On the homotopy classification problem

Volume 4

Homotopy theory of differential algebras

by
Hans Joachim Baues

Volume 4 consists of a lecture and of the chapters
C1, C2 and D.

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This volume is the fourth part of a study of the homotopy classification problem. There are at the moment five parts:

- Volume 1: Abstract homotopy theory with an Appendix on extensions of categories (Chapter I,II,III,IV,V and chapter Ext).
- Volume 2: Combinatorial homotopy theory of CW - complexes (The first, the second and the third part of chapter A).
- Volume 3: Low dimensional homotopy theory (Chapter B, in preparation).
- Volume 4: Homotopy theory of differential algebras (Chapter C1, C2 and chapter D and a lecture).
- Volume 5: Homotopy theory of topological mapping cones (Chapter E and chapter F).

Introduction of Volume 4:

Differential algebras appear in algebraic topology frequently. For example the singular chain complex of a topological group or of a loop space has the structure of a differential algebra the multiplication of which is induced by the multiplication on the space. In this volume we study the algebraic homotopy theory of differential algebras with coefficients in a principal ideal domain. Then we study connections with topology which are obtained by the chains on a loop space.

For a complete discussion of the contents see chapter introductions. Also the following lecture is an introduction for this volume. Chapter C2 is an analogue of the first part of chapter A. An analogue of the second part of chapter A can also be worked out; this and a continuation of chapter D are in preparation.

Bonn, October 17, 1983

Chapter C1: The cofibration - category of chain algebras

Chapter C2: Homotopy theory of chain algebras

(D) The chains on a loop space

The chains on the loops and 4 - dimensional homotopy types

Hans Joachim Baues

In this lecture I want to describe some results on a problem of J.H.C. Whitehead:

> Classify algebraically the homotopy types of 4 - dimensional polyhedra!

In 1948 J.H.C. Whitehead solved this problem for simply connected 4 - dimensional polyhedra; he first used the 'enriched cohomology ring' [7] and later the 'certain exact sequence' [9] as the classifying invariants which determine the homotopy type. On the other hand he has shown in [8] that two 3 - dimensional polyhedra are homotopy equivalent iff the cellular chain complexes of the universal coverings are homotopy equivalent. These results rely on the following facts (a) and (b) respectively:

(a) Each simply connected 4 - dimensional CW - complex X is homotopy equivalent to the mapping cone C_f of a map f between 1 - point unions of 2 - dimensional and 3 - dimensional spheres,

$$f: \bigvee_A S^3 \vee \bigvee_B S^2 \longrightarrow \bigvee_C S^3 \vee \bigvee_D S^2$$

Here A, B, C and D denote index sets.

(b) Each chain map between chain complexes of universal coverings is geometrically realizable up to dimension 3.

For 4 - dimensional polyhedra which are not simply connected there is no analogue of (a) and we cannot use (b). Therefore the solution of the general problem has to be different from the solutions in the special cases described by J.H.C. Whitehead.

We are not yet able to solve the problem completely but we describe a result which solves the problem 'up to the prime 2'. Moreover, we solve the problem if the second homotopy group π_2 satisfies the condition that multiplication by 2 is an isomorphism or that $\pi_2 = \mathbb{Z}$ is the group of integers.

Our classifying invariant is the chain algebra given by the chains on the loop space. In fact, for all connected polyhedra this chain algebra determines the chains of the universal covering, see (24). Therefore this invariant is more powerful than the chains of the universal covering used by Whitehead.

Also we describe a small model of this chain algebra, which extends the result of Adams and Hilton [1] to the non simply connected case.

The results described in this lecture are just a few items in my forhtcoming book 'On the homotopy classification problem'. This book will contain all proofs and the explicit constructions.

We say a CW - complex X is reduced if the 0 - skeleton of X consists of a single point, which is the base point of X . All maps and homotopies which we consider are basepoint preserving. Let $\mathcal{CW}$ be the category of reduced CW - complexes and of cellular maps. Its homotopy category $\mathcal{CW}/\simeq$ is equivalent to the homotopy category of all path connected CW - spaces. Let

(1) $$\hat{C}_* : \mathit{CW} \longrightarrow \mathit{Chain}$$

be the functor of <u>cellular chains on the universal covering</u>. Here $\mathit{Chain}^{\wedge} = \mathit{Chain}^{\wedge}_{\mathbb{Z}}$ is the following category. Objects are pairs (π, C) where π is a group and where C is a positive chain complex of free (right) $\mathbb{Z}[\pi]$-modules with $C_0 = \mathbb{Z}[\pi]$. A morphism in $\mathit{Chain}^{\wedge}$ is a pair $(\varphi, f):$ $(\pi, C) \longrightarrow (\pi', C')$ where $\varphi: \pi \longrightarrow \pi'$ is a homomorphism between groups and where f is a φ-equivariant chain map, that is $df = fd$ and $f(x \cdot \alpha) = (fx) \cdot (\varphi\alpha)$ for $x \in C$, $\alpha \in \pi$; $f_0 = \mathbb{Z}[\varphi]$. The functor in (1) is given by

$$\hat{C}_*(X) = (\pi_1 X, C_* \hat{X})$$

where $\hat{X}$ is the universal covering of X in which we fixed a basepoint $*$. Thus for each map $f: X \longrightarrow Y$ in CW there is a unique basepoint preserving covering map $\hat{f}: \hat{X} \longrightarrow \hat{Y}$. This map is cellular and induces a $\pi_1(f)$-equivariant chain map, $\hat{f}_*: C_*\hat{X} \longrightarrow C_*\hat{Y}$, between the cellular chain complexes. The functor $\hat{C}_*$ in (1) carries the map f to the pair $(\pi_1(f), \hat{f}_*)$. Two morphisms (φ, f), (φ', f') in $\mathit{Chain}^{\wedge}$ are homotopic if $\varphi = \varphi'$ and if there is a φ-equivariant map $\alpha: C \longrightarrow C'$ of degree $+1$ with $\alpha_0 = 0$ and $d\alpha + \alpha d = f' - f$. The functor (1) induces the functor

(2) $$\hat{C}_*: \mathit{CW}/\simeq \longrightarrow \mathit{Chain}^{\wedge}/\simeq$$

between homotopy categories. As a variant of the Whitehead theorem we have

(3) A map f in CW is a homotopy equivalence if and only if $\hat{C}_* f$ is a homotopy equivalence in $\mathit{Chain}^{\wedge}$.

Moreover, for the full subcategory $\mathcal{CW}^n$ of n-dimensional complexes in $\mathcal{CW}$ Whitehead proves in [8]:

(4) On $\mathcal{CW}^3/\simeq$ the functor $\hat{C}_*$ is full and on $\mathcal{CW}^2/\simeq$ the functor $\hat{C}_*$ is full and faithful.

We derive from (3) and (4)

(5) Two 3-dimensional complexes X, Y in $\mathcal{CW}$ are homotopy equivalent if and only if their chain complexes $\hat{C}_*X$, $\hat{C}_*Y$ are homotopy equivalent in $\mathit{Chain}^\wedge$.

Such a result is not true for 4-dimensional complexes, but we get

(6) <u>Theorem</u>: *Let X, Y be 4-dimensional complexes in $\mathcal{CW}$ and let*

$$F=(\varphi,f):\ \hat{C}_*X \longrightarrow \hat{C}_*Y$$

be a map in $\mathit{Chain}^\wedge$. Then we associate with the triple (X,Y,F) an obstruction element

$$O_{X,Y}(F) \in \hat{H}^4(X,\varphi^*\Gamma(\pi_2Y))$$

with the following property:
The homotopy class of F in $\mathit{Chain}^\wedge/\simeq$ is realizable by a map in $\mathcal{CW}$ if and only if $O_{X,Y}(F)=0$.

Here $\hat{H}^4$ denotes the cohomology with local coefficients and Γ is the quadratic construction of Whitehead [9]; Γ is a functor so that $\Gamma(\pi_2Y)$ is a $\pi_1(Y)$-module which via φ^* is considered as a $\pi_1(X)$-module.

If X and Y are simply connected we can derive from the theorem the result of Whitehead in [7]. In the general case we get:

(7) <u>**Corollary:**</u> *Two 4-dimensional complexes X, Y in CW are homotopy equivalent if and only if there is a homotopy equivalence*

$$F = (\varphi, f) : \hat{C}_* X \simeq \hat{C}_* Y$$

in $\mathrm{Chain}^\wedge$ *with* $O_{X,Y}(F) = 0$.

Thus we have to compute the obstruction $O_{X,Y}(F)$ in terms of invariants of X and Y respectively. This is not yet completely done. We know however, how to compute the image of this obstruction under the following homomorphism:

$$\tau_* : \tilde{H}^4(X, \varphi^* \Gamma(\pi_2 Y)) \longrightarrow \tilde{H}^4(X, \varphi^*(\pi_2 Y \otimes \pi_2 Y)) \ . \tag{8}$$

Here $\tau : \Gamma(A) \longrightarrow A \otimes A$ is the canonical homomorphism associated to the quadratic map $q : A \longrightarrow A \otimes A$, $q(a) = a \otimes a$. Clearly, if $A = \pi_2 Y$ is a $\pi_1 Y$-module then τ is a homomorphism of $\pi_1(Y)$-modules. It is well known that τ is an isomorphism if $A = \mathbb{Z}$ or that τ admits a natural retraction if multiplication by 2 is an isomorphism on A . Thus in these cases τ_* in (8) is injective and therefore the image $\tau_* O_{X,Y}(F)$ vanishes iff $O_{X,Y}(F)$ vanishes. We can describe the element $\tau_* O_{X,Y}(F)$ in terms of F and the chain algebras $C_* \Omega X$ and $C_* \Omega Y$ which are given by the cubical chains on the loop spaces ΩX and ΩY respectively. For the computation of $\tau_* O_{X,Y}(F)$ we only need to know small models of these chain algebras.

To this end we describe the connection of the chain algebra $C_* \Omega X$

and of the chain complex $\hat{C}_*X$.

Let R be a subring of the rationals $\mathbb{Q}$. We introduce the following diagram of categories and functors:

$$\begin{tikzcd}
\mathit{CW} \arrow[r, "C_*\Omega\otimes R"] \arrow[d, "\hat{C}_*\otimes R"] & \mathit{DA}_R \arrow[d, "\hat{B}"] \\
\mathit{Chain}^{\wedge}_R \arrow[r, "\hat{a}"] & \mathit{DF}\hat{M}_R
\end{tikzcd} \tag{9}$$

The functor $\hat{C}_*$ was defined in (1); if we replace in the definition of $\mathit{Chain}^{\wedge}$ the ring $\mathbb{Z}$ by the subring R of $\mathbb{Q}$ we obtain the category $\mathit{Chain}^{\wedge}_R$.

The subcategory DA_R of chain algebras is defined as follows:

A <u>chain algebra</u> A is a graded, associative algebra A with unit together with a differential $d\colon A \longrightarrow A$ of degree -1 and an augmentation $\varepsilon\colon A \longrightarrow R$ such that

(a) A , as a module, is a free R-module,

(b) (A,d) is a chain complex with $A_i = 0$ for $i < 0$,

(c) ε is an algebra homomorphism and a chain map,

(d) the multiplication $\mu\colon A\otimes A \longrightarrow A$ is a chain map.

We say A is <u>good</u> if

(e) the homology in degree 0 , H_0A , is free as an R-module.

Let $\mathcal{D}A_R$ be the category of good chain algebras. The maps are of degree zero and preserve μ, d and ε.

Clearly, $A = C_*\Omega X \otimes R$ is a chain algebra in $\mathcal{D}A_R$. Here ΩX is the Moore loop space which has an associative multiplication. The homology in degree 0 is

(10) $$H_0(C_*\Omega X \otimes R) = R[\pi_1 X] .$$

For a chain algebra we have the canonical projection

(11) $$\lambda: A \longrightarrow H_0 A = H$$

which is trivial in degree >1. Via λ the algebra $H = H_0 A$ is an A-module. Let BA be the reduced bar construction and let $\tau: BA \longrightarrow A$ be the canonical twisting cochain. Then via (11) the two-sided bar construction

(12) $$\hat{B}A = H \otimes_\tau BA \otimes_\tau H$$

is defined, see [3], [4]. This is an object in the following category $\mathcal{D}F\hat{M}_R$. Objects of $\mathcal{D}F\hat{M}_R$ are pairs $(H, \hat{C})$ where H is an augmented (non graded) algebra with unit which is free as an R-module and where $\hat{C}$ is a chain complex of H-bimodules with the following properties:

(a) $\hat{C}_n$ is a free H-bimodule and the differential d is a map of H-bimodules, $n \in \mathbb{Z}$,

(b) $\hat{C}_0 = H \otimes H$, $\hat{C}_n = 0$ for $n < 0$, A map $\sigma = (\varphi, \sigma): (H, \hat{C}) \longrightarrow (H', \hat{C}')$ is a homomorphism $\varphi: H \longrightarrow H'$ between augmented (non graded) algebras together with a φ-biequivariant chain map $\sigma: \hat{C} \longrightarrow \hat{C}'$ with $\sigma_0 = \varphi \otimes \varphi$ in degree 0.

The functor $\hat{\alpha}$ in (9) carries the object (π, C) to $(H, \hat{C})$ with

(13) $$H = R[\pi] \quad \text{and} \quad \hat{C} = C \otimes_H \bar{\xi}^* (H \otimes H^{op}) \quad .$$

We identify an H-bimodule with a right $H \otimes H^{op}$-module. Here H^{op} is the opposite algebra and $H \otimes H^{op}$ is the enveloping algebra, see [2]. If $H = R[\pi]$ we have the canonical homomorphism

$$\xi: H \longrightarrow H^{op} \ , \ \xi[\alpha] = [\alpha^{-1}]$$

which yields

(14) $$\bar{\xi}: H \longrightarrow H \otimes H^{op} \ , \ \bar{\xi}(x) = x \otimes \xi x \ .$$

Via $\bar{\xi}$ the algebra $H \otimes H^{op}$ is a left H-module which is denoted by $\bar{\xi}^* (H \otimes H^{op})$. The functor $\hat{\alpha}$ is defined on maps in the obvious way.

Now all categories and functors in diagram (9) are defined. We introduce homotopy categories by localizing with respect to weak equivalences:

(15) <u>Definition</u>: We call $f: X \longrightarrow Y$ in $\mathcal{CW}$ a <u>twisted R-equivalence</u> if f induces isomorphisms

$$f_*: \pi_1 X \cong \pi_1 Y \qquad \text{and}$$

$$f_*: \pi_n X \otimes R \cong \pi_n Y \otimes R \ , \quad n \geq 2 \ .$$

Let $Ho_R \mathcal{CW}$ be the category obtained by localizing $\mathcal{CW}$ with respect to twisted R-equivalences.

In $\mathcal{DA}_R$ a weak equivalence is a map which induces isomorphisms in homology. In $\mathit{Chain}_R^{\wedge}$ and $\mathcal{DF}\hat{M}_R$ weak equivalences are pairs (φ,f) where φ is an isomorphism and where f induces isomorphisms is homology. Moreover, twisted R-equivalences are the weak equivalences in $\mathcal{CW}$. Then we get:

(16) **Proposition:** *All functors in (9) carry weak equivalences to weak equivalences.*

For $\hat{C}_*\otimes R$ this is part of the R-local Whitehead theorem, for $\hat{B}$ this is proven in [5].

Moreover, we have the following important result:

(17) **Theorem:** *(A) A map f in $\mathcal{CW}$ is a twisted R-equivalence if and only if the induced map $\hat{C}_*f\otimes R$ is a weak equivalence in Chain_R.*

(B) A map g in $\mathcal{DA}_R$ is a weak equivalence if and only if the induced map $\hat{B}g$ is a weak equivalence in $\mathcal{DF}\hat{M}_R$.

Part (A) is a variant of the Whitehead theorem, see (3), part (B) seems to be new. For A in $\mathcal{DA}_R$ the homology of $\hat{B}A$ is denoted by

$$\mathrm{Tor}_A(H_oA,H_oA) = H_*(\hat{B}A) \quad . \tag{18}$$

Therefore, (B) is equivalent to

(19) **Theorem:** *A map $f: A \longrightarrow B$ in $\mathcal{DA}_R$ induces isomorphisms in homology if and only if*

$$Tor_f(H_of,H_of): Tor_A(H_oA,H_oA) \longrightarrow Tor_B(H_oB,H_oB)$$

is an isomorphism.

This result is well known in the connected case $H_0A = H_0B = R$. We now localize all categories in diagram (9) with respect to weak equivalences. By (16) we obtain the diagram of functors:

(20)
$$\begin{tikzcd}
Ho_R\mathit{CW} \arrow[rr, "C_*\Omega\otimes R"] \arrow[dd, "\hat{C}_*\otimes R"] \arrow[dr, dashed, "A_R"] & & Ho\mathit{DA}_R \arrow[dd, "\hat{B}"] \\
& \hat{\alpha}^* Ho\mathit{DA}_R \arrow[ur, dashed] \arrow[dl, dashed] \arrow[r, phantom, "\text{pull}"] & \\
Ho\mathit{Chain}^{\wedge}_R \arrow[rr, "\hat{\alpha}"'] & & Ho\mathit{DF}\hat{M}_R
\end{tikzcd}$$

In extension of equation (10) we get:

(21) <u>Theorem</u>: *Diagram (20) commutes, that is, there is a natural equivalence*

$$t:\ \hat{B}(C_*\Omega X\otimes R) \xrightarrow{\sim} \hat{\alpha}(\hat{C}_*X\otimes R)$$

in $Ho\mathit{DF}\hat{M}_R$.

Let $\hat{\alpha}^* Ho\mathit{DA}_R$ be the pull back category. Then commutativety of the diagram yields the functor A_R in (20). The functor $\hat{\alpha}$ is faithful so that the pull back category can be considered as a subcategory of $Ho\mathit{DA}_R$.

(22) <u>Remark</u>: The result seems only to be known in the case $X = K(\pi,1)$. In this case we have the equivalence

$$C_*\Omega X \otimes R \sim R[\pi]$$

in $\mathcal{DA}_R$ and

$$\hat{B}(R[\pi]) = \hat{B}(R[\pi], R[\pi])$$

is the normalized bar construction of the non graded algebra $R[\pi]$. From (21) we deduce the classical equations

$$\hat{H}_*(R[\pi], \Gamma) = \hat{H}_*(K(\pi,1), {}_\xi\Gamma) \quad ,$$

$$\hat{H}^*(R[\pi], \Gamma) = \hat{H}^*(K(\pi,1), \Gamma_\xi) \quad .$$

Here Γ is a $R[\pi]$-bimodule and the left side is the Hochschild (co-) homology of the algebra $R[\pi]$. The right side is the (co-) homology of the group π, compare chapter X, theorem 5.5 in [5]. //

For an algebra A in $\mathcal{DA}_R$ we have the one-sided bar construction (see [3], [4]):

(23) $$BA \otimes_\tau H \quad , \quad H = H_0 A \quad ,$$

where H is an A-module by $\lambda: A \longrightarrow H_0 A$. From (21) we deduce

(24) Corollary: *There is a natural equivalence*

$$B(C_*\Omega X \otimes R) \otimes_\tau H \xrightarrow{\sim} \hat{C}_* X \otimes R$$

in $\mathit{HoChain}^\wedge$ *where* $H = R[\pi] = H_0 \Omega X$.

Proof: For $A = C_*\Omega X \otimes R$ and $C = \hat{C}_* X \otimes R$ let

$$1 \otimes \varepsilon: \Lambda = H \otimes H^{op} \longrightarrow H \otimes R = H \quad .$$

Then we get

$$BA\otimes_{\tau}H = \hat{B}(A)\otimes_{\Lambda}(1\otimes\varepsilon)^{*}H$$
$$\sim (\tilde{\alpha}C)\otimes_{\Lambda}(1\otimes\varepsilon)^{*}H = C \quad .$$

//

The corollary shows that the chain algebra $C_*\Omega X\otimes R$ determines up to weak equivalence the chain complex of the universal covering as mentioned in the introduction.

We now are ready to state our results on the classification problem for 4-dimensional polyhedra:

(25) **Theorem:** *Let* X *and* Y *be* CW*-complexes of dimension* ≤ 4 *and let* $1/2\in R\subset \mathbb{Q}$ *. Then* X *and* Y *are equivalent in* $Ho_R CW$ *if and only if* $A_R(X)$ *and* $A_R(Y)$ *are equivalent in the pull back category* $\tilde{\alpha}^{*}HoDA_R$.

(26) **Theorem:** *Let* X *and* Y *be* CW*-complexes of dimension* ≤ 4 *and assume that* $\pi_2 Y=\mathbb{Z}$ *or that multiplication by* 2 *is an isomorphism on* $\pi_2 Y$ *. Then the complexes* X *and* Y *are homotopy equivalent in* CW *if and only if* $A_{\mathbb{Z}}(X)$ *and* $A_{\mathbb{Z}}(Y)$ *are equivalent in* $\tilde{\alpha}^{*}HoDA_{\mathbb{Z}}$.

(27) **Remark on the proof of (25) and (26):** Let $A=C_*\Omega X\otimes R$ and let $B=C_*\Omega Y\otimes R$. We show that for a map

$$F=(\varphi,f):\ \hat{C}_*X\otimes R \longrightarrow \hat{C}_*Y\otimes R$$

there is an obstruction

$$O_{A,B}(\tilde{\alpha}F)\in \tilde{H}^{4}(X,\varphi^{*}(\pi_2 Y\otimes\pi_2 Y)\otimes R)$$

with the following properties. We have:

$$O_{A,B}(\widehat{\alpha F}) = 0$$

if and only if there is $\bar{F}: A \longrightarrow B$ in $\mathrm{Ho}\mathcal{DA}_R$ such that for the equivalence t in (21)

$$t(\widehat{B\bar{F}}) = (\widehat{\alpha F})t$$

in $\mathrm{Ho}\mathcal{DF}\hat{M}_R$. Moreover, for τ_* in (8) we have

$$\tau_* O_{X,Y}(F) = O_{A,B}(\widehat{\alpha F}) \quad .$$

This shows that (26) is a consequence of (7), since the assumptions in (26) imply that τ_* is injective. Similarly we prove (25). //

Next we show that the localized categories in diagram (20) can be replaced by homotopy categories. We introduced already the notion of homotopy in $\mathit{Chain}^{\wedge}$ in (2). We have an isomorphism of categories

(28) $$\mathrm{Ho}\mathit{Chain}^{\wedge}_R = \mathit{Chain}^{\wedge}_R/\simeq \quad .$$

Similarly, we have

(29) $$\mathrm{Ho}\mathcal{DF}\hat{M}_R = \mathcal{DF}\hat{M}_R/\simeq$$

where two maps $(\varphi,\sigma),(\varphi',\sigma')$ in $\mathcal{DF}\hat{M}_R$ are homotopic if $\varphi=\varphi'$ and if there is a φ-biequivariant map α of degree $+1$ with $\alpha_0=0$ and $d\alpha+\alpha d=\sigma'-\sigma$.

We now consider $\mathrm{Ho}_R\mathcal{CW}$. We say that a space X is a <u>twisted</u> R-<u>space</u> if X is a complex in $\mathcal{CW}$ for which the universal covering is an R-space, that is $\pi_n X = \pi_n\hat{X} = \pi_n\hat{X}\otimes R$, $n \geq 2$. Let $\mathcal{CW}_R$ be the full subcategory of $\mathcal{CW}$ consisting of twisted R-spaces. Then we have the canonical equivalence of categories:

(30) $$\mathrm{Ho}_R\mathcal{CW} \sim \mathrm{Ho}_R\mathcal{CW}_R = \mathcal{CW}_R/\simeq \quad .$$

The equivalence is induced by the inclusion $\mathcal{CW}_R \subset \mathcal{CW}$.

Moreover, let $\mathcal{DFA}_R$ be the full subcategory of $\mathcal{DA}_R$ consisting of chain algebras A for which the underlying algebra is a free associative algebra the generators of which have augmentation 0 , we write $A=(T(V), d)$ where V is the set of generators of A . We introduce the notion of homotopy on $\mathcal{DFA}_R$ as follows: Two maps $f,g\colon A \longrightarrow B$ are homotopic if there is a map $\alpha\colon A \longrightarrow B$ of degree 1 of the underlying graded modules with

$$\begin{aligned}&\alpha d + d\alpha = g - f\\ &\alpha(xy) = (\alpha x)(gy) + (-1)^{|x|}(fx)(\alpha y) \quad .\end{aligned}$$

Homotopy '$\simeq$' is a natural equivalence relation on $\mathcal{DFA}_R$.

(31) Theorem: *We have the canonical equivalence of categories*

$$\mathit{Ho}\mathcal{DA}_R \sim \mathit{Ho}\mathcal{DFA}_R = \mathcal{DFA}_R/\simeq \quad .$$

The equivalence is induced by the inclusion $\mathcal{DFA}_R \subset \mathcal{DA}_R$. This result seems to be known only in case R is a field, see [6]. It is in fact available if R is a principle ideal domain.

By use of the equivalences in (28),(29),(30) and (31) we can replace all categories in diagram (20) by homotopy categories. This is important for computations, in particular the chain algebra $C_*\Omega X \otimes R$ can be replaced by a free chain algebra $A=(T(V),d)$ with a small number of generators:

(32) Theorem: *Let X be a CW-complex in CW with cellular chain complexes C_*X. (Here $C_nX = H_n(X^n, X^{n-1})$ is the free abelian group generated by the n-cells of X.) For*

$$V_n = \begin{cases} s^{-1}C_1X \oplus s^{-1}C_1X & , \quad n=0 \ , \\ C_1X \oplus s^{-1}C_2X & , \quad n=1 \ , \\ s^{-1}C_{n+1}X & , \quad n \geqq 2 \end{cases}$$

there is a differential d on the tensoralgebra $T(V)$ such that the chain algebra $A = (T(V),d)$ is equivalent to $C_*\Omega X$ in $HoDA_{\mathbb{Z}}$.

This result shows that theorem (25) and (26) can be used for explicit computations.

Theorem (32) is known for the special case that X has trivial 1-skeleton $X^1 = *$. Then we have $C_1X = 0$ and $V = s^{-1}\tilde{C}_*(X)$, and the differential of the theorem is the one constructed by Adams and Hilton in [1]. The method of Adams and Hilton relies on the Moore comparison theorem for spectral sequences which is not available in the non simply connected case. Our proof of theorem (32) is totally different and uses a new technique. All details and many more facts related to the results above will be contained in my forthcoming book 'On the homotopy classification problem'.

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Chapter C: The homotopy theory of chain algebras

Introduction:

In the first part, chapter C1, we show that the category of chain algebras is a cofibration category and we discuss the basic homotopy theory in this category. In particular, we consider homotopy groups and homotopy groups of function spaces for chain algebras. These groups are related to the homology of the chain complex of derivations, see (C1.§ 6) . The arguments and constructions in this chapter are to some extent typical for the homotopy theory in various algebraic categories, compare the examples in (I.§ 5) .

In the second part, chapter C2, the homotopy category of chain algebras is approximated by a tower of categories in a similar way as this was done in chapter A for the homotopy category of CW - complexes. Therefore, many results of combinatorial homotopy theory are now available for chain algebras. It turns out that twisted cohomology for chain algebras is the same as the Hochschild cohomology. The Whitehead theorem for CW - complexes corresponds to the result that a map $f: A \longrightarrow B$ between chain algebras (with $H_o A$ and $H_o B$ free as modules) is a weak equivalence if and only if the induced map between the Hochschild homology groups is an isomorphism, see (C2.2.18). Moreover, we will see that the theory on finiteness obstructions of C.T.C. Wall [8] for CW - complexes has a strict analogue for chain algebras. This result is based on model constructions, in particular minimal models are constructed (these generalize those in [2], see (C2.§ 9)). We point out that in the literature homotopy theory of chain algebras usually considers only connected chain algebras. Most of the results here also deal with chain algebras which are not connected.

Chapter C1: The cofibration - category of chain algebras

§ 0 Notation

Let R be a fixed principal ideal domain of coefficients with unit $1 \in R$.

A (graded) module V is a sequence of R-modules, $V = \{V_n : n \in \mathbb{Z}\}$. V is free if all V_n are free R-modules and of finite type if all V_n are finitely generated R-modules. An element $x \in V$ has degree $|x| = n$ iff $x \in V_n$. A map $f: V \longrightarrow V'$ of degree r $(r \in \mathbb{Z})$ is a sequence of R-linear maps $f_n: V_n \longrightarrow V_{n+r}$, $n \in \mathbb{Z}$. A graded module V is positive if $V_n = 0$ for $n < 0$.

A chain complex V is a positive module V with a map $d: V \longrightarrow V$ of degree -1 satisfying $dd = 0$. The homology HV is the graded module kernel d / image d. A chain map $f: V \longrightarrow V'$ is a map of degree 0 with $df = fd$. f is a weak equivalence if f induces an isomorphism $f_*: HV \cong HV'$ in homology. We consider a graded module as being a chain complex with trivial differential $d = 0$.

We denote by $*$ the trivial chain complex which is R in degree 0 and is 0 in all other degrees. An augmentation is a chain map $\varepsilon: V \longrightarrow *$. $\tilde{V}$ denotes the kernel of ε and $H\tilde{V} = \tilde{H}V$ is the reduced homology. An augmented chain complex V is pointed if we have a chain map $i: * \longrightarrow V$ with $\varepsilon i = 1$. A map f is pointed if $fi = i$ and $\varepsilon f = \varepsilon$.

The direct sum of chain complexes M and N is $M \oplus N$, $(M \oplus N)_n = M_n \oplus N_n$, with the differential $d(x + y) = dx + dy$ for $x \in M_n$,

$y \in N_n$. Clearly, $H(M \otimes N) = HM \otimes HN$. For a pointed chain complex V we have $V = * \otimes \tilde{V}$.

