{pmatrix} \mathbf{1} & B \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \;\middle|\; B = \begin{pmatrix} b & b' \\ b' & -b \end{pmatrix},\, b,\, b' \in \mathbb{Z} \right\}.$$

Transformations of $\mathbf{L}$ by a matrix $M \in \mathit{\Lambda}_\infty(\mathbf{S})$ with parameters $b,\, b' \in \mathbb{Z}$ clearly maps a point of $\mathbf{L}$ with coordinates $x,\, y,\, t$ to the point with the coordinates of the form $x + b,\, y + b',\, t$. It follows that the domain of integration in (5.26) can be considered as a fundamental domain of the group $\mathit{\Lambda}_\infty(\mathbf{S})$ on $\mathbf{L}$,

$$\left\{ \begin{pmatrix} x & y \\ y & -x \end{pmatrix} + it\mathbf{1} \in \mathbf{L} \;\middle|\; |x| \le 1/2,\, |y| \le 1/2,\, t > 0 \right\} = \mathit{\Lambda}_\infty(\mathbf{S})\backslash\mathbf{L}. \tag{5.28}$$

By Theorem 3.1, function G satisfies $G|_k M = G$ for all $M \in \Lambda(S) \subset \Lambda$. It follows that $G(M\langle Z\rangle) = G(Z)$ for all $M \in \Lambda_\infty(\mathbf{S})$ and $Z \in \mathbf{L}$. Therefore, in view of absolute convergence, the integral (5.26) is independent of choice of fundamental domain of the group $\mathit{\Lambda}_\infty(\mathbf{S})$ on $\mathbf{L}$ and can be written in the form

$$\int_{\mathit{\Lambda}_\infty(\mathbf{S})\backslash\mathbf{L}} G\left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix} + it\mathbf{1} \right) t^{s-1} dxdydt. \tag{5.29}$$

Further transformations of the integral is based on automorphic properties of the integrand function G. The target of transformations is to replace the tube domain of integration $\mathit{\Lambda}_\infty(\mathbf{S})\backslash\mathbf{L}$ having infinite volume by a subset with finite invariant volume. The larger transformation group of the integrand, the smaller integration domain can be obtained. As we know, the function G with arbitrary parameter matrix satisfies $G|_k M = G$ for all $M \in \Lambda(\mathbf{S})$. By a special choice of the parameter matrix L in the definition of G the transformation group can be increased. We shall assume from now on that $L = 1_2 = \mathbf{1}$. Then, for $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Upsilon(\mathbf{S})$ with $D = \left(\begin{smallmatrix} d & d' \\ -d' & d \end{smallmatrix}\right)$, we have

$$L[D] = {}^tDD = (d^2 + (d')^2)\mathbf{1} = \det D \cdot \mathbf{1}$$

and so, by Proposition 1.4, the function

$$G_{\mathbf{1}} = F|\mathcal{T}(\chi,\, \mathbf{1}) \tag{5.30}$$

satisfies relations

$$\begin{aligned} G_{\mathbf{1}}|_k M &= G_{\mathbf{1}}|_k \varrho(D) = \chi(\det D)(F|_k \varrho(D))|\mathcal{T}(\chi,\, \mathbf{1}) \\ &= \chi(\det D) G_{\mathbf{1}} \qquad \left(\forall\, M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Upsilon(\mathbf{S})\right). \end{aligned} \tag{5.31}$$

Now, let

$$\Upsilon(\mathbf{S}) = \bigcup_{M \in \Upsilon_\infty(\mathbf{S}) \backslash \Upsilon(\mathbf{S})} \Upsilon_\infty(\mathbf{S}) M$$

be a decomposition of $\Upsilon(\mathbf{S})$ into left cosets modulo the subgroup $\Upsilon_\infty(\mathbf{S})$. If $D(\Upsilon(\mathbf{S}))$ is a fundamental domain for the group $\Upsilon(\mathbf{S})$ on $\mathbf{L}$, then the union of the sets $M\langle D(\Upsilon(\mathbf{S}))\rangle$ with $M \in \Upsilon_\infty(\mathbf{S}) \backslash \Upsilon(\mathbf{S})$ is a fundamental domain for the group $\Upsilon_\infty(\mathbf{S})$. On the other hand, since the set (5.28) is also a fundamental domain for $\Upsilon_\infty(\mathbf{S})$ and the integrand in (5.29) is invariant under all transformations of $\Upsilon_\infty(\mathbf{S})$, we can write down (at least formally) the relation

$$\begin{aligned} &\int_{\Lambda_\infty(\mathbf{S}) \backslash \mathbf{L}} G_{\mathbf{1}} \left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix} + it\mathbf{1} \right) t^{s-1} dx dy dt \\ &= \sum_{M \in \Upsilon_\infty(\mathbf{S}) \backslash \Upsilon(\mathbf{S})} \int_{M\langle D(\Upsilon(\mathbf{S}))\rangle} G_{\mathbf{1}} \left(\begin{pmatrix} x & y \\ y & -x \end{pmatrix} + it\mathbf{1} \right) t^{s-1} dx dy dt, \end{aligned} \tag{5.32}$$

which can be justified in every half-plane of absolute and uniform convergence, say for Re $s > k+1$.

