COMPLEX VECTOR BUNDLES AND JACOBI FORMS

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Abstract. The elliptic genus (EG) of a compact complex manifold was introduced as a holomorphic Euler characteristic of some formal power series with vector bundle coefficients. EG is an automorphic form in two variables only if the manifold is a Calabi–Yau manifold. In physics such a function appears as the partition function of $N = 2$ superconformal field theories. In these notes we define the modified Witten genus or the automorphic correction of elliptic genus. It is an automorphic function in two variables for an arbitrary holomorphic vector bundle over a compact complex manifold. This paper is an exposition of the talks given by the author at Symposium "Automorphic forms and L-functions" at RIMS, Kyoto (January, 27, 1999) and at Arbeitstagung in Bonn (June, 20, 1999).

Introduction

In these notes we present a link between the theory of automorphic forms and geometry. For an arbitrary compact spin manifold one can define its elliptic genus. It is a modular form in one variable with respect to a congruence subgroup of level 2 (see, for example, [W1], [L], [HBJ]). For a compact complex manifold one can define its elliptic genus $\phi(M;\tau,z)$ as a function in two complex variables (see, for example, [W2], [EOTY], [Hö], [FY], [KYY]). In the last case the elliptic genus is the holomorphic Euler characteristic of a formal power series with vector bundle coefficients. If the first Chern class $c_1(M)$ of the complex manifold is equal to zero in $H^2(M,\mathbb{R})$, then the elliptic genus is a weak Jacobi modular form (with integral Fourier coefficients) of weight 0 and index $d/2$, where $d = \dim_{\mathbb{C}}(M)$. The same modular form appears in physic as the partition function of $N = 2$ super-symmetric sigma model whose target space is the given Calabi–Yau manifold. We note that any "good" partition function has appeared in physic is an automorphic form with respect to some group. This reflects the fact that physical models have some additional symmetries. If $c_1(M) \neq 0$, then the elliptic genus $\phi(M;\tau,z)$ of M is not an automorphic form. In these notes we define *the modified Witten genus* (MWG) or *the automorphic correction of elliptic genus* of an arbitrary holomorphic vector bundle over a compact complex manifold and we briefly study its properties. This new object is always an automorphic form in two variables.

We are going to present here automorphic aspects of the theory. In the proof of the theorem that the modified Witten genus is a Jacobi form we use a nice formula which relates the Jacobi theta-series, its logarithmic derivative, the quasi-modular Eisenstein series $G_2(\tau)$ and all the derivatives of the Weierstrass $\wp$-function (see Proposition 1.4

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bellow). The fact that MWG of a vector bundle is a Jacobi modular form has a number of applications in the theory of vector bundles. Among them we have relations between $\hat{A}$-genera and other cohomological invariants. One can get from this some obstructions if the rank of the vector bundle is smaller that the dimension of the manifold. The special values of the MWG have also some nice arithmetic properties.

One can see from the definition that the coefficients of MWG are rationals with bounded denominators. To get applications to the theory of complex manifolds we study the $\mathbb{Z}$-structure of the bigraded ring

$$J^{w,\mathbb{Z}}_{*,*/2} = \bigoplus_{k\in\mathbb{Z},\, m\in\frac{1}{2}\mathbb{Z}} J^{w,\mathbb{Z}}_{k,m}$$

of all weak Jacobi forms of all weights and indices with integral Fourier coefficients. We prove that this ring has 14 generators over $\mathbb{Z}$ (see Theorem 2.3):

$$J^{w,\mathbb{Z}}_{*,*/2} = \mathbb{Z}[E_4, E_6, \Delta, E_{4,1}, E_{4,2}, E_{4,3}, E_{6,1}, E_{6,2}, E'_{6,3}, \phi_{0,1}, \phi_{0,\frac{3}{2}}, \phi_{0,2}, \phi_{0,4}, \phi_{-1,\frac{1}{2}}],$$

where E_4, E_6 and Δ are $SL_2(\mathbb{Z})$-generators, $E_{4,1}, \dots E_{6,3}$ are the Eisenstein–Jacobi series and $E'_{6,3}$ is a modification of $E_{6,3}$. The four functions $\phi_{0,1}, \dots, \phi_{0,4}$ of weight 0 generate the graded $\mathbb{Z}$-ring $J^{w,\mathbb{Z}}_{0,*/2}$ of the Jacobi forms of weight 0 with integral Fourier coefficients. These Jacobi forms are related to Calabi–Yau manifolds of dimension $d = 2,\ 3,\ 4,\ 8$. The Jacobi forms $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3} = \phi^2_{0,3/2}$ are the generating functions for the multiplicities of positive roots of the three main generalized Lorentzian Kac–Moody Lie algebras of Borcherds type constructed in [GN1–GN4] (see also §4 of this paper). The form $\phi_{0,4}$ determines the "most odd" even Siegel theta-constant. (see [G4, (2.11)]).

The q^0-term of the Fourier expansion ($q = e^{2\pi i\tau}$) of the elliptic genus $\phi(M;\tau,z)$ is essentially equal to the Hirzebruch χ_y-genus of the manifold. Thus we can analyze the arithmetic properties of the χ_y-genus of the complex manifold with $c_1(M) = 0$ and its special values such as the signature ($y = 1$) and the Euler number ($y = -1$) in terms of Jacobi forms. For example, we prove that the Euler number of a Calabi-Yau manifold M_d of dimension d satisfies

$$e(M_d) \equiv 0 \mod 8 \qquad \text{if} \quad d \equiv 2 \mod 8$$

(see Proposition 2.6). The special values of the generators of the Jacobi ring at $z = \frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ are related to Hauptmoduls of the fields of modular functions. Using this fact we prove that

$$\chi_{y=\zeta_3}(M_d) \equiv 0 \mod 9 \qquad \text{if} \quad d \equiv 2 \mod 6$$

(see Proposition 2.7). The special values of MWG are related with $\hat{A}$-genera. Some other constructions (for example, $\hat{A}^{(2)}_2$-genus, the second quantized elliptic genus) and other applications to the theory of vector bundles one can find in my course of lectures given at RIMS, Kyoto University, at our joint seminar with K. Saito in 1998. I would like to take this opportunity to express my gratitude to all members of K. Saito's seminar.

§1. Automorphic correction of elliptic genus

Let M be a complex compact manifold M of (complex) dimension d and let E be a complex vector bundle over M. Let us fix two formal variables $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi iz)$, where $\tau \in \mathbb{H}_1$ (the upper half-plane) and $z \in \mathbb{C}$. One defines a formal power series $\mathbf{E}_{q,y} \in K(M)[[q, y^{\pm 1}]]$

$$\mathbf{E}_{q,y} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-y^{-1}q^n} E^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-yq^n} E \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_M^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_M, \tag{1.1}$$

where T_M denotes the holomorphic tangent bundle of M and

$$\bigwedge\nolimits_x E = \sum_{k\geq 0} (\wedge^k E) x^k, \qquad S_x E = \sum_{k\geq 0} (S^k E) x^k$$

are formal power series with exterior powers and symmetric powers of a bundle E as coefficients. We propose the following

Definition 1.1. *Modified Witten genus* (MWG) of a complex vector bundle E of rank r over a compact complex manifold M of dimension d is defined as follows

$$\chi(M, E; \tau, z) = q^{(r-d)/12} y^{r/2} \int_M \exp\Bigl(\frac{1}{2}\bigl(c_1(E) - c_1(T_M)\bigr)\Bigr)\cdot$$

$$\exp\Bigl(\bigl(p_1(E) - p_1(T_M)\bigr)\cdot G_2(\tau)\Bigr) \exp\Bigl(-\frac{c_1(E)}{2\pi i}\frac{\vartheta_z(\tau,z)}{\vartheta(\tau,z)}\Bigr) \operatorname{ch}(\mathbf{E}_{q,y}) \operatorname{td}(T_M),$$

where $c_1(E)$ and $p_1(E)$ are the first Chern and Pontryagin class of E, td is the Todd class, $\operatorname{ch}(\mathbf{E}_{q,y})$ is the Chern character applied to each coefficient of the formal power series and the integral $\int_M$ denotes the evaluation of the top degree differential form on the fundamental cycle of the manifold.

In the definition we use Jacobi theta-series of level two $\vartheta(\tau, z) = -i\vartheta_{11}(\tau, z)$:

$$\vartheta(\tau, z) = \sum_{n\equiv 1 \, mod\, 2} (-1)^{\frac{n-1}{2}} q^{\frac{n^2}{8}} y^{\frac{n}{2}} = -q^{1/8} y^{-1/2} \prod_{n\geq 1} (1 - q^{n-1}y)(1 - q^n y^{-1})(1 - q^n),$$

$\vartheta_z(\tau, z) = \dfrac{\partial\vartheta}{\partial z}(\tau, z)$ and $G_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)\, q^n$ is a quasi-modular Eisenstein series of weight 2, where $\sigma_1(n)$ is the sum of all positive divisors of n.