The tensor product $M \otimes N$ of chain complexes is given by

(0.1)
$$(M \otimes N)_n = \bigoplus_{r+s=n} M_r \otimes N_s$$
$$d(x \otimes y) = (dx) \otimes y + (-1)^{|x|} x \otimes dy \ .$$

Here $\otimes$ is the tensor product of R-modules over R . For the differential d on $M \otimes N$ we also write $d = d \otimes 1_N + 1_M \otimes d$, ($1 = 1_M$ denotes the identity of M). Here we use the Koszul convention for signs, namely, the interchange isomorphism

(0.2)
$$T: M \otimes N \longrightarrow N \otimes M$$

is given by $T(x \otimes y) = (-1)^{|x||y|} y \otimes x$ and any calculation which involves the interchange map will be made accordingly. For instance $(1 \otimes d)(x \otimes y) = (-1)^{|x|} x \otimes dy$ since d has degree -1 . We have

(0.3)
$$j: HV \otimes HW \longrightarrow H(V \otimes W)$$

by $j(\{x\} \otimes \{y\}) = \{x \otimes y\}$. Here $\{x\} \in HV$ denotes the homology class represented by the cycle x . The homomorphism j enters in the Künneth formula.

(0.4) Definition: A (graded) algebra A is a positive module A together with a map $\mu: A \otimes A \longrightarrow A$ of degree 0 and an element $1 \in A_0$ such that the multiplication $x \cdot y = \mu(x \otimes y)$ is associative and 1 is the neutral element, $(1 \cdot x = x \cdot 1 = x)$. A (non graded) algebra is a graded algebra A which is concentrated in degree 0, that is $A_0 = A$.

A map $f: A \longrightarrow B$ between algebras is a map of degree 0 with $f(1) = 1$ and $f(x \cdot y) = f(x) \cdot f(y)$. An augmentation of an algebra A is a map $\varepsilon: A \longrightarrow *$ between algebras. An augmented algebra is pointed by $i: * \longrightarrow A$, $i(1) = 1$. The quotient module

$$QA = \tilde{A} / \tilde{A} \cdot \tilde{A}$$

is the module of indecomposables of A. Here $\tilde{A} \cdot \tilde{A}$ is the image of $\mu: \tilde{A} \otimes \tilde{A} \longrightarrow \tilde{A}$. A map $f: A \longrightarrow B$ between algebras with $\varepsilon f = \varepsilon$ induces a map $Qf: QA \longrightarrow QB$ between modules. //

þ.5) <u>Definition</u>: For a graded module V we have the <u>tensor algebra</u>

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$$

where $V^{\otimes n} = V \otimes \ldots \otimes V$ is the n-fold product, $V^{\otimes 0} = R$. We have inclusions and projections of graded modules

$$V^{\otimes n} \xrightarrow[i_n]{} T(V) \xrightarrow[p_n]{} V^{\otimes n} .$$

The tensor algebra is an algebra with multiplication given by $V^{\otimes n} \otimes V^{\otimes m} = V^{\otimes(n+m)}$. The algebra is pointed by $i = i_0$, $\varepsilon = p_0$. We clearly have $QT(V) = V$. For a map $\alpha: V \longrightarrow W$ of degree 0 let

$$T(\alpha): T(V) \longrightarrow T(W)$$

be given by $T(\alpha)(x_1 \otimes \ldots \otimes x_n) = \alpha x_1 \otimes \ldots \otimes \alpha x_n$. Then $QT(\alpha) = \alpha$. //

6) <u>Definition</u>: An algebra A is <u>free</u> if there is a submodule $V \subset A$ with the following properties:

(1) V is a free module.

(2) The homomorphism $T(V) \longrightarrow A$ of algebras given by $V \subset A$ is an isomorphism.

In this case V generates the free algebra A. If E is a basis of V the free monoid $Mon(E)$, generated by E, is a basis of the free R-module A. Moreover, the composition $V \subset \tilde{A} \longrightarrow QA$ is an isomorphism of free modules.

We say, an algebra A is connected if $A_o = R$ is the coefficient ring.

(0.7) Lemma: *Let $F: A \longrightarrow A'$ be a homomorphism between connected free algebras, then F is an isomorphism if and only if $QF: QA \longrightarrow QA'$ is an isomorphism.*

Proof: We can use an inductive argument over the degree. The assumption of connectedness is crucial for the direction '$\Leftarrow$'. □

For algebras A and B we have the free product $A \amalg B$ which is the push out of $A \longleftarrow * \longrightarrow B$ in the category of algebras. Such free products exist. In particular, the free product $B \amalg T(V)$ of B with the free algebra $T(V)$ is given by

$$B \amalg T(V) = \bigoplus_{n \geq 0} B \otimes (V \otimes B)^{\otimes n} . \tag{0.8}$$

Here the multiplication is defined by

$$(a_o \otimes v_1 \otimes \ldots \otimes v_n \otimes a_n)(b_o \otimes w_1 \otimes \ldots \otimes w_m \otimes b_m) =$$

$$a_o \otimes v_1 \otimes \ldots \otimes v_n \otimes (a_n \cdot b_o) \otimes w_1 \otimes \ldots \otimes w_m \otimes b_m$$

with $a_i, b_j \in B$ and $v_i, w_j \in V$. The product $a_n \cdot b_o$ is taken in the algebra B.

If B is augmented by $\epsilon_B : B \longrightarrow *$ we obtain the augmentation $\epsilon : B \amalg T(V) \longrightarrow *$ by $\epsilon(b) = \epsilon_B(b)$ for $b \in B$ and $\epsilon(v) = 0$ for $v \in V$.

§ 1 The category of chain algebras

(1.1) Definition: A chain algebra A is a graded algebra A together with a differential $d: A \longrightarrow A$ of degree -1 and an augmentation $\varepsilon: A \longrightarrow *$ such that

(a) A ,as a module,is a free R-module,

(b) (A,d) is a chain complex,

(c) ε is an algebra homomorphism and a chain map,

(d) the multiplication $\mu: A \otimes A \longrightarrow A$ is a chain map, that is

$$d(x\cdot y) = (dx)y + (-1)^{|x|}x(dy) .$$

A map $f: A \longrightarrow B$ between chain algebras is a map of degree 0 with $f(1)=1$, $\varepsilon f=\varepsilon$, $df=fd$ and $f(x\cdot y)=fx\cdot fy$. Let $\mathcal{DA}$ be the category of chain algebras and of such maps. //

The trivial chain complex $*$ is also a chain algebra which is the initial and the final object in the category $\mathcal{DA}$ of chain algebras.

The homology HA of a chain algebra A is an augmented algebra with the multiplication

(1.2) $$HA \otimes HA \xrightarrow[j]{} H(A \otimes A) \xrightarrow[\mu_*]{} H(A)$$

and with the augmentation $\varepsilon_*: HA \longrightarrow H* = *$. The unit in HA is

$1 = \{1\} \in H_0A$. Moreover, the homology H_0A in degree 0 is a (non graded) algebra with augmentation ε_* .

(1.3) __Definition__: We say a chain algebra A is __free__ if A is free as an augmented algebra, see (0.5) and (0.6). //

We now introduce the structure of a cofibration category for $\mathcal{D}A$.

(1) A map $f\colon B \longrightarrow A$ in $\mathcal{D}A$ is a __weak equivalence__ iff f induces an isomorphism $f : HB \longrightarrow HA$ in homology.

(2) A map $B \longrightarrow A$ in $\mathcal{D}A$ is a __cofibration__ if there is a submodule V of A with the following properties:

(a) V is a free module with $\varepsilon(V) = 0$.

(b) the homomorphism $B \amalg T(V) \longrightarrow A$ of algebras, given by $B \longrightarrow A$ and $V \subset A$, is an isomorphism of algebras.

We call V a __module of generators__ for $B \rightarrowtail A$.

__Remark__: A chain algebra A is free iff $* \longrightarrow A$ is a cofibration. Thus the subcategory of free chain algebras is the category $(\mathcal{D}A)_c$ of cofibrant objects in $\mathcal{D}A$. Clearly, V in (2) is free by (1.1)(a) since R is a principal ideal domain. //

1.4) __Theorem__: *The category* $\mathcal{D}A$ *with the structure (1), (2) above is a cofibration category in which all objects are fibrant.*

__Proof__: The composition of cofibrations is a cofibration since we have

$$(B \amalg T(V)) \amalg T(W) = B \amalg (T(V) \amalg T(W)) = B \amalg T(V \oplus W) \quad .$$

Thus the composition axiom C1 is clearly satisfied. The push out axiom C2 is proven in § 2. Axiom C3 is proven in (1.5) below and C4 is proven in § 4. □

<u>Remark</u>: In fact, most arguments needed for the proof of (1.4) are implicitely contained in the paper of Adams and Hilton [1]. Since, however, these arguments are quite hidden in the context of [1], we prefer to give a complete proof of (1.4) in the following sections.

The following proposition is the factorization axiom C3 in $\mathcal{D}A$.

(1.5) <u>Proposition</u>: *Let* $f\colon B \longrightarrow X$ *be a map in* $\mathcal{D}A$ *. Then there exists a cofibration* i *and a weak equivalence* g *with* $f = gi\colon B \rightarrowtail A \xrightarrow{\sim} X$.

(1.6) <u>Definition</u>: We call a cofibration $B \subset A = B \amalg T(V)$ a <u>simple cofibration</u> if the differential has the property $dV \subset B$. In this case we write $A = B[V]$.

(1.7) <u>Lemma</u>: *For a cofibration* $B \subset A$ *there is a filtration*

$$B \subset A^0 \subset A^1 \subset \ldots\ldots \subset A$$

of simple cofibrations $B \subset A^0$ *and* $A^i \subset A^{i+1}$ $(i \geq 0)$ *with* $A = \lim A^i$.

<u>Proof</u>: For $A = B \amalg T(V)$ let $V^i = \{x \in V : |x| \leq i\}$. Then $A^i = B \amalg T(V^i)$ is a chain subalgebra of A with $A^{i+1} = A^i[V_{i+1}]$. □

(1.8) <u>Lemma</u>: *Let* B *be a chain algebra with differential* d_B *and let* V *be a positive module. For a homomorphism* $d\colon V \longrightarrow B$ *of degree* -1 *with* $d_B d = 0$ *there exists a unique differential* d_A *on* $A =$

$B \amalg T(V)$ which extends d such that $B \subset A$ is a simple cofibration.

Proof: Clearly, by the derivation formula there is at most one d_A . Also d_A is well defined since the derivation formula is compatible with the associativity law, that is

$$\begin{aligned} d((xy)z) &= d(xy)\cdot z + (-1)^{|xy|}xy\,dz \\ &= (dx)yz + (-1)^{|x|}x\,d(yz) \\ &= d(x(yz)) \quad . \end{aligned}$$

$\square$

Lemma (1.7) and (1.8) yield an inductive construction of cofibrations. This is useful in the construction of the following proof.

Proof of (1.5): We construct inductively simple cofibrations

$$B = A^{-1} \subset A^{o} \subset \ldots \subset \lim A^{i} = A$$

and extensions $g_n : A^n \longrightarrow X$ of f such that $g_{n*} : H_i A^n \longrightarrow H_i X$ is an isomorphism for $i < n$ and is surjective for $i = n$. This is true for $n = -1$. Assume we have constructed g_n . Then we choose $V = V_{n+1}$ and d such that (Z denotes the cycles)

$$V \xrightarrow[d]{} (ZA^n)_n \xrightarrow[p]{} H_n A^n$$

maps surjectively onto kernel$(g_n)_*$. Therefore we can choose g' such that

$$\begin{tikzcd} A^n_n \arrow[r, "g_n"] & X_n \\ V \arrow[u, "d"] \arrow[r, "g'"] & X_{n+1} \arrow[u, "d"'] \end{tikzcd}$$

commutes. Now g' yields the map

$$g' : A' = A^n \amalg T(V) \longrightarrow X$$

of chain algebras which extends g_n .

Clearly, $H_i A' = H_i A^n$ for $i < n$. For $i = n$ we have $H_n A' = (ZA^n)_n / d'A'_{n+1}$ where

$$d' : A'_{n+1} = A^n_{n+1} \oplus A^n_o \otimes V \otimes A^n_o \longrightarrow (ZA^n)_n$$

as follows by (2.1). By construction of V the map pd' maps surjectively onto kernel g_{n*} . Therefore g' induces the isomorphism $H_n A' \cong H_n X$. Now we choose $W = W_{n+1}$ and g'' such that the composition

$$W \xrightarrow{g''} (ZX)_{n+1} \longrightarrow H_{n+1} X$$

is surjective. We set $dW = 0$. Then the extension

$$g_{n+1} : A^{n+1} = A' \amalg T(W) \longrightarrow X$$

of g' given by g'' induces an isomorphism $H_i g_{n+1}$ for $i \leq n$ and a surjection $H_{n+1} g_{n+1}$. $\square$

§ 2 A spectral sequence for cofibrations and the push out axiom

We first observe that for a cofibration i and a map f there exists the push out diagram in $\mathcal{DA}$

(2.1)
$$\begin{tikzcd}
A \arrow[r, "\bar{f}"] & X = A \cup_f Y \\
B \arrow[u, tail, "i"'] \arrow[r, "f"] & Y \arrow[u, tail, "\bar{i}"']
\end{tikzcd}$$

where $\bar{i}$ is a cofibration.

Proof: By (1.7) it is enough to prove the existence of the push out (2.1) for a simple cofibration $B \rightarrowtail A = B[V]$. For such a cofibration we define

$$X = Y[V] \ , \ d_X = fd_A : V \longrightarrow B \longrightarrow Y \ ,$$

see (1.8). $\square$

For the proof of the push out axiom C2 in $\mathcal{DA}$ it remains to show:

(2.2) Proposition: *If f is a weak equivalence then $\bar{f}$ in (2.1) is a weak equivalence.*

(2.3) Remark: By (1.7) and a degree argument it is enough to prove (2.2) for a simple cofibration i in (2.1). This simplifies the spectral sequence below a little bit.

For the proof of (2.2) we use the following spectral sequence of a cofibration: Let $B \rightarrowtail A = B \amalg T(V)$ be a cofibration generated by V. We introduce the double degree $|x| = (p,q)$ of a typical element

$$x = b_o v_1 b_1 \dots v_n b_n \in A$$

by $p = \sum_{i=1}^{n} |v_i|$ and $q = \sum_{i=0}^{n} |b_i|$, compare (0.8).

Let $A_{p,q}$ be the module of elements in A of bidegree (p,q) and let

(2.4) $$F_p A = \bigoplus_{\substack{i \leq p \\ j \geq 0}} A_{i,j} \quad .$$

The modules $F_p A$ form a filtration of subchain complexes in A which is bounded above since $A_n \subset F_n A$. This yields the spectral sequence $\{E^n_{p,q} A, d^n\}$ which converges to HA . We obtain the E^1 - term as follows: We have

$$E^o_{p,q} = (F_p A / F_{p-1} A)_{p+q} = A_{p,q} \quad .$$

Using the interchange isomorphism T we get

$$A_{p,q} \cong \bigoplus_{n \geq 0} (V^{\otimes n})_p \otimes (B^{\otimes (n+1)})_q \quad ,$$

compare (0.8). Moreover, the differential d^o is given by the commutative diagram

$$\begin{tikzcd}
E^o_{p,q} \arrow[d, "d^o"] \arrow[r, phantom, "\cong"] & \bigoplus_{n \geq 0} (V^{\otimes n})_p \otimes (B^{\otimes (n+1)})_q \arrow[d, "\bigoplus_{n \geq 0} 1 \otimes d_B"] \\
E^o_{p,q-1} \arrow[r, phantom, "\cong"] & \bigoplus_{n \geq 0} (V^{\otimes n})_p \otimes (B^{\otimes (n+1)})_{q-1}
\end{tikzcd} \quad .$$

Here d_B is the differential on $B^{\otimes (n+1)}$ determined by the differential on B . Since V is a free R-module we derive

$$(2.5)\qquad \begin{aligned} E^1_{p,q} &= H(E^0_{p,q}, d^0) \\ &\cong \bigoplus_{n\geq 0} (V^{\otimes n})_p \otimes H_q(B^{\otimes (n+1)}) \ . \end{aligned}$$

We now use the comparison theorem for spectral sequences for the proof of (2.2). Since (2.1) is a push out diagram we have $X = Y \amalg T(V)$ and $\bar{f}(u) = v$ for $v \in V$. Therefore $\bar{f}$ induces a map between spectral sequences such that for the E^1-term the following diagram commutes:

$$(2.6)\qquad \begin{tikzcd} E^1_{p,q}A \arrow[d, "\bar{f}_*"] & \cong & \bigoplus_{n\geq 0} (V^{\otimes n})_p \otimes H_q(B^{\otimes (n+1)}) \arrow[d, "\bigoplus_{n\geq 0} 1 \otimes H_q(f^{\otimes (n+1)})"] \\ E^1_{p,q}X & = & \bigoplus_{n\geq 0} (V^{\otimes n})_p \otimes H_q(Y^{\otimes (n+1)}) \end{tikzcd}$$

Since f is a weak equivalence we know that $\bar{f}_*$ in (2.6) is an isomorphism by the Künneth formula. Now the comparison theorem shows that also $\bar{f}_*$: $HA \cong HX$ is an isomorphism and (2.2) is proven. □

§ 3 Cylinders

In a cofibration category we use the of a cylinder $I_B A$ of a cofibration $B \rightarrowtail A$. In the category $\mathcal{D}A$ we have an explicit construction for such a cylinder:

(3.1) **Definition:** Let $B \rightarrowtail A$ be a cofibration in $\mathcal{D}A$. We define a cylinder

$$\begin{tikzcd} A \cup_B A \arrow[r, tail, "(i_o, i_1)"'] & I_B A \arrow[r, "p"'] & A \end{tikzcd}$$

as follows: As in (3) of (1.4) we choose a generating module W of the cofibration $B \subset A$ so that $A = B \amalg T(W)$. The underlying algebra of $I_B A$ is

(1) $$I_B A = B \amalg T(W' \oplus W'' \oplus sW) \quad .$$

Here W' and W'' are two copies of the graded module W and sW is the graded module with $(sW)_n = W_{n-1}$. We define i_o and i_1 by the identity on B and by

(2) $$i_o x = x' \;,\; i_1 x = x'' \quad \text{for} \quad x \in W \;.$$

Here $x' \in W'$ and $x'' \in W''$ are the elements which correspond to $x \in W = W' = W''$. Moreover, we define p by the identity on B and by

(3) $$\begin{cases} px' = px'' = x & \text{for} \quad x' \in W' \;,\; x'' \in W'' \;, \\ p(sx) = 0 & \text{for} \quad sx \in sW \;. \end{cases}$$

The differential d on $I_B A$ is given by

(4) $$\begin{cases} dx' = i_o dx \;, & dx'' = i_1 dx \\ dsx = x'' - x' - Sdx & . \end{cases}$$

where $x \in W$. Here $S: A \longrightarrow I_B A$ is the unique map of degree +1 between graded modules which satisfies

$$(5) \quad \begin{cases} Sb = 0 & \text{for} \quad b \in B \quad , \\ Sx = sx & \text{for} \quad x \in W \quad , \\ S(xy) = (Sx)y'' + (-1)^{|x|}x'(Sy) & \text{for} \quad x,y \in A \quad . \end{cases} \quad //$$

(3.2) Lemma: *(a) The map* S *is welldefined by (5).*

(b) The differential is welldefined by (4) and (5) and satisfies $dd = 0$.

(c) The inclusions i_0, i_1 *satisfy*

$$i_1 - i_0 = Sd + dS$$

(d) i_0, i_1 *and* p *are chain maps with* $pi_0 = pi_1 = \text{identity}$.

(e) (i_0, i_1) *is a cofibration and* p *is a weak equivalence.*

By (d) and (e) we see that $I_B A$ is a factorization of the folding map $(1,1): A \cup_B A \longrightarrow A$. Thus, $I_B A$ satisfies the conditions for a cylinder in a cofibration category, compare axiom C3.

Proof: For (a) it is enough to check, by (0.8), that S is compatible with the associativity law of the multiplication, that is

$$S((xy)z) = S(x(yz)) =$$
$$= (Sx)y''z'' + (-1)^{|x|}x'(Sy)z'' + (-1)^{|x|+|y|}x'y'(Sz) \cdot$$

Moreover, (b) follows from (c). Now (c) is clear on B and on $w \in W$ we obtain (c) by

$$Sdw + dSw = Sdw + dsw = (w'' - w') ,$$

see (4) in (3.1). Assume that (c) holds on $x , y \in A$. Then (c) holds on the product $x \cdot y$ since

$$\begin{aligned} & Sd(xy) + dS(xy) = \\ & \quad = (Sdx + dSx) \cdot y'' + x'(Sdy + dSy) = \\ & \quad = (x'' - x')y'' + x'(y'' - y') = x''y'' - x'y' \quad . \end{aligned}$$

Moreover, (d) is clear since $pS = 0$ by (5) in (3.1). By definition in (1) of (3.1) we directly see that (i_o, i_1) is a cofibration. It remains to prove, that p is a weak equivalence ; see (3.5) below. $\square$

(3.3) Definition: Let $B \subset A$ be a cofibration and let $f,g\colon A \longrightarrow X$ be maps between chain algebras. We say f and g are $\mathcal{DA}$ - homotopic relative B if $f|_B = g|_B$ and if there is a map

$$H\colon I_B A \longrightarrow X$$

of chain algebras with $f = Hi_o , g = Hi_1$. We call H a $\mathcal{DA}$ - homotopy from f to g rel B . Equivalently we call the map

$$\alpha = HS\colon A \longrightarrow X$$

of degree +1 a $\mathcal{DA}$ - homotopy from f to g (rel B) , since α determines H . A map $\alpha\colon A \longrightarrow X$ of degree +1 is given by a homotopy H iff the following holds:

(1) $\alpha(b) = 0$ for $b \in B$,

(2) $\alpha d + d\alpha = g - f$,

(3) $\alpha(xy) = (\alpha x)(gy) + (-1)^{|x|}(fx)(\alpha y)$.

By (2) we see that a $\mathcal{D}A$-homotopy is a chain homotopy. Theorem (1.4) implies that '$\mathcal{D}A$-homotopic rel B' is actually an equivalence relation.

(3.4) **Lemma**: *Assume the cofibration* $B \rightarrowtail A = B \amalg T(W)$ *is generated by* W. *Then a map* $\alpha: A \longrightarrow X$ *of degree* $+1$ *is a* $\mathcal{D}A$*-homotopy from* f *to* g *if (1) and (3) in (3.3) hold and if (2) in (3.3) is satisfied on generators* $w \in W$.

Proof: We prove inductively that (2) in (3.3) is satisfied. Assume (2) is satisfied on x and y , x,y ∈ A , that is

$$\alpha dx + d\alpha x = gx - fx \quad ,$$

$$\alpha dy + d\alpha y = gy - fy \quad .$$

Then (2) is also satisfied on x·y since we have

$$\begin{aligned} \alpha d(xy) + d\alpha(xy) &= \alpha((dx)y + \bar{x}(dy)) + d((\alpha x)gy + (f\bar{x})\alpha y) \\ &= (\alpha dx + d\alpha x)(gy) + fx(\alpha dy + d\alpha y) \\ &= (gx - fx)gy + fx(gy - fy) \\ &= gxgy - fxfy = g(xy) - f(xy) \quad . \end{aligned}$$

Here we set $\bar{x} = (-1)^{|x|}x$. □

(3.5) **Proposition**: *p*: $I_B A \longrightarrow A$ *is a weak equivalence.*

Proof: By the following argument it is enough to prove (3.5) for a simple cofibration $B \rightarrowtail A$. We consider the following diagram with notation as in (1.7):

(1)
$$\begin{tikzcd}
I_B A^n \arrow[r, "\bar{p}"] & I_{A^{n-1}} A^n \arrow[r, "p"] & A^n \\
I_B A^{n-1} \arrow[u, tail] \arrow[r, "p_{n-1}"'] \arrow[ur, phantom, "\text{push}"] & A^{n-1} \arrow[u, tail] &
\end{tikzcd}$$

Here $p_n = p\bar{p}$ is the projection p of $I_B A^n$. Assume (3.5) is true for all simple cofibrations $A^{n-1} \rightarrowtail A^n$. Then we see by (1) inductively that p_n is a weak equivalence for all n. Here we use the fact that (1) is a push out diagram and that $\bar{p}$ is a weak equivalence if p_{n-1} is one by (2.2). We now prove (3.5) for the simple cofibration

$$B \rightarrowtail A = B \amalg T(W) \ , \ dW \subset B \ .$$

In this case the differential on $I_B A$ is given by

(2) $$dsw = w'' - w' \quad ,$$

compare (4) and (5) in (3.1). For $i_o: A \rightarrowtail I_B A$ we have $pi_o = 1_A$. On the other hand we construct a $\mathcal{D}A$-homotopy

(3) $$\alpha: I_B A \longrightarrow I_B A \ , \ \alpha: 1 \simeq i_o p \quad ,$$

from the identity 1 on $I_B A$ to $i_o p$. This shows that p is a weak equivalence. We define α on generators w', w'' and sw $(w \in W)$ by

(4) $$\begin{cases} \alpha sw = 0 & , \\ \alpha w' = 0 & , \\ \alpha w'' = sw & , \end{cases}$$

and we set $\alpha(b) = 0$ for $b \in B$. Then (2) in (3.3) is satisfied on generators of the cofibration $B \rightarrowtail I_B A$ since we have

$$(5) \qquad d\alpha(sw) + \alpha d(sw) = 0 + \alpha(w" - w')$$
$$= sw - 0 = (1 - i_o p)(sw) \ ,$$

$$(6) \qquad d\alpha(w') + \alpha d(w') = 0 = (1-i_o p)(w') \ ,$$

$$(7) \qquad d\alpha(w") + \alpha d(w") = dsw + 0$$
$$= w" - w' = (1-i_o p)(w") \ .$$

Now (3.4) shows that α is welldefined by (4). $\square$

§ 4 The relative lifting theorem

We consider the diagram in $\mathcal{D}A$

$$\begin{tikzcd}
B \arrow[r, "\bar{f}"] \arrow[d, tail, "i"] & X \arrow[d, "p", "\sim"'] \\
A \arrow[r, "g"] \arrow[ur, dashed, "f"] & Y
\end{tikzcd} .$$

(4.1) **Proposition:** *Let* i *be a cofibration and let* p *be a weak equivalence and let* $p\bar{f} = gi$ *. Then there exists* f *such that* $if = \bar{f}$ *and such that* pf *and* g *are* $\mathcal{D}A$*-homotopic relative* B .

(4.2) **Corollary:** *Each object in* $\mathcal{D}A$ *is a fibrant model and therefore the axiom on fibrant models, (C4), is satisfied in* $\mathcal{D}A$.

Proof of (4.2): Consider in (4.1) the case where $\bar{f} = 1$ and $g = 1$ and $p = i$. $\square$

Remark: By (II.1.11) we know that (4.1) is actually a consequence of the axioms (C1),...,(C4). Thus it would be enough to prove (4.2). The proof of (4.1) however is conceptually as simple as the proof of (4.2). //

Proof: We construct f and the $\mathcal{DA}$-homotopy $\alpha: pf \simeq g$ inductively. Let $W^n = \{x \in W : |x| \leq n\}$. Then we have the filtration of cofibrations

$$B = A^{-1} \subset A^0 \subset \dots \subset A = B \amalg T(W)$$

where $A^n = B \amalg T(W^n)$. We define $f_n: A^n \longrightarrow X$ and $\mathcal{DA}$-homotopies $\alpha_n: pf_n \simeq g|_{A^n} = g_n$ as follows:

For $n = -1$ let $f_{-1} = \bar{f}$, $\alpha_{-1} = 0$. Assume now f_n and α_n are defined and let a be an element in a basis of W_{n+1}. Then $da \in A^n$ and by

$$\alpha_n d + d\alpha_n = g_n - pf_n$$

we have

$$\begin{aligned} pf_n(da) &= g_n(da) - d\alpha_n(da) \\ &= d(g_{n+1}(a) - \alpha_n(da)) \quad . \end{aligned}$$

Since p is a weak equivalence there is $x \in X$ with $dx = f_n(da)$. Moreover, $g_{n+1}(a) - \alpha_n(da) - px$ is a cycle in Y. Thus there is a cycle $z \in X$ and an element $y \in Y$ such that

$$pz + dy = g_{n+1}(a) - \alpha_n da - px \quad .$$

We now define the extension f_{n+1} of f_n by $f_{n+1}(a) = x + z$ and we define the extension α_{n+1} of α_n by $\alpha_{n+1}(a) = y$. Thus we obtain

$$\begin{aligned} df_{n+1}(a) &= dx = f_n(da) = f_{n+1}(da) \\ g_{n+1}(a) - pf_{n+1}(a) &= g_{n+1}(a) - p(x) - p(z) \\ &= dy + \alpha_n da = d\alpha_{n+1}(a) + \alpha_{n+1}d(a) \quad . \end{aligned}$$

Therefore α_{n+1} is a $\mathcal{D}A$-homotopy relative B from g_{n+1} to pf_{n+1}. We set $f = \lim f_n$ and $\alpha = \lim \alpha_n$. $\square$

§ 5 The cylinder functor

The cylinder in the category $\mathcal{DA}$ which we described in (3.1) is natural in the following sense. A <u>pair</u> (A,B) in $\mathcal{DA}$ is a cofibration $B \rightarrowtail A$ in $\mathcal{DA}$ and a <u>pair map</u> $f: (A,B) \longrightarrow (X,Y)$ is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & X \\
B \arrow[u, tail] \arrow[r] & Y \arrow[u, tail]
\end{tikzcd}$$

in $\mathcal{DA}$. Such a pair map induces the map

.1) $$If: I_B A \longrightarrow I_Y X$$

as follows: Let $A = B \amalg T(V)$. Then we define

$$(*) \qquad \begin{cases} (If)i_o a = i_o fa & \text{for } a \in A \\ (If)i_1 a = i_1 fa & \text{for } a \in A \\ (If)sv = S_X fv & \text{for } v \in V \, . \end{cases}$$

Here $S_X = S: X \longrightarrow I_Y X$ is the map of degree +1 in (3.1)(5).

