Max-Planck-Institut für Mathematik
Bonn

The Nielsen numbers of iterations of maps on infra-solvmanifolds of type (R) and periodic points

by

Alexander Fel'shtyn
Jong Bum Lee

Max-Planck-Institut für Mathematik
Preprint Series 2014 (23)

The Nielsen numbers of iterations of maps on infra-solvmanifolds of type (R) and periodic points

Alexander Fel'shtyn
Jong Bum Lee

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Instytut Matematyki
Uniwersytet Szczecinski
ul. Wielkopolska 15
70-451 Szczecin
Poland

Department of Mathematics
Sogang University
Seoul 121-742
Korea

MPIM 14-23

THE NIELSEN NUMBERS OF ITERATIONS OF MAPS ON INFRA-SOLVMANIFOLDS OF TYPE (R) AND PERIODIC POINTS

ALEXANDER FEL'SHTYN AND JONG BUM LEE

ABSTRACT. Utilizing the arguments employed mainly in [2] and [19, Chap. III] for the Lefschetz numbers of iterations, we study the asymptotic behavior of the sequence of the Nielsen numbers $\{N(f^k)\}$, the essential periodic orbits of f and the homotopy minimal periods of f by using the Nielsen theory of maps f on infra-solvmanifolds of type (R).

CONTENTS

1. INTRODUCTION

Let $f : X \to X$ be a map on a finite complex X. A point $x \in X$ is a fixed point of f if $f(x) = x$; a periodic point of f with period n if $f^n(x) = x$. The smallest period of x is called the minimal period. We will use the following notations:

$$\begin{aligned}
\mathrm{Fix}(f) &= \{x \in X \mid f(x) = x\},\\
\mathrm{Per}(f) &= \text{the set of all minimal periods of } f,\\
P_n(f) &= \text{the set of all periodic points of } f \text{ with minimal period } n,\\
\mathrm{HPer}(f) &= \bigcap_{g \simeq f} \{n \in \mathbb{N} \mid P_n(g) \neq \emptyset\}\\
&= \text{the set of all homotopy minimal periods of } f.
\end{aligned}$$

Let $p : \tilde{X} \to X$ be the universal cover of X and $\tilde{f} : \tilde{X} \to \tilde{X}$ a lift of f, i.e., $p \circ \tilde{f} = f \circ p$. Two lifts $\tilde{f}$ and $\tilde{f}'$ are called *conjugate* if there is a $\gamma \in \Gamma \cong \pi_1(X)$ such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}(f)$ is called the *fixed point class* of f determined by the lifting class $[\tilde{f}]$. A fixed point class is called *essential* if its index is nonzero. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$ [20].

The Nielsen number is always finite and is a homotopy invariant lower bound for the number of fixed points of f. In the category of compact, connected polyhedra the

Date: March 30, 2014.

2000 *Mathematics Subject Classification.* 37C25, 55M20.

Key words and phrases. Infra-solvmanifold, Nielsen number, Nielsen zeta function, periodic point.

Supported in part by Basic Science Researcher Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No. 2013R1A1A2058693) and by the Sogang University Research Grant of 2010 (10022).

Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.

From the dynamical point of view, it is natural to consider the Nielsen numbers $N(f^k)$ of all iterations of f simultaneously. For example, N. Ivanov [18] introduced the notion of the asymptotic Nielsen number, measuring the growth of the sequence $N(f^k)$ and found the basic relation between the topological entropy of f and the asymptotic Nielsen number. Later on, it was suggested in [9, 30, 10, 11] to arrange the Nielsen numbers $N(f^k)$ of all iterations of f into the Nielsen zeta function

$$N_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k\right).$$

The Nielsen zeta function $N_f(z)$ is a nonabelian analogue of the Lefschetz zeta function

$$L_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{L(f^k)}{k} z^k\right),$$

where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \mathrm{tr}\left[f^n_{*k} : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q})\right]$$

is the Lefschetz number of the iterate f^n of f.

Nice analytical properties of $N_f(z)$ [12, 11, 5, 13] indicate that the numbers $N(f^k)$ are closely interconnected. Another manifestations of this are Gauss congruences

$$\sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d) \equiv 0 \mod k,$$

for any $k > 0$, where f is a map on an infra-solvmanifold of type (R) [13].

The fundamental invariants of f used in the study of periodic points are the Lefschetz numbers $L(f^k)$, and their algebraic combinations, the Nielsen numbers $N(f^k)$ and the Nielsen-Jiang periodic numbers $NP_n(f)$ and $N\Phi_n(f)$.

The study of periodic points by using the Lefschetz theory has been done extensively by many authors in the literatures such as [20], [7], [2], [19], [29]. A natural question is to know how much information we can get about the set of essential periodic points of f or about the set of (homotopy) minimal periods of f from the study of the sequence $\{N(f^k)\}$ of the Nielsen numbers of iterations of f. Utilizing the arguments employed mainly in [2] and [19, Chap. III] for the Lefschetz numbers of iterations, we will study the asymptotic behavior of the sequence $\{N(f^k)\}$, the essential periodic orbits of f and the homotopy minimal periods of f by using the Nielsen theory of maps f on infra-solvmanifolds of type (R).

Acknowledgments. The first author is indebted to the Max-Planck-Institute for Mathematics(Bonn) for the support and hospitality and the possibility of the present research during his visit there.

2. Nielsen numbers $N(f^k)$

Let S be a connected and simply connected solvable Lie group. A discrete subgroup Γ of S is a *lattice* of S if $\Gamma\backslash S$ is compact, and in this case, we say that the quotient space $\Gamma\backslash S$ is a *special* solvmanifold. Let $\Pi \subset \mathrm{Aff}(S)$ be a torsion-free finite extension of the lattice $\Gamma = \Pi \cap S$ of S. That is, Π fits the short exact sequence

$$\begin{array}{ccccccccc}
1 & \longrightarrow & S & \longrightarrow & \mathrm{Aff}(S) & \longrightarrow & \mathrm{Aut}(S) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Pi/\Gamma & \longrightarrow & 1
\end{array}$$

Then Π acts freely on S and the manifold $\Pi\backslash S$ is called an *infra-solvmanifold*. The finite group $\Phi = \Pi/\Gamma$ is the *holonomy group* of Π or $\Pi\backslash S$. It sits naturally in $\mathrm{Aut}(S)$. Thus every infra-solvmanifold $\Pi\backslash S$ is finitely covered by the special solvmanifold $\Gamma\backslash S$. An infra-solvmanifold $M = \Pi\backslash S$ is of type (R) if S is of *type* (R) or *completely solvable*, i.e., if $\mathrm{ad}\, X : \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for all X in the Lie algebra $\mathfrak{S}$ of S.

Recall that a connected solvable Lie group S contains a sequence of closed subgroups

$$1 = N_1 \subset \cdots \subset N_k = S$$

such that N_i is normal in N_{i+1} and $N_{i+1}/N_i \cong \mathbb{R}$ or $N_{i+1}/N_i \cong S^1$. If the groups $N_1, \cdots, N_k$ are normal in S, the group S is called *supersolvable*. The supersolvable Lie groups are the Lie groups of type (R).

Lemma 2.1 ([33, Lemma 4.1], [14, Lemma 2.4]). *For a connected Lie group S, the following are equivalent:*

(1) *S is supersolvable.*
(2) *All elements of* $\mathrm{Ad}(S)$ *have only positive eigenvalues.*
(3) *S is of type* (R).

Recall [33, Theorem 1] in which it is proved that every infra-solvmanifold is modeled in a canonical way on a supersolvable Lie group. Hence whenever we deal with infra-solvmanifolds, we may assume that we are given infra-solvmanifolds $M = \Pi\backslash S$ of type (R).

In this paper, we shall assume that $f : M \to M$ is a continuous map on an infra-solvmanifold $M = \Pi\backslash S$ of type (R) with holonomy group Φ. Then f has an affine homotopy lift $(d, D) : S \to S$, and so f^k has an affine homotopy lift $(d, D)^k = (d', D^k)$ where $d' = dD(d) \cdots D^{k-1}(d)$. By the averaging formula for the Nielsen number $N(f^k)$ [24, Theorem 4.2], we have

$$N(f^k) = \frac{1}{\#\Phi} \sum_{A\in\Phi} |\det(I - A_* D_*^k)|. \tag{AV}$$

Concerning the Nielsen numbers $N(f^k)$ of all iterates of f, we recall the following results:

Theorem 2.2 ([13, Theorem 11.4]). *Let $f : M \to M$ be a map on an infra-solvmanifold M of type* (R). *Then*

$$\sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d) \equiv 0 \mod k \tag{DN}$$

for all $k > 0$.

Consider the sequences of algebraic multiplicities $\{A_k(f)\}$ and Dold multiplicities $\{I_k(f)\}$ associated to the sequence $\{N(f^k)\}$:

$$A_k(f) = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d), \quad I_k(f) = \sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d).$$

Then $I_k(f) = kA_k(f)$ and all $A_k(f)$ are integers by Theorem 2.2. From the Möbius inversion formula, we immediately have

$$N(f^k) = \sum_{d|k} d\, A_d(f).$$

Theorem 2.3 ([5, Theorem 4.5]). *Let $f : M \to M$ be a map on an infra-solvmanifold M of type* (R). *Then the Nielsen zeta function of f*

$$N_f(z) = \exp\left(\sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k\right)$$

is a rational function.

In fact, it is well-known that

$$L_f(z) = \prod_{k=0}^{m} \det(I - z \cdot f_{*k})^{(-1)^{k+1}}$$

where $f_{*k} : H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})$. Hence $L_f(z)$ is a rational function with coefficients in $\mathbb{Q}$. In [5, Theorem 4.5], it is shown that $N_f(z)$ is either

$$N_f(z) = L_f((-1)^n z)^{(-1)^{p+n}}$$

or

$$N_f(z) = \left(\frac{L_{f_+}((-1)^n z)}{L_f((-1)^n z)} \right)^{(-1)^{p+n}}$$

where p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1. Here f_+ is a lift of f to a certain 2-fold covering of M which has the same affine homotopy lift (d, D) as f. Consequently, $N_f(z)$ is a rational function with coefficients in $\mathbb{Q}$.

On the other hand, since $N_f(0) = 1$ by definition, $z = 0$ is not a zero nor a pole of the rational function $N_f(z)$. Thus we can write

$$N_f(z) = \frac{u(z)}{v(z)} = \frac{\prod_i (1 - \beta_i z)}{\prod_j (1 - \gamma_j z)} = \prod_{i=1}^{r} (1 - \lambda_i z)^{-\rho_i}$$

with all λ_i distinct nonzero algebraic integers and ρ_i nonzero integers. Taking log on both sides of the above identity, we obtain

$$\sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k = \sum_{i=1}^{r} -\rho_i \log(1 - \lambda_i z) = \sum_{i=1}^{r} \rho_i \left(\sum_{k=1}^{\infty} \frac{(\lambda_i z)^k}{k} \right) = \sum_{k=1}^{\infty} \frac{\sum_{i=1}^{r} \rho_i \lambda_i^k}{k} z^k.$$

This induces

$$(N1) \qquad N(f^k) = \sum_{i=1}^{r(f)} \rho_i \lambda_i^k.$$

Note that $r(f)$ is the number of zeros and poles of $N_f(z)$. Since $N_f(z)$ is a homotopy invariant, so is $r(f)$. This argument tells us that whenever we have a rational expression of $N_f(z)$, we can write down all $N(f^k)$ directly from the expression. However even though we can compute all $N(f^k)$ using the averaging formula, it can be rather complicated to write down the rational expression of $N_f(z)$, see [8].