In order to simplify forthcoming computations of the integrals we parametrize the space $\mathbf{L}$ by associating with a matrix

$$Z = \begin{pmatrix} x & y \\ y & -x \end{pmatrix} + \begin{pmatrix} it & 0 \\ 0 & it \end{pmatrix} = \begin{pmatrix} x + it & y \\ y & -x + it \end{pmatrix} \in \mathbf{L}$$

a point

$$u = u(Z) = (x + iy,\, t) = (w,\, t) \tag{5.33}$$

of the three-dimensional hyperbolic space, called *Lobachevski space*,

$$\mathbb{L} = \mathbb{L}^3 = \{u = (w,\, t) \mid\ w = x + iy \in \mathbb{C},\, t > 0\}.$$

The space $\mathbb{L}$ is an homogeneous space for the group $\Sigma = SL_2(\mathbb{C})$ operating as a transitive transformation group by the rule

$$\begin{aligned} &\Sigma \ni \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \\ &\qquad u = (w,\, t) \mapsto \sigma\langle u\rangle = \left(\frac{(\alpha w + \beta)(\overline{\gamma w} + \overline{\delta}) + \alpha\overline{\gamma} t^2}{\Delta(\sigma,\, u)},\, \frac{t}{\Delta(\sigma,\, u)} \right), \end{aligned} \tag{5.34}$$

where

$$\Delta(\sigma,\, u) = |\gamma w + \delta|^2 + |\gamma|^2 t^2. \tag{5.35}$$

The composition of transformations corresponds to the product of matrices, and the function $\Delta(\sigma,\, u)$ is an automorphy factor, i.e., it is not vanish on $\Sigma \times \mathbb{L}$ and satisfies relations

$$\Delta(\sigma\tau,\, u) = \Delta(\sigma,\, \tau\langle u\rangle)\Delta(\tau,\, u) \qquad (\sigma, \tau \in \Sigma,\, u \in \mathbb{L}). \tag{5.36}$$

Note, finally, that $\Sigma-$invariant element of volume on $\mathbb{L}$ is

$$du = t^{-3} dxdydt \qquad (u = (x + iy,\, t)), \tag{5.37}$$

where dx, dy, dt are the euclidean element of volume on the real line. It turns out that the pairs $(\mathbf{S},\, \mathbf{L})$ and $(\Sigma,\, \mathbb{L})$ are naturally isomorphic. If $M \in \mathbf{S}$ is a matrix of the form (5.27) we set

$$\sigma(M) = \begin{pmatrix} a + ia' & b' + ib \\ c' - ic & d - id' \end{pmatrix} \in \mathbb{C}_2^2,$$

and define the mapping $u : \mathbf{L} \mapsto \mathbb{L}$ by (5.33). Then by [3, Theorem 3.4.2], the map σ is an isomorphism of the real Lie groups, the map u is an analytic isomorphism, the map u is compatible with the actions of $\mathbf{S}$ on $\mathbf{L}$ and Σ on $\mathbb{L}$, that is for any $Z \in \mathbf{L}$ and $M \in \mathbf{S}$ we have the relation $u(M\langle Z\rangle) = \sigma(M)\langle u(Z)\rangle$, and under the given maps the automorphy factor of the pair $(\mathbf{S},\, \mathbf{L})$ goes into the automorphy factor of the pair $(\Sigma,\, \mathbb{L})$, that is

$$\det(CZ + D) = \Delta(\sigma(M),\, u(Z)) \qquad \left(M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } Z \in \mathbf{L}\right). \tag{5.38}$$

It follows from definition that images of discrete subgroups $\Gamma(\mathbf{S})$, $\Upsilon(\mathbf{S})$, and $\Upsilon_\infty(\mathbf{S})$ of $\mathbf{S}$ under the isomorphism σ are discrete subgroups of Σ of the form

$$\begin{aligned}
&\sigma\left(\Gamma(\mathbf{S})\right) = SL_2(\mathcal{O}(\mathfrak{G})), \\
&\sigma\left(\Upsilon(\mathbf{S})\right) = \left\{\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathcal{O}(\mathfrak{G})) \;\middle|\; \gamma \equiv 0 \pmod{m^2}\right\}, \\
&\sigma\left(\Upsilon_\infty(\mathbf{S})\right) = \left\{\sigma = \pm\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \;\middle|\; \beta \in \mathcal{O}(\mathfrak{G})\right\},
\end{aligned}$$

where $\mathcal{O}(\mathfrak{G}) = \mathbb{Z}[\sqrt{-1}]$ is the ring of gaussian integers. We shall denote these groups by

$$\Pi = \sigma\left(\Gamma(\mathbf{S})\right), \quad \Pi_0(m^2) = \sigma\left(\Upsilon(\mathbf{S})\right), \quad \text{and} \quad \Pi_\infty = \sigma\left(\Upsilon_\infty(\mathbf{S})\right).$$

The identity (5.26) for $G = G_{\mathbf{1}}$ with the integral written in the form (5.32) takes the shape

$$(4\pi)^{-s}\Gamma(s)R_{G_{\mathbf{1}}}(s) \tag{5.39}$$

$$= \sum_{M\in\Upsilon_\infty(\mathbf{S})\backslash\Upsilon(\mathbf{S})} \int_{M\langle D(\Upsilon(\mathbf{S}))\rangle} G_{\mathbf{1}}\left(\begin{pmatrix} x & y \\ y & -x\end{pmatrix} + it\mathbf{1}\right) t^{s-1} dx dy dt \qquad (\operatorname{Re} s > k+1).$$

