Energy Functionals of Knots II

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ABSTRACT. We study an energy functional of knots, $e_j{}^p$ ($jp > 2$), that is finite valued for embedded circles and takes $+\infty$ for circles with double points. We show that for any $b \in \mathbb{R}$ there are finitely many solid tori $T_1, \cdots, T_m$ such that for any knot with $e_j{}^p \leq b$ can be contained in some T_i in a good manner. Then we can show the existence of a minimizer of $e_j{}^p$ in each knot type.

0. INTRODUCTION

An *energy functional of knots* is a functional on the space of immersed circles in $\mathbb{R}^3$ which is finite valued for embeddings, and which blows up to $+\infty$ for circles with double points.

Let $S^1 = \mathbb{R}/\mathbb{Z}$. Put

$$\mathcal{I} = \{f : S^1 \to \mathbb{R}^3 \,|\, C^1\text{-immersion such that } |f'(t)| = 1 \text{ for all } t \in S^1\}.$$

Let $|x-y|_{S^1}$ ($x, y \in S^1$) denote the minimum of the arc-lengths on S^1 between x and y;

$$|x-y|_{S^1} = \min\{|x-y|, 1-|x-y|\}.$$

Then it is equal to the minimum of the arc-length on $f(S^1)$ between $f(x)$ and $f(y)$ for any $f \in \mathcal{I}$.

Define a functional $e_j{}^p : \mathcal{I} \to \mathbb{R} \cup \{+\infty\}$ $(0 < j, p < +\infty)$ by the following integral.

$$e_j{}^p(f) = \frac{1}{j}\left[\int_0^1 \int_0^1 \left\{\frac{1}{|f(x)-f(y)|^j} - \frac{1}{|x-y|_{S^1}{}^j}\right\}^p dxdy\right]^{\frac{1}{p}}. \tag{0.1}$$

Since the integrand is non-negative $e_j{}^p(f) \geq 0$ or $e_j{}^p(f) = \infty$ for any $f \in \mathcal{I}$.

We studied $e_j{}^p$ with $0 < j \leq 2$ and $1 \leq p < +\infty$ in [O2]. The arguments there also hold for $j > 2$. The basic properties of $e_j{}^p$ depend only on whether $jp > 2$, $jp = 2$, or $jp < 2$.

For instance, $e_j{}^p$ is an energy functional of knots if and only if

$$p \geq \frac{2}{j} \quad (0 < j \leq 2) \quad \text{or} \quad \frac{1}{j-2} > p \geq \frac{2}{j} \quad (2 < j < 4) \tag{0.2}$$

(Theorem 1.1).

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In particular, ${e_2}^1$ is a regularization of potential energy of charged knots when we assume that the repulsive force is inversely proportional to cubic of distance. (Therefore it is different from the usual Newton potential energy.) This special case was formerly defined as *the energy* E in [O1] with $E = {e_2}^1 - 2$.

A *knot type* is an ambient isotopy classes of embedded circles in $\mathbb{R}^3$. In [O2], we showed that only finitely many knot types occur below any finite ${e_j}^p$ threshold if $jp > 2$.

In [Fr-H], Freedman and He studied the energy E on the space of rectifiable curves, and showed that the finiteness of knot types also holds for any finite E threshold.

As for the minimizers for the energy E, Freedman and He showed that there is a C^1 planar convex circle that realizes the infimum of E in the class of all simple closed curves. They also defined E for embedded lines in $\mathbb{R}^3$ and showed that for each prime knot type K, there is a proper rectifiable line γ_K with "*knot type*" K that realizes the infimum of E amoung all proper rectifiable lines with the same "*knot type*" K.[1]

In this paper we study ${e_j}^p$ with (0.2) and $jp > 2$. The contribution of $jp \neq 2$ is as follows. If ${e_j}^p(f)$ with $jp \geq 2$ is finite, then f is a bilipschitz embedding with uniform Lipschitz norm ([O2]). If, furthermore, ${e_j}^p(f)$ with $jp > 2$ is finite, then f is $C^{1,\frac{jp-2}{2(p+2)}}$-embedding with uniform $\frac{jp-2}{2(p+2)}$- Hölder norm on f' (Theorem 1.11). Thus, ${e_j}^p$ with $jp > 2$ is more restrictive in the sense that if ${e_j}^p$ with $jp > 2$ is finite, then the "pull-tight" phenomena are excluded, which may occur below finite value of ${e_j}^p$ with $jp = 2$ (Theorem 3.1 of [O2]).

We show that only finitely many "shapes" of knots occur below any finite ${e_j}^p$ threshold. That is, for any j, p and $b > 0$, there is a set of finite solid tori $\{T_1, \cdots, T_m\}$ such that any $f(S^1)$ with ${e_j}^p(f) \leq b$ can be contained in some T_i "in a good manner" after a congruent translation of $\mathbb{R}^3$ (Theorem 2.3). Then we can use the argument in [Fr-H] to show the existence of the minimizers of ${e_j}^p$ in any knot type. That is, for any j, p and for any knot type K, there is an embedded circle $f_{j,p,K}$ with knot type K that realizes the infimum of ${e_j}^p$ amoung all embedded circles of the same knot type K (Theorem 3.2).

We also show that the number of knot types which have representatives with ${e_j}^p \leq b$ is less than $\exp(Cb^{\frac{2p}{jp-2}})$ for some $C > 0$ (Corollary 2.7), and that the *thickness* of a knot is greater than $C'({e_j}^p)^{-\frac{p}{jp-2}}$ for some $C' > 0$ (Proposition 4.2).

Remark 0.1. Suppose f is a C^1-immersion whose $|f'|$ is not necessarily 1. Let L_f be the total length of $f(S^1)$;

$$L_f = \int_0^1 |f'(t)| dt,$$

and $D_f(f(x), f(y))$ be the minimum of the arc-lengths on $f(S^1)$ between $f(x)$ and

[1] M.H.Freedman, Z-X.He, Z.Wang, and S.Bryson announced that they showed that E is Möbius invariant, and using it they showed that $E(f_0) \leq E(f)$ for any f if and only if $f_0(S^1)$ is the round planar circle, and that in every prime knot type there exist minimizers of E, which turn out to be of class $C^{1,1}$.