On the other hand, we can show that

$$N_f(z) = \prod_{k=1}^{\infty} (1 - z^k)^{-A_k(f)}.$$

This identity shows that ± 1 are possible zeros or poles of $N_f(z)$. Indeed, this identity follows from the following observation

$$\begin{aligned}
\log \left(\prod_{m=1}^{\infty} (1 - z^m)^{-A_m(f)} \right) &= \sum_{m=1}^{\infty} -A_m(f) \log(1 - z^m) = \sum_{m=1}^{\infty} A_m(f) \left(\sum_{n=1}^{\infty} \frac{(z^m)^n}{n} \right) \\
&= \sum_{k=1}^{\infty} \left(\sum_{k=mn} \frac{A_m(f)}{n} (z^m)^n \right) = \sum_{k=1}^{\infty} \left(\sum_{m \mid k} \frac{m A_m(f)}{k} \right) z^k \\
&= \sum_{k=1}^{\infty} \frac{N(f^k)}{k} z^k.
\end{aligned}$$

Consider another generating function associated to the sequence $\{N(f^k)\}$:

$$S_f(z) = \sum_{k=1}^{\infty} N(f^k) z^{k-1}.$$

Then it is easy to see that

$$S_f(z) = \frac{d}{dz} \log N_f(z).$$

Moreover,

$$S_f(z) = \sum_{k=1}^{\infty} \sum_{i=1}^{r(f)} \rho_i \lambda_i^k z^{k-1} = \sum_{i=1}^{r(f)} \frac{\rho_i \lambda_i}{1 - \lambda_i z}$$

is a rational function with simple poles and integral residues, and 0 at infinity. The rational function $S_f(z)$ can be written as $S_f(z) = u(z)/v(z)$ where the polynomials $u(z)$ and $v(z)$ are of the form

$$u(z) = N(f) + \sum_{i=1}^{s} a_i z^i, \quad v(z) = 1 + \sum_{j=1}^{t} b_j z^j$$

with a_i and b_j integers, see (3) $\Rightarrow$ (5), Theorem 2.1 in [2] or [19, Lemma 3.1.31]. Let $\tilde{v}(z)$ be the conjugate polynomial of $v(z)$, i.e., $\tilde{v}(z) = z^t v(1/z)$. Then the numbers $\{\lambda_i\}$ are the roots of $\tilde{v}(z)$, and $r(f) = t$.

The following can be found in the proof of (3) $\Rightarrow$ (5), Theorem 2.1 in [2].

Lemma 2.4. *If λ_i and λ_j are roots of the rational polynomial $\tilde{v}(z)$ which are algebraically conjugate (i.e., λ_i and λ_j are roots of the same irreducible polynomial), then $\rho_i = \rho_j$.*

Proof. Let $\Sigma = \mathbb{Q}(\lambda_1, \cdots, \lambda_r) \subset \mathbb{C}$ be the field of the rational polynomial $\tilde{v}(z)$ and let σ be an automorphism of Σ over $\mathbb{Q}$, i.e., σ is the identity on $\mathbb{Q}$. The group of all such automorphisms is called the *Galois group* of Σ. Since the $\sigma(\lambda_i)$ are again the roots of $\tilde{v}(z)$, we have $\sigma(\lambda_i) = \lambda_{\sigma(i)}$. That is, σ induces a permutation σ on $\{1, \cdots, r\}$. Applying σ to the sequence $\{N(f^k)\}$, we obtain

$$\sigma\left(N(f^k)\right) = \sigma\left(\sum_{i=1}^{r} \rho_i \lambda_i^k\right) = \sum_{i=1}^{r} \rho_i \sigma(\lambda_i)^k = \sum_{i=1}^{r} \rho_i \lambda_{\sigma(i)}^k = \sum_{i=1}^{r} \rho_{\sigma^{-1}(i)} \lambda_i^k.$$

Since the $N(f^k)$ are integers, $\sigma(N(f^k)) = N(f^k)$ and consequently

$$\sum_{i=1}^{r} \rho_i \lambda_i^k = \sum_{i=1}^{r} \rho_{\sigma^{-1}(i)} \lambda_i^k.$$

As a matrix form, we can write

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_r \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_r \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_r^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^r & \lambda_2^r & \cdots & \lambda_r^r \end{bmatrix} \begin{bmatrix} \rho_{\sigma^{-1}(1)} \\ \rho_{\sigma^{-1}(2)} \\ \vdots \\ \rho_{\sigma^{-1}(r)} \end{bmatrix}.$$

Since the λ_i are distinct, the matrices in this equation are nonsingular (the Vandermonde determinant). Thus $\rho_i = \rho_{\sigma^{-1}(i)}$ for all $i = 1, \cdots, r$. On the other hand, it is known that the Galois group acts transitively on the set of algebraically conjugate roots. Since λ_i and λ_j are conjugate roots of $\tilde{v}(z)$, we can choose σ in the Galois group so that $\sigma(\lambda_i) = \lambda_j$. Hence $\sigma(i) = j$ and so $\rho_i = \rho_j$. $\square$

Let $\tilde{v}(z)=\prod_{\alpha=1}^{s}\tilde{v}_\alpha(z)$ be the decomposition of the monic integral polynomial $\tilde{v}(z)$ into irreducible polynomials $\tilde{v}_\alpha(z)$ of degree r_α. Of course, $r=r(f)=\sum_{\alpha=1}^{s}r_\alpha$ and

$$\begin{aligned}\tilde{v}(z)&=z^r+b_1z^{r-1}+b_2z^{r-2}+\cdots+b_{r-1}z+b_r\\&=\prod_{\alpha=1}^{s}(z^{r_\alpha}+b_1^\alpha z^{r_\alpha-1}+b_2^\alpha z^{r_\alpha-2}+\cdots+b_{r_\alpha-1}^\alpha z+b_{r_\alpha}^\alpha)=\prod_{\alpha=1}^{s}\tilde{v}_\alpha(z).\end{aligned}$$

If $\{\lambda_i^{(\alpha)}\}$ are the roots of $\tilde{v}_\alpha(z)$, then the associated ρ's are the same ρ_α. Consequently, we can rewrite $(N1)$ as

$$\begin{aligned}N(f^k)&=\sum_{\alpha=1}^{s}\rho_\alpha\left(\sum_{i=1}^{r_\alpha}(\lambda_i^{(\alpha)})^k\right)\\&=\sum_{\rho_\alpha>0}\rho_\alpha^+\left(\sum_{i=1}^{r_\alpha}(\lambda_i^{(\alpha)})^k\right)-\sum_{\rho_\alpha<0}\rho_\alpha^-\left(\sum_{i=1}^{r_\alpha}(\lambda_i^{(\alpha)})^k\right).\end{aligned}$$

Consider the $r_\alpha\times r_\alpha$-integral square matrices

$$M_\alpha=\begin{bmatrix}0&0&\cdots&0&-b_{r_\alpha}^\alpha\\1&0&\cdots&0&-b_{r_\alpha-1}^\alpha\\\vdots&\vdots&&\vdots&\vdots\\0&0&\cdots&0&-b_2^\alpha\\0&0&\cdots&1&-b_1^\alpha\end{bmatrix}.$$

The characteristic polynomial is $\det(zI-M_\alpha)=\tilde{v}_\alpha(z)$ and therefore $\{\lambda_i^{(\alpha)}\}$ are the eigenvalues of M_α. This implies that $N(f^k)=\sum_{\alpha=1}^{s}\rho_\alpha\operatorname{tr}M_\alpha^k$. Set

$$M_+=\bigoplus_{\rho_\alpha>0}\rho_\alpha^+M_\alpha,\qquad M_-=\bigoplus_{\rho_\alpha<0}\rho_\alpha^-M_\alpha.$$

Then

$$(N2)\qquad N(f^k)=\operatorname{tr}M_+^k-\operatorname{tr}M_-^k=\operatorname{tr}(M_+\bigoplus -M_-)^k.$$

We will show in Proposition 5.4 that if $A_k(f)\neq 0$ then $N(f^k)\neq 0$ and hence f has an essential periodic point of period k. In the following we investigate some other necessary conditions under which $N(f^k)\neq 0$. Recall that

$$N(f^k)=\text{ the number of essential fixed point classes of } f^k.$$

If $\mathbb{F}$ is a fixed point class of f^k, then $f^k(\mathbb{F})=\mathbb{F}$ and the *length* of $\mathbb{F}$ is the smallest number p for which $f^p(\mathbb{F})=\mathbb{F}$, written $p(\mathbb{F})$. We denote by $\langle\mathbb{F}\rangle$ the f-orbit of $\mathbb{F}$, i.e., $\langle\mathbb{F}\rangle=\{\mathbb{F},f(\mathbb{F}),\cdots,f^{p-1}(\mathbb{F})\}$ where $p=p(\mathbb{F})$. If $\mathbb{F}$ is essential, so is every $f^i(\mathbb{F})$ and $\langle\mathbb{F}\rangle$ is an *essential* periodic orbit of f with length $p(\mathbb{F})$ and $p(\mathbb{F})\mid k$. These are variations of Corollaries 2.3, 2.4 and 2.5 of [2].

Corollary 2.5. *If $r(f)\neq 0$, then $N(f^i)\neq 0$ for some $1\le i\le r(f)$. In particular, f has at least $N(f^i)$ essential periodic points of period i and an essential periodic orbit with the length $p\mid i, i\le r(f)$.*

Proof. Recall that

$$S_f(z)=\sum_{i=1}^{r(f)}\frac{\rho_i\lambda_i}{1-\lambda_iz}=\sum_{k=1}^{\infty}N(f^k)z^{k-1}.$$

Assume that $N(f)=\cdots=N(f^{r(f)})=0$. For simplicity, write the above identity as $S_f(z)=u(z)/v(z)=s(z)$ where $v(z)$ is a polynomial of degree $r(f)$ and $s(z)$ is the series of the right-hand side. Then $u(z)=v(z)s(z)$. A simple calculation shows that the higher order derivative of $v(z)s(z)$ up to order $r(f)-1$ at 0 are all zero. Since $u(z)$ is a polynomial of degree $r(f)-1$, it shows that $u(z)=0$, a contradiction. $\square$

Recalling the identity $N(f^k)=\sum_{i=1}^{r(f)}\rho_i\lambda_i^k$, we define

$$\rho(f)=\sum_{i=1}^{r(f)}\rho_i,\quad M(f)=\max\left\{\sum_{\rho_i\ge 0}\rho_i,-\sum_{\rho_j<0}\rho_j\right\}.$$

Corollary 2.6. *If $\rho(f)=0$ and $r(f)\ge 1$, then $r(f)\ge 2$ and $N(f^i)\neq 0$ for some $1\le i<r(f)$. In particular, f has at least $N(f^i)$ essential periodic points of period i and an essential periodic orbit with the length $p\mid i, i\le r(f)-1$.*

Proof. The conditions $\rho(f)=0$ and $r(f)\ge 1$ immediately implies that $r(f)\ge 2$. Since $r(f)\neq 0$ by the previous corollary there exists $i\in\{1,\cdots,r(f)\}$ such that $N(f^i)\neq 0$. Assume $N(f)=\cdots=N(f^{r(f)-1})=0$. So, $N(f^{r(f)})\neq 0$. As in the proof of the above corollary, we consider $u(z)/v(z)=s(z)$ where $u(z)$ is a polynomial of degree $r(f)-1\ge 1$ and $s(z)$ is a power series starting from the nonzero term $N(f^{r(f)})z^{r(f)-1}$. The derivative of order $r(f)-1$ on both sides of the identity $u(z)=v(z)s(z)$ yields that $N(f^{r(f)})=0$, a contradiction. $\square$

Corollary 2.7. *If $r(f)>0$, then $N(f^i)\neq 0$ for some $1\le i\le M(f)$. In particular, f has at least $N(f^i)$ essential periodic points of period i and an essential periodic orbit with the length $p\mid i, i\le M(f)$.*