1.2. Witten genus. As a particular case of the definition above one obtains the Witten genus (see [W1], [W2], [L], [HBJ]). Let assume that M admits a spin structure (i.e., the second Witney-Stiefel class $w_2(M)$ is zero or $c_1(T_M) \equiv 0 \mod 2$) and $p_1(M) = 0$. Let $E = \mathbb{C}^r$ be the trivial vector bundle of rank r over M. Then $\operatorname{ch}(\bigwedge_x E) = (1 + x)^r$ and

$$q^{r/12} y^{r/2} \operatorname{ch}\Bigl(\bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-y^{-1}q^n} E^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-yq^n} E\Bigr) = \Bigl(\frac{\vartheta(\tau,z)}{\eta(\tau)}\Bigr)^r.$$

Thus

$$q^{d/12}\chi(M,\mathbb{C}^r;\tau,z)=\frac{\vartheta(\tau,z)^r}{\eta(\tau)^r}\int_M\prod_{i=1}^{d}\frac{x_i/2}{\sinh(x_i/2)}\prod_{n=1}^{\infty}\frac{1}{(1-q^ne^{x_i})(1-q^ne^{-x_i})}$$
$$=\hat{A}\big(M,\bigotimes_{n=1}^{\infty}S_{q^n}(T_M\oplus T_M{}^*)\big)\frac{\vartheta(\tau,z)^r}{\eta(\tau)^r}=\text{Witten genus}\,(M)\frac{\vartheta(\tau,z)^r}{\eta(\tau)^{r+2d}}.$$

If we take the trivial vector bundle of rank 0, then

$$\chi(M,0;\tau,z)=\chi(M;\tau)=\frac{\text{Witten genus}\,(M)}{\eta(\tau)^{2d}}.$$

This is an automorphic function in τ with respect to $SL_2(\mathbb{Z})$.

1.3. Elliptic genus of a Calabi–Yau manifold. This case is of some interest in physics. Let $E=T_M$ and $c_1(T_M)=0$. Then there are no correction terms of type $\exp(\dots)$ in Definition 1.1. Thus the MEG of T_M is, up to the factor $y^{d/2}$, the Euler–Poincaré characteristic of the element $\mathbf{E}_{q,y}$. This function is called *elliptic genus* of the Calabi–Yau manifold M or genus one partition function of the super-symmetric $(2,2)$-sigma model whose target space is M:

$$\chi(M,T_M;\tau,z)=\text{Elliptic genus}\,(M;\tau,z)=y^{d/2}\int_M\operatorname{ch}(\mathbf{E}_{q,y})\operatorname{td}(T_M).$$

According to the Riemann–Roch–Hirzebruch theorem one can see that the q^0-term of $\chi(M;\,\tau,z)$ is essentially the Hirzebruch χ_y-genus of the manifold M (with interchanging of y with $-y$):

$$\chi(M;\,\tau,z)=\sum_{p=0}^{d}(-1)^p\chi_p(M)\,y^{\frac{d}{2}-p}+ \tag{1.2}$$
$$q\Big(\sum_{p=-1}^{d+1}(-1)^py^{-p}\big(\chi_p(M,T_M^*)-\chi_{p-1}(M,T_M^*)+\chi_p(M,T_M)-\chi_{p+1}(M,T_M)\big)\Big)+\dots$$

where $\chi(M,E)=\sum_{q=0}^{d}(-1)^q\dim H^q(M,E)$ and $\chi_p(M,E)=\chi(M,\wedge^pT_M^*\otimes E)$ or, for a Kähler manifold, $\chi_p(M)=\sum_q(-1)^qh^{p,q}(M)$. We remark that in this case every Fourier coefficient of the elliptic genus is equal to the index of the Dirac operator twisted with a corresponding vector bundle coefficient of the formal power series $\mathbf{E}_{q,y}$.

It is known that the elliptic genus of a Calabi–Yau manifold is a modular form in variables τ and z (see [Hö], [KYY], [Li]), i.e., it is a weak Jacobi form of weight 0 and index $d/2$. *If $c_1(T_M)\neq 0$, then the elliptic genus of M defined above is not a modular form.* We add the three correction factors in Definition 1.1 in order to obtain a function with a good behavior with respect to the modular transformations in τ and z. If $E=T_M$ and $c_1(T_M)\neq 0$, then the integral in Definition 1.1 contains the only correction term

$$\exp\Big(-\frac{c_1(T_M)}{2\pi i}\frac{\vartheta_z}{\vartheta}(\tau,z)\Big).$$

Thus the elliptic genus of M (as a function in two variables) is equal to the zeroth term in a sum of $d+1$ summands of the modified genus. These summands correspond to all powers of the first Chern class of M

$$\chi(M,T_M;\tau,z) = \text{Elliptic genus}\,(M;\tau,z) + \sum_{n=1}^{d}\Big(\int_M c_1(M)^n(\dots)\Big).$$

In general the elliptic genus is not an automorphic form but the modified elliptic genus is. The main result of this section is

Theorem 1.2. *Let E be a holomorphic vector bundle of rank r over a compact complex manifold M of dimension d. Let $\chi(M,E;\tau,z)$ be the modified Witten genus. Then the product*

$$\chi(M,E;\tau,z)\left(\frac{\vartheta(\tau,z)}{\eta(\tau)}\right)^{d-r}$$

is a weak Jacobi form of weight 0 *and index* $d/2$.

Remark. We note that the definition of MWG, Theorem 1 and its proof are valid for an arbitrary compact $Spin^c$-manifold M and a Hermitian C^∞-vector bundle E over M. In particular, we have the same result for an almost complex manifolds.

First we recall the definition of Jacobi forms of the type we need in this paper. Let $t \ge 0$ and k be integral or half-integral. Let v be a character of finite order (or a multiplier system for half-integral k) of $SL_2(\mathbb{Z})$. A holomorphic function $\phi(\tau,z)$ on $\mathbb{H}_1 \times \mathbb{C}$ is called a *weak Jacobi form of weight k and index t with character v* if it satisfies the functional equations

$$\phi\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = v(\gamma)(c\tau+d)^k\, e^{2\pi i t\frac{cz^2}{c\tau+d}}\,\phi(\tau,z) \qquad \left(\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})\right) \quad (1.3a)$$

and

$$\phi(\tau, z+\lambda\tau+\mu) = (-1)^{2t(\lambda+\mu)}\, e^{-2\pi i t(\lambda^2\tau+2\lambda z)}\phi(\tau,z) \qquad (\lambda,\mu\in\mathbb{Z}) \quad (1.3b)$$

and $\phi(\tau,z)$ has the Fourier expansion of the type

$$\phi(\tau,z) = \sum_{n\ge 0\ \ l\in t+\mathbb{Z}} f(n,l)\,q^n y^l.$$

We denote the space of all week Jacobi forms of weight k, index t and character (or multiplier system) v by $J^w_{k,t}(v)$. The space $J^w_{k,t}(v)$ is finite dimensional (see [EZ]). The only difference with [EZ] is that we admit Jacobi forms of half-integral index.

Corollary 1.3. *In the conditions of Theorem 1.2 we have that*

$$\chi(M,E;\tau,z)\in J^w_{0,\frac{r}{2}}(v_\eta^{2(r-d)}) \qquad r\ge d$$

or

$$\chi(M,E;\tau,z)\in J^{mer}_{0,\frac{r}{2}}(v_\eta^{2(r-d)}) \qquad r< d$$

with a possible pole of order not bigger than $d-r$ along $z=0$.

Proof of Theorem 1.2. To prove the theorem we represent $\chi(M,E;\tau,z)$ in terms of the theta-series. Let $c(E)$ be the total Chern class of the vector bundle E

$$c(E)=\sum_{i=0}^{r}c_i(E)=\prod_{i=1}^{r}(1+x_i)$$

where $x_i=2\pi i\xi_i$ $(1\le i\le r)$ are the formal Chern roots of E. We denote by $x'_j=2\pi i\zeta_j$ $(1\le j\le d)$ the Chern roots of T_M. We recall that

$$\mathrm{ch}\,(\textstyle\bigwedge_t E)=\displaystyle\prod_{i=1}^{r}(1+te^{x_i}),\qquad \mathrm{ch}\,(S_tE)=\prod_{i=1}^{r}\frac{1}{1-te^{x_i}}.$$

According to the last formulae we have

$$\mathrm{ch}\,(\mathbf{E}_{q,y})\,\mathrm{td}\,(T_M)=\prod_{n=1}^{\infty}\prod_{j=1}^{d}\prod_{i=1}^{r}\frac{(1-q^{n-1}y^{-1}e^{-x_i})(1-q^nye^{x_i})}{(1-q^{n-1}e^{-x'_j})(1-q^ne^{x'_j})}x'_j\,.$$