$f(y)$;

$$D_f(f(x),f(y)) = \min\{\int_x^y |f'(t)|dt, \int_y^{x+1} |f'(t)|dt\}.$$

We can define $\tilde{e}_j{}^p(f)$ by

$$\frac{1}{j}L_f{}^{\frac{jp-2}{p}}\left[\int_0^1\int_0^1\left\{\frac{1}{|f(x)-f(y)|^j}-\frac{1}{D_f(f(x),f(y))^j}\right\}^p|f'(x)||f'(y)|dxdy\right]^{\frac{1}{p}}.$$

Then, $\tilde{e}_j{}^p(f) = e_j{}^p(f)$ if $f \in \mathcal{I}$, and $\tilde{e}_j{}^p$ does not depend on the parametrizations or affine similarities of $\mathbb{R}^3$.

1. Basic properties of $e_j{}^p$

Theorem 1.1. *The functional $e_j{}^p$ satisfies the following two conditions if and only if*

$$p \geq \frac{2}{j} \quad (0 < j \leq 2) \quad \text{or} \quad \frac{1}{j-2} > p \geq \frac{2}{j} \quad (2 < j < 4). \tag{1.1}$$

(1) If $f(S^1)$ has a double point then $e_j{}^p(f) = \infty$.
(2) If f is a C^∞-embedding, then $e_j{}^p(f) < \infty$.

Proof. Since the proofs of Theorems 1.9 and 2.3 of [O2] also hold for $j > 2$, the condition (1) is satisfied if and only if $jp \geq 2$.

Suppose f is a C^∞-embedding. Since

$$|f(x)-f(y)|^2 = |x-y|_{S^1}{}^2 - \frac{|f''(x)|^2}{12}|x-y|_{S^1}{}^4 + o(|x-y|_{S^1}{}^4)$$

near the diagonal, the integrand of (0.1) is $O(|x-y|_{S^1}{}^{(2-j)p})$ near the diagonal. Hence $e_j{}^p(f) < \infty$ if and only if $(2-j)p > -1$. □

Definition 1.2. When the condition of Theorem 1.1 is satisfied, we say that $e_j{}^p$ is an *energy functional of knots.*

Remarks 1.3. ([O2]) Suppose f is a C^2-embedding. Then

(1)

$$\lim_{p\to\infty} e_2{}^p(f) = \sup_{x,y\in S^1}\frac{1}{2}\{\frac{1}{|f(x)-f(y)|^2} - \frac{1}{|x-y|_{S^1}{}^2}\}$$
$$\geq \frac{1}{24}\max_{x\in S^1}|f''(x)|^2,$$

where $|f''(x)|$ is the curvature of f at x.

(2)

$$\lim_{j\to 0}\lim_{p\to\infty} e_j{}^p(f) = \log\left(\sup_{x,y\in S^1 x\neq y}\frac{|x-y|_{S^1}}{|f(x)-f(y)|}\right)$$
$$= \log(Distor(f)),$$

where $Distor(f)$ denotes Gromov's distortion of f ([Gr]).

We can weaken the condition (2) of Theorem 1.1 to the following form.

Proposition 1.4. *Suppose $e_j{}^p$ is an energy functional of knots. If f is a $C^{1,\alpha}$-embedding, where $\alpha > \frac{jp-1}{2p}$, then $e_j{}^p(f) < \infty$.*

Proof. If f is of class $C^{1,\alpha}$, then for some $C > 0$,

$$|f(x) - f(y)| \geq |x-y|_{S^1} - C|x-y|_{S^1}{}^{2\alpha+1}$$

for all $x, y \in S^1$. Hence the integrand of (0.1) is bounded above by $C'|x-y|_{S^1}{}^{(2\alpha-j)p}$ for some $C' > 0$. □

In the following of this paper we work with fixed j and p with (1.1) and $jp > 2$, i.e.

$$p > \frac{2}{j} \quad (0 < j \leq 2) \qquad \text{or} \qquad \frac{1}{j-2} > p > \frac{2}{j} \quad (2 < j < 4).$$

Remark. In this paper we denote constants which can be given explicitly as continuous functions of j and p by C_i with capital C, and other constants by c_j.

We show that if $e_j{}^p(f) < \infty$ then f is a $C^{1,\frac{jp-2}{2(p+2)}}$-embedding.

We improve Theorem 2.4 of [O2] to obtain

Lemma 1.5. *There exists a constant $C_1 > 0$ such that for any $b > 0$ if $e_j{}^p(f) \leq b$ then*

$$|f(x) - f(y)| \geq C_1 b^{-\frac{p}{jp-2}} \left(1 - \frac{|f(x)-f(y)|}{|x-y|_{S^1}}\right)^{\frac{p+2}{jp-2}}$$

for all $x, y \in S^1$.

Proof. Assume $e_j{}^p(f) \leq b$. Fix $x, y \in S^1$ and put

$$\delta = |x-y|_{S^1},\ d = |f(x)-f(y)|,\ \beta = \delta - d, \text{ and } h = 1 - \frac{d}{\delta}.$$

Suppose $0 \leq s, t \leq \frac{\beta}{8}$. Then

$$|f(x+s) - f(y-t)| \leq d + s + t,$$
$$|(x+s) - (y-t)|_{S^1} \geq d + \frac{3}{4}\beta.$$

Therefore

$$\{j e_j{}^p(f)\}^p \geq \int_0^{\frac{\beta}{8}} \int_0^{\frac{\beta}{8}} \left\{ \frac{1}{(d+s+t)^j} - \frac{1}{(d+\frac{3}{4}\beta)^j} \right\}^p ds dt$$
$$= \int_0^{\frac{\beta}{8}} \int_0^{\frac{\beta}{8}} \left\{ 1 - \left(\frac{d+s+t}{d+\frac{3}{4}\beta} \right)^j \right\}^p (d+s+t)^{-jp} ds dt.$$

Since

$$\frac{d+s+t}{d+\frac{3}{4}\beta} \leq \frac{d+\frac{1}{4}\beta}{d+\frac{3}{4}\beta} \leq 1 - \frac{h}{2},$$

$$(jb)^p \geq \left\{ 1 - \left(1 - \frac{h}{2}\right)^j \right\}^p \int_0^{\frac{\beta}{8}} \int_0^{\frac{\beta}{8}} (d+s+t)^{-jp} ds dt.$$

Similarly

$$\int_0^{\frac{\beta}{8}} (d+s+t)^{-jp}\,ds = \frac{1}{jp-1}\left\{1-\left(\frac{d+t}{d+t+\frac{\beta}{8}}\right)^{jp-1}\right\}(d+t)^{1-jp}$$
$$\geq \frac{1}{jp-1}\left\{1-\left(1-\frac{h}{8}\right)^{jp-1}\right\}(d+t)^{1-jp},$$

and therefore

$$\int_0^{\frac{\beta}{8}}\int_0^{\frac{\beta}{8}} (d+s+t)^{-jp}\,dsdt$$
$$\geq \frac{1}{(jp-1)(jp-2)}\left\{1-\left(1-\frac{h}{8}\right)^{jp-1}\right\}\left\{1-\left(1-\frac{h}{8}\right)^{jp-2}\right\}d^{2-jp}.$$

