On an inverse problem of determining a connection on a vector bundle

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Abstract

We consider the problem of determining a connection on a vector bundle over a compact Riemannian manifold with boundary from the known parallel transport between boundary points along geodesics. The main result is the local uniqueness theorem: if two connections ∇' and ∇'' are C-close to a given connection ∇ whose curvature tensor is sufficiently small, then coincidence of parallel transports with respect to ∇' and ∇'' implies existence of an automorphism of the bundle which is identical on the boundary and transforms ∇' to ∇''. A linearized version of the problem is also considered.

1 Posing the problem and formulating the result

Starting with some inverse problem for the Schrödinger equation with magnetic potential, G. Uhlmann posed the following matrix tomography problem.

Let $M \subset \mathbf{R}^n$ be a closed convex bounded domain with smooth boundary. Let n matrix functions

$$\Gamma_i(x) = \left(\Gamma^\alpha_{i\beta}(x)\right)^m_{\alpha,\beta=1} \qquad (1 \le i \le n, x \in M)$$

of order m are defined in M and smoothly depend on $x \in M$. One has to recover these functions from the following information that is measured on the boundary ∂M of the domain M. Given two boundary points $x_0, x_1 \in \partial M$, we parameterize the straight line segment $\gamma = [x_0, x_1]$ with endpoints x_0 and x_1 in such the way as

$$\gamma(t) = x_0 + t(x_1 - x_0), \quad 0 \le t \le 1$$

and consider the following system of ordinary differential equations along the segment:

$$\dot u^\alpha + \dot\gamma^i(t)\Gamma^\alpha_{i\beta}(\gamma(t))u^\beta = 0. \tag{1.1}$$

Hereafter the following agreement is used: summation from 1 to n (m) is assumed over repeating different level roman (greek) indices. Assume that, for an arbitrary initial value

$$u_0 = u(0) \tag{1.2}$$

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at the point x_0, we can measure the final value $u_1 = u(1)$ at the point x_1 of the solution to the Cauchy problem (1.1)–(1.2). In other words, we are given the fundamental matrix $U(x_0, x_1)$ of system (1.1)

$$u_1 = U(x_0, x_1)u_0 \tag{1.3}$$

for all $x_0, x_1 \in \partial M$. One has to recover the coefficients $\Gamma^\alpha_{i\beta}$ of system (1.1) from the known fundamental matrix $U(x_0, x_1)$.

A similar matrix tomography problem was investigated by L. B. Wertgeim [W], but instead of (1.1) he considered the system

$$\dot{u}^\alpha + \Gamma^\alpha_\beta(\gamma(t))u^\beta = 0. \tag{1.4}$$

The difference between (1.1) and (1.4) is as follows: while coefficients of system (1.4) depend only on a current point $\gamma(t)$ of the segment $\gamma = [x_0, x_1]$, coefficients of system (1.1) depend linearly on the direction $\dot{\gamma}$ of the segment also. This distinction allows us formulate our problem in pure geometric terms.

The problem can be slightly generalized in the following way. Instead of a domain in Euclidean space we may introduce a compact Riemannian manifold $(M, \tilde{g})$ with boundary ∂M and consider equation (1.1) along geodesics γ, of the metric $\tilde{g}$, joining boundary points. In what follows we restrict ourselves to considering simple Riemannian manifolds. Let us recall that $(M, \tilde{g})$ is called a *simple compact Riemannian manifold* if the boundary is strictly convex with respect to the metric $\tilde{g}$ and, for every two points $x, y \in M$, there exists a unique geodesic joining the points which depends smoothly on these points.

A solution to the posed problem is not unique. Indeed, let us transform system (1.1) by the linear change of variables

$$u^\alpha = a^\alpha_\beta(\gamma(t))u'^\beta, \tag{1.5}$$

where $a = a(x) = (a^\alpha_\beta(x))$ is a nondegenerate $m \times m$-matrix depending smoothly on $x \in M$ and meeting the boundary condition

$$a|_{\partial M} = 0. \tag{1.6}$$

Under change (1.5), system (1.1) transforms to the system of the same kind

$$\dot{u}'^\alpha + \dot{\gamma}^i \Gamma'^\alpha_{i\beta}(\gamma(t))u'^\beta = 0 \tag{1.1$'$}$$

with the coefficients

$$\Gamma'_i = a^{-1}\Gamma_i a + a^{-1}\frac{\partial a}{\partial x^i}. \tag{1.7}$$

By (1.6), the fundamental matrices of systems (1.1) and (1.1$'$) coincide:

$$U(x_0, x_1) = U'(x_0, x_1) \qquad (x_0, x_1 \in \partial M).$$

Uhlmann's conjecture is the conversion of the latter statement: if the fundamental matrices of systems (1.1) and (1.1$'$) coincide on ∂M, then their coefficients can be obtained one from another by transformation (1.7).

Our approach to the posed problem is based on the evident analogy between formulas (1.1), (1.7) and the corresponding formulas of connection theory. Indeed, equation (1.1) means that the vector field $u = (u^\alpha)$ is parallel along the geodesic γ with respect to the connection, on the trivial vector bundle $\eta = M \times \mathbf{C}^m$, whose Christoffel symbols are equal to $\Gamma^\alpha_{i\beta}$. After introducing the connection form

$$\Gamma = \Gamma_i dx^i, \tag{1.8}$$

formula (1.7) becomes the known formula

$$\Gamma' = a^{-1}\Gamma a + a^{-1}da \tag{1.9}$$

of transforming the connection form under the automorphism a of the vector bundle η. We thus arrive at the following equivalent formulation of Uhlmann's conjecture.

Let $(M, \tilde g)$ be a simple compact Riemannian manifold, η be a smooth complex vector bundle over M, ∇' and ∇'' be two connections on η. Given points $x, y \in M$, let η_x be the fiber of the bundle η over x, and $I'_{x,y} : \eta_x \to \eta_y$ ($I''_{x,y} : \eta_x \to \eta_y$) be the parallel transport with respect to the connection ∇' (∇'') along the geodesic of the metric $\tilde g$ joining the points. Assume that $I'_{x,y} = I''_{x,y}$ for every two boundary points $x, y \in \partial M$. Does this imply existence of an automorphism $a : \eta \to \eta$ of the bundle η which is identical on the boundary, $a|_{\partial M} = \mathrm{Id}$, and transforms the connection ∇' to ∇''?

In some important cases there is a priori information on the coefficients of system (1.1). For instance, in the above-mentioned inverse problem for the Schrödinger equation the matrices Γ_i are skew-hermitian. In this case the fundamental matrix (1.3) is unitary. From the geometrical viewpoint this means that a Hermitian metric is introduced on the bundle η, and we consider a connection compatible with the metric.

The main result of the paper gives positive answer to some local version of the above-posed problem. Namely, we shall prove the conjecture for connections ∇' and ∇'' that are close enough to a given connection ∇. The result is obtained under some assumption of smallness of the curvature tensors of the connection ∇ and of the metric $\tilde g$.

To formulate the condition of smallness of curvature tensors, we need some definitions of connection theory.

For a manifold M, by $\tau_M = (TM, p, M)$ we denote the tangent bundle; by T_xM we denote the tangent space at a point $x \in M$, and by T^*_xM, the dual space. Points of the manifold TM will be denoted by pairs (x, ξ), where $x \in M$ and $\xi \in T_xM$.

Let η be a complex vector bundle over a manifold M. Recall that a *connection* on the bundle η is a mapping

$$\nabla : C^\infty(\tau_M) \times C^\infty(\eta) \to C^\infty(\eta), \qquad \nabla : (v, u) \mapsto \nabla_v u$$

possessing the following properties:

(1) $\nabla_{\varphi_1 v_1 + \varphi_2 v_2} u = \varphi_1 \nabla_{v_1} u + \varphi_2 \nabla_{v_2} u \quad (\varphi_i \in C^\infty(M))$;

(2) $\nabla_v(u_1 + u_2) = \nabla_v u_1 + \nabla_v u_2$;

(3) $\nabla_v(\varphi u) = \varphi \nabla_v u + v\varphi \cdot u$.

Let $(x^1, \dots, x^n)$ be a local coordinate system with domain $U \subset M$, and $(e_1, \dots, e_m)$ be a trivialization of the bundle η over U. The *Christoffel symbols* $\Gamma^\alpha_{i\beta} \in C^\infty(U)$ of the connection are introduced by the formula

$$\nabla_{\partial_i} e_\beta = \Gamma^\alpha_{i\beta} e_\alpha \quad \left(\partial_i = \frac{\partial}{\partial x^i}\right). \tag{1.10}$$

The Christoffel symbols do not constitute components of a tensor field. Nevertheless, if ∇ and ∇' are two connections with Christoffel symbols $\Gamma^\alpha_{i\beta}$ and $\Gamma'^\alpha_{i\beta}$ respectively, then the differences $(\Gamma'^{\alpha}_{i\beta} - \Gamma^{\alpha}_{i\beta})$ constitute the tensor field. This means that the set of all connections on a given vector bundle is an affine space. This affine space is endowed in a natural way with the C^k-topology for every $0 \le k \le \infty$ because the corresponding linear space is the space of sections of the bundle $\tau^*_M \otimes \eta \otimes \eta^*$. Speaking about closeness of two connections, we will always mean the closeness in the sense of topology of this affine space.

The *curvature tensor* of the connection ∇ is the linear operator

$$R : T_xM \otimes \eta_x \to T^*_xM \otimes \eta_x, \quad (R(v \otimes u))^{\alpha}_j = R^{\alpha}_{ij\beta} v^i u^{\beta} \tag{1.11}$$

whose components are expressed through the Christoffel symbols by the formula

$$R^{\alpha}_{ij\beta} = \frac{\partial \Gamma^{\alpha}_{j\beta}}{\partial x^i} - \frac{\partial \Gamma^{\alpha}_{i\beta}}{\partial x^j} + \Gamma^{\alpha}_{i\gamma}\Gamma^{\gamma}_{j\beta} - \Gamma^{\alpha}_{j\gamma}\Gamma^{\gamma}_{i\beta}. \tag{1.12}$$

A *Hermitian metric* on a complex vector bundle is a Hermitian inner product in fibers of the bundle smoothly depending on a base point. A complex vector bundle η endowed with a Hermitian metric g is called the *Hermitian vector bundle*. Such a bundle will be denoted by (η, g) or sometimes simply by η. A connection on η is called *compatible with a Hermitian metric* if the parallel transport in the sense of the connection conserves the inner product.

For a Riemannian manifold $(M, \tilde g)$, by $\Omega M = \{(x, \xi) \in TM \mid |\xi| = 1\}$ we denote the manifold of unit tangent vectors. Given a point $(x, \xi) \in \Omega M$, by $\gamma_{x,\xi} : [\tau_-(x, \xi), \tau_+(x, \xi)] \to M$ we denote the maximal geodesic meeting the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot\gamma_{x,\xi}(0) = \xi$. For a simple Riemannian manifold, the functions $\tau_-(x, \xi)$ and $\tau_+(x, \xi)$ are finite on ΩM and smooth on $\Omega M \setminus \Omega(\partial M)$.

In the case of a connection on a Hermitian vector bundle (η, g) over a Riemannian manifold $(M, \tilde g)$, we can define the norm $|R(x)|$ of operator (1.11). Defining the norm, we assume η_x to be endowed with the inner product g, and $T_xM \cong T^*_xM$, with the inner product $\tilde g$. Assuming $(M, \tilde g)$ to be a simple manifold, we define

$$\rho(M, \tilde g, \eta, g, \nabla) = \sup_{x,\xi \in \Omega M} \int\limits_0^{\tau_+(x,\xi)} t|R(\gamma_{x,\xi}(t))|\, dt. \tag{1.13}$$

The similar quantity

$$\tilde\rho(M, \tilde g) = \sup_{x,\xi \in \Omega M} \int\limits_0^{\tau_+(x,\xi)} t|\tilde R(\gamma_{x,\xi}(t))|\, dt \tag{1.14}$$

is defined for the curvature tensor

$$\tilde R : T_xM \otimes T_xM \to T^*_xM \otimes T^*_xM, \quad (\tilde R(v \otimes w))_{ik} = \tilde R_{ijkl} v^j w^l$$

of the Riemannian manifold $(M, \tilde g)$.

We can now formulate the main result of the present article.

Theorem 1.1 *Let $(M, \tilde g)$ be a simple compact n-dimensional Riemannian manifold, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric. Assume the quantities (1.13) and (1.14) to satisfy the inequalities*

$$\rho(M, \tilde g, \eta, g, \nabla) < \frac{1}{36}\sqrt{n - 3/2} \tag{1.15}$$

and

$$\widetilde{\rho}(M, \widetilde{g}) < 1/6. \tag{1.16}$$

There exists a C-neighborhood of the connection ∇ such that the following statement is valid for every two connections ∇' and ∇'' belonging the neighborhood: if the parallel transport with respect to ∇' between every two boundary points along geodesics of metric $\widetilde{g}$ coincides with the same with respect to the connection ∇'', then there exists an automorphism a of the bundle η which is identical on the boundary ∂M and transforms ∇' to ∇''. If the connections ∇' and ∇'' are compatible with the metric g, then a is an automorphism of the Hermitian bundle (η, g).

A connection is called *flat* if its curvature tensor is identical zero. Let us recall that, in the case of a flat connection on a vector bundle η over a simply connected manifold M, the absolute parallelism is defined, i.e., the result of parallel transport from a point $x \in M$ to another point $y \in M$ is independent of the choice of a curve along which the transport is realized. In particular, for a simple Riemannian manifold $(M, \widetilde{g})$, the parallel transport along every geodesic triangle with vertices in ∂M is the identical transform. We shall show that the latter property characterizes flat connections under some condition of smallness of the curvature tensors.

Theorem 1.2 *Let $(M, \widetilde{g})$ be a simple compact n-dimensional Riemannian manifold, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric. Assume inequalities (1.15) and (1.16) to be satisfied. If the result of parallel transport, with respect to the connection ∇ along every geodesic triangle with vertices in ∂M, is the identical transform, then the connection ∇ is flat.*

This theorem is a special case of the following more general result. Let us recall that a section $f \in C^\infty(\eta)$ is called *absolutely parallel* if $\nabla f = 0$ identically.

Theorem 1.3 *Assume $(M, \widetilde{g})$, (η, g) and ∇ to be as in Theorem 1.2; inequalities (1.15) and (1.16) to be satisfied. Let $f \in C^\infty(\eta; \partial M)$ be a section of the bundle η over the boundary ∂M such that, for every two points $x, y \in \partial M$, the result of parallel transport of the vector $f(x)$ to the point y along the geodesic of the metric $\widetilde{g}$ coincides with $f(y)$. Then f can be extended to an absolutely parallel section on the whole manifold M.*

Similar results are valid for a real vector bundle endowed with a Riemannian metric and a connection compatible with the metric.

Let us show that Theorem 1.3 implies Theorem 1.2. Indeed, let hypotheses of Theorem 1.2 be satisfied. We choose a point $x_0 \in \partial M$ and an orthonormal basis $(f_1(x_0), \dots, f_m(x_0))$ of the fiber η_{x_0}. We move this basis to all boundary points $x \in \partial M$ by the parallel transport along geodesics. In such the way we obtain the orthonormal basis $(f_1(x), \dots, f_m(x))$ smoothly depending on $x \in \partial M$. For every $1 \le \alpha \le m$, the section $f_\alpha \in C^\infty(\eta; \partial M)$ satisfies the hypothesis of Theorem 1.3. Applying this theorem, we get a trivialization $(f_1(x), \dots, f_m(x))$ of the bundle η over the whole manifold M which consists of orthonormal and absolutely parallel bases, and Theorem 1.2 is proved.

In sections 2–5 we develop some tensor analysis machinery that is used in the proofs of these theorems. This tensor analysis is a natural generalization of the techniques exposed in Chapter 4 of [Sh] which can be considered as the special case of our situation corresponding to $\eta = \tau_M$. Sections 6–7 contain the proofs of Theorems 1.1 and 1.3. In Sections 8–9 we discuss the linearized version of the problem.

2 The tensor algebra associated to a vector bundle

Let η be a smooth complex vector bundle over a manifold M. By η we denote the bundle itself as well as its total space; and by η_x, the fiber of η over a point $x \in M$. As usually, η^* denotes the dual bundle.

Let us recall the definition of the conjugate bundle $\bar\eta$. The manifold $\bar\eta$ is a copy of the manifold η; for $u \in \eta$, the corresponding element of the set $\bar\eta$ is denoted by $\bar u$. Fibers of the bundle $\bar\eta$ are copies of the corresponding fibers of η; and the vector space structure is introduced in fibers of $\bar\eta$ in such the way that the following formulas are valid: $\bar u + \bar v = \overline{u+v}, \quad \bar\alpha\bar u = \overline{\alpha u}$ for $u, v \in \eta_x,\ \alpha \in \mathbf{C}$. The identical mapping

$$\eta \to \bar\eta, \qquad u \mapsto \bar u \tag{2.1}$$

is an anti-isomorphism of complex vector bundles. It defines the anti-isomorphism

$$C^\infty(\eta) \to C^\infty(\bar\eta), \quad u \mapsto \bar u \tag{2.2}$$

of $C^\infty(M)$-modulus which meets the rule $\overline{\varphi v} = \bar\varphi\bar v$ for $\varphi \in C^\infty(M)$ and $v \in C^\infty(\eta)$.

There is the canonical isomorphism $(\bar\eta)^* \cong (\eta^*)^-$, therefore each of these bundles is denoted by $\bar\eta^*$.