Proof. Assume that $N(f^k)=0$ for all $k=1,\cdots,M(f)$. From $(N2)$, we have $J_k:=\operatorname{tr}M_+^k=\operatorname{tr}M_-^k$. For simplicity, suppose

$$m(f)=\sum_{\rho_i>0}\rho_i\le-\sum_{\rho_j<0}\rho_j=M(f).$$

Then the matrix M_+ has size $m(f)$ and M_- has size $M(f)$. We write the eigenvalues of M_+ and M_- respectively as

$$\mu_1,\cdots,\mu_{m(f)};\ \tilde\mu_1,\cdots,\tilde\mu_{M(f)}.$$

Of course, $\{\mu_1,\cdots,\mu_{m(f)},\tilde\mu_1,\cdots,\tilde\mu_{M(f)}\}=\{\lambda_1,\cdots,\lambda_r\}$ as a set. Now the identities $\operatorname{tr}M_+^k=\operatorname{tr}M_-^k$ yield the $M(f)$ equations

$$\mu_1^k+\cdots+\mu_{m(f)}^k+0+\cdots+0=\tilde\mu_1^k+\cdots+\tilde\mu_{M(f)}^k$$

where $\mu_j=0$ when $m(f)<j\le M(f)$. By [3, p.72, Corollary], there exists a permutation σ on $\{1,\cdots,M(f)\}$ such that $\mu_i=\tilde\mu_{\sigma(i)}$. If $m(f)<M(f)$ then $0=\mu_{M(f)}=\tilde\mu_{\sigma(M(f))}=\lambda_j$ for some j, a contradiction. Hence $m(f)=M(f)$ and the λ_i's associated to $\rho_i>0$ and the λ_j's associated with $\rho_j<0$ are the same. This implies that the rational function $N_f(z)$ has the same poles and zeros of equal multiplicity and hence $N_f(z)\equiv 1$, contradicting that $r(f)>0$. $\square$

3. Radius of convergence of $N_f(z)$

From the Cauchy-Hadamard formula, we can see that the radii R of convergence of the infinite series $N_f(z)$ and $S_f(z)$ are the same and given by

$$\frac{1}{R}=\limsup_{k\to\infty}\left(\frac{N(f^k)}{k}\right)^{1/k}=\limsup_{k\to\infty}N(f^k)^{1/k}.$$

We will understand the radius R of convergence from the identity $N(f^k)=\sum_{i=1}^{r(f)}\rho_i\lambda_i^k$. Recall that the λ_i^{-1} are the poles or the zeros of the rational function $N_f(z)$. We define

$$\lambda(f)=\max\{|\lambda_i|\mid i=1,\cdots,r(f)\}.$$

Remark 3.1. In this paper, the number $\lambda(f)$ will play a similar role as the "essential spectral radius" in [19] or the "reduced spectral radius" in [2]. Theorem 3.2 below shows that $1/\lambda(f)$ is the "radius" of the Nielsen zeta function $N_f(z)$. Note also that $\lambda(f)$ is a homotopy invariant.

If $r(f) = 0$, i.e., if $N(f^k) = 0$ for all $k > 0$, then $N_f(z) \equiv 1$ and $1/R = 0$. In this case, we define customarily $\lambda(f) = 0$. We shall assume now that $r(f) \neq 0$. In what follows, when $\lambda(f) > 0$, we consider another homotopy invariant:

$$n(f) = \#\{i \mid |\lambda_i| = \lambda(f)\}.$$

First we can observe easily the following:

(1) $\limsup z_k^{1/k} = \limsup (r_k e^{i\theta_k})^{1/k} = \limsup r_k^{1/k} e^{i\theta_k/k} = \limsup r_k^{1/k}$.
(2) $\limsup (\lambda^k)^{1/k} = |\lambda|$ by taking $z_k = \lambda^k$ in (1).
(3) When $\lim z_k = 0$ in (1), $\limsup z_k^{1/k} = 0$.
(4) $\lim (z_k + \rho)^{1/k} = \lim z_k^{1/k}$ when $\lim z_k = \infty$. For, in this case (1) induces

$$\lim \left(\frac{z_k + \rho}{z_k}\right)^{1/k} = 1.$$

Assume $|\lambda_j| \neq \lambda(f)$ for some j; then we have

$$\frac{N(f^k)}{\lambda_j^k} = \sum_{i \neq j} \rho_i \left(\frac{\lambda_i}{\lambda_j}\right)^k + \rho_j, \quad \lim \sum_{i \neq j} \rho_i \left(\frac{\lambda_i}{\lambda_j}\right)^k = \infty.$$

It follows from the above observations that $1/R = \limsup (\sum_{i \neq j} \rho_i \lambda_i^k)^{1/k}$. Consequently, we may assume that $N(f^k) = \sum_j \rho_j \lambda_j^k$ with all $|\lambda_j| = \lambda(f)$ and then we have

$$\frac{1}{R} = \limsup \left(\sum_{|\lambda_j| = \lambda(f)} \rho_j \lambda_j^k\right)^{1/k}.$$

Remark that if $\lambda(f) < 1$ then $N(f^k) = \sum_{|\lambda_j| = \lambda(f)} \rho_j \lambda_j^k \to 0$ and so the sequence of integers are eventually zero, i.e., $N(f^k) = 0$ for all k sufficiently large. This shows that $1/R = 0$ and furthermore, $N_f(z)$ is the exponential of a polynomial. Hence the rational function $N_f(z)$ has no poles and zeros. This forces $N_f(z) \equiv 1$; hence $\lambda(f) = 0$. If $\lambda(f) > 1$, then $N(f^k) \to \infty$ and by L'Hopital's rule we obtain

$$\limsup_{k \to \infty} \frac{\log N(f^k)}{k} = \limsup_{k \to \infty} \frac{\log \left(\sum_j \rho_j \lambda_j^k\right)}{k} = \log \lambda(f) \ \Rightarrow \ \frac{1}{R} = \lambda(f).$$

If $\lambda(f) = 1$, then $N(f^k) \leq \sum_j |\rho_j| < \infty$ is a bounded sequence and so it has a convergent subsequence. If $\limsup N(f^k) = 0$, then $N(f^k) = 0$ for all k sufficiently large and so by the same reason as above, $\lambda(f) = 0$, a contradiction. Hence $\limsup N(f^k)$ is a finite nonzero integer and so $1/R = 1 = \lambda(f)$.

Summing up, we have obtained that

Theorem 3.2. *Let f be a map on an infra-solvmanifold of type* (R). *Let R denote the radius of convergence of the Nielsen zeta function $N_f(z)$ of f. Then $\lambda(f) = 0$ or $\lambda(f) \geq 1$, and*

$$\frac{1}{R} = \lambda(f). \tag{R1}$$

In particular, $R > 0$.

By Theorem 3.2, we see that the sequence $N(f^k)$ is either bounded or exponentially unbounded.

Remark 3.3. Recall that

$$S_f(z) = \sum_{i=1}^{r(f)} \frac{\rho_i\lambda_i}{1-\lambda_i z},$$
$$N_f(z) = \prod_{i=1}^{r(f)} (1-\lambda_i z)^{-\rho_i} = \frac{\prod_{\rho_j<0}(1-\lambda_j z)^{-\rho_j}}{\prod_{\rho_i>0}(1-\lambda_i z)^{\rho_i}}.$$

These show that all of the $1/\lambda_i$ are the poles of $S_f(z)$, whereas the $1/\lambda_i$ with corresponding $\rho_i > 0$ are the poles of $N_f(z)$. The radius of convergence of a power series centered at a point a is equal to the distance from a to the nearest point where the power series cannot be defined in a way that makes it holomorphic. Hence the radius of convergence of $S_f(z)$ is $1/\lambda(f)$ and the radius of convergence of $N_f(z)$ is $1/\max\{|\lambda_i| \mid \rho_i > 0\}$. In particular, we have shown that

$$\lambda(f) = \max\{|\lambda_i| \mid i = 1, \cdots, r(f)\} = \max\{|\lambda_i| \mid \rho_i > 0\}.$$

Notice this identity in Example 3.6.

On the other hand, we can understand the radius R of convergence using the averaging formula. Compare our result with [13, Theorem 7.10]. Let $\{\mu_1, \cdots, \mu_m\}$ be the eigenvalues of D_*, counted with multiplicities, where m is the dimension of the manifold M. We denote by $\mathrm{sp}(A)$ the *spectral radius* of the matrix A which is the largest modulus of an eigenvalue of A. From the definition, we have

$$\mathrm{sp}(D_*) = \max\{|\mu_j| \mid i = 1, \cdots, m\},$$
$$\mathrm{sp}\left(\bigwedge D_*\right) = \begin{cases} \sum_{|\mu|>1} |\mu| & \text{when } \mathrm{sp}(D_*) > 1; \\ 1 & \text{when } \mathrm{sp}(D_*) \le 1. \end{cases}$$

Note $|\det(I - D_*^k)| = \prod_{j=1}^m |1-\mu_j^k|$. If some $\mu_j = 1$, then $\det(I - D_*^k) = 0$ for all $k > 0$ and hence $\limsup |\det(I - D_*^k)|^{1/k} = 0$. Assume now all $\mu_j \neq 1$; then $\det(I - D_*^k) \neq 0$ for all $k > 0$. Remark further from [13, Theorem 4.1] that

$$\begin{aligned}\log\left(\limsup_{k\to\infty} |\det(I - D_*^k)|^{1/k}\right) &= \limsup_{k\to\infty} \sum_{j=1}^m \frac{\log|1-\mu_j^k|}{k} \\ &= \begin{cases} \sum_{|\mu|>1} \log|\mu| & \text{when } \mathrm{sp}(D_*) > 1 \\ 0 & \text{when } \mathrm{sp}(D_*) \le 1. \end{cases}\end{aligned}$$

Now we ascertain that if D_* has no eigenvalue 1, then

$$(R2) \qquad \frac{1}{R} = \limsup_{k\to\infty} |\det(I - D_*^k)|^{1/k} = \mathrm{sp}\left(\bigwedge D_*\right).$$

From the averaging formula, we have $N(f^k) \ge |\det(I - D_*^k)|/|\Phi|$. This induces

$$\begin{aligned}\frac{1}{R} = \limsup_{k\to\infty} N_k^{1/k} &\ge \limsup_{k\to\infty} \left(\frac{|\det(I - D_*^k)|}{|\Phi|}\right)^{1/k} \\ &= \limsup_{k\to\infty} |\det(I - D_*^k)|^{1/k}.\end{aligned}$$

Furthermore, for any $A \in \Phi$, we obtain (see the proof of [13, Theorem 4.3])

$$|\det(I - A_* D_*^k)| \le \prod_{j=1}^m (1 + |\mu_j|^k)$$

and hence from the averaging formula

$$N(f^k) \le \prod_{j=1}^{m}(1+|\mu_j|^k) \ \Rightarrow \frac{1}{R} \le \prod_{|\mu|>1} |\mu|.$$

This finishes the proof of our assertion.

Following from $(R1)$ and $(R2)$, we immediately have:

Theorem 3.4. *Let f be a map on an infra-solvmanifold of type* (R) *with an affine homotopy lift (d,D). Let R denote the radius of convergence of the Nielsen zeta function of f. If D_* has no eigenvalue 1, then*

$$\frac{1}{R} = \mathrm{sp}\left(\bigwedge D_*\right) = \lambda(f).$$

We recall that the asymptotic Nielsen number of f is defined to be

$$N^\infty(f) : \max\left\{1, \limsup_{k\to\infty} N(f^k)^{1/k}\right\}.$$

We also recall that the most widely used measure for the complexity of a dynamical system is the topological entropy $h(f)$. A basic relation between these two numbers is $h(f) \ge \log N^\infty(f)$, which was found by Ivanov in [18]. There is a conjectural inequality $h(f) \ge \log(\mathrm{sp}(f))$ raised by Shub [31]. This conjecture was proven for all maps on infra-solvmanifolds of type (R), see [27, 28] and [13]. Now we can state about relations between $N^\infty(f)$, $\lambda(f)$ and $h(f)$.

Corollary 3.5. *Let f be a map on an infra-solvmanifold of type* (R) *with an affine homotopy lift (d,D). If D_* has no eigenvalue 1, then*

$$N^\infty(f) = \lambda(f), \quad h(f) \ge \log \lambda(f).$$

Proof. From [13, Theorem 4.3] and Theorem 3.4, we have $N^\infty(f) = \mathrm{sp}(\bigwedge D_*) = \lambda(f)$. Hence by Ivanov's inequality, we obtain that $h(f) \ge \log N^\infty(f) = \log \lambda(f)$. □

The following example shows that the assumption in Theorem 3.4 and its Corollary that 1 is not in the spectrum of D_* is essential.