Hence

$$(jb)^p \geq \frac{1}{(jp-1)(jp-2)}$$
$$\left\{1-\left(1-\frac{h}{2}\right)^{j}\right\}^p\left\{1-\left(1-\frac{h}{8}\right)^{jp-1}\right\}\left\{1-\left(1-\frac{h}{8}\right)^{jp-2}\right\}d^{2-jp}.$$

As $(1-\xi)^a \leq 1-\min\{1,a\}\xi$ for $0\leq \xi \leq 1$ and $a>0$,

$$d^{jp-2} \geq \frac{(\min\{1,j\})^p \min\{1,jp-2\}}{2^{p+6}j^p(jp-1)(jp-2)}b^{-p}h^{p+2}. \qquad \square$$

Proposition 1.6. *Let $b>0$. Put*

$$\phi_b(d) = \frac{d}{1-C_1^{-\frac{jp-2}{p+2}}b^{\frac{p}{p+2}}d^{\frac{jp-2}{p+2}}}, \quad (0\leq d < C_1 b^{-\frac{p}{jp-2}}),$$

where C_1 is given in Lemma 1.5. Put

$$\mathcal{D}_b = \{(\delta,d)|0\leq d\leq \delta \leq \frac{1}{2} \text{ such that } \delta \leq \phi_b(d) \text{ if } d < C_1 b^{\frac{p}{jp-2}}\}.$$

(See Figure 1.1.) Then if $e_j{}^p(f)\leq b$ then

$$(|x-y|_{S^1}, |f(x)-f(y)|) \in \mathcal{D}_b.$$

In particular, if $e_j{}^p(f)<\infty$ then f is an embedding.

Proof. Suppose $e_j{}^p(f)\leq b$ $(b>0)$. Put $\delta_0 = |x_0-y_0|_{S^1}$ and $d_0 = |f(x_0)-f(y_0)|$. By Lemma 1.5, (δ_0,d_0) lies in the (δ,d)-plane above the the curve γ given by

$$\gamma = \{(\frac{1}{1-h}C_1 b^{-\frac{p}{jp-2}} h^{\frac{p+2}{jp-2}}, C_1 b^{-\frac{p}{jp-2}} h^{\frac{p+2}{jp-2}}) | 0 \leq h < 1\}$$
$$= \{\delta = \phi_b(d) | 0 \leq d < C_1 b^{-\frac{p}{jp-2}}\}. \qquad \square$$

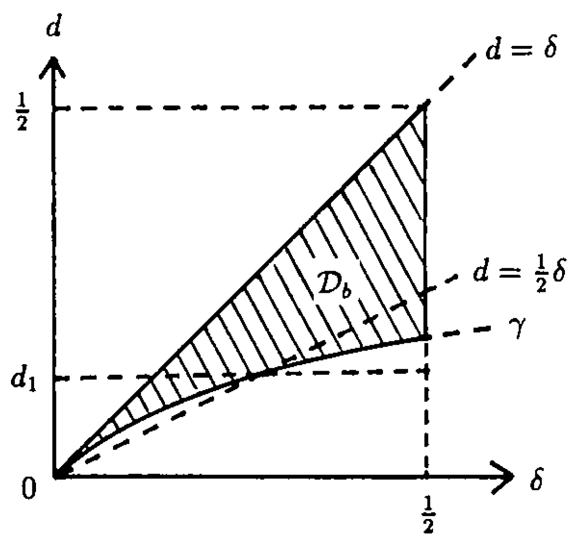

FIGURE 1.1

Put

$$d_1 = d_1(b) = (\frac{1}{2})^{\frac{p+2}{jp-2}} C_1 b^{-\frac{p}{jp-2}}, \tag{1.2}$$

(see Figure 1.1). Then if $0 \leq d \leq d_1$ then

$$\phi_b(d) \leq d(1 + 2{C_1}^{-\frac{jp-2}{p+2}} b^{\frac{p}{p+2}} d^{\frac{jp-2}{p+2}}) \leq 2d. \tag{1.3}$$

Hence if $(\delta, d) \in \mathcal{D}_b$ then either $d \geq \frac{\delta}{2}$ or $d \geq d_1$, which means;

Corollary 1.7. *For any $b > 0$ if ${e_j}^p(f) \leq b$ then either*

$$|f(x) - f(y)| \geq \frac{1}{2}|x - y|_{S^1}$$

or

$$|f(x) - f(y)| \geq (\frac{1}{2})^{\frac{p+2}{jp-2}} C_1 b^{-\frac{p}{jp-2}}$$

for all $x, y \in S^1$, where C_1 is given in Lemma 1.5.

Corollary 1.8. *For any $f \in \mathcal{I}$,*

$$e_j{}^p(f) \geq (1 - \frac{1}{\sqrt{2}})^{\frac{p+2}{p}} (2C_1)^{\frac{jp-2}{p+2}},$$

where C_1 is given in Lemma 1.5.

Proof. By the proof of Lemma 3.1 of [Fr-H], for any $f \in \mathcal{I}$, either

$$|f(0) - f(\frac{1}{2})| \leq \frac{1}{2\sqrt{2}}$$

or

$$|f(\frac{1}{4}) - f(\frac{3}{4})| \leq \frac{1}{2\sqrt{2}}.$$

Suppose $e_j{}^p(f) = b < (1 - \frac{1}{\sqrt{2}})^{\frac{p+2}{p}} (2C_1)^{\frac{jp-2}{p+2}}$. Then $\phi_b(d) < \sqrt{2}d$ for $0 \leq d \leq \frac{1}{2}$, and hence $(\frac{1}{2}, \frac{1}{2\sqrt{2}})$ is not in $\mathcal{D}_b$, which is a contradiction. □

Let $\angle v_1 \cdot v_2$ denote the angle between two vectors v_1 and v_2.

We improve Proposition 2.5 of [O2] by replacing the angle $\frac{\pi}{4}$ by θ and using Proposition 1.6 instead of Theorems 2.3 and 2.4 of [O2] to obtain

Lemma 1.9. *There exists a constant $C_2 > 0$ such that for any $b > 0$, f with $e_j{}^p(f) \leq b$, and θ $(0 \leq \theta < \pi)$, if*

$$|x - y|_{S^1} \leq C_2 b^{-\frac{p}{jp-2}} \theta^{\frac{2(p+2)}{jp-2}}$$

then $\angle f'(x) \cdot f'(y) \leq \theta$.