For nonnegative integers $(r, \rho, \lambda; s, \sigma, \mu)$, we put

$$\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta =$$

$$= \underbrace{\tau\otimes\ldots\otimes\tau}_{r}\otimes\underbrace{\tau^*\otimes\ldots\otimes\tau^*}_{s}\otimes\underbrace{\eta\otimes\ldots\otimes\eta}_{\rho}\otimes\underbrace{\eta^*\otimes\ldots\otimes\eta^*}_{\sigma}\otimes\underbrace{\bar\eta\otimes\ldots\otimes\bar\eta}_{\lambda}\otimes\underbrace{\bar\eta^*\otimes\ldots\otimes\bar\eta^*}_{\mu}, \tag{2.3}$$

where τ is the complexification of the tangent bundle τ_M, tensor products are taken over $\mathbf{C}$. We call (2.3) the *bundle of η-tensors of degree $(r, s, \lambda; \rho, \sigma, \mu)$ over M*, and its sections, *η-tensor fields.*

Let us list the algebraic operations defined on η-tensors and η-tensor fields.

Since $C^\infty(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta)$ is a $C^\infty(M)$-module, η-tensor fields of the same degree can be summed and multiplied by smooth functions.

A permutation of each of the six sets

$$\{1,\ldots,r\}, \quad \{1,\ldots,s\}, \quad \{1,\ldots,\rho\}, \quad \{1,\ldots,\sigma\}, \quad \{1,\ldots,\lambda\}, \quad \{1,\ldots,\mu\}$$

determines the automorphism of the bundle $\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta$ by the corresponding permutation of factors of one of the six groups on the right-hand side of (2.3). These automorphisms are called *transpositions of indices.*

For $1 \le k \le r$ and $1 \le l \le s$, the canonical pairing of the k-th factor of the first group in (2.3) with the l-th factor of the second group determines the homomorphism

$$C^k_l : \tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta \to \tau^{(r-1,\rho,\lambda)}_{(s-1,\sigma,\mu)}\eta.$$

In a similar way the homomorphisms

$$C'^k_l : \tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta \to \tau^{(r,\rho-1,\lambda)}_{(s,\sigma-1,\mu)}\eta \quad (1 \le k \le \rho,\ 1 \le l \le \sigma),$$

$$C''^k_l : \tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta \to \tau^{(r,\rho,\lambda-1)}_{(s,\sigma,\mu-1)}\eta \quad (1 \le k \le \lambda,\ 1 \le l \le \mu)$$

are defined. These homomorphisms are called *contractions* with respect to corresponding indices.

The tensor product

$$C^\infty(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta)\times C^\infty(\tau^{(r',\rho',\lambda')}_{(s',\sigma',\mu')}\eta)\to C^\infty(\tau^{(r+r',\rho+\rho',\lambda+\lambda')}_{(s+s',\sigma+\sigma',\mu+\mu')}\eta)$$

is defined which turns $C^\infty(\tau^{(*,*,*)}_{(*,*,*)}\eta)=\sum\limits_{r,s,\rho,\sigma,\lambda,\mu=0}^{\infty}C^\infty(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta)$ into a sixfold graded $C^\infty(M)$-algebra.

Since τ is the complexification of the real bundle τ_M, the anti-automorphism $\tau\to\tau,\ v\mapsto\bar v$ is defined and determines, together with (2.1), the bundle anti-isomorphism

$$\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\to\tau^{(r,\lambda,\rho)}_{(s,\mu,\sigma)}\eta,\quad u\mapsto\bar u.$$

The latter, in its turn, allows us to define the anti-isomorphism

$$C^\infty(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta)\to C^\infty(\tau^{(r,\lambda,\rho)}_{(s,\mu,\sigma)}\eta),\quad u\mapsto\bar u \tag{2.4}$$

of $C^\infty(M)$-algebras.

We will often use coordinate representation of η-tensor fields. Let $(x^1,\dots,x^n)$ be a local coordinate system in M with domain $U\subset M$, and $(e_1,\dots,e_m)$ be a trivialization of the bundle η over U; this means that $e_\alpha\in C^\infty(\eta;U)$ and the vectors $e_1(x),\dots,e_m(x)$ constitute a basis of the fiber η_x at every point $x\in U$. We denote by $\theta^1(x),\dots,\theta^m(x)$ the dual basis of η^*_x. Then $(\theta^1,\dots,\theta^m)$ is a trivialization of η^* over U, $(\bar e_1,\dots,\bar e_m)$ is a trivialization of $\bar\eta$ over U, and $(\bar\theta^1,\dots,\bar\theta^m)$ is a trivialization of $\bar\eta^*$ over U. A section $u\in C^\infty(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta;U)$ of bundle (2.3) over U is uniquely represented in the form

$$u=u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s}\otimes$$

$$\otimes e_{\alpha_1}\otimes\dots\otimes e_{\alpha_\rho}\otimes\theta^{\beta_1}\otimes\dots\otimes\theta^{\beta_\sigma}\otimes\bar e_{\gamma_1}\otimes\dots\otimes\bar e_{\gamma_\lambda}\otimes\bar\theta^{\delta_1}\otimes\dots\otimes\bar\theta^{\delta_\mu}. \tag{2.5}$$

As usual, we will abbreviate record (2.5) to the following one:

$$u=\left(u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}\right). \tag{2.6}$$

Let, along with the local coordinate system $(x^1,\dots,x^n)$ and trivialization $(e_1,\dots,e_m)$ with the domain U, we are given "new" coordinates $(x'^1,\dots,x'^n)$ and trivialization $(e'_1,\dots,e'_m)$ defined in a domain U'. Then in $U\cap U'$ the following relations are valid:

$$x'^i=x'^i(x^1,\dots,x^n),\quad e'_\alpha=a^\beta_\alpha e_\beta,\quad \theta'^\alpha=b^\alpha_\beta\theta^\beta,\quad \bar e'_\alpha=\bar a^\beta_\alpha\bar e_\beta,\quad \bar\theta'^\alpha=\bar b^\alpha_\beta\bar\theta^\beta, \tag{2.7}$$

where (a^α_β) is the transformation matrix from the basis (e_α) to (e'_α), and $(b^\alpha_\beta)=(a^\alpha_\beta)^{-1}$. These relations imply the following formula for transforming components of an η-tensor field under changing coordinates and trivialization:

$$u'^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}=$$

$$=\frac{\partial x'^{i_1}}{\partial x^{k_1}}\dots\frac{\partial x'^{i_r}}{\partial x^{k_r}}\frac{\partial x^{l_1}}{\partial x'^{j_1}}\dots\frac{\partial x^{l_s}}{\partial x'^{j_s}}b^{\alpha_1}_{\nu_1}\dots b^{\alpha_\rho}_{\nu_\rho}a^{\pi_1}_{\beta_1}\dots a^{\pi_\sigma}_{\beta_\sigma}\bar b^{\gamma_1}_{\varepsilon_1}\dots\bar b^{\gamma_\lambda}_{\varepsilon_\lambda}\bar a^{\theta_1}_{\delta_1}\dots\bar a^{\theta_\mu}_{\delta_\mu}u^{k_1\dots k_r\nu_1\dots\nu_\rho|\varepsilon_1\dots\varepsilon_\lambda}_{l_1\dots l_s\pi_1\dots\pi_\sigma|\theta_1\dots\theta_\mu}. \tag{2.8}$$

The anti-isomorphism (2.4) is expressed in coordinate form by the equality

$$\overline{u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}} = \bar u^{i_1\dots i_r\gamma_1\dots\gamma_\lambda|\alpha_1\dots\alpha_\rho}_{j_1\dots j_s\delta_1\dots\delta_\mu|\beta_1\dots\beta_\sigma}. \tag{2.9}$$

An η-tensor field of degree $(r,\rho,\rho;s,\sigma,\sigma)$ is called *Hermitian* if $\bar u = u$. By (2.9), this is expressed in coordinate form by the equality

$$\overline{u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\rho}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\sigma}} = u^{i_1\dots i_r\gamma_1\dots\gamma_\rho|\alpha_1\dots\alpha_\rho}_{j_1\dots j_s\delta_1\dots\delta_\sigma|\beta_1\dots\beta_\sigma}.$$

The typical example of a Hermitian η-tensor field is an Hermitian metric. Given a Hermitian vector bundle (η, g), the metric tensor $g = (g_{\alpha|\beta})$ is Hermitian, i.e., $\overline{g_{\alpha|\beta}} = g_{\beta|\alpha}$. It determines the Hermitian inner product

$$\langle u|v\rangle = g_{\alpha|\beta}u^{\alpha|}\overline{v^{\beta|}} = g_{\alpha|\beta}u^{\alpha|}\bar v^{|\beta} \quad (u, v\in\eta_x)$$

on fibers of the bundle η. This inner product is extendible to fibers of the bundle $\tau^{(0,\rho,\lambda)}_{(0,\sigma,\mu)}\eta$ by the formula

$$\langle u|v\rangle = g_{\alpha_1|\varepsilon_1}\dots g_{\alpha_\rho|\varepsilon_\rho}g_{\nu_1|\gamma_1}\dots g_{\nu_\lambda|\gamma_\lambda}g^{\beta_1|\theta_1}\dots g^{\beta_\sigma|\theta_\sigma}g^{\pi_1|\delta_1}\dots g^{\pi_\mu|\delta_\mu}u^{\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}\bar v^{\nu_1\dots\nu_\lambda|\varepsilon_1\dots\varepsilon_\rho}_{\pi_1\dots\pi_\mu|\theta_1\dots\theta_\sigma}, \tag{2.10}$$

where $(g^{\alpha|\beta})$ is the inverse matrix to $(g_{\alpha|\beta})$, i.e., $g^{\alpha|\gamma}g_{\beta|\gamma} = \delta^\alpha_\beta$.

In the case of an Hermitian vector bundle, the canonical isomorphisms

$$\varphi:\eta\cong\bar\eta^*, \quad \psi:\bar\eta\cong\eta^* \tag{2.11}$$

are defined by the equalities

$$(\varphi u)(\bar v) = \langle u|v\rangle, \quad (\psi\bar u)(v) = \langle v|u\rangle \quad (u,v\in\eta_x).$$

We will consider isomorphisms (2.11) as identifications. They are expressed in coordinate form as follows: every vector $u\in\eta$ has the contravariant coordinates $u = (u^{\alpha|})$ and the covariant coordinates $u = (u_{|\alpha})$ that are related by the equalities

$$u_{|\alpha} = g_{\beta|\alpha}u^{\beta|}, \quad u^{\alpha|} = g^{\alpha|\beta}u_{|\beta}.$$

Similarly, every vector $u\in\bar\eta$ has the contravariant coordinates $u = (u^{|\alpha})$ and the covariant coordinates $u = (u_{\alpha|})$ that are related by the equalities

$$u_{\alpha|} = g_{\alpha|\beta}u^{|\beta}, \quad u^{|\alpha} = g^{\beta|\alpha}u_{\beta|}.$$

Isomorphisms (2.11) allow us also to establish the identifications

$$\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta \cong \tau^{(r,\rho+\mu,\lambda+\sigma)}_{(s,0,0)}\eta \cong \tau^{(r,0,0)}_{(s,\lambda+\sigma,\rho+\mu)}\eta$$

that are expressed in coordinate form by the following rules of raising and lowing greek indices of an η-tensor:

$$u^{i_1\dots i_r}_{j_1\dots j_s\alpha_1\dots\alpha_\rho|\beta_1\dots\beta_\sigma} = g_{\alpha_1|\delta_1}\dots g_{\alpha_\rho|\delta_\rho}g_{\gamma_1|\beta_1}\dots g_{\gamma_\sigma|\beta_\sigma}u^{i_1\dots i_r\gamma_1\dots\gamma_\sigma|\delta_1\dots\delta_\rho}_{j_1\dots j_s},$$

$$u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\beta_1\dots\beta_\sigma}_{j_1\dots j_s} = g^{\alpha_1|\delta_1}\dots g^{\alpha_\rho|\delta_\rho}g^{\gamma_1|\beta_1}\dots g^{\gamma_\sigma|\beta_\sigma}u^{i_1\dots i_r}_{j_1\dots j_s\gamma_1\dots\gamma_\sigma|\delta_1\dots\delta_\rho}.$$

Using these identifications, formula (2.10) can be rewritten as follows:

$$\langle u|v\rangle = u^{\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\bar{v}_{\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma} = u_{\alpha_1\ldots\alpha_\sigma|\beta_1\ldots\beta_\rho}\bar{v}^{\alpha_1\ldots\alpha_\sigma|\beta_1\ldots\beta_\rho}. \tag{2.12}$$

If the manifold M is endowed with a Riemannian metric $\tilde{g} = (\tilde{g}_{ij})$, then we can also raise and low latin indices of an η-tensor and establish in this way the isomorphisms

$$\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta \cong \tau^{(r+s,\rho+\mu,\lambda+\sigma)}_{(0,0,0)}\eta \cong \tau^{(0,0,0)}_{(r+s,\lambda+\sigma,\rho+\mu)}\eta.$$

Formula (2.12) is generalized to the case of η-tensors of arbitrary degree

$$\langle u|v\rangle = u^{i_1\ldots i_r\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\bar{v}_{i_1\ldots i_r\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma} = u_{i_1\ldots i_r\alpha_1\ldots\alpha_\sigma|\beta_1\ldots\beta_\rho}\bar{v}^{i_1\ldots i_r\alpha_1\ldots\alpha_\sigma|\beta_1\ldots\beta_\rho}. \tag{2.13}$$

3 Covariant differentiation

The definition of a connection on a complex vector bundle η over a manifold M was given in Section 1. In local coordinates a connection is expressed as follows. Let $(x^1,\ldots,x^n)$ be a local coordinate system with domain $U \subset M$, and $(e_1,\ldots,e_m)$ be a trivialization of the bundle η over U. The *Christoffel symbols* $\Gamma^\alpha_{i\beta} \in C^\infty(U)$ are introduced by formula (1.10). If $v = (v^i) \in C^\infty(\tau_M;U)$ and $u = (u^{\alpha|}) \in C^\infty(\eta;U)$, then

$$\nabla_v u = v^i \nabla_i u^{\alpha|}\cdot e_\alpha,$$

where

$$\nabla_i u^{\alpha|} = \frac{\partial u^{\alpha|}}{\partial x^i} + \Gamma^\alpha_{i\beta}u^{\beta|}. \tag{3.1}$$

One can easily show that the latter formula implies the following rule of transforming Christoffel symbols under change (2.7) of coordinates and trivialization:

$$\Gamma'^\alpha_{i\beta} = \frac{\partial x^j}{\partial x'^i}b^\alpha_\gamma\Gamma^\gamma_{j\delta}a^\delta_\beta + b^\alpha_\gamma\frac{\partial a^\gamma_\beta}{\partial x'^i}. \tag{3.2}$$

Distinguish the case when the coordinate system does not change, i.e., $x^i = x'^i$. In this case formula (3.2) takes the form

$$\Gamma'^\alpha_{i\beta} = b^\alpha_\gamma\Gamma^\gamma_{i\delta}a^\delta_\beta + b^\alpha_\gamma\frac{\partial a^\gamma_\beta}{\partial x^i} \tag{3.3}$$

that is equivalent to formula (1.9).

A connection ∇ on a bundle η determines the first order differential operator (that is denoted by the same letter)

$$\nabla : C^\infty(\eta) = C^\infty(\tau^{(0,1,0)}_{(0,0,0)}\eta) \to C^\infty(\tau^{(0,1,0)}_{(1,0,0)}\eta) = C^\infty(\tau^*\otimes\eta) \tag{3.4}$$

by the formula $(\nabla u)(v) = \nabla_v u$. The operator is called the *covariant differentiation* with respect to the given connection. A section $u \in C^\infty(\eta)$ is called *absolutely parallel* if $\nabla u = 0$.

To extend differential operator (3.4) to η-tensor fields of arbitrary degree, we need also a connection on the manifold M.

Theorem 3.1 *Let a connection*

$$\widetilde{\nabla} : C^\infty(\tau_M) \to C^\infty(\tau'_M \otimes \tau_M) \tag{3.5}$$

be given on a manifold M, and let ∇ be a connection on a complex vector bundle η over M. For all $(r,s,\lambda;\rho,\sigma,\mu)$, there exist uniquely determined first order differential operators

$$\nabla : C^\infty(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta) \to C^\infty(\tau^{(r,\rho,\lambda)}_{(s+1,\sigma,\mu)}\eta) \tag{3.6}$$

satisfying the following conditions.