.2) <u>Lemma</u>: *There is a unique map* If *in* $\mathcal{DA}$ *which satisfies* (*) *above and we have*

$$(**) \qquad (If)S_A = S_X f \quad .$$

<u>Proof</u>: Clearly, there is a unique map between algebras which satisfies

(*). We now check (**). On $b \in B$ equation (**) is true since $S_A b = 0$ and since for $fb \in Y$ also $S_X fb = 0$. Moreover, by definition, (**) is true on $v \in V$. Assume now (**) is true on $x,y \in A$. Then we have

$$\begin{aligned}(If)S(xy) &= (If)((Sx)y'' + (-1)^{|x|}x'Sy)\\ &= ((If)(Sx))\cdot(fy)'' + (-1)^{|x|}(fx)'\cdot((If)(Sy))\\ &= Sf(xy) \quad ,\end{aligned}$$

compare (3.1)(5). This shows that (**) is also true on xy and thus (**) is proven. From (**) we derive that If is a chain map:

$$\begin{aligned}(If)d(sv) &= (If)(v'' - v' - Sdv)\\ &= (fv)'' - (fv)' - Sfdv\\ &= dS(fv)\\ &= d(If)(sv) \quad ,\end{aligned}$$

compare (3.1)(4). Now lemma (5.2) is proven. □

(5.3) <u>Corollary</u>: *For a composition of pair maps* $fg: (D,E) \longrightarrow (A,B) \longrightarrow (X,Y)$ *in* $\mathcal{DA}$ *we have*

$$(***) \qquad I(fg) = (If)(Ig) \ .$$

<u>Proof</u>: Let $D = E \amalg T(W)$. Then we have for $w \in W$

$$\begin{aligned}(If)(Ig)(sw) &= (If)(Sgw)\\ &= S(fgw) && \text{by } (**)\\ &= (I(fg))(sw) && \text{by } (*) \ .\end{aligned}$$

□

(5.4) <u>Corollary</u>: *The cylinder* $I_B A$ *of a cofibration* $B \rightarrowtail A$ *in* $\mathcal{DA}$ *is* *welldefined up to canonical isomorphism.*

<u>Proof</u>: The definition of $I_B A$ in (3.1) depends on the choice of generators V for $B \rightarrowtail A$. Let V' be a different choice and let $I'_B A$ be the cylinder given by V' . Then we have the canonical isomorphism in $\mathcal{D}A$

$$I1 : I_B A \longrightarrow I'_B A$$

which is induced by the identity on A . That $I1$ is an isomorphism follows from the functorial property in (***). $\square$

The cylinder functor (5.1) is compatible with push-outs as follows: Let

$$\begin{tikzcd}
\bar{A} \arrow[r, "\bar{f}"] & \bar{X} = \bar{A} \cup_f X \\
A \arrow[u, tail] \arrow[r, "f"] \arrow[ur, phantom, "\text{push}"] & X \arrow[u, tail]
\end{tikzcd}$$

be a push out diagram with f as in (5.1). Then the diagram

5.5)
$$\begin{tikzcd}
I_B\bar{A} \arrow[r, "I\bar{f}"] & I_Y\bar{X} \\
I_B A \arrow[u, tail] \arrow[r, "If"] \arrow[ur, phantom, "\text{push}"] & I_Y X \arrow[u, tail]
\end{tikzcd}$$

is also a push out diagram.

Proof: Let $\bar{A} = A \amalg T(W)$. Since for $w \in W$ we have

$$(I\bar{f})sw = S(\bar{f}w) = Sw = sw$$

we derive (5.4) from the construction of a push out in (2.1). $\square$

Let B be an object in $\mathcal{D}A$. Then we have the category $(\mathcal{D}A)^B_c$. The objects of this category are the pairs (A,B) in $\mathcal{D}A$, the maps are the maps under B in $\mathcal{D}A$. We obtain by (5.3) and (5.4) a functor

$$I: (\mathcal{D}A)^B_c \longrightarrow (\mathcal{D}A)^B_c$$

with $I(A,B) = (I_B A, B)$. We say a map $f: (A,B) \longrightarrow (A',B)$ in $(\mathcal{D}A)^B_c$ is a cofibration if $f: A \longrightarrow A'$ is a cofibration in $\mathcal{D}A$. The initial object of $(\mathcal{D}A)^B_c$ is the pair (B,B) given by the identity of B . For this structure of $(\mathcal{D}A)^B_c$ we have the

(5.6) **Proposition:** *The category $(\mathcal{D}A)^B_c$ is an I-category in the sense of § 3 in chapter I.*

In particular the full subcategory of free objects in $\mathcal{D}A$, namely $(\mathcal{D}A)_c = (\mathcal{D}A)^*_c$ is an I-category, see § 0 of the following chapter.

Proof: We have to check the axioms (I1),...,(I5). (I1) is obvious by (***) and (I2) follows from (5.5). Now (I3) follows from (1.4) and (II.4.10). Moreover, (I4) is clear by the definition of the cylinder in (3.1). The interchange axiom (I5) follows by (1.4), in fact, by (4.1) we can choose a lifting T for the diagram

$$\begin{tikzcd}
\partial I_B I_B A \arrow[r, "\partial T"] \arrow[d, tail] & \partial I_B I_B A \arrow[r, tail] & I_B I_B A \arrow[d, "PP", "\sim"'] \\
I_B I_B A \arrow[rru, dashed, "T"'] \arrow[rr, "PP"'] & & A
\end{tikzcd} \quad . \qquad \square$$

§ 6 Homotopy groups

Let $B \rightarrowtail A = B \amalg T(W)$ be a cofibration in $\mathcal{DA}$ and let $u: B \longrightarrow U$ be a map in $\mathcal{DA}$. We denote by

$$DA(A,U)^u$$

the set of all maps $A \longrightarrow U$ which extend u. On this set we have the homotopy relation rel B in (3.3) which is an equivalence relation. This gives us the set of homotopy classes

$$[A,U]^u = DA(A,U)^u/(\simeq \text{ rel } B) \quad . \tag{6.1}$$

If $B = *$ and $A = T(W)$ is free, we set

$$[A,U] = DA(A,U)/(\simeq \text{ rel } *) \quad . \tag{6.2}$$

We denote by $\{x\}$ the homotopy class represented by x.

For the cylinder

$$I_B A = B \amalg T(W' \oplus W'' \oplus sW)$$

in (3.1) we have the homotopy addition map

$$m: I_B A \longrightarrow I_B A \cup_A I_B A = DZ \tag{6.3}$$

see (II.2.5). Here we have

$$DZ = B \amalg T(W' \oplus W'' \oplus W''' \oplus sW_o \oplus sW_1) \tag{6.4}$$

with $W' = W'' = W''' = W_o = W_1 = W$.

There is no explicit description of the map m since this map depends on the differential of A .

We describe the following <u>inductive construction of the map</u> m .

The map m in (6.4) is a homotopy $i' \simeq i'''$ of the inclusions i', $i''': A \longrightarrow DZ$. Therefore m is given by the homotopy (see (3.3)):

6.5) $$M: A \longrightarrow DZ \qquad \text{with}$$

(1) $$M(b) = 0 \qquad \text{for } b \in B ,$$

(2) $$m(sw) = M(w) \qquad \text{for } w \in W ,$$

(3) $$Md + dM = i''' - i' \qquad ,$$

(4) $$M(xy) = (Mx)y''' + (-1)^{|x|}x'(My) \qquad \text{for } x,y \in A \quad .$$

On the other hand we obtain from the canonical inclusions

$$j_o, j_1: I_B A \longrightarrow DZ$$

$$j_o w' = w' , \; j_o w'' = w'' , \; j_o sw = sw_o$$

$$j_1 w' = w'' , \; j_1 w'' = w''' , \; j_1 sw = sw_1$$

the homotopies

$$S_o = j_o S , \; S_1 = j_1 S: A \longrightarrow DZ$$

where S is defined in (3.1). These homotopies satisfy

(5) $$\begin{aligned} S_o d + dS_o &= i'' - i' \\ S_1 d + dS_1 &= i''' - i'' \quad . \end{aligned}$$

Therefore we have the operator

(6) $$\hat{M} = M - S_o - S_1$$

which satisfies the following equations:

(7) $$d\hat{M} + \hat{M}d = 0$$

(8) $$\hat{M}(b) = 0 \quad \text{for} \quad b \in B$$

(9) $$\hat{M}(xy) = (\hat{M}x)y''' + (-1)^{|x|}x'(\hat{M}y) + (S_o x)(y''' - y'') - (-1)^{|x|}(x'' - x')(S_1 y) \quad .$$

We now construct $\hat{M}$ inductively: Let $v \in W$ be given with $|v| = n$ and assume we know $\hat{M}(w)$ for all $w \in W$ with $|w| < n$. Then by (9) we have a formula for $\hat{M}dv$. By (7) this element is a boundary $\hat{M}dv = d\xi$. We choose such a ξ and we set

(10) $$\hat{M}v = -\xi \ , \ Mv = S_o v + S_1 v - \xi \quad .$$

(6.6) Example: We consider the example which in topology corresponds to a product $X = S^{n+1} \times S^{m+1}$ of two spheres ($n,m \geq 1$). Let v,w and $v \times w$ be elements of degree n,m and $n+m+1$ respectively and let $A(X) = T(v,w,v \times w)$ be the free chain algebra generated by these elements with the differential

(1) $$\begin{cases} dv = dw = 0 \text{ and for } [v,w] = vw - (-1)^{nm}wv \text{ let} \\ d(v \times w) = (-1)^n[v,w] \quad . \end{cases}$$

We construct $\hat{M}$ on $A(X)$. First we set $\hat{M}v = \hat{M}w = 0$. Since $dS_o v = v'' - v'$ and $dS_1 w = w''' - w''$ we obtain from (9) in (6.5) the equation

$$\begin{aligned} \hat{M}(vw) &= (S_o v)(w''' - w'') - (-1)^n(v'' - v')(S_1 w) \\ &= (-1)^{n+1}d((S_o v)\cdot(S_1 w)) \quad . \end{aligned}$$

Similarly we get

$$\hat{M}(wv) = (-1)^{m+1}d((S_o w)\cdot(S_1 v)) \quad .$$

Therefore we see that $\hat{M}d(v\times w) = d\xi$ with

(2) $$-\xi = (S_o v)(S_1 w) + (-1)^{(n+1)(m+1)}(S_o w)(S_1 v) \quad .$$

Now we can define M by

$$Mv = S_o v + S_1 v \quad , \quad Mw = S_o w + S_1 w$$

$$M(v\times w) = S_o(v\times w) + S_1(v\times w) + (S_o v)(S_1 w) + (-1)^{(n+1)(m+1)}(S_o w)(S_1 v) \quad .$$ //

For a map $u\colon A \longrightarrow U$ we consider the homotopy groups of function spaces $(n \geq 1)$

(6.7) $$\pi_n(U^{A|B},u) = [\Sigma_B^n A, U]^u \quad ,$$

compare (II. § 6 , § 10). We know that π_1 is a group and that π_n, for $n \geq 2$, is an abelian group. For $n=1$ the group structure on π_1 is induced by the comultiplication m on $\Sigma_B A$ which makes the diagram

$$\begin{tikzcd}
I_B A \arrow[r, "m"] \arrow[d, "p"] & I_B A \cup_A I_B A \arrow[d, "p \cup p"] \\
\Sigma_B A \arrow[r, "m"'] \arrow[d, equal] & \Sigma_B A \cup_A \Sigma_B A \arrow[d, equal] \\
A \amalg T(sW) & A \amalg T(sW_o \oplus sW_1)
\end{tikzcd}$$

commutative. Here p is the canonical identification map with $pw' = pw''' = w$, $psw = sw$ for $w \in W$. From (III.7.10) and (C2.0.7) we derive:

(6.8) <u>**Theorem:**</u> *Let $B \rightarrowtail A = B \amalg T(W)$ be a cofibration and assume W is finite dimensional. Then $\pi_1(U^{A|B},u)$ is a nilpotent group.*

From the definition of $\Sigma_B^n A$ we derive:

(6.9) $$\Sigma_B^n A = B \amalg T(W \oplus s^n W) = A \amalg T(s^n W) \quad .$$

The differential $\underline{d}$ on $\Sigma_B^n A$ $(n \geq 1)$ is given as follows. Let

(6.10) $$S^n : A \longrightarrow \Sigma_B^n A$$

be the map of degree n with

$$\begin{cases} S^n b = 0 & \text{for } b \in B \ , \\ S^n w = s^n w & \text{for } w \in W \ , \\ S^n(xy) = (S^n x)\cdot y + (-1)^{n|x|} x\cdot(S^n y) & \text{for } x,y \in A \ . \end{cases}$$

Then $\underline{d}$ is defined on W by d_A and on $s^n W$ by

(6.11) $$\underline{d} s^n w = (-1)^n S^n dw \quad .$$

One easily checks that this is compatible with the definition of $\Sigma_B^n A$ in chapter II, § 5 .

If $B = *$ (and $A = T(W)$ is a free chain algebra) we define the <u>suspension</u> ΣA by the push out

$$\begin{tikzcd}
\Sigma_* A \arrow[r] & \Sigma A \\
A \arrow[u, tail] \arrow[r] & * \arrow[u, tail]
\end{tikzcd}$$

or equivalently $\Sigma A = \Sigma_* A /\!/ A$, compare (2.4). We derive from (6.11)

(6.12) $$\begin{cases} \Sigma A = (T(sW), d) & \text{with} \\ d(sw) = -sd_Q w & \text{for } w \in W \ . \end{cases}$$

Here d_Q is the differential on $QA = W$, see (0.6). Clearly, the n-fold suspension is

$$\Sigma^n A = \Sigma_*^n A /\!\!/ A = (T(s^n W), d)$$

with $ds^n w = (-1)^n s^n d_Q w$.

The homotopy set $[\Sigma^n A, U]$ has a group structure which we obtain as a special example of (6.7) since we have

$$[\Sigma^n A, U] = [\Sigma_*^n A, U]^0 \tag{6.13}$$

where $0: A \longrightarrow * \longrightarrow U$ is the trivial map. The following example shows that the group $[\Sigma A, U]$,in general, is not abelian.

(6.14) Example: Let $A = A(S^{n+1} \times S^{m+1})$ be the chain algebra of (6.6). Then we have for $x = sv$, $y = sw$, $z = s(v \times w)$ the degrees $|z| = |x| + |y|$, see (6.6). Moreover, $\Sigma A = T(x,y,z)$ has trivial differential and the comultiplication

$$m: \Sigma A \longrightarrow \Sigma A \amalg \Sigma A = T(x_0, x_1, y_0, y_1, z_0, z_1)$$

is given by

$$(1) \qquad \begin{cases} mx = x_o + x_1 & , \\ my = y_o + y_1 & , \\ mz = z_o + z_1 + x_o y_1 + (-1)^{|x||y|} y_o x_1 & . \end{cases}$$

For the homotopy set (see (6.19) below)

$$[\Sigma A, U] = H_{|x|}(U) \times H_{|y|}(U) \times H_{|x|+|y|}(U)$$

we obtain by (1) the multiplication + :

$$(2) \qquad (\alpha_o, \beta_o, \gamma_o) + (\alpha_1, \beta_1, \gamma_1) =$$
$$(\alpha_o + \alpha_1, \beta_o + \beta_1, \gamma_o + \gamma_1 + \alpha_o \beta_1 + (-1)^{|x||y|} \beta_o \alpha_1) \ .$$

Here $\alpha_o\beta_1$ and $\beta_o\alpha_1$ are products in the algebra H_*U .

The commutator of the elements

$$\bar{\alpha} = (\alpha,0,0) \ , \ \bar{\beta} = (0,\beta,0) \in [\Sigma A,U]$$

is by (2) the element

$$-\bar{\alpha} - \bar{\beta} + \bar{\alpha} + \bar{\beta} = (0,0,[\alpha,\beta]) \quad . \tag{3}$$

Thus, if the Lie bracket

$$[\alpha,\beta] = \alpha\cdot\beta - (-1)^{|\alpha||\beta|}\beta\cdot\alpha$$

of the algebra H_*U is not trivial, the group $[\Sigma A,U]$ is non abelian. //

By use of derivations there is an alternative description of the homotopy groups in (6.7) and (6.13).

(6.15) Definition: Let $B \subset A$ be a cofibration and let $u\colon A \longrightarrow U$ be a map in $\mathcal{D}A$. An $(A|B,u)$ - derivation of degree n $(n \in \mathbf{Z})$ is a map

$$F\colon A \longrightarrow U$$

of degree n of the underlying graded modules such that

$$F(b) = 0 \qquad \text{for } b \in B \quad , \tag{1}$$

$$F(xy) = (Fx)(uy) + (-1)^{n|x|}(ux)(Fy) \qquad \text{for } x,y \in A \quad . \tag{2}$$

The set of all such derivations:

$$\mathrm{Der}_n = \mathrm{Der}_n(U^{A|B},u) \tag{3}$$

is a module by $(F+G)(x) = (Fx) + (Gx)$. We define a boundary operator

(4)
$$\partial\colon \mathrm{Der}_n \longrightarrow \mathrm{Der}_{n-1}$$
$$\partial(F) = F\circ d - (-1)^n d\circ F$$

where d denotes the differential in A and U respectively. One easily checks that $\partial(F)$ is an element of Der_{n-1} and that $\partial\partial = 0$. Thus we have for all $n \in \mathbb{Z}$ the homology

(5)
$$H_n \mathrm{Der}_*(U^{A|B},u) = \ker(\partial) / \mathrm{im}(\partial)$$

of the chain complex of derivations. //

(6.16) **Proposition**: *Let $B \subset A = B \amalg T(W)$ be a cofibration in $\mathcal{D}A$. Then we have for $n \geq 1$ a canonical bijection*

$$\pi_n(U^{A|B},u) = H_n \mathrm{Der}_*(U^{A|B},u) \quad .$$

For $n \geq 2$ this is an isomorphism of abelian groups.

Since for $n = 1$ the example (6.14) shows that π_1 needs not to be abelian the bijection in (6.16) cannot be an isomorphism of groups for $n = 1$. It is an interesting problem to compute the group structure of π_1.

Proof: Let $A(\Sigma_B^n A, U)^u$ be the set of all algebra maps

$$\bar{F}\colon \Sigma_B^n A \longrightarrow U$$

between the underlying graded algebras with $\bar{F}|_A = u$. We have the bijection

(1)
$$A(\Sigma_B^n A, U)^u = \mathrm{Der}_n$$
$$\bar{F} \longrightarrow \bar{F}S^n = F$$

with S^n in (6.10). Now $\bar{F}$ is a chain algebra map ($d\bar{F} = \bar{F}d$) if and only if $\partial F = 0$. Therefore (1) gives us the bijection (see (6.1))

(2) $$\mathcal{DA}(\Sigma^n{}_B A, U)^u = \text{kernel}(\partial) .$$

It remains to show that for

$$\bar{F}_1, \bar{F}_2 \in \mathcal{DA}(\Sigma^n{}_B A, U)^u$$

we have

(*) $$\bar{F}_1 \simeq \bar{F}_2 \text{ rel } A \Leftrightarrow \exists\, G \in \mathrm{Der}_{n+1} \text{ with } \partial G = F_2 - F_1 .$$

Then clearly (2) gives us the bijection in (6.16), see (6.1) and (6.7).

Now a $\mathcal{DA}$-homotopy $\bar{G}$ from $\bar{F}_1$ to $\bar{F}_2$ (rel A) is given by a map of degree +1

$$\bar{G}: \Sigma^n{}_B A \longrightarrow U$$

with (see (3.4)):

(3) $$\bar{G}x = 0 \quad \text{for } x \in A ,$$

(4) $$\bar{G}d + d\bar{G} = \bar{F}_2 - \bar{F}_1 ,$$

(5) $$\bar{G}(xy) = (\bar{G}\bar{x})(\bar{F}_2\bar{y}) + (-1)^{|\bar{x}|}(\bar{F}_1\bar{x})(\bar{G}\bar{y})$$

for $x, y \in \Sigma^n{}_B A$. Therefore the map

(6) $$G = \bar{G}S^n$$

is of degree $n+1$ and satisfies by (6.10)

(7) $$Gb = 0 \quad \text{for } b \in B ,$$

$$(8) \qquad G(xy) = \bar{G}((S^n x)y + (-1)^{n|x|} x(S^n y))$$

$$= (Gx)(uy) + (-1)^{(n+1)|x|}(ux)(Gy)$$

for $x,y \in A$. Here we use (3) and (5). By (7) and (8) we know that $G \in \mathrm{Der}_{n+1}$. Moreover, by (4) and (6.11) we have for $v \in W$

$$(9) \qquad \partial G(v) = \bar{G}S^n dv - (-1)^{n+1} d\bar{G}S^n v$$

$$= (-1)^n \bar{G} ds^n v + (-1)^n d\bar{G}s^n v$$

$$= (-1)^n (\bar{F}_2 - \bar{F}_1) s^n v$$

$$= (-1)^n (F_2 - F_1)(v) \quad .$$

Therefore $\partial(-1)^n G = F_2 - F_1$. The other direction of (*) can be proven in a similar way.

It follows from (6.23) and (6.22) below that the bijection is actually an isomorphism of groups for $n \geq 2$. ▯

(6.17) <u>Definition</u>: For chain complexes V,W let

$$\mathrm{Hom}_n = \mathrm{Hom}_n(V,W) \qquad (n \in \mathbb{Z})$$

be the module of linear maps of degree n from V to W . Let $\partial: \mathrm{Hom}_n \longrightarrow \mathrm{Hom}_{n-1}$ be defined by

$$\partial f = fd - (-1)^n df \quad .$$

Then, since $\partial\partial = 0$, we have the homology

$$H_n \mathrm{Hom}_*(V,W)$$

of the chain complex $(\mathrm{Hom}_*, \partial)$.

<u>Remark</u>: Let $s^n V$ be the chain complex with $ds^n v = (-1)^n s^n dv$. Then $H_n \mathrm{Hom}_*(V,W)$ is just the set of homotopy classes of chain maps $s^n V \longrightarrow W$.

From (6.16) we derive the following special case:

(6.18) <u>Corollary</u>: *Let A be a free chain algebra and let QA be the chain complex of indecomposables of A . Then we have for $n \geq 1$ the canonical bijection*

$$[\Sigma^n A, U] = H_n Hom_*(QA, U) \quad .$$

For $n \geq 2$ this is an isomorphism of abelian groups.

(6.19) <u>Addendum</u>: *If the differential on QA is trivial we have*

$$H_n Hom_*(QA, U) = Hom_n(QA, H_* U) \ ,$$

compare (III.4.3) in [6].

<u>Proof of (6.18)</u>: In fact, we have for $0: A \longrightarrow * \longrightarrow U$ the isomorphism of chain complexes

$$\mathrm{Der}_*(U^{A|*}, 0) = \mathrm{Hom}_*(QA, U) \quad .$$

Since QA is a free module we obtain (6.19). $\square$

(6.20) <u>Definition</u>: A cofibration $B \rightarrowtail A$ in $\mathcal{DA}$ with

$$A = B \amalg T(W) = \bigoplus_{j \geq 0} B \otimes (W \otimes B)^{\otimes j}$$

is <u>of filtration $\leq n$ (with respect to W)</u> if

$$dW \subset \bigoplus_{j=0}^{n} B \otimes (W \otimes B)^{\otimes j} \qquad ,$$

compare (0.8).

The simple cofibrations which we considered in (1.6) are just those of filtration 0. We now consider cofibrations of filtration 1 with

(6.21) $$dW \subset B \oplus B \otimes W \otimes B \quad .$$

(6.22) Example: Let $Y \subset X = Y \amalg T(V)$ be a cofibration in $\mathcal{DA}$. Then for $n \geq 1$

$$X \subset \Sigma^n_Y X = X \amalg T(s^n V)$$

is a cofibration of filtration ≤ 1 with respect to $s^n V$. In fact, by (6.11) we see

$$\underline{d}s^n V \subset X \oplus s^n V \otimes X \quad .$$

(6.23) Proposition: *Let $B \subset A = B \amalg T(W)$ be a cofibration of filtration ≤ 1. Then the comultiplication m on $\Sigma_B A$ in (6.8) is given by*

$$m(a) = a \qquad \text{for } a \in A \ ,$$

$$m(sw) = sw_0 \dotplus sw_1 \qquad \text{for } w \in W \ .$$

(6.24) Corollary: *If $B \subset A$ is of filtration ≤ 1 then the bijection*

$$\pi_1(U^{A|B}, u) = H_1 Der_*(U^{A|B}, u) \quad ,$$

is an isomorphism of abelian groups, see (6.16).

Proof of (6.23): For $w \in W$ we have

(1) $$dw = d_1 w + d_2 w$$

with $d_1 w \in B$ and $d_2 w \in B \otimes W \otimes B$. This shows that for $\hat{M}$ in (6) of

(6.5) we have

(2) $$\hat{M}(dw) = \hat{M}(d_2w) \quad ,$$

see (8) in (6.5). Now (8) and (9) in (6.5) show us that for $\alpha \otimes v \otimes \beta$, $(\alpha, \beta \in B\, ,\, v \in W)$, we have

(3) $$\hat{M}(\alpha \otimes v \otimes \beta) = (-1)^k \alpha' \otimes (\hat{M}v) \otimes \beta''' \ , \ k = |\alpha| \quad .$$

Assume now we have constructed $\hat{M}$ with $\hat{M}v = 0$ for $|v| < n$. Then for w , $|w| = n+1$, the equation (1), (2) and (3) above show

(4) $$\hat{M}(dw) = 0 \quad .$$

Therefore we can choose $\xi = 0$ in (10) of (6.5). Thus for all $v \in W$ we have $\hat{M}v = 0$ or equivalently

(5) $$m(sv) = Mv = S_o v + S_1 v = sv_o + sv_1 \quad . \quad \square$$

Let T be the free chain algebra generated by the generator t in degree 0 , clearly $dt = 0$. Then $\Sigma^n T = T(s^n t)$ has trivial differential and the homotopy groups are

(6.25) $$\pi_n^T(A) = [\Sigma^n T, A] = H_n(A)$$

for $n \geq 1$, see (II. § 6) and (6.18). Moreover, the relative homotopy groups of a pair (A,B) in $\mathcal{DFA}$ are

(6.26) $$\pi_n^T(A,B) = H_n(A\, ,\, B) = H_n(A/B) \quad .$$

Here A/B is the quotient chain complex of the underlying chain complexes of B and A , see (II.7.7). The <u>exact homotopy sequence</u> (II.7.8) for the functor π_n^T is just the long exact homology sequence

$$(6.27)\qquad \cdots \xrightarrow[\partial]{} H_nB \xrightarrow[i]{} H_nA \xrightarrow[j]{} H_n(A,B) \xrightarrow[\partial]{} H_{n-1}B \longrightarrow \cdots$$

(6.28) <u>Remark:</u> T corresponds in topology to the 1-sphere S^1 so that $\pi_n^T(A)$ corresponds to the homotopy groups $\pi_{n+1}(X)$, $n \geq 1$, of a space X . Clearly, the <u>homotopy groups of spheres</u> $\pi_n^T(\Sigma^m T)$ can be easily computed in the category $\mathcal{DA}$.

Chapter C2: Homotopy theory of chain algebras

§ 0 Free models of chain algebras

We have seen in chapter C1 that the category $\mathcal{D}A$ of chain algebras is a cofibration category. We consider the full subcategory

(0.1) $$\mathcal{D}FA \subset \mathcal{D}A \quad .$$

Here $\mathcal{D}FA$ consists of free chain algebras, that is of differential free algebras. We know that $\mathcal{D}FA = (\mathcal{D}A)_c$ is the subcategory of cofibrant objects in $\mathcal{D}A$ which is an I-category by (C1.5.6). Let $\mathcal{D}FA/\simeq$ be the homotopy category of $\mathcal{D}FA$. We derive from (II.3.6) the following equivalence of categories which is induced by the inclusion (0.1).

(0.2) **Theorem:**

$$Ho(\mathcal{D}A) \sim \mathcal{D}FA/\simeq \quad .$$

Here Ho denotes localization with respect to weak equivalences in

(0.3) Remark: The equivalence (0.2) corresponds in topology to $Ho(Top_o) \sim CW/$ see (A.0.2). We have the functor

$$C_*\Omega : Top_o \longrightarrow \mathcal{D}A$$

which carries a space X to the singular chains of the loopspace ΩX , see chapter D. This functor carries the 'homotopy theory in Top_o' to t 'homotopy theory in $\mathcal{D}A$'. We will study this functor in more detail in chapter D. In this chapter we analyze the purely algebraic homotopy theo of $\mathcal{D}A$. We proceed in close analogy to the combinatorial homotopy theor for CW-complexes in chapter A. It will be very helpful to the reader t compare our results and constructions here with the corresponding result and constructions for CW-complexes in chapter A. For example, the following corollary is the analogue of the Whitehead theorem (A.0.3). //

(0.3) <u>Corollary</u>: *A map in* $\mathcal{D}FA$ *is a homotopy equivalence iff it is a weak equivalence.*

For each algebra A we choose by the factorization axiom (C3) a <u>free model</u> MA , which is a free chain algebra together with

$$* \rightarrowtail MA \xrightarrow{\sim} A \quad .$$

The equivalence in (0.2) is induced by the <u>model functor</u>

$$\mathcal{D}A \xrightarrow[M]{} \mathcal{D}FA/\simeq \tag{0.4}$$

which associates with $f\colon A \longrightarrow X$ in $\mathcal{D}A$ the unique homotopy class Mf for which

$$\begin{tikzcd}
 & * \arrow[dl, tail] \arrow[dr, tail] & \\
MA \arrow[rr, dashed, "Mf"'] \arrow[d, "\sim"] & & MX \arrow[d, "\sim"'] \\
A \arrow[rr, "f"'] & & X
\end{tikzcd}$$

homotopy commutes. Here Mf is given by (4.1). We know that different choices of models yield <u>canonically</u> isomorphic model functors. Compare the proof of (II.3.6).