Therefore

$$q^{(r-d)/12}y^{r/2}\exp\Bigl(\frac12(c_1(E)-c_1(T_M))\Bigr)\mathrm{ch}\,(\mathbf{E}_{q,y})\mathrm{td}\,(T_M)=$$

$$(-1)^{r-d}\prod_{i=1}^{r}\frac{\vartheta(\tau,-z-\xi_i)}{\eta(\tau)}\prod_{j=1}^{d}\frac{\eta(\tau)}{\vartheta(\tau,-\zeta_j)}(2\pi i\zeta_j).\qquad(1.4)$$

Putting the last expression under the integral we obtain the following formula for the modified Witten genus

$$\chi(M,E;\tau,z)=\int_M\prod_{i=1}^{r}\exp\Bigl(-4\pi^2G_2(\tau)\xi_i^2-\frac{\vartheta_z}{\vartheta}(\tau,z)\xi_i\Bigr)\,\frac{\vartheta(\tau,z+\xi_i)}{\eta(\tau)}\times$$

$$\prod_{j=1}^{d}\exp\bigl(4\pi^2G_2(\tau)\zeta_i^2\bigr)\,\frac{\eta(\tau)}{\vartheta(\tau,\zeta_j)}(2\pi i\zeta_j).\qquad(1.5)$$

We shall calculate the top differential form under the integral using Proposition 1.4 bellow. In order to formulate it we need to recall the definition of the Weierstrass $\wp$-function

$$\wp(\tau,z)=z^{-2}+\sum_{\substack{\omega\in\mathbb{Z}\tau+\mathbb{Z}\\ \omega\ne0}}\bigl((z+\omega)^{-2}-\omega^{-2}\bigr)\in J_{2,0}^{mer}\qquad(1.6)$$

which is a meromorphic Jacobi form of weight 2 and index 0 with pole of order 2 along $z\in\mathbb{Z}\tau+\mathbb{Z}$.

Proposition 1.4. *The following formula is valid*

$$\exp\Big(-4\pi^2 G_2(\tau)\xi^2 - \frac{\vartheta_z}{\vartheta}(\tau,z)\xi\Big)\vartheta(\tau,z+\xi) = \vartheta(\tau,z)\exp\Big(-\sum_{n\ge 2}\wp^{(n-2)}(\tau,z)\frac{\xi^n}{n!}\Big)$$

where $\wp^{(n)}(\tau,z) = \dfrac{\partial^n}{\partial z^n}\wp(\tau,z)$.

Proof. One of the main examples of weak Jacobi forms of half-integral index with trivial SL_2-character is

$$\phi_{-1,1/2}(\tau,z) = \vartheta(\tau,z)/\eta(\tau)^3 = (y^{1/2}-y^{-1/2}) + q(\dots) \in J^w_{-1,\frac12}.$$

The Jacobi form $\phi_{-1,\frac12}$ has the following exponential representation as a Weierstrass σ-function (see, for example, review [Sk])

$$\phi_{-1,\frac12}(\tau,z) = \frac{\vartheta(\tau,z)}{\eta(\tau)^3} = (2\pi i z)\exp\Big(-\sum_{k\ge 1}\frac{2}{(2k)!}G_{2k}(\tau)(2\pi i z)^{2k}\Big) \tag{1.7}$$

where $G_{2k}(\tau) = -B_{2k}/4k + \sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n$ is the Eisenstein series of weight $2k$. (For each $\tau\in\mathbb{H}_1$ the product is normally convergent in $z\in\mathbb{C}$.) Since one can obtain the Weierstrass $\wp$-function as the second derivative of the Jacobi theta-series $\dfrac{\partial^2}{\partial z^2}\log\vartheta(\tau,z) = -\wp(\tau,z) + 8\pi^2 G_2(\tau)$, the identity (1.7) implies that

$$\wp^{(n-2)}(\tau,z) = \frac{(-1)^n(n-1)!}{z^n} + 2\sum_{k\ge 2,\,2k\ge n}(2\pi i)^{2k}G_{2k}(\tau)\frac{z^{2k-n}}{(2k-n)!}.$$

After that the formula of the lemma follows by direct calculation.

Now we can finish the proof of Theorem 1.2. According to the formula of Proposition 1.4 we can split the Chern roots x_i $(1\le i\le r)$ of the vector bundle E and the Chern roots x'_j $(1\le j\le d)$ of the manifold M under the integral in (1.5), i.e.,

$$\chi(M,E;\tau,z) = \frac{\vartheta^r}{\eta^{r+2d}}\int_M P(E;\tau,z)\cdot W(M;\tau). \tag{1.8}$$

The first factor depends only on the vector bundle E

$$P(E;\tau,z) = \exp\Big(-\sum_{n\ge 2}\frac{\wp^{(n-2)}(\tau,z)}{(2\pi i)^n n!}\big(\sum_{i=1}^r x_i^n\big)\Big).$$

The second factor is the Witten factor

$$W(M;\tau) = \exp\Big(2\sum_{k\ge 2}\frac{G_{2k}(\tau)}{(2k)!}\big(\sum_{j=1}^d {x'_j}^{2k}\big)\Big)$$

which determines the Witten genus of the manifold M as a function in one variable τ (see §1.3). The derivation of order $(n-2)$ of the Weierstrass $\wp$-function is a meromorphic Jacobi form of weight n and index 0 with pole of order n along $z=0$. Thus the coefficient of a monomial in x_i, x'_j of the total degree d in (1.8) is a meromorphic Jacobi form of weight 0 and index $r/2$ with pole of order not bigger than $(d-r)$. Therefore the product $\eta(\tau)^{r-d}\vartheta(\tau,z)^{d-r}\chi(M,E,\tau,z)$ is holomorphic on $\mathbb{H}_1\times\mathbb{C}$ and it has trivial character. It is a weak Jacobi form since its Fourier expansion does not contain negative powers of q. Theorem 1.2 is proved.

We note that the special cocycle defined by the logarithmic derivative of the Jacobi theta-series is very natural in the following context.

Proposition 1.5. *Let $\phi\in J^{mer}_{k,m}$ be a (meromorphic) Jacobi form. The following functions*

$$\Psi(\tau,z)=\exp\bigl(-8\pi^2 m\,G_2(\tau)z^2\bigr)\phi(\tau,z)$$

and

$$\Phi(\tau,z,\xi)=\exp\left(-8\pi^2 m\,G_2(\tau)\xi^2-\frac{\phi_z(\tau,z)}{\phi(\tau,z)}\xi\right)\phi(\tau,z+\xi). \tag{1.9}$$

where $\phi_z=\dfrac{\partial\psi}{\partial z}$, transform like a modular form in τ and a Jacobi form of index m in τ and z respectively. More exactly they satisfy the equations

$$\Psi(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d})=(c\tau+d)^k\Psi(\tau,z)$$

$$\Phi(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d},\frac{\xi}{c\tau+d})=v(\gamma)(c\tau+d)^k e^{2\pi i m\frac{cz^2}{c\tau+d}}\Phi(\tau,z,\xi)$$

$$\Phi(\tau,z+\lambda\tau+\mu,\xi)=(-1)^{(\lambda+\nu)}e^{-2\pi i m(\lambda^2\tau+2\lambda z)}\Phi(\tau,z,\xi)$$

for any $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbb{Z})$ and $\lambda,\mu\in\mathbb{Z}$.

As a direct corollary we obtain that the coefficients $g_n(\tau)$ and $f_n(\tau,z)$ in the Taylor expansion of Ψ in z and Φ in ξ

$$\Psi(\tau,z)=\sum_{n\in\mathbb{Z}}g_n(\tau)z^n,\qquad \Phi(\tau,z,\xi)=\sum_{n\in\mathbb{Z}}f_n(\tau,z)\xi^n \tag{1.10}$$

are a modular forms of weight $n+k$ and a (meromorphic) Jacobi form of weight $k+n$ and index m respectively. Proposition 1.4 gives us the exact "additional" formula for $\vartheta(\tau,z+\xi)$.