1. *$\nabla\varphi = d\varphi$ for $\varphi \in C^\infty(M) = C^\infty(\tau^{(0,0,0)}_{(0,0,0)}\eta)$.*
2. *∇ coincides with operator (3.4) in the case of $(r,\rho,\lambda;s,\sigma,\mu) = (0,1,0;0,0,0)$.*
3. *∇ coincides with operator (3.5) in the case of $(r,\rho,\lambda;s,\sigma,\mu) = (1,0,0;0,0,0)$.*
4. *The operator ∇ commutes with all contractions.*
5. *∇ is a differentiation with respect to the tensor product.*
6. *∇ commutes with anti-isomorphism (2.4).*

In coordinate form this operator is expressed as follows. If $u = \left(u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}\right) \in C^\infty\left(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right)$, then

$$\nabla u = \nabla_k u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}\frac{\partial}{\partial x^{i_1}}\otimes\dots\otimes\frac{\partial}{\partial x^{i_r}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_s}\otimes dx^k\otimes$$

$$\otimes e_{\alpha_1}\otimes\dots\otimes e_{\alpha_\rho}\otimes\theta^{\beta_1}\otimes\dots\otimes\theta^{\beta_\sigma}\otimes\bar{e}_{\gamma_1}\otimes\dots\otimes\bar{e}_{\gamma_\lambda}\otimes\bar{\theta}^{\delta_1}\otimes\dots\otimes\bar{\theta}^{\delta_\mu},$$

where

$$\nabla_k u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} = \frac{\partial}{\partial x^k}u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}+$$

$$+\sum_{m=1}^{r}\widetilde{\Gamma}^{i_m}_{kp}u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}-\sum_{m=1}^{s}\widetilde{\Gamma}^{p}_{kj_m}u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}+$$

$$+\sum_{\kappa=1}^{\rho}\Gamma^{\alpha_\kappa}_{k\varepsilon}u^{i_1\dots i_r\alpha_1\dots\alpha_{\kappa-1}\varepsilon\alpha_{\kappa+1}\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}-\sum_{\kappa=1}^{\sigma}\Gamma^{\varepsilon}_{k\beta_\kappa}u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_{\kappa-1}\varepsilon\beta_{\kappa+1}\dots\beta_\sigma|\delta_1\dots\delta_\mu}+$$

$$+\sum_{\kappa=1}^{\lambda}\bar{\Gamma}^{\gamma_\kappa}_{k\varepsilon}u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_{\kappa-1}\varepsilon\gamma_{\kappa+1}\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}-\sum_{\kappa=1}^{\mu}\bar{\Gamma}^{\varepsilon}_{k\delta_\kappa}u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_{\kappa-1}\varepsilon\delta_{\kappa+1}\dots\delta_\mu}. \tag{3.8}$$

Here $\Gamma^\alpha_{i\beta}$ are the Christoffel symbols of the connection ∇, and $\widetilde{\Gamma}^k_{ij}$ are the Christoffel symbols of the connection $\widetilde{\nabla}$.

We omit the proof of the theorem which can be done in the full analogy with the proof of Theorem 3.2.1 of [Sh].

Let now η be endowed with a Hermitian metric $g \in C^\infty(\tau^{(0,0,0)}_{(0,1,1)}\eta)$. One can easily show that a connection ∇ is compatible with the metric if and only if $\nabla g = 0$.

4 Semibasic η-tensor fields

Let $p : TM \to M$ be the projection of the tangent bundle, and η be a complex vector bundle over M. The bundle

$$\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta = p^*\left(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right)$$

over TM is called the *bundle of semibasic η-tensors*, and its sections are called *semibasic η-tensor fields*. For $(r,\rho,\lambda;s,\sigma,\mu) = (0,1,0;0,0,0)$, we use the term *semibasic sections of the bundle η*.

Given a local coordinate system and trivialization of η with domain $U \subset M$, a semibasic η-tensor field $u \in C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta; p^{-1}(U)\right)$ can be represented in form (2.5) with components depending on $(x,\xi) \in p^{-1}(U) \subset TM$. In $C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right)$, there can be distinguished the subspace of *basic fields* whose components are independent of ξ; this subspace can be identified with $C^\infty\left(\tau^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right)$. Under changing coordinates and trivialization, the components of a semibasic η-tensor field are transformed by the same formula (2.8) as components of an ordinary (= basic) η-tensor field.

The following operations defined above for η-tensor fields are extendible to semibasic η-tensor fields in an evident way: contractions, transpositions of indices, tensor product, the anti-isomorphism $u \mapsto \bar{u}$. Unlike basic η-tensor fields, semibasic η-tensor fields can be multiplied by smooth functions $\varphi(x,\xi)$ depending on $(x,\xi) \in TM$, i.e., $C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right)$ is a $C^\infty(TM)$-module.

The *vertical covariant derivative*

$$\overset{v}{\nabla} : C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right) \to C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s+1,\sigma,\mu)}\eta\right)$$

is defined by the equality

$$\overset{v}{\nabla}_k u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} = \frac{\partial}{\partial\xi^k} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu}.$$

Given a connection $\widetilde{\nabla}$ on M and connection ∇ on η, we define the *horizontal derivative*

$$\overset{h}{\nabla} : C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)}\eta\right) \to C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s+1,\sigma,\mu)}\eta\right)$$

by the formula

$$\begin{aligned}\overset{h}{\nabla}_k u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} &= \frac{\partial}{\partial x^k} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} - \widetilde{\Gamma}^p_{kq}\xi^q\frac{\partial}{\partial\xi^p} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} + \\ &+ \sum_{m=1}^{r}\widetilde{\Gamma}^{i_m}_{kp} u^{i_1\dots i_{m-1}pi_{m+1}\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} - \sum_{m=1}^{s}\widetilde{\Gamma}^{p}_{kj_m} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_{m-1}pj_{m+1}\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} + \\ &+ \sum_{\kappa=1}^{\rho}\Gamma^{\alpha_\kappa}_{k\varepsilon} u^{i_1\dots i_r\alpha_1\dots\alpha_{\kappa-1}\varepsilon\alpha_{\kappa+1}\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} - \sum_{\kappa=1}^{\sigma}\Gamma^{\varepsilon}_{k\beta_\kappa} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_{\kappa-1}\varepsilon\beta_{\kappa+1}\dots\beta_\sigma|\delta_1\dots\delta_\mu} + \\ &+ \sum_{\kappa=1}^{\lambda}\bar{\Gamma}^{\gamma_\kappa}_{k\varepsilon} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_{\kappa-1}\varepsilon\gamma_{\kappa+1}\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_\mu} - \sum_{\kappa=1}^{\mu}\bar{\Gamma}^{\varepsilon}_{k\delta_\kappa} u^{i_1\dots i_r\alpha_1\dots\alpha_\rho|\gamma_1\dots\gamma_\lambda}_{j_1\dots j_s\beta_1\dots\beta_\sigma|\delta_1\dots\delta_{\kappa-1}\varepsilon\delta_{\kappa+1}\dots\delta_\mu}. \end{aligned} \tag{4.1}$$

Here $\Gamma^\alpha_{i\beta}$ are the Christoffel symbols of the connection ∇, and $\widetilde{\Gamma}^k_{ij}$ are the Christoffel symbols of the connection $\widetilde{\nabla}$.

We establish commutation formulas for $\overset{v}{\nabla}$ and $\overset{h}{\nabla}$.

Lemma 4.1 *The following formulas are valid:*

$$[\overset{v}{\nabla}_k, \overset{v}{\nabla}_l] = 0, \tag{4.2}$$

$$[\overset{h}{\nabla}_k, \overset{v}{\nabla}_l] = 0, \tag{4.3}$$

$$\begin{aligned}
&[\overset{h}{\nabla}_k, \overset{h}{\nabla}_l] u^{i_1 \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_\mu} = -\widetilde{R}^p_{qkl} \xi^q \overset{v}{\nabla}_p u^{i_1 \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_\mu} + \\
&+ \sum_{m=1}^{r} \widetilde{R}^{i_m}_{pkl} u^{i_1 \dots i_{m-1} p i_{m+1} \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_\mu} - \sum_{m=1}^{s} \widetilde{R}^p_{j_m kl} u^{i_1 \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_{m-1} p j_{m+1} \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_\mu} \\
&+ \sum_{\kappa=1}^{\rho} R^{\alpha_\kappa |}_{kl\varepsilon |} u^{i_1 \dots i_r \alpha_1 \dots \alpha_{\kappa-1} \varepsilon \alpha_{\kappa+1} \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_\mu} - \sum_{\kappa=1}^{\rho} R^{\varepsilon |}_{kl \beta_\kappa |} u^{i_1 \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_{\kappa-1} \varepsilon \beta_{\kappa+1} \dots \beta_\sigma | \delta_1 \dots \delta_\mu} + \\
&+ \sum_{\kappa=1}^{\lambda} \bar{R}^{|\gamma_\kappa}_{kl|\varepsilon} u^{i_1 \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_{\kappa-1} \varepsilon \gamma_{\kappa+1} \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_\mu} - \sum_{\kappa=1}^{\mu} \bar{R}^{|\varepsilon}_{kl|\delta_\kappa} u^{i_1 \dots i_r \alpha_1 \dots \alpha_\rho | \gamma_1 \dots \gamma_\lambda}_{j_1 \dots j_s \beta_1 \dots \beta_\sigma | \delta_1 \dots \delta_{\kappa-1} \varepsilon \delta_{\kappa+1} \dots \delta_\mu},
\end{aligned} \tag{4.4}$$

where $(R^{\alpha|}_{ij\beta|} = R^{\alpha}_{ij\beta})$ *is the curvature tensor of connection* ∇ *defined by formula (1.11), and* $(\widetilde{R}^i_{jkl})$ *is the curvature tensor of the connection* $\widetilde{\nabla}$ *defined by the formula*

$$\widetilde{R}^i_{jkl} = \frac{\partial \widetilde{\Gamma}^i_{jl}}{\partial x^k} - \frac{\partial \widetilde{\Gamma}^i_{jk}}{\partial x^l} + \widetilde{\Gamma}^i_{kp} \widetilde{\Gamma}^p_{jl} - \widetilde{\Gamma}^i_{lp} \widetilde{\Gamma}^p_{jk}.$$

Remark. In section 1 we denoted the components of the curvature tensor of the connection ∇ by $R^{\alpha}_{ij\beta}$ because the sense of vertical bars in indices was not explained at that moment. Now we use the more precise notation $R^{\alpha|}_{ij\beta|}$.

We omit the proof that is quite similar to the proof of Theorem 3.5.2 of [Sh].

The operator

$$H : C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)} \eta\right) \to C^\infty\left(\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)} \eta\right)$$

is defined by the equality $H = \xi^i \overset{h}{\nabla}_i$.

5 The Pestov identity

Let $(M, \widetilde{g})$ be a Riemannian manifold, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric. The metrics g and $\widetilde{g}$ allow us to define the operations of raising and lowing all, greek and roman, indices of a semibasic η-tensor field in the same way as we did in Section 2 for basic η-tensor fields; and also to introduce the inner product in fibers of the bundle $\beta^{(r,\rho,\lambda)}_{(s,\sigma,\mu)} \eta$ which is expressed by formula (2.13). We use the Levi-Civita connection $\widetilde{\nabla}$ of the Riemannian manifold $(M, \widetilde{g})$ in the definition of the horizontal derivative.

Lemma 5.1 *(the Pestov identity) Let* $(M, \widetilde{g})$ *be a Riemannian manifold, and* (η, g) *be a Hermitian vector bundle over* M *endowed with a connection* ∇ *compatible with the metric. For a semibasic* η*-tensor field* $u \in C^\infty(\beta^{(r,\rho,\sigma)}_{(0,0,0)} \eta)$ *the following identity is valid:*

$$2Re\,\langle\overset{h}{\nabla}u|\overset{v}{\nabla}Hu\rangle = |\overset{h}{\nabla}u|^2+\overset{h}{\nabla}_k v^k+\overset{v}{\nabla}_k w^k-\tilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{i_1\ldots i_r\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\cdot\overset{v}{\nabla}{}^l\bar{u}_{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}+$$

$$+Re\left[\left(\sum_{m=1}^{r}\tilde{R}^{i_m}_{pkl}u^{i_1\ldots i_{m-1}pi_{m+1}\ldots i_r\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}+\sum_{\kappa=1}^{\rho}R^{\alpha_\kappa|}_{kl\varepsilon|}u^{i_1\ldots i_r\alpha_1\ldots\alpha_{\kappa-1}\varepsilon\alpha_{\kappa+1}\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}+\right.\right.$$

$$\left.\left.+\sum_{\kappa=1}^{\sigma}\bar{R}^{|\beta_\kappa}_{kl|\varepsilon}u^{i_1\ldots i_r\alpha_1\ldots\alpha_\rho|\beta_1\ldots\beta_{\kappa-1}\varepsilon\beta_{\kappa+1}\ldots\beta_\sigma}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\right], \tag{5.1}$$

where

$$v^k = Re\left(\xi^k\overset{h}{\nabla}{}^l u^{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\cdot\overset{v}{\nabla}_l\bar{u}_{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}-\right.$$

$$\left.-\,\xi^l\overset{v}{\nabla}{}^k u^{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\cdot\overset{h}{\nabla}_l\bar{u}_{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\right), \tag{5.2}$$

$$w^k = Re\left(\xi^l\overset{h}{\nabla}{}^k u^{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\cdot\overset{h}{\nabla}_l\bar{u}_{i_1\ldots i_r\alpha\ldots\alpha_\rho|\beta_1\ldots\beta_\sigma}\right). \tag{5.3}$$

Proof. We will prove the identity in the case of $(r,\rho,\sigma)=(2,2,2)$. For other values of (r,ρ,σ) the proof is similar.

By the definition of the operator H,

$$2\langle\overset{h}{\nabla}u|\overset{v}{\nabla}Hu\rangle = 2\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\left(\xi^l\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}\right) = 2|\overset{h}{\nabla}u|^2+2\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}. \tag{5.4}$$

We transform the second summand on the right-hand side of the latter relation. To this end we define a function φ by the equality

$$2\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta} = \overset{v}{\nabla}_k\left(\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}\right)+$$

$$+\overset{h}{\nabla}_l\left(\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\bar{u}_{ij\alpha\beta|\gamma\delta}\right)-\overset{h}{\nabla}{}^k\left(\xi^l\overset{v}{\nabla}_k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}\right)-\varphi. \tag{5.5}$$

Let us show that the real part of the function φ is independent of second-oder derivatives of the field u. Indeed, expressing the derivatives of the products on the right-hand side of (5.5) through the derivatives of the factors, we obtain

$$\varphi = -2\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}+|\overset{h}{\nabla}u|^2+$$

$$+\xi^l\overset{v}{\nabla}_k\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}+\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}+$$

$$+\xi^l\overset{h}{\nabla}_l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\bar{u}_{ij\alpha\beta|\gamma\delta}+\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}_l\overset{v}{\nabla}_k\bar{u}_{ij\alpha\beta|\gamma\delta}-$$

$$-\xi^l\overset{h}{\nabla}{}^k\overset{v}{\nabla}_k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}-\xi^l\overset{v}{\nabla}_k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}{}^k\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}.$$

After evident transformations, this equality takes the form

$$\varphi = |\overset{h}{\nabla}u|^2+\xi^l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot[\overset{h}{\nabla}_l,\overset{v}{\nabla}_k]\bar{u}_{ij\alpha\beta|\gamma\delta}+\xi^l[\overset{v}{\nabla}_k,\overset{h}{\nabla}{}^k]u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}+$$

$$+\xi^l\overset{h}{\nabla}_l\overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{v}{\nabla}_k\bar{u}_{ij\alpha\beta|\gamma\delta}-\xi^l\overset{v}{\nabla}_k u^{ij\alpha\beta|\gamma\delta}\cdot\overset{h}{\nabla}{}^k\overset{h}{\nabla}_l\bar{u}_{ij\alpha\beta|\gamma\delta}.$$

By (4.2) and (4.3), the second and third terms on the right-hand side are equal to zero, and the equality simplifies to the following one:

$$\varphi = |\overset{h}{\nabla} u|^2 + \xi^l \overset{h}{\nabla}_l \overset{h}{\nabla}^k u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}_k \bar{u}_{ij\alpha\beta|\gamma\delta} - \xi^l \overset{h}{\nabla}^k \overset{h}{\nabla}_l \bar{u}_{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}_k u^{ij\alpha\beta|\gamma\delta}.$$

Taking the real parts, we obtain

$$2\mathrm{Re}\,\varphi = 2|\overset{h}{\nabla} u|^2 + \xi^l \overset{h}{\nabla}_l \overset{h}{\nabla}^k u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}_k \bar{u}_{ij\alpha\beta|\gamma\delta} + \xi^l \overset{h}{\nabla}_l \overset{h}{\nabla}^k \bar{u}^{ij\gamma\delta|\alpha\beta} \cdot \overset{v}{\nabla}_k u_{ij\gamma\delta|\alpha\beta} -$$

$$- \xi^l \overset{h}{\nabla}^k \overset{h}{\nabla}_l \bar{u}_{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}_k u^{ij\alpha\beta|\gamma\delta} - \xi^l \overset{h}{\nabla}^k \overset{h}{\nabla}_l u_{ij\gamma\delta|\alpha\beta} \cdot \overset{v}{\nabla}_k \bar{u}^{ij\gamma\delta|\alpha\beta}.$$

We raise and low indices to obtain

$$2\mathrm{Re}\,\varphi = 2|\overset{h}{\nabla} u|^2 + \xi^l \overset{h}{\nabla}_l \overset{h}{\nabla}_k u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}^k \bar{u}_{ij\alpha\beta|\gamma\delta} + \xi^l \overset{h}{\nabla}_l \overset{h}{\nabla}_k \bar{u}^{ij\gamma\delta|\alpha\beta} \cdot \overset{v}{\nabla}^k u_{ij\gamma\delta|\alpha\beta} -$$

$$- \xi^l \overset{h}{\nabla}_k \overset{h}{\nabla}_l \bar{u}^{ij\gamma\delta|\alpha\beta} \cdot \overset{v}{\nabla}^k u_{ij\gamma\delta|\alpha\beta} - \xi^l \overset{h}{\nabla}_k \overset{h}{\nabla}_l u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}^k \bar{u}_{ij\alpha\beta|\gamma\delta}.$$