Using the isomorphism of the pairs $(\mathbf{S}, \mathbf{L})$ and $(\Sigma, \mathbb{L})$ given by the mappings σ and u, we can rewrite the sum on the right of (5.39) as

$$\begin{aligned} &= \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \int_{\sigma\langle D(\Pi_0(m^2))\rangle} \widetilde{G}_{\mathbf{1}}(u) t^{s+2} du \\ &= \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \int_{D(\Pi_0(m^2))} \widetilde{G}_{\mathbf{1}}(\sigma\langle u\rangle) t(\sigma\langle u\rangle)^{s+2} d\sigma\langle u\rangle, \end{aligned} \tag{5.40}$$

where $D(\Pi_0(m^2))$ is a fundamental domain of the group $\Pi_0(m^2)$ on $\mathbb{L}$, du is the invariant element of volume (5.37), and where for a function G on $\mathbb{H}^2$, we denote by $\widetilde{G}(u)$ the function on $\mathbb{L}$ satisfying the condition

$$\widetilde{G}(u(Z)) = G(Z) \qquad \text{for all } Z \in \mathbf{L}. \tag{5.41}$$

If $\sigma = \sigma(M)$ with $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Psi(\mathbf{S})$, then by (5.40) and (1.1) we have

$$\widetilde{G}_{\mathbf{1}}(\sigma\langle u\rangle) = \widetilde{G}_{\mathbf{1}}(\sigma(M)\langle u\rangle) = \widetilde{G}_{\mathbf{1}}(u(M\langle Z\rangle)) = G_{\mathbf{1}}(M\langle Z\rangle) = \det(CZ+D)^k\, G_{\mathbf{1}}|_k M,$$

which, by (5.31) and (5.38) is equal to

$$\chi(\det D)\det(CZ+D)^k\, G_{\mathbf{1}}(Z) = \mathbf{x}(\sigma)\Delta(\sigma,\, u)^k \widetilde{G}_{\mathbf{1}}(u),$$

where, for $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \Pi_0(m^2)$,

$$\mathbf{x}(\sigma) = \mathbf{x}\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\right) = \mathbf{x}(\delta) = \chi(\delta\bar{\delta}), \tag{5.42}$$

denotes the natural extension of the character $\mathbf{x}$ on $\Pi_0(m^2)$. Besides, by (5.34) we have $t(\sigma\langle u\rangle) = \Delta(\sigma,\, u)^{-1}t$. Therefore, the sum (5.40) is equal in every half-plane of absolute and uniform convergence to the sum

$$\sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \int_{D(\Pi_0(m^2))} \mathbf{x}(\sigma)\Delta(\sigma,\, u)^k \widetilde{G}_{\mathbf{1}}(u)(\Delta(\sigma,\, u)^{-1}t)^{s+2} du$$

$$\int_{D(\Pi_0(m^2))} t^{s-k+2} \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \frac{\mathbf{x}(\sigma)}{\Delta(\sigma,\, u)^{s-k+2}} t^k \widetilde{G}_{\mathbf{1}}(u) du.$$

Thus, finally, we obtain the integral representation

$$\begin{aligned} &(4\pi)^{-s}\Gamma(s) R_{G_{\mathbf{1}}}(s) \\ &\quad = \int_{D(\Pi_0(m^2))} E(u,\, s-k+2,\, \mathbf{x}) t^k \widetilde{G}_{\mathbf{1}}(u) du \qquad (\operatorname{Re} s > k+1), \end{aligned} \tag{5.43}$$

where the series

$$E(u,\, s,\, \mathbf{x}) = y^s \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \frac{\mathbf{x}(\sigma)}{\Delta(\sigma,\, u)^s} \tag{5.44}$$

is called *Eisenstein series for the group* $\Pi_0(m^2)$ *with character* $\mathbf{x}$.

Eisenstein series for $\Pi_0(m^2)$ and theta-series. Here we consider analytical properties of Eisenstein series (5.44). Among well-known properties of the Eisenstein series (see, e.g., [10] or [9]) we mention here that it converges absolutely for Re $s > 2$ and, by (5.36), satisfies the relations

$$E(\tau\langle u\rangle,\, s,\, \mathbf{x}) = \left(\frac{t}{\Delta(\tau;\, u)}\right)^{s} \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \frac{\mathbf{x}(\sigma)}{\Delta(\sigma,\, \tau\langle u\rangle)^{s}} \tag{5.45}$$

$$= t^{s} \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \frac{\mathbf{x}(\sigma)}{\Delta(\sigma\tau;\, u)^{s}} = t^{s} \sum_{\sigma\in\Pi_\infty\backslash\Pi_0(m^2)} \frac{\mathbf{x}(\sigma\tau\tau^{-1})}{\Delta(\sigma\tau;\, u)^{s}}$$

$$= \mathbf{x}(\tau^{-1})E(u,\, s,\, \mathbf{x}) = \overline{\mathbf{x}}(\tau)E(u,\, s,\, \mathbf{x}) \qquad \big(\tau\in\Pi_0(m^2)\big).$$

Further, two matrices

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \sigma' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in \Pi_0(m^2)$$

belong to the same left coset modulo Π_∞ if and only if

$$\sigma'\sigma^{-1} = \begin{pmatrix} \alpha'\delta-\beta'\gamma & -\alpha'\beta+\beta'\alpha \\ \gamma'\delta-\delta'\gamma & -\gamma'\beta+\delta'\alpha \end{pmatrix} \in \Pi_0(m^2),$$

which is equivalent with the conditions

$$\gamma'\delta-\delta'\gamma = 0 \quad \text{and} \quad -\gamma'\beta+\delta'\alpha = \pm 1_2.$$