Examples for dimension $d=2,3,4$ and $d=6$, $r=2$. For $d=2$ and $d=3$ we consider an arbitrary vector bundle E over M. It follows from the representation (1.8) that

$$\chi(M_2,E_r;\tau,z)=\frac{1}{2(2\pi i)^2}\left(\sum_{i=1}^r x_i^2\right)[M]\frac{\vartheta^r}{\eta^{r+4}}\wp=\frac{1}{24}(c_1(E)^2-2c_2(E))\,\phi_{0,1}\left(\frac{\vartheta}{\eta}\right)^{r-2}$$

where $\phi_{0,1}(\tau,z)=(y+10+y^{-1})+q(\dots)$ is the unique, up to a constant, weak Jacobi form of weight 0 and index 1 (see §2). Let $\phi_{0,\frac{3}{2}}(\tau,z)=\vartheta(\tau,2z)/\vartheta(\tau,z)$. For $d=3$ we have

$$\chi(M_3,E_r;\tau,z)=\frac{1}{6}\big(c_1(E)^3-3c_1(E)c_2(E)+3c_3(E)\big)\phi_{0,\frac{3}{2}}(\tau,z)\left(\frac{\vartheta}{\eta}\right)^{r-3}.$$

Let us consider the case $d=4$, $r\ge 4$ with $p_1(E_4)=p_1(M_4)$ ($c_1(E_4)$ is arbitrary). We have the following formula for MWG

$$\chi(M_4,E_r;\tau,z)=\big[\hat{A}(M_4)\psi^{(2)}_{0,2}(\tau,z)+\big(r\hat{A}(M_4)-\hat{A}(M_4,E_r)\big)\phi_{0,2}(\tau,z)\big]\left(\frac{\vartheta(\tau,z)}{\eta(\tau)}\right)^{r-4}, \tag{1.11}$$

where $\phi_{0,2}(\tau,z)=y+4+y^{-1}+q(\dots)$ is defined in (2.1) and $\psi^{(2)}_{0,2}=\phi_{0,1}^2-24\phi_{0,2}$ (see Theorem 2.2).

Let us consider the case $d=6$, $r=2$, $p_1(E)=p_1(T_M)$. We assume also that MWG is holomorphic. In the case under consideration this condition is equivalent to the relation $< c_1(E)^4c_2(E)-\frac{1}{6}c_1(E)^6 > [M]=0$. Then

$$\chi(M_6,E_2;\tau,z)=\big[\hat{A}(M)E_6(\tau)\phi_{-2,1}+(2\hat{A}(M)-\hat{A}(M,E))E_{4,1}(\tau,z)\big]\eta(\tau)^{-8}, \tag{1.12}$$

where $\phi_{-2,1}=\vartheta^2/\eta^6$, $E_6(\tau)$ is the Eisenstein series of weight 6 and $E_{4,1}(\tau,z)$ is the Eisenstein–Jacobi series of weight 4 and index 1. ($E_{4,1}$ is equal to the Jacobi theta-series of the lattice $\mathbb{E}_8$.) For more details and examples see [G2].

Theorem 1.2 shows us that some possible "**strange periodicity**" of MWG could exist. We note that if $p_1(E)=p_1(M)$ and $c_1(E)=0$, then MWG is the partition function of $(0,2)$-symmetric non-linear sigma-model (see [KM]).

Corollary 1.6. *Let us assume that* $\chi(M_1,E_2,\tau,z)\equiv\chi(M_1,E_2,\tau,z)$. *Then* $r_1=r_2$ *and* $d_1\equiv d_2 \mod 12$.

(See also Question 2 and 3 after Theorem 2.3.)

§2. $\mathbb{Z}$-structure of the graded ring of Jacobi forms and the special values of the elliptic genus

The structure over $\mathbb{C}$ of the bigraded ring of all weak Jacobi forms was determined in [EZ]. The elliptic genus of a Calabi–Yau manifold is a weak Jacobi form of weight 0 with integral Fourier coefficients. Thus one can put a question about the $\mathbb{Z}$-structure of the graded ring

$$J^{\mathbb{Z}}_{0,*}=\bigoplus_{m\in\mathbb{Z}}J^{\mathbb{Z}}_{0,m}$$

of all Jacobi forms of weight 0 of integral weight with integral Fourier coefficients. For an arbitrary vector bundle E, the form $\chi(M,E;\tau,z)$ can be written, up to a η-factor, as a linear combination of weak Jacobi forms of positive weight (see (1.12)). Thus we have also a question about the $\mathbb{Z}$-structure of the bigraded ring of all weak Jacobi forms of arbitrary weight and index with integral Fourier coefficients. We introduce an ideal of $J^{\mathbb{Z}}_{0,*}$

$$J^{\mathbb{Z}}_{0,*}(q)=\{\phi\in J^{\mathbb{Z}}_{0,*}\,|\,\phi(\tau,z)=\sum_{n\ge1,\ l\in\mathbb{Z}}a(n,l)q^ny^l\}$$

consisting of the Jacobi forms without q^0-term. Using standard considerations with divisors of one can prove

Lemma 2.1. **A**. *Let m be integral, then we have*

$$J^{\mathbb{Z}}_{2k,m+\frac12}=\phi_{0,\frac32}\cdot J^{\mathbb{Z}}_{2k,m-1},\qquad J^{\mathbb{Z}}_{2k+1,m+\frac12}=\phi_{-1,\frac12}\cdot J^{\mathbb{Z}}_{2k+2,m}$$

where $\phi_{0,\frac32}(\tau,z)=\vartheta(\tau,2z)/\vartheta(\tau,z)$ and $\phi_{-1,\frac12}=\vartheta(\tau,z)/\eta(\tau)^3$.

B. *The ideal $J^{\mathbb{Z}}_{0,*}(q)$ is principal. It is generated by a weak Jacobi form of weight* 0 *and index* 6

$$\xi_{0,6}(\tau,z)=\Delta(\tau)\phi_{-1,\frac12}(\tau,z)^{12}=\frac{\vartheta(\tau,z)^{12}}{\eta(\tau)^{12}}=q(y^{\frac12}-y^{-\frac12})^{12}+q^2(\dots).$$

There exists only one (up to a constant) weak Jacobi form of weight 0 and index 1

$$\phi_{0,1}(\tau,z)=-\frac{3}{\pi^2}\frac{\wp(\tau,z)\vartheta(\tau,z)^2}{\eta(\tau)^6}=(y+10+y^{-1})+q(10y^{\pm2}-88y^{\pm1}-132)+\dots$$

(see [EZ]). In the theory of generalized Lorentzian Kac–Moody algebras (see [GN1–GN4]) we defined the following important Jacobi forms of small index:

$$\begin{aligned}\phi_{0,2}(\tau,z)&=\tfrac12\eta(\tau)^{-4}\sum_{m\,,n\in\mathbb{Z}}(3m-n)\left(\frac{-4}{m}\right)\left(\frac{12}{n}\right)q^{\frac{3m^2+n^2}{24}}y^{\frac{m+n}{2}}\\&=(y+4+y^{-1})+q(y^{\pm3}-8y^{\pm2}-y^{\pm1}+16)+q^2(\dots),\qquad(2.1)\\ \phi_{0,3}(\tau,z)&=\phi^2_{0,\frac32}(\tau,z)=(y+2+y^{-1})+q(-2y^{\pm3}-2y^{\pm2}+2y^{\pm1}+4)+q^2(\dots),\\ \phi_{0,4}(\tau,z)&=\frac{\vartheta(\tau,3z)}{\vartheta(\tau,z)}=(y+1+y^{-1})-q(y^{\pm4}+y^{\pm3}-y^{\pm1}-2)+q^2(\dots).\qquad(2.2)\end{aligned}$$

We have the following interpretation of these forms as elliptic genus

$$2\phi_{0,1}(\tau,z)=EG(K3,\tau,z),\qquad \phi_{0,\frac32}=EG(CY_3^{(e=2)},\tau,z)$$

($CY_3^{(e=2)}$ is a Calabi–Yau 3-fold with Euler number equals 2.) The form $\xi_{0,6}$ is the elliptic genus of the trivial vector bundle of rank 12 over a point

$$\xi_{0,6}(\tau,z)=\chi(\cdot,\mathbb{C}^{12};\tau,z).$$

One can also represent these functions as symmetric polynomials in the quotients of the Jacobi theta-series $\vartheta_{ab}(\tau,z)/\vartheta_{ab}(\tau,0)$ of level 2. Let us put

$$\xi_{00}=\frac{\vartheta_{00}(\tau,z)}{\vartheta_{00}(\tau,0)},\quad \xi_{10}=\frac{\vartheta_{10}(\tau,z)}{\vartheta_{10}(\tau,0)},\quad \xi_{01}=\frac{\vartheta_{01}(\tau,z)}{\vartheta_{01}(\tau,0)}.$$

Then we have

$$\phi_{0,1}(\tau,z)=4(\xi_{00}^2+\xi_{10}^2+\xi_{01}^2),\qquad \phi_{0,\frac32}(\tau,z)=4\xi_{00}\xi_{10}\xi_{01}$$

$$\phi_{0,2}(\tau,z)=2\big((\xi_{00}\xi_{10})^2+(\xi_{00}\xi_{01})^2+(\xi_{10}\xi_{01})^2\big).$$

(To check these formulae one should compare only q^0-terms of corresponding Jacobi forms.)

In the next theorem we construct a basis of the module $J^{\mathbb{Z}}_{0,m}/J^{\mathbb{Z}}_{0,m}(q)$ and we find generators of the graded ring $J_{0,*}$.