Example 3.6. Let $f : M \to M$ be a map of type (r,ℓ,q) on the Klein bottle M induced by an affine map $(d,D) : \mathbb{R}^2 \to \mathbb{R}^2$. Recall from [22, Theorem 2.3] and its proof that r is odd and $q = 0$, and

$$(d,D) = \begin{cases} \left(-\frac{1}{2}\begin{bmatrix} * \\ \ell \end{bmatrix}, \begin{bmatrix} r & 0 \\ 0 & q \end{bmatrix}\right) & \text{when } r \text{ is odd;} \\ \left(\begin{bmatrix} * \\ * \end{bmatrix}, \begin{bmatrix} r & 0 \\ 2\ell & 0 \end{bmatrix}\right) & \text{when } r \text{ is even and } q=0, \end{cases}$$

$$N(f^k) = \begin{cases} |q^k(1-r^k)| = \begin{cases} q^k(r^k-1) & \text{when } r \text{ is odd and } qr>0 \\ (-1)^k q^k(r^k-1) & \text{when } r \text{ is odd and } qr<0 \end{cases} \\ |1-r^k| = \begin{cases} 1 & \text{when } q=r=0 \\ r^k-1 & \text{when } r>0 \text{ and } q=0 \\ (-1)^k(r^k-1) & \text{when } r<0 \text{ and } q=0. \end{cases} \end{cases}$$

A simple calculation shows that

q, r	$N(f^k)$	$N_f(z)$	$S_f(z)$	$(\lambda(f), \mathrm{sp}(\bigwedge D_*))$
$r=1$	0	1	0	$(0, \max\{1, \lvert q\rvert\})$
r odd, $qr>0$	$(qr)^k - q^k$	$\dfrac{1-qz}{1-qrz}$	$\dfrac{qr}{1-qrz} - \dfrac{q}{1-qz}$	(qr, qr)
r odd, $qr<0$	$(-qr)^k - (-q)^k$	$\dfrac{1+qz}{1+qrz}$	$-\dfrac{qr}{1+qrz} + \dfrac{q}{1+qz}$	$(-qr, -qr)$
$q=0$, $r=0$	1	$\dfrac{1}{1-z}$	$\dfrac{1}{1-z}$	$(1,1)$
$q=0$, $r>0$	r^k-1	$\dfrac{1+z}{1-rz}$	$\dfrac{r}{1-rz} - \dfrac{1}{1+z}$	(r,r)
$q=0$, $r<0$	$(-r)^k - (-1)^k$	$\dfrac{1+z}{1+rz}$	$-\dfrac{r}{1+rz} + \dfrac{1}{1+z}$	$(-r,-r)$

These observations show that when one of the eigenvalues is 1, the invariants $N_f(z)$, $\mathrm{sp}(\bigwedge D_*)$ and $\lambda(f)$ still strongly depend upon the other eigenvalue. Remark also in this example that the identity $\lambda(f) = \max\{|\lambda_i| \mid \rho_i > 0\}$ holds.

4. Asymptotic behavior of the sequence $\{N(f^k)\}$

In this section, we study the asymptotic behavior of the Nielsen numbers of iterates of maps on infra-solvmanifolds of type (R). Compared to the asymptotic behavior of the Lefschetz numbers, see [2, Theorem 2.6] or [19, Theorem 3.1.53], the Nielsen numbers on infra-solvmanifolds have rather very restrictive asymptotic behavior. For example, the case (b) of [2, Theorem 2.6] does not occur.

Theorem 4.1. *For a map f of an infra-solvmanifold of type* (R)*, one of the following two possibilities holds:*

(1) $\lambda(f)=0$*, which occurs if and only if* $N_f(z)\equiv 1$.
(2) *The sequence* $\{N(f^k)/\lambda(f)^k\}$ *has the same limit points as a periodic sequence* $\{\sum_j \alpha_j\epsilon_j^k\}$ *where* $\alpha_j\in\mathbb{Z}, \epsilon_j\in\mathbb{C}$ *and* $\epsilon_j^q = 1$ *for some* $q>0$.

Proof. For simplicity, we denote $\lambda(f)$ by λ_0. Recall that $\lambda_0 = 0$ if and only if all $N(f^k)=0$ and otherwise, $\lambda_0\ge 1$. Suppose that $\lambda_0\ge 1$. Let

$$\lambda_1 = \lambda_0 e^{2i\pi\theta_1}, \cdots, \lambda_{n(f)} = \lambda_0 e^{2i\pi\theta_{n(f)}}$$

be all the λ_i of modulus λ_0 in the identity $N(f^k) = \sum_{i=1}^{r(f)} \rho_i\lambda_i^k$.

First we observe that the sequence $\{N(f^k)/\lambda_0^k\}_k$ has the same asymptotic behavior as the sequence $\{\sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}\}_k$. Indeed, we can write $N(f^k) = \Gamma_k + \Omega_k$, where

$$\Gamma_k = \lambda_0^k\left(\sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}\right),\quad \Omega_k = \sum_{i=n(f)+1}^{r(f)} \rho_i\lambda_i^k \text{ with } |\lambda_i|<\lambda_0.$$

Consequently,

$$\Big|\frac{N(f^k)}{\lambda_0^k} - \sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}\Big| \le \sum_{i=n(f)+1}^{r(f)} |\rho_i|\frac{|\lambda_i|^k}{\lambda_0^k} \to 0,$$

since $|\lambda_i|<\lambda_0$.

Suppose all $\theta_j = p_j/q_j$ $(j=1,\cdots,n(f))$ are rational. Then every $\lambda_j/\lambda_0 = e^{2i\pi\theta_j}$ is a q_jth root of unity, and thus all $\lambda_j/\lambda_0 = e^{2i\pi\theta_j}$ are roots of unity of degree $q = \mathrm{lcm}(q_1,\cdots,q_{n(f)})$, and hence the sequence $\{\sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}\}_k$ is periodic of period q. This proves (2).

Now we claim that all θ_j must be rational. For the given map $f:\Pi\backslash S\to\Pi\backslash S$ on the infra-solvmanifold $\Pi\backslash S$, let $\varphi:\Pi\to\Pi$ be the homomorphism induced by f and let $(d,D):S\to S$ be an affine map such that $\varphi(\alpha)(d,D) = (d,D)\alpha$ for all $\alpha\in\Pi\subset S\rtimes\mathrm{Aut}(S)$. Then φ induces a function $\hat\varphi:\Phi\to\Phi$ on the holonomy group Φ satisfying that $\hat\varphi(A)D = DA$ for all $A \in \Phi$. Recall from [6, Lemma 3.1] the following: Given $A \in \Phi$, we choose a sequence $B_1 = A$ and $B_{i+1} = \hat\varphi(B_i)$ so that there exist $j,k \geq 1$ such that $B_{j+k} = B_j$. Furthermore,

(1) $\forall i \in \mathbb{N}$, $\det(I - \hat\varphi(B_i)_* D_*) = \det(I - \hat\varphi(B_{i+1})_* D_*)$,
(2) $\exists \ell \in \mathbb{N}$ such that $(\hat\varphi(B_j)_* D_*)^\ell = D_*^\ell$.

Let $\mu_1, \cdots, \mu_m$ be the eigenvalues of D_*, and $\nu_1, \cdots, \nu_m$ the eigenvalues of $\hat\varphi(B_j)_* D_*$. Since $(\hat\varphi(B_j)_* D_*)^\ell = D_*^\ell$, $(\hat\varphi(B_j)_* D_*)^\ell$ has the eigenvalues $\{\mu_1^\ell, \cdots, \mu_m^\ell\} = \{\nu_1^\ell, \cdots, \nu_m^\ell\}$. We may assume that $|\mu_i|^\ell = |\nu_i|^\ell$ and so $|\mu_i| = |\nu_i|$ for all $i = 1, \cdots, m$. Now, we have

$$\begin{aligned}\det(I - A_* D_*) &= \det(I - D_* A_*) = \det(I - \hat\varphi(B_j)_* D_*)\\ &= \prod_{i=1}^m (1 - \nu_i)\\ &= 1 - \sum_{i=1}^m \nu_i + \sum_{1 \leq i < j \leq n} \nu_i \nu_j - \cdots + (-1)^m \nu_1 \cdots \nu_m.\end{aligned}$$

Let $\mu_1, \cdots, \mu_s$ be all the eigenvalues with absolute value > 1 and let $\mu_{s+1}, \cdots, \mu_{s+t}$ $(s + t \leq m)$ be the eigenvalues of modulus 1, which are roots of unity by a well-known result of Kronecker: Every nonzero algebraic integer that lies with its algebraic conjugates in the closed unit disc is a root of unity. In the above expression for $\det(I - A_* D_*)$, the largest modulus of a term is $\prod_{|\mu| \geq 1} |\mu| = \mathrm{sp}(\bigwedge D_*)$ and the terms of this modulus are of the form $\prod_{i=1}^s \nu_i \prod_{J \subset \{s+1, \cdots, s+t\}} \nu_J$. Remark that $\det(I - A_* D_*)$ is a real number (in fact, an integer). Hence

$$N(f) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \pm \left(1 - \sum_{i=1}^m \nu_i + \sum_{1 \leq i < j \leq n} \nu_i \nu_j - \cdots + (-1)^m \nu_1 \cdots \nu_m\right),$$

$$\Gamma_1 = \sum_{j=1}^{n(f)} \rho_j \lambda_j = \frac{1}{|\Phi|} \sum_{A \in \Phi} \pm \left(\prod_{i=1}^s \nu_i \prod_{J \subset \{s+1, \cdots, s+t\}} \nu_J\right).$$

Since $\lambda(f) \geq 1$, by Lemma 4.2 below $\lambda(f) = \mathrm{sp}(\bigwedge D_*)$. Consequently, we must have

$$\{\lambda_j \mid |\lambda_j| = \lambda(f)\} = \left\{\prod_{i=1}^s \nu_i \prod_{J \subset \{s+1, \cdots, s+t\}} \nu_J \mid A \in \Phi\right\}.$$

Remark that ν_{s+j} $(j = 1, \cdots, t)$ is a root of unity because $|\nu_{s+j}| = |\mu_{s+j}|$. Remark also that if some ν_i $(i = 1, \cdots, s)$ is a complex number then its complex conjugate $\bar\nu_i$ is ν_j for some $j \in \{1, \cdots, s\}$. That is, the collection $\{\nu_1, \cdots, \nu_s\}$ with modulus > 1 contains complex conjugate pairs. This shows that $\prod_{i=1}^s \nu_i$ is a real number. Hence it follows that the θ_j associated to λ_j is a rational number. □

In Theorem 3.4, we showed that if D_* has no eigenvalue 1 then $\lambda(f) = \mathrm{sp}(\bigwedge D_*)$. In Example 3.6, we have seen that when D_* has an eigenvalue 1, there are maps f for which $\lambda(f) \neq \mathrm{sp}(\bigwedge D_*)$ with $\lambda(f) = 0$, and $\lambda(f) = \mathrm{sp}(\bigwedge D_*)$ with $\lambda(f) \geq 1$. In fact, we prove in the following that the latter case is always true.

Lemma 4.2. *Let f be a map on an infra-solvmanifold of type* (R) *with an affine homotopy lift (d, D). If $\lambda(f) \geq 1$, then $\lambda(f) = \mathrm{sp}(\bigwedge D_*)$.*

Proof. Since $\lambda(f) \geq 1$, by Corollary 2.7, $N(f^k) \neq 0$ for some $k \geq 1$ and then by the averaging formula, there is $B \in \Phi$ such that $\det(I - B_* D_*^k) \neq 0$. Choose $\beta \in \Pi$ of the form $\beta = (b, B)$. Then $\beta(d, D)^k$ is another homotopy lift of f^k. We have observed above that there are numbers $\nu_1, \cdots, \nu_m$ such that $\det(I - B_* D_*^k) = \prod_{i=1}^m (1 - \nu_i)$ and $\{(\mu_i^k)^\ell\} = \{\nu_i^\ell\}$ for some $\ell > 0$. Since $\det(I - B_* D_*^k) \neq 0$, $B_* D_*^k$ has no eigenvalue 1.