Proof. Suppose $e_j{}^p(f) \leq b$. Put $q = \frac{jp-2}{p+2} (< 2)$.

Suppose $|f(x_0) - f(y_0)| \leq \frac{d_1}{2}$ $(x_0 < y_0)$. Then for any x, y with $x_0 \leq x < y \leq y_0$

$$d = |f(x) - f(y)| \leq |x_0 - y_0|_{S^1} \leq d_1,$$

by (1.2), and hence by (1.3)

$$|x - y|_{S^1} \leq d + c_3 d^{1+q},$$

where $c_3 = c_3(b) = 2C_1^{-\frac{jp-2}{p+2}} b^{\frac{p}{p+2}}$.

Therefore the curve segment $f([x, y])$ is contained in the solid cylinder with axis $\overleftrightarrow{f(x)f(y)}$ and radius $r_b(d)$ given by

$$r_b(d) = \frac{\sqrt{2c_3 + c_3{}^2 d_1{}^q}}{2} d^{1+\frac{q}{2}} = \frac{\sqrt{6}}{2} C_1{}^{-\frac{q}{2}} b^{\frac{p}{2(p+2)}} d^{1+\frac{q}{2}}.$$

Put $c_4 = c_4(b) = \frac{\sqrt{6}}{2} C_1{}^{-\frac{q}{2}} b^{\frac{p}{2(p+2)}} d^{1+\frac{q}{2}}$.

The proof of Lemma 2.6 of [O2] goes parallel if we replace the angle $\frac{\pi}{4}$ by θ to yield

Sublemma 1.10. *Suppose $0 < \theta < \pi$. Then there exists a constant $c_5 = c_5(\theta) > 0$ such that if a C^1-immersion $h : (x_0 - \epsilon, y_0 + \epsilon) \to \mathbb{R}^3$ ($\epsilon > 0, x_0 < y_0$) with $|h(x_0) - h(y_0)| = 1$ satisfies the following condition (*), then $\angle h'(x_0) \cdot h'(y_0) \le \theta$.*

(*) For any x, y with $x_0 \le x < y \le y_0$, the curve segment $h([x,y])$ is contained in the solid cylinder with axis $\overleftrightarrow{h(x)h(y)}$ and radius $c_5|h(x) - h(y)|^{1+\frac{q}{2}}$.

Proof. Put $c_5 = 2^{-4}(2^{\frac{q}{2}} - 1)\theta$. □

Define a constant $C_2 > 0$ by

$$\min\{\frac{d_1}{2}, (\frac{c_5}{c_4})^{\frac{2}{q}}\} = \min\{2^{-1-\frac{1}{q}}\pi^{-\frac{2}{q}}, 2^{-\frac{6}{q}}3^{-\frac{1}{q}}(2^{\frac{q}{2}} - 1)^{\frac{2}{q}}\}C_1 b^{-\frac{p}{jp-2}}\theta^{\frac{2(p+2)}{jp-2}}$$
$$= C_2 b^{-\frac{p}{jp-2}}\theta^{\frac{2(p+2)}{jp-2}}.$$

If $|x_0 - y_0|_{S^1} \le C_2 b^{-\frac{p}{jp-2}}\theta^{\frac{2(p+2)}{jp-2}}$ then the homothety

$$|f(x_0) - f(y_0)|^{-1} f : (x_0 - \epsilon, y_0 + \epsilon) \to \mathbb{R}^3$$

for some small $\epsilon > 0$ satisfies the condition (*) of Sublemma 1.10, hence $\angle f(x_0) \cdot f(y_0) \le \theta$. □

Theorem 1.11. *There exists a constant $C_0 > 0$ such that for any $b > 0$ if $e_j{}^p(f) \le b$ then f is a $C^{1,\frac{jp-2}{2(p+2)}}$- embedding such that*

$$|f'(x) - f'(y)| \le \angle f'(x) \cdot f'(y) \le C_0 b^{\frac{p}{2(p+2)}} |x - y|_{S^1}{}^{\frac{jp-2}{2(p+2)}}$$

for all $x, y \in S^1$.

Proof. Follows from Proposition 1.6 and Lemma 1.9 with $C_0 = C_2^{-\frac{jp-2}{2(p+2)}}$.

Remark 1.12. The author does not know whether $e_j{}^p(f)$ is finite when f is a $C^{1,\alpha}$-embedding with $\frac{jp-2}{2(p+2)} \le \alpha \le \frac{jp-1}{2p}$.

2. Finiteness of shapes of knots

Fix any $b > 0$.

Definition 2.1. A homeomorphism $F : D^2 \times S^1 \to T \subset \mathbb{R}^3$ is a *good (ϵ-) torus* for b ($\epsilon > 0$), or *good (ϵ-) torus* for short, if the following three conditions are satisfied;

(1) Let $f(t) = F(0,t)$. Then $f \in \mathcal{I}$ and $e_j{}^p(f) \le b$.

(2) $T \supset N_\epsilon(f(S^1)) = \{P \in \mathbb{R}^3 | \mathrm{dist}(P, f(S^1)) \le \epsilon\}$.

(3) If $e_j{}^p(g) \le b$ ($g \in \mathcal{I}$) and $g(S^1) \subset T$, then $\mathrm{Pr}_2 \circ F^{-1} \circ g : S^1 \to S^1$ is a homeomorphism, where $\mathrm{Pr}_2 : D^2 \times S^1 \to S^1$ is a projection.

Definition 2.2. (1) An embedding $f \in \mathcal{I}$ *can be contained in a good torus* F if there is an orientation preserving congruent translation of $\mathbb{R}^3$, U, such that

$$U \circ f(S^1) \subset F(D^2 \times S^1).$$

(2) A set of finite good ϵ-solid tori $\mathcal{S} = \{F_1, \cdots, F_m\}$ ($m \in \mathbb{N}$) is a *complete (ϵ-) system* if any $f \in \mathcal{I}$ with $e_j{}^p(f) \le b$ can be contained in some $F_i \in \mathcal{S}$.

Theorem 2.3. *There exists a complete ϵ-system for some $\epsilon > 0$.*

In order to prove Theorem 2.3, it suffices to work in the following class $\mathcal{I}_0^*$;

$$\mathcal{I}_0^* = \{f \in \mathcal{I} | f(0) = 0, f'(0) = (1,0,0), \text{ and } e_j{}^p(f) \leq b\}.$$

Roughly speaking, we construct a complete system of "thickened PL-knots".

Definition 2.4. Let $N \in \mathbb{N}$ and $r > 0$.