Theorem 2.2. **1**. *Let m be a positive integer. The module*

$$J_{0,m}^{\mathbb{Z}}/J_{0,m}^{\mathbb{Z}}(q)=\mathbb{Z}\,[\psi_{0,m}^{(1)},\dots,\psi_{0,m}^{(m)}]$$

is a free $\mathbb{Z}$-module of rank m. Moreover we can chose a basis with the following q^0-terms

$$[\psi_{0,m}^{(n)}(\tau,z)]_{q^0}=y^n-n^2y+(2n^2-2)-n^2y^{-1}+y^{-n}\qquad (2\le n\le m),$$
$$[\psi_{0,m}^{(1)}]_{q^0}=\frac{1}{(12,m)}\left(my+(12-2m)+my^{-1}\right)$$

where $(12,m)$ is the greatest common divisor of 12 and m.

2. *The graded ring of all weak Jacobi forms of weight 0 with integral Fourier coefficients is finitely generated*

$$J_{0,*}^{\mathbb{Z}}=\bigoplus_m J_{0,m}^{\mathbb{Z}}=\mathbb{Z}\,[\phi_{0,1},\phi_{0,2},\phi_{0,3},\phi_{0,4}]$$

where $\phi_{0,1}$, $\phi_{0,2}$, $\phi_{0,3}$ are algebraicly independent and

$$4\phi_{0,4}=\phi_{0,1}\phi_{0,3}-\phi_{0,2}^2.$$

The second claim of the theorem is a corollary of the first part which on can prove by induction on m and n. We give here only the formulae for the most important exceptional Jacobi forms having the q^0-term of type $y+c+y^{-1}$ (see a detailed proof of Theorem 2.2 in [G1]):

$$\phi_{0,6}(\tau,z)=\phi_{0,2}\phi_{0,4}-\phi_{0,3}^2=(y+y^{-1})+q(\dots),$$
$$\phi_{0,8}(\tau,z)=\phi_{0,2}\phi_{0,6}-\phi_{0,4}^2=(2y-1+2y^{-1})+q(\dots),$$
$$\phi_{0,12}(\tau,z)=\phi_{0,4}\phi_{0,8}-2\phi_{0,6}^2=(y-1+y^{-1})+q(\dots).$$

We note also that

$$\xi_{0,6}=-\phi_{0,1}^2\phi_{0,4}+9\phi_{0,1}\phi_{0,2}\phi_{0,3}-8\phi_{0,2}^3-27\phi_{0,3}^2.\qquad (2.3)$$

To prove that $\phi_{0,1}$, $\phi_{0,2}$ and $\phi_{0,3}$ are algebraicly independent one has to consider values at $z=\frac{1}{2}$. We have

$$\phi_{0,2}(\tau,\frac{1}{2})\equiv 2,\quad \phi_{0,3}(\tau,\frac{1}{2})\equiv 0,\quad \phi_{0,4}(\tau,\frac{1}{2})\equiv -1.$$

(The two last identities follow from definition and the first one is a corollary of the torsion relation of the theorem.) The restriction of

$$\phi_{0,1}(\tau,\frac{1}{2})=\alpha(\tau)=8+2^8q+2^{11}q^2+11\cdot 2^{10}q^3+3\cdot 2^{14}q^4+359\cdot 2^9q^5+\dots$$

is a modular function with respect to $\Gamma_0(2)$ with a character of order 2.

Generators $\phi_{0,1},\dots,\phi_{0,4}$ and Tate curves. One can associate with a plane cubic model of an elliptic curve the quantities b_2, b_4, b_6, b_8 (see [T]). Theorem 2.2 shows us that the generators of the graded ring of the weak Jacobi forms are analytic analogies of the Tate's parameters since $4b_8 = b_2b_6 - b_4^2$. The formula (2.3) is the expression for the discriminant Δ of the Tate curve in terms of b_i.

Minimal models. As we mentioned above the elliptic genus appeared in physics as the partition function of a $N = 2$ superconformal field theory (SCFT). Among them we have the elliptic genus a sigma model with a Calabi–Yau manifold as targent space, the elliptic genus of Landau–Ginzburg models and of $N = 2$ minimal models (see [W2], [FY], [KYY]).

Let us consider SCFT with central charge $\hat c$ ($\hat c = c/3$). Then its elliptic genus $Z(\tau, z)$ satisfies the functional equations

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = e^{\pi i\hat c\frac{cz^2}{c\tau+d}}\, Z(\tau,z) \qquad (\gamma = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z}))$$

and

$$Z(\tau, z+\lambda\tau+\mu) = (-1)^{2\hat c(\lambda+\mu)}\, e^{-2\pi i\hat c(\lambda^2\tau+2\lambda z)} Z(\tau,z) \qquad (\lambda,\mu\in h\mathbb{Z})$$

where $h\hat c$ is integer. (One can determine h in terms of the $U(1)$ charges of the chiral ring elements in some sector.) We would like to note that the transformation $z \to hz$ is a Hecke type operator which multiplies the index by h^2:

$$\phi(\tau,z)\in J_{k,m} \to \phi(\tau,hz)\in J_{k,h^2m}.$$

(See [GN4, §1.4] where such operators were used in construction of Siegel theta-series.) Therefore

$$Z(\tau, hz) \in J_{0,h^2\hat c}. \tag{2.4}$$

Let us consider the elliptic genus $Z_k^{(min)}(\tau,z)$ of the $N=2$ minimal model with $\hat c = \frac{k}{k+2}$, where $k = 1, 2, \dots$. Thus

$$Z_k^{(min)}(\tau,(k+2)z) \in J^{w,\mathbb{Z}}_{0,\frac12 k(k+2)}.$$

In particular, for $k = 1$ the elliptic genus of $N = 2$ minimal model is equal, up to a constant, to the Jacobi form $\phi_{0,3/2}$ which is the elliptic genus of a Calabi–Yau 3-fold. If $k = 2$, then the elliptic genus is a linear combination of the four forms $\phi_{0,4}$, $\psi^{(2)}_{0,4}$, $\psi^{(3)}_{0,4}$ and $\psi^{(4)}_{0,4}$ constructed in Theorem 2.2. The corresponding coefficients are determined by q^0-term of the elliptic genus.

For $k = 3$ the index is equal to $\frac{15}{2}$. Thus the elliptic genus is a combination of the seven forms $\phi_{0,3/2}\cdot\psi^{(m)}_{0,6}$ ($m = 1,\dots,6$) and $\phi_{0,3/2}\cdot\xi_{0,6}$. This is the first case when one needs an information about one coefficient of the q^1-term in order to find the elliptic genus. We note that if k is odd, then $Z_k^{(min)}(\tau,(k+2)z)$ is divisible by $\phi_{0,3/2}$.

Cobordism ring. G. Höhn was calculated the cobordism ring $\Omega_*^{SU}\otimes\mathbb{Q}$ over $\mathbb{Q}$ of SU-manifolds (see [Hö]). Using Theorem 2.2 above we can solve this problem over $\mathbb{Z}$.

We have also a result about the structure of the bigraded ring of all integral weak Jacobi forms

$$J^{w,\mathbb{Z}}_{*,*/2}=\bigoplus_{k\in\mathbb{Z},\,m\in\frac12\mathbb{Z}} J^{w,\mathbb{Z}}_{k,m}.$$

Theorem 2.3.

$$J^{w,\mathbb{Z}}_{*,*/2}=\mathbb{Z}[E_4,E_6,\Delta,E_{4,1},E_{4,2},E_{4,3},E_{6,1},E_{6,2},E'_{6,3},\phi_{0,1},\phi_{0,2},\phi_{0,\frac32},\phi_{0,4},\phi_{-1,\frac12}]$$

where $E_4(\tau)$, $E_6(\tau)$ and $\Delta(\tau)$ are generators of the graded ring of $SL_2(\mathbb{Z})$-modular forms, $\phi_{-2,1}=\vartheta^2/\eta^6$, $E_{4,1},\dots E_{6,3}$ are the Eisenstein–Jacobi series with the zeroth Fourier coefficient equals to 1 *and $E'_{6,3}=E_{6,3}+\frac{22}{61}\Delta_{12}\phi^3_{-2,1}$.*

As we mentioned at the end of §1, the MWG is a partition function of $(0,2)$-symmetric sigma model. We would like to formulate a problem about existence of some special vector bundles with an interesting $(0,2)$-partition function. We assume that M_d is a complex compact manifold of complex dimension d, E_r is a holomorphic vector bundle of rank r over M. Moreover

$$p_1(E_r)=p_1(M_d)\quad\text{ and }\quad c_1(E_r)=0\ (\text{over }\mathbb{R}).$$

Question 1. *The Eisenstein type.* Let $d=6$, $r=2$ and

$$\hat A(M_6)=0,\qquad \hat A(M_6,E_2)=-2.$$

Similar questions for dimensions 8 and 10:

$$\hat A(M_8)=\hat A(M_8,E_4)=0,\quad \hat A(M_8,\wedge^2E_4)=-2$$

and

$$\hat A(M_{10})=\hat A(M_{10},E_6)=\hat A(M_{10},\wedge^2E_6)=0,\ \hat A(M_{10},\wedge^3E_6)=-2.$$

The MWG of a vector bundle with invariant as above is an Eisenstein-Jacobi series from Theorem 2.3.