Hence by Theorem 3.4, we have $\lambda(f^k)=\mathrm{sp}(\bigwedge B_*D_*^k)$. Recall that $N(f^k)=\sum_{i=1}^{r(f)}\rho_i\lambda_i^k$ and $\lambda(f)=\max\{|\lambda_i|\}$. Since $\lambda(f)\ge 1$, it follows that $\lambda(f^k)=\lambda(f)^k$. Observe further that $\mathrm{sp}(\bigwedge B_*D_*^k)=\prod_{|\mu_i|\ge1}|\nu_i|=\prod_{|\mu_i|\ge1}|\mu_i^k|=\mathrm{sp}(\bigwedge D_*)^k$. Consequently, we obtain the required identity $\lambda(f)=\mathrm{sp}(\bigwedge D_*)$. □

Example 4.3. Consider the 3-dimensional orientable flat manifold with fundamental group $\mathfrak{G}_2$ generated by $\{t_1,t_2,t_3,\alpha\}$ where

$$\begin{aligned}
&t_1=\left(\begin{bmatrix}1\\0\\0\end{bmatrix},I\right),\ t_2=\left(\begin{bmatrix}0\\1\\0\end{bmatrix},I\right),\ t_3=\left(\begin{bmatrix}0\\0\\1\end{bmatrix},I\right),\\
&\alpha=(a,A)=\left(\begin{bmatrix}\frac12\\0\\0\end{bmatrix},\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\right).
\end{aligned}$$

Thus,

$$\mathfrak{G}_2=\left\langle t_1,t_2,t_3,\alpha\mid [t_i,t_j]=1,\alpha^2=t_1,\alpha t_2\alpha^{-1}=t_2^{-1},\alpha t_3\alpha^{-1}=t_3^{-1}\right\rangle.$$

Let $\varphi:\mathfrak{G}_2\to\mathfrak{G}_2$ be any homomorphism. Every element of $\mathfrak{G}_2$ is of the form $\alpha^k t_2^m t_3^n$. Thus φ has the form

$$\varphi(t_2)=\alpha^{k_2}t_2^{m_2}t_3^{n_2},\ \varphi(t_3)=\alpha^{k_3}t_2^{m_3}t_3^{n_3},\ \varphi(\alpha)=\alpha^k t_2^m t_3^n.$$

The relations $\alpha t_2\alpha^{-1}=t_2^{-1}$ and $\alpha t_3\alpha^{-1}=t_3^{-1}$ yield that $k_2=k_3=0$, $(-1)^k m_i=-m_i$ and $(-1)^k n_i=-n_i$. Hence when k is even, we have $m_i=n_i=0$. Further, $\varphi(t_1)=t_1^k$.

Now we shall determine an affine map (d,D) satisfying $\varphi(\beta)(d,D)=(d,D)\beta$ for all $\beta\in\mathfrak{G}_2$.

Case $k=2\ell$.
In this case, we have $\varphi(t_2)=\varphi(t_3)=1$ and $\varphi(\alpha)=\alpha^k t_2^m t_3^n=t_1^\ell t_2^m t_3^n$, and hence we need to determine (d,D) satisfying

$$\begin{aligned}
&(d,D)=(d,D)(e_2,I)\Rightarrow D(e_2)=\mathbf{0},\\
&(d,D)=(d,D)(e_3,I)\Rightarrow D(e_3)=\mathbf{0},\\
&(\ell e_1+me_2+ne_3,I)(d,D)=(d,D)(a,A)\\
&\qquad\Rightarrow \ell e_1+me_2+ne_3+d=d+D(a),\ D=DA.
\end{aligned}$$

Hence the second and the third columns of D must be $\mathbf{0}$ and so $D=DA$ is automatically satisfied and the first column of D is $2\begin{bmatrix}\ell&m&n\end{bmatrix}^t$. That is,

$$(d,D)=\left(\begin{bmatrix}*\\ *\\ *\end{bmatrix},\begin{bmatrix}2\ell&0&0\\2m&0&0\\2n&0&0\end{bmatrix}\right).$$

The eigenvalues of D are 0 (multiple) and 2ℓ, and $N(f^k)=|(2\ell)^k-1|$ and

$$N_f(z)=\begin{cases}\frac{1}{1-z} & \text{when } \ell=0;\\ \frac{1-z}{1-2\ell z} & \text{when } \ell\ge1;\\ \frac{1+z}{1+2\ell z} & \text{when } \ell\le-1.\end{cases}$$

It follows that $\lambda(f)=\max\{1,2|\ell|\}$ and $r(f)=1$.

Case $k=2\ell+1$.
In this case, we have $\varphi(t_2)=t_2^{m_2}t_3^{n_2}$, $\varphi(t_3)=t_2^{m_3}t_3^{n_3}$ and $\varphi(\alpha)=\alpha^k t_2^m t_3^n=\alpha t_1^\ell t_2^m t_3^n$,

and hence we need to determine (d,D) satisfying

$$\begin{aligned}
&\varphi(t_2)(d,D)=(d,D)t_2 \Rightarrow D(e_2)=m_2e_2+n_2e_3,\\
&\varphi(t_3)(d,D)=(d,D)t_3 \Rightarrow D(e_3)=m_3e_2+n_3e_3,\\
&\varphi(\alpha)(d,D)=(d,D)\alpha\\
&\qquad\Rightarrow a+A(\ell e_1+me_2+ne_3)+A(d)=d+D(a),\ AD=DA.
\end{aligned}$$

These yield

$$(d,D)=\left(\begin{bmatrix}*\\-\frac{m}{2}\\-\frac{n}{2}\end{bmatrix},\begin{bmatrix}2\ell+1&0&0\\0&m_2&m_3\\0&n_2&n_3\end{bmatrix}\right).$$

Now we consider some explicit examples of such D. First we take D to be

$$D=\begin{bmatrix}-1&0&0\\0&d&e\\0&-e&d\end{bmatrix}.$$

Then D has eigenvalues -1 and $\mu=d\pm ei$. Thus

$$\begin{aligned}
&\det(I-D)=2((1-d)^2+e^2),\ \det(I-AD)=2((1+d)^2+e^2),\\
&N(f)=((1-d)^2+e^2)+((1+d)^2+e^2)=2(1+d^2+e^2)=2(1+|\mu|^2).
\end{aligned}$$

Clearly $N(f^k)=0$ for all even integers $k>0$. Now for an odd k, D^k has eigenvalues -1 and μ^k and consequently $N(f^k)=2(1+|\mu|^{2k})$. This yields that

$$\begin{aligned}
N_f(z)&=\exp\left(\sum_{k=1}^\infty\frac{2(1+|\mu|^{2(2k-1)})}{2k-1}z^{2k-1}\right)\\
&=\exp\left(\sum_{k=1}^\infty\frac{2}{2k-1}z^{2k-1}+\sum_{k=1}^\infty\frac{2}{2k-1}(|\mu|^2z)^{2k-1}\right)\\
&=\exp\left(\log\frac{1+z}{1-z}+\log\frac{1+|\mu|^2z}{1-|\mu|^2z}\right)\\
&=\frac{(1+z)(1+|\mu|^2z)}{(1-z)(1-|\mu|^2z)}.
\end{aligned}$$

Moreover, $\lambda(f)=\max\{1,|\mu|^2\}$, and $r(f)=2$ when $\lambda(f)>1$ and $r(f)=1$ when $\lambda(f)\le 1$.

Secondly, we take D to be

$$D=\begin{bmatrix}-1&0&0\\0&2&1\\0&1&1\end{bmatrix}.$$

Then the eigenvalues of D are -1 and $\mu_i=(3\pm\sqrt5)/2$ and $N(f^k)=(1-(-1)^k)(\mu_1^k+\mu_2^k)$. We then have the case (2) of Theorem 4.1. Observe also that $N_f(z)=(1+\mu_1z)(1+\mu_2z)/(1-\mu_1z)(1-\mu_2z)$ and so $\lambda(f)=\max\{|\mu_i|\}=(3+\sqrt5)/2$ with $r(f)=2$. Another remark about the eigenvalues of AD is: The eigenvalues of AD are -1 and $\nu_i=(-3\pm\sqrt5)/2$; hence $|\mu_i|=|\nu_i|$ but $\mu_i\ne\nu_i$.

It is important to know not only the rate of growth of the sequence $\{N(f^k)\}$ but also the frequency with which the largest Nielsen number is encountered. The following theorem shows that this sequence grows relatively dense. The following are variations of Theorem 2.7, Proposition 2.8 and Corollary 2.9 of [2].

Theorem 4.4. *Let $f:M\to M$ be a map on an infra-solvmanifold of type* (R). *If $\lambda(f)\ge 1$, then there exist $\gamma>0$ and a natural number N such that for any $m>N$ there is an $\ell\in\{0,1,\cdots,n(f)-1\}$ such that $N(f^{m+\ell})/\lambda(f)^{m+\ell}>\gamma$.*

Proof. As in the proof of Theorem 4.1, for any $k > 0$, we can write $N(f^k) = \Gamma_k + \Omega_k$, where

$$\Gamma_k = \lambda(f)^k \left(\sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}\right), \quad \Omega_k = \sum_{i=n(f)+1}^{r(f)} \rho_i \lambda_i^k \text{ with } |\lambda_i| < \lambda(f).$$

Then $\Gamma_k/\lambda(f)^k = \sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}$. Consider the following $n(f)$ consecutive identities

$$\frac{\Gamma_{k+\ell}}{\lambda(f)^{k+\ell}} = \sum_{j=1}^{n(f)} \left(\rho_j e^{2i\pi(k\theta_j)}\right) e^{2i\pi(\ell\theta_j)}, \quad \ell = 0, \cdots, n(f)-1.$$

Let $W = W(\theta_1, \cdots, \theta_{n(f)})$ be the Vandermonde operator on $\mathbb{C}^{n(f)}$

$$W(\theta_1, \cdots, \theta_{n(f)}) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{2i\pi\theta_1} & e^{2i\pi\theta_2} & \cdots & e^{2i\pi\theta_{n(f)}} \\ e^{2i\pi(2\theta_1)} & e^{2i\pi(2\theta_2)} & \cdots & e^{2i\pi(2\theta_{n(f)})} \\ \vdots & \vdots & & \vdots \\ e^{2i\pi(n(f)-1)\theta_1} & e^{2i\pi(n(f)-1)\theta_2} & \dots & e^{2i\pi(n(f)-1)\theta_{n(f)}} \end{bmatrix}$$

and let $2\gamma = \|W^{-1}\|^{-1}$. Then the vector $\vec{\rho} = \left(\rho_1 e^{2i\pi(k\theta_1)}, \cdots, \rho_{n(f)} e^{2i\pi(k\theta_{n(f)})}\right)$ satisfies $\|W\vec{\rho}\| \geq \|W^{-1}\|^{-1}\|\vec{\rho}\| = 2\gamma\|\vec{\rho}\| \geq 2\gamma\sqrt{n(f)}$. Thus there is at least one of the coordinates of the vector $W\vec{\rho}$ whose modulus is $\geq 2\gamma$. That is, there is an $\ell \in \{0, 1, \cdots, n(f)-1\}$ such that $|\Gamma_{k+\ell}|/\lambda(f)^{k+\ell} \geq 2\gamma$.

On the other hand, since all $|\lambda_i| < \lambda(f)$, we have $\Omega_k/\lambda(f)^k \to 0$. Thus we can choose N so large that $m > N \Rightarrow |\Omega_m|/\lambda(f)^m < \gamma$.