We see that our equality can be rewritten in the form

$$2\mathrm{Re}\,\varphi = 2|\overset{h}{\nabla} u|^2 + \xi^l [\overset{h}{\nabla}_l, \overset{h}{\nabla}_k] u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}^k \bar{u}_{ij\alpha\beta|\gamma\delta} + \xi^l [\overset{h}{\nabla}_l, \overset{h}{\nabla}_k] \bar{u}^{ij\gamma\delta|\alpha\beta} \cdot \overset{v}{\nabla}^k u_{ij\gamma\delta|\alpha\beta}.$$

The last two terms are conjugate to one other, therefore

$$\mathrm{Re}\,\varphi = |\overset{h}{\nabla} u|^2 + \mathrm{Re}\,\left(\xi^l [\overset{h}{\nabla}_l, \overset{h}{\nabla}_k] u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}^k \bar{u}_{ij\alpha\beta|\gamma\delta}\right).$$

Using commutation formula (4.4), we obtain

$$\mathrm{Re}\,\varphi = |\overset{h}{\nabla} u|^2 + \widetilde{R}_{pqkl} \xi^q \xi^l \overset{v}{\nabla}{}^p u^{ij\alpha\beta|\gamma\delta} \overset{v}{\nabla}{}^k \bar{u}_{ij\alpha\beta|\gamma\delta} -$$

$$-\mathrm{Re}\left[\xi^l \left(\widetilde{R}^i_{pkl} u^{pj\alpha\beta|\gamma\delta} + \widetilde{R}^j_{pkl} u^{ip\alpha\beta|\gamma\delta} + R^\alpha_{kl\varepsilon} u^{ij\varepsilon\beta|\gamma\delta} + \right.\right.$$

$$\left.\left. + R^\beta_{kl\varepsilon} u^{ij\alpha\varepsilon|\gamma\delta} + \bar{R}^\gamma_{kl\varepsilon} u^{ij\alpha\beta|\varepsilon\delta} + \bar{R}^\delta_{kl\varepsilon} u^{ij\alpha\beta|\gamma\varepsilon}\right) \overset{v}{\nabla}{}^k \bar{u}_{ij\alpha\beta|\gamma\delta}\right].$$

Substituting this value into (5.5), we obtain

$$2\mathrm{Re}\,\left(\xi^l \overset{h}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta} \cdot \overset{v}{\nabla}_k \overset{h}{\nabla}_l \bar{u}_{ij\alpha\beta|\gamma\delta}\right) = -|\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_k v^k + \overset{v}{\nabla}_k w^k -$$

$$-\widetilde{R}_{kplq} \xi^p \xi^q \overset{v}{\nabla}{}^k u^{ij\alpha\beta|\gamma\delta} \overset{v}{\nabla}{}^l \bar{u}_{ij\alpha\beta|\gamma\delta} + \mathrm{Re}\left[\left(\widetilde{R}^i_{pkl} u^{pj\alpha\beta|\gamma\delta} + \widetilde{R}^j_{pkl} u^{ip\alpha\beta|\gamma\delta} + \right.\right.$$

$$\left.\left. + R^\alpha_{kl\varepsilon} u^{ij\varepsilon\beta|\gamma\delta} + R^\beta_{kl\varepsilon} u^{ij\alpha\varepsilon|\gamma\delta} + \bar{R}^\gamma_{kl\varepsilon} u^{ij\alpha\beta|\varepsilon\delta} + \bar{R}^\delta_{kl\varepsilon} u^{ij\alpha\beta|\gamma\varepsilon}\right) \xi^l \overset{v}{\nabla}{}^k \bar{u}_{ij\alpha\beta|\gamma\delta}\right]$$

with v and w defined by (5.2) and (5.3). Finally, substituting this value into (5.4), we arrive at (5.1). The lemma is proved.

6 Proof of Theorem 1.1

We recall the notation $T^0M = \{(x,\xi) \in TM \mid \xi \neq 0\}$.

Let $(M, \tilde{g})$ be a simple Riemannian manifold, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric. Let connections ∇' and ∇'' on η be C-close to the connection ∇ (the degree of the closeness will be specified later). Assume that parallel transports, between boundary points along geodesics of the metric $\tilde{g}$, coincide for connections ∇' and ∇''.

Given a point $(x,\xi) \in T^0M$, we define the automorphism $a(x,\xi) : \eta_x \to \eta_x$ of the vector space η_x in the following way. Let $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), \tau_+(x,\xi)] \to M$ be the maximal geodesic, of the metric $\tilde{g}$, satisfying the initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Then the point $y = \gamma(\tau_-(x,\xi))$ belongs to the boundary ∂M. Let $I' : \eta_y \to \eta_x$ be the parallel transport along γ with respect to the connection ∇', and $I'' : \eta_y \to \eta_x$ be the parallel transport along γ with respect to the connection ∇''. We put

$$a(x,\xi) = I''(I')^{-1}.$$

We have thus constructed the section a of the bundle $\beta^{(0,1,0)}_{(0,1,0)}\eta$ over the domain T^0M. First of all, we should discuss the smoothness properties of the section. One can easily see that the smoothness of $a(x,\xi)$ is the same as the smoothness of the function $\tau_-(x,\xi)$. Using the smoothness properties of the latter function which are exposed in Section 4.1 of [Sh], we see that a is continious on T^0M and (infinitely) smooth on $T^0M \setminus T^0(\partial M)$. Some of derivatives of the field a can be unbounded at the set $T^0(\partial M)$. Consequently, some of integrals considered below are improper and we have to verify their convergence. The verification is performed in a quite similar way as in Section 4.6 of [Sh], since the singularities of a are due only to the singularities of the function $\tau_-(x,\xi)$. Therefore, in order to simplify the exposition, we will not pay attention to the singularities of a and will deal in such a way as a would be a smooth semibasic η-tensor field, $a \in C^\infty(\beta^{(0,1,0)}_{(0,1,0)}\eta; T^0M)$. We will prove that, under hypotheses of Theorem 1.1, a is independent of ξ. In particular, this will imply smoothness of a.

By the construction, the field $a(x,\xi)$ is positively homogeneous of zero degree in the second argument

$$a(x,t\xi) = a(x,\xi) \qquad (t > 0), \tag{6.1}$$

and, by the hypothesis on coincidence of parallel transports in the sense of ∇' and ∇'', satisfies the boundary condition

$$a|_{\partial(T^0M)} = \mathrm{Id}. \tag{6.2}$$

We will derive a differential equation for the field a. Let us fix a point $(x,\xi) \in T^0M$ and a vector $u_0 \in \eta_x$, and construct the vector field $u'(t)$ parallel along the geodesic $\gamma(t) = \gamma_{x,\xi}(t)$ with respect to the connection ∇' and satisfying the initial condition $u'(0) = u_0$. Let us then construct the vector field $u''(t)$ parallel along $\gamma(t)$ with respect to the connection ∇'' and satisfying the initial condition $u''(\tau_-(x,\xi)) = u'(\tau_-(x,\xi))$. By the definition of the operator a, the equality

$$a(\gamma(t), \dot{\gamma}(t))u'(t) = u''(t)$$

holds for every $t \in [\tau_-(x,\xi), \tau_+(x,\xi)]$. We write down the equality in coordinate form using a local coordinate system and trivialization

$$a^{\alpha|}_{\beta|}(\gamma(t), \dot{\gamma}(t))u'^{\beta|}(t) = u''^{\alpha|}(t).$$

We differentiate the equality with respect to t

$$\dot\gamma^i\frac{\partial a^{\alpha|}_{\beta|}}{\partial x^i}u'^{\beta|}+\ddot\gamma^i\frac{\partial a^{\alpha|}_{\beta|}}{\partial \xi^i}u'^{\beta|}+a^{\alpha|}_{\beta|}\dot u'^{\beta|}=\dot u''^{\alpha|}.$$

By the equation for geodesics

$$\ddot\gamma^i=-\widetilde\Gamma^i_{jk}\dot\gamma^j\dot\gamma^k.$$

The vector fields u' and u'' are parallel along γ with respect to connections ∇' and ∇'' respectively; in coordinate form this is written as follows:

$$\dot u'^{\alpha|}=-\dot\gamma^i\Gamma'^{\alpha}_{i\beta}u'^{\beta|},\qquad \dot u''^{\alpha|}=-\dot\gamma^i\Gamma''^{\alpha}_{i\beta}u''^{\beta|}=-\dot\gamma^i\Gamma''^{\alpha}_{i\beta}a^{\beta|}_{\varepsilon|}u'^{\varepsilon|},$$

where $\Gamma'^{\alpha}_{i\beta}$ and $\Gamma''^{\alpha}_{i\beta}$ are the Christoffel symbols of the connections ∇' and ∇'' respectively. Substituting these values into the previous equation, we obtain

$$\dot\gamma^i\left(\frac{\partial a^{\alpha|}_{\beta|}}{\partial x^i}-\widetilde\Gamma^p_{iq}\dot\gamma^q\frac{\partial a^{\alpha|}_{\beta|}}{\partial \xi^p}-\Gamma'^{\varepsilon}_{i\beta}a^{\alpha|}_{\varepsilon|}\right)u'^{\beta|}=-\dot\gamma^i\Gamma''^{\alpha}_{i\varepsilon}a^{\varepsilon|}_{\beta|}u'^{\beta|}.$$

Putting $t=0$ in the latter equality and remembering that $\dot\gamma(0)=\xi$, we obtain

$$\xi^i\left(\frac{\partial a^{\alpha|}_{\beta|}}{\partial x^i}-\widetilde\Gamma^p_{iq}\xi^q\frac{\partial a^{\alpha|}_{\beta|}}{\partial \xi^p}-\Gamma'^{\varepsilon}_{i\beta}a^{\alpha|}_{\varepsilon|}\right)u'^{\beta|}_0=-\xi^i\Gamma''^{\alpha}_{i\varepsilon}a^{\varepsilon|}_{\beta|}u'^{\beta|}_0.$$

Since u_0 is an arbitrary vector, the latter relation implies

$$\xi^i\left(\frac{\partial a^{\alpha|}_{\beta|}}{\partial x^i}-\widetilde\Gamma^p_{iq}\xi^q\frac{\partial a^{\alpha|}_{\beta|}}{\partial \xi^p}-\Gamma'^{\varepsilon}_{i\beta}a^{\alpha|}_{\varepsilon|}\right)=-\xi^i\Gamma''^{\alpha}_{i\varepsilon}a^{\varepsilon|}_{\beta|}. \tag{6.3}$$

In order to write equation (6.3) in a covariant form, we introduce the notations

$$f^{\alpha|}_{i\beta|}=\Gamma^{\alpha}_{i\beta}-\Gamma''^{\alpha}_{i\beta},\quad h^{\alpha|}_{i\beta|}=\Gamma'^{\alpha}_{i\beta}-\Gamma^{\alpha}_{i\beta},$$

where $\Gamma^{\alpha}_{i\beta}$ are the Christoffel symbols of the connection ∇. As was mentioned in Section 1, $(f^{\alpha|}_{i\beta|})$ and $(h^{\alpha|}_{i\beta|})$ are well defined η-tensor fields of degree $(0,1,0;1,1,0)$. By the hypothesis of the theorem on closeness of the connections, we can assume the inequalities

$$|f|<\varepsilon,\qquad |h|<\varepsilon \tag{6.4}$$

to hold with arbitrary small positive ε. Adding the expression $\xi^i\Gamma^{\alpha}_{i\varepsilon}a^{\varepsilon|}_{\beta|}+\xi^i(\Gamma'^{\varepsilon}_{i\beta}-\Gamma^{\varepsilon}_{i\beta})a^{\alpha|}_{\varepsilon|}$ to the both parts of equation (6.3), we obtain

$$\xi^i\left(\frac{\partial a^{\alpha|}_{\beta|}}{\partial x^i}u'^{\beta|}-\widetilde\Gamma^p_{iq}\xi^q\frac{\partial a^{\alpha|}_{\beta|}}{\partial \xi^p}+\Gamma^{\alpha}_{i\varepsilon}a^{\varepsilon|}_{\beta|}-\Gamma^{\varepsilon}_{i\beta}a^{\alpha|}_{\varepsilon|}\right)=\xi^i(\Gamma^{\alpha}_{i\varepsilon}-\Gamma''^{\alpha}_{i\varepsilon})a^{\varepsilon|}_{\beta|}+\xi^i(\Gamma'^{\varepsilon}_{i\beta}-\Gamma^{\varepsilon}_{i\beta})a^{\alpha|}_{\varepsilon|}.$$

Comparing the left-hand side of this relation with definition (4.4) of the horizontal derivative, we see that the equation can be written in the covariant form

$$Ha^{\alpha|}_{\beta|}=\xi^i f^{\alpha|}_{i\varepsilon|}a^{\varepsilon|}_{\beta|}+\xi^i h^{\varepsilon|}_{i\beta|}a^{\alpha|}_{\varepsilon|}, \tag{6.5}$$

where $H = \xi^i \overset{h}{\nabla}_i$ is the operator with respect to the connection ∇.

We have thus got equation (6.5), boundary condition (6.2), and homogeneity condition (6.1) for the field a. From now on we can forget the connections ∇' and ∇''. All we need are inequalities (6.4) for the fields f and h. We will show that, if the number ε on (6.4) is sufficiently small, then relations (6.1), (6.2), (6.4) and (6.5) imply

$$\overset{v}{\nabla} a = 0. \tag{6.6}$$

After this equality is proved, equation (6.3) implies the relation

$$\frac{\partial a^{\alpha|}_{\beta|}}{\partial x^i} - \Gamma'^{\varepsilon}_{i\beta} a^{\alpha|}_{\varepsilon|} = -\Gamma''^{\alpha}_{i\varepsilon} a^{\varepsilon|}_{\beta|}$$

that can be rewritten in matrix form as follows:

$$\Gamma' = a^{-1}\Gamma'' a + a^{-1}\frac{\partial a}{\partial x}.$$

The latter equality means that the automorphism a transform the connection ∇'' into ∇', and the theorem is proved.

So, our aim is proving (6.6). To this end we write the Pestov identity (5.1) for $(u^{i\alpha|\beta}) = (\overset{v}{\nabla}{}^i a^{\alpha|\beta}) \in C^\infty(\beta^{(1,1,1)}_{(0,0,0)}\eta; T^0M)$:

$$\begin{aligned}
2\mathrm{Re}\,\langle \overset{h}{\nabla}\overset{v}{\nabla} a | \overset{v}{\nabla} H \overset{v}{\nabla} a\rangle = |\overset{h}{\nabla}\overset{v}{\nabla} a|^2 + \overset{h}{\nabla}_k v^k + \overset{v}{\nabla}_k w^k - \widetilde{R}_{kplq}\xi^p\xi^q \overset{v}{\nabla}{}^k \overset{v}{\nabla}{}^i a^{\alpha|\beta} \cdot \overset{v}{\nabla}{}^l \overset{v}{\nabla}_i \bar{a}_{\alpha|\beta} + \\
+\mathrm{Re}\left[\left(\widetilde{R}^i_{pkl}\overset{v}{\nabla}{}^p a^{\alpha|\beta} + R^{\alpha|}_{kl\varepsilon|}\overset{v}{\nabla}{}^i a^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}\overset{v}{\nabla}{}^i a^{\alpha|\varepsilon}\right)\xi^l \overset{v}{\nabla}{}^k \overset{v}{\nabla}_i \bar{a}_{\alpha|\beta}\right],
\end{aligned} \tag{6.7}$$

where

$$v^k = \mathrm{Re}\,\left(\xi^k \overset{h}{\nabla}{}^l \overset{v}{\nabla}{}^i a^{\alpha|\beta} \cdot \overset{v}{\nabla}_l \overset{v}{\nabla}_i \bar{a}_{\alpha|\beta} - \xi^l \overset{v}{\nabla}{}^k \overset{v}{\nabla}{}^i a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}_i \bar{a}_{\alpha|\beta}\right), \tag{6.8}$$

$$w^k = \mathrm{Re}\,\left(\xi^l \overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^i a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}_i \bar{a}_{\alpha|\beta}\right). \tag{6.9}$$

We consider the left-hand side of (6.7) in order to distinguish a divergent term. By the commutation formula

$$\overset{v}{\nabla} H - H \overset{v}{\nabla} = \overset{h}{\nabla} \tag{6.10}$$

and equation (6.5), we have

$$H\overset{v}{\nabla}_l \bar{a}_{\alpha|\beta} = \overset{v}{\nabla}_l H \bar{a}_{\alpha|\beta} - \overset{h}{\nabla}_l \bar{a}_{\alpha|\beta} = -\overset{h}{\nabla}_l \bar{a}_{\alpha|\beta} + \overset{v}{\nabla}_l \left(\xi^i \bar{f}^{\varepsilon|}_{i\alpha|}\bar{a}_{\varepsilon|\beta} + \xi^i \bar{h}^{|\varepsilon}_{i|\beta}\bar{a}_{\alpha|\varepsilon}\right). \tag{6.11}$$

Taking into account that f and h are independent of ξ, we obtain

$$H\overset{v}{\nabla}_l \bar{a}_{\alpha|\beta} = -\overset{h}{\nabla}_l \bar{a}_{\alpha|\beta} + \bar{f}^{\varepsilon|}_{l\alpha|}\bar{a}_{\varepsilon|\beta} + \bar{h}^{|\varepsilon}_{l|\beta}\bar{a}_{\alpha|\varepsilon} + \xi^i \bar{f}^{\varepsilon|}_{i\alpha|}\overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \xi^i \bar{h}^{|\varepsilon}_{i|\beta}\overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon}.$$