(1) An embedding $f \in \mathcal{I}_0^*$ is *r-captured by a sequence of N points in $\mathbb{R}^3$*, $L = (0, P_1, \cdots, P_{N-1})$, if

$$f(\frac{j}{N}) \in B_r(P_j) = \{P \in \mathbb{R}^3 | |P - P_j| \leq r\}$$

for all j $(1 \leq j \leq N-1)$.

(2) $L = (0, P_1, \cdots, P_{N-1})$ is an *admissible (N, r)-polygon* if there is an $f \in \mathcal{I}_0^*$ which is r-captured by L.

(3) A set of finite admissible (N, r)-polygons

$$S = \{L^i = (0, P_1^i, \cdots, P_{N-1}^i) | 1 \leq i \leq m\}$$

is a *complete family of (N, r)-polygons* if any $f \in \mathcal{I}_0^*$ is r-captured by some L^i.

Lemma 2.5. *For any $N \in \mathbb{N}$ and $r > 0$, there exists a complete family of (N, r)-polygons.*

Proof. Put $j_0 = l_0 = 1$ and $P_{j_0} = 0$. Put

$$W_{j_0} = \{f(\frac{1}{N}) \in \mathbb{R}^3 | \exists f \in \mathcal{I}_0^*\}.$$

Since W_{j_0} is bounded, there are finite points $\{P_{j_0 1}, P_{j_0 2}, \cdots, P_{j_0 l_{j_0}}\}$ such that;

(1) For any j_1 $(1 \leq j_1 \leq l_{j_0})$ there is an $f_{j_0 j_1} \in \mathcal{I}_0^*$ such that $f_{j_0 j_1}(\frac{1}{N}) \in B_r(P_{j_0 j_1})$,

(2) $W_{j_0} \subset \cup_{j_1=1}^{l_{j_0}} B_r(P_{j_0 j_1})$.

Inductively, for a sequence of k points $(P_{j_0}, P_{j_0 j_1}, \cdots, P_{j_0 j_1 \cdots j_{k-1}})$ thus constructed with $2 \leq k \leq N-1$ and $1 \leq j_i \leq l_{j_0 j_1 \cdots j_{i-1}}$ $(1 \leq i \leq k-1)$, put

$$W_{j_0 j_1 \cdots j_{k-1}} = \{f(\frac{k}{N}) \in \mathbb{R}^3 | \exists f \in \mathcal{I}_0^* \text{ such that } f(\frac{i}{N}) \in B_r(P_{j_0 j_1 \cdots j_i}) \text{ for all } i \, (1 \leq i \leq k-1)\}.$$

Since $W_{j_0 j_1 \cdots j_{k-1}}$ is bounded , there are finite points $\{P_{j_0 j_1 \cdots j_{k-1} 1}, \cdots, P_{j_0 j_1 \cdots j_{k-1} l_{j_0 j_1 \cdots j_{k-1}}}\}$ such that

(1) For any j_k $(1 \leq j_k \leq l_{j_0 j_1 \cdots j_{k-1}})$ there is an $f_{j_0 j_1 \cdots j_k} \in \mathcal{I}_0^*$ such that $f_{j_0 j_1 \cdots j_k}(\frac{i}{N}) \in B_r(P_{j_0 j_1 \cdots j_i})$ for all i $(1 \leq i \leq k)$,

(2) $W_{j_0 j_1 \cdots j_{k-1}} \subset \cup_{j_k=1}^{l_{j_0 \cdots j_{k-1}}} B_r(P_{j_0 j_1 \cdots j_k})$.

The set $\{(P_{j_0}, P_{j_0 j_1}, \cdots P_{j_0 j_1 \cdots j_{N-1}})\}$ thus constructed is a complete family of (N, r)-polygons. $\square$

Proof of Theorem 2.3. The proof of Theorem 2.3 reduces to the following lemma;

Lemma 2.6. *There exist $N_0 \in \mathbb{N}$, $r_0 > 0$, and $\epsilon_0 > 0$ such that for any admissible (N_0, r_0)-polygon L, there exists a homeomorphism $\hat{F}_L : D^2 \times S^1 \to T \subset \mathbb{R}^3$ such that;*

(1) If $f \in \mathcal{I}_0^*$ is r_0-captured by L, then $N_{\epsilon_0}(f(S^1)) \subset T$.

(2) If $g \in \mathcal{I}_0^*$ satisfies $g(S^1) \subset T$, then $\mathrm{Pr}_2 \circ \hat{F}_L{}^{-1} \circ g : S^1 \to S^1$ is a bilipschitz map.

Suppose L is an admissible (N_0, r_0)-polygon, and $f \in \mathcal{I}_0^*$ is r_0-captured by L. We can deform $\hat{F}_L$ to obtain a good ϵ_0-solid torus F_L such that

$$F_L(\{0\} \times S^1) = f(S^1),$$
$$F_L(D^2 \times \{t\}) = \hat{F}_L(D^2 \times \{t\}) \qquad \text{for all } t \in S^1.$$

Therefore for any complete family of (N_0, r_0)-polygons $\{L_1, \cdots L_m\}$, we can construct a complete ϵ_0-system $\{F_{L_1}, \cdots, F_{L_m}\}$. Then Lemma 2.5 implies Theorem 2.3. $\square$

Proof of Lemma 2.6. (1) Take $N_1 \in \mathbb{N}$ such that

$$\frac{1}{N_1} \leq \left(\frac{\pi}{200C_0}\right)^{\frac{2(p+2)}{jp-2}} b^{-\frac{p}{jp-2}}, \tag{2.1}$$

where C_0 is given in Theorem 1.12.

Then if $0 \leq y_0 - x_0 \leq \frac{1}{N_1}$, then for any $f \in \mathcal{I}_0^*$ and for any $x, y \in [x_0, y_0]$,

$$\angle f'(x) \cdot f'(y) \leq \frac{\pi}{100}, \tag{2.2}$$

$$\angle f'(x) \cdot \overrightarrow{f(x_0)f(y_0)} \leq \frac{\pi}{100}. \tag{2.3}$$

Assume $N \geq N_1$. Put

$$r = r(N) = \frac{\pi}{400N}. \tag{2.4}$$

Suppose $L = (P_0, \cdots, P_{N-1})$ is an (N, r)-polygon, and $f \in \mathcal{I}_0^*$ is r-captured by L, i.e.

$$|f(\frac{i}{N} - P_i)| \leq r \qquad \text{for all } i. \tag{2.5}$$

Since by (2.2)

$$\frac{1}{N} \geq |f(\frac{i}{N}) - f(\frac{i+1}{N})| \geq \frac{1}{N} \cos \frac{\pi}{100} \qquad \text{for all } i,$$