In all, whenever $m > N$ there is an $\ell \in \{0, 1, \cdots, n(f)-1\}$ such that

$$\frac{N(f^{m+\ell})}{\lambda(f)^{m+\ell}} \geq \frac{|\Gamma_{m+\ell}|}{\lambda(f)^{m+\ell}} - \frac{|\Omega_{m+\ell}|}{\lambda(f)^{m+\ell}} > 2\gamma - \gamma = \gamma.$$

This finishes the proof. □

Proposition 4.5. *Let $f : M \to M$ be a map on an infra-solvmanifold of type* (R) *such that $\lambda(f) > 1$. Then for any $\epsilon > 0$, there exists N such that if $N(f^m)/\lambda(f)^m \geq \epsilon$ for $m > N$, then the Dold multiplicity $I_m(f)$ satisfies*

$$|I_m(f)| \geq \frac{\epsilon}{2}\lambda(f)^m.$$

Proof. From the definition of Dold multiplicity $I_k(f)$, we have

$$|I_k(f)| = \Big|\sum_{d|k} \mu\left(\frac{k}{d}\right) N(f^d)\Big| \geq N(f^k) - \Big|\sum_{d|k, d\neq k} \mu\left(\frac{k}{d}\right) N(f^d)\Big|.$$

Let C be any number such that $2M(f) \leq C$. Then for any $d > 0$

$$N(f^d) \leq \sum_{i=1}^{r(f)} |\rho_i|\lambda(f)^d \leq 2M(f)\lambda(f)^d \leq C\lambda(f)^d.$$

Thus we have

$$\begin{aligned} |I_k(f)| &\geq N(f^k) - C\sum_{d|k, d\neq k} \lambda(f)^d \geq N(f^k) - C\tau(k)\lambda(f)^{k/2} \\ &= N(f^k) - C\frac{\tau(k)}{\lambda(f)^{k/2}}\lambda(f)^k \end{aligned}$$

where $\tau(k)$ is the number of divisors of k. Since $\tau(k) \leq 2\sqrt{k}$, see [19, Ex 3.2.17], and since $\lambda(f) > 1$, we have $\lim_{k\to\infty} \tau(k)/\lambda(f)^{k/2} = 0$, and so there exists an integer N such that $C\tau(k)/\lambda(f)^{k/2} < \epsilon/2$ for all $k > N$. Let $m > N$ such that $N(f^m)/\lambda(f)^m \geq \epsilon$. The above inequality induces the required inequality

$$|I_m(f)| \geq \left(\frac{N(f^m)}{\lambda(f)^m} - C\frac{\tau(m)}{\lambda(f)^{m/2}}\right)\lambda(f)^m \geq \frac{\epsilon}{2}\lambda(f)^m. \qquad \square$$

Theorem 4.4 and Proposition 4.5 imply immediately the following:

Corollary 4.6. *Let $f : M \to M$ be a map on an infra-solvmanifold of type* (R) *such that $\lambda(f) > 1$. Then there exist $\gamma > 0$ and a natural number N such that if $m \geq N$ then there exists ℓ with $0 \leq \ell \leq n(f) - 1$ such that $|I_{m+\ell}(f)|/\lambda(f)^{m+\ell} \geq \gamma/2$. In particular $I_{m+\ell}(f) \neq 0$ and so $A_{m+\ell}(f) \neq 0$.*

Remark 4.7 (Compare with [19, Remark 3.1.60])**.** We state a little bit more about the density of the set of algebraic periods $\mathcal{A}(f) = \{m \in \mathbb{N} \mid A_m(f) \neq 0\}$. We consider the notion of the *lower density* $\mathrm{DA}(f)$ of the set $\mathcal{A}(f) \subset \mathbb{N}$:

$$\mathrm{DA}(f) = \liminf_{k\to\infty} \frac{\#(\mathcal{A}(f) \cap [1,k])}{k}.$$

By Corollary 4.6, when $\lambda(f) > 1$, we have $\mathrm{DA}(f) \geq 1/n(f)$. On the other hand, when $\lambda(f) \geq 1$ by Theorem 4.1, the sequence $\{N(f^k)/\lambda(f)^k\}$ has the same limit points as the periodic sequence $\{\sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)}\}$ of period $q = \mathrm{lcm}(q_1, \cdots, q_{n(f)})$. Hence by Theorem 4.4, we have $\mathrm{DA}(f) \geq 1/q$.

5. Essential periodic orbits

In this section, we shall give an estimate from below the number of *essential periodic orbits* of maps on infra-solvmanifolds of type (R).

First of all, we recall the following:

Theorem 5.1 ([32])**.** *If $f : M \to M$ is a C^1-map on a smooth compact manifold M and $\{L(f^k)\}$ is unbounded, then the set of periodic points of f, $\bigcup_k \mathrm{Fix}(f^k)$, is infinite.*

This theorem is not true for continuous maps. Consider the one-point compactification of the map of the complex plane $f(z) = 2z^2/\|z\|$. This is a continuous degree two map of S^2 with only two periodic points. But $L(f^k) = 2^{k+1}$.

However, when M is an infra-solvmanifold of type (R), the theorem is true for all continuous maps f on M. In fact, using the averaging formula, we obtain

$$|L(f^k)| \leq \frac{1}{|\Phi|}\sum_{A\in\Phi} |\det(I - A_*D_*^k)| = N(f^k).$$

If $L(f^k)$ is unbounded, then so is $N(f^k)$ and hence the number of essential fixed point classes of all f^k is infinite.

Remark 5.2. The inequality $|L(f)| \leq N(f)$ for any maps f on an infra-solvmanifold was proved in [34]. On the other hand, we can prove this inequality using averaging formula for infra-solvmanifolds of type (R) as above and then using the fact [33, Theorem 1] that every infra-solvmanifold is modeled in a canonical way on a solvable Lie group of type (R). In fact, the term supersolvability is used in [33]. But it can be seen easily from Lemma 2.1 that the supersolvable groups are the Lie groups of type (R).

Recall that any map f on an infra-solvmanifold of type (R) is homotopic to a map $\bar{f}$ induced by an affine map (d, D). By [13, Proposition 9.3], every essential fixed point class of $\bar{f}$ consists of a single element with index sign $\det(I - df_x)$. Hence $N(f) = N(\bar{f})$ is the number of essential fixed point classes of $\bar{f}$. It is a classical fact that a homotopy between f and $\bar{f}$ induces a one-one correspondence between the fixed point classes of f and those of $\bar{f}$, which is index preserving. Consequently, we obtain

$$|L(f^k)| \leq N(f^k) \leq \#\mathrm{Fix}(f^k).$$

This induces the following conjectural inequality (see [31, 32]) for infra-solvmanifolds of type (R):

$$\limsup_{k\to\infty} \frac{1}{k}\log|L(f^k)| \le \limsup_{k\to\infty}\frac{1}{k}\log\#\mathrm{Fix}(f^k).$$

We denote by $\mathcal{O}(f,k)$ the set of all essential periodic orbits of f with length $\le k$. Thus

$$\mathcal{O}(f,k)=\{\langle\mathbb{F}\rangle \mid \mathbb{F} \text{ is a essential fixed point class of } f^m \text{ with } m\le k\}.$$

Theorem 5.3. *Let f be a map on an infra-solvmanifold of type* (R)*. Suppose that the sequence $N(f^k)$ is unbounded. Then there exists a natural number N_0 such that*

$$k\ge N_0 \Longrightarrow \#\mathcal{O}(f,k)\ge \frac{k-N_0}{r(f)}.$$

Proof. As mentioned earlier, we may assume that every essential fixed point class $\mathbb{F}$ of any f^k consists of a single element $\mathbb{F}=\{x\}$. Denote by $\mathrm{Fix}_e(f^k)$ the set of essential fixed point (class) of f^k. Thus $N(f^k)=\#\mathrm{Fix}_e(f^k)$. Recalling also that f acts on the set $\mathrm{Fix}_e(f^k)$ from the proof of [13, Theorem 11.4], we have

$$\mathcal{O}(f,k)=\{\langle x\rangle \mid x \text{ is a essential periodic point of } f \text{ with length} \le k\}.$$

Observe further that if x is an essential periodic point of f with least period p, then $x\in\mathrm{Fix}_e(f^q)$ if and only if $p\mid q$. The length of the orbit $\langle x\rangle$ of x is p, and

$$\mathrm{Fix}_e(f^k)=\bigcup_{d\mid k}\mathrm{Fix}_e(f^d),$$
$$\mathrm{Fix}_e(f^d)\bigcap\mathrm{Fix}_e(f^{d'})=\mathrm{Fix}_e(f^{\gcd(d,d')}).$$

Recalling that

$$A_m(f)=\frac{1}{m}\sum_{k\mid m}\mu\Big(\frac{m}{k}\Big)\,N(f^k)=\frac{1}{m}\sum_{k\mid m}\mu\Big(\frac{m}{k}\Big)\,\#\mathrm{Fix}_e(f^k),$$

we define $A_m(f,\langle x\rangle)$ for any $x\in\bigcup_i\mathrm{Fix}_e(f^i)$ to be

$$A_m(f,\langle x\rangle)=\frac{1}{m}\sum_{k\mid m}\mu\Big(\frac{m}{k}\Big)\,\#\Big(\langle x\rangle\cap\mathrm{Fix}_e(f^k)\Big).$$

Then we have

$$A_m(f)=\sum_{\substack{\langle x\rangle\\ x\in\mathrm{Fix}_e(f^m)}}A_m(f,\langle x\rangle).$$

We begin with new notation. For a given integer $k>0$ and $x\in\bigcup_m\mathrm{Fix}_e(f^m)$, let

$$\mathcal{A}(f,k)=\{m\le k\mid A_m(f)\ne 0\},$$
$$\mathcal{A}(f,\langle x\rangle)=\{m\mid A_m(f,\langle x\rangle)\ne 0\}.$$

Remark that if $A_m(f)\ne 0$ then there exists an essential periodic point x of f with period m such that $A_m(f,\langle x\rangle)\ne 0$. Consequently, we have

$$\mathcal{A}(f,k)\subset\bigcup_{\langle x\rangle\in\mathcal{O}(f,k)}\mathcal{A}(f,\langle x\rangle)$$

Since $N(f^k)$ is unbounded, we have that $\lambda(f)>1$, see the observation just above Theorem 3.2. By Corollary 4.6, there is N_0 such that if $n\ge N_0$ then there is i with $n\le i\le n+n(f)-1$ such that $A_i(f)\ne 0$. This leads to the estimate

$$\#\mathcal{A}(f,k)\ge\frac{k-N_0}{n(f)}\quad\forall k\ge N_0.$$

Assume that x has least period p. Then we have

$$A_m(f,\langle x\rangle)=\frac{1}{m}\sum_{p|n|m}\mu\Big(\frac{m}{n}\Big)\,\#\langle x\rangle=\frac{p}{m}\sum_{p|n|m}\mu\Big(\frac{m}{n}\Big).$$

Thus if m is not a multiple of p then by definition $A_m(f,\langle x\rangle)=0$. It is clear that $A_p(f,\langle x\rangle)=\mu(1)=1$, i.e., $p\in\mathcal{A}(f,\langle x\rangle)$. Because $p\mid n\mid rp\Leftrightarrow n=r'p$ with $r'\mid r$, we have $A_{rp}(f,\langle x\rangle)=1/r\sum_{p|n|rp}\mu(rp/n)=1/r\sum_{r'|r}\mu(r/r')$ which is 0 when and only when $r>1$. Consequently, $\mathcal{A}(f,\langle x\rangle)=\{p\}$.

In all, we obtain the required inequality

$$\frac{k-N_0}{r(f)}\le\#\mathcal{A}(f,k)\le\#\mathcal{O}(f,k).$$ □

We consider the set of periodic points of f with minimal period k

$$P_k(f)=\mathrm{Fix}(f^k)-\bigcup_{d|k,d<k}\mathrm{Fix}(f^d).$$

It is clear that $\mathrm{Fix}(f)\subset\mathrm{Fix}(f^2)$, i.e., any fixed point class of f is naturally contained in a unique fixed point class of f^2. It is also known that $\mathrm{Fix}_e(f)\subset\mathrm{Fix}_e(f^2)$. We define

$$\mathrm{E}P_k(f)=\mathrm{Fix}_e(f^k)-\bigcup_{d|k,d<k}\mathrm{Fix}_e(f^d),$$

the set of *essential periodic points* of f with minimal period k. Because

$$\mathrm{Fix}_e(f^k)=\coprod_{d|k}\mathrm{E}P_d(f),$$

we have

$$N(f^k)=\#\mathrm{Fix}_e(f^k)=\sum_{d|k}\#\mathrm{E}P_d(f).$$

Proposition 5.4. *For every $k>0$, we have*

$$\#\mathrm{E}P_k(f)=\sum_{d|k}\mu\left(\frac{k}{d}\right)N(f^d)=I_k(f).$$

In particular, if $I_k(f)\neq 0$ then $N(f^k)\neq 0$.