Applying the operator $\overset{v}{\nabla}$ to the latter equality, we obtain

$$\overset{v}{\nabla}_k H \overset{v}{\nabla}_l \bar{a}_{\alpha|\beta} = -\overset{h}{\nabla}_l \overset{v}{\nabla}_k \bar{a}_{\alpha|\beta} + \bar{f}^{\varepsilon|}_{l\alpha|} \overset{v}{\nabla}_k \bar{a}_{\varepsilon|\beta} + \bar{h}^{|\varepsilon}_{l|\beta} \overset{v}{\nabla}_k \bar{a}_{\alpha|\varepsilon} +$$

$$+ \bar{f}^{\varepsilon|}_{k\alpha|} \overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \bar{h}^{|\varepsilon}_{k|\beta} \overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon} + \xi^i \bar{f}^{\varepsilon|}_{i\alpha|} \overset{v}{\nabla}_k \overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \xi^i \bar{h}^{|\varepsilon}_{i|\beta} \overset{v}{\nabla}_k \overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon}. \tag{6.12}$$

Therefore

$$\langle \overset{h}{\nabla} \overset{v}{\nabla} a | \overset{v}{\nabla} H \overset{v}{\nabla} a \rangle = \overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{v}{\nabla}_k H \overset{v}{\nabla}_l \bar{a}_{\alpha|\beta} = -\overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}_k \bar{a}_{\alpha|\beta} +$$

$$+ \overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^l a^{\alpha|\beta} \Big(\bar{f}^{\varepsilon|}_{l\alpha|} \overset{v}{\nabla}_k \bar{a}_{\varepsilon|\beta} + \bar{f}^{\varepsilon|}_{k\alpha|} \overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \bar{h}^{|\varepsilon}_{l|\beta} \overset{v}{\nabla}_k \bar{a}_{\alpha|\varepsilon} + \bar{h}^{|\varepsilon}_{k|\beta} \overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon} +$$

$$+ \xi^i \bar{f}^{\varepsilon|}_{i\alpha|} \overset{v}{\nabla}_k \overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \xi^i \bar{h}^{|\varepsilon}_{i|\beta} \overset{v}{\nabla}_k \overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon} \Big).$$

Introducing the notation

$$\mathcal{F}[a] = 2\mathrm{Re} \Big[\overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^l a^{\alpha|\beta} \Big(\bar{f}^{\varepsilon|}_{l\alpha|} \overset{v}{\nabla}_k \bar{a}_{\varepsilon|\beta} + \bar{f}^{\varepsilon|}_{k\alpha|} \overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \bar{h}^{|\varepsilon}_{l|\beta} \overset{v}{\nabla}_k \bar{a}_{\alpha|\varepsilon} + \bar{h}^{|\varepsilon}_{k|\beta} \overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon} +$$

$$+ \xi^i \bar{f}^{\varepsilon|}_{i\alpha|} \overset{v}{\nabla}_k \overset{v}{\nabla}_l \bar{a}_{\varepsilon|\beta} + \xi^i \bar{h}^{|\varepsilon}_{i|\beta} \overset{v}{\nabla}_k \overset{v}{\nabla}_l \bar{a}_{\alpha|\varepsilon} \Big) \Big], \tag{6.13}$$

we write the preceding relation in the form

$$\mathrm{Re} \langle \overset{h}{\nabla} \overset{v}{\nabla} a | \overset{v}{\nabla} H \overset{v}{\nabla} a \rangle = -\mathrm{Re} \Big(\overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}_k \bar{a}_{\alpha|\beta} \Big) + \frac{1}{2} \mathcal{F}[a]. \tag{6.14}$$

In order to distinguish a divergent term, we transform the first term on the right-hand side on (6.14) as follows:

$$\overset{h}{\nabla}{}^k \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}_k \bar{a}_{\alpha|\beta} = \overset{h}{\nabla}{}^k \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}_k \bar{a}_{\alpha|\beta} \Big) - \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}{}^k \overset{h}{\nabla}_l \overset{v}{\nabla}_k \bar{a}_{\alpha|\beta} =$$

$$= \overset{h}{\nabla}_k \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} \Big) - \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_k \overset{h}{\nabla}_l \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} =$$

$$= \overset{h}{\nabla}_k \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} \Big) - \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{h}{\nabla}_k \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} - \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot [\overset{h}{\nabla}_k, \overset{h}{\nabla}_l] \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} =$$

$$= \overset{h}{\nabla}_k \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} \Big) - \overset{h}{\nabla}_l \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_k \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} \Big) +$$

$$+ \overset{h}{\nabla}_l \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_k \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} - \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot [\overset{h}{\nabla}_k, \overset{h}{\nabla}_l] \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} =$$

$$= \overset{h}{\nabla}_k \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} - \overset{v}{\nabla}{}^k a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^l \bar{a}_{\alpha|\beta} \Big) +$$

$$+ \overset{h}{\nabla}_k \overset{v}{\nabla}{}^k a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^l \bar{a}_{\alpha|\beta} - \overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot [\overset{h}{\nabla}_k, \overset{h}{\nabla}_l] \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta}.$$

Introducing the notations

$$\tilde{v}^k = \mathrm{Re} \Big(\overset{v}{\nabla}{}^l a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^k \bar{a}_{\alpha|\beta} - \overset{v}{\nabla}{}^k a^{\alpha|\beta} \cdot \overset{h}{\nabla}_l \overset{v}{\nabla}{}^l \bar{a}_{\alpha|\beta} \Big) \tag{6.15}$$

and

$$z^{\alpha|\beta} = \overset{h}{\nabla}_k \overset{v}{\nabla}{}^k a^{\alpha|\beta}, \tag{6.16}$$

we write the result in the form

$$\mathrm{Re}\left(\overset{h}{\nabla}{}^k\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{h}{\nabla}_l\overset{v}{\nabla}_k\bar a_{\alpha|\beta}\right)=|z|^2+\overset{h}{\nabla}_k\widetilde v^k-\mathrm{Re}\left(\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot[\overset{h}{\nabla}_k,\overset{h}{\nabla}_l]\overset{v}{\nabla}{}^k\bar a_{\alpha|\beta}\right).$$

Inserting the latter value into (6.14) and using commutation formula (4.4), we obtain

$$\mathrm{Re}\langle\overset{h}{\nabla}\overset{v}{\nabla}a|\overset{v}{\nabla}H\overset{v}{\nabla}a\rangle=-|z|^2-\overset{h}{\nabla}_k\widetilde v^k+\frac{1}{2}\mathcal F[a]-$$

$$-\mathrm{Re}\left[\overset{v}{\nabla}{}^l a^{\alpha|\beta}\left(\widetilde R^p_{qkl}\xi^q\overset{v}{\nabla}_p\overset{v}{\nabla}{}^k\bar a_{\alpha|\beta}-\widetilde R^k_{pkl}\overset{v}{\nabla}{}^p\bar a_{\alpha|\beta}+R^{\varepsilon|}_{kl\alpha|}\overset{v}{\nabla}{}^k\bar a_{\varepsilon|\beta}+\bar R^{|\varepsilon}_{kl|\beta}\overset{v}{\nabla}{}^k\bar a_{\alpha|\varepsilon}\right)\right].$$

We now substitute the latter expression into the left-hand side of the Pestov identity (6.7) to obtain

$$|\overset{h}{\nabla}\overset{v}{\nabla}f|^2+2|z|^2=-\overset{h}{\nabla}_k(v^k+2\widetilde v^k)-\overset{v}{\nabla}_k w^k+\widetilde{\mathcal R}[a]-\mathcal R[a]+\mathcal F[a],\tag{6.17}$$

where $\widetilde{\mathcal R}[a]$ denotes the sum of terms dependent on the curvature tensor $\widetilde R$

$$\widetilde{\mathcal R}[a]=\widetilde R_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^i a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\overset{v}{\nabla}_i\bar a_{\alpha|\beta}+$$

$$+\mathrm{Re}\left[2\overset{v}{\nabla}{}^l a^{\alpha|\beta}\left(-\widetilde R^p_{qkl}\xi^q\overset{v}{\nabla}_p\overset{v}{\nabla}{}^k\bar a_{\alpha|\beta}+\widetilde R^k_{pkl}\overset{v}{\nabla}{}^p\bar a_{\alpha|\beta}\right)-\widetilde R^i_{pkl}\xi^l\overset{v}{\nabla}{}^p a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^k\overset{v}{\nabla}_i\bar a_{\alpha|\beta}\right],\tag{6.18}$$

and $\mathcal R[a]$ denotes the sum of terms dependent on the curvature tensor R

$$\mathcal R[a]=\mathrm{Re}\left[2\overset{v}{\nabla}{}^l a^{\alpha|\beta}\left(R^{\varepsilon|}_{kl\alpha|}\overset{v}{\nabla}{}^k\bar a_{\varepsilon|\beta}+\bar R^{|\varepsilon}_{kl|\beta}\overset{v}{\nabla}{}^k\bar a_{\alpha|\varepsilon}\right)+\left(R^{\alpha|}_{kl\varepsilon|}\overset{v}{\nabla}{}^i a^{\varepsilon|\beta}+\bar R^{|\beta}_{kl|\varepsilon}\overset{v}{\nabla}{}^i a^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\overset{v}{\nabla}_i\bar a_{\alpha|\beta}\right].\tag{6.19}$$

We integrate equality (6.17) over ΩM and transform the integrals of the divergent terms by the Gauss — Ostrogradskiĭ formulas (Theorem 3.6.3 of [Sh])

$$\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+2\|z\|^2=-\int\limits_{\partial\Omega M}\langle v+2\widetilde v|\nu\rangle d\Sigma^{2n-2}-(n-2)\int\limits_{\Omega M}\langle w|\xi\rangle d\Sigma+$$

$$+\int\limits_{\Omega M}\widetilde{\mathcal R}[a]d\Sigma-\int\limits_{\Omega M}\mathcal R[a]d\Sigma+\int\limits_{\Omega M}\mathcal F[a]d\Sigma.\tag{6.20}$$

Hereafter we use the notation

$$\|u\|^2=\int\limits_{\Omega M}|u|^2\,d\Sigma$$

for a semibasic η-tensor field u.

It follows from boundary condition (6.2) and definitions (6.8), (6.15) of the fields v and $\widetilde v$ that $v|_{\partial\Omega M}=\widetilde v|_{\partial\Omega M}=0$. Therefore the first integral on the right-hand side of (6.20) is equal to zero.

By (6.9), the integrand of the second integral on (6.20) is

$$\langle w|\xi\rangle=\mathrm{Re}\left(\xi_i\overset{h}{\nabla}{}^i\overset{v}{\nabla}_k a^{\alpha|\beta}\cdot\xi^j\overset{h}{\nabla}_j\overset{v}{\nabla}{}^k\bar a_{\alpha|\beta}\right)=|H\overset{v}{\nabla}a|^2.$$

Substituting this value into (6.20), we obtain

$$\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+(n-2)\|H\overset{v}{\nabla}a\|^2+2\|z\|^2=\int\limits_{\Omega M}\widetilde{\mathcal R}[a]d\Sigma-\int\limits_{\Omega M}\mathcal R[a]d\Sigma+\int\limits_{\Omega M}\mathcal F[a]d\Sigma.\tag{6.21}$$

We will estimate the right-hand side integrals on (6.21) by the left-hand side of this equality.

We start with the first integral. Expression (6.18) for its integrand can be slightly simplified. To this and we rewrite (6.18) in the form

$$\widetilde{\mathcal{R}}[a]=\tilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^i a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\overset{v}{\nabla}_i\bar{a}_{\alpha|\beta}+\mathrm{Re}A, \tag{6.22}$$

where

$$A=2\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}-2\tilde{R}^p_{qkl}\xi^q\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}_p\overset{v}{\nabla}{}^k\bar{a}_{\alpha|\beta}-\tilde{R}^i_{pkl}\xi^l\overset{v}{\nabla}{}^p a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^k\overset{v}{\nabla}_i\bar{a}_{\alpha|\beta}.$$

After changing summation indices in the last term, this equality takes the form

$$A=2\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}-(2\tilde{R}_{pqkl}+\tilde{R}_{klpq})\xi^q\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}.$$

Using the symmetry $\tilde{R}_{klpq}=\tilde{R}_{pqkl}$ of the curvature tensor, we obtain

$$A=2\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}-3\tilde{R}_{pqkl}\xi^q\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}.$$

We transform this formula once more by distinguishing a divergence term. Since the curvature tensor is independent of ξ, we can write

$$A=2\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}-3\overset{v}{\nabla}{}^k\left(\tilde{R}_{pqkl}\xi^q\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}\right)+$$

$$+3\tilde{R}_{pqkl}\tilde{g}^{kq}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}+3\tilde{R}_{pqkl}\xi^q\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}.$$

The last term is equal to zero because $\tilde{R}_{pqkl}$ is skew-symmetric in the indices k,l while $\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^l a^{\alpha|\beta}$ is symmetric in these indices. Therefore the latter relation takes the form

$$A=-\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}-3\overset{v}{\nabla}{}^k\left(\tilde{R}_{pqkl}\xi^q\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}\right).$$

Inserting this expression into (6.22), we obtain

$$\widetilde{\mathcal{R}}[a]=\tilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^i a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\overset{v}{\nabla}_i\bar{a}_{\alpha|\beta}-$$

$$-\mathrm{Re}\left[\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}+3\overset{v}{\nabla}{}^k\left(\tilde{R}_{pqkl}\xi^q\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}\right)\right].$$

Integrating this equality and transforming the last integral by the Gauss — Ostrogradskiĭ formula, we obtain

$$\int\limits_{\Omega M}\widetilde{\mathcal{R}}[a]d\Sigma=\int\limits_{\Omega M}\Big[\tilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^i a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\overset{v}{\nabla}_i\bar{a}_{\alpha|\beta}+$$

$$+3(n-2)\tilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\bar{a}_{\alpha|\beta}-\tilde{R}^k_{pkl}\overset{v}{\nabla}{}^l a^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^p\bar{a}_{\alpha|\beta}\Big]d\Sigma. \tag{6.23}$$

We have omitted the sign *Re* because the integrand on the right-hand side is real.

We stop our calculations for a moment to note some possibility that is not realized in the present paper. Observe that the right-hand side integrand on (6.23) depends only on the sectional curvature and Ricci curvature of the Riemannian manifold $(M,\tilde g)$. Therefore the statement of Theorem 1.1 can be changed in such a way that the hypothesis on $(M,\tilde g)$ would be expressed in terms of the sectional and Ricci curvatures. We do not use the possibility because it makes the statement of Theorem more cumbersome.

The module of the right-hand side integrand on (6.23) can be estimated by the quantity

$$|\tilde R||\overset{v}{\nabla}\overset{v}{\nabla}a\wedge\xi|^2+3(n-2)|\tilde R||\overset{v}{\nabla}a\wedge\xi|^2+(n-1)|\tilde R||\overset{v}{\nabla}a|^2\le|\tilde R|\left(|\overset{v}{\nabla}\overset{v}{\nabla}a|^2+(4n-7)|\overset{v}{\nabla}a|^2\right).$$

Therefore

$$\left|\int\limits_{\Omega M}\widetilde{\mathcal R}[a]d\Sigma\right|\le\int\limits_{\Omega M}|\tilde R|\left(|\overset{v}{\nabla}\overset{v}{\nabla}a|^2+(4n-7)|\overset{v}{\nabla}a|^2\right)d\Sigma. \tag{6.24}$$

By (6.2), the boundary conditions

$$\overset{v}{\nabla}a|_{\partial\Omega M}=0,\quad \overset{v}{\nabla}\overset{v}{\nabla}a|_{\partial\Omega M}=0$$

hold and, consequently, the Poincaré inequality (Lemma 4.5.1 of [Sh]) can be applied to these fields. Estimating the right-hand integrals on (6.24) with the help of the Poincaré inequality, we obtain

$$\left|\int\limits_{\Omega M}\widetilde{\mathcal R}[a]\,d\Sigma\right|\le\tilde\rho\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+(4n-7)\tilde\rho\|H\overset{v}{\nabla}a\|^2, \tag{6.25}$$

where the number $\tilde\rho=\tilde\rho(M,\tilde g)$ is defined by formula (1.14).

We now estimate the second integral on (6.21). As is seen from (6.19), the integrand admits the estimate

$$|\mathcal R[a]|\le 2|R||\overset{v}{\nabla}a|\cdot|\overset{v}{\nabla}\overset{v}{\nabla}a|+4|R||\overset{v}{\nabla}a|^2\le|R|\left(\lambda|\overset{v}{\nabla}\overset{v}{\nabla}a|^2+(4+1/\lambda)|\overset{v}{\nabla}a|^2\right),$$

where λ is an arbitrary positive number. Integrating this relation and using the Poincaré inequality, we obtain

$$\left|\int\limits_{\Omega M}\mathcal R[a]\right|\le\lambda\rho\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+(4+1/\lambda)\rho\|H\overset{v}{\nabla}a\|^2, \tag{6.26}$$

where the number $\rho=\rho(M,\tilde g,\eta,g,\nabla)$ is defined by (1.13).