Since δ, γ as well as δ', γ' are coprime, the first relation implies that δ divides δ' and vice versa, that is $\delta' = \epsilon\delta$, where ϵ is a unit of the ring $\mathcal{O} = \mathcal{O}(\mathfrak{G})$. Hence, the first relation is equivalent with relations $\gamma' = \epsilon\gamma$, $\delta' = \epsilon\delta$, and the second relation means that $\epsilon(-\gamma\beta+\delta\alpha) = \pm 1_2$, that is $\epsilon = \pm 1_2$. Hence, $(\gamma', \delta') = \pm(\gamma, \delta)$. Conversely, if $(\gamma', \delta') = \pm(\gamma, \delta)$, then clearly $\sigma'\sigma^{-1}\in\Pi_\infty$. Then it follows from the condition (5.42), relating characters of $\Pi_0(m^2)$ to corresponding characters with the same notation of the ring $\mathcal{O}$, that the Eisenstein series (5.44) can be written in the form

$$E(u,\, s,\, \mathbf{x}) = \frac{y^{s}}{2} \sum_{\substack{\gamma,\,\delta\in\mathcal{O},\,\gamma\mathcal{O}+\delta\mathcal{O}=\mathcal{O},\\ \gamma\equiv 0 \pmod{m^2}}} \frac{\mathbf{x}(\delta)}{(|\gamma w+\delta|^2+|\gamma|^2t^2)^{s}}. \tag{5.46}$$

On the other hand, since each ideal of the ring $\mathcal{O} = \mathcal{O}(\mathfrak{G})$ is principal, and $\mathbf{x}(\alpha\mathcal{O}) = \mathbf{x}(\alpha) = \chi(\alpha\overline{\alpha})$ for all $\alpha\in\mathcal{O}$, it follows that the same characters can be considered also as characters of the ideal semigroup of $\mathcal{O}$, and so the L-series (5.14) of the ring can be written in the form

$$L_{\mathcal{O}}(s,\, \mathbf{x}) = \sum_{(\alpha)\subset\mathcal{O},\,(\alpha)\neq(0)} \frac{\mathbf{x}((\alpha))}{N((\alpha))^{s}} = \frac{1}{4} \sum_{\alpha\in\mathcal{O},\,\alpha\neq 0} \frac{\mathbf{x}(\alpha)}{N(\alpha)^{s}}, \tag{5.47}$$

where (α) in the first sum ranges over all nonzero ideals $(\alpha) = \alpha\mathcal{O}$ of the ring $\mathcal{O}$. Then we obtain

$$L_{\mathcal{O}}(s,\,\mathbf{x})E(u,\,s,\,\mathbf{x}) \tag{5.48}$$
$$= \frac{t^s}{8} \sum_{\alpha\in\mathcal{O},\,\alpha\neq 0} \sum_{\substack{\gamma,\,\delta\in\mathcal{O},\,\gamma\mathcal{O}+\delta\mathcal{O}=\mathcal{O},\\ \gamma\equiv 0 \pmod{m^2}}} \frac{\mathbf{x}(\alpha)\mathbf{x}(\delta)}{N(\alpha)^s(|\gamma w+\delta|^2+|\gamma|^2t^2)^s}$$
$$= \frac{t^s}{8} \sum_{\substack{\alpha,\,\gamma,\,\delta\in\mathcal{O},\,\alpha\neq 0,\\ \alpha\gamma\mathcal{O}+\alpha\delta\mathcal{O}=\alpha\mathcal{O},\,\gamma\equiv 0 \pmod{m^2}}} \frac{\mathbf{x}(\alpha\delta)}{(|\alpha\gamma w+\alpha\delta|^2+|\alpha\gamma|^2t^2)^s}$$
$$= \frac{t^s}{8} \sum_{\substack{\gamma,\,\delta\in\mathcal{O},\,(\gamma,\delta)\neq(0,0)\\ \gamma\equiv 0 \pmod{m^2}}} \frac{\mathbf{x}(\delta)}{(|\gamma w+\delta|^2+|\gamma|^2t^2)^s}.$$

In order to obtain analytic continuation of the Eisenstein series $E(u,\,s,\,\mathbf{x})$ we use an integral representation of the series by means of suitable theta-series. For $u = (w,\,t) \in \mathbb{L}^3$ and $v > 0$, we introduce in former notation the theta-series

$$\Theta(v,\,u,\,\mathbf{x}) = \sum_{\substack{\gamma,\,\delta\in\mathcal{O},\\ \gamma\equiv 0 \pmod{m^2}}} \mathbf{x}(\delta)\exp\left(-\frac{\pi v}{t}(|\gamma w+\delta|^2+|\gamma|^2t^2)\right). \tag{5.49}$$

Since, by our assumption, $m > 1$, it follows that $\chi(0) = 0$, and so the constant term of this theta-series is zero. Then it directly follows from (5.48) and the Euler formula (4.23) that

$$\pi^{-s}\Gamma(s)L_{\mathcal{O}}(s,\,\mathbf{x})E(u,\,s,\,\mathbf{x}) = \frac{1}{8}\int_0^\infty v^{s-1}\Theta(v,\,u,\,\mathbf{x})dv \qquad (\operatorname{Re} s > 2). \tag{5.50}$$