(0.5) <u>Example</u>: By use of the <u>bar construction</u> B and the <u>cobar construction</u> Ω (see § 2 below) we have the functor

$$\Omega B\colon \mathcal{DA} \longrightarrow \mathcal{DFA}\ .$$

together with a natural map

$$\alpha\colon \Omega B(A) \xrightarrow{\sim} A$$

which is a weak equivalence. This is proven in [5]. Thus $\Omega B(A)$ is a model of $A \in Ob(\mathcal{DA})$ and for each model $MA \xrightarrow{\sim} A$ we have a homotopy equivalence

$$MA \simeq \Omega BA$$

which is welldefined up to homotopy. In fact, ΩB is a model functor and two model functors as in (0.4) are isomorphic.

(0.6) Remark:

In the literature there is the following different approach of describing the localized category $Ho(\mathcal{DA})$ in terms of a homotopy category. Let $\mathcal{DASH}$ be the category with the same objects as $\mathcal{DA}$ but with morphisms

$$\mathcal{DASH}(A,X) = \mathcal{DA}(\Omega B(A),X)\ .$$

Composition in $\mathcal{DASH}$ is defined by

$$g\bullet f = g\circ\Omega\bar{f}\colon \Omega BA \longrightarrow \Omega BX \longrightarrow Y\ .$$

Here $\bar{f}\colon BA \longrightarrow BX$ is the adjoint of $f\colon \Omega BA \longrightarrow X$. Homotopy in $\mathcal{DASH}$ is defined by homotopy rel $*$ in $\mathcal{DA}$. By the equivalence α in (0.5) we clearly have the bijection

$$\alpha_*\colon [\Omega BA,\Omega BX] \xrightarrow{\approx} \mathcal{DASH}(A,X)/_{\simeq}\ .$$

We thus derive from (0.2) the equivalence of categories

(0.11) <u>Corollary</u>:

$$\mathcal{DASH}/_{\simeq} \sim \mathrm{Ho}(\mathcal{DA})$$

In the case, where R is a field, this result was obtained by Munkholm [7].

Each free chain algebra $A = (T(V),d)$ in $\mathcal{DFA}$ is a complex in $\mathcal{DA}$ as follows, compare (III.§ 3):

Let $V^n = \{x \in V: |x| \leq n\}$. Then $A^n = (T(V^n),d)$ is a sub chain algebra of A which we call the <u>n - skeleton</u> of A . In particular $A^o = T(V_o)$ is just the tensor algebra on $V_o = V^o$ with trivial differential. We have the filtration of chain algebras

$$A^o \subset A^1 \subset \ldots\ldots \subset A = \varinjlim A^n \tag{0.7}$$

Each inclusion $A^n \subset A^{n+1}$ is a principal cofibration:

(0.8)

$$\begin{tikzcd} CD_{n+1} \arrow[r] & A^{n+1} \\ D_{n+1} \arrow[u, phantom, "\cup"] \arrow[r, "f_{n+1}"'] \arrow[ur, phantom, "\text{push}"] & A^n \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Here $D_{n+1} = (T(s^{-1}V_{n+1}),d=0)$ is a tensor algebra with trivial differential. The <u>attaching map</u> f_{n+1} is given by the differential d in A , namely

$$f_{n+1}(s^{-1}v) = d(v) \quad , \quad v \in V_{n+1} \quad .$$

Moreover, $CD_{n+1} = (T(s^{-1}V_{n+1} \oplus V_{n+1}),d)$, with $d(s^{-1}v) = 0$ and $d(v) = s^{-1}v$ for $v \in V_{n+1}$, is the cone on D_{n+1}. Clearly, (0.8) is a push out diagram in $\mathcal{DA}$.

By (0.8) and (0.7) we see that the class of objects in $\mathcal{DFA}$ is the clas $\mathfrak{X}$, of complexes in $\mathcal{DA}$ with the following properties: $X \in \mathfrak{X}$ iff (a) and (b) hold:

(a) X^0 is a free algebra concentrated in degree 0.

(b) X has attaching maps $f_{n+1}: D_{n+1} \longrightarrow X^n$ where D_{n+1} is a free algebra generated by elements in degree n.

By a degree argument all maps in $\mathcal{DFA}$ are filtration preserving. Therefore we have the isomorphism of categories

$$\mathcal{DFA} = \mathit{Complex}(\mathfrak{X}) \quad . \tag{0.9}$$

Recall that $\mathit{Complex}(\mathfrak{X})$ is the full subcategory of the category $\mathit{Complex}$ in (III.3.6) consisting of objects in $\mathfrak{X}$.

We consider two natural equivalence relations, $\sim$ and $\simeq$, on the category $\mathcal{DFA}$. Let $f,g: A \longrightarrow B$ be maps in $\mathcal{DFA}$ and let IA be the cylinder on $* \rightarrowtail A$, see (C1.§ 3) .

(0.10) Definition: We set $f \sim g$ if there is a homotopy $H: f \simeq g$, $H: IA \longrightarrow B$, such that H restricts to $H^n: IA^n \longrightarrow B^n$ for all n. Then H is a 0-homotopy in the sense of (III.1.13). //

(0.11) Definition: We set $f \simeq g$ if there is a homotopy $H: f \simeq g$, $H: IA \longrightarrow B$, as in (C1.3.3). Clearly, $H: A^n \cup IA^{n-1} \cup A^n \longrightarrow B^n$ since H is a map of degree 0 between graded modules. This shows that H is a <u>1 - homotopy</u> in the sense of (III.1.13). //

We obtain the homotopy categories

(0.12) $$\mathcal{D}FA / \overset{1}{\simeq} = \mathcal{D}FA / \simeq \sim \mathrm{Ho}(\mathcal{D}A)$$

and we have quotient functors

(0.13) $$\mathcal{D}FA \longrightarrow \mathcal{D}FA / \sim \longrightarrow \mathcal{D}FA / \simeq \quad .$$

We will describe 'towers of categories' which approximate the categories $\mathcal{D}FA / \sim$ and $\mathcal{D}FA / \simeq$ respectively. These correspond in topology to the towers for $CW / \sim$ and $CW / \simeq$, see chapter A.

We derive these towers for $\mathcal{D}FA$ from the general twisted towers in (V.§ 6) since we have the following crucial lemma:

(0.14) <u>Lemma</u>: *The class, $\mathfrak{X}$, of free chain algebras in $\mathcal{D}A$ is a very good class of complexes in the sense of (V.6.1).*

For the proof of the lemma we first observe the following equations: Let D_{n+1} be a free chain algebra generated by the set Z of elements in degree n. Then we have

(0.15) $$D_{n+1} = \bigvee_{Z} \Sigma^n T$$

where T is the free chain algebra generated by the generator t in degree 0. We therefore get the following equations for the homotopy set (compare (C1.6.18) and (C1.6.25)):

(0.16) $$\begin{aligned}[D_{n+1},U] &= [\bigvee_{Z} \Sigma^n T,U] \\ &= \mathop{\times}_{Z} [\Sigma^n T,U] = \mathop{\times}_{Z} \pi_n^T(U) = \mathop{\times}_{Z} H_n(U) \quad .\end{aligned}$$

By use of (0.16) we see that (0.14) is an immediate consequence of the following lemmata; compare the proof of (A.0.12).

(0.17) <u>Lemma</u>: *Let* V *be a free module concentrated in degree* n *and let* A *be a free chain algebra. Then the inclusion* $A^k \subset A$ *of the* k*-skeleton induces the map*

$$H_n(T(V) \vee A^k)_2 \longrightarrow H_n(T(V) \vee A)_2$$

which is surjecitve for $k = 0$ *and which is an isomorphism for* $k \geq 1$.

Recall that $H_n(X \vee Y)_2$ is the kernel of $H_n(0,1)$ where $(0,1): X \vee Y \longrightarrow Y$ is the projection.

<u>Proof of (0.17)</u>: The lemma is clear for n=0. For $n \geq 1$ we show

(1) $$H_n(T(V) \vee A)_2 = H \otimes V \otimes H$$

where $H = H_0 A$. In fact, we have

(2) $$(T(V) \vee A)_n = A^0 \otimes V \otimes A^0 \oplus A_n$$

and (0,1) on $(T(V) \vee A)_n$ is the projection onto A_n . Let $x \in A^0 \otimes V \otimes A^0$ and $y \in A_n$. Then $x+y$ represents $\{x+y\}$ $H_n(T(V) \vee A)_2$ if $d(x+y) = 0$ and if $y = dy'$, $y' \in A_{n+1}$. This shows $d(x) = 0$ and $\{x\} = \{x+y\}$. Now we obtain equation (1) by (2) and by

(3) $$(T(V) \vee A)_{n+1} = A_1 \otimes V \otimes A^0 \oplus A^0 \otimes V \otimes A_1 \oplus A_{n+1} \ . \ \square$$

(0.18) <u>Corollary</u>: *Let* $n \geq 1$. *Then the partial suspension* E *yields the commutative diagram*

$$\begin{tikzcd}
H_n(T(V) \vee A)_2 \arrow[d, "E"', "\approx"] \arrow[r, phantom, "="] & H \otimes V \otimes H \arrow[d, "\bar{s}"] \\
H_{n+1}(\Sigma T(V) \vee A)_2 \arrow[r, phantom, "="] & H \otimes sV \otimes H
\end{tikzcd}$$

where $\bar{s}$ *is the* H*-biequivariant map with* $\bar{s}(v)=sv$ *for* $v \in V$ *. For* $n=0$ *the partial suspension* E *is surjecitve and is induced by the composition:*

$$T(V) \vee A^o \xrightarrow[pr]{} A^o \otimes V \otimes A^o \xrightarrow[\bar{s}]{} H \otimes sV \otimes H \quad .$$

Here pr *is the projection onto a direct summand of* $T(V) \vee A^o$ *and* $\bar{s}(\alpha \otimes v \otimes \beta) = \{\alpha\} \otimes sv \otimes \{\beta\}$.

(0.19) **Lemma**: *Let* $V=s^{-1}V_{n+1}$ *and let* $f=f_{n+1}\colon T(V) \longrightarrow A^n$ *be the attaching map in (0.8). Then the map* $(\pi_{f},1)_*$ *:*

$$H_{n+1}(CT(V) \vee A^n, T(V) \vee A^n) \longrightarrow H_{n+1}(A^{n+1}, A^n) \; ,$$

which is induced by the pair map (0.8) is surjective for n=0 and is an isomorphsim for $n \geq 1$ *. For* $n \geq 1$ *we have*

$$H_{n+1}(A^{n+1}, A^n) = H \otimes V_{n+1} \otimes H$$

where $H=H_oA^n=H_oA$ *. For n=0 see (3.2) below.*

Proof: We have $(A^{n+1}/A^n)_n = 0$ and

$$(A^{n+1}/A^n)_{n+1} = A^o \otimes V_{n+1} \otimes A^o \quad .$$

This shows that $(\pi_f,1)_*$ is surjective. Now let $n \geq 1$. Then we have

$$(A^{n+1}/A^n)_{n+2} = A_1 \otimes V_{n+1} \otimes A^o \oplus A^o \otimes V_{n+1} \otimes A_1 \quad . \square$$

Remark: The n - skeleton of a free chain algebra corresponds in topology to the (n+1) - skeleton of a CW - complex. Therefore the lemmata above correspond to the lemmata in (A.0.16),...,(A.0.18) with a shift in degree by +1. For the proof of (0.14) we use the lemmata here only in case $n \geq 1$. Therefore we do not need the analogue of (A.0.19). Still an analogue of (A.0.19) is true if we restrict to chain algebras for which the homology in degree 0 is a free R - module, compare the proof of (4.7) below. //

§ 1 Chains and twisted chains and the Hochschild (co)homology

For a chain algebra A the homology in degree 0, $H = H_o A$, is an augmented algebra (non graded). The augmentation

(1.1) $$\varepsilon : H \longrightarrow R$$

is induced by the augmentation of A. We say M is an H-bimodule if M is an R-module together with actions of H on M from the right and from the left, that is, we have a map of R-modules

$$H \otimes M \otimes H \xrightarrow{\mu} M \quad ,$$

which we write $\mu(\alpha \otimes m \otimes \beta) = \alpha \cdot m \cdot \beta$ and which satisfies $1 \cdot m \cdot \beta = m \cdot \beta$, $\alpha \cdot m \cdot 1 = \alpha \cdot m$ and $(\alpha \cdot m) \cdot \beta = \alpha \cdot (m \cdot \beta)$. If V is a free R-module then $M = H \otimes V \otimes H$ is the free H-bimodule generated by V (with the obvious action of H from the left and the right).

(1.2) Remark: Let H^{op} be the opposite algebra of H. As modules, $H^{op} = H$. We denote by $\mu^* \in H^{op}$ the element corrsponding to $\mu \in H$. Then the multiplication in H^{op} is defined by

$$\mu^* \cdot \lambda^* = (\lambda \cdot \mu)^* \quad , \quad \lambda, \mu \in H \quad .$$

The algebra $H \otimes H^{op}$ with $(\lambda \otimes \mu^*) \cdot (\lambda_1 \otimes \mu_1^*) = \lambda\lambda_1 \otimes (\mu_1 \cdot \mu)^*$ is called the enveloping algebra of H, see Cartan Eilenberg [3]. An H-bimodule M may be regarded as a right module over the algebra $H \otimes H^{op}$ by setting

$$m\cdot(\lambda\otimes\mu^{*}) = \mu\cdot m\cdot\lambda$$

for $m\in M$.

For H - bimodules we have the following functors Hom_{H-H} and $\otimes_{H-H}$. Let M and N be H - bimodules, then

(1.3)a $$\mathrm{Hom}_{H-H}(M,N)$$

is the R - module of H - biequivariant maps $F\colon M \longrightarrow N$, (that is $F(\alpha\cdot m\cdot\beta) = \alpha\cdot Fm\cdot\beta$).

Moreover, the 'bimodule tensor product' of M and N is the R - module

(1.3)b $$M\otimes_{H-H}N$$

which is obtained from the tensor product $M\otimes N$ of R - modules by the identifications

$$m\beta\otimes n = m\otimes\beta n \quad , \quad \alpha,\beta\in H \quad ,$$

$$\alpha m\otimes n = m\otimes n\alpha \qquad m\in M \, , \, n\in N \, .$$

Compare X§4 in [6].

If $M=H\otimes V\otimes H$ is a free bimodule we have

(1.4) $$\mathrm{Hom}_{H-H}(H\otimes V\otimes H,N) = \mathrm{Hom}(V,N)$$
$$(H\otimes V\otimes H)\otimes_{H-H}N = V\otimes N$$

Let G_1,G_2 be algebras and let N be a (G_1,G_2) - <u>bimodule</u>, that is a left G_1 - module and a right G_2 - module with $(g_1\cdot n)\cdot g_2 = g_1\cdot(n\cdot g_2)$ for $g_1\in G_1$, $g_2\in G_2$, $n\in N$. If

$$\varphi: H \longrightarrow G_1 \ , \ \psi: H \longrightarrow G_2$$

are algebra homomorphisms, we obtain an H-bimodule structure on N by

$$\alpha\cdot n\cdot\beta = (\varphi\alpha)\cdot n\cdot(\psi\beta) \quad .$$

We denote this H-bimodule by

(1.5) $$(\varphi,\psi)^* N \quad .$$

If $G_1 = G_2$ and $\varphi = \psi$ we write

$$\varphi^* N = (\varphi,\varphi)^* N \quad .$$

In particular we have the H-bimodule $\varepsilon^* R$ with ε in (1.1). For an H-bimodule M and the (G_1,G_2)-bimodule N we say $F: M \longrightarrow N$ is <u>(φ,ψ)-equivariant</u> if $F(\alpha\cdot m\cdot\beta) = (\varphi\alpha)\cdot(Fm)\cdot(\psi\beta)$. Clearly,

(1.6) $$\mathrm{Hom}_{H-H}(M, (\varphi,\psi)^* N) = \mathrm{Hom}_{\varphi,\psi}(M,N)$$

is the R-module of (φ,ψ)-equivariant maps. By (1.4) we have the isomorphism of R-moduls

(1.7) $$\begin{aligned} \mathrm{Hom}(V,N) &= \mathrm{Hom}_{\varphi,\psi}(H \otimes V \otimes H, N) \\ f &\mapsto \varphi \otimes f \otimes \psi \end{aligned}$$

with $(\varphi\otimes f\otimes\psi)(\alpha\otimes v\otimes\beta) = (\varphi\alpha)\cdot(fv)\cdot(\psi\beta)$.

If $\varphi = \psi$ we call a (φ,φ)-equivariant map also a <u>φ-biequivariant</u> map.

For the category $\mathcal{DFA}$ of free chain algebras we introduce the following commutative diagram of categories and functors:

(1.8)

$$\begin{tikzcd}
 & \mathcal{D}F\widehat{M} \arrow[dd, "\varepsilon"] \\
\mathcal{D}FA \arrow[ur, "\hat{b}"] \arrow[dr, "b"'] & \\
 & \mathcal{D}FM
\end{tikzcd}$$

We call b the chains and $\hat{b}$ the twisted chains on $\mathcal{D}FA$. Diagram (1.8) corresponds to diagram (A.1.24).

(1.9) Definition of $\mathcal{D}FM$:

Objects are the free chain complexes (or differential free modules), C, with the following properties:

(a) C_n is a free R-module, $n \geq 0$,

(b) $C_o = R$,

(c) $d_1 = 0$.

Maps are chain maps σ, for which σ_o is the identity of R in degree 0. //

(1.10) Definition of $\mathcal{D}F\widehat{M}$:

Objects are pairs $(H,\hat{C})$ where H is an augmented non graded algebra and where $\hat{C}$ is a chain complex of H-bimodules with the following properties:

(a) $\hat{C}_n$ is a free H-bimodule and $d_n: \hat{C}_n \longrightarrow \hat{C}_{n-1}$ is biequivariant,

(b) $\hat{C}_o = H \otimes H$ (let $\mu: \hat{C}_o \longrightarrow H$ be the multiplication),

(c) $0 = \mu d_1: \hat{C}_1 \longrightarrow \hat{C}_o \longrightarrow H$.

A map $\sigma = (\varphi,\sigma): (H,\hat{C}) \longrightarrow (H',\hat{C}')$ is a homomorphism, $\varphi: H \longrightarrow H'$, between augmented non graded algebras together with a φ-biequivariant chain map, $\sigma: \hat{C} \longrightarrow \hat{C}'$, with $\sigma_o = \varphi \otimes \varphi$ in degree 0. //

Via the augmentation ε we have an H-bimodule structure on R. The functor ε in (1.8) carries the object $(H,\hat{C})$ in $\mathit{DF}\hat{M}$ to the object

(1.11) $$\hat{C} \otimes_{H-H} \varepsilon^{*} R \quad \text{in} \quad \mathit{DFM} \quad .$$

Clearly, for a free H-bimodule $M = H \otimes V \otimes H$ the module $M \otimes_{H-H} \varepsilon^{*} R = V$ is a free R-module. This shows that the functor ε in (1.8) is well-defined by (1.11).

We now define the functor of chains b on DFA in (1.8). For the free chain algebra $A = (T(V),d)$ let

$$\tilde{A} = \text{kernel}(\varepsilon : A \longrightarrow R)$$

be the augmentation ideal. Then $\tilde{A} \cdot \tilde{A} \subset \tilde{A}$ are sub chain complexes of A. The quotient chain complex

(1.12) $$QA = \tilde{A}/\tilde{A}\cdot\tilde{A} = (V,d_Q)$$

is the chain complex of indecomposables, see (C1.0.6). We define the differential d on $sQA = sV$ by $dsv = -sd_Q v$, $v \in V$. This gives us the chain functor b in (1.8) with

(1.13) $$bA = R \otimes sQA = (R \otimes sV,d) \quad .$$

Here R is concentrated in degree 0. The functor b is defined on maps f in DFA in the obvious way: $bf = 1_R \otimes sQf$.

Next we define the functor $\hat{b}$ of equivariant chains on DFA . This functor carries a free chain algebra, $A = (T(V),d)$ to

(1.14) $$\hat{b}A = (H, \hat{b}A) \ , \ H = H_oA \ ,$$

where $\hat{b}A$, as a free graded H - bimodule, is given by

$$\hat{b}A = H \otimes (R \oplus sV) \otimes H$$

or equivalently by

$$\hat{b}_nA = \begin{cases} H \otimes sV_{n-1} \otimes H & , \quad n \geq 1 \ , \\ H \otimes H & , \quad n = 0 \ . \end{cases}$$

For the definition of the differential $\hat{d}$ on $\hat{b}A$ we use the map Λ in (1.16) below. Let

(1.15) $$\lambda \colon A \longrightarrow H = H_oA$$

be the obvious projection with $\lambda x = 0$ for $|x| > 0$ and $\lambda x = \{x\}$ for $|x| = 0$. Here $\{x\}$ denotes the homology class of $x \in A_o$. Clearly, all elements of A_o are cycles. By λ we obtain the map of degree 0

(1.16) $$\Lambda \colon A = T(V) \longrightarrow H \otimes V \otimes H$$

with

$$\Lambda(1) = 0 \ ,$$

$$\Lambda(v) = 1 \otimes v \otimes 1 \qquad \text{for} \quad v \in V \ ,$$

$$\Lambda(xy) = (\lambda x)(\Lambda y) + (\Lambda x)(\lambda y) \qquad \text{for} \quad x, y \in A \ .$$

Now the differential $\hat{d}$ of $\hat{b}A$ in (1.14) is the H - biequivariant map of degree -1 with $(v \in V)$:

(1.17) $$\hat{d}sv = \begin{cases} -\bar{s}\Lambda dv & |v| \geq 1 \ , \\ (\lambda v) \otimes 1 - 1 \otimes (\lambda v) & |v| = 0 \ . \end{cases}$$

Here $\bar{s}: H \otimes V \otimes H \longrightarrow H \otimes sV \otimes H$ is the biequivariant map which carries v to sv.

(1.18) <u>Remark:</u> For $n \geq 1$ the diagram

$$\begin{tikzcd}
(A^n)_n \arrow[ddrr, bend right, "\Lambda"'] & = & T(V_0) \otimes V_n \otimes T(V_0) \otimes (A^{n-1})_n \arrow[d, "\text{projection}"] \\
 & & T(V_0) \otimes V_n \otimes T(V_0) \arrow[d, "\lambda \otimes 1 \otimes \lambda"] \\
 & & H \otimes V_n \otimes H
\end{tikzcd}$$

commutes. For $n = 0$ the map

$$\Lambda: (A^0)_0 = T(V_0) \longrightarrow H \otimes V_0 \otimes H$$

is the unique map with

$$\Lambda(1) = 0$$

$$\Lambda(v) = 1 \otimes v \otimes 1 \qquad \text{for } v \in V_0$$

$$\Lambda(xy) = (\lambda x)(\Lambda y) + (\Lambda x)(\lambda y) \quad .$$

//

By (1.18) we have a direct way of computation for Λdv in (1.17) since for $|v| = n+1$ we have $dv \in (A^n)_n$.

The functor $\hat{b}$ carries a map $F: A \longrightarrow B$ in $\mathcal{DFA}$ to the map $(\varphi, \hat{F}_*)$ in $\mathcal{DF}\hat{\mathcal{M}}$. Here

$$\varphi = F_*: H = H_0A \longrightarrow G = H_0B$$

is the induced map in homology and

$$\hat{F}_*: \hat{b}A \longrightarrow \hat{b}B$$

is the φ-biequivariant chain map with

(1.19)
$$\hat{F}_0 = \varphi \otimes \varphi$$
$$\hat{F}_n(sv) = \bar{s}\Lambda Fv \quad , v \in V_{n-1}, n \geq 1 \ .$$

By (1.17) and (1.19) we obtain the commutative diagram of chain maps

(1.20)
$$\begin{tikzcd} A \arrow[r, "F"] \arrow[d, "\bar{s}\Lambda"] & B \arrow[d, "\bar{s}\Lambda"] \\ \hat{b}A/\hat{b}_0A \arrow[r, "\hat{F}_*"'] & \hat{b}B/\hat{b}_0B \end{tikzcd} \ .$$

Here $\bar{s}\Lambda$ is a chain map of degree +1. We will see in (D.4.24) that $\bar{s}\Lambda$ corresponds in topology to the 'homology suspension' map.

(1.21) Lemma: *$\hat{b}$ is a welldefined functor on $\mathcal{DFA}$ and we have the canonical equivalence of functors in A*

$$\hat{b}A \otimes_{H-H} \varepsilon^*R = bA \ .$$

This shows that diagram (1.8) commutes.

We leave it to the reader to check all details for (1.21).

Next we consider homotopies.

(1.22) <u>Definition:</u> Two maps $\sigma, \sigma' \colon C \longrightarrow C'$ are <u>homotopic in $\mathcal{DFM}$</u> ($\sigma \simeq \sigma'$) if there are maps $\alpha = \alpha_j \colon C_j \longrightarrow C'_{j+1}$ of R-modules with

(a) $$\alpha_o = 0$$

(b) $$d\alpha + \alpha d = \sigma' - \sigma \quad .$$

Two maps $(\varphi, \sigma), (\varphi', \sigma') \colon (H, \hat{C}) \longrightarrow (H', \hat{C}')$ are <u>homotopic in $\mathcal{DF}\hat{\mathcal{M}}$</u> if $\varphi = \varphi'$ and if there are φ-biequivariant maps $\alpha = \alpha_j \colon \hat{C}_j \longrightarrow \hat{C}'_{j+1}$ $(j \geq 0)$ which satisfy (a) and (b). //

<u>Remark:</u> A map $\sigma \colon C \longrightarrow C'$ in $\mathcal{DFM}$ is a <u>weak equivalence</u> iff $\sigma_* \colon HC \cong HC'$ is an isomorphism. A map $(\varphi, \sigma) \colon (H, \hat{C}) \longrightarrow (H', \hat{C}')$ in $\mathcal{DF}\hat{\mathcal{M}}$ is a <u>weak equivalence</u> iff $\varphi \colon H \cong H'$ and $\sigma_* \colon H\hat{C} \cong H\hat{C}'$ are isomorphisms.

In $\mathcal{DFM}$ and $\mathcal{DF}\hat{\mathcal{M}}$ respectively we have:

(1) homotopy is a natural equivalence relation,

(2) weak equivalences are exactly the homotopy equivalences. //

The functors in (1.8) induce the following functors between homotopy categories:

(1.23)
$$\begin{tikzcd}
 & \mathcal{DF}\hat{\mathcal{M}}/\simeq \arrow[dd, "\varepsilon"] \\
\mathcal{DFA}/\simeq \arrow[ur, "\hat{b}"] \arrow[dr, "b"'] & \\
 & \mathcal{DFM}/\simeq
\end{tikzcd}$$

(1.24) **Lemma**: *The functors in (1.23) are welldefined.*

Proof: We consider the cylinders in the categories DFM and $\mathit{DF\hat{M}}$ which yield the homotopy relations in these categories respectively. For $C = (R \oplus W, d)$ in DFM we have the cylinder

$$IC = R \oplus W' \oplus W'' \oplus \hat{W} \tag{1}$$

where $W' = W'' = W$ and $\hat{W} = sW$ with $d\hat{w} = w'' - w' - \widehat{dw}$ for $w \in W$. We have the canonical isomorphism

$$\tau\colon IbA = bIA \tag{2}$$

where for $A = (T(V), d)$ in DFA the cylinder $IA = (T(V' \oplus V'' \oplus \hat{V}), d)$ is defined by (C1.3.1) with $B = *$. Thus we have

$$IbA = R \oplus (sV)' \oplus (sV)'' \oplus \widehat{sV} \tag{3}$$

$$bIA = R \oplus s(V' \oplus V'' \oplus \hat{V})' . \tag{4}$$

The isomorphism τ in (2) is the identity $(sV)' = sV'$, $(sV)'' = sV''$ and is defined on $\widehat{sV}$ by

$$\tau(\widehat{sv}) = -s\hat{v} , \quad v \in V . \tag{5}$$

Here $v \mapsto sv$ and $v \mapsto \hat{v}$ are two maps of degree +1 which are interchanged. Therefore we have the sign in the definition of τ. One easily checks by (C1.3.1) that τ is a chain map.

From (2) we derive that b induces a functor between homotopy categories in (1.23). Similarly we proceed for $\hat{b}$. We define a cylinder $I(H, \hat{C}) = (H, I\hat{C})$ in $\mathit{DF\hat{M}}$ by

(6) $$I\hat{C} = H\bullet(R\oplus W'\oplus W''\oplus \hat{W})\bullet H$$

where $\hat{C} = H\bullet(R\oplus W)\bullet H$. On $\hat{W}$ the differential in $I\hat{C}$ is given by

$$d\hat{w} = w'' - w' - \widehat{dw}$$

where now $\hat{\ } : H\bullet W\bullet H \longrightarrow H\bullet\hat{W}\bullet H$ is the biequivariant map which maps w to $\hat{w}$. This cylinder gives us the homotopy relation on $\mathcal{D}F\hat{M}$ in (1.22). The functor $\hat{b}$ in (1.23) is welldefined since we have the isomorphism in $\mathcal{D}F\hat{M}$

(7) $$\tau : I\hat{b}A = \hat{b}IA \quad .$$

Here τ is the biequivariant map which is defined on generators in the same way as τ in (1.23). We have to check that τ is a chain map:

<u>Proof of (7):</u> For $A = (T(V),d)$ we have $\hat{b}A = H\bullet(R\oplus sV)\bullet H$, $sV = W$. For $w = sv\in W$ we get

$$\begin{aligned} d\widehat{sv} &= (sv)' - (sv)'' - \widehat{dsv} \in I\hat{b}A \\ \tau\,\widehat{sv} &= -s\hat{v} \qquad\qquad \in \hat{b}IA \\ ds\hat{v} &= -\bar{s}\Lambda d\hat{v} \\ &= -\bar{s}\Lambda(v'' - v' - Sdv) \in I\hat{b}A \quad . \end{aligned}$$

Therefore it is enough to check

(8) $$\tau\widehat{dsv} = \bar{s}\Lambda Sdv \quad .$$

Here $dsv = -\bar{s}\Lambda dv$. We deduce (8) from

(9) $$\hat{\Lambda}x = \Lambda Sx \quad \text{for } x\in A = T(V) \quad .$$

Now (9) is clear for $x=1$ and $x \in V$. Assume (9) is true for x and $y \in A$ then we get by (1.16) and $\lambda S = 0$:

$$\begin{aligned} \Lambda S(x \cdot y) &= \Lambda((Sx)\cdot y'' + (-1)^{|x|} x' \cdot (Sy)) \\ &= (\Lambda Sx)(\lambda y'') + (-1)^{|x|}(\lambda x')(\Lambda Sy) \end{aligned}$$

where $\lambda x' = 0$ if $|x| \neq 0$. On the other hand

$$\widehat{\Lambda(xy)} = (\lambda x)\widehat{\Lambda y} + (\widehat{\Lambda x})(\lambda y) \quad .$$

Since $\lambda y'' = \lambda y$, $\lambda x' = \lambda x$ equation (9) is proved. □

Similarly as in (A.1.26) we define <u>homology</u> and <u>cohomology</u>.