We estimate the last integrand on (6.21). By (6.4) and (6.13), the integrand admits the estimate

$$|\mathcal F[a]|\le 2\varepsilon\left(3|\overset{h}{\nabla}\overset{v}{\nabla}a|^2+|\overset{v}{\nabla}\overset{v}{\nabla}a|^2+2|\overset{v}{\nabla}a|^2\right).$$

Integrating the inequality, we obtain

$$\left|\int\limits_{\Omega M}\mathcal F[a]\,d\Sigma\right|\le 2\varepsilon\left(3\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+\|\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+2\|\overset{v}{\nabla}a\|^2\right). \tag{6.27}$$

Estimating the right-hand side integrals on (6.21) by (6.25)–(6.27), we arrive at the inequality

$$\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+(n-2)\|H\overset{v}{\nabla}a\|^2\leq 2\varepsilon\left(3\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+\|\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+2\|\overset{v}{\nabla}a\|^2\right)+$$

$$+(\lambda\rho+\widetilde{\rho})\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+\left((4+1/\lambda)\rho+(4n-7)\widetilde{\rho}\right)\|H\overset{v}{\nabla}a\|^2. \tag{6.28}$$

We are going to eliminate the quantities $\|\overset{v}{\nabla}a\|$, $\|\overset{v}{\nabla}\overset{v}{\nabla}a\|$ and $\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|$ from the inequality (6.28).

First of all, with the help of the Poincaré inequality, we obtain

$$\|\overset{v}{\nabla}a\|^2\leq\frac{d^2}{2}\|H\overset{v}{\nabla}a\|^2, \tag{6.29}$$

$$\|\overset{v}{\nabla}\overset{v}{\nabla}a\|^2\leq\frac{d^2}{2}\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2, \tag{6.30}$$

where $d=d(M,\widetilde{g})$ is the diameter of $(M,\widetilde{g})$.

With the help of commutation formula (6.10), equation (6.12) gives

$$H\overset{v}{\nabla}_k\overset{v}{\nabla}_l a_{\alpha|\beta}=(\overset{v}{\nabla}_k H-\overset{h}{\nabla}_k)\overset{v}{\nabla}_l a_{\alpha|\beta}=-\overset{h}{\nabla}_k\overset{v}{\nabla}_l a_{\alpha|\beta}-\overset{h}{\nabla}_l\overset{v}{\nabla}_k a_{\alpha|\beta}+$$

$$+f_{l|\beta}^{|\varepsilon}\overset{v}{\nabla}_k a_{\alpha|\varepsilon}+f_{k|\beta}^{|\varepsilon}\overset{v}{\nabla}_l a_{\alpha|\varepsilon}+h_{l\alpha|}^{\varepsilon|}\overset{v}{\nabla}_k a_{\varepsilon|\beta}+h_{k\alpha|}^{\varepsilon|}\overset{v}{\nabla}_l a_{\varepsilon|\beta}+\xi^i f_{i|\beta}^{|\varepsilon}\overset{v}{\nabla}_k\overset{v}{\nabla}_l a_{\alpha|\varepsilon}+\xi^i h_{i\alpha|}^{\varepsilon|}\overset{v}{\nabla}_k\overset{v}{\nabla}_l a_{\varepsilon|\beta}.$$

This implies the estimate

$$|H\overset{v}{\nabla}\overset{v}{\nabla}a|\leq 2|\overset{h}{\nabla}\overset{v}{\nabla}a|+2\varepsilon|\overset{v}{\nabla}\overset{v}{\nabla}a|+4\varepsilon|\overset{v}{\nabla}a|.$$

Squaring this inequality, we obtain

$$|H\overset{v}{\nabla}\overset{v}{\nabla}a|^2\leq 4|\overset{h}{\nabla}\overset{v}{\nabla}a|^2+4\varepsilon^2|\overset{v}{\nabla}\overset{v}{\nabla}a|^2+16\varepsilon^2|\overset{v}{\nabla}a|^2+$$

$$+8\varepsilon|\overset{h}{\nabla}\overset{v}{\nabla}a|\cdot|\overset{v}{\nabla}\overset{v}{\nabla}a|+16\varepsilon|\overset{h}{\nabla}\overset{v}{\nabla}a|\cdot|\overset{v}{\nabla}a|+16\varepsilon^2|\overset{v}{\nabla}\overset{v}{\nabla}a|\cdot|\overset{v}{\nabla}a|.$$

Estimating the last tree terms with the help of the inequality between arithmetical and geometrical means, we obtain

$$|H\overset{v}{\nabla}\overset{v}{\nabla}a|^2\leq(4+2\mu)|\overset{h}{\nabla}\overset{v}{\nabla}a|^2+(12+16/\mu)\varepsilon^2|\overset{v}{\nabla}\overset{v}{\nabla}a|^2+(24+64/\mu)\varepsilon^2|\overset{v}{\nabla}a|^2$$

with an arbitrary positive number μ. Integrating this inequality, we obtain

$$\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2\leq(4+2\mu)\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+(12+16/\mu)\varepsilon^2\|\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+(24+64/\mu)\varepsilon^2\|\overset{v}{\nabla}a\|^2. \tag{6.31}$$

Estimating the last two terms on (6.31) with the help of (6.29) and (6.30), we obtain

$$\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2\leq(4+2\mu)\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+(6+8/\mu)d^2\varepsilon^2\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2+(12+32/\mu)d^2\varepsilon^2\|H\overset{v}{\nabla}a\|^2.$$

This equality, being rewritten in the form

$$(1-(6+8/\mu)d^2\varepsilon^2)\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2\leq(4+2\mu)\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+(12+32/\mu)d^2\varepsilon^2\|H\overset{v}{\nabla}a\|^2,$$

gives the estimate

$$\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2\leq\frac{4+2\mu}{1-(6+8/\mu)d^2\varepsilon^2}\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2+\frac{(12+32/\mu)d^2\varepsilon^2}{1-(6+8/\mu)d^2\varepsilon^2}\|H\overset{v}{\nabla}a\|^2. \tag{6.32}$$

Inequalities (6.30) and (6.32) imply the estimate

$$\|\overset{v}{\nabla}\overset{v}{\nabla}a\|^2 \leq \frac{(2+\mu)d^2}{1-(6+8/\mu)d^2\varepsilon^2}\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2 + \frac{(6+16/\mu)d^4\varepsilon^2}{1-(6+8/\mu)d^2\varepsilon^2}\|H\overset{v}{\nabla}a\|^2. \tag{6.33}$$

Estimating the quantities $\|H\overset{v}{\nabla}\overset{v}{\nabla}a\|^2$, $\|\overset{v}{\nabla}\overset{v}{\nabla}a\|^2$ and $\|\overset{v}{\nabla}a\|^2$ on the right-hand side of (6.28) with the help of (6.29), (6.32) and (6.33), we obtain

$$\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2 + (n-2)\|H\overset{v}{\nabla}a\|^2 \leq$$

$$\leq 2\varepsilon\left(3\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2 + \frac{(2+\mu)d^2}{1-(6+8/\mu)d^2\varepsilon^2}\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2 + \frac{(6+16\mu)d^4\varepsilon^2}{1-(6+8/\mu)d^2\varepsilon^2}\|H\overset{v}{\nabla}a\|^2 + d^2\|H\overset{v}{\nabla}a\|^2\right) +$$

$$+(\lambda\rho+\widetilde{\rho})\left(\frac{4+2\mu}{1-(6+8/\mu)d^2\varepsilon^2}\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2 + \frac{(12+32/\mu)d^2\varepsilon^2}{1-(6+8/\mu)d^2\varepsilon^2}\|H\overset{v}{\nabla}a\|^2\right) +$$

$$+\left((4+1/\lambda)\rho + (4n-7)\widetilde{\rho}\right)\|H\overset{v}{\nabla}a\|^2.$$

This inequality can be rewritten in the form

$$\alpha\|\overset{h}{\nabla}\overset{v}{\nabla}a\|^2 + \beta\|H\overset{v}{\nabla}a\|^2 \leq 0, \tag{6.34}$$

where

$$\alpha = \alpha(\rho,\widetilde{\rho},\lambda,\mu,\varepsilon) = 1 - \frac{(4+2\mu)(\lambda\rho+\widetilde{\rho})}{1-(6+8/\mu)d^2\varepsilon^2} - \frac{(4+2\mu)d^2\varepsilon}{1-(6+8/\mu)d^2\varepsilon^2} - 6\varepsilon$$

and

$$\beta = \beta(\rho,\widetilde{\rho},\lambda,\mu,\varepsilon) =$$

$$= n-2-(4+1/\lambda)\rho-(4n-7)\widetilde{\rho} - \frac{(12+32/\mu)d^4\varepsilon^3}{1-(6+8/\mu)d^2\varepsilon^2} - d^2\varepsilon - \frac{(\lambda\rho+\widetilde{\rho})(12+32/\mu)d^2\varepsilon^2}{1-(6+8/\mu)d^2\varepsilon^2}.$$

In order to include the two-dimensional case into the scope of our result, we rewrite inequality (6.34) in a slightly different form. To this end we introduce the semibasic η-tensor field $y = (y^{ij\alpha|\beta}) \in C^\infty\left(\beta^{(2,1,1)}_{(0,0,0)}\eta; T^0M\right)$ by the equality

$$\overset{h}{\nabla}{}^i\overset{v}{\nabla}{}^j a^{\alpha|\beta} = y^{ij\alpha|\beta} + \frac{1}{|\xi|^2}\xi^i H\overset{v}{\nabla}{}^j a^{\alpha|\beta}.$$

The summands on the right-hand side of this equality are orthogonal one to other. Therefore, for $|\xi| = 1$,

$$|\overset{h}{\nabla}\overset{v}{\nabla}a|^2 = |y|^2 + |H\overset{v}{\nabla}a|^2.$$

So, inequality (6.34) can be rewritten in the form

$$\alpha\|y\|^2 + (\alpha+\beta)\|H\overset{v}{\nabla}a\|^2 \leq 0. \tag{6.35}$$

We have to prove that both the coefficients on (6.35) are positive,

$$\alpha(\rho,\widetilde{\rho},\lambda,\mu,\varepsilon) > 0, \quad \alpha(\rho,\widetilde{\rho},\lambda,\mu,\varepsilon) + \beta(\rho,\widetilde{\rho},\lambda,\mu,\varepsilon) > 0$$

under suitable choosing the positive numbers λ, μ and ε. Since these coefficients depend continuously on ε, it suffices to prove the inequalities

$$\alpha(\rho, \tilde{\rho}, \lambda, \mu, 0) > 0, \quad \alpha(\rho, \tilde{\rho}, \lambda, \mu, 0) + \beta(\rho, \tilde{\rho}, \lambda, \mu, 0) > 0.$$

Since $\alpha(\rho, \tilde{\rho}, \lambda, \mu, 0)$ and $\beta(\rho, \tilde{\rho}, \lambda, \mu, 0)$ depend continuously on μ, it suffices to prove the inequalities

$$\alpha(\rho, \tilde{\rho}, \lambda, 0, 0) > 0, \quad \alpha(\rho, \tilde{\rho}, \lambda, 0, 0) + \beta(\rho, \tilde{\rho}, \lambda, 0, 0) > 0.$$

Substituting the values of α and β, we arrive at the system

$$4(\lambda\rho + \tilde{\rho}) < 1, \quad (4 + 4\lambda + 1/\lambda)\rho + (4n - 3)\tilde{\rho} < n - 1. \tag{6.36}$$

If this system is satisfied, then both the coefficients on (6.35) are positive for sufficiently small positive μ and ε. It is easy to check that system (6.36) can be satisfied by some positive λ under conditions (1.15) and (1.16).

So the both coefficients on (6.35) are positive, and this inequality gives

$$H\overset{v}{\nabla} a = 0.$$

The latter equation together with the boundary condition

$$\overset{v}{\nabla} a|_{\partial\Omega M} = 0$$

gives

$$\overset{v}{\nabla} a = 0.$$

As was mentioned above, this proves the theorem.

7 Proof of Theorem 1.3

The proof is very similar to the content of the previous section. Therefore our exposition here will be brief.

Let a Riemannian manifold $(M, \tilde{g})$, Hermitian bundle (η, g), connection ∇, and section $f \in C^\infty(\eta; \partial M)$ of the bundle η over the boundary ∂M satisfy hypotheses of Theorem 1.3. For a point $(x, \xi) \in T^0M$, let $\gamma = \gamma_{x,\xi} : [\tau_-(x, \xi), \tau_+(x, \xi)] \to M$ be the maximal geodesic, of the metric $\tilde{g}$, satisfying the initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$. Then the point $y = \gamma(\tau_-(x, \xi))$ belongs to the boundary ∂M. We define the vector $f(x, \xi) \in \eta_x$ as the result of the parallel transport of the vector $f(y)$ from y to x along the geodesic γ. We have thus constructed a semibasic section $f \in C^\infty(\beta^{(0,1,0)}_{(0,0,0)}\eta; T^0M)$. By the construction, it satisfies the equation

$$Hf = 0, \tag{7.1}$$

the boundary condition

$$\overset{v}{\nabla} f|_{\partial\Omega M} = 0, \tag{7.2}$$

and is positively homogeneous of zero degree in ξ

$$f(x, t\xi) = f(x, \xi) \qquad (t > 0). \tag{7.3}$$

We write the Pestov identity (5.1) for $u = \overset{v}{\nabla} f \in C^\infty(\beta^{(0,1,0)}_{(1,0,0)}\eta; T^0M)$:

$$2\mathrm{Re}\,\langle \overset{h}{\nabla}\overset{v}{\nabla} f | \overset{v}{\nabla} H \overset{v}{\nabla} f\rangle = |\overset{h}{\nabla}\overset{v}{\nabla} f|^2 + \overset{h}{\nabla}_k v^k + \overset{v}{\nabla}_k w^k - \widetilde{R}_{kplq}\xi^p\xi^q \overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^i f^{\alpha|}\cdot \overset{v}{\nabla}{}^l\overset{v}{\nabla}_i \bar{f}_{\alpha|} +$$

$$+\mathrm{Re}\left[\left(\widetilde{R}^i_{pkl}\overset{v}{\nabla}{}^p f^{\alpha|} + R^{\alpha|}_{kl\varepsilon|}\overset{v}{\nabla}{}^i f^{\varepsilon|}\right)\xi^l \overset{v}{\nabla}{}^k\overset{v}{\nabla}_i \bar{f}_{\alpha|}\right], \tag{7.4}$$

where

$$v^k = \mathrm{Re}\,\left(\xi^k \overset{h}{\nabla}{}^i\overset{v}{\nabla}_j f^{\alpha|}\cdot \overset{v}{\nabla}_i\overset{v}{\nabla}{}^j \bar{f}_{\alpha|} - \xi^i \overset{v}{\nabla}{}^k\overset{v}{\nabla}_j f^{\alpha|}\cdot \overset{h}{\nabla}_i\overset{v}{\nabla}{}^j \bar{f}_{\alpha|}\right), \tag{7.5}$$

$$w^k = \mathrm{Re}\,\left(\xi^i \overset{h}{\nabla}{}^k\overset{v}{\nabla}_j f^{\alpha|}\cdot \overset{h}{\nabla}_i\overset{v}{\nabla}{}^j \bar{f}_{\alpha|}\right). \tag{7.6}$$

By the commutation formula (6.10) and equation (7.1),

$$H\overset{v}{\nabla} f = \overset{v}{\nabla} H f - \overset{h}{\nabla} f = -\overset{h}{\nabla} f. \tag{7.7}$$

Hence

$$\overset{v}{\nabla}_i H \overset{v}{\nabla}_j f^{\alpha|} = -\overset{v}{\nabla}_i\overset{h}{\nabla}_j f^{\alpha|} = -\overset{h}{\nabla}_j\overset{v}{\nabla}_i f^{\alpha|}. \tag{7.8}$$

Therefore

$$\langle \overset{h}{\nabla}\overset{v}{\nabla} f | \overset{v}{\nabla} H \overset{v}{\nabla} f\rangle = -\overset{h}{\nabla}{}^i\overset{v}{\nabla}{}^j f^{\alpha|}\cdot \overset{h}{\nabla}_j\overset{v}{\nabla}_i \bar{f}_{\alpha|}.$$

We transform the latter expression in full analogy with the paragraph after formula (6.14) of the previous section. In such the way we obtain

$$\mathrm{Re}\,\langle \overset{h}{\nabla}\overset{v}{\nabla} f | \overset{v}{\nabla} H \overset{v}{\nabla} f\rangle = -|z|^2 - \overset{h}{\nabla}_k \tilde{v}^k -$$

$$-\mathrm{Re}\left[\overset{v}{\nabla}{}^l f^{\alpha|}\left(\widetilde{R}^p_{qkl}\xi^q \overset{v}{\nabla}_p\overset{v}{\nabla}{}^k \bar{f}_{\alpha|} - \widetilde{R}^k_{pkl}\overset{v}{\nabla}{}^p \bar{f}_{\alpha|} + R^{\varepsilon|}_{kl\alpha|}\overset{v}{\nabla}{}^k \bar{f}_{\varepsilon|}\right)\right],$$

where

$$z^{\alpha|} = \overset{h}{\nabla}_k\overset{v}{\nabla}{}^k f^{\alpha|}$$

and

$$\tilde{v}^k = \mathrm{Re}\left(\overset{v}{\nabla}{}^l f^{\alpha|}\cdot \overset{h}{\nabla}_l\overset{v}{\nabla}{}^k \bar{f}_{\alpha|} - \overset{v}{\nabla}{}^k f^{\alpha|}\cdot \overset{h}{\nabla}_l\overset{v}{\nabla}{}^l \bar{f}_{\alpha|}\right). \tag{7.9}$$

Substituting the latter value into the left-hand side of (7.4), we obtain

$$|\overset{h}{\nabla}\overset{v}{\nabla} f|^2 + 2|z|^2 = -\overset{h}{\nabla}_k(v^k + 2\tilde{v}^k) - \overset{v}{\nabla}_k w^k + \widetilde{\mathcal{R}}[f] - \mathcal{R}[f], \tag{7.10}$$

where

$$\widetilde{\mathcal{R}}[f] = \widetilde{R}_{kplq}\xi^p\xi^q \overset{v}{\nabla}{}^k\overset{v}{\nabla}{}^i f^{\alpha|}\cdot \overset{v}{\nabla}{}^l\overset{v}{\nabla}_i \bar{f}_{\alpha|} +$$

$$+\mathrm{Re}\left[2\overset{v}{\nabla}{}^l f^{\alpha|}\left(-\widetilde{R}^p_{qkl}\xi^q \overset{v}{\nabla}_p\overset{v}{\nabla}{}^k \bar{f}_{\alpha|} + \widetilde{R}^k_{pkl}\overset{v}{\nabla}{}^p \bar{f}_{\alpha|}\right) - \widetilde{R}^i_{pkl}\xi^l \overset{v}{\nabla}{}^p f^{\alpha|}\cdot \overset{v}{\nabla}{}^k\overset{v}{\nabla}_i \bar{f}_{\alpha|}\right] \tag{7.11}$$

and

$$\mathcal{R}[f] = \mathrm{Re}\left(2R^{\varepsilon|}_{kl\alpha|}\overset{v}{\nabla}{}^l f^{\alpha|}\cdot \overset{v}{\nabla}{}^k \bar{f}_{\varepsilon|} + R^{\alpha|}_{kl\varepsilon|}\xi^l \overset{v}{\nabla}{}^i f^{\varepsilon|}\cdot \overset{v}{\nabla}{}^k\overset{v}{\nabla}_i \bar{f}_{\alpha|}\right). \tag{7.12}$$

We integrate equality (7.10) over ΩM and transform the integrals of the divergent terms by the Gauss — Ostrogradskiĭ formulas

$$\|\overset{h}{\nabla}\overset{v}{\nabla}f\|^2+2\|z\|^2=-\int\limits_{\partial\Omega M}\langle v+2\tilde{v}|\nu\rangle d\Sigma^{2n-2}-(n-2)\int\limits_{\Omega M}\langle w|\xi\rangle d\Sigma+$$
$$+\int\limits_{\Omega M}\widetilde{\mathcal{R}}[f]d\Sigma-\int\limits_{\Omega M}\mathcal{R}[f]d\Sigma. \tag{7.13}$$

It follows from boundary condition (7.2) and definitions (7.5), (7.9) of the fields v and $\tilde{v}$ that $v|_{\partial\Omega M}=\tilde{v}|_{\partial\Omega M}=0$. Therefore the first integral in the right-hand side of (7.13) is equal to zero.