Let us now turn to investigation of this theta-series. Consider auxiliary theta-series of the form

$$\theta(v;\,u,\,(\sigma,\sigma')) = \sum_{\gamma,\,\delta\in\mathcal{O}} \exp\left(-\frac{\pi v}{t}(|(\gamma+\sigma)w+(\delta+\sigma')|^2+|\gamma+\sigma|^2t^2)\right), \tag{5.51}$$

where $\sigma,\,\sigma' \in \mathbb{C}$. It is easy to check that, for all $u = (w,\,t) = (x+iy,\,t) \in \mathbb{L}$ and $\rho = \rho_1 + i\rho_2$, $\rho' = \rho_3 + i\rho_4 \in \mathbb{C}$, the following identity holds

$$\frac{1}{t}\left(|\rho w+\rho'|^2+|\rho|^2t^2\right) = {}^tRQR, \tag{5.52}$$

where

$$R = \begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \rho_4\end{pmatrix} \quad \text{and} \quad Q = \frac{1}{t}\begin{pmatrix} x^2+y^2+t^2 & 0 & x & y\\ 0 & x^2+y^2+t^2 & -y & x\\ x & -y & 1 & 0\\ y & x & 0 & 1\end{pmatrix}. \tag{5.53}$$

One directly checks that $\det Q = 1$ and the matrix Q is positive definite. It follows that the theta-series (5.51) can be written in the form

$$\theta(v;\, u,\, (\sigma,\sigma')) = \theta(v;\, u,\, Q,\, S) = \sum_{N\in\mathbb{Z}^4} \exp\left(-\pi v Q[N+S]\right), \tag{5.54}$$

where we use the notation $\gamma = n_1 + in_2$, $\delta = n_3 + in_4$, $N = {}^t(n_1, n_2, n_3, n_4)$, $\sigma = \sigma_1 + i\sigma_2$, $\sigma' = \sigma_3 + i\sigma_4$, $S = {}^t(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and $Q[T] = {}^tTQT$.

According to the classical inversion formula for the theta-series (5.54) (see, e.g. [11, Chapter 6, Proposition 23], the following formula holds

$$v^2\theta(v;\, u,\, (\sigma,\sigma')) = v^2\theta(v;\, u,\, Q,\, S) = \sum_{N\in\mathbb{Z}^4} \exp\Big(2\pi i\, {}^tNS - \frac{\pi}{v}Q^{-1}[N]\Big). \tag{5.55}$$

By direct computation we obtain the formulas

$$Q^{-1} = \frac{1}{y}\begin{pmatrix} 1 & 0 & -x & -y \\ 0 & 1 & y & -x \\ -x & y & x^2+y^2+t^2 & 0 \\ -y & -x & 0 & x^2+y^2+t^2 \end{pmatrix} = {}^tJ_2 Q' J_2, \tag{5.56}$$

where the matrix $J_2 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$, and the matrix Q' is obtained from Q by change $y \mapsto -y$. With replacement of y by $-y$ in (5.52) we get the identity

$$\frac{1}{t}\left(|\rho\overline{w} + \rho'|^2 + |\rho|^2t^2\right) = {}^tRQ'R.$$

Substituting here $R = J_2N = {}^t(n_3, n_4, -n_1, -n_2)$, we obtain the identity

$$\begin{aligned} {}^tNQ^{-1}N = {}^tN\,{}^tJ_2Q'J_2N &= \frac{1}{t}\left(|(n_3+in_4)\overline{w} - (n_1+in_2)|^2 + |n_3+in_4|^2t^2\right) \\ &= \frac{1}{t}\left(|\delta\overline{w} - \gamma|^2 + |\delta|^2t^2\right). \end{aligned}$$

Thus, since

$$2\,{}^tNS = \sum_{1\le j\le 4} n_j\sigma_j = \gamma\overline{\sigma} + \overline{\gamma}\sigma + \delta\overline{\sigma}' + \overline{\delta}\sigma' = 2\mathrm{Re}\,(\gamma\overline{\sigma} + \delta\overline{\sigma}'),$$

the inversion formula (5.55) can be rewritten in the form

$$\begin{aligned} v^2\theta(v;\, u,\, (\sigma,\sigma')) &= v^2 \sum_{\gamma,\,\delta\in\mathcal{O}} \exp\Big(-\frac{\pi v}{t}(|(\gamma+\sigma)w + (\delta+\sigma')|^2 + |\gamma+\sigma|^2t^2)\Big) \\ &= \sum_{\gamma,\,\delta\in\mathcal{O}} \exp\Big(2\pi i\mathrm{Re}\,(\gamma\overline{\sigma} + \delta\overline{\sigma}') - \frac{\pi}{vt}(|\delta\overline{w} - \gamma|^2 + |\delta|^2t^2)\Big). \end{aligned}$$

On replacing of γ by δ and vise versa, this formula takes the form

$$v^2\theta(v;\, u,\, (\sigma,\sigma')) = \sum_{\gamma,\,\delta\in\mathcal{O}} \exp\Big(2\pi i \mathrm{Re}\ (\gamma\overline{\sigma}' + \delta\overline{\sigma}) - \frac{\pi}{vt}(|\gamma\overline{w} - \delta|^2 + |\gamma|^2 t^2)\Big) \qquad (5.57)$$