(2.5) implies

$$\frac{1}{N} + 2r \geq |P_i - P_{i+1}| \geq \frac{1}{N} \cos \frac{\pi}{100} - 2r \qquad \text{for all } i, \tag{2.6}$$

where suffixes are taken modulo N. Then by (2.4), (2.5), and (2.6)

$$\angle \overrightarrow{f(\frac{i}{N})f(\frac{i+1}{N})} \cdot \overrightarrow{P_iP_{i+1}} \leq \arcsin \frac{2r}{|P_i - P_{i+1}|} \leq \frac{\pi}{100} \quad \text{for all } i. \tag{2.7}$$

Hence by (2.3) and (2.7)

$$\angle f'(x) \cdot \overrightarrow{P_iP_{i+1}} \leq \frac{2\pi}{100} \quad \text{if } x \in [\frac{i}{N}, \frac{i+1}{N}] \quad \text{for all } i, \tag{2.8}$$

and therefore,

$$\angle \overrightarrow{P_{i-1}P_i} \cdot \overrightarrow{P_iP_{i+1}} \leq \frac{4\pi}{100} \quad \text{for all } i. \tag{2.9}$$

Let V_+^i (V_-^i) be a 'forward' ('backward' *resp.*) cone with axis $\overleftrightarrow{P_iP_{i+1}}$ and angle $\frac{2\pi}{100}$ which is tangent to $B_r(P_i)$ ($B_r(P_{i+1})$ *resp.*), where $B_r(P_i)$ is a 3-ball with center P_i and radius r. Put $\partial D^i = V_+^i \cap V_-^i$, where D^i is a 2-disc. Let U_i be the union of $B_r(P_i)$, $B_r(P_{i+1})$, and the intersection of the two closed cones of V_+^i and V_-^i sandwiched between $B_r(P_i)$ and $B_r(P_{i+1})$. (See Figure 2.1.)

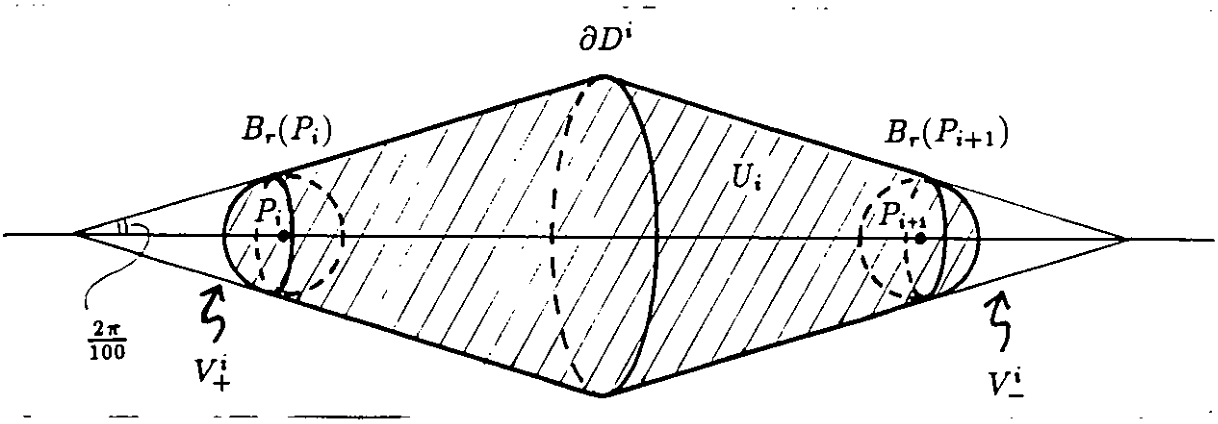

FIGURE 2.1

Then by (2.5) and (2.8) $f([\frac{i}{N}, \frac{i+1}{N}]) \subset U_i$ for all i.

Suppose $0 < \rho \leq \frac{2}{N}$. Let $\Pi_i = \Pi_i(\rho)$ denote a solid cylinder with axis $\overleftrightarrow{P_iP_{i+1}}$ and radius ρ sandwiched between two flat surfaces $\Sigma_i = \Sigma_i(\rho)$ and $\Sigma_{i+1} = \Sigma_{i+1}(\rho)$ where Π_i meets Π_{i-1} and Π_{i+1}. (See Figure 2.2.) Put $T_L(\rho) = \cup_{i=0}^{N-1} \Pi_i(\rho)$.

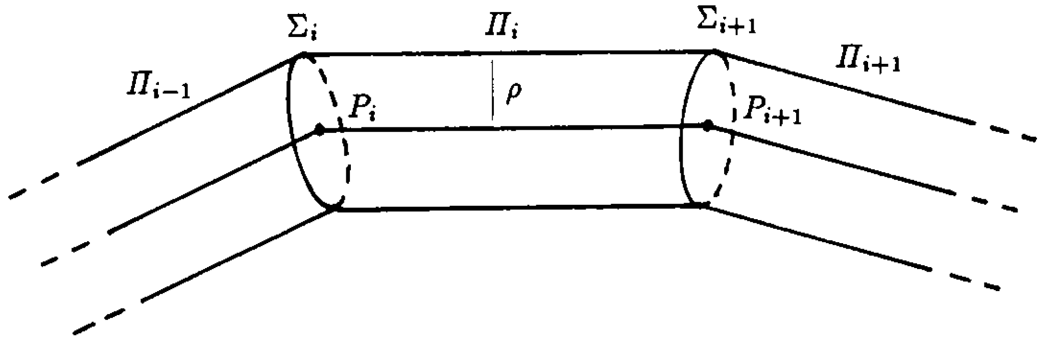

FIGURE 2.2

Put

$$R = R(N) = r(N) + \left(\frac{\frac{1}{N} + 2r(N)}{2} + r(N)\sin\frac{2\pi}{100}\right)\tan\frac{2\pi}{100},$$

then $R < \frac{1}{20N}$. By (2.6) the radius of D^i is not greater than R for all i. Hence

(2.10) $$f(S^1) \subset \cup_{i=0}^{N-1} U_i \subset \cup_{i=0}^{N-1} \Pi_i(R) = T_L(R).$$

We look for N ($N \geq N_1$) and ϵ ($0 < \epsilon \leq \frac{39}{20N}$) such that for any admissible $(N, r(N))$-polygon L, $T_L(R(N) + \epsilon)$ is a solid torus.