(1.25) <u>Definition</u>: Let A be a free chain algebra with $H = H_o A$ and let Γ be an H-bimodule. The functor $\hat{b}$ gives us the chain complex of R-modules

$$\hat{b}_*(A,\Gamma) = \hat{b}(A) \otimes_{H-H} \Gamma$$

the homology of which is $\hat{H}_*(A;\Gamma)$. Moreover, we obtain the cochain complex

$$\hat{b}^*(A,\Gamma) = \mathrm{Hom}_{H-H}(\hat{b}(A),\Gamma)$$

the homology of which is $\hat{H}^*(A;\Gamma)$. //

By the standard arguments we see that homotopic maps in $\mathcal{DFA}$ induce the same homomorphisms in homology and cohomology. Therefore the homology and cohomology groups of A are homotopy invariants. This allows us to define for <u>any chain algebra</u> X in $\mathcal{DA}$ and any H_oX-bimodule Γ the modules:

$$(1.26)\qquad \begin{aligned} \hat{H}^*(X;\Gamma) &= \hat{H}^*(MX;\Gamma) \\ \hat{H}_*(X;\Gamma) &= \hat{H}_*(MX;\Gamma) \end{aligned}$$

where MX is a model of X in $\mathcal{D}FA$, see §0 . In fact, any homotopy functor on $\mathcal{D}FA$ gives us a nice functor on $\mathcal{D}A$ by the equivalence in (0.2). We again point out that models MX of X are unique up to a canonical homotopy equivalence.

By (0.9) we can take for the model MX of X the canonical model $\Omega BX \xrightarrow{\sim} X$. We set

$$\hat{B}X = \hat{b}\Omega BX$$

and thus we obtain equivalently to (1.26):

$$(1.26)'\qquad \begin{aligned} \hat{H}_*(X;\Gamma) &= H_*(\hat{B}X \otimes_{H-H} \Gamma) \\ \hat{H}^*(X;\Gamma) &= H^*\mathrm{Hom}_{H-H}(\hat{B}X,\Gamma) \quad . \end{aligned}$$

The advantage of (1.26) is that we can take any model MX of X for the computation of these (co)homology groups. In the next section we study $\hat{B}X$ in more detail.

(1.27) Remark: Slightly more general we define the homology and cohomology of pairs in $\mathcal{D}FA$. Let $B \rightarrowtail A$ be a cofibration in $\mathcal{D}FA$ with $B = T(W)$, $A = T(W \oplus V)$. The generators W of B determine a sub chain complex of H - bimodules in $\hat{b}A$, namely

$$H \otimes (R \otimes sW) \otimes H \subset \hat{b}A \quad ,$$

which gives us the quotient chain complex:

$$\hat{b}(A,B) = \hat{b}(A) \,/\, (H \otimes (R \otimes sW) \otimes H) \quad .$$

If we replace $\hat{b}(A)$ by $\hat{b}(A,B)$ in (1.25) we obtain the relative versions of homology and cohomology, $\hat{H}_*(A,B;\Gamma)$ and $\hat{H}^*(A,B;\Gamma)$ respectively. //

(1.28) Remark: Let X be a (non graded) algebra which is free as an R-module and let Γ be an X-bimodule. We can consider X as being an object in $\mathcal{D}A$ concentrated in degree 0 (with trivial differential and $H_0X=X$). Therefore we have by (1.26) homology and cohomology groups

$$(1) \qquad \hat{H}_*(X;\Gamma) \quad , \quad \hat{H}^*(X;\Gamma)$$

which are invariants of the homotopy type of X in $Ho(\mathcal{D}A)$. On the other hand we have the homology and cohomology groups of Hochschild, see (2.15),

$$(2) \qquad H_*(X;\Gamma) \quad , \quad H^*(X;\Gamma)$$

as defined in Cartan-Eilenberg [3] and Mac Lane [6]. We shall prove in § 2 that the homology and cohomology groups respectively in (1) and (2) coincide. Thus Hochschild (co)homology is an invariant of the homotopy type of X in $Ho(\mathcal{D}A)$. We call the groups in (1.26)' the generalized Hochschild (co)homology.

(1.29) Remark: Let A be an object in $\mathcal{D}A$ and let M be an R-module. Then $M=\varepsilon^*M$ is an H_0A-bimodule via the augmentation $\varepsilon: H_0A \longrightarrow R$. Let $X \xrightarrow{\sim} A$ be a model of A in $\mathcal{D}FA$. Then we have

$$(*) \qquad \begin{aligned} \hat{H}^n(A,\varepsilon^*M) &= H^n Hom_R(bX,M) \\ \hat{H}_n(A,\varepsilon^*M) &= H_n(bX \otimes_R M) \quad . \end{aligned}$$

This is the <u>stage 1 (co)homology</u> of A with coefficients in M as defined by Mac Lane in chapter 10, § 11 [6], see § 2 . We have to distinguish this (co)homology from the '<u>stage 0 (co)homology</u>' which is $H_n(A;M)$ and $H^n(A;M)$ defined by the underlying chain complex of A . Therefore we use $\dot{H}$. Clearly, (*) is again an invariant in $Ho(\mathcal{D}A)$, compare theorem 11.2 in chapter X of [6]. //

We now compare the Hochschild cohomology with the twisted cohomology defined in (III.§ 5) , see (1.32). In fact, we discovered the chain complex $\hat{b}(A)$ above by considering the twisted chain complex $\check{K}(A) = E\check{k}(A)$ in (III.6.3). For the free chain algebra A , which is a complex in $\mathcal{D}A$ as in (0.9), we have the attaching maps

$$f_n\colon D_n = T(s^{-1}V_n) \longrightarrow A^{n-1}$$

given by the differential in $A = (T(V),d)$, $f_n(s^{-1}v) = dv$. These attaching maps yield the twisted chain complex $k^V(A)$ with the boundaries $(j\colon A^{n-1} \subset A)$

(1.30) $$d_n = (1 \vee j)\nabla f_n \in [D_n, \Sigma D_{n-1} \vee A]_2 \quad ,$$

see (III.6.3). Here d_n is also defined for $n=1$ by use of the comultiplication

$$\mu\colon T(V_o) \longrightarrow T(V_o) \vee T(V_o) = T(V'_o \oplus V''_o)$$

where $V'_o = V''_o = V_o$ and where $\mu(v) = v' + v''$ for $v \in V_o$. For the chain complex $\check{K}(A) = Ek^V(A)$ with boundaries Ed_n we get:

(1.31) <u>Lemma</u>: *For a basis element* $v \in V_n$, $n \geq 1$, *we have*

$$(Ed_n)(v) \in H_n(\Sigma^2 D_{n-1} \vee A)_2 = H \otimes sV_{n-1} \otimes H$$

by (0.16) and (0.18). This element satisfies the equation

$$(Ed_n)(v) = \bar{s}\Lambda dv \quad ,$$

see (1.17).

<u>Proof</u>: First let $n > 1$. We have

$$dv \in A^{n-1} = A^0 \otimes V_{n-1} \otimes A^0 \oplus (A^{n-2})_{n-1}$$

with $dv = x+y$, $y \in (A^{n-2})_{n-1}$ and

$$x = \Sigma\, \alpha_i \otimes v_i \otimes \beta_i \in A^0 \otimes V_{n-1} \otimes A^0 \quad .$$

where $\alpha_i, \beta_i \in A^0$, $v_i \in V_{n-1}$. The map

$$\begin{tikzcd}
A^{n-1} \arrow[r, "i_2+i_1"'] & \Sigma D_{n-1} \vee A^{n-1} \arrow[d, equal] \\
 & A^0 \otimes (V'_{n-1} \oplus V''_{n-1}) \otimes A^0 \oplus (A^{n-2})_{n-1}
\end{tikzcd}$$

is given by the identity on $(A^{n-2})_{n-1}$ and by $v \mapsto v'' + v'$ for $v \in V_{n-}$
This shows that $(\nabla f_n)(v)$ is represented by

$$\begin{aligned}(\nabla f_n)(v) &= - \Sigma\, \alpha_i \otimes v''_i \otimes \beta_i + \Sigma\, \alpha_i \otimes (v''_i + v'_i) \otimes \beta_i \\ &= \Sigma\, \alpha_i \otimes v'_i \otimes \beta_i \quad .\end{aligned}$$

Therefore we get

$$(E\nabla f_n)(v) = \bar{s}\, \Sigma\, \{\alpha_i\} \otimes v'_i \otimes \{\beta_i\} = \Lambda dv \quad ,$$

see (1.18). Now let $n = 1$. Then we have for $v \in V_1$ the element

$$d(v) = \sum_I v_{i_1} \dots v_{i_s} \in T(V_0) \quad , \quad v_{i_k} \in V_0 \quad .$$

Now $(\nabla f_1)(v)$ is given by

$$(\nabla f_1)(v) = - \sum_I v''_{i_1} \dots v''_{i_s} + \sum_I (v''_{i_1} + v'_{i_1}) \dots (v''_{i_s} + v'_{i_s}) .$$

By (0.18) we know that the partial suspension projects all words to zero with more than two factors in V''_o . Therefore $(E\nabla f_1)(v)$ is represented by

$$\sum_I \sum_{k=1}^{s} v'_{i_1} \dots v'_{i_{k-1}} (sv'') v'_{i_{k+1}} \dots v'_{i_s} \quad .$$

This element represents Λdv , see (1.18). ▯

From the lemma (1.31) we derive:

(1.32) **Theorem**: *Let* A *be a free chain algebra and let* $u: A \longrightarrow U$ *be a map in* $\mathcal{D}A$. *Then there is a natural isomorphism*

$$\check{H}^n_k(KA,u) = \hat{H}^n(A,u^* H_{n+k} U)$$

for $n \geq 1$, $n+k \geq 1$.

Here $H_{n+k}U$ is the $H_o(U)$ - bimodule given by the algebra structure on $H_*(U)$ and $u^*H_{n+k}U$ is the $H_o(A)$ - bimodule induced by $u_*: H_oA \longrightarrow H_oU$, see (1.5).

§ 1 Appendix: The spectral sequence for homotopy groups of function spaces

Let X and Y be free chain algebras and let $Y \rightarrowtail X$ be a cofibration, that is $Y = T(W)$ and $X = T(W \oplus V)$. We set

$$X_p = T(W \oplus V^{p-1}) \quad , \quad p \geq 0 \quad ,$$

with $V^p = \{v \in V : |v| \leq p\}$. Then X_p is a subchain algebra of X and we have $Y = X_o \subset \ldots \subset X$.

Let U be a chain algebra in $\mathcal{DA}$ and let $u: X \longrightarrow U$ be a map in $\mathcal{DA}$. We consider the homotopy groups

(1) $$\pi_n(U^{X|Y}, u) \quad , \quad n \geq 1 \quad ,$$

see (C1. § 6). These are abelian groups for $n \geq 2$, but for $n = 1$ this group, in general, is not abelian. In § 6 of chapter C1 we discussed the relationship of the groups (1) with the homology of the derivation complex.

The inclusion $X_p \subset X$ induces a homomorphism between groups

$$\alpha_p : \pi_q(U^{X|Y}, u) \longrightarrow \pi_q(U^{X_p|Y}, u_p)$$

where $u_p : X_p \longrightarrow U$ is the restriction of u . For $K_{p,q} = \text{kernel } \alpha_p$ we have the filtration

(2) $$\ldots \subset K_{p,q} \subset K_{p-1,q} \subset \ldots \subset K_{0,q} = \pi_q(U^{X|Y}, u)$$

of subgroups. We say (X,Y) is finite dimensional if there is N with $X_N = X$. In this case we have $K_{N,q} = 0$.

The spectral sequence below describes the associated graded group of the filtration (7.3) which is defined by the quotients

(3) $$G^{p,q} = K_{p-1,q} / K_{p,q} \quad , \quad q \geq 1 \ .$$

These are abelian groups. Moreover, the spectral sequence can be used for the computation of the set $(p \geq 2)$

$$G^{p,0} = (i_p^*)^{-1}\{u_{p-1}\}$$

where $i_p^*: [X_p, U]^v \longrightarrow [X_{p-1}, U]^v$ is induced by $i_p: X_{p-1} \subset X_p$. Here $v = u_0$ is the restriction of u . The set $G^{p,0}$ has a canonical abelian group structure with $\{u_{p-1}\}$ as zero element, see (III.7.6).

In the following theorem we use the cohomology in (1.25) with the following coefficients. The map $u: X \longrightarrow U$ induces

$$\varphi = u_*: H_0X \longrightarrow H_0U \quad .$$

Since H_nU , $n \geq 1$, is a H_0U - bimodule we have the induced coefficients $\varphi^* H_nU = (\varphi,\varphi)^* H_nU$ as in (1.5). We denote these coefficients as well by $u^* H_nU$, see (1.32).

) **Theorem:** *There is a spectral sequence*

$$E_r^{p,q} = (E_r^{p,q}(X,Y,u) \ , \ d_r)$$

with the following properties $(r \geq 2)$*:*

(E1) $E_r^{p,q} = 0$ *for* $p < 1$, $q < 0$ *and* $d_r: E_r^{p,q} \longrightarrow E_r^{p\ r,q-1}$ *is a homomorphism of bidegree* $(r,-1)$.

(E2) *For* $p,q \geq 1$ *we have an isomorphism* $E_\infty^{p,q} = G_{p,q}$. *If* (X,Y)

is finite dimensional with $X_N = X$ *we have* $E_\infty^{p,q} = E_N^{p,q}$. *Moreover, for* $q=0$ *and* $p \geq 2$ *we have* $E_p^{p,0} = G^{p,0}$.

(E3) *The* E_2 *- term is given by the Hochschild cohomology*

$$E_2^{p,q} = \hat{H}^p(X,Y;u^* H_{p+q-1} U) \quad , \quad p+q \geq 2 \ , \ q \geq 1 \ .$$

For $q=0$ *and* $p \geq 2$ *we have*

$$E_2^{p,0} = \hat{H}^p(X_p, Y; u_p^* H_{p-1} U) \quad .$$

(E4) *The spectral sequence is natural in the obvious way.*

Proof: By (1.32) this is essentially a special case of (III.7.7). In degree $q \geq 2$ the result also can be derived from a filtration of the chain complex $\mathrm{Der}_*(U^{X|Y}, u)$, see (C1.§ 6) . ▯

The spectral sequence here is the analogue of the one for CW - complexes in (III.9.9).

§ 2 The bar and the cobar construction

We introduce the notion of a graded coalgebra which is dual to the notion of a graded algebra in (C1.0.4). We only consider connected graded coalgebras.

(2.1) Definition: A (connected graded) coalgebra C is a positive module C together with a map $\Delta: C \longrightarrow C \otimes C$ such that

(a) $C_0 = R$,

(b) Δ is associative: $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$,

(c) The map $\varepsilon: C \longrightarrow R$, which is the identity in degree 0, is a counit, $(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta$.

A map $f: C \longrightarrow D$ between coalgebras is a map of degree 0 with $(f \otimes f)\Delta = \Delta f$ and $\varepsilon f = \varepsilon$. The inclusion $i: R = C_0 \subset C$ is a map between coalgebras. We have

$$\bar{C} = C/R = \text{cokernel}(i) \ ,$$

$$\bar{\Delta}: \bar{C} \longrightarrow \bar{C} \otimes \bar{C} \quad \text{induced by } \Delta \ ,$$

$PC = \text{kernel}(\bar{\Delta})$ is the module of primitives in C . The map f induces $Pf: PC \longrightarrow PD$.

//

(2.2) Definition: For a positive module V with $V_0 = 0$ the tensor coalgebra is

$$T'(V) = \bigoplus_{n \geq 0} V^{\otimes n} \quad .$$

The diagonal is defined by

$$\Delta(v_1 \otimes \ldots \otimes v_k) = \sum_{i=0}^{k} (v_1 \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes v_k)$$

with $v_1, \ldots, v_k \in V$. We have $i = i_o$, $\varepsilon = p_o$ as in (C1.0.5) and $PT'(V) = V$. We say that a coalgebra C is free if there is a map $p: C \longrightarrow V$ of degree 0 which induces the isomorphism $C \cong T'(V)$ of coalgebras. //

(2.3) Definition: A chain coalgebra C is a (connected graded) coalgebra C together with a differential $d: C \longrightarrow C$ of degree -1 such that

(a) C is free as an R-module

(b) (C,d) is a chain complex: $dd = 0$.

(c) ε and Δ are chain maps.

A map $f: C \longrightarrow D$ between chain coalgebras is a map between coalgebras with $df = fd$. Let $\mathcal{DC}$ be the category of chain coalgebras. We say a chain coalgebra is free if the underlying coalgebra C is free. Let $\mathcal{DFC}$ be the full subcategory of $\mathcal{DC}$ consisting of free coalgebras. //

We now define the bar construction B and the cobar construction Ω . These constructions are functors

$$\mathcal{DA} \xrightarrow{B} \mathcal{DFC} \quad , \quad \mathcal{DC} \xrightarrow{\Omega} \mathcal{DFA} \quad . \tag{2.4}$$

The cobar construction $\Omega(C)$ is the chain algebra

$$\Omega(C) = (T(s^{-1}\bar{C}), d_\Omega)$$

with the differential d_Ω whose restriction to generators $i_1: s^{-1}\bar{C} \subset T(s^{-1}\bar{C})$ is defined by, see (2.1), (2.5) $$d_{\Omega} i_1 = -i_1(s^{-1} ds) + i_2(s^{-1} \otimes s^{-1}) \bar{\Delta} s \quad .$$

The <u>bar construction</u> $B(A)$ is the chain coalgebra

$$B(A) = (T'(s\tilde{A}), d_B)$$

with the differential d_B whose component $p_1 d_B$ for $p_1 \colon T'(s\tilde{A}) \longrightarrow s\tilde{A}$ is given by, see (C1.0.4),

(2.6) $$p_1 d_B = -(sds^{-1})p_1 + s\tilde{\mu}(s^{-1} \otimes s^{-1})p_2 \quad .$$

(2.7) <u>Remark</u>: We give a more explicit description for the boundary d_B on BA and for the boundary d_{Ω} on ΩC. Write a typically generator

(1) $$(s \otimes \ldots \otimes s)(a_1 \otimes \ldots \otimes a_n) \quad , \quad a_i \in \tilde{A} \quad ,$$

of BA as $[a_1 | \ldots | a_n]$. With the Koszul sign convention we have

(2) $$[a_1 | \ldots | a_n] = (-1)^{\tau}(sa_1) \otimes \ldots \otimes (sa_n) \quad ,$$

$$\tau = \sum_{j=1}^{n} (n-j)|a_j| \; .$$

From (2.6) we derive the formula

(3) $$d_B[a_1 | \ldots | a_n] = \sum_{j=1}^{n-1} (-1)^j [a_1 | \ldots | a_j a_{j+1} | \ldots | a_n]$$

$$+ \sum_{j=1}^{n} (-1)^{n+\delta(j)} [a_1 | \ldots | da_j | \ldots | a_n]$$

with $\delta(j) = \sum_{k=1}^{j-1} |a_k|$. The formula for the diagonal in (2.2) is replaced by

$$\Delta[a_1 | .. | a_n] = \sum_{j=0}^{n} (-1)^{\tau(j)} [a_1 | .. | a_j] \otimes [a_{j+1} | .. | a_n]$$

where $\tau(j) = (n-j) \sum_{k=1}^{j} |a_k|$.

On the other hand we have the following explicit formula for the cobar construction ΩC . The boundary d_Ω on ΩC is given on generators $s^{-1}x$, $x \in \bar{C}$, by

(5) $$d_\Omega(s^{-1}x) = -s^{-1}(dx) + \sum_i (-1)^{|x_i'|}(s^{-1}x_i') \otimes (s^{-1}x_i'')$$

where $\bar{\Delta}x = \sum_i x_i' \otimes x_i'' \in \bar{C} \otimes \bar{C}$. //

The bar construction B is a right adjoint functor for the cobar construction Ω . The *adjunction maps*

(2.8) $$\begin{aligned} \alpha &: \Omega B(A) = T(s^{-1}\overline{BA}) \longrightarrow A \\ \beta &: C \longrightarrow T'(s\tilde{\Omega}C) = B\Omega(C) \end{aligned}$$

are just the algebra and coalgebra extensions respectively of the projection or inclusion

$$\begin{aligned} s^{-1}(p_1) &: s^{-1}\overline{BA} \longrightarrow s^{-1}s\tilde{A} = \tilde{A} \\ s(i_1) &: \bar{C} = ss^{-1}\bar{C} \longrightarrow s\tilde{\Omega}C \quad . \end{aligned}$$

In [5] it is proven that α *and* β *induce isomorphisms in homology*. Therefore $\alpha: \Omega B(A) = MA \xrightarrow{\sim} A$ is a model of A .

We now consider the functors b and $\hat{b}$ in (1.8). One easily verifies the equations in $\mathcal{DFM}$:

$$\begin{aligned} C &= b\Omega(C) \qquad \text{for } C \in \mathcal{DC} \\ B(A) &= b\Omega B(A) \qquad \text{for } A \in \mathcal{DA} \quad . \end{aligned}$$

Moreover, we define the *two sided constructions*

(2.10)
$$\begin{aligned} \hat{C} &= \hat{b}\Omega(C) = (H \otimes C \otimes H, \hat{d}) \quad , \quad H = H_o \Omega C \\ \hat{B}(A) &= \hat{b}\Omega B(A) = (H \otimes BA \otimes H, \hat{d}) \quad , \quad H = H_o A \end{aligned}$$

which are chain complexes in $DF\hat{M}$. For $\hat{B}(A)$ we use the canonical isomorphism

$$\alpha_*: H = H_o(\Omega BA) = H_o A$$

given by α in (2.8).

(2.11) __Lemma__: *Let* $\bar{\lambda}: \bar{C} \longrightarrow H = H_o(\Omega C)$ *be given by* $\bar{\lambda}x = 0$ *for* $|x| > 1$ *and* $\bar{\lambda}x = \{s^{-1}x\}$ *for* $|x| = 1$. *Then the* H*-biequivariant differential* $\hat{d}$ *on* $\hat{C}$ *is given by ,* $x \in \bar{C}$ *,*

$$\hat{d}x = (\bar{\lambda}x) \otimes 1 - 1 \otimes (\bar{\lambda}x) \in H \otimes H \qquad \text{for} \quad |x| = 1$$

$$\hat{d}x = 1 \otimes dx \otimes 1 + \sum_i [(\bar{\lambda}x_i') \otimes x_i'' \otimes 1 + (-1)^{|x|} 1 \otimes x_i' \otimes \bar{\lambda}x_i''] \qquad \text{for} \quad |x| \geq 2 \quad ,$$

see (5) in (2.7).

This is an easy consequence of (1.17). For $C = BA$ we obtain the special case:

(2.12) __Corollary__: *The differential* $\hat{d}$ *on* $\hat{B}A$ *is the* H*-biequivariant map with (see (2.7)):*

$$\begin{aligned} \hat{d}[a_1 | \dots | a_n] = 1 \otimes d_B[a_1 | \dots | a_n] \otimes 1 \; + \\ \lambda(a_1) \otimes [a_n | .. | a_n] \otimes 1 \; + \\ (-1)^n 1 \otimes [a_1 | \dots | a_{n-1}] \otimes \lambda(a_n) \end{aligned}$$

where $\lambda: A \longrightarrow H = H_o A$ *is defined in (1.15). This formula is* *also true for* $n=1$ *since the empty bracket* $[\ ]$ *is the unit* $1 \in R$.

We point out that $\varepsilon\lambda a_i = 0$ in (2.12) since $a_i \in \tilde{A}$. Therefore $\varepsilon(\hat{B}A) = BA$ for the functor ε in (1.8).

(2.13) <u>**Proposition**</u>: *Let* $A \in \mathcal{D}A$ *and let* $MA \xrightarrow{\sim} A$ *be a model of* A *. Then we have the canonical homotopy equivalence*

$$bMA \simeq BA \quad \text{in} \quad \mathcal{D}FM$$
$$\hat{b}MA \simeq \hat{B}A \quad \text{in} \quad \mathcal{D}F\hat{M} \ .$$

<u>**Proof**</u>: In fact, we have the canonical homotopy equivalence $\Omega BA \simeq MA$ in $\mathcal{D}FA$. Now we apply the functor b and $\hat{b}$ in (1.23). ▯

In particular, if A is a free chain algebra we can take $MA = A$. This shows:

(2.14) <u>**Corollary**</u>: *Let* $A \in \mathcal{D}FA$ *, then we have canonical homotopy equivalences*

$$b\alpha\colon BA \simeq bA \quad \text{in} \quad \mathcal{D}FM$$
$$\hat{b}\alpha\colon \hat{B}A \simeq \hat{b}A \quad \text{in} \quad \mathcal{D}F\hat{M} \ .$$

This is clear since $\alpha\colon \Omega BA \longrightarrow A$ in (2.8) is a homotopy equivalence, see (2.9) and (2.10).

The map $b\alpha$ in (2.14) is induced by the projection $q\colon \tilde{A} \longrightarrow QA = \tilde{A}/\tilde{A}\cdot\tilde{A}$. More precisely, the diagram

$$\begin{tikzcd}
BA/R \arrow[rr, "b\alpha"] \arrow[d, equal] & & bA/R \arrow[d, equal] \\
T'(s\tilde{A})/R \arrow[r, "p_1"] & s\tilde{A} \arrow[r, "s(q)"] & sQA
\end{tikzcd}$$

commutes, see (2.8) and (1.13).

(2.15) Remark: If $X = X_0$ is an algebra in $\mathcal{DA}$ (which is concentrated in degree 0) we have $X = H_0X = H$. In this case BX is the reduced bar resolution and $\hat{B}X = B(X,X)$ is the normalized bar resolution of X. Compare Mac Lane [6], chapter X.

The Hochschild homology and cohomology of X with coefficients in the X-bimodule Γ is defined by

$$H_n(X;\Gamma) = H_n(B(X,X) \otimes_{X-X} \Gamma)$$

$$H^n(X;\Gamma) = H^n(\mathrm{Hom}_{X-X}(B(X,X);\Gamma))$$

respectively, see [6], chapter X, § 3, § 4.

We have the homotopy equivalence

$$MX \simeq \Omega BX$$

which induces the homotopy equivalence

$$\hat{b}MX \simeq \hat{b}\Omega BX = \hat{B}X = B(X,X) \quad \text{in} \quad \mathcal{DF}\hat{\mathcal{M}} .$$

This shows that Hochschild (co)homology coincides with the (co)homology in (a) of (1.28). //

(2.16) Remark: Let A be a chain algebra and let M and N be left and right differential A-modules. Then the two sided bar construction

$$N \otimes_\tau BA \otimes_\tau M$$

is defined, the homology of which is

$$(*) \qquad \mathrm{Tor}_A(N,M) = H_*(N \otimes_\tau BA \otimes_\tau N) \quad ,$$

compare [5] and [4]. The homology H_oA is a left and right A-modul by $\lambda: A \longrightarrow H$ which we denote by λ^*H . One can check that there is a canonical isomorphism of chain complexes

$$\hat{B}A = \lambda^*H \otimes_\tau BA \otimes_\tau \lambda^*H \quad .$$

Therefore the homology of $\hat{B}A$ is

$$(**) \qquad \mathrm{Tor}_A(\lambda^*H, \lambda^*H) = H_*\hat{B}A \quad .$$

Now let M and N be left and right H-modules respectively. Then $M \otimes N$ is an H-bimodule in the obvious way. Via the map $\lambda: A \longrightarrow H_oA$ the modules M and N are also left and right A-modules which we denote by λ^*M and λ^*N respectively. In this case we get slightly more generally than (**) above the natural equation

$$(***) \qquad \hat{H}_*(A, M \otimes N) = \mathrm{Tor}_A(\lambda^*N, \lambda^*M) \quad .$$

The advantage of this equation is the fact that the left side can be computed by any free model MA of A , see (1.26). This is in general not true for Tor_A in (*).

(2.17) <u>Proposition</u>: *Let F be a weak equivalence in DA . Then Bf and $\hat{B}f$ are homotopy equivalences in DFM and $DF\hat{M}$ respectively. In particular, Bf and $\hat{B}f$ induce isomorphisms in homology.*

In particular Bf *and* $\hat{B}f$ *induce isomorphisms in homology.*

<u>Remark</u>: For Bf the result in (2.17) was proven in theorem 11.2 of chapter X [6]. For $\hat{B}f$ the result in (2.17) was obtained by Eilenberg-Moore in case $f: A \longrightarrow X$ is a weak equivalence where H_oA, H_oX are R-flat, see 2.6_* in [4] and (2.16).