Let us now return to the series (5.49) and express it in terms of the series (5.51). We obtain

$$\Theta(v,\, u,\, \mathbf{x}) = \sum_{\substack{\gamma,\,\delta\in\mathcal{O},\\ \gamma\equiv 0 \pmod{m^2}}} \mathbf{x}(\delta)\exp\Big(-\frac{\pi v}{t}(|\gamma w + \delta|^2 + |\gamma|^2 t^2)\Big). \qquad (5.58)$$

$$= \sum_{\rho\in\mathcal{O}/m^2\mathcal{O}} \chi((m^2\delta+\rho)\overline{(m^2\delta+\rho)}) \sum_{\gamma,\delta\in\mathcal{O}} \exp\Big(-\frac{\pi v}{t}(|m^2\gamma w + m^2\delta + \rho|^2) + |m^2\gamma|^2 t^2\Big)$$

$$= \sum_{\rho\in\mathcal{O}/m^2\mathcal{O}} \chi(\rho\overline{\rho}) \sum_{\gamma,\delta\in\mathcal{O}} \exp\Big(-\frac{\pi m^4 v}{t}(|\gamma w + \delta + \rho/m^2|^2) + |\gamma|^2 t^2\Big)$$

$$= \sum_{\rho\in\mathcal{O}/m^2\mathcal{O}} \mathbf{x}(\rho)\theta\left(m^4 v,\, u,\, \left(0,\, \rho/m^2\right)\right).$$

According to (5.57), for each of the theta-series in the right side of this relation, we can apply the inversion formula

$$\theta\left(m^4 v,\, u,\, \left(0,\, \rho/m^2\right)\right) = \frac{1}{v^2 m^8} \sum_{\gamma,\,\delta\in\mathcal{O}} \exp\Big(2\pi i \mathrm{Re}\ (\gamma\overline{\rho}/m^2) - \frac{\pi}{m^4 vt}(|\gamma\overline{w} - \delta|^2 + |\gamma|^2 t^2)\Big),$$

Therefore, we came to the formula

$$\Theta(v,\, u,\, \mathbf{x}) = \sum_{\gamma,\,\delta\in\mathcal{O}} \mathbf{x}(\delta)\exp\left(-\frac{\pi v}{y} t(|m^2\gamma w + \delta|^2 + m^4|\gamma|^2 y t^2)\right) \qquad (5.59)$$

$$= \frac{1}{v^2 m^8} \sum_{\rho\in\mathcal{O}/m^2\mathcal{O}} [\chi](\rho) \sum_{\gamma,\,\delta\in\mathcal{O}} \exp\Big(2\pi i \mathrm{Re}\ (\gamma\overline{\rho}/m^2) - \frac{\pi}{m^4 vt}(|\gamma\overline{w} - \delta|^2 + |\gamma|^2 t^2)\Big)$$

$$= \frac{1}{v^2 m^8} \sum_{\gamma,\,\delta\in\mathcal{O}} G'(\mathbf{x},\, \gamma)\exp\Big(-\frac{\pi}{m^4 vt}(|\gamma\overline{w} - \delta|^2 + |\gamma|^2 t^2\Big),$$

where

$$G'(\mathbf{x},\, \gamma) = \sum_{\rho\in\mathcal{O}/m^2\mathcal{O}} \mathbf{x}(\rho)\exp\left(\frac{2\pi i}{m^2}\mathrm{Re}\ (\gamma\overline{\rho})\right) = \sum_{\rho\in\mathcal{O}/m^2\mathcal{O}} \mathbf{x}(\rho)\exp\left(\frac{2\pi i}{m^2}\mathrm{Re}\ (\gamma\rho)\right).$$

is a Gaussian sum for the ring $\mathcal{O}$ modulo m^2. Since χ is a character modulo m, it easily follows that $\mathbf{x}$ considered as a character of the ring $\mathcal{O}$ is again a character modulo m . It allows us to express the Gaussian sums $G'(\mathbf{x},\, \gamma)$ through a Gaussian sums modulo m.

Lemma 5.2. *The Gaussian sum* $G'(\mathbf{x},\,\gamma)$ *can be written in the form*

$$G'(\mathbf{x},\,\gamma)=\begin{cases} m^2\sum_{\rho\in\mathcal{O}/m\mathcal{O}}\mathbf{x}(\rho)\exp\left(\frac{2\pi i}{m}\mathrm{Re}\;((\gamma/m)\rho)\right) & \text{if } \gamma\equiv 0 \pmod m,\\ 0 & \text{otherwise}\;. \end{cases}$$

Proof. It is clear that the sum $\rho'+m\rho''$ ranges a full system of residues of the ring $\mathcal{O}$ modulo m^2, when ρ' and ρ'' run full systems of residues modulo m. Hence we have

$$\begin{aligned} G'(\mathbf{x},\,\gamma)&=\sum_{\rho',\,\rho''\in\mathcal{O}/m\mathcal{O}}\mathbf{x}(\rho'+m\rho'')\exp\left(\frac{2\pi i}{m^2}\mathrm{Re}\;(\gamma(\rho'+m\rho''))\right)\\ &=\sum_{\rho'\in\mathcal{O}/m\mathcal{O}}\mathbf{x}(\rho')\exp\left(\frac{2\pi i}{m^2}\mathrm{Re}\;(\gamma\rho')\right)\sum_{\rho''\in\mathcal{O}/m\mathcal{O}}\exp\left(\frac{2\pi i}{m}\mathrm{Re}\;(\gamma\rho'')\right). \end{aligned}$$