By (7.6) and (7.7),

$$\langle w|\xi\rangle=\mathrm{Re}\left(\xi_i\overset{h}{\nabla}{}^i\overset{v}{\nabla}_k f^{\alpha|}\cdot\xi^j\overset{h}{\nabla}_j\overset{v}{\nabla}{}^k\bar{f}_{\alpha|}\right)=|H\overset{v}{\nabla}f|^2=|\overset{h}{\nabla}f|^2.$$

Substituting this value into (7.13), we obtain

$$\|\overset{h}{\nabla}\overset{v}{\nabla}f\|^2+(n-2)\|\overset{h}{\nabla}f\|^2+2\|z\|^2=\int\limits_{\Omega M}\widetilde{\mathcal{R}}[f]d\Sigma-\int\limits_{\Omega M}\mathcal{R}[f]d\Sigma. \tag{7.14}$$

Repeating the corresponding arguments of the previous section, we obtain the following analogs of estimates (6.25) and (6.26):

$$\left|\int\limits_{\Omega M}\widetilde{\mathcal{R}}[f]\,d\Sigma\right|\le\tilde{\rho}\|H\overset{v}{\nabla}\overset{v}{\nabla}f\|^2+(4n-7)\tilde{\rho}\|H\overset{v}{\nabla}f\|^2, \tag{7.15}$$

$$\left|\int\limits_{\Omega M}\mathcal{R}[f]\,d\Sigma\right|\le\frac{1}{4}\lambda\rho\|H\overset{v}{\nabla}\overset{v}{\nabla}f\|^2+(2+1/\lambda)\rho\|H\overset{v}{\nabla}f\|^2 \tag{7.16}$$

with an arbitrary positive number λ.

It follows from (6.10) and (7.7) that

$$H\overset{v}{\nabla}_i\overset{v}{\nabla}_j f^{\alpha|}=(H\overset{v}{\nabla}_i)\overset{v}{\nabla}_j f^{\alpha|}=(\overset{v}{\nabla}_i H-\overset{h}{\nabla}_i)\overset{v}{\nabla}_j f^{\alpha|}=$$
$$=\overset{v}{\nabla}_i(H\overset{v}{\nabla}_j f^{\alpha|})-\overset{h}{\nabla}_i\overset{v}{\nabla}_j f^{\alpha|}=-\overset{v}{\nabla}_i\overset{h}{\nabla}_j f^{\alpha|}-\overset{h}{\nabla}_i\overset{v}{\nabla}_j f^{\alpha|}=-\overset{h}{\nabla}_i\overset{v}{\nabla}_j f^{\alpha|}-\overset{h}{\nabla}_j\overset{v}{\nabla}_i f^{\alpha|}.$$

Therefore

$$|H\overset{v}{\nabla}\overset{v}{\nabla}f|\le 2|\overset{h}{\nabla}\overset{v}{\nabla}f|. \tag{7.17}$$

With the help of (7.7) and (7.17), estimates (7.15) and (7.16) take the form

$$\left|\int\limits_{\Omega M}\widetilde{\mathcal{R}}[f]\,d\Sigma\right|\le 4\tilde{\rho}\|\overset{h}{\nabla}\overset{v}{\nabla}f\|^2+(4n-7)\tilde{\rho}\|\overset{h}{\nabla}f\|^2, \tag{7.18}$$

$$\left|\int\limits_{\Omega M}\mathcal{R}[f]\,d\Sigma\right|\le\lambda\rho\|\overset{h}{\nabla}\overset{v}{\nabla}f\|^2+(2+1/\lambda)\rho\|\overset{h}{\nabla}f\|^2. \tag{7.19}$$

Estimating integrals on the right-hand side of (7.14) with the help of (7.18) and (7.19), we arrive to the following analog of inequality (6.34):

$$(1-\lambda\rho-4\tilde{\rho})\|\overset{h}{\nabla}\overset{v}{\nabla}f\|^2+(n-2-(2+1/\lambda)\rho-(4n-7)\tilde{\rho})\|\overset{h}{\nabla}f\|^2\le 0. \tag{7.20}$$

We introduce the semibasic η-tensor field $y = (y^{ij\alpha|}) \in C^\infty\left(\beta^{(2,1,0)}_{(0,0,0)}\eta; T^0M\right)$ by the equality

$$\overset{h}{\nabla}{}^i\overset{v}{\nabla}{}^j f^{\alpha|} = y^{ij\alpha|} - \frac{1}{|\xi|^2}\xi^i\overset{h}{\nabla}{}^j f^{\alpha|}.$$

By (7.7), the summands on the right-hand side of this equality are orthogonal one to other. Therefore, for $|\xi| = 1$,

$$|\overset{h}{\nabla}\overset{v}{\nabla} f|^2 = |y|^2 + |\overset{h}{\nabla} f|^2.$$

So, inequality (7.20) can be rewritten in the form

$$(1 - \lambda\rho - 4\tilde\rho)\|y\|^2 + (n - 1 - (2 + \lambda + 1/\lambda)\rho - (4n - 3)\tilde\rho)\|\overset{h}{\nabla} f\|^2 \le 0. \tag{7.21}$$

Under conditions (1.15) and (1.16), we can choose $\lambda > 0$ such that both the coefficients in (7.21) are positive. In such the case (7.21) implies

$$\overset{h}{\nabla} f = 0, \quad \overset{h}{\nabla}\overset{v}{\nabla} f = 0.$$

Together with boundary condition (7.2), these equations give

$$\overset{h}{\nabla} f = 0, \quad \overset{v}{\nabla} f = 0.$$

These equalities mean that the section f is basic and absolutely parallel. The theorem is proved.

8 The Uhlmann ray transform

Let $(M, \tilde g)$ be a simple Riemannian manifold, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric.

We represent the boundary $\partial\Omega M$ of the manifold ΩM as the union of the two submanifolds

$$\partial_\pm\Omega M = \{(x,\xi) \in \Omega M \mid x \in \partial M, \pm\langle\xi, \nu(x)\rangle \ge 0\}$$

of outer and inner vectors; here $\nu(x)$ is the unit normal to the boundary ∂M. Let $p : \partial_+\Omega M \to \partial M$ be the restriction of the projection of the tangent bundle. By $\pi^{(0,1,0)}_{(1,1,0)}\eta$ we denote the bundle over $\partial_+\Omega M$ which is defined by the equality

$$\pi^{(0,1,0)}_{(1,1,0)}\eta = p^*(\tau^{(0,1,0)}_{(1,1,0)}\eta).$$

The *Uhlmann ray transform* is the linear operator

$$U : C^\infty(\tau^{(0,1,0)}_{(1,1,0)}\eta) \to C^\infty(\pi^{(0,1,0)}_{(0,1,0)}\eta) \tag{8.1}$$

defined by the formula

$$(Uf)(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} I^t\langle f(\gamma(t)), \dot\gamma(t)\rangle\, dt \qquad ((x,\xi) \in \partial_+\Omega M), \tag{8.2}$$

where $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is the maximal geodesic of the metric $\tilde{g}$ satisfying the initial condition $\gamma(0) = x$ and $\dot{\gamma}(0) = \xi$, the angle brackets

$$\langle \cdot, \xi\rangle : \tau^{(0,1,0)}_{(1,1,0)}\eta \to \tau^{(0,1,0)}_{(0,1,0)}\eta, \qquad \langle f, \xi\rangle^{\alpha|}_{\beta|} = \xi^i f^{\alpha|}_{i\beta|}$$

mean the contraction with a vector $\xi \in TM$, and I^t is the parallel transport along γ from the point $\gamma(t)$ to the point $x = \gamma(0)$ with respect to the connection ∇.

In coordinate form the Uhlmann ray transform can be written as follows. Let us choose a parallel along γ basis $(e_1(t), \dots, e_m(t))$ of the bundle η. In this basis

$$(Uf)^{\alpha|}_{\beta|} = \int\limits_{\tau_-(x,\xi)}^{0} \dot{\gamma}^i(t) f^{\alpha|}_{i\beta|}(\gamma(t))dt. \tag{8.3}$$

Of course, the similar operator

$$U : C^\infty(\tau^{(r,\rho,\lambda)}_{(s+1,\sigma,\mu)}\eta) \to C^\infty(\pi^{(r,s,\lambda)}_{(s,\sigma,\mu)}\eta)$$

can be defined for arbitrary $(r, \rho, \lambda; s, \sigma, \mu)$. Nevertheless we restrict ourselves to considering operator (8.1) because the problem of inverting this operator is a natural linearization of the above-considered problem of determining a connection, as we shall show.

A smooth one-parameter family ∇^τ $(-\varepsilon < \tau < \varepsilon)$ of connections on η is called the *deformation* of the connection ∇ if $\nabla^0 = \nabla$. For such a deformation, the derivative

$$f = \frac{d\nabla^\tau}{d\tau}\bigg|_{\tau=0}$$

is a well defined η-tensor field $f \in C^\infty(\tau^{(0,1,0)}_{(1,1,0)}\eta)$ which is called the *tangent field* of the deformation ∇^τ. A deformation is called *trivial* if there exists a one-parameter family of automorphisms $a^\tau : \eta \to \eta$ of the bundle η such that $a^\tau|_{\partial M} = \mathrm{Id}$ and $\nabla^\tau = (a^\tau)^*\nabla$. One can easily check that the tangent field f of a trivial deformation $\nabla^\tau = (a^\tau)^*\nabla$ satisfies the equation

$$\nabla v = f \tag{8.4}$$

with the η-tensor field $v = da^\tau/d\tau|_{\tau=0} \in C^\infty(\tau^{(0,1,0)}_{(0,1,0)}\eta)$ meeting the boundary condition

$$v|_{\partial M} = 0. \tag{8.5}$$

An η-tensor field f is called *potential* if the boundary value problem (8.4)–(8.5) has a solution. So, potential η-tensor fields of degree $(0, 1, 0; 0, 1, 0)$ are just tangent fields of trivial deformations of the connection ∇.

Let ∇^τ be a deformation of a connection ∇. Given a point $(x,\xi) \in \partial_+\Omega M$, we put $y = \gamma_{x,\xi}(\tau_-(x,\xi))$ and denote by $I^\tau_{x,\xi} : \eta_x \to \eta_y$ the parallel transport from x to y along $\gamma_{x,\xi}$ with respect to the connection ∇^τ. Then the derivative

$$J_{x,\xi} = \frac{dI^\tau_{x,\xi}}{d\tau}\bigg|_{\tau=0} : \eta_x \to \eta_y$$

is a linear operator smoothly dependendig on $(x,\xi) \in \partial_+\Omega M$. We will show that this operator determines the Uhlmann ray transform of the tangent field f of the deformation

∇^τ. Therefore the problem of inverting the Uhlmann ray transform turns out to be a linearization of our inverse problem of determining a connection.

Choosing a coordinate system and trivialization, we write the equation for the parallel transport along $\gamma = \gamma_{x,\xi}$ with respect to the connection ∇^τ

$$\dot{w}^{\alpha|} + \dot{\gamma}^i(t)\overset{\tau}{\Gamma}{}^{\alpha}_{i\beta}(\gamma(t))w^{\beta|} = 0, \tag{8.6}$$

where $\overset{\tau}{\Gamma}{}^{\alpha}_{i\beta}$ are the Christoffel symbols of the connection ∇^τ. Let us fix a vector $w_0 = (w_0^{\alpha|}) \in \eta_x$ and denote by $w(t,\tau) = (w^{\alpha|}(t,\tau))$ the solution to equation (8.6) meeting the initial condition

$$w(0,\tau) = w_0. \tag{8.7}$$

Then, by the definition of the operator $J_{x,\xi}$,

$$\frac{\partial w}{\partial \tau}(\tau_-(x,\xi),0) = J_{x,\xi}w_0. \tag{8.8}$$

Differentiating equation (8.6) with respect to τ and putting then $\tau = 0$, we obtain

$$\dot{u}^\alpha + \dot{\gamma}^i\Gamma^\alpha_{i\beta}u^{\beta|} = -\dot{\gamma}^i f^{\alpha|}_{i\beta|}w^{\beta|},$$

where $u = \partial w/\partial\tau|_{\tau=0}$, $\Gamma^\alpha_{i\beta}$ are the Christoffel symbols of the connection ∇, f is the tangent field of the deformation ∇^τ, and $w = w(t) = w(t,0)$ is a vector field parallel along γ with respect to ∇. This equation can be written in covariant form

$$\left(\frac{Du}{dt}\right)^{\alpha|} = -\dot{\gamma}^i f^{\alpha|}_{i\beta|}w^{\beta|}, \tag{8.9}$$

where $D/dt = \dot{\gamma}^i\nabla_i$ is the total derivative along γ. By (8.7) and (8.8),

$$u(0) = 0, \qquad u(\tau_-(x,\xi)) = J_{x,\xi}w_0. \tag{8.10}$$

Let us choose a parallel along γ basis of the bundle η and write down equation (8.9) with respect to the basis

$$\dot{u}^{\alpha|} = -\dot{\gamma}^i f^{\alpha|}_{i\beta|}w^{\beta|}. \tag{8.11}$$

The vector field $w = (w^{\beta|})$ is parallel along γ, i.e., $w^{\beta|} = \mathrm{const}$ in the chosen basis. Relations (8.10)–(8.11) imply

$$(J_{x,\xi}w)^{\alpha|} = u^{\alpha|}(\tau_-(x,\xi)) - u^{\alpha|}(0) = w^{\beta|}\int\limits_{\tau_-(x,\xi)}^{0}\dot{\gamma}^i f^{\alpha|}_{i\beta|}\,dt.$$

Comparing the latter equality with (8.3), we see that

$$J_{x,\xi} = Uf(x,\xi).$$

Therefore the problem of inverting the Uhlmann ray transform is just the linearization of the above-considered inverse problem of determining a connection.

The space of potential fields is a subspace of the kernel Ker$U$ of the Uhlmann ray transform. Indeed, if $f = \nabla v$, then the integrand on (8.3) is equal to $d(v^{\alpha|}_{\beta|}(\gamma(t))/dt$. Therefore $U(\nabla v) = 0$ if v vanishes on the boundary.

We will use the Sobolev spaces $H^k(\alpha)$ of sections of a vector bundle α over a compact manifold. The Uhlmann ray transform (8.1) has the bounded extension

$$U : H^k(\tau^{(0,1,0)}_{(1,1,0)}\eta) \to H^k(\pi^{(0,1,0)}_{(0,1,0)}\eta) \tag{8.12}$$

for every $k \ge 0$. This fact is proved quite similarly to Theorem 4.2.1 of [Sh].