<u>Proof of (2.17)</u>: We have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f", "\sim"'] & Y \\
\Omega BX \arrow[u, "\sim"'] \arrow[r, "\Omega Bf"] & \Omega BY \arrow[u, "\sim"']
\end{tikzcd} .$$

Thus $\Omega B(f)$ is a weak equivalence in $\mathcal{D}FA$ and therefore a homotopy equivalence. This shows that $Bf = b(\Omega Bf)$ and $\hat{B}f = \hat{b}(\Omega Bf)$ are homotopy equivalences, see (1.23). $\square$

(2.18) <u>Remark</u>: Let A and X be chain algebras such that H_oA and H_oX are <u>free</u> as R-modules. In (8.13) we will prove that in addition to (2.17) the following statements are equivalent:

(1) $f: A \longrightarrow X$ is a weak equivalence in $\mathcal{D}A$.

(2) $\hat{B}f: \hat{B}A \longrightarrow \hat{B}X$ is a homotopy equivalence in $\mathcal{D}F\hat{M}$.

(3) $\hat{B}f: \hat{B}A \longrightarrow \hat{B}X$ is a weak equivalence in $\mathcal{D}F\hat{M}$.

(4) The induced map $f_* = \mathrm{Tor}_f(H_of, H_of)$,

$$f_*: \mathrm{Tor}_A(\lambda^* H_o A, \lambda^* H_o A) \longrightarrow \mathrm{Tor}_X(\lambda^* H_o X, \lambda^* H_o X) \quad ,$$

is an isomorphism.

For (2) ⇔ (3) see the remark following (1.22) and for (3) ⇔ (4) compare (**) in (2.16) where we use the following result (2.19). //

(2.19) Theorem: *Let A be a free chain algebra for which $H_o A = H$ is free as an R-module. Then the sequence*

$$\hat{b}_2 A \xrightarrow{\hat{d}} \hat{b}_1 A \xrightarrow{\hat{d}} \hat{b}_o A \xrightarrow{\mu} H \longrightarrow 0 \quad ,$$

given by the chain complex $\hat{b}A$, is exact. Here $\hat{b}_o A = H \otimes H$ and $\mu: H \otimes H \longrightarrow H$ is the algebra multiplication in H. Equivalently we have

$$H_1 \hat{b} A = 0 \quad \text{and} \quad \mu_*: H_o \hat{b} A \cong H_o A \quad .$$

Remark: This result corresponds in topology to the fact that the universal cover of a CW-complex X is connected and simply connected. See (A.1.18). In addition to (2.19) we show in (4.6) below that $H_2 \hat{b} A = H_1 A$ if $H_o A$ is free as an R-module. This corresponds to the Hurewicz isomorphism $H_2 \hat{X} = \pi_1 \Omega X$.

(2.20) Corollary: *Let X be a chain algebra such that the underlying R-module of $H_o X$ is a free R-module. Then we have isomorphisms of $H_o X$-bimodules*

$$\mathrm{Tor}_X(H_o X, H_o X)_o = H_o \hat{B} X = H_o X \quad ,$$

$$Tor_X(H_0X,H_0X)_1 = H_1\hat{B}X = 0 \quad ,$$

$$Tor_X(H_0X,H_0X)_2 = H_2\hat{B}X = H_1X \quad .$$

Proof of (2.20): We use (2.13), (2.19) and (4.6), and (2.16). □

Proof of (2.19): For $\lambda\colon A \longrightarrow H = H_0A$ we have the diagram in $\mathcal{DA}$

(1)
$$\begin{tikzcd}
\Omega BH \arrow[r, "\alpha", "\sim"'] & H \\
\Omega BA \arrow[u, "\Omega B\lambda"'] \arrow[r, "\alpha", "\simeq"'] & A \arrow[u, "\lambda"'] \arrow[lu, dashed, "g_1"']
\end{tikzcd}$$

where α is the adjunction map which is a weak equivalence. Since A and ΩBA are objects in $\mathcal{DFA}$ we can define the map g_1 in (1) by a homotopy inverse α^{-1} of α,

(2) $$g_1 = (\Omega B\lambda)\alpha^{-1} \quad .$$

Now g_1 induces an isomorphism in H_0 and is surjective in H_1 since this is the case for $\lambda\colon A \longrightarrow H$.

We construct a model M of g_1,

(3) $$A \underset{i}{\rightarrowtail} M \underset{g}{\xrightarrow{\sim}} \Omega BH$$

as in (C1.1.5). Thus

(4) $$M = A \amalg T(W) = T(V \oplus W) \quad ,$$

where we have $W_0 = W_1 = 0$, since we can choose g_1 in the proof of (C1.1.5) to be g_1 in (2).

We know that $\hat{B}H = \hat{b}\Omega\hat{B}H = B(H,H)$ is the normalized bar resolution of H, see (2.15). Therefore, see [6],

(5)
$$H_n\hat{B}H = 0 \quad \text{for} \quad n \geq 1 \quad \text{and}$$
$$\mu_*: H_o\hat{B}H \xrightarrow{\cong} H \quad \text{is an isomorphism} \quad .$$

Since g in (3) is a homotopy equivalence in $\mathcal{DFA}$ we see that $\hat{b}g$ is a homotopy equivalence in $\mathcal{DF\hat{M}}$. Therefore

(6)
$$(\hat{b}g)_*: H_n\hat{b}M \cong H_n\hat{B}H \quad \text{for all} \quad n \;,$$

compare (2.17).

We now consider the inclusion $A \subset M$ in (3). Since $W_o = W_1 = 0$ we see that

(7)
$$\hat{b}_nA = \hat{b}_nM \quad \text{for} \quad n \leq 2 \; .$$

Therefore $\hat{b}A \longrightarrow \hat{b}M$ induces the isomorphism

$$H_1\hat{b}A = H_1\hat{b}M = 0 \qquad \text{by (6) and (5)}$$
$$H_o\hat{b}A = H_o\hat{b}M = H_o\hat{B}H = H \quad \text{by (6) and (5)} \; .$$

Thus the proposition is proved.

§ 3 Homotopy systems for chain algebras

We introduce homotopy systems for chain algebras in the same way as this was done by J.H.C. Whitehead for CW - complexes, see (A.§ 2).

Let $A=(T(V),d)$ be a free chain algebra with skeleta $A^n = (T(V^n),d)$ where $V^n=\{x\in V:\ |x|\leq n\}$. The projection $p_o=\varepsilon: A \longrightarrow R$ is the augmentation of A. We set

(i)
$$\rho_n=\rho_n A = \begin{cases} H_n(A^n,A^{n-1}) & , \ n\geq 1 \\ H_o(A^o) & , \ n=0 \ . \end{cases}$$

Clearly, $H_o(A^o) = T(V^o)$ is a free algebra with augmentation $\varepsilon=p_o$. We have boundary maps

(ii)
$$d_n:\ \rho_n \longrightarrow \rho_{n-1} \quad , \quad n\geq 1 \ .$$

Here $d_1=\partial:\ H_1(A^1,A^o) \longrightarrow H_oA^o$ and $d_n=j\partial:\ H_n(A^n,A^{n-1}) \longrightarrow H_{n-1}A^{n-1} \longrightarrow H_{n-1}(A^{n-1},A^{n-2})$ for $n\geq 2$. The operators ∂ and j are taken form the exact homology sequence of a pair of chain complexes. All ρ_n are R-modules but $\rho_o=T(V_o)$ is a free algebra. Moreover, all ρ_n are ρ_o-bimodules by

$$\begin{array}{ccc} \rho_o\otimes\rho_n\otimes\rho_o & \longrightarrow & \rho_n \\ \alpha\otimes\{x\}\otimes\beta & \mapsto & \{\alpha\cdot x\cdot\beta\} \end{array}$$

Clearly, if $x\in(A^n)_n$ is a cycle relative $(A^n)_{n-1}$ then also $\alpha\cdot x\cdot\beta$ is a cylce relative $(A^n)_{n-1}$. In particular,

(iii) ρ_1 is a ρ_o-bimodule.

The image $d_1(\rho_1)$ is an ideal in $\tilde{\rho}_o$ = kernel ε such that

$$\bar{\rho}_o = \rho_o / d_1\rho_1 = H_oA = H_oA^n \quad , \quad n \geq 1$$

is an augmented algebra. Moreover,

(iv) ρ_n is a $\bar{\rho}_o$ - bimodule for $n \geq 2$.

Here the action of $\bar{\rho}_o$ is induced by the action of ρ_o on ρ_n . These data (i),...,(iv) have the following properties:

(a) $dd = 0$, $\epsilon d = 0$.

(b) ρ_o is a free algebra.

(c) ρ_1 is a crossed (ρ_o, d_1) - bimodule.

This means that ρ_1 is a ρ_o - bimodule and $d_1 : \rho_1 \longrightarrow \rho_o$ satisfies

$$d_1(\alpha \cdot x \cdot \beta) = \alpha \cdot (d_1x) \cdot \beta \quad , \alpha, \beta \in \rho_o \ , \ x \in \rho_1$$

$$(d_1x) \cdot y = x \cdot (d_1y) \quad , x, y \in \rho_1 \ .$$

Moreover, ρ_1 is a free crossed (ρ_o, d_1) - bimodule, see (3.2).

(d) ρ_n is a free $\bar{\rho}_o$ - bimodule for $n \geq 2$.

(e) d_n for $n \geq 3$ is a map of $\bar{\rho}_o$ - bimodules and $d_2 : \rho_2 \longrightarrow$ kernel d_1 is a map of $\bar{\rho}_o$ - bimodules by the action of ρ_o . This follows from (c).

Motivated by the definition of homotopy systems for CW - complexes by J.H.C. Whitehead we now introduce the category of homotopy systems for chain algebras:

(3.1) Definition: A system of R - modules ρ_n , $n \geq 0$, with boundary maps d_n and actions as in (i),...,(iv) is called a homotopy system for

$\mathcal{D}A$ if the conditions (a),...,(e) are satisfied. By a homomorphism between such homotopy systems,

$$f: \rho \longrightarrow \rho' \quad ,$$

we mean a family of R - linear maps $f_n: \rho_n \longrightarrow \rho'_n$ $(n \geq 0)$ such that

(1) $fd = df$.

(2) f_o is an augmented map between free algebras, f_1 is f_o - biequivariant and f_n , $n \geq 2$, is $\bar{f}_o$ - equivariant, where $\bar{f}_o: \bar{\rho}_o \longrightarrow \bar{\rho}'_o$ is induced by $f_o: \rho_o \longrightarrow \rho'_o$.

Let $H = H(\mathcal{D}A)$ be the category of homotopy systems and of homomorphisms between them. //

It is convenient to fix for a homotopy system ρ a <u>basis</u> V_n <u>in each free object</u> ρ_n of (b),(c) and (d) respectively. For example, if $\rho = \rho(A)$ with $A = T(V)$, then V_n is a basis of the free object $\rho_n(A)$. We consider this in more detail:

(3.2) <u>Remark</u>: We already pointed out that

$$(*) \qquad \rho_o A = H_o(A^o) = T(V_o)$$

is the <u>free algebra</u> generated by V_o . We obtain $\rho_1 A$ by the chain complex A^1 / A^o with

$$\begin{tikzcd}
(A^1/A^o)_2 \arrow[r, "\partial"] \arrow[d, equal] & (A^1/A^o)_1 \arrow[r] \arrow[d, equal] & (A^1/A^o)_o \arrow[d, equal] \\
T(V_o) \otimes V_1 \otimes T(V_o) \otimes V_1 \otimes T(V_o) \arrow[r] & T(V_o) \otimes V_1 \otimes T(V_o) \arrow[r] & 0
\end{tikzcd}$$

Here ∂ is given by d in A, namely

$$\partial(\alpha\otimes x\otimes\beta\otimes y\otimes\gamma) = (\alpha\cdot dx\cdot\beta)\otimes y\otimes\gamma - \alpha\otimes x\otimes(\beta\cdot dy\cdot\gamma)$$

for $x,y\in V_1$, $\alpha,\beta,\gamma\in T(V_0)$. This in fact shows that

(**) $$\rho_1 A = H_1(A^1,A^0) = T(V_0)\otimes V_1\otimes T(V_0)/\operatorname{im}\partial$$

is the free crossed (ρ_1,d_1)-bimodule generated by V_1, compare the equations in (c).

Moreover,

(***) $$\rho_n A = H_n(A^n,A^{n-1}) = \bar{\rho}_0\otimes V_n\otimes\bar{\rho}_0 \qquad (n\geq 2)$$

is the free $\bar{\rho}_0$-bimodule generated by V_n. This is easily checked by computation of $H_n(A^n/A^{n-1})$, see (0.19). //

We now compare the homotopy system $\rho_n(A)$ with the chain complex of equivariant chains $\hat{b}(A)$ which was defined in (1.14). For this we introduce maps, $n\geq 0$,

(3.3) $$h_n\colon \rho_n(A) \longrightarrow \hat{b}_{n+1}(A) = H\otimes sV_n\otimes H\ .$$

Definition of h_n: For $n\geq 2$ let $h_n=\bar{s}$ be the suspension:

$$h_n=\bar{s}\colon H\otimes V_n\otimes H \xrightarrow{\approx} H\otimes sV_n\otimes H$$

where $\bar{\rho}_0 = H = H_0A$. Let $\lambda\colon T(V_0)=\rho_0 \longrightarrow H$ be the quotient map. Then h_1 is the λ-biequivariant map with $h_1(v)=sv$, $v\in V_1$. Moreover let $h_0=\bar{s}\Lambda$ be the map given by Λ in (1.18). //

The maps h_n fit into the diagram

(3.4)
$$\begin{tikzcd}
{} \arrow[r] & \hat{b}_3A \arrow[r, "\hat{d}"] & \hat{b}_2A \arrow[r, "\hat{d}"] & \hat{b}_1A \arrow[r, "\hat{d}"] & \hat{b}_0A \\
{} \arrow[r] & \rho_2A \arrow[u, "h_2"', "\approx"] \arrow[r, "d_2"'] & \rho_1A \arrow[u, two heads, "h_1"'] \arrow[r, "d_1"'] & \rho_0A \arrow[u, "h_0"'] &
\end{tikzcd}$$

for which we have the equation

$$\hat{d}h = -hd \quad .$$

This is clear by definition of $\hat{d}$ in (1.17). Since h_1 is surjective the equivariant chain complex $\hat{b}A$ is essentially given by the homotopy system ρA . This leads to the following commutative diagram of functors:

(3.5)
$$\begin{tikzcd}
& \mathit{DFA} \arrow[dl, "\hat{b}"'] \arrow[dr, "\rho"] & \\
\mathit{DF\hat{M}} & & \mathit{H} \arrow[ll, "C"]
\end{tikzcd}$$

<u>Remark</u>: Diagram (3.4) above cooresponds to diagram (A .2.3)(3) and the functors in (3.5) correspond to the functors in (A .2.2). //

The functor $\hat{b}$ in (3.5) was defined in (1.14). The functor ρ is given by $\rho(A)$ in (i) above. Moreover, we define the functor C in such a way that (3.5) commutes:

(3.6) Definition of the functor C :

For a homotopy system ρ in $\mathcal{H}$ let

$$C(\rho) = (H, C_*) \quad \text{in} \quad \mathcal{DF}\hat{M}$$

be given by $H = \bar{\rho}_o = \rho_o / d_1\rho_1$ and by

(1) $$C_n = \begin{cases} H \otimes H & , n = 0 \\ H \otimes sV_{n-1} \otimes H & , n \geq 1 \end{cases}$$

where V_n is a basis of the free object ρ_n $(n \geq 0)$. We have maps

(2) $$h_n: \rho_n \longrightarrow C_{n+1} \quad (n \geq 0)$$

which are defined in the same way as h_n in (3.4). In particular h_n is the suspension $\bar{s}$ for $n \geq 2$. For the quotient map $\lambda: \rho_o \longrightarrow \bar{\rho}_o = H$ the map h_1 is the λ-biequivariant map with $h_1(v) = 1 \otimes sv \otimes 1$ for $v \in V_1$. Moreover $h_o: T(V_o) = \rho_o \longrightarrow H \otimes sV_o \otimes H$ is given by $h_o = \bar{s}\Lambda$ with Λ in (1.18). We define the boundary $\hat{d}: C_n \longrightarrow C_{n-1}$ in the diagram

(3) $$\begin{tikzcd} {} \arrow[r] & C_3 \arrow[r, "\hat{d}"] & C_2 \arrow[r, "\hat{d}"] & C_1 \arrow[r, "\hat{d}"] & C_o \\ {} \arrow[r] & \rho_2 \arrow[r] \arrow[u, "\approx"', "h_2"] & \rho_1 \arrow[r] \arrow[u, "h_1"] & \rho_o \arrow[u, "h_o"] & \end{tikzcd}$$

by $\hat{d}h = -hd$. On C_1 let $\hat{d}$ be the biequivariant map with

(4) $$\hat{d}(sv) = (\lambda v) \otimes 1 - 1 \otimes \lambda v \quad \text{for} \quad v \in V_o .$$

Compare (1.17). For a homomorphism $f: \rho \longrightarrow \rho'$ the induced map is $f_*: C\rho \longrightarrow C\rho'$ which is $\bar{f}_o$-biequivariant and which satisfies

(5) $$\begin{aligned} (f_*)_o &= \bar{f}_o \otimes \bar{f}_o && \text{in degree } 0 \\ h'_n f_n &= (f_*)_n h_n && \text{in degree } n \quad . \end{aligned}$$

//

(3.7) **Proposition**: *C is a welldefined functor and diagram (3.5) commutes.*

Proof: We first check that $C(\rho) = (H,C_*)$ in (3.6) is a welldefined object in $\mathcal{DF}\hat{\mathcal{M}}$. For this we have the show (1) and (2):

(1) $$h_0 d_1(\text{kernel } h_1) = 0 \quad ,$$

(2) $$\tilde{d} h_0 d_1 = 0 \quad .$$

For (1) we consider the commutative diagram

$$\begin{tikzcd}
H \otimes V_1 \otimes H \arrow[r, "-\tilde{d}"] & H \otimes V_0 \otimes H \arrow[r, "\tilde{d}"] & H \otimes H \\
H_1(A^1,A^0) \arrow[u, "h_1"] \arrow[r, "d_1"'] & T(V_0) \arrow[u, "h_0 = \Lambda"'] & \\
T(V_0) \otimes V_1 \otimes T(V_0) \arrow[u] \arrow[ur, "d"'] \arrow[uu, bend left=60, "\lambda \otimes 1 \otimes \lambda"] & &
\end{tikzcd}$$

We know by definition of λ that

(3) $$\lambda d_1 = 0 \quad .$$

Therefore we see that for $\alpha,\beta \in T(V_0)$, $u \in V_1$ we have

$$\begin{aligned}
h_0 d_1(\alpha\cdot v\cdot\beta) &= \Lambda d(\alpha \otimes v \otimes \beta) \\
&= \Lambda(\alpha\cdot dv\cdot\beta) \\
&= \Lambda(\alpha)\cdot\lambda(dv)\cdot\lambda(\beta) + \lambda(\alpha)\cdot\Lambda(dv)\cdot\lambda(\beta) \\
&\qquad + \lambda(\alpha)\cdot\lambda(dv)\cdot\Lambda(\beta) \\
&= \lambda(\alpha)\cdot(\Lambda dv)\cdot\lambda(\beta) \quad .
\end{aligned}$$

Therefore, $h_o d$ in the diagram above is λ-biequivariant. This implies (1). For the proof of (2) we show

$$\hat{d}h_o(\alpha) = (\lambda\alpha) \bullet 1 - 1 \bullet (\lambda\alpha) \quad , \quad \alpha \in T(V_o) \quad . \tag{5}$$

Then clearly $\hat{d}h_o d_1 = 0$ by (3).

Clearly, for $\alpha = v \in V_o$ we have by definition of $\hat{d}$ and Λ

$$\hat{d}h_o(v) = \hat{d}(1 \bullet v \bullet 1) = (\lambda v) \bullet 1 - 1 \bullet (\lambda v) \ .$$

Now assume equation (5) is true for $\alpha = x$ and $\alpha = y$, $x, y \in T(V_o)$. Then

$$\begin{aligned} \hat{d}h_o(x \cdot y) &= \hat{d}(\lambda x \cdot \Lambda y + \Lambda x \cdot \lambda y) \\ &= \lambda x(\hat{d}\Lambda y) + (\hat{d}\Lambda x)\lambda y \\ &= \lambda x(\hat{d}h_o y) + (\hat{d}h_o x)\lambda y \\ &= \lambda x(\lambda y \bullet 1 - 1 \bullet \lambda y) + (\lambda x \bullet 1 - 1 \bullet \lambda x)\lambda y \\ &= \lambda(x \cdot y) \bullet 1 - \lambda x \bullet \lambda y + \lambda x \bullet \lambda y - 1 \bullet \lambda(x \cdot y) \ . \end{aligned}$$

Thus (5) is also true for $x \cdot y$. This proves (5). Next we have to check that (5) in (3.6) gives us a welldefined map f_* in DFR. The map $f_{*2}\colon C_2 \longrightarrow C'_2$ is the $\bar{f}_o$-biequivariant map with

$$f_{*2}(v) = h'_1 f_1(v) \quad \text{for} \quad v \in V_1 \subset \rho_1 \ . \tag{6}$$

Similarly $f_{*1}\colon C_1 \longrightarrow C'_1$ is the $\bar{f}_o$-biequivariant map with

$$f_{*1}\ (v) = h'_o f_o(v) \quad \text{for} \quad v \in V_o \subset \rho_o \quad . \tag{7}$$

Now we can check $h'f = f_* h$ and $\hat{d}f_* = f_*\hat{d}$.

Moreover, we have to check that for an arbitrary homotopy system ρ in $\mathcal{H}$ the chain complex $C\rho = \hat{C}$ satisfies condition (c) in (1.10). This is clear since $\mu\hat{d}_{|V_0} = 0$. □

We have the cylinder $IA = T(V' \oplus V'' \oplus sV)$ of $A = T(V)$ in $\mathcal{DFA}$. We define $I\rho$ in $\mathcal{H}$ in such a way that $\rho IA = I\rho$ for $\rho = \rho A$. This leads to the following notion of homotopy in $\mathcal{H}$:

(3.8) Definition: Let $f,g: \rho \longrightarrow \rho'$ be homomorphisms in $\mathcal{H}$. We write $f \simeq g$ if there is a sequence $\alpha_n: \rho_n \longrightarrow \rho'_{n+1}$ $(n \geq 0)$ of linear maps with the following properties:

(a) $$\alpha_0(x \cdot y) = (\alpha_0 x)(g_0 y) + (f_0 x)(\alpha_0 y) \quad , \; x,y \in \rho_0 \; ,$$

(b) α_n is a homomorphism which is λf_0 biequivariant for $n = 1$ and which is $\bar{f}_0$ - biequivariant for $n > 1$. (We have $\lambda f_0 = \lambda g_0$ and $\bar{f}_0 = \bar{g}_0$ by the following condition)

(c) $$-f_0 + g_0 = d\alpha_0$$

(d) $$-f_n + g_n = d\alpha + \alpha d \qquad (n \geq 1)$$

We call $\alpha = \{\alpha_n\}$ a homotopy from f to g . Let $\mathcal{H}/_{\simeq}$ be the homotopy category of $\mathcal{H}$. //

The functors in (3.5) induce a commutative diagram of homotopy categories and functors

(3.9)
$$\begin{tikzcd}
 & \mathcal{DFA}/\simeq \arrow[dl, "\hat{b}"'] \arrow[dr, "\rho"] & \\
\hat{\mathcal{DFM}}/\simeq & & \mathcal{H}/\simeq \arrow[ll, "C"]
\end{tikzcd}$$

It is in fact easily seen that for the cylinder $I\rho$ in H we have $CI\rho = IC\rho$. This shows that the functor C is compatible with the homotopy relations.

In the topological situation, (A.2.6), we have seen that C is a full and faithful functor. We do not know whether this is also true for C in (3.9). We can prove, however, the following result which relies on theorem (2.19). Let $H_{(f)}$ be the full subcategory of H consisting of homotopy systems ρ for which $\overline{\rho_o} = \rho_o/d_1\rho_1$ is <u>free as an R-module</u>.

(3.10) **<u>Theorem</u>**: *The functor*

$$C\colon H_{(f)}/\simeq \longrightarrow DF\hat{M}/\simeq$$

is full and faithful.

<u>Proof of (3.10)</u>: We derive the result, in the same way as in (A.2.7), from the fact that C restricted to the categories of 1-dimensional objects,

$$C\colon H^1_{(f)}/\simeq \longrightarrow DF\hat{M}^1/\simeq$$

is full and faithful, see (C3.1.11). We need the assumption on $H_{(f)}$ in the proof of (4.7) below, compare (6) in the proof of (A.2.7). ▯

§ 4 The exact sequence of J.H.C. Whitehead for chain algebras

In (A.§ 3) we described the exact sequence of Whitehead for CW-complexes. By similar arguments we now obtain an exact sequence for chain algebras.

Let X be a free chain algebra $X=(T(V),d)$ with skeleta $X^n=(T(V^n),d)$. Let $H=H_oX$ be the subalgebra of degree 0 in the graded algebra H_*X. Clearly, H_nX is an H-bimodule by the algebra structure in H_*X. As we have seen in (C1.6.25) the homology H_nX can be interpreted as a homotopy group:

(4.1) $$\pi_n^T(X) = H_n(X) \quad .$$

The map $\bar{s}\Lambda\colon X \longrightarrow \hat{b}X/\hat{b}_oX$ of degree +1 in (1.20) induces

(4.2) $$j\colon H_nX \longrightarrow H_{n+1}\hat{b}X \quad , \quad n \geq 1 \; .$$

This is a biequivariant map between H-bimoudles. It corresponds in topology to the Hurewicz-homomorphism (A.3.4)

We define the Γ-functor of Whitehead on the category $\mathcal{DFA}$ by

(4.3) $$\Gamma_n^T(X) = \mathrm{Image}(i\colon H_n(X^{n-1}) \longrightarrow H_n(X^n)) \quad .$$

Here i is induced by the inclusion $X^{n-1} \subset X^n$, compare (A.3.5). Clearly,

(4.4) $$\Gamma_o^T = 0 \quad \text{and} \quad \Gamma_1^T = 0 \; .$$

Moreover, Γ_n^T is an H-bimodule for $n \geq 2$.

One easily checks that $\pi_n^T = H_n$, Γ_n^T and $H_{n+1}\hat{b}$ $(n \geq 1)$ are functors defined on the homotopy category $\mathcal{DFA}/\simeq$, and thus on $\mathrm{Ho}(\mathcal{DA})$, see §0 .

(4.5) <u>**Proposition:**</u> *Assume $H = H_0X$ is free as an R-module. Then we have the natural exact sequence of H-bimodules, $k \geq 1$,*

$$\xrightarrow{b} \Gamma_k^T X \xrightarrow{i} H_k X \xrightarrow{j} H_{k+1}\hat{b}X \xrightarrow{b} \Gamma_{k-1}^T X \quad .$$

The sequence terminates with, see (4.4),

$$H_2X \twoheadrightarrow H_3\hat{b}X \longrightarrow 0 \longrightarrow H_1X \xrightarrow{\cong} H_2\hat{b}X \longrightarrow 0 \ .$$

Naturality means that the sequence is a functor which carries objects in $\mathcal{DFA}$ to exact sequences in the category $\hat{M}$. Here $.M.$ is the category with objects $(H;M)$ for which H is a non graded algebra and for which M is an H-bimodule.
Morphisms are pairs (α,β) where $\beta: M \longrightarrow M'$ is α-biequivariant. By (4,2) and (4.3) we have functors

$$\mathcal{DFA}/\simeq \xrightarrow{H_k\, ,\, \Gamma_k^T\, ,\, H_k\hat{b}} .M. \quad .$$

We call $.M.$ the <u>category of bimodules</u>.

The result (4.5) shows that for any free chain algebra X , for which H_0X is free as an R-module, we have the isomorphism of H_0X-bimodules:

$$H_1X = H_2\hat{b}X \quad . \qquad (4.6)$$

In topology this corresponds to the Hurewicz isomorphism $\pi_1\Omega X = H_2\hat{X}$.

(4.6) is essentially a consequence of proposition (4.7) below.