(2.11) Let $h_i = h_i(\rho) : D^2 \times [0,1] \to \Pi_i(\rho)$ be a homeomorphism such that

(1) $h_i(D^2 \times \{0\}) = \Sigma_i(\rho)$, $h_i(D^2 \times \{1\}) = \Sigma_{i+1}(\rho)$,

(2) $h_i(0,t) = tP_{i+1} + (1-t)P_i$,

(3) $h_i(D^2 \times \{t\})$ is a flat surface whose forward normal vector $v_i(t)$ changes affinely as t moves from 0 to 1.

Then by (2.9)

(2.12) $$\angle v_i(t) \cdot \overrightarrow{P_iP_{i+1}} \leq \frac{2\pi}{100} \qquad \text{for all } t \in [0,1].$$

Suppose $R(N) \leq \rho \leq \frac{2}{N}$. For each i, we can define $\tau_i : \Pi_i(\rho) \to (\frac{i-1}{N}, \frac{i+2}{N}) \subset S^1$ by

$$\mathrm{Pr}_2 \circ {h_i}^{-1}(X) = \mathrm{Pr}_2 \circ {h_i}^{-1} \circ f(\tau_i(X)) \qquad X \in \Pi_i(\rho).$$

Then for any $X \in \Pi_i(\rho)$, by (2.12),(2.2), and (2.8)

(2.13) $$|X - f(\tau_i(X))| \leq \frac{R+\rho}{\cos\frac{2\pi}{100}},$$

(2.14) $$\angle f'(\tau_i(X)) \cdot \overrightarrow{P_iP_{i+1}} \leq \frac{3\pi}{100}.$$

Suppose $\Pi_i(\rho) \cap \Pi_j(\rho) \ni X$ $(i < j)$. Then there is k with $i < k < j$ such that

$$\angle \overrightarrow{P_iP_{i+1}} \cdot \overrightarrow{P_kP_{k+1}} > \frac{\pi}{2}.$$

Put $t_i = \tau_i(X)$, $t_j = \tau_j(X)$, and $t_k = \frac{k}{N}$. Then $t_i < t_k < t_j$, and $\angle f'(t_i) \cdot f'(t_k) \geq \frac{\pi}{2} - \frac{5\pi}{100}$ by (2.8) and (2.14). Then by Theorem 1.11

$$|t_i - t_k| \geq C_0{}^{-\frac{2(p+2)}{jp-2}} \left(\frac{\pi}{2} - \frac{5\pi}{100}\right)^{\frac{2(p+2)}{jp-2}} b^{-\frac{p}{jp-2}}.$$

By (2.13) and Corollary 1.7,

$$\begin{aligned}
\frac{2(R+\rho)}{\cos\frac{2\pi}{100}} &\geq |X - f(t_i)| + |X - f(t_j)| \\
&\geq |f(t_i) - f(t_j)| \\
&\geq \begin{cases} \frac{1}{2}|t_i - t_j| \\ \text{or} \\ (\frac{1}{2})^{\frac{p+2}{jp-2}} C_1 b^{-\frac{p}{jp-2}} \end{cases} \\
&\geq \min\{\frac{1}{2}C_0{}^{-\frac{2(p+2)}{jp-2}}(\frac{\pi}{2} - \frac{5\pi}{100})^{\frac{2(p+2)}{jp-2}}, (\frac{1}{2})^{\frac{p+2}{jp-2}} C_1\} b^{-\frac{p}{jp-2}}.
\end{aligned}$$

Let the last term be denoted by $C_6 b^{-\frac{p}{jp-2}}$. Take $N_2 \in \mathbb{N}$ such that

$$\frac{1}{N_2} \leq \frac{1}{4} C_6 b^{-\frac{p}{jp-2}}. \tag{2.15}$$

Put $N_0 = \max\{N_1, N_2\}$, $r_0 = r(N_0)$, $\epsilon_0 = \frac{1}{2N_0}$, and $\rho_0 = R(N_0) + \epsilon_0$. Then $\rho_0 < \frac{2}{N_0}$ and

$$\frac{2(R(N_0) + \rho_0)}{\cos\frac{2\pi}{100}} < C_6 b^{-\frac{p}{jp-2}}.$$

Hence $\Pi_i(\rho_0) \cap \Pi_j(\rho_0) = \emptyset$ if $i \neq j$. Therefore for any admissible (N_0, r_0)-polygon $L = (P_0, \cdots, P_{N_0-1})$, $T = T_L(\rho_0)$ is a solid torus. By gluing h_i's constructed in (2.11) together, we can obtain a homeomorphism $\hat{F}_L : D^2 \times S^1 \to T_L(\rho_0)$. Then by (2.10) $T_L(\rho_0) \supset N_{\epsilon_0}(f(S^1))$ for any $f \in \mathcal{I}_0^*$ that is r_0-captured by L.

Thus the condition (1) is verified.

(2) Suppose for some $g \in \mathcal{I}_0^*$ and $t_0 \in S^1$, $g(t_0) \in \Pi_i(\rho_0)$ and

$$\frac{7}{10}\pi > \angle g'(t_0) \cdot \overrightarrow{P_iP_{i+1}} > \frac{3}{10}\pi.$$

If $t_0 - \frac{1}{N_0} \leq t \leq t_0 + \frac{1}{N_0}$, then $g(t)$ is contained in a cone V with "axis" $g'(t_0)$ and angle $\frac{\pi}{100}$, and

$$|(g(t) - g(t_0), g'(t_0))| \geq \cos\frac{\pi}{100} \cdot |t - t_0|,$$

where (,) is the inner product.

By (2.9) the angle between any cone ray of V and $\overrightarrow{P_jP_{j+1}}$ ($j = i-1$ or $i+1$) is not smaller than $\frac{\pi}{4}$. Since $\rho_0 < \frac{11}{20N_0}$,

$$\frac{1}{N_0}\sin\frac{\pi}{4} \geq \rho_0,$$

$$\frac{1}{N_0}\cos\frac{\pi}{4} \leq |P_j - P_{j+1}| - \rho_0 \tan\frac{2\pi}{100}, \qquad j = i-1 \text{ or } i+1$$

by (2.6). Therefore g must pass through $\partial\Pi_{i-1} \cup \partial\Pi_i \cup \partial\Pi_{i+1}$ at some $t \in [t_0 - \frac{1}{N_0}, t_0 + \frac{1}{N_0}]$. Hence for any $g \in \mathcal{I}_0^*$ such that $g(S^1) \subset T_L(\rho_0)$

$$0 \leq \angle g'(t) \cdot \overrightarrow{P_iP_{i+1}} \leq \frac{3}{10}\pi \text{ or } \frac{7}{10}\pi \leq \angle g'(t) \cdot \overrightarrow{P_iP_{i+1}} \leq \pi$$

if $g(t) \in \Pi_i(\rho_0)$. This, together with (2.9), implies the condition (2). □