Proof. We apply the Möbius inversion formula to the above identity and then we obtain $\#\mathrm{E}P_k(f)=\sum_{d|k}\mu\big(\frac{k}{d}\big)\,N(f^d)$, which is exactly $I_k(f)$ by its definition. □

Definition 5.5. We consider the mod 2 reduction of the Nielsen number $N(f^k)$ of f^k, written $N^{(2)}(f^k)$. A positive integer k is a $N^{(2)}$*-period* of f if $N^{(2)}(f^{k+i})=N^{(2)}(f^i)$ for all $i\ge 1$. We denote the minimal $N^{(2)}$-period of f by $\alpha^{(2)}(f)$.

Proposition 5.6 ([29, Proposition 1])**.** *Let p be a prime number and let A be a square matrix with entries in the field $\mathbb{F}_p$. Then there exists k with $(p,k)=1$ such that*

$$\mathrm{tr}\,A^{k+i}=\mathrm{tr}\,A^i$$

for all $i\ge 1$.

Recalling $(N2)$: $N(f^k)=\mathrm{tr}\,M_+^k-\mathrm{tr}\,M_-^k=\mathrm{tr}\,(M_+\oplus -M_-)^k$, we can see easily that the minimal $N^{(2)}$-period $\alpha^{(2)}(f)$ always exists and must be an odd number.

Now we obtain a result which resembles [29, Theorem 2].

Theorem 5.7. *Let f be a map on an infra-solvmanifold of type* (R). *Let $k>0$ be an odd number. Suppose that $\alpha^{(2)}(f)^2 \mid k$ or $p \mid k$ where p is a prime such that $p \equiv 2^i \mod \alpha^{(2)}(f)$ for some $i \geq 0$. Then*

$$\#\{\langle x\rangle \mid x \in \mathrm{E}P_k(f)\} = \#\mathrm{E}P_k(f)/k$$

is even.

Proof. By Proposition 5.4, $\#\mathrm{E}P_k(f) = I_k(f)$. Hence it is sufficient to show that $I_k(f)$ is even.

Let $\alpha = \alpha^{(2)}(f)$. Consider the case where $\alpha^2 \mid k$. If $d \mid k$ and $\mu(k/d) \neq 0$ then it follows that $\alpha \mid d$. By the definition of α, $N^{(2)}(f^d) = N^{(2)}(f^\alpha)$. This induces that

$$I_k(f) \equiv \sum_{d\mid k} \mu\left(\frac{k}{d}\right) N^{(2)}(f^d) = N^{(2)}(f^\alpha) \sum_{d\mid k} \mu\left(\frac{k}{d}\right) = 0 \mod 2.$$

Assume p is a prime such that $p \mid k$ and $p \equiv 2^i \mod \alpha$ for some $i \geq 0$. Write $k = p^j r$ where $(p,r)=1$. Then

$$\begin{aligned} I_k(f) &= \sum_{d\mid r} \mu\left(\frac{r}{d}\right) \left(\sum_{e \mid p^j} \mu\left(\frac{p^j}{e}\right) N((f^d)^e)\right) \\ &= \sum_{d\mid r} \mu\left(\frac{r}{d}\right) \left(\mu(1)\, N((f^d)^{p^j}) + \mu(p)\, N((f^d)^{p^{j-1}})\right) \\ &= I_r(f^{p^j}) - I_r(f^{p^{j-1}}). \end{aligned}$$

Since α is a $N^{(2)}$-period of f, it follows that the sequence $\{I_r(f^i)\}_i$ is α-periodic in its mod 2 reduction, i.e., $I_r(f^{j+\alpha}) \equiv I_r(f^j) \mod 2$ for all $j \geq 1$. Since $p \equiv 2^i \mod \alpha$, we have $I_r(f^{p^s}) \equiv I_r(f^{2^{is}}) \mod 2$ for all $s \geq 0$. Recall [4, Proposition 5]: For any square matrix B with entries in the field $\mathbb{F}_p$ and for any $j \geq 0$, we have $\mathrm{tr}\, B^{p^j} = \mathrm{tr}\, B$. Due to this result, we obtain

$$N^{(2)}(f^{2^j}) \equiv \mathrm{tr}\,(M_+ \oplus -M_-)^{2^j} \equiv \mathrm{tr}\,(M_+ \oplus -M_-) \equiv N^{(2)}(f) \mod 2$$

and it follows that $I_r(f^{2^{is}}) \equiv I_r(f) \mod 2$. Consequently, we have

$$I_k(f) = I_r(f^{p^j}) - I_r(f^{p^{j-1}}) \equiv I_r(f^{2^{ij}}) - I_r(f^{2^{i(j-1)}}) \equiv I_r(f) - I_r(f) = 0 \mod 2.$$

This finishes the proof. □

6. Homotopy minimal periods

In this section, we study (homotopy) minimal periods of maps f on infra-solvmanifolds of type (R). We like to determine $\mathrm{HPer}(f)$ only from the knowledge of the sequence $\{N(f^k)\}$. This approach was used in [1, 15, 21] for torus maps, in [19] for maps of compact nilmanifolds and certain solvmanifolds and in [25] for expanding maps on infra-nilmanifolds. Recalling that $N(f^k) = \sum_{i=1}^{r(f)} \rho_i \lambda_i^k$ and $\lambda(f) = \max\{|\lambda_i| \mid i = 1, \cdots, r(f)\}$, we define

$$N^{|\lambda|}(f^k) = \sum_{|\lambda_i|=|\lambda|} \rho_i \lambda_i^k, \quad \tilde{N}^{|\lambda|}(f^k) = \frac{1}{|\lambda|^k} N^{|\lambda|}(f^k).$$

Lemma 6.1. *If $\lambda(f) \geq 1$, then we have*

$$\limsup_{k\to\infty} \frac{N(f^k)}{\lambda(f)^k} = \limsup_{k\to\infty} |\tilde{N}^{\lambda(f)}(f^k)|.$$

Proof. We have

$$\frac{N(f^k)}{\lambda(f)^k} = \tilde{N}^{\lambda(f)}(f^k) + \frac{1}{\lambda(f)^k} \sum_{|\lambda_i|<\lambda(f)} \rho_i \lambda_i^k.$$

Since for $|\lambda_i| < \lambda(f)$, $\lim \lambda_i^k/\lambda(f)^k = 0$, it follows that the proof is completed. □

Theorem 6.2. *Let f be a map on an infra-solvmanifold of type* (R). *Suppose that the sequence $N(f^k)$ is unbounded. Then there exist m and an infinite sequence $\{p_i\}$ of primes such that $\{mp_i\} \subset \mathrm{Per}(f)$. Furthermore, $\{mp_i\} \subset \mathrm{HPer}(f)$.*

Proof. Since the sequence $N(f^k)$ is unbounded, by Theorem 4.1, there exists q such that all $\lambda_i/|\lambda_i|$ with $|\lambda_i| = \lambda(f)$ are roots of unity of degree q, and the sequence $\{\tilde{N}^{\lambda(f)}(f^k)\}$ is periodic and nonzero, because $\limsup_{k\to\infty} |\tilde{N}^{\lambda(f)}(f^k)| > 0$ by Lemma 6.1. Consequently, there exists m with $1 \le m \le q$ such that $\tilde{N}^{\lambda(f)}(f^m) \neq 0$.

Let $h = f^m$. Then $\lambda(h) = \lambda(f^m) = \lambda(f)^m \ge 1$. The periodicity $\tilde{N}^{\lambda(f)}(f^{m+\ell q}) = \tilde{N}^{\lambda(f)}(f^m)$ induces $\tilde{N}^{\lambda(h)}(h^{1+\ell q}) = \tilde{N}^{\lambda(h)}(h)$ for all $\ell > 0$. By Lemma 6.1 or Theorem 4.1, we can see that there exists $\gamma > 0$ such that $N(h^{1+\ell q}) \ge \gamma\lambda(h)^{1+\ell q} > 0$ for all ℓ sufficiently large. From Proposition 4.5 it follows that the Dold multiplicity $I_{1+\ell q}(h)$ satisfies $|I_{1+\ell q}(h)| \ge (\gamma/2)\lambda(h)^{1+\ell q}$ when ℓ is sufficiently large.

According to Dirichlet prime number theorem, since $(1, q) = 1$, there are infinitely many primes p of the form $1+\ell q$. Consider all primes p_i satisfying $|I_{p_i}(h)| \ge (\gamma/2)\lambda(h)^{p_i}$. Remark that for a prime p,

$$\begin{aligned} I_p(h) &= \sum_{d|p} \mu\left(\frac{p}{d}\right) N(h^d) = \mu(p)N(h) + \mu(1)N(h^p) = N(h^p) - N(h) \\ &= \#\mathrm{Fix}_e(h^d) - \#\mathrm{Fix}_e(h) = \#(\mathrm{Fix}_e(h^p) - \mathrm{Fix}_e(h)) \end{aligned}$$

where the last identity follows from that fact that $\mathrm{Fix}_e(h) \subset \mathrm{Fix}_e(h^p)$. Since p is a prime, the set $\mathrm{Fix}_e(h^p) - \mathrm{Fix}_e(h)$ consists of essential periodic points of h with minimal period p.

Because $|I_{p_i}(h)| > 0$, each p_i is the minimal period of some essential periodic point of h. Thus mp_i is a period of f. This means that $m_i p_i$ is the minimal period of f for some m_i with $m_i \mid m$. Choose a subsequence $\{m_{i_k}\}$ of the sequence $\{m_i\}$ bounded by m which is constant, say m_0. Consequently, the infinite sequence $\{m_0 p_{i_k}\}$ consists of minimal periods of f, or $\{mp_i\} \subset \mathrm{Per}(f)$.

These arguments also work for all maps homotopic to f. Hence $\{mp_i\} \subset \mathrm{HPer}(f)$, which completes the proof. □

In the proof of Theorem 6.2, we have shown the following, which proves that the algebraic period is a homotopy minimal period when it is a prime number.

Corollary 6.3. *Let f be a map on an infra-solvmanifold of type* (R). *For all primes p, if $A_p(f) \neq 0$ then $p \in \mathrm{HPer}(f)$.*

Corollary 6.4. *Let f be a map on an infra-solvmanifold of type* (R). *If the sequence $\{N(f^k)\}$ is strictly monotone increasing, then there exists N such that the set* $\mathrm{HPer}(f)$ *contains all primes larger than N.*

Proof. By the assumption, we have $\lambda(f) > 1$. Thus by Theorem 4.4, there exist $\gamma > 0$ and N such that if $k > N$ then there exists $\ell = \ell(k) < r(f)$ such that $N(f^{k-\ell})/\lambda(f)^{k-\ell} > \gamma$. Then for all $k > N$, the monotonicity induces

$$\frac{N(f^k)}{\lambda(f)^k} \ge \frac{N(f^{k-\ell})}{\lambda(f)^k} = \frac{N(f^{k-\ell})}{\lambda(f)^{k-\ell}\lambda(f)^\ell} \ge \frac{\gamma}{\lambda(f)^\ell} \ge \frac{\gamma}{\lambda(f)^{r(f)}}.$$

Applying Proposition 4.5 with $\epsilon = \gamma/\lambda(f)^{r(f)}$, we see that $I_k(f) \neq 0$ and so $A_k(f) \neq 0$ for all k sufficiently large. Now our assertion follows from Corollary 6.3. □

We next recall the following:

Definition 6.5. A map $f : M \to M$ is *essentially reducible* if any fixed point class of f^k being contained in an essential fixed point class of f^{kn} is essential, for any positive integers k and n. The space M is *essentially reducible* if every map on M is essentially reducible.