Set $\gamma=a_1+ia_2$, $\rho''=b_1+ib_2$, then the inner sum on the right can be computed in the form

$$\sum_{\rho''\in\mathcal{O}/m\mathcal{O}}\exp\left(\frac{2\pi i}{m}\mathrm{Re}\;(\gamma\rho'')\right)=\sum_{b_1,\,b_2\in\mathbb{Z}/m\mathbb{Z}}\exp\left(\frac{2\pi i}{m}(a_1b_1-a_2b_2)\right)$$

$$=\begin{cases} m^2 & \text{if } a_1\equiv a_2\equiv 0 \pmod m \Leftrightarrow \gamma\equiv 0 \pmod m,\\ 0 & \text{otherwise.}\end{cases}\qquad\square$$

This lemma implies that the inversion formula (5.59) can be rewritten in the form

$$\Theta(v,\,u,\,\mathbf{x})=\sum_{\substack{\gamma,\,\delta\in\mathcal{O},\\ \gamma\equiv 0 \pmod{m^2}}}\mathbf{x}(\delta)\exp\left(-\frac{\pi v}{t}(|\gamma w+\delta|^2+|\gamma|^2t^2)\right). \tag{5.60}$$

$$\frac{1}{v^2m^6}\sum_{\substack{\gamma,\,\delta\in\mathcal{O},\\ \gamma\equiv 0 \pmod m}}G(\mathbf{x},\,\gamma)\exp\left(-\frac{\pi}{m^4vt}(|\gamma\overline{w}-\delta|^2+|\gamma|^2t^2\right),$$

where

$$G(\mathbf{x},\,\gamma)=\sum_{\rho\in\mathcal{O}/m\mathcal{O}}\mathbf{x}(\rho)\exp\left(\frac{2\pi i}{m^2}\mathrm{Re}\;(\gamma\rho)\right).$$

Analytic continuation of 1−ray series. Here we shall outline the proof of the following theorem.

Theorem 5.3. *Let* $G_{\mathbf{1}}=F|\mathcal{T}(\chi,\,\mathbf{1})$ *be the twist of a cusp form* $F\in\mathfrak{N}_k(\Gamma)$ *with a primitive Dirichlet character* χ *modulo* $m>1$ *and the unit p−matrix, and let*

$$R_{G_{\mathbf{1}}}(s)=\sum_{r=1}^{\infty}\frac{\chi(2r)f(r\mathbf{1})}{r^s}\qquad(\mathrm{Re}\;s>k+1), \tag{5.61}$$

where $f(N)$ are Fourier coefficients of F, be the $\mathbf{1}-$ray series of $G_{\mathbf{1}}$. Then the function

$$\Psi_F(s,\,\chi)=\Gamma(s)\Gamma(s-k+2)L_{\mathcal{O}}(s-k+2,\,\mathbf{x})R_{G_{\mathbf{1}}}(s), \tag{5.62}$$

where $L_{\mathcal{O}}(s,\,\mathbf{x})$ is the $L-$function (5.14) of the ring $\mathcal{O}=\mathcal{O}(\mathfrak{G})$, can be continued analytically to the whole $s-$plane as a meromorphic function having at most one simple pole at the point $s=k$.

Proof. With the help of integral representations (5.43) and (5.50) we obtain the identity

$$\begin{aligned}
&2^{3-2s}\pi^{k-2s-2}\Psi_F(s,\,\chi)\\
&=\int_{D(\Pi_0(m^2))}\left(\int_0^{\infty}v^{s-k+1}\Theta(v,\,u,\,\mathbf{x})dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du \qquad (5.63)\\
&=\int_{D(\Pi_0(m^2))}\left(\int_0^{1/m^2}v^{s-k+1}\Theta(v,\,u,\,\mathbf{x})dv+\int_{1/m^2}^{\infty}v^{s-k+1}\Theta(v,\,u,\,\mathbf{x})dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du\\
&=I_1(s)+I_2(s),
\end{aligned}$$

where

$$I_1(s)=\int_{D(\Pi_0(m^2))}\left(\int_0^{1/m^2}v^{s-k+1}\Theta(v,\,u,\,\mathbf{x})dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du,$$

$$I_2(s)=\int_{D(\Pi_0(m^2))}\left(\int_{1/m^2}^{\infty}v^{s-k+1}\Theta(v,\,u,\,\mathbf{x})dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du,$$

valid if Re s is sufficiently large, say, Re $s>k+1$. With the replacement of v by $1/m^4v$ and using the inversion formula (5.60) with $1/m^4v$ in place of v, we can rewrite the first integral in the form

$$\begin{aligned}
I_1(s)&=m^{4k-4s-8}\int_{D(\Pi_0(m^2))}\left(\int_{1/m^2}^{\infty}v^{k-s-3}\Theta(1/m^4v,\,u,\,\mathbf{x})dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du\\
&=m^{4k-4s-6}\int_{D(\Pi_0(m^2))}\Bigg(\sum_{\substack{\gamma,\,\delta\in\mathcal{O},\\ \gamma\equiv 0 \pmod m}}G(\mathbf{x},\,\gamma)\int_{1/m^2}^{\infty}v^{k-s-1}\\
&\times\exp\left(-\frac{\pi v}{t}(|\gamma\overline{w}-\delta|^2+|\gamma|^2t^2)\,dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du\\
&=m^{4k-4s-6}\sum_{\substack{\gamma,\,\delta\in\mathcal{O},\,(\gamma,\delta)\neq(0,0),\\ \gamma\equiv 0 \pmod m}}\int_{D(\Pi_0(m^2))}\Bigg(G(\mathbf{x},\,\gamma)\int_{1/m^2}^{\infty}v^{k-s-1}\\
&\times\exp\left(-\frac{\pi v}{t}(|\gamma\overline{w}-\delta|^2+|\gamma|^2t^2)\,dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du\\
&+m^{4k-4s-6}G(\mathbf{x},\,0)\int_{D(\Pi_0(m^2))}\left(\int_{1/m^2}^{\infty}v^{k-s-1}dv\right)t^k\widetilde{G}_{\mathbf{1}}(u)du\\
&=I_1'(s)+\frac{m^{2k-2s-6}G(\mathbf{x},\,0)}{s-k}\int_{D(\Pi_0(m^2))}t^k\widetilde{G}_{\mathbf{1}}(u)du, \qquad (5.64)
\end{aligned}$$