In order to specify the problem of inverting the Uhlmann ray transform, we introduce the notion of potential and solenoidal parts of an η-tensor field. This is done by the following

Theorem 8.1 *Let $(M, \widetilde{g})$ be a compact Riemannian manifold with boundary, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric. For every η-tensor field $f \in H^k(\tau^{(0,1,0)}_{(1,1,0)}\eta)$ $(k \ge 1)$, there exist uniquely determined $\widetilde{f} \in H^k(\tau^{(0,1,0)}_{(1,1,0)}\eta)$ and $v \in H^{k+1}(\tau^{(0,1,0)}_{(0,1,0)}\eta)$ such that*

$$f = \widetilde{f} + \nabla v, \qquad \nabla^i \widetilde{f}^{\alpha|}_{i\beta|} = 0, \qquad v|_{\partial M} = 0. \tag{8.13}$$

The estimates

$$\|\widetilde{f}\|_{H^k} \le C\|f\|_{H^k}, \quad \|v\|_{H^{k+1}} \le C\|f\|_{H^k}$$

hold with a constant C independent of f. In particular, $\widetilde{f}$ and v are smooth if f is smooth.

We call $\widetilde{f}$ and ∇v the *solenoidal* and *potential* parts of the field f. The proof is omitted because it is quite similar to the proof of Theorem 3.3.2 of [Sh].

As was mentioned above, the Uhlmann ray transform does not pay attention to the potential part. So, given Uf, one can hope to recover at most the solenoidal part $\widetilde{f}$ of the field f. We will prove that the recovering is possible under some conditions on the manifold and connection.

Let $(M, \widetilde{g})$ be a Riemannian manifold. For a point $x \in M$ and a two-dimensional subspace $\sigma \subset T_xM$, by $K(x, \sigma)$ we denote the sectional curvature at the point x in the two-dimensional direction σ. For $(x, \xi) \in \Omega M$, we put

$$K(x, \xi) = \sup_{\sigma \ni \xi} K(x, \sigma), \qquad K^+(x, \xi) = \max\{0, K(x, \xi)\}. \tag{8.14}$$

For a simple Riemannian manifold $(M, \widetilde{g})$, we introduce the following characteristic:

$$k^+(M, \widetilde{g}) = \sup_{(x,\xi)\in\partial_-\Omega M} \int\limits_0^{\tau_+(x,\xi)} tK^+(\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t))dt. \tag{8.15}$$

We recall that here $\gamma_{x,\xi} : [0, \tau_+(x, \xi)] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \xi$. In particular $k^+(M, \widetilde{g}) = 0$ if the sectional curvature is nonpositive.

We can now formulate our main result on the Uhlmann ray transform.

Theorem 8.2 *Let $(M, \widetilde{g})$ be a simple n-dimensional Riemannian manifold, and (η, g) be a Hermitian vector bundle over M endowed with a connection ∇ compatible with the metric. Assume quantities (1.13) and (8.15) to satisfy the inequalities*

$$\rho(M, \widetilde{g}, \eta, g, \nabla) < \frac{1}{4}\sqrt{n/2}, \tag{8.16}$$

$$k^+(M,\tilde g) \le 1/4. \tag{8.17}$$

For every η-tensor field $f \in H^1(\tau^{(0,1,0)}_{(1,1,0)}\eta)$, the solenoidal part $\tilde f$ is uniquely determined by the Uhlmann ray transform Uf and the conditional stability estimate

$$\|\tilde f\|^2_{L_2} \le C\left(\|f\|_{H^1}\cdot\|Uf\|_{L_2} + \|Uf\|^2_{H^1}\right) \tag{8.18}$$

holds with a constant C independent of f.

The remarks given after statement of Theorem 4.3.3 of [Sh] are valid for this theorem too.

Let us now discuss the case of a flat connection ∇ on the trivial vector bundle $\eta = M\times \mathbf{C}^m$ over a simple Riemannian manifold $(M,\tilde g)$. In this case the Uhlmann ray transform can be reduced to the (longitudinal) ray transform of a vector field

$$I : C^\infty(\tau_M) \to C^\infty(\partial_+\Omega M),\quad (If)(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} \dot\gamma^i_{x,\xi}(t) f_i(\gamma_{x,\xi}(t))dt \tag{8.19}$$

that was considered in Chapter 4 of [Sh]. Indeed, we can choose a global trivialization of the bundle η consisting of orthonormal and absolutely parallel bases. In formula (8.3) being written with respect to the trivialization, the components of a field f are independent of the geodesic γ. Therefore we can fix indices α and β and define the vector field $f = (f_i) \in C^\infty(\tau_M)$ by putting $f_i = f^{\alpha|}_{i\beta|}$. After this definition, formulas (8.3) and (8.19) coincide. In this case Theorem 8.2 coincides with Theorem 4.3.3 of [Sh] with the only exception that condition (8.17) can be replaced with the following weaker inequality: $k^+(M,\tilde g) < 1/2$.

Finally, consider the case when ∇ is a flat connection, M is a domain in Euclidean space, and the metric $\tilde g$ coincides with the Euclidean one. In this case there is the explicit inversion formula that expresses the solenoidal part $\tilde f$ of a field f through the Uhlmann ray transform Uf. See Theorem 2.12.2 and Section 2.14 of [Sh].

9 The proof of Theorem 8.2

Theorem 8.2 can be reduced to the following special case.

Lemma 9.1 *Let hypotheses of Theorem 8.2 be satisfied. For an η-tensor field $f \in C^\infty(\tau^{(0,1,0)}_{(1,1,0)}\eta)$ satisfying the condition*

$$\nabla^i f^{\alpha|}_{i\beta|} = 0, \tag{9.1}$$

the estimate

$$\|f\|^2_{L_2} \le C\left(\|j_\nu f|_{\partial M}\|_{L_2}\cdot\|Uf\|_{L_2} + \|Uf\|^2_{H^1}\right) \tag{9.2}$$

holds with a constant C independent of f. Here j_ν is the contraction with the unit vector ν of outer normal to the boundary: $(j_\nu f)^{\alpha|}_{\beta|} = \nu^i f^{\alpha|}_{i\beta|}$.

The lemma implies Theorem 8.2 by the arguments quite similar to those presented after statement of Lemma 4.3.4 of [Sh].

Proof of Lemma 8.1. We define the semibasic η-tensor field $u \in C^\infty(\beta^{(0,1,1)}_{(0,0,0)}\eta; T^0M)$ on T^0M by the formula

$$u(x,\xi) = \int\limits_{\tau_-(x,\xi)}^{0} I^t\langle f(\gamma(t)), \dot\gamma(t)\rangle, \tag{9.3}$$

where $\gamma = \gamma_{x,\xi} : [\tau_-(x,\xi), 0] \to M$ is the maximal geodesic satisfying the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = \xi$. The difference between formulas (8.2) and (9.3) is that (8.2) is considered only for $(x,\xi) \in \partial_+\Omega M$ while (9.3), for all $(x,\xi) \in T^0M$.

The field u satisfies the differential equation

$$Hu^{\alpha|\beta} = \xi^i f_i^{\alpha|\beta}, \tag{9.4}$$

the boundary conditions

$$u|_{\partial_-\Omega M} = 0 \tag{9.5}$$

and

$$u|_{\partial_+\Omega M} = Uf, \tag{9.6}$$

and is positively homogeneous of zero degree in the second argument

$$u(x,t\xi) = u(x,\xi) \qquad (t>0). \tag{9.7}$$

We write the Pestov identity for the field u

$$2\mathrm{Re}\,\langle \overset{h}{\nabla} u | \overset{v}{\nabla} Hu\rangle = |\overset{h}{\nabla} u|^2 + \overset{h}{\nabla}_k v^k + \overset{v}{\nabla}_k w^k - \widetilde R_{kplq}\xi^p\xi^q \overset{v}{\nabla}{}^k u^{\alpha|\beta}\cdot \overset{v}{\nabla}{}^l \bar u_{\alpha|\beta} +$$

$$+\mathrm{Re}\left[\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar R^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l \overset{v}{\nabla}{}^k \bar u_{\alpha|\beta}\right], \tag{9.8}$$

where

$$v^k = \mathrm{Re}\left(\xi^k \overset{h}{\nabla}{}^l u^{\alpha|\beta}\cdot \overset{v}{\nabla}_l \bar u_{\alpha|\beta} - \xi^l \overset{v}{\nabla}{}^k u^{\alpha|\beta}\cdot \overset{h}{\nabla}_l \bar u_{\alpha|\beta}\right), \tag{9.9}$$

$$w^k = \mathrm{Re}\left(\xi^l \overset{h}{\nabla}{}^k u^{\alpha|\beta}\cdot \overset{h}{\nabla}_l \bar u_{\alpha|\beta}\right). \tag{9.10}$$

Let us show that the left-hand side of (9.8) is of divergent form. Indeed, by (9.4)

$$\langle \overset{h}{\nabla} u | \overset{v}{\nabla} Hu\rangle = \overset{h}{\nabla}{}^k u^{\alpha|\beta}\cdot \overset{v}{\nabla}_k(\xi^i \bar f_{i\alpha|\beta}) = \overset{h}{\nabla}{}^k u^{\alpha|\beta}\cdot \bar f_{k\alpha|\beta} = \overset{h}{\nabla}{}^k(u^{\alpha|\beta}\bar f_{k\alpha|\beta}) - u^{\alpha|\beta}\nabla^k \bar f_{k\alpha|\beta}.$$

By (9.1), the last term on the right-hand side is equal to zero; and we obtain

$$\mathrm{Re}\,\langle \overset{h}{\nabla} u | \overset{v}{\nabla} Hu\rangle = \overset{h}{\nabla}_k \tilde v^k, \tag{9.11}$$

where

$$\tilde v^k = \mathrm{Re}\left(u^{\alpha|\beta}\bar f^k_{\alpha|\beta}\right). \tag{9.12}$$

We substitute the value (9.11) into the left-hand side of the Pestov Identity (9.8)

$$|\overset{h}{\nabla}u|^2 = \overset{h}{\nabla}_k(2\widetilde{v}^k - v^k) - \overset{v}{\nabla}_k w^k + \widetilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{\alpha|\beta} \cdot \overset{v}{\nabla}{}^l \bar{u}_{\alpha|\beta} -$$
$$-\mathrm{Re}\left[\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{\alpha|\beta}\right].$$

Integrating this equality over ΩM and transforming the integrals of the divergent terms by Gauss — Ostrogradskiĭ, we obtain

$$\|\overset{h}{\nabla}u\|^2 = \int\limits_{\partial\Omega M}\langle 2\widetilde{v} - v, \nu\rangle d\Sigma^{2n-2} - n\int\limits_{\Omega M}\langle w, \xi\rangle d\Sigma + \int\limits_{\Omega M}\widetilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{\alpha|\beta} \cdot \overset{v}{\nabla}{}^l\bar{u}_{\alpha|\beta}\, d\Sigma -$$
$$-\mathrm{Re}\int\limits_{\Omega M}\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{\alpha|\beta}\, d\Sigma.$$

By (9.10), $\langle w, \xi\rangle = |Hu|^2$, and our equality takes the form

$$\|\overset{h}{\nabla}u\|^2 + n\|Hu\|^2 = \int\limits_{\partial\Omega M}\langle 2\widetilde{v} - v, \nu\rangle d\Sigma^{2n-2} + \int\limits_{\Omega M}\widetilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{\alpha|\beta} \cdot \overset{v}{\nabla}{}^l\bar{u}_{\alpha|\beta}\, d\Sigma -$$
$$-\mathrm{Re}\int\limits_{\Omega M}\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{\alpha|\beta}\, d\Sigma. \tag{9.13}$$

We will estimate each of the right-hand side integrals on (9.13).

We start with the third integral. The module of its integrand is estimated by the quantity

$$2|R| \cdot |u| \cdot |\overset{v}{\nabla}u| \le |R|\left(\lambda|\overset{v}{\nabla}u|^2 + \frac{1}{\lambda}|u|^2\right)$$

with an arbitrary positive number λ. Therefore

$$\left|\int\limits_{\Omega M}\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{\alpha|\beta}\, d\Sigma\right| \le \int\limits_{\Omega M}|R|\left(\lambda|\overset{v}{\nabla}u|^2 + \frac{1}{\lambda}|u|^2\right)d\Sigma. \tag{9.14}$$

By (9.5), both the fields u and $\overset{v}{\nabla}u$ vanish on $\partial_-\Omega M$, and the Poincaré inequality can be applied to these fields. Estimating the right-hand side integrals on (9.14) with the help of the Poincaré inequality, we obtain

$$\left|\int\limits_{\Omega M}\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{\alpha|\beta}\, d\Sigma\right| \le \rho\left(\lambda\|H\overset{v}{\nabla}u\|^2 + \frac{1}{\lambda}\|Hu\|^2\right), \tag{9.15}$$

where $\rho = \rho(M, \widetilde{g}, \eta, g, \nabla)$ is defined by formula (1.13).

The next step is estimating the norm $\|H\overset{v}{\nabla}u\|$ through $\|\overset{h}{\nabla}u\|$ and $\|Hu\|$. Applying the operator $\overset{v}{\nabla}$ to equation (9.4) and using commutation formula (6.10), we obtain (remember that f is independent of ξ)

$$H\overset{v}{\nabla}u = f - \overset{h}{\nabla}u.$$

Therefore

$$|H\overset{v}{\nabla}u|^2 = |\overset{h}{\nabla}u|^2 + |f|^2 - 2\mathrm{Re}\langle\overset{h}{\nabla}u, f\rangle \le 2|\overset{h}{\nabla}u|^2 + 2|f|^2.$$

Integrating this inequality, we obtain

$$\|H\overset{v}{\nabla}u\|^2 \le 2\|\overset{h}{\nabla}u\|^2 + 2\|f\|^2. \tag{9.16}$$

On the other hand, the norm $\|f\|$ can be easily expressed through $\|Hu\|$. Indeed, squaring equation (9.4) and integrating the result, we obtain

$$\|Hu\|^2 = \int\limits_M f_i^{\alpha|\beta}\bar{f}_{j\alpha|\beta}\left(\int\limits_{\Omega_x M}\xi^i\xi^j\,d\omega_x(\xi)\right)dV^n(x) = \frac{1}{n}\|f\|^2. \tag{9.17}$$

With the help of the latter equality, (9.16) gives

$$\|H\overset{v}{\nabla}u\|^2 \le 2\|\overset{h}{\nabla}u\|^2 + 2n\|Hu\|^2. \tag{9.18}$$

Combining estimates (9.15) and (9.18), we obtain

$$\left|\int\limits_{\Omega M}\left(R^{\alpha|}_{kl\varepsilon|}u^{\varepsilon|\beta} + \bar{R}^{|\beta}_{kl|\varepsilon}u^{\alpha|\varepsilon}\right)\xi^l\overset{v}{\nabla}{}^k\bar{u}_{\alpha|\beta}\,d\Sigma\right| \le 2\lambda\rho\|\overset{h}{\nabla}u\|^2 + (2n\lambda + 1/\lambda)\,\rho\|Hu\|^2. \tag{9.19}$$

We now estimate the second integral on the right-hand side of (9.13). Its integrand admits the estimate

$$\widetilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\bar{u}_{\alpha|\beta}\,d\Sigma \le K^+(x,\xi)|\overset{v}{\nabla}u|^2,$$

where $K^+(x,\xi)$ is defined by formula (8.14). Integrating this estimate and using the Poincaré inequality, we obtain

$$\int\limits_{\Omega M}\widetilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\bar{u}_{\alpha|\beta}\,d\Sigma \le k^+\|H\overset{v}{\nabla}u\|^2$$

with $k^+ = k^+(M,\tilde{g})$ defined by (8.15). Combining the latter estimate with (9.18), we obtain

$$\int\limits_{\Omega M}\widetilde{R}_{kplq}\xi^p\xi^q\overset{v}{\nabla}{}^k u^{\alpha|\beta}\cdot\overset{v}{\nabla}{}^l\bar{u}_{\alpha|\beta}\,d\Sigma \le 2k^+\|\overset{h}{\nabla}u\|^2 + 2nk^+\|Hu\|^2. \tag{9.20}$$

The first integral on the right-hand side of (9.13) can be estimated as follows:

$$\left|\int\limits_{\partial\Omega M}\langle 2\tilde{v} - v, \nu\rangle d\Sigma^{2n-2}\right| \le C\left(\|j_\nu f|_{\partial M}\|_{L_2}\cdot\|Uf\|_{L_2} + \|Uf\|^2_{H^1}\right). \tag{9.21}$$

This estimate can be proved, on the base of boundary conditions (9.5)–(9.6) and definitions (9.9) and (9.12), in full analogy with the arguments exposed at the beginning of Section 4.7 of [Sh]; we do not reproduce these arguments here.

Estimating the right-hand side integrals on (9.13) with the help of (9.19)–(9.21), we arrive at the inequality

$$\alpha\|\overset{h}{\nabla}u\|^2 + \beta\|Hu\|^2 \le C\left(\|j_\nu f|_{\partial M}\|_{L_2}\cdot\|Uf\|_{L_2} + \|Uf\|^2_{H^1}\right) \tag{9.22}$$

with the coefficients

$$\alpha = \alpha(\rho, k^+, \lambda) = 1 - 2\lambda\rho - 2k^+,$$

$$\beta = \beta(n, \rho, k^+, \lambda) = n - (2n\lambda + 1/\lambda)\rho - 2nk^+.$$

Under conditions (8.16) and (8.17), there exists a positive λ such that

$$\alpha \geq 0, \qquad \beta > 0.$$

In such the case (9.22) implies the inequality

$$\|Hu\|^2 \leq C' \left(\|j_\nu f|_{\partial M}\|_{L_2} \cdot \|Uf\|_{L_2} + \|Uf\|^2_{H^1} \right). \tag{9.23}$$

Finally, (9.17) and (9.23) imply estimate (9.2). The lemma is proved.

References

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