By (2.1) and (2.15) we can take

$$N_0 = \left[\max\left\{\left(\frac{200C_0}{\pi}\right)^{\frac{2(p+2)}{jp-2}}, \frac{4}{C_6}\right\} b^{\frac{p}{jp-2}}\right] + 1,$$

where [ ] is Gauss's symbol. By Corollary 1.8 $N_0 \leq C_7 b^{\frac{p}{jp-2}}$ for some $C_7 > 0$. By Lemmas 2.5 and 2.6, every knot with $e_j{}^p \leq b$ is ambient istotopic to a PL-knot with N_0 vertices. Since the number of knot types of PL-knots with N_0 vertices is less than $5^{\frac{N_0(N_0-3)}{2}}$, we obtain

Corollary 2.7. *There exists a constant $C_8 > 0$ such that for any $b > 0$ the number of knot types which have representatives with $e_j{}^p \leq b$ is less than $\exp(C_8 b^{\frac{2p}{jp-2}})$.*

3. Existence of minimizers of $e_j{}^p$

Definition 3.1. Let K be a knot type. Define $e_j{}^p(K)$ by $e_j{}^p(K) = \inf_{f\in K} e_j{}^p(f)$.

The argument given in Lemma 4.2 of [Fr-H] can be combined with Theorems 1.11 and 2.3 to show that $e_j{}^p(K)$ is realized by a minimizer $f_{j,p,K}$.

Theorem 3.2. *For any knot type K, there exists a $C^{1,\frac{jp-2}{2(p+2)}}$-embedding $f_{j,p,K}$ with knot type K such that $e_j{}^p(f_{j,p,K}) \leq e_j{}^p(f)$ for any $f \in \mathcal{I}$ of the same knot type K.*

Proof. Let $\{f_i\} \subset \mathcal{I}$ be a sequence of embeddings of knot type K with

$$\lim_{i\to\infty} e_j{}^p(f_i) = e_j{}^p(K).$$

By Theorem 2.3 there is a good solid torus for $e_j{}^p(K)+1$, F, and a subsequence of $\{f_i\}$, still denoted by $\{f_i\}$, such that f_i can be contained in F for all i. By Theorem 1.11

$$|f_i{}'(x) - f_i{}'(y)| \leq C_0 b^{\frac{p}{2(p+2)}} |x-y|_{S^1}{}^{\frac{jp-2}{2(p+2)}} \qquad \text{for all } x, y \in S^1$$

for all i. Hence all $f_i{}'$'s are uniformly continuous. Since $f_i{}'(S^1) \subset S^2$ for all i, by Ascoli-Arzelà's theorem, there is a subsequence of $\{f_i{}'\}$, again denoted by $\{f_i{}'\}$, which converges uniformly to a $\frac{jp-2}{2(p+2)}$-Hölder map $g : S^1 \to S^2$.

Since

$$\int_0^1 g(t)dt = 0,$$

$g = f'_\infty$ for some $C^{1,\frac{jp-2}{2(p+2)}}$-immersion $f_\infty \in \mathcal{I}$. As the integrand of (0.1) converges pointwise as $i \to \infty$, by Fatou's lemma

$$e_j{}^p(f_\infty) \leq \liminf_{i\to\infty} e_j{}^p(f_i) = e_j{}^p(K).$$

Since f_∞ can clearly be contained in F and $e_j{}^p(f_\infty) \leq e_j{}^p(K)$, $f_\infty \in K$ by Definition 2.1. Therefore $e_j{}^p(f_\infty) = e_j{}^p(K)$. □

Remark. Some studies and computer simulations on polygonal knots minimizing some *energy-like* functionals are given in [A],[B-O],[B-S],[Fu],[Gu], and [O3].

4. Thickness of a knot

We give a notion of the thickness of a knot in connection with the argument nof solid tori containing knots in §2.

Definition 4.1. (1) The *Thickness of a C^1-embedding* $f : S^1 \to \mathbb{R}^3$ is defined by

$$Tk(f) = \frac{1}{L_f} \sup\{\epsilon_0 | N_\epsilon(f(S^1)) \text{ is a solid torus for all } \epsilon \text{ with } 0 < \epsilon < \epsilon_0\},$$

where L_f is the total length of $f(S^1)$, and $N_\epsilon(f(S^1))$ is an ϵ-neighborhood of $f(S^1)$. Definition 2.1.

(2) The *Thickness of a knot type* K is defined by

$$Tk(K) = \sup_{f\in K}\{Tk(f)\}.$$

Proposition 4.2. *There exists a constant $C_9 > 0$ such that for any $f \in \mathcal{I}$ with $e_j{}^p(f) < \infty$,*

$$Tk(f) \geq C_9 e_j{}^p(f)^{-\frac{p}{jp-2}}.$$

Proof. Let $e_j{}^p(f) = b > 0$. Put $t_0 = (\frac{\pi}{8C_0})^{\frac{2(p+2)}{jp-2}} b^{-\frac{p}{jp-2}}$. Suppose $0 < y_0 - x_0 \leq t_0$ and $z \in [x_0, y_0]$. Since $\angle f'(x) \cdot \overrightarrow{f(x_0)f(y_0)} \leq \frac{\pi}{4}$ for any $x \in [x_0, y_0]$, $f(z)$ lies in the intersection of forward and backward solid cones with axis $\overleftrightarrow{f(x_0)f(y_0)}$, angle $\frac{\pi}{4}$, and vertices $f(x_0)$ and $f(y_0)$. Therefore for any point P in the straight line segment with endpoints $f(x_0)$ and $f(y_0)$,

$$|P - f(z)| \leq \max\{|P - f(x_0)|, |P - f(y_0)|\}.$$

Put

$$r_1 = \frac{1}{2}\min\left\{\frac{1}{2}\left(\frac{\pi}{8C_0}\right)^{\frac{2(p+2)}{jp-2}}, \left(\frac{1}{2}\right)^{\frac{p+2}{jp-2}} C_1\right\} b^{-\frac{p}{jp-2}}.$$

Then by Corollary 1.7, if $|x-y|_{S^1} > t_0$ then $|f(x)-f(y)| > 2r_1$. Assume $N_r(f(S^1))$ is not a solid torus for some r $(0 < r < r_1)$. Then there are x_1, y_1, z_1 $(x_1 < z_1 < y_1)$, and $P_1 \in N_r(f(S^1))$ such that $|P_1 - f(x_1)| \leq r$, $|P_1 - f(y_1)| \leq r$, and $|P_1 - f(z_1)| > r$, which is a contradiction. Hence $Tk(f) \geq r_1$. □

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