Question 2. *The "trivial" type.* Let $d=12$, $r=2$ and

$$\hat A(M_{12})=\hat A(M_{12},E_2)=0,\qquad \hat A(M_{12},E_2\otimes T_M)\neq 0.$$

For a $K3$-surface we can formulate

Question 3. *The K3 type.* Let $d=14$, $r=2$,

$$\hat A(M)=\hat A(M,E)=0,\qquad \hat A(M,E\otimes T_M)=-10.$$

The last condition is equivalent to $\hat A(M,S^2E)+2\hat A(M,T_M)=2$.

Question 4. *The Kac–Moody type.* Let $d = 14$, $r = 2$.

$$\hat{A}(M) = 0, \quad \hat{A}(M, E) = -2$$

and

$$\hat{A}(M, E \otimes T_M) = -68.$$

The last condition is equivalent to $\hat{A}(M, S^2E) + 2\hat{A}(M, T_M) = 0$. Such a vector bundle (if it exist) will be related to the simplest hyperbolic Kac–Moody algebra. (See §4 bellow.)

Using Theorems 2.2 and 2.3 we can analyze the value of the elliptic genus at the following special points $z = 0$ (Euler number), $z = \frac{1}{2}$ (signature), $z = \frac{\tau+1}{2}$ ($\hat{A}$-genus) and $z = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$. For this end we have to study the restriction of the generators of the graded ring of the integral week Jacobi forms. A special value of a Jacobi form is a modular form in τ. In the next lemma we give a little more precise statement than in [EZ, Theorem 1.3].

Lemma 2.4. *Let $\phi \in J_{0,t}$ $(t \in \mathbb{Z}/2)$ and $X = (\lambda, \mu) \in \mathbb{Q}^2$. Then*

$$\phi|_X(\tau, 0) = \phi(\tau, \lambda\tau + \mu) \exp(2\pi i t(\lambda^2\tau + \lambda\mu))$$

is an automorphic form of weight 0 *with a character with respect to the subgroup*

$$\Gamma_X = \{M \in SL_2(\mathbb{Z}) \mid XM - X \in \mathbb{Z}^2\}.$$

It is easy to see that if $\phi \in J^{\mathbb{Z}}_{k,m}$ with integral m, then the form $\phi(\tau, \frac{1}{N})$ still has integral Fourier coefficients if $N = 1, \dots, 6$. In particular, the value of $\xi_6(\tau, z)$ at these points is related to the "Hauptmodul" for the corresponding group $\Gamma_0(N)$:

$$\xi_6(\tau, \frac{1}{2}) = 2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)}, \qquad \xi_6(\tau, \frac{1}{4}) = 2^6\left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/2},$$
$$\xi_6(\tau, \frac{1}{3}) = 3^6\left(\frac{\Delta(3\tau)}{\Delta(\tau)}\right)^{1/2}, \qquad \xi_6(\tau, \frac{1}{6}) = \left(\frac{\Delta(\tau)\Delta(6\tau)}{\Delta(2\tau)\Delta(3\tau)}\right)^{1/2}.$$

Let us analyze the corresponding values of the four generators $\phi_{0,n}$ of the graded ring $J^{\mathbb{Z}}_{0,*}$. From the definition (see (2.1)–(2.2)) and the identity $4\phi_{0,4} = \phi_{0,1}\phi_{0,3} - \phi_{0,2}^2$ we obtain

$$\phi_{0,1}(\tau, 0) = 12, \quad \phi_{0,2}(\tau, 0) = 6, \quad \phi_{0,3}(\tau, 0) = 4, \quad \phi_{0,4}(\tau, 0) = 3 \tag{2.5}$$

and

$$\begin{array}{lll} \phi_{0,1}(\tau, \frac{1}{2}) = \alpha(\tau) & \phi_{0,1}(\tau, \frac{1}{3}) = \beta^2(\tau) & \phi_{0,1}(\tau, \frac{1}{4}) = \dfrac{\gamma(\tau)^4 + 4}{\gamma(\tau)} \\ \phi_{0,2}(\tau, \frac{1}{2}) = 2 & \phi_{0,2}(\tau, \frac{1}{3}) = \beta(\tau) & \phi_{0,2}(\tau, \frac{1}{4}) = 4\gamma^2(\tau) \\ \phi_{0,3}(\tau, \frac{1}{2}) = 0 & \phi_{0,3}(\tau, \frac{1}{3}) = 1 & \phi_{0,3}(\tau, \frac{1}{4}) = 2\gamma(\tau) \\ \phi_{0,4}(\tau, \frac{1}{2}) = -1 & \phi_{0,4}(\tau, \frac{1}{3}) = 0 & \phi_{0,4}(\tau, \frac{1}{4}) = 1. \end{array} \tag{2.6}$$

The automorphic functions $\alpha(\tau)$, $\beta(\tau)$ and $\gamma(\tau)$ are automorphic forms of weight 0 with respect to the group Γ_0, $\Gamma_0^{(1)}(3)$ and $\Gamma_0^{(1)}(4)$ respectively. These functions have integral Fourier coefficients. The identity (2.3) gives us the following relations between the automorphic functions α, β and γ

$$2^{12}\frac{\Delta(2\tau)}{\Delta(\tau)} = \alpha(\tau)^2 - 64, \qquad 3^6\left(\frac{\Delta(3\tau)}{\Delta(\tau)}\right)^{1/2} = \beta(\tau)^3 - 27$$
$$2^6\left(\frac{\Delta(4\tau)}{\Delta(\tau)}\right)^{1/2} = 4(\left(\frac{\gamma(\tau)}{2}\right)^2 - \left(\frac{2}{\gamma(\tau)}\right)^2).$$

It follows that

$$\alpha(\tau) - 8 \equiv 0 \mod 2^8, \qquad \beta(\tau) - 3 \equiv 0 \mod 3^3 \tag{2.7}$$

(compare with the formula for $\phi_{0,1}(\tau, \frac{1}{2})$ above). Using the definition of $\phi_{0,3}$ and $\gamma(\tau)$ and the relations between the Jacobi theta-series ϑ_{ab} of level 2 we have

$$\gamma(\tau) = \frac{\vartheta_{00}(2\tau)}{\vartheta_{01}(2\tau)} = \frac{\vartheta_{00}(2\tau, 0)}{\vartheta_{01}(2\tau, 0)}.$$

One can check that $\phi_{0,1}(\tau, 2z) = \phi_{0,2}^2(\tau, z) - 8\phi_{0,4}(\tau, z)$. Thus

$$\alpha(\tau) = 16\gamma(\tau)^4 - 8 = 16\frac{\vartheta_{00}^4(2\tau)}{\vartheta_{01}^4(2\tau)} - 8.$$

In particular *all Fourier coefficients of* $\gamma(\tau)$ *and* $\alpha(\tau)$ *are positive.*

Example 2.5. $\hat{A}$-genus. Let $X = (\frac{1}{N}, \frac{1}{N})$. Then Γ_X (see Lemma 2.4) contains the principle congruence subgroup $\Gamma_1(N)$. In some cases Γ_X will be strictly larger. For example, if $X_2 = (\frac{1}{2}, \frac{1}{2})$, then

$$\phi|_{X_2}(\tau, 0) = \phi(\tau, \frac{\tau+1}{2})\exp(\frac{\pi i}{2}(\tau+1))$$

is an automorphic form with respect of the so-called theta-group

$$\Gamma_\theta = \left\{M = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}) \mid M \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \mod 2\right\}.$$

The corresponding character is given by $\epsilon_2(M) = \exp(2\pi i m(d+b-a-c)/4) = \pm 1$. This character is trivial if index m of Jacobi form is even. Let us consider Γ_θ-automorphic function

$$\hat{\phi}_m(\tau) = q^{-\frac{m}{4}}\phi_{0,m}(\tau, -\frac{\tau+1}{2}).$$

We have

$$\hat{\phi}_3 = 0, \quad \hat{\phi}_4 = -1, \quad \hat{\phi}_2 = -2, \quad \hat{\xi}_6 = \hat{\phi}_1^2 + 64 = \left(\frac{\vartheta_{00}}{\eta}\right)^{12}$$

where
$$\hat\phi_1(\tau)=4\frac{\vartheta_{10}^4-\vartheta_{01}^4}{\vartheta_{01}^2\vartheta_{10}^2}=-q^{-\frac14}+20q^{\frac14}+\dots\in\mathfrak{M}_0^{\mathbb Z}(\Gamma_0(2),\epsilon_2).$$

Now we analyze some special values of the elliptic genus. As it easy follows from (1.2) we get Euler number of a Calabi–Yau manifold M_d for $z=0$ (d is arbitrary) and and its signature for $z=\frac12$ (d is even):

$$\chi(M_d,\tau,0)=e(M_d),$$
$$\chi(M_d,\tau,\frac12)=\sigma_M(\tau)=(-1)^{\frac d2}s(M_d)+q(\dots)\in\mathfrak{M}_0^{\mathbb Z}(\Gamma_0(2),v_2),\quad v_2(\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))=e^{\pi i m\frac c2}.$$

The formulae (2.5) gives us some divisibility of Euler number of Calabi–Yau manifolds. We note that the quotient $e(M)/24$ appears in physics as obstruction to cancelling the tadpole (see [SVW] where it was proved that $e(M_4)\equiv 0 \mod 6$).