Lemma 6.6 ([1, Proposition 2.2])**.** *Let $f : M \to M$ be an essentially reducible map. If*

$$\sum_{\frac{m}{k}:\ \text{prime}} N(f^k) < N(f^m),$$

then any map which is homotopic to f has a periodic point with minimal period m, i.e., $m \in \mathrm{HPer}(f)$.

Lemma 6.7. *Every infra-solvmanifold of type* (R) *is essentially reducible.*

Proof. Let $f : M \to M$ be a map on an infra-solvmanifold $M = \Pi\backslash S$ of type (R). Then Π fits a short exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \Pi \longrightarrow \Phi \longrightarrow 1$$

where $\Gamma = \Pi \cap S$ and the holonomy group Φ of Π naturally sits in $\mathrm{Aut}(S)$. By [24, Lemma 2.1], we know that Π has a fully invariant subgroup Λ of finite index and $\Lambda \subset \Gamma$. Therefore $\Lambda \subset \Gamma \subset S$ and $\bar{M} = \Lambda\backslash S$ is a special solvmanifold which covers M. Since Λ is a fully invariant subgroup of Π, it follows that any map $f : M \to M$ has a lifting $\bar{f} : \bar{M} \to \bar{M}$, and $\bar{M}$ is a regular covering of M. By [16, Corollary 4.5], $\bar{f}$ is essentially reducible and then by [25, Proposition 2.4], f is essentially reducible. □

We can strengthen Corollary 6.4 as follows:

Proposition 6.8. *Let f be a map on an infra-solvmanifold of type* (R)*. Suppose that the sequence $\{N(f^k)\}$ is strictly monotone increasing. Then:*

1. *All primes belong to* $\mathrm{HPer}(f)$*.*
2. *There exists N such that if p is a prime $> N$ then $\{p^n \mid n \in \mathbb{N}\} \subset \mathrm{HPer}(f)$.*

We not only extend but also strengthen Corollary 6.4.

Proof. Observe that for any prime p

$$N(f^p) - \sum_{\frac{p}{k}:\ \text{prime}} N(f^k) = N(f^p) - N(f) = I_p(f).$$

The strict monotonicity implies $A_p(f) = pI_p(f) > 0$, and hence $p \in \mathrm{HPer}(f)$, which proves (1).

Under the same assumption, we have shown in the proof of Corollary 6.4 that there exists N such that $k > N \Rightarrow I_k(f) > 0$. Let p be a prime $> N$ and $n \in \mathbb{N}$. Then

$$N(f^{p^n}) - \sum_{\frac{p^n}{k}:\ \text{prime}} N(f^k) = \sum_{i=0}^{n} I_{p^i}(f) - N(f^{p^{n-1}}) = I_{p^n}(f) > 0.$$

By Lemma 6.6, we have $p^n \in \mathrm{HPer}(f)$, which proves (2). □

In Remark 4.7, we observed about the lower density $\mathrm{DA}(f)$ of the set of algebraic periods $\mathcal{A}(f) = \{m \in \mathbb{N} \mid A_m(f) \neq 0\}$. We can consider as well the lower densities of $\mathrm{Per}(f)$ and $\mathrm{HPer}(f)$, see also [17] and [26]:

$$\mathrm{DP}(f) = \liminf_{k\to\infty} \frac{\#(\mathrm{Per}(f) \cap [1,k])}{k},$$

$$\mathrm{DH}(f) = \liminf_{k\to\infty} \frac{\#(\mathrm{HPer}(f) \cap [1,k])}{k}.$$

Since $I_k(f) = \#\mathrm{E}P_k(f)$ by Proposition 5.4, it follows that $\mathcal{A}(f) \subset \mathrm{HPer}(f) \subset \mathrm{Per}(f)$. Hence we have $\mathrm{DA}(f) \leq \mathrm{DH}(f) \leq \mathrm{DP}(f)$.

Corollary 6.9. *Let f be a map on an infra-solvmanifold of type* (R). *Suppose that the sequence $\{N(f^k)\}$ is strictly monotone increasing. Then* $\mathrm{HPer}(f)$ *is cofinite and* $\mathrm{DA}(f) = \mathrm{DH}(f) = \mathrm{DP}(f) = 1$.

Proof. Under the same assumption, we have shown in the proof of Corollary 6.4 that there exists N such that if $k > N$ then $I_k(f) > 0$. This means $\mathrm{E}P_k(f)$ is nonempty by Proposition 5.4 and hence $k \in \mathrm{HPer}(f)$. □

Now we can prove the main result of [25].

Corollary 6.10 ([23, Theorem 4.6], [25, Theorem 3.2]). *Let f be an expanding map on an infra-nilmanifold. Then* $\mathrm{HPer}(f)$ *is cofinite.*

Proof. Since f is expanding, we have that $\lambda(f) = \mathrm{sp}(\bigwedge D_*) > 1$. For any $k > 0$, we can write $N(f^k) = \Gamma_k + \Omega_k$, where

$$\Gamma_k = \lambda(f)^k \left(\sum_{j=1}^{n(f)} \rho_j e^{2i\pi(k\theta_j)} \right), \quad \Omega_k = \sum_{i=n(f)+1}^{r(f)} \rho_i \lambda_i^k \ \text{ with } |\lambda_i| < \lambda(f).$$

Here $\Omega_k \to 0$ and $\Gamma_k \to \infty$ as $k \to \infty$. This implies that $N(f^k)$ is eventually strictly monotone increasing. We can use Corollary 6.4 and then Corollary 6.9 to conclude the assertion. □

References

[1] L. Alsedà, S. Baldwin, J. Llibre, R. Swanson and W. Szlenk, Minimal sets of periods for torus maps via Nielsen numbers, Pacific J. Math., **169** (1995), 1–32.

[2] I. K. Babenko and S. A. Bogatyĭ, The behavior of the index of periodic points under iterations of a mapping, Izv. Akad. Nauk SSSR Ser. Mat., **55** (1991), 3–31 (Russian); translation in Math. USSR-Izv., **38** (1992), 1-26.

[3] N. Bourbaki, Algebra II. Chapters 4-7, Translated from the 1981 French edition by P. M. Cohn and J. Howie, Reprint of the 1990 English edition [Springer, Berlin], Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2003.

[4] F. E. Browder, The Lefschetz fixed point theorem and asymptotic fixed point theorems, *Partial differential equations and related topics* (Program, Tulane Univ., New Orleans, La., 1974), pp. 96–22, Lecture Notes in Math., Vol. 446, Springer, Berlin, 1975.

[5] K. Dekimpe and G.-J. Dugardein, Nielsen zeta functions for maps on infra-nilmanifolds are rational, arXiv:1302.5512.

[6] K. Dekimpe, B. de Rock and W. Malfait, The Anosov relation for Nielsen numbers of maps of infra-nilmanifolds, Monatsh. Math., **150** (2007), 1–10.

[7] A. Dold, Fixed point indices of iterated maps, Invent. Math., **74** (1983), 419–435.

[8] G.-J. Dugardein, Nielsen zeta functions on infra-nilmanifolds up to dimension three, to appear in Topology Appl..

[9] A. L. Fel'shtyn, New zeta function in dynamic, in Tenth Internat. Conf. on Nonlinear Oscillations, Varna, Abstracts of Papers, B, 1984.

[10] A. L. Fel'shtyn, New zeta functions for dynamical systems and Nielsen fixed point theory, Lecture Notes in Math., **1346**, Springer, 1988, 33–55.

[11] A. Fel'shtyn, Dynamical zeta functions, Nielsen theory and Reidemeister torsion, Mem. Amer. Math. Soc., **699**, Amer. Math. Soc., Providence, R.I., 2000.

[12] A. L. Fel'shtyn and R. Hill, The Reidemeister zeta function with applications to Nielsen theory and a connection with Reidemeister torsion, K-theory, **8** (1994), 367–393.

[13] A. Fel'shtyn and J. B. Lee, The Nielsen and Reidemeister numbers of maps on infra-solvmanifolds of type (R), to appear in Topology Appl. (arXiv:1303.0784).

[14] K. Y. Ha and J. B. Lee, Averaging formulas of the Lefschetz and Nielsen numbers of maps between infra-solvmanifolds, Submitted for publication.

[15] B. Halpern, Periodic points on tori, Pacific J. Math., **83** (1979), 117–133.

[16] P. Heath and E. Keppelmann, Fibre techniques in Nielsen periodic point theory on nil and solvmanifolds. I, Topology Appl., **76** (1997), 217–247.

[17] J. W. Hoffman, Z. Liang, Y. Sakai and X. Zhao, Homotopy minimal period self-maps on flat manifolds, Adv. Math., **248** (2013), 324–334.
[18] N. V. Ivanov, Entropy and the Nielsen numbers. Dokl. Akad. Nauk SSSR 265 (2) (1982), 284-287 (in Russian); English transl.: Soviet Math. Dokl. 26 (1982), 63–66.
[19] J. Jezierski and W. Marzantowicz, Homotopy methods in topological fixed and periodic points theory, Topological Fixed Point Theory and Its Applications, 3, Springer, Dordrecht, 2006.
[20] B. Jiang, Lectures on Nielsen fixed point theory, Contemporary Math., **14**, Amer. Math. Soc., Providence, R.I., 1983.
[21] B. Jiang and J. Llibre, Minimal sets of periods for torus maps, Discrete Contin. Dynam. Systems, **4** (1998), 301–320.
[22] H. J. Kim, J. B. Lee and W. S. Yoo, Computation of the Nielsen type numbers for maps on the Klein bottle, J. Korean Math. Soc., **45** (2008), 1483–1503.
[23] J. B. Lee and K. B. Lee, Lefschetz numbers for continuous maps, and periods for expanding maps on infra-nilmanifolds, J. Geom. Phys., **56** (2006), 2011–2023.
[24] J. B. Lee and K. B. Lee, Averaging formula for Nielsen numbers of maps on infra-solvmanifolds of type (R), Nagoya Math. J., **196** (2009), 117–134.
[25] J. B. Lee and X. Zhao, Homotopy minimal periods for expanding maps on infra-nilmanifolds, J. Math. Soc. Japan, **59** (2007), 179–184.
[26] J. B. Lee and X. Zhao, Density of the homotopy minimal periods of maps on infra-solvmanifolds, Preprint.
[27] W. Marzantowicz and F. Przytycki, Entropy conjecture for continuous maps of nilmanifolds, Israel J. Math., **165** (2008), 349–379.
[28] W. Marzantowicz and F. Przytycki, Estimates of the topological entropy from below for continuous self-maps on some compact manifolds, Discrete Contin. Dyn. Syst., **21** (2008), 501–512.
[29] T. Matsuoka, The number of periodic points of smooth maps, Ergod. Th. & Dynam. Sys., **9** (1989), 153–163.
[30] V. B. Pilyugina and A. L. Fel'shtyn, The Nielsen zeta function, Funktsional. Anal. i Prilozhen., **19** (1985) 61–67 (in Russian); English transl.: Functional Anal. Appl., **19**, (1985) 300–305.
[31] M. Shub, Dynamical systems, filtrations and entropy, Bull. Amer. Math. Soc., **80** (1974), 27–41.
[32] M. Shub and D. Sullivan, A remark on the Lefschetz fixed point formula for differentiable maps, Topology, **13** (1974), 189–191.
[33] B. Wilking, Rigidity of group actions on solvable Lie groups, Math. Ann., **317** (2000), 195–237.
[34] P. Wong, Reidemeister number, Hirsch rank, coincidences on polycyclic groups and solvmanifolds, J. reinde angew. Math., **524** (2000), 185–204.

Instytut Matematyki, Uniwersytet Szczecinski, ul. Wielkopolska 15, 70-451 Szczecin, Poland
E-mail address: `felshtyn@ihes.fr, fels@wmf.univ.szczecin.pl`

Department of mathematics, Sogang University, Seoul 121-742, KOREA
E-mail address: `jlee@sogang.ac.kr`