<u>Proof of (4.5)</u>: We apply (A.3.13). We use similar definitions for C_n and A_n as in (A.3.14). That is

(1) $$C_n = \rho_n \text{ and } A_n = H_n X^n \quad \text{for } n \geq 0$$

and $C_n = A_n = 0$ for $n < 0$. The homomorphisms j and β in (A.3.12) are the obvious operators of the homology exact sequence of a pair. We thus obtain by (A.3.13) the exact sequence ($k \in \mathbb{Z}$)

(3) $$\xrightarrow{b} \Gamma^T_k X \xrightarrow{i} H_k X \xrightarrow{j} H_k \rho X \xrightarrow{b} \Gamma_{k-1} X \quad .$$

Here $H_k \rho X = H_k = \text{kernel } d_n / \text{image } d_{n+1}$ is defined by the homotopy system $(\rho X, d)$ in § 3 . The exact sequence in (4.5) relies on the fact that we have canonical isomorphisms

(3) $$H_n \rho X = H_{n+1} \hat{b} X \quad , \quad n \geq 1 \; ,$$

(4) $$H_o \rho X = \rho_o / \text{Image } d_1 = H = H_o X \quad .$$

We prove (3) in the following proposition (4.7). □

(4.7) <u>Proposition</u>: *Let* ρ *be a homotopy system in* $H_{(f)}$. *Then* h_n *in (3) of (3.7) induces the isomorphism of* H*-bimodules:*

$$h_*: H_n \rho \cong H_{n+1} C(\rho) \quad (n \geq 1) \; .$$

<u>Proof</u>: Let ρ^1 be the homotopy system with $\rho^1_k = 0$ for $k \geq 2$ which coincides with ρ in dimension ≤ 1 . We call ρ^1 the 1 - skeleton of ρ . Consider the diagram

$$\begin{tikzcd}
\text{kernel } \hat{d}_2 \arrow[r, phantom, "\subset"] & C_2 \arrow[r, "\hat{d}_2"] & C_1 \arrow[r] & C_o \\
\text{kernel } d_1 \arrow[u] \arrow[r, phantom, "\subset"] & \rho_1 \arrow[u, "h_1"] \arrow[r, "d_1"] & \rho_o \arrow[u, "h_o"] &
\end{tikzcd}$$

given by ρ^1 , see (3.6). We prove that

(1) $$h_1 : \text{kernel } d_1 \xrightarrow{\cong} \text{kernel } \hat{d}_2$$

is an isomorphism of H-bimodules, $H = \rho_o/d_1\rho_1$. Let V_1 be a basis of ρ_1 and let $\rho_o = T(V_o)$. Then $d_1 : V_1 \longrightarrow T(V_o)$ yields the chain algebra

(2) $$A = (T(V_o \oplus V_1), d)$$

with $dv = d_1 v$ for $v \in V_1$. Clearly, we have

(3) $$\rho(A) = \rho^1 \quad .$$

As in the proof of (2.19) we choose a model M

(4) $$A \rightarrowtail M \xrightarrow{\sim} \Omega BH$$

in the category $\mathcal{DA}$. We know

(5) $$M^1 = A^1 \quad ,$$

(6) $$H_n \hat{b} M = 0 \quad \text{and} \quad H_n M = 0 \quad \text{for} \quad n \geq 1 \; .$$

We now consider diagram (3.4) for M . By (5) we obtain

(7)

$$\begin{tikzcd}
\hat{b}_4M \arrow[r] & \hat{b}_3M \arrow[r] & C_2 \arrow[r] & C_1 \\
\rho_3M \arrow[r] \arrow[u, "\approx"] & \rho_2M \arrow[r] \arrow[u, "\approx"] & \rho_1 \arrow[r] \arrow[u, two heads, "h_1"'] & \rho_o \arrow[u]
\end{tikzcd}$$

By (6) we know that the rows of this diagram are exact, (in fact, $H_2\rho M = 0$ and $H_1\rho M = 0$ by (2) in the proof of (4.5)). Since diagram (7) commutes exactness of the rows implies that h_1 in (1) is an isomorphism. This proves (4.7). $\square$

§ 5 On 1 - dimensional chain algebras

We define the dimension of a free chain algebra.

(5.1) Definition: A free chain algebra $A = (T(V),d)$ has dimension $\leq n$ if $V_k = 0$ for $k > n$. Let $\mathcal{DFA}^n$ be the full subcategory of $\mathcal{DFA}$ consisting of n - dimensional chain algebras.//

Chain algebras of dimension 1 are closely related to presentations of (non graded) algebras. In fact, by choosing generators of a non graded algebra H we find a sequence

(5.2) $$V_1 \xrightarrow{f} T(V_0) \overset{\lambda}{\twoheadrightarrow} H$$

with the following properties

(a) V_1 and V_0 are free R - modules,

(b) λ is a surjective map between algebras,

(c) $f(V_1)$ generates kernel(λ) as an ideal in $T(V_0)$.

We call such a sequence a presentation of H . Property (c) is satisfied iff the sequence

(5.3) $$T(V_0) \otimes V_1 \otimes T(V_0) \xrightarrow[d_f]{} T(V_0) \xrightarrow[\lambda]{} H$$

is exact, $d_f(\alpha \otimes v \otimes \beta) = \alpha \cdot fv \cdot \beta$.

(5.4) Remark: Presentations of H are in 1-1 correspondence to 1 - dimensional free chain algebras A with $H_0 A = H$. In fact, f in (5.2) gives us the chain algebra $C_f = A = (T(V_0,V_1),d)$ with $d(v) = f(v)$, $v \in V_1$, which satisfies $H_oA = H$. Here C_f is the mapping cone of the map

$$f: T(s^{-1}V_1) \longrightarrow T(V_o)$$

in $\mathcal{DA}$ given by f in (5.2), $f(s^{-1}v) = f(v)$. Thus '1 - dimensional chain algebras' and 'presentations of algebras' are in similar connection to each other as, in topology, 2 - dimensional 'CW - complexes' and 'presentations of groups'.//

Let C_g be a further chain algebra of dimension 1 with $g: W_1 \longrightarrow T(W_o)$. A map

$$F: C_f \longrightarrow C_g$$

in $\mathcal{DFA}^1$ induces the homomorphism

(5.5) $$\varphi = F_*: H_oC_f \longrightarrow H_oC_g \quad .$$

Moreover, F yields a commutative diagram

(5.6) $$\begin{tikzcd} V_1 \arrow[r, "\xi"'] \arrow[d] & T(W_o) \otimes W_1 \otimes T(W_o) \arrow[d, "d_g"] \\ T(V_o) \arrow[r, "\eta"'] \arrow[d] & T(W_o) \arrow[d, "\lambda"] \\ H_oC_f \arrow[r, "\varphi"'] & H_oC_g \end{tikzcd}$$

where $\eta = F_o$ is an algebra map and where ξ is the linear map, $\xi = F_1|V_1$. The map ξ is welldefined since we have in degree 1, $(C_g)_1 = T(W_o) \otimes W_1 \otimes T(W_o)$. On the other hand we can choose for each algebra map φ in (5.6) a pair (ξ, η) such that the diagram (5.6) commutes. In this case we say that (ξ,η) <u>induces</u> φ . Diagram (5.6) corresponds in topology to diagra (A.4.3).

Similar as in (A.4.6) we derive from (5.6) the following <u>model categor</u> of $\mathcal{DFA}^1$.

(5.7) <u>Definition</u>: The objects of $\mathcal{M}^1$ are linear maps $f: V_1 \longrightarrow T(V_o)$ where V_o and V_1 are free R-modules. The morphisms $(\xi,\eta): f \longrightarrow g$ in $\mathcal{M}^1$ are pairs (ξ,η) for which diagram (5.6) commutes. Composition is define by

$$(\xi,\eta)\circ(\xi',\eta') = ((\eta\oplus\xi\oplus\eta)\circ\xi',\eta\circ\eta') \quad ,$$

see (1.7). Clearly, we have the canonical equivalence of categories

$$\mathcal{M}^1 = \mathcal{DFA}^1$$

which carries f to C_f and (ξ,η) to F with $F_o = \eta$, $F_1|V_1 = \xi$ as above. //

The category $\mathcal{H}^1$ of 1-dimensional homotopy systems is a quotient cat gory of $\mathcal{M}^1$. We obtain the natural equivalence relation $\sim$, for which t diagram

(5.8)
$$\begin{tikzcd}
\mathcal{M}^1 \arrow[r, equal] \arrow[d, "p"] & \mathcal{DFA}^1 \arrow[d, "\rho"] \\
\mathcal{M}^1/\sim \arrow[r, equal] & \mathcal{H}^1
\end{tikzcd}$$

commutes, as follows: Let

$$q_g: T(W_o)\oplus W_1\oplus T(W_o) \longrightarrow \rho_1 C_g$$

be the quotient map in (**) of (3.2). Then we define, for morphisms $f \longrightarrow g$ in M^1,

$$(\xi,\eta) \sim (\bar\xi,\bar\eta) \iff \eta=\bar\eta \text{ and } q_g\xi=q_g\bar\xi \ .$$

One easily checks that diagram (5.8) commutes.

We obtain the homotopy relations in $M^1 = \mathcal{D}FA^1$ by the homotopy relation in $\mathcal{D}FA^1$, see (0.10), (0.11). Moreover, we have by (3.8) the homotopy relation in H^1.

(5.9) <u>Proposition</u>:

$$\mathcal{D}FA^1/\sim \; = \; M^1/\sim \; = \; H^1 \quad ,$$

$$\mathcal{D}FA^1/\simeq \; = \; M^1/\simeq \; = \; H^1/\simeq \quad .$$

Let $M^1_{(f)} = \mathcal{D}FA^1_{(f)}$ be the full subcateogry of objects C_g in $\mathcal{D}FA^1$ for which the homology in degree 0, H_oC_g, is <u>free as an R-module</u>. For this subcategory of $\mathcal{D}FA^1$ the homotopy relation can be described in a simpler fashion:

(5.10) <u>Lemma</u>: *For morphisms* $(\xi,\eta),(\xi',\eta'): f \longrightarrow g$ *in* $M^1_{(f)}$ *we have* $(\xi,\eta) \simeq (\xi',\eta')$ *iff there is a linear map*

$$\alpha\colon V_o \longrightarrow T(W_o)\otimes W_1\otimes T(W_o)$$

with

$$\text{(1)} \qquad \eta'|V_o - \eta|V_o = d_g\alpha \quad \text{in} \quad Hom(V_o,T(W_o)) \ ,$$

$$\text{(2)} \qquad \bar E_g\xi' - \bar E_g\xi = (\varphi\theta\bar E_g\alpha\theta\varphi)\Lambda f \quad .$$

Here (1) implies that (ξ,η) *and* (ξ',η') *induce the same map*

$$\varphi\colon H_oC_f = H \longrightarrow H_oC_g = H' \quad .$$

The map $\Lambda: T(V_o) \longrightarrow H \otimes V_o \otimes H$ *is defined in (1.8) and*

$$\bar{E}_g = \lambda \otimes 1 \otimes \lambda: T(W_o) \otimes W_1 \otimes T(W_o) \longrightarrow H' \otimes W_1 \otimes H'$$

is given by the quotient map $\lambda: T(W_o) \longrightarrow H'$ *, (2) is an equation in* $Hom(V_1, H' \otimes W_1 \otimes H')$.

Remark: We have checked that $\bar{s}\Lambda f = (1 \vee j)E(\nabla f)$, compare (1.31). Therefo

(5.10) corresponds to the homotopy relation in (A.4.8). //

Proof of (5.10): We consider the diagram

$$\begin{tikzcd}
 & (C_g)_2 \arrow[d, "d"] & H' \otimes W_1 \otimes H' \\
V_1 \arrow[ur, dashed, "\alpha_1"] \arrow[r, "\xi{,}\bar{\xi}"'] \arrow[d, "f"'] & T(W_o) \otimes W_1 \otimes T(W_o) \arrow[r, "p"'] \arrow[d, "d_g"] \arrow[ur, dash dot, "E_g"] & \rho'_1 \arrow[u, "h_1"'] \arrow[d, "d_1"] \\
T(V_o) \arrow[ur, dashed, "\alpha_o"] \arrow[r, "\eta{,}\bar{\eta}"'] & T(W_o) \arrow[r, phantom, "="] & \rho'_o
\end{tikzcd}$$.

Here $\rho' = \rho(C_g)$ is the homotopy system of the 1-dimensional algebra C_g . p is the quotient map in (**) of (3.2). The map α_o is given by a map

$$\alpha: V_o \longrightarrow T(W_o) \otimes W_1 \otimes T(W_o)$$

and has the property

$$(*) \qquad \begin{aligned} \alpha_o(v) &= \alpha(v) \quad \text{for} \quad v \in V_o \ , \\ \alpha_o(x \cdot y) &= (\alpha_o x)(\eta' y) + (\eta x)(\alpha_o y). \end{aligned}$$

By definition of homotopy in *DFA* we know that $(\xi,\eta) \simeq (\bar{\xi},\bar{\eta})$ are homotopic maps in *DFA* iff there is α and α_1 such that

(1) $$\bar{\eta} - \eta = d_g \alpha_o \quad ,$$

(2) $$\bar{\xi} - \xi = d\alpha_1 + \alpha_o f \quad .$$

Now (1) is satisfied iff

(1)' $$\bar{\eta}|_{V_o} - \eta|_{V_o} = d_g \alpha \quad .$$

Moreover, since image d = kernel p we have $(\xi,\eta) \simeq (\bar{\xi},\bar{\eta})$ in *DFA* iff there is α with (1)' and

(2)' $$p(\bar{\xi} - \xi - \alpha_o f) = 0 \quad .$$

We have

$$\begin{aligned} d_1 p(\bar{\xi} - \xi - \alpha_o f) &= (\bar{\eta} - \eta) f - d_1 p \alpha_o f \\ &= d_g \alpha_o f - d_g \alpha_o f = 0 \quad . \end{aligned}$$

Thus $p(\bar{\xi} - \xi - \alpha_o f)$ maps to kernel(d_1) . By (1) in the proof of (4.7) we know that h_1 is injective on kernel(d_1) . Therefore (2)' is satisfied iff for $E_g = h_1 p$ we have

(2)" $$E_g(\bar{\xi} - \xi - \alpha_o f) = 0 \quad .$$

By (*) above we have

(3) $$E_g \alpha_o = (\varphi \otimes E_g \otimes \varphi)\Lambda \quad .$$

Therefore (1)' and (2)" are equivalent to (1) and (2) in (5.10). This proves the proposition. $\square$

(5.11) <u>Proposition</u>: *The functor*

$$\hat{b}\colon \mathit{DFA}^{1}_{(f)}/\simeq \longrightarrow \mathit{DF}\hat{M}/\simeq$$

is full and faithful.

<u>Proof</u>: By use of (5.10) this can be proved in the same way as (A.4.30). W describe a slightly different method of proof as follows. We leave it to the reader to check the details. $\square$

Let $[f,g]_{\varphi}$ be the set of homotopy classes of maps $f \longrightarrow g$ in M^1 which induce $\varphi\colon H_oC_f \longrightarrow H_oC_g$. We have a <u>transitive and effective act</u> of the abelian group $\hat{H}_2(C_f,\varphi^* H_1C_g)$ on $[f,g]_{\varphi}$, see (6.4) below.

On the other hand let $[\hat{b}C_f,\hat{b}C_g]_{\varphi}$ be the set of homotopy classes of φ-biequivariant maps in $\mathit{DF}\hat{M}/\simeq$. Then we have similarly a <u>transitive an effective action</u> of $\hat{H}_2(C_f,\varphi^* H_2\hat{b}C_g)$ on this set. Moreover, the function

$$\hat{b}\colon [f,g]_{\varphi} \longrightarrow [\hat{b}C_f,\hat{b}C_g]_{\varphi}$$

is j_*-equivariant, where

$$j_*\colon \hat{H}_2(C_f,\varphi^* H_1C_g) \longrightarrow \hat{H}_2(C_f,\varphi^* H_2\hat{b}C_g)$$

is induced by

$$j\colon H_1C_g \longrightarrow H_2\hat{b}C_g \quad .$$

Thus $\hat{b}$ is a bijection since j is an isomorphism, see (4.6). $\square$

§ 6 On n-dimensional algebras

We consider the homotopy category of n-dimensional chain algebras, $\mathcal{DFA}^n/\simeq$, and a quadratic natural group action on $\mathcal{DFA}^n/\simeq$. An n-dimensional chain algebra $X = X^n$ gives us the principal cofibration

$$X^{n-1} \subset X^n = X^{n-1} \amalg T(V_n) \ .$$

The attaching map $f: T(s^{-1}V_n) \longrightarrow X^{n-1}$ of $X^n = C_f$ is given by the differential d in X^n, $f(s^{-1}v) = d(v)$, see (0.8).

We have the group action

$$[X^n,U] \times E(X^n,U) \longrightarrow [X^n,U] \tag{6.1}$$

on the homotopy set $[C_f,U]$, $U \in \mathcal{DA}$. See (III.3.6). Here we use the abelian group

$$\begin{aligned} E(X^n,U) &= [\Sigma T(s^{-1}V_n),U] \\ &= \mathrm{Hom}(V_n,H_nU) = \mathrm{Hom}_{\varphi,\varphi}(\hat{b}_{n+1}\ X^n,H_nU) \\ &= \hat{b}^{n+1}(X^n,\varphi^* H_nU) \quad , \end{aligned} \tag{6.2}$$

see (1.25). Here we can take for φ any algebra map $\varphi: H_0X \longrightarrow H_0U$.

The action (6.1) can be described by

$$\{F\} + \alpha = \{F + \tilde{\alpha}\}$$

where $F + \tilde{\alpha}: X^n \longrightarrow U$ is the map in $\mathcal{DA}$ with $(F+\tilde{\alpha})_k = F_k$ for $k < n$ and

$$(F+\tilde{\alpha})_n \mid V_n = (F_n|V_n) + \tilde{\alpha} \ . \tag{6.3}$$

Here we choose $\tilde{\alpha}$ such that

$$\begin{tikzcd}
V_n \arrow[rr, "\tilde{\alpha}"] \arrow[dr, "\alpha"'] & & Z_nU \subset U_n \arrow[dl] \\
& H_nU &
\end{tikzcd}$$

commutes. Z_nU denotes the cycles in U of degree n.

Let $[X^n,U]_\varphi$ be the subset of all homotopy classes in $[X^n,U]$ which induce φ in H_o. This set can be empty. If it is non empty, the coboundaries in $\hat{b}^{n+1}(X^n,\varphi^* H_n U)$ are elements of the isotropy group in each $u_n \in [X^n,U]_\varphi$. Therefore (6.1) induces the action

(6.4)
$$[X^n,U]_\varphi \times \hat{H}^{n+1}(X^n,\varphi^* H_n U) \xrightarrow{+} [X^n,U]_\varphi$$

Only for n=1 this is a transitive and effective action. For $n>1$ the spectral sequence in the appendix of § 1 yields the isotropy group $I(u_n)$ of the action (6.4) in u_n :

(6.5)
$$E_{n+1}^{n+1,0}(u_n) = \hat{H}^{n+1}(X^n,\varphi^* H_n U) / I(u_n) \quad .$$

By (6.1) and (6.4) we have a natural group action on $\mathit{DFA}^n/\simeq$ which has a quadratic distributivity law. The mixed term $\alpha * \beta$ is given similarly as in (A.5.9) by composition of cocycles. We describe this in more detail.

Let X,Y,Z be objects in DFA^n and let

$$F \in [X,Y]_\varphi \quad , \quad G \in [Z,X]_\psi$$
$$\alpha \in \hat{H}^{n+1}(X,\varphi^* H_n Y) \quad , \quad \beta \in \hat{H}^{n+1}(Z,\psi^* H_n X) \quad .$$

Then composition in $\mathit{DFA}^n/\simeq$ satisfies the distributivity law

(6.6) $$(F+\alpha)(G+\beta) = FG + F_*\beta + G^*\alpha + \alpha * \beta$$

in $[Z,Y]_{\varphi\psi}$. The induced functions F_* and G^* are the obvious homomorphisms for cohomology. The mixed term is the element

$$\alpha * \beta \in \hat{H}^{n+1}(Z,\psi^*\varphi^*H_nY)$$

with

$$\alpha * \beta = \{\hat{b}_{n+1}Z \xrightarrow[b]{} H_nX \xrightarrow[j]{} \hat{H}_{n+1}\hat{b}X \subset \hat{b}_{n+1}X \xrightarrow[a]{} H_nY\} \ .$$

Here j is defined as in (4.2) and $b \in \beta$, $a \in \alpha$ are cocycles which represent the cohomology classes α and β respectively. We have for $X = X^n$ that

(6.7) $$\text{kernel } j = \Gamma_n^T(X)$$

by the exact sequence (4.5). The inclusion of H_oX - bimodules,

$$j\colon \Gamma_n^T(X) \subset H_nX \quad ,$$

induces the homomorphism

$$j\colon \hat{H}^{n+1}(Z,\psi^*\Gamma_n^TX) \longrightarrow \hat{H}^{n+1}(Z,\psi^*H_nX)$$

via which we define the action

(6.8) $$[Z,X]_\psi \times \hat{H}^{n+1}(Z,\psi^*\Gamma_n^TX) \xrightarrow{+} [Z,X]_\psi$$

$$\{F\} + \gamma = \{F\} + j(\gamma) \quad , \text{ see } (6.4).$$

This, again, is a natural group action on $\mathcal{D}FA^n/\simeq$ which we denote by $\Gamma +$. Moreover, (6.7) and (6.6) show that $\Gamma +$ is a linear action which gives rise to the linear extension

(6.9) $$\Gamma + \longrightarrow \mathcal{D}FA^n/\simeq \longrightarrow (\mathcal{D}FA^n/\simeq)/\Gamma$$

of categories. In the next section we give a different description of the category $(\mathcal{DFA}^n/\simeq)/\Gamma$, see (8.12).

(6.10) Remark: We can develop this section further in close analogy to (A.§ 5). We leave it to the reader to introduce Postnikov functors for $\mathcal{DFA}/\simeq$; and to introduce the categories $\mathcal{T}_n$, $\mathcal{T}_n^n$ for $\mathcal{DFA}^n/\simeq$ by use of k-invariants, see (A.5.14). We even think that an analogue of Whitehead's result (A.5.28) should be true for chain algebras.//

§ 7 Homotopy systems of order n , $n \geq 3$

Almost the same arguments and definitions which we used in chapter A for the construction of the CW - tower are now available for the construction of the DA - tower. Similarly as in (A.§ 6) we first define the category of homotopy systems of order n.

A free chain algebra $X = (T(V),d)$ has attaching maps

(7.1) $$f_n : D_n = T(s^{-1}V_n) \longrightarrow X^{n-1}$$

with $X^n = C_{f_n}$, see (0.8). The homotopy class of f_n is given by the map

(7.2) $$f_n : s^{-1}V_n \longrightarrow H_{n-1}X^{n-1}$$

which carries $s^{-1}v$ to $\{dv\}$, $v \in V_n$. Here d is the differential in X , compare (0.16).

Let $H = H_oX^{n-1} = H_oX$, $n \geq 2$. The homotopy class of f_n is as well given by the H - biequivariant map f_n which makes the following diagram commutative

(7.3) $$\begin{tikzcd} \hat{b}_{n+1}X \arrow[r, equal] \arrow[dd, dashed, "f_n"] & H \otimes sV_n \otimes H \\ & H \otimes V_n \otimes H \arrow[u, "\approx"', "\bar{s}"] \arrow[d, equal] \\ H_{n-1}(X^{n-1}) & H_n(X^n,X^{n-1}) \arrow[l, "\partial"'] \end{tikzcd}$$

(7.4) **Remark:** The element f_n is a cocycle in the cochain complex

$$\mathrm{Hom}_{H-H}(\hat{b}_* X, H_{n-1}(X^{n-1})) \quad ,$$

that is $f_n \hat{d}_{n+2} = 0$, see (1.17). This follows for example from property (a) in (V.5.2), compare also (A.6.4).//

Let $Y = (T(W), d)$ be a further free chain algebra and let $F: X \longrightarrow$ be a map in $\mathcal{DA}$. By naturality of diagram (7.3) the map F induces the commutative diagram $(n \geq 2)$

(7.5)
$$\begin{tikzcd}
\hat{b}_{n+1}X \arrow[r, "\xi_{n+1}"'] \arrow[d, "f_n"] & \hat{b}_{n+1}Y \arrow[d, "g_n"] \\
H_{n-1}(X^{n-1}) \arrow[r, "\eta_*"'] & H_{n-1}(Y^{n-1})
\end{tikzcd} \quad .$$

Here g_n is the attaching map for Y defined in the same way as in (7.3 The map $\xi_{n+1} = \hat{b}_{n+1}F$ is induced by F , see (1.19), and $\eta: X^{n-1} \longrightarrow Y^{n-}$ is the restriction of F . By (1.20) we see that diagram (7.5) is a commutative diagram. The maps ξ_{n+1} and η_* are φ-biequivariant where $\varphi = H_o(\eta): H_o X \longrightarrow H_o Y$.

We deduce from f_n in (7.3) the boundary $\hat{d}_{n+1}$ by the following commutative diagram

(7.6)
$$\begin{tikzcd}
\hat{b}_{n+1}X \arrow[r, "f_n"'] \arrow[dd, dashed, "\hat{d}_{n+1} = \partial(f_n)"] & H_{n-1}X^{n-1} \arrow[d, "j"] \\
& H_n \hat{b}X^{n-1} \arrow[d, phantom, "\cap"] \\
\hat{b}_n X \arrow[r, phantom, "="] & \hat{b}_n X^{n-1}
\end{tikzcd} \quad .$$

Here j is the map in (4.2) which corresponds to the Hurewicz homomorphism.

We now introduce the category of homotopy systems similarly as in (A.6.7).

(7.7) Definition: Let $n \geq 2$. A homotopy system of order $(n+1)$ in $\mathcal{DA}$ is a triple

$$(\hat{b}, f_n, X^{n-1})$$

where X^{n-1} is a free chain algebra of dimension $\leq n-1$ in $\mathcal{DFA}$ and where $(H_o X^{n-1}, \hat{b})$ is an object in $\mathcal{DF}\hat{M}$, see (1.10), which coincides with $\hat{b}X^{n-1}$ in degree $\leq n$. Moreover,

$$f_n : \hat{b}_{n+1} \longrightarrow H_{n-1}X^{n-1}$$

is an $(H_o X^{n-1})$ - biequivariant map with

(1) $$d_{n+1} = \partial(f_n) \quad .$$

Here d_{n+1} is the boundary in $\hat{b}$ and $\partial(f_n)$ is given as in (7.6).

A homomorphism or a map between homotopy systems of order $n+1$ is a pair (ξ, η) which we write

$$(\xi,\eta): (\hat{b}, f_n, X^{n-1}) \longrightarrow (\hat{b}', g_n, Y^{n-1})$$

where $\eta: X^{n-1} \longrightarrow Y^{n-1}$ is a map in $\mathcal{DFA}/\sim$, see (0.10), and where $\xi: \hat{b} \longrightarrow \hat{b}'$ is a $H_o(\eta)$ - biequivariant map which coincides with $\hat{b}_*\eta$ in degree $\leq n$. Moreover, (ξ,η) induces the commutative diagram

(2) $$\begin{tikzcd} \hat{b}_{n+1} \arrow[r, "\xi_{n+1}"'] \arrow[d, dashed, "f_n"] & \hat{b}'_{n+1} \arrow[d, "g_n"] \\ H_{n-1}X^{n-1} \arrow[r, "\eta_*"] & H_{n-1}Y^{n-1} \end{tikzcd}$$

Let H_{n+1} be the category of homotopy systems of order (n+1) in $\mathcal{D}A$ and of maps as above. Clearly, composition is defined by $(\xi,\eta)(\bar{\xi},\bar{\eta}) = (\xi\bar{\xi},\eta\bar{\eta})$ .//

We have obvious functors $(n \geq 2)$

(7.8)
$$\mathcal{D}FA/\sim \xrightarrow[r_{n+1}]{} H_{n+1} \xrightarrow[\lambda]{} H_n$$

with $\lambda r_{n+1} = r_n$, $n \geq 3$. Here we set

$$r_{n+1}(X) = (\hat{b}_* X, f_n, X^{n-1})$$

where f_n is given as in (7.5). The functor r_{n+1} carries the 0-homotop class of $F: X \longrightarrow Y$ to $(\hat{b}_* F,\eta)$ where $\eta: X^{n-1} \longrightarrow Y^{n-1}$ is the restriction of F , see (0.10). By (7.5) and (7.6) we see that r_{n+1} is a welldefined functor.

We define λ by

$$\lambda(\hat{b}, f_n, X^{n-1}) = (\hat{b}, f_{n-1} X^{n-2})$$

where f_{n-1} is the attaching map for X^{n-1} .

(7.9) **Lemma**: *Let* $\mathfrak{X}$ *be the class of free chain algebras, considered as a class of complexes in* $\mathcal{D}A$ *, see (0.9). Then the category of twisted* n*-system admits the canonical isomorphism*

$$H_{n+1} = T_n(\mathfrak{X}) = K_n^{\vee}(\mathfrak{X}) \qquad (n \geq 2) \quad .$$

The functors in (7.8) correspond to those in (V.1.16) and (V.4.18). See also (V.6.2).

Proof: The lemma is a consequence of the definitions where we use (1.31) and (0.14). ▯

(7.10) **Lemma**: *There is a canonical isomorphism of categories* $\lambda: H^3 \cong H$, *where* H *is the category of homotopy systems in* § 3 . *(Compare (A.7.8)).*

Proof: We define λ by

$$\lambda(\hat{b},f_2,X^1) = (\rho_*,d)$$

where

$$(1) \qquad \rho_i = \begin{cases} \hat{b}_{i+1} & , \ i \geq 2 \\ \rho_i(X^1) & , \ i = 0,1 \ . \end{cases}$$

The boundary in $d: \rho_{i+1} \longrightarrow \rho_i$ is given by d in $\hat{b}$ for $i \geq 2$ and is given by $\rho(X^1)$ for $i=0$. For $i=1$ we obtain d by the composition

$$d: \rho_2 = \hat{b}_3 \xrightarrow[f_2]{} H_1X^1 \subset H_1(X^1,X^0) = \rho_1(X^1) = \rho_1 \ .$$

The functor λ carries a map (ξ,η) to the obvious induced map. On the other hand if (ρ_*,d) is given we obtain $(\hat{b},f_2,X^1)$ with

$$(2) \qquad \lambda(\hat{b},f_2,X^1) = (\rho_*,d)$$

as follows: Let $X^1 = (T(V_o \oplus V_1),d)$ be given by the boundary

$$V_1 \subset \rho_1 \longrightarrow \rho_o = T(V_o)$$

where V_i is the fixed basis in ρ_i . We get f_2 by $d: \rho_2 \longrightarrow \rho_1$ since

$$d(\rho_2) \subset \text{kernel } d = H_1X^1 \ .$$

By (3.4) and (1) in (7.7) we see that (2) is satisfied. By (5.9) we see that the functor λ is an isomorphism of categories. ▯

Clearly, for λ in (7.10) the diagram

(7.11)
$$\begin{tikzcd} DFA/\sim \arrow[rr, "r_3"'] \arrow[dr, "\rho"'] & & H_3 \arrow[dl, "\lambda", "\cong"'] \\ & H & \end{tikzcd}$$

commutes, compare (3.4) and (3.5).

We define the homotopy relation $\simeq$ on H_{n+1} in such a way that r_{n+1} in (7.8) induces a functor between homotopy categories. We follows the definition of $\simeq$ on $K_n^V(\mathfrak{X})$ in (V.1.21) and (V.1.21)'.