Proposition 2.6. *Let M_d be an almost complex manifold of complex dimension d such that $c_1(M)=0$ in $H^2(M,\mathbb R)$. Then*
$$d\cdot e(M_d)\equiv 0\mod 24.$$
If $c_1(M)=0$ in $H^2(M,\mathbb Z)$, then we have a more strong congruence
$$e(M)\equiv 0\mod 8\qquad \text{if } d\equiv 2\mod 8.$$

Remark. More generally, we have the following relation for $\hat A$-genus of a spin bundle. Let us consider the case
$$d=r,\quad p_1(E)=p_1(T_M),\quad c_1(E)=0\ (\text{over }\mathbb Z).$$
We consider spin bundle $\Delta^+(E)-\Delta^-(E)$ such that $\mathrm{ch}(\Delta^+(E)-\Delta^-(E))=\prod_{i=1}^r(e^{x_i/2}-e^{-x_i/2})$. Then
$$\hat A(M,\Delta^+(E)-\Delta^-(E))\equiv\begin{cases}0\mod 8 & d\equiv 1,2,5\mod 8\\ 0\mod 4 & d\equiv 6,7\mod 8\\ 0\mod 2 & d\equiv 3,4\mod 8\\ 0\mod 3 & d\not\equiv 0\mod 3.\end{cases}$$

Proof. The first fact follows simply from (2.5). If $d\equiv 2\mod 8$ one can write the elliptic genus as a polynomial over $\mathbb Z$ in the generators ϕ
$$e(M_d)\equiv P(\phi_{0,1},\phi_{0,2},\phi_{0,3},\phi_{0,4})|_{z=0}\equiv c_{1,m}(\phi_{0,1}|_{z=0})(\phi_{0,4}|_{z=0})^{\frac{d-2}{8}}\mod 8.$$
If one put $z=-\frac{\tau+1}{2}$, i.e., $y=-q^{1/2}$, then one can see that the series
$$\mathbb E_{q,-q^{1/2}}=\bigotimes_{n\ge1}\bigwedge\nolimits_{q^{n/2}}T_M\otimes\bigotimes_{n\ge1}\bigwedge\nolimits_{q^{n/2}}T_M^*\otimes\bigotimes_{n\ge1}S_{q^n}(T_M\oplus T_M^*)$$

is $*$-symmetric. According to the Serre duality all Fourier coefficients of $\hat{\chi}(M_d,\tau)$ are even. The constant $c_{1,m}$ from the last congruence is equal to the coefficient of $\hat{\chi}(M_d,\tau)$ at the minimal negative power of q. Therefore $c_{1,m}$ is even and we obtain divisibility of $e(M_{8m+2})$ by 8.

We note that divisibility of $d\cdot e(M)$ by 3 was proved by F. Hirzebruch in 1960. For a hyper-Kähler compact manifold the claim of the proposition above was proved by S. Salamon in [S]. After my talk on the elliptic genus at a seminar of MPI in Bonn in April 1997 Professor F. Hirzebruch informed me that the result of Proposition 2.6 was known for him (non-published). Using some natural examples he also proved that this property of divisibility of the Euler number modulo 24 is strict (see [H2]).

Formulae (2.6) provide us with a formula for the signature $\chi(M_d;\tau,\frac{1}{2})$ as a polynomial in $\alpha(\tau)$. As a corollary of (2.6) and Theorem 2.2 we have that for an arbitrary Jacobi form of integral index

$$\phi_{0,4m}(\tau,\frac{1}{2})=c+2^{13}q(\dots)\qquad \phi_{0,4m+1}(\tau,\frac{1}{2})=8c+2^{8}q(\dots)$$
$$\phi_{0,4m+2}(\tau,\frac{1}{2})=2c+2^{12}q(\dots)\qquad \phi_{0,4m+3}(\tau,\frac{1}{2})=16c+2^{9}q(\dots).$$

Similar to the proof of Proposition 2.4 we obtain a better congruence for the signature of a manifold with $\dim\equiv 2 \mod 8$ and $c_1(M)=0$:

$$\chi(M_{8m+2};\tau,z)=16c+2^{9}q(\dots). \tag{2.9}$$

It is interesting that the values of the Hirzebruch y-genus at $y=e^{2\pi i/3}$ and $y=i$ also have some properties of divisibility. For $z=\frac{1}{3}$ (resp. $z=\frac{1}{4}$) we can write $\phi_{0,m}(\tau,\frac{1}{3})$ (resp. $\phi_{0,m}(\tau,\frac{1}{4})$) as a polynomial in $\beta(\tau)=3+27(q+\dots)$ (resp. in $\gamma(\tau)^{\pm 1}$). This gives us the following results

$$\phi_{0,3m}(\tau,\frac{1}{3})=c+3^{6}q(\dots),\qquad \phi_{0,3m+1}(\tau,\frac{1}{3})=9c+3^{4}q(\dots)$$
$$\phi_{0,3m+2}(\tau,\frac{1}{3})=3c+3^{3}q(\dots).$$

Thus we have

Proposition 2.7. *If $c_1(M)=0$ (over $\mathbb{R}$), then*

$$\chi(M_{6m};\tau,\frac{1}{3})\equiv c_1 \mod 3^{6},\qquad \chi(M_{6m+2};\tau,\frac{1}{3})\equiv 9c_2 \mod 3^{4},$$
$$\chi(M_{6m+4};\tau,\frac{1}{3})\equiv 3c_3 \mod 3^{3}.$$

where $c_1,c_2,c_3\in\mathbb{Z}$. For $z=\frac{1}{4}$ we have:

$$\chi(M_{8m+2};\tau,\frac{1}{4})=4c+2^{4}q(\dots),\qquad \phi_{0,4m+2}(\tau,\frac{1}{4})=4c+2^{5}q(\dots)$$
$$\phi_{0,4m+3}(\tau,\frac{1}{4})=2c+2^{8}q(\dots).$$

§3. $\hat{A}_2^{(2)}$-GENUS.

The formal power series $\mathbf{E}_{q,y}$ over $K(M)$ (see (1.1)) is a geometric analog of the Jacobi theta-series $\vartheta(\tau, z)$ which is the Weyl–Kac denominator function of the affine Lie algebra $\hat{A}_1^{(1)}$. A similar construction we can propose for an arbitrary affine Lie algebra. We are going to consider a general case in a separate publication. In this section we consider the case of the affine Lie algebra $\hat{A}_2^{(2)}$.

Let us define

$$\mathbf{E}^{(2)}_{q,y} = \bigwedge\nolimits_{y^{-1}} E^* \otimes \bigotimes_{n\ge 1} \bigwedge\nolimits_{-q^n y^{-2}} \Psi_2(E^*) \otimes \bigotimes_{n\ge 1} \bigwedge\nolimits_{-q^n y^{2}} \Psi_2(E)$$
$$\otimes \bigotimes_{n\ge 1} S_{q^n y^{-1}} E^* \otimes \bigotimes_{n\ge 1} S_{q^n y} E \otimes \bigotimes_{n=1}^{\infty} S_{q^n}(T_M \oplus T_M^*)$$

where $\Psi_2(E)$ is the second Adams operation on vector bundle E. We remind that

$$\operatorname{ch}(\Psi_2(E)) = \operatorname{ch}(E\otimes E) - \operatorname{ch}(\wedge^2 E) = \sum_{i=1}^{r} e^{2x_i}.$$

The series $\mathbf{E}_{q,y}$ is a geometric variant of the Jacobi triple product and $\mathbf{E}^{(2)}_{q,y}$ is a geometric analog of the quintiple product

$$\vartheta_{3/2}(\tau, z) = q^{\frac{1}{24}} r^{-\frac{1}{2}} \prod_{n\ge 1} (1+q^{n-1}r)(1+q^n r^{-1})(1-q^{2n-1}r^2)(1-q^{2n-1}r^{-2})(1-q^n).$$

We note that

$$\vartheta_{3/2}(\tau, z) = \frac{\eta(\tau)\vartheta(\tau, 2z)}{\vartheta(\tau, z)} \in J_{\frac{1}{2},\frac{3}{2}}(v_\eta).$$

For $\hat{A}_2^{(2)}$ we can give the following definition (compare with Definition 1.1).