where

$$I_1'(s) = m^{4k-4s-6} \sum_{\substack{\gamma,\,\delta\in\mathcal{O},\,(\gamma,\delta)\neq(0,0),\\ \gamma\equiv 0 \pmod m}} \int_{D(\Pi_0(m^2))} \Big(G(\mathbf{x},\,\gamma) \int_{1/m^2}^{\infty} v^{k-s-1} \\ \times \exp\Big(-\frac{\pi v}{t}(|\gamma\overline{w}-\delta|^2 + |\gamma|^2 t^2\Big)\, dv \Big) t^k \widetilde{G}_{\mathbf{1}}(u)du.$$

It follows from relations (5.63) and (5.64) that

$$2^{3-2s}\pi^{k-2s-2}\Psi_F(s,\,\chi) \\ = I_1'(s) + I_2(s) + \frac{m^{2k-2s-6}G(\mathbf{x},\,0)}{s-k} \int_{D(\Pi_0(m^2))} t^k \widetilde{G}_{\mathbf{1}}(u)du \quad (5.65)$$

if, say, Re $s > k+1$. In the same way as it was done in the paper [3, §3.8] for the case $m = 1$ one can check that all integrals to the right of this relation are absolutely and uniformly convergent for all s and are therefore holomorphic functions of s: it follows from an estimate analogous to [3, Proposition 3.7.1(II)] that the inner integrals in $I_1'(s)$ and $I_2(s)$ are finite for all s, and if $t \to 0$ (respectively, ∞) and $|w|$ is bounded, they tend to infinity not faster than t^{-c} (respectively, t^c), where c is a positive constant; the fundamental domain $D(\Pi_0(m^2))$ is a union of a compact set and finitely many neighborhoods of parabolic vertices, that is, points where $D(\Pi_0(m^2))$ goes out to the boundary of $\mathbb{L} = \mathbb{L}^3$ at inequivalent parabolic fixed points of the group $\Pi_0(m^2)$; since $\widetilde{G}_{\mathbf{1}}$ together with F is a cusp form it follows from the definition of cusp forms that, as u tends from within of $D(\Pi_0(m^2))$ to one of its parabolic vertices, $\widetilde{G}_{\mathbf{1}}$ tends to zero like $\exp(-c'/t)$ (respectively, $\exp(-c't)$ if this vertex lies in the plane $t = 0$ (respectively, at infinity); hence all of the integral in (5.65) are bounded on the fundamental domain, and since it has finite invariant volume the integral are finite for all s. □

On analytic continuation of L-functions of cusp forms for genus 2.

Theorem 5.4. *Let $F \in \mathfrak{N}_k(\Gamma)$ be an eigenfunction for all $m-$regular Hecke operators for the group Γ, and χ – a primitive Dirichlet character modulo $m > 1$. Let us suppose that the $\mathbf{1}-$ray series (5.61) of the twist $G_{\mathbf{1}} = F|\mathcal{T}(\chi,\,\mathbf{1})$ of F with character χ and the unit p−matrix is not identical zero. Then the L−function (5.6) of F with the character χ has analytic continuation on the whole s−plane as a meromorphic function, and the function*

$$\Gamma(s)\Gamma(s-k+2)L_F(s,\,\chi) \quad (5.66)$$

where Γ is the gamma-function, is a meromorphic function having at most one simple pole at the point $s = k$.

Proof. By Theorem 5.1 for $G = G_{\mathbf{1}}$ we obtain the identity

$$g(\mathbf{1})\Gamma(s)\Gamma(s-k+2)L_F(s,\,\chi) \\ = \Gamma(s)\Gamma(s-k+2)L_{\mathcal{O}(\mathfrak{G})}(s-k+2,\,\mathbf{x})R_{G_{\mathbf{1}}}(s)g(\mathbf{1})$$

valid in a right half-plane. By Theorem 5.3, the last function can be continued analytically to the whole $s-$plane as a meromorphic function having at most one simple pole at the point $s = k$. Hence, the same is true and for the function (5.66), since the assumption $R_{G_{\mathbf{1}}}(s) \not\equiv 0$ is equivalent to the condition $g(\mathbf{1}) \neq 0$. □

Remarks. The restriction $R_{G_{\mathbf{1}}}(s) \not\equiv 0$ is not critical one and was selected just to simplify calculations. In order to avoid it one has no other choice, but to consider in the same way an arbitrary $N-$ray series (5.10), as it was done in [3] for the case $m = 1$. Another natural question, the question on functional equations of $L-$functions when $m > 1$ seems to be more enigmatic. Something prevents to get a functional equation of usual form even with all simplifying assumptions.

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