(7.12) <u>Definition</u>: Let $n \geq 2$ and let

$$(\xi,\eta),(\xi',\eta'): (\hat{b},f_n,X^{n-1}) \longrightarrow (\hat{b}',g_n,Y^{n-1})$$

be maps in H_{n+1}. We set $(\xi,\eta) \simeq (\xi',\eta')$ if $H_0\eta = H_0\eta'$ $(=\varphi)$ and if there exist φ-biequivariant maps

$$d_{j+1}: \hat{b}_j \longrightarrow \hat{b}'_{j+1} \quad (j \geq n)$$

such that

(a) $$\{\eta\} + g_n\alpha_{n+1} = \{\eta'\} \quad \text{in} \quad DFA/\simeq \quad ,$$

(b) $$\xi'_k - \xi_k = \alpha_k d_k + d_{k+1}\alpha_{k+1} \quad , \quad k \geq n+1 \quad .$$

The action $+$ in (a) is defined in (6.1), $\{\eta\}$ denotes the homotopy clas of η in $[X^{n-1},Y^{n-1}]$. We write $\alpha: (\xi,\eta) \simeq (\xi',\eta')$. //

One can check that $\simeq$ is a natural equivalence relation on H_{n+1}. Moreover, we have

(7.13) <u>Proposition</u>: *The isomorphisms in (7.9) and (7.10) induce isomorphisms* *between homotopy categories* $(n \geq 2)$

$$H_{n+1}/\simeq \; = T_n(\mathfrak{X})/\simeq \; = K_n^{\vee}(\mathfrak{X})/\simeq \, ,$$

$$H_3/\simeq \; = H/\simeq \quad , \quad \textit{see } (3.8).$$

We now are ready to apply the result on the twisted tower in (V.§ 6) .

§ 8 The main result: the DA - tower of categories

We show that the categories $\mathit{DFA}/\sim$ and $\mathit{DFA}/\simeq$ can be approximated by towers of categories. These towers are examples of the twisted tower in (V.§ 6), they correspond in topology to the CW - towers in (A.§ 7). In fact, as we will prove in chapter D there is a canonical map which carri the CW - tower to the DA - tower. This map is given by the chains on the loop space, see (O.3). We here describe the purely algebraic properties of the DA - towers which are constructed by use of the categories $\mathcal{H}_n$ an $\mathcal{H}_n/\simeq$ in § 7.

Whitehead's functor Γ_n^T in (4.3) factors over the category $\mathcal{H}_{n+1}$, that is we have a commutative diagram ($n \geq 2$)

(8.1)
$$\begin{tikzcd}
\mathit{DFA}/\sim \arrow[rr, "r_{n+1}"'] \arrow[dr, "\Gamma_n^T"'] & & \mathcal{H}_{n+1} \arrow[dl, "\Gamma_n^T"] \\
& .\mathcal{M}. &
\end{tikzcd}$$

where $.\mathcal{M}.$ is the category of bimodules, see (4.5). We define $\Gamma_n^T X$ for a object $X = (\hat{b}, f_n, X^{n-1})$ in $\mathcal{H}_{n+1}$ by

$$\Gamma_n^T X = \text{Image } i_{f*}: H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(C_f)$$

where f is a map which represents f_n, f is given by choosing a bas in $\hat{b}_{n+1}$. For a map

$$(\xi, \eta): X = (\hat{b}, f_n, X^{n-1}) \longrightarrow Y = (\hat{b}', g_n, Y^{n-1})$$

in $\mathcal{H}_{n+1}$ there is a map

(8.2) $$F: C_f \longrightarrow C_g \quad , \quad g \in \mathcal{g}_n \quad ,$$

which is associated to (ξ_{n+1}, η) . Therefore the induced map

$$(\xi,\eta)_* = F_*: \Gamma_n^T X \longrightarrow \Gamma_n^T Y$$

is welldefined. Clearly, $(\xi,\eta)_*$ depends only on the homotopy class of (ξ,η) in $\mathcal{H}_{n+1}/\simeq$.

We define a natural system $Z^{n+1}\Gamma_n^T$ on $\mathcal{H}_{n+1}$ as follows: For (ξ,η) as above let

(8.3) $$Z^{n+1}\Gamma_n^T(\xi,\eta) \subset \mathrm{Hom}_{\varphi,\varphi}(\hat{b}_{n+1}, \Gamma_n^T Y)$$

be the subgroup of cocycles in the cochain complex

(8.4) $$\mathrm{Hom}_{\varphi,\varphi}(\hat{b}, \Gamma_n^T Y) = \mathrm{Hom}_{H-H}(\hat{b}, \varphi^* \Gamma_n^T Y) \quad ,$$

compare (1.25). Here $H = H_o X$ and $\varphi: H_o X \longrightarrow H_o Y$, $\varphi = \eta_*$, is induced by η .

We define the natural system $H^k\Gamma_n^T$ on $\mathcal{H}_{n+1}$ by the cohomology of (8.4):

(8.5) $$H^k\Gamma_n^T(\xi,\eta) = \hat{H}^k(\hat{b}, \varphi^* \Gamma_n^T Y)$$

with φ as above. For k=n+1 we have the quotient map $Z^{n+1}\Gamma_n^T \longrightarrow H^{n+1}\Gamma_n^T$. The induced maps are the obvious homomorphisms given by the functor Γ_n^T in (8.1) and given by chain maps. Clearly, $H^k\Gamma_n^T$ is also a natural system on $\mathcal{H}_{n+1}/\simeq$.

The following result is a special case of (IV.6.3):

(8.6) <u>Theorem</u> *(the DA - tower): We have towers of categories which converge to the homotopy categories* $\mathcal{D}FA/\sim$ *and* $\mathcal{D}FA/\simeq$ *respectively, see (0.10 and (0.11). These towers are constructed by the categories* H_n *and* $H_n/\simeq$ *of homotopy systems,* $n \geq 3$ *, and we have a canonical quotient between these towers as indicated in the following diagram, compare notation in (Ext.§ 6).*

$$\begin{tikzcd}
 & \mathcal{D}FA/\sim \arrow[rr, "p"'] \arrow[d, "r"] & & \mathcal{D}FA/\simeq \arrow[d, "r"] & \\
 & \vdots & & \vdots & \\
Z^{n+1}\Gamma^T_n \arrow[r, "+"] & H_{n+2} \arrow[d, "\lambda"] & H^{n+1}\Gamma^T_n \arrow[r, "+"] & H_{n+2}/\simeq \arrow[d, "\lambda"] & \\
 & H_{n+1} \arrow[r, "0"] \arrow[d] & H^{n+2}\Gamma^T_n & H_{n+1}/\simeq \arrow[r, "0"] \arrow[d] & H^{n+2}\Gamma^T_n \\
 & \vdots & & \vdots & \\
 & H_3 \arrow[rr, "p"'] & & H_3/\simeq & \\
 & H \arrow[u, equal] & & H/\simeq \;\supset\; H_{(f)}/\simeq \arrow[u, equal] \arrow[d, "C"] & \\
 & & & \mathcal{D}F\hat{M}/\simeq \;. &
\end{tikzcd}$$

Here H is the category of homotopy systems in § 3 and C is the funct (3.9). The composition of all functors in the left column is the functor ρ , see (3.5). The composition of all functors in the column to the right

is the twisted chain functor $\hat{b}$ in (1.23).

By (3.10) we know that the functor C is full and faithful on the full subcategory $H_{(f)}/\simeq$ in $H/\simeq$.

The quotient functor p is a map between towers for which the diagram

(8.7)
$$\begin{tikzcd}
Z^{n+1}\Gamma_n^T \arrow[r, "+"] \arrow[d, "p"] & H_{n+2} \arrow[r] \arrow[d, "p"] & H_{n+1} \arrow[r, "0"] \arrow[d, "p"] & H^{n+2}\Gamma_n^T \arrow[d, equal, "1"] \\
H^{n+1}\Gamma_n^T \arrow[r, "+"] & H_{n+2}/\simeq \arrow[r] & H_{n+1}/\simeq \arrow[r, "0"] & H^{n+2}\Gamma_n^T
\end{tikzcd}$$

is a commutative diagram with exact rows, see (Ext.5.7).

We define the obstruction operator 0 and the action $Z^{n+1}\Gamma_n^T +$ in (8.7) as follows: Let X,Y be objects in H_{n+2} and let $(\xi,\eta): \lambda X \longrightarrow \lambda Y$ be a map in H_{n+1}. Then there is a map $F: X^n \longrightarrow Y^n$ as in (8.2) which is associated to (ξ_{n+1},η). This map yields the element

(8.8)
$$0(F) = -g_{n+1}\xi_{n+2} + (F_*)f_{n+1}$$
$$\in \mathrm{Hom}_{\varphi,\varphi}(\hat{b}_{n+2}, H_nY^n) \quad .$$

The homomorphism $0(F)$ factors actually over $\Gamma_n^TY \subset H_nY^n$ and is a cocycle in

$$\mathrm{Hom}_{\varphi,\varphi}(\hat{b}_{n+2}, \Gamma_n^TY) \quad .$$

Therefore $0(F)$ determines the cohomology class

(8.9)
$$0(\xi,\eta) = \{0(F)\} \in \hat{H}^{n+2}(\hat{b},\varphi^*\Gamma_n^TY) \quad ,$$

see (8.5). This is the obstruction operator in (8.7).

Now let $(\xi,\eta): X \longrightarrow Y$ be a map in $\mathcal{H}_{n+2}$ and let

$$\alpha \in Z^{n+1}\Gamma_n^T(\lambda(\xi,\eta)) \subset \mathrm{Hom}_{\varphi,\varphi}(\hat{b}_{n+1}, \Gamma_n^T Y) \quad .$$

With the inclusion $i: \Gamma_n^T Y \subset H_n Y^n$ we define the natural action $Z^{n+1}\Gamma_n^T +$ on $\mathcal{H}_{n+2}$ by

(8.10) $$(\xi,\eta) + \alpha = (\xi, \eta + \widetilde{i\alpha})$$

where $\eta + \widetilde{i\alpha}$ is defined as in (6.3).

(8.11) Proof of (8.6): We leave it to the reader to give a direct proof of (8.6 As in (A.7.12) the result is as well a direct consequence of (V.6.3) by (0.14). By (1.31) we see that $\Gamma_n +$ on $\mathcal{T}_n(\mathfrak{X})$ coincides with $Z^{n+1}\Gamma_n^T +$ on $\mathcal{H}_{n+1}$. Also the natural systems $H_0^n\Gamma$, $H_{-1}^{n+1}\Gamma$ on $\mathcal{T}_n(\mathfrak{X})/\simeq$ coincide with $H^{n+1}\Gamma_n^T$ and $H^{n+2}\Gamma_n^T$ on $\mathcal{H}_{n+1}/\simeq$. Now the theorem follows from (V.6.3) by using (7.13) and (7.9). $\square$

We describe explicitely some properties of the DA - tower. These properties are immediate consequences of the discussion in (Ext.§ 6) . The similarity with the results in (A.§ 7) is striking.

Clearly, the functor r_{n+1} is full and faithful on the category $\mathcal{D}FA^{n-1}/\sim$ and $\mathcal{D}FA^{n-1}/\simeq$ respectively, compare the definition of r_{n+1} . Let $\mathcal{H}_{n+1}^{n+1}$ be the full subcategory of $\mathcal{H}_{n+1}$ consisting of objects $(\hat{b}, f_n, X^{n-1})$ with $\hat{b}_i = 0$ for $i \geq n+2$. Then the quotient category $(\mathcal{D}FA^n/\simeq)/\Gamma$ in (6.9) can be described by the equivalence of categories

(8.12) $$(\mathcal{D}FA^n/\simeq)/\Gamma \xrightarrow{\sim} \mathcal{H}_{n+1}^{n+1}/\simeq$$

which is induced by r_{n+1} . This follows from (8.6) since $\Gamma +$ in (6.9) corresponds to $H^{n+1}\Gamma_n^T +$ in (8.7), see (6.8).

From (Ext.6.6) and (8.6) and (3.10) we derive:

(8.13) <u>Theorem</u>: *Let* $f\colon X \longrightarrow Y$ *be a map in* $\mathcal{D}FA$.

(A) *The restrictions* $f^n\colon X^n \longrightarrow Y^n$ *are equivalences in* $\mathcal{D}FA/\sim$ *for all* n *if and only if the induced map* $\rho f\colon \rho X \longrightarrow \rho Y$ *in* H *is an isomorphism.*

(B) *The map* f *is a homotopy equivalence in* $\mathcal{D}FA/\simeq$ *if and only if the induced map* $\rho f\colon \rho X \longrightarrow \rho Y$ *is a homotopy equivalence in* $H/\simeq$.

(C) *Assume that the underlying modules of* H_oA *and* H_oB *are free. Then the map* f *is a homotopy equivalence in* $\mathcal{D}FA$ *if and only if the induced map* $\hat{b}_*f\colon \hat{b}_*X \longrightarrow \hat{b}_*Y$ *is a homotopy equivalence in* $\mathcal{D}F\hat{M}/\simeq$.

This result is known if $H_oA = H_oB = 0$ is trivial. We don't know of a reference for this theorem in case H_oA is non trivial, see also (2.18). Part (B) of the result above shows that homotopy systems in § 3 have their own significance and have better properties then $\hat{b}$ or $\hat{B}$.

We consider the <u>homotopy classification problem</u> for maps in $\mathcal{D}FA/\simeq$. Let X and Y be free chain algebras; we want to compute the set [X,Y] of homotopy classes of maps from X to Y in $\mathcal{D}FA/\simeq$. The DA-tower yields the following method:

Let r_nX , r_nY be the restrictions in H_n and let

$$[X,Y]^n_\varphi = [r_nX, r_nY]_\varphi \tag{8.14}$$

be the set of morphisms $r_nX \longrightarrow r_nY$ in $H_n/\simeq$ which induce φ . For each algebra homomorphism φ the DA-tower yields the following diagram:

(8.15)

$$\begin{tikzcd}
 & {[X,Y]_{\varphi}} \arrow[d, "r"] & \\
 & \vdots & \\
\hat{H}^{n}(X,\varphi^{*}\Gamma^{T}_{n-1}Y)+ \arrow[r, "D"] & {[X,Y]^{n+1}_{\varphi}} \arrow[d, "\lambda"] & \\
 & {[X,Y]^{n}_{\varphi}} \arrow[r, "\mathit{0}"] \arrow[d, "\lambda"] & \hat{H}^{n+1}(X,\varphi^{*}\Gamma^{T}_{n-1}Y) \\
 & \vdots & \\
\hat{H}^{3}(X,\varphi^{*}\Gamma^{T}_{2}Y)+ \arrow[r, "D"] & {[X,Y]^{4}_{\varphi}} \arrow[d, "\lambda"] & \\
 & {[X,Y]^{3}_{\varphi}} \arrow[r, "\mathit{0}"] \arrow[d, "C"] & \hat{H}^{4}(X,\varphi^{*}\Gamma^{T}_{2}Y) \\
 & {[\hat{b}X,\hat{b}Y]_{\varphi}} &
\end{tikzcd}$$

Here C is a bijective map if H_0X and H_0Y are free R-modules. The arrow D+ denotes the action of the corresponding group on the homotopy set. The arrows $r,\lambda,\mathit{0},C$ denote maps between sets. Each sequence $D,\lambda,\mathit{0}$ is exact in the sense that

$$\begin{aligned}
&\text{Image } \lambda = \mathit{0}^{-1}(0) \quad , \\
&\text{Orbits of } D \text{ in } F = \lambda^{-1}(\lambda F) \quad .
\end{aligned} \tag{8.16}$$

If X has dimension $\leq n$ the tower (8.15) is finite with

$$[X,Y]^{n+1}_{\varphi} = [X,Y]_{\varphi} \quad .$$

The isotropy group of the action D+ can be computed by use of the spectral sequence in the appendix of § 1. Here we use the same method as in (A.7.29).

The tower of sets in (8.15) leads to the following <u>obstruction theory</u>:

(8.17) <u>Definition</u>: Let X and Y be free chain algebras and let $\varphi: H_0X \longrightarrow H_0Y$ be a homomorphism of algebras. We say that a φ-biequivariant chain map

$$d: \hat{b}X \longrightarrow \hat{b}Y$$

is <u>n-realizable</u> if the homotopy class $\{d\}$ lies in the image of

$$\hat{b}: [X,Y]^{n}_{\varphi} \longrightarrow [\hat{b}X,\hat{b}Y]_{\varphi} \quad , \quad n \geq 3 \ .$$

We define the <u>higher order obstruction</u> to be the subset

$$\mathcal{O}^{n+1}(d) = \mathcal{O}\hat{b}^{-1}\{d\} \subset \hat{H}^{n+1}(X,\varphi^{*}\Gamma^{T}_{n-1}Y) \quad . \ //$$

Clearly,

$$d \text{ is } n\text{-realizable} \Longleftrightarrow \mathcal{O}^{n+1}(d) \text{ is non empty,}$$

$$d \text{ is } (n+1)\text{-realizable} \Longleftrightarrow 0 \in \mathcal{O}^{n+1}(d) \quad .$$

If H_0X and H_0Y are free R-modules each d is 3-realizable.

(8.18) <u>Theorem</u>: *Let X and Y be free chain algebras for which H_0X and H_0Y are free R-modules. We suppose that X is finite dimensional. Then X and Y are homotopy equivalent in DFA / ≃ if and only if there exists a pair (φ, d) with the following properties:*

(a) $\varphi: H_0X \cong H_0Y$ is an isomorphism of algebras.

(b) $d: \hat{b}X \longrightarrow \hat{b}Y$ is a φ-biequivariant chain map which induces iso-

morphisms in homology.

(c) $O \in O^n(d)$ *for* $n \geq 4$.

It seems likely that the algebraic properties of the obstructions i (8.18) are very similar to the algebraic properties of the obstruction in the topological case, see (A.7.31). In fact, the analogy between th DA - tower and the CW - tower might lead to a deeper understanding of th homotopy classification problems in topology, compare chapter (D).

Next we consider the group $E(X)$ of homotopy equivalences $X \longrightarrow$ in $\mathit{DFA}/\simeq$. We denote by $E_n(X)$ the group of equivalences of r_nX i $\mathit{H}_n/\simeq$. Moreover, let $E(\hat{b}X)$ be the group of homotopy equivalences in $\mathit{DF\hat{M}}/\simeq$; $\mathrm{Aut}(H_oX)$ denotes the group of automorphisms of the algebra H_oX .

The DA - tower gives us the following tower of groups with $H_o = H_oX$

$$\begin{tikzcd}
 & E(X) \arrow[d, "r"] & \\
 & \vdots & \\
\hat{H}^n(X, \Gamma^T_{n-1}X) \arrow[r, "1^+"] & E_{n+1}(X) \arrow[d, "\lambda"] & \\
 & E_n(X) \arrow[r, "\mathcal{O}"] \arrow[d] & \bigcup\limits_{\varphi \in \mathrm{Aut}\, H_o} \hat{H}^{n+1}(X, \varphi^* \Gamma^T_{n-1}X) \\
 & \vdots & \\
\hat{H}^3(X, \Gamma^T_2 X) \arrow[r, "1^+"] & E_4(X) \arrow[d, "\lambda"] & \\
 & E_3(X) \arrow[r, "\mathcal{O}"] \arrow[d, "C"] & \bigcup\limits_{\varphi \in \mathrm{Aut}\, H_o} \hat{H}^4(X, \varphi^* \Gamma^T_2 X) \\
 & E(\hat{b}X) \arrow[d, "H_o"] & \\
 & \mathrm{Aut}\, H_o &
\end{tikzcd} \tag{8.19}$$

If H_oX is free as an R-module then C in (8.19) is an isomorphism of groups. The function $\mathcal{O}$ is the <u>obstruction operator</u> with

$$\mathcal{O}(u) \in \hat{H}^{n+1}(X, \varphi^* \Gamma^T_{n-1}X)$$

for $u \in E_n(X)$ and $\varphi \colon H_oX \longrightarrow H_oX$ induced by u.

The obstruction is a derivation, that is

$$O(u\ v) = u_{*}O(v) + v^{*}O(u) \quad .$$

Thus the kernel of O is a subgroup of E_nX . We define the homomorphis 1^+ in (8.19) by the action $D+$ and by the identity 1 in $E_{n+1}X$, tha is

$$1^+(\alpha) = 1 + \alpha \quad , \quad \alpha \in \hat{H}^n(X, \Gamma^T_{n-1}X) \quad .$$

The kernel of this homomorphism can be computed by the spectral sequence in the appendix of § 1 .

The arrows $r, 1^+, \lambda, C, H_o$ of diagram (8.19) are homomorphisms of group Moreover, the sequences $(1^+, \lambda, O)$ are exact, that is

$$\begin{aligned} \text{kernel } O &= \text{image } \lambda \quad , \\ \text{kernel } \lambda &= \text{image } 1^+ \quad . \end{aligned} \tag{8.20}$$

By exactnes we obtain the group extension

$$0 \longrightarrow \text{image } 1^+ \longrightarrow E_{n+1}X \xrightarrow{\lambda} \text{kernel } O \longrightarrow 0$$

The associated homomorphism

$$\text{kernel } O \xrightarrow{h} \text{Aut}(\text{image } 1^+)$$

is induced by the homomorphism

$$\begin{aligned} &E_nX \xrightarrow{h} \text{Aut}(\hat{H}^n(X, \Gamma^T_{n-1}X)) \\ &h(u)(\alpha) = u_{*}(u^{-1})^{*}(\alpha) = (u^{-1})_{*}u^{*}(\alpha) \quad , \end{aligned} \tag{8.21}$$

compare (Ext.§ 4).

All these results are immediate consequences of theorem (8.6), compare

(Ext.§ 6) . Moreover, by (Ext.6.24) and (Ext.6.25) we get

(8.22) **Theorem:** *Let* X *be a finite dimensional free chain algebra for which* H_oX *is free as an* R*-module. Then the kernel of*

$$\hat{b}_*: E(X) \longrightarrow E(\hat{b}_*X)$$

is a solvable group. If G *is a sugbroup of* $E(X)$ *such that* $\hat{b}_*(G)$ *is a nilpotent group and such that* G *acts nilpotently on* $\hat{H}^{n+1}(X,\Gamma_n^T X)$ *via (8.21) for all* $n \geq 2$ *then* G *is a nilpotent group.*

Compare the result in (A.7.45). We leave it to the reader to formulate the obvious results which correspond to (A.7.48) and (A.7.50).

§ 9 Finiteness obstructions and minimal models

We here show that the results in (A.§ 8) on CW-complexes are in a similar way available for free chain algebras. In particular the results of C.T.C. Wall on finiteness obstructions for CW-complexes have a direct analogue for chain algebras. As in (A.§ 8) the crucial result is the foll wing:

(9.1) <u>Theorem</u>: *The functor* $\lambda: H_{n+1}/\simeq \longrightarrow H_n/\simeq$ *of the DA-tower satisfies the strong sufficiency condition for* $n \geq 3$ *, compare (A.8.1).*

By (A.8.1) and by the DA-tower (8.6) we thus have:

(9.2) <u>Corollary</u>: *The functor* $\lambda^{n-3}: H_n/\simeq \longrightarrow H_3/\simeq$ *and the functor*

$$\rho: DFA^n/\simeq \longrightarrow H/\simeq$$

satisfy the strong sufficiency condition.

By a limit argument we even have:

(9.3) <u>Theorem</u>: *The functor*

$$\rho: DFA/\simeq \longrightarrow H/\simeq$$

satisfies the strong sufficiency condition.

For convenience of the reader we give the essential part of the proof of (9.1). It is a strict analogue of the proof of (A.8.4).

<u>Proof of (9.1)</u>: Let $G: \check{X} \longrightarrow \check{Y}$ be a homotopy equivalence in $H_n/\simeq$, $n \geq 3$, and assume we have an object Y in H_{n+1} with $\lambda Y = \check{Y}$. We have t construct an object X together with a map $F: X \longrightarrow Y$ in H_{n+1} such

that $\lambda X = \check{X}$ and $\lambda F = G$. Let $\check{X} = (b, \check{f}, X^{n-2})$ where

$$(1) \qquad \check{f}\colon b_n \longrightarrow H_{n-2}X^{n-2} \quad .$$

Here b_n is a free H_oX^{n-2} - bimodule. For a generating module sV_{n-1} of b_n we choose a map of R - modules:

$$(2) \qquad V_{n-1} \longrightarrow Z_{n-2}X^{n-2} \subset (X^{n-2})_{n-1}$$

which represents (1). Let X^{n-1} be the mapping cone of this map. Let $Y = (b', g, Y^{n-1})$ and let

$$(3) \qquad G = (\xi, \eta)\colon \check{X} \longrightarrow \lambda Y = \check{Y}$$

be the homotopy equivalence in the assumption of the theorem. We can choose a map

$$(4) \qquad G_1\colon X^{n-1} \longrightarrow Y^{n-1} \quad \text{in} \quad \mathcal{D}FA^{n-1}$$

which is associated to (ξ_n, η) in (3). We derive the following commutative diagram of unbroken arrows:

$$\begin{tikzcd}
 & b_n \arrow[d, phantom, "\cup" description] \arrow[ddd, bend right=70, "\xi_n"'] & b_{n+1} \arrow[l, "d"'] \arrow[dl, dashed, "d"] \arrow[d, dashed, "f_o"] \arrow[ddd, bend left=60, "\xi_{n+1}"] \\
\Gamma^T_{n-2}X^{n-1} \arrow[d, "G_{1*}", "\cong"'] & H_n\hat{b}X^{n-1} \arrow[l, "\partial"'] \arrow[d, "\hat{b}G_{1*}"] & H_{n-1}X^{n-1} \arrow[l, "j"'] \arrow[d, "G_{1*}"] \\
\Gamma^T_{n-2}Y^{n-1} & H_n\hat{b}Y^{n-1} \arrow[l, "\partial"'] \arrow[d, phantom, "\cap" description] & H_{n-1}Y^{n-1} \arrow[l, "j"'] \\
 & b'_n & b'_{n+1} \arrow[l, "d'"'] \arrow[u, "g"']
\end{tikzcd}$$

The rows of this diagram are exact, they are part of the exact sequenc of J.H.C. Whitehead. Moreover, since G in (3) is a homotopy equivalence the induced map

(6)
$$\begin{tikzcd}
\Gamma_k^T X^{n-1} \arrow[r, "G_{1*}"'] \arrow[d, equal] & \Gamma_k^T Y^{n-1} \arrow[d, equal] \\
\Gamma_k^T \check{X} \arrow[r, "\cong", "G_*"'] & \Gamma_k^T \check{Y}
\end{tikzcd}$$

is an isomorphism for $k<n$.

Diagram (5) and (6) show that there is a map f_o in diagram (5) with

(7) $$jf_o = d \ .$$

In fact, since b_{n+1} is a free H_oX-bimodule, we obtain f_o by showing $\partial d = 0$. This is the case since

$$G_{1*}\partial d = \partial j g \xi_{n+1} = 0$$

where $\partial j = 0$. We use (6). The map f_o in (7) gives us the object

(8) $$X_{(o)} = (b, f_o, X^{n-1})$$

with $\lambda X_{(o)} = \check{X}$. Therefore the obstruction

(9) $$\mathcal{O}_{X_{(o)},Y}(G) \in \hat{H}^{n+1}(\check{X}, \varphi^* \Gamma_{n-1}^T \check{Y})$$

is defined; $\varphi = G_{1*}: H_oX^{n-1} \longrightarrow H_oY^{n-1}$. The isomorphism of coefficient G_* in (6) gives us the element

(10) $$\{\alpha\} = G_*^{-1}\mathcal{O}_{X_{(o)},Y} \in \hat{H}^{n+1}(\check{X}, \Gamma_{n-1}^T \check{X}) \quad .$$

Let a be a map given by the composition

(11) $$a: b_{n+1} \xrightarrow{\alpha} \Gamma_{n-1}^T \check{X} = \Gamma_{n-1}^T X^{n-1} \xrightarrow[i]{} a_{n-1}X^{n-1} \quad .$$

Here α is a cocycle which represents (10). We define

$$(12) \qquad \begin{cases} f = f_o - a\colon b_{n+1} \longrightarrow H_{n-1}X^{n-1} , \\ X = (b, f, X^{n-1}) . \end{cases}$$

We have $\lambda X = X_{(o)} = \check{X}$ and

$$(13) \qquad \mathcal{O}_{X,Y}(G) = 0 .$$

Therefore G is realizable by a map $F\colon X \longrightarrow Y$ in $\mathcal{H}_{n+1}/\simeq$ with $\lambda F = G$ in $\mathcal{H}_n/\simeq$. Clearly, F is a homotopy equivalence since λ satisfies the sufficiency condition.

We have to check that (13) is satisfied. The cohomology class in (13) is represented by the cocycle, see (8.8),

$$G_{1*}f - g\xi_{n+1}\colon b_{n+1} \longrightarrow \Gamma^T_{n-1}\check{Y} \subset H_{n-1}Y^{n-1} .$$

By definition of f we get

$$\begin{aligned} \mathcal{O}_{X,Y}(G) &= \{G_{1*}(f_o - a) - g\xi_{n+1}\} \\ &= \{(G_{1*}f_o - g\xi_{n+1}) - G_{1*}a\} \\ &= \{G_{1*}f_o - g\xi_{n+1}\} - G_{1*}\{\alpha\} \\ &= \mathcal{O}_{X_{(o)},Y}(G) - G_{1*}\{\alpha\} \\ &= 0 \quad , \text{ by } (10). \ \square \end{aligned}$$

We proceed in the same way as in (A.§ 8) :

(9.4) **Definition**: An object $C = (H, C)$ in $\mathcal{DF}\hat{\mathcal{M}}$ is n-realizable if there is a free chain algebra X and an isomorphism in $\mathcal{DF}\hat{\mathcal{M}}$:

$$(H, C^n) \cong (H_o X, (\hat{b}X)^n) .$$

Here C^n and