Definition 3.1. $\hat{A}_2^{(2)}$-*genus* of a complex vector bundle E of rank r over a compact complex manifold M of dimension d is defined as follows

$$\alpha(M, E; \tau, z) = q^{-d/12} y^{r/2} \int_M \exp\Big(\frac{1}{2}\big(c_1(E) - c_1(T_M)\big)\Big)$$
$$\exp\Big(\big(3p_1(E) - p_1(T_M)\big)\cdot G_2(\tau)\Big) \exp\Big(-\frac{c_1(E)}{2\pi i}\frac{\partial}{\partial z}\log(\vartheta_{3/2}(\tau, z))\Big) \operatorname{ch}(\mathbf{E}^{(2)}_{q,y}) \operatorname{td}(T_M).$$

Theorem 3.1. *Let E be a holomorphic vector bundle of rank r over a compact complex manifold M of dimension d. Let $\alpha(M, E; \tau, z)$ be the $\hat{A}_2^{(2)}$-genus. Then*

$$\eta(\tau)^{d+r} \vartheta_{3/2}(\tau, z)^{d-r} \alpha(M, E; \tau, z) \in J^w_{d, \frac{3}{2}d}$$

is a weak Jacobi form of weight d and index $\frac{3d}{2}$.

The proof is similar to the proof of Theorem 1.2.

§4. SQEG and hyperbolic root systems

We can consider n-fold symmetric product of the manifold M, i.e., the orbifold space $M^{[n]} = M^n/S_n$, where S_n is the symmetric group of n elements. This is a singular manifold but one can define the string orbifold elliptic genus of $M^{[n]}$ (see for details the talk of R. Dijkgraaf at ICM-1998 in Berlin [D]). Using some arguments from the conformal field theory on orbifolds it was proved in [DVV] and [DMVV] that the string elliptic genus of the second quantization $\cup_{n\ge 1} M^{[n]}$ of a Calabi–Yau manifold M coincides with the second quantized elliptic genus of the given manifold:

$$\sum_{n=0}^{\infty} p^n \chi_{orb}(M^{[n]};\, q, y) = \prod_{m\ge 0,\, l,\, n>0} \frac{1}{(1-q^m y^l p^n)^{f(mn,l)}}, \tag{4.1}$$

where $\chi(M,\tau,z) = \sum_{m\ge 0,\ l\in\mathbb{Z}\ (or\,\mathbb{Z}/2)} f(m,l) q^m y^l$ is the elliptic genus of M.

For a $K3$ surface, the product in the left hand side of (4.1) is essentially the power -2 of the infinite product expansion of the product of all even theta-constants has found in [GN1]. Following [DVV, §4] we call the product in (4.1) *second-quantized elliptic genus* (SQEG) of the manifold M.

Theorem 4.1. *(See* [GN4], [G1]*.) Let $M = M_d$ be a compact complex manifold of dimension d with trivial $c_1(M)$,*

$$\chi(M;\,\tau,z) = \sum_{m\ge 0,\ l\in\mathbb{Z}\ (or\,\mathbb{Z}/2)} f(m,l) q^m y^l$$

be its elliptic genus and $\mathrm{SQEG}(M;\,Z)$ *($Z\in\mathbb{H}_2$) be its second quantized elliptic genus. We define a factor*

$$H(M;\,Z) = \begin{cases} \eta(\tau)^{-\frac12(e-3\chi'_{d_0})} \prod_{p=1}^{d_0} \bigl(\vartheta(\tau,pz)\, e^{\pi i p^2\omega}\bigr)^{-\chi'_{d_0-p}} & \text{if } d=2d_0 \\ \eta(\tau)^{-\frac12 e} \prod_{p=1}^{d_0} \bigl(\vartheta(\tau,\frac{2p-1}{2}z)\, e^{\frac14\pi i(2p-1)^2\omega}\bigr)^{-\chi'_{d_0-p+1}} & \text{if } d=2d_0+1 \end{cases}$$

where $e=e(M)$ is Euler number of M and $\chi'_p = (-1)^p\chi_p(M)$ (see (1.2)*). Then the product*

$$E(M;\,Z) = \Psi(M;\,Z)\cdot \mathrm{SQEG}(M;\,Z) \qquad (d=2d_0)$$
$$E^{(2)}(M;\,Z) = (E|\Lambda_2)(M;\,Z) \qquad (d=2d_0+1)$$

determines a Siegel automorphic form of weight $-\frac12\chi'_{d_0}(M)$ if d is even and of of weight 0 if d is odd with a character or a multiplier system of order $24/(24,e)$ with respect to a double extension of the paramodular group Γ_d^+ (resp. Γ_{2d}^+), if d is even (resp. d is odd).

The case of $\mathbf{CY_4}$. The basic Jacobi modular forms for this dimension are the Jacobi forms $\phi_{0,2}$ and $\psi_{0,2}^{(2)}$ (see Theorem 2.2, part 1). They correspond to the following cusp forms for the paramodular group Γ_2 (see [GN1] and [GN4]):

$$\Delta_2(Z) = \text{Exp-Lift}(\phi_{0,2}(\tau,z)) = \mathrm{Lift}(\eta(\tau)^3\vartheta(\tau,z))$$
$$= \sum_{N\ge 1} \sum_{\substack{n,\,m>0,\,l\in\mathbb{Z}\\ n,\,m\equiv 1\,mod\,4\\ 2nm-l^2=N^2}} N\Bigl(\frac{-4}{Nl}\Bigr) \sum_{a\,|\,(n,l,m)} \Bigl(\frac{-4}{a}\Bigr) q^{n/4} y^{l/2} s^{m/2} \in \mathfrak{M}_2^{cusp}(\Gamma_2, v_\eta^6\times v_H)$$

and

$$\Delta_{11}(Z) = \operatorname{Lift}(\eta(\tau)^{21}\vartheta(\tau, 2z)) = \operatorname{Exp-Lift}(\psi^{(2)}_{0,2}(\tau, z)) \in \mathfrak{N}_{11}(\Gamma_2).$$

For an arbitrary Calabi–Yau 4-fold M_4 we have the following formula for its SQEG

$$E(M_4;\, Z) = \Delta_{11}(Z)^{-\chi_0(M)}\Delta_2(Z)^{\chi_1(M)}. \tag{3.2}$$

We note that $\Delta_2(Z)^4$ is the first Γ_2-cusp form with trivial character and $\Delta_{11}(Z)$ is the first cusp form of odd weight with respect to Γ_2.

The Fourier expansion of the cusp forms $\Delta_2(Z)$, $\Delta_{11}(Z)$ and $\frac{\Delta_{11}(Z)}{\Delta_2(Z)}$ coincide with the Weyl–Kac–Borcherds denominator formula of generalized Kac–Moody super-algebras with a system of simple real roots of hyperbolic type determined by Cartan matrix $A_{1,II}$, $A_{2,II}$ and $A_{2,0}$ respectively:

$$A_{2,II} = \begin{pmatrix} 2 & -2 & -6 & -2 \\ -2 & 2 & -2 & -6 \\ -6 & -2 & 2 & -2 \\ -2 & -6 & -2 & 2 \end{pmatrix}, \quad A_{2,0} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}, \quad A_{2,I} = \begin{pmatrix} 2 & -2 & -4 & 0 \\ -2 & 2 & 0 & -4 \\ -4 & 0 & 2 & -2 \\ 0 & -4 & -2 & 2 \end{pmatrix}$$

(see [GN1]–[GN4]). Thus, the formula (3.2) gives us three particular cases of Calabi–Yau 4-folds of Kac–Moody type when the second quantized elliptic genus is a power of the denominator function of the corresponding Lorentzian Kac–Moody algebra:

$$\begin{aligned} E(M_4;\, Z) &= \Delta_{11}(Z)^{-\chi_0} && \text{if } \chi_1 = 0 \\ E(M_4;\, Z) &= \left(\frac{\Delta_{11}(Z)}{\Delta_2(Z)}\right)^{-\chi_0} && \text{if } \chi_0(M) = -\chi_1(M) \\ E(M_4;\, Z) &= \Delta_2(Z)^{\chi_1} && \text{if } \chi_0(M) = 0. \end{aligned}$$

For more details and for the cases of $d > 4$ see [G1]. In [GN2–GN4] we started to classify all generalized Lorentzian Kac–Moody algebras whose denominator function is an automorphic form on the Siegel upper-half plane of genus two.

Conjecture. *The Weyl–Kac–Borcherds denominator function of arbitrary automorphic Lorentzian Kac–Moody algebra with a non-empty system of real simple roots of rank* 3 *is related to the second quantized elliptic genus of some holomorphic vector bundle.*

The Igusa modular form $\Delta_{35}(Z)$ determines the automorphic correction of the simplest hyperbolic Kac–Moody algebra with the Cartan matrix

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 2 \\ -1 & -2 & 2 \end{pmatrix}.$$

The form $\Delta_{35}(Z)$ would be the SQEG of a vector bundle of rank 2 over a manifold of dim$= 14$ (if it would exist) with invariants given in Question 3 in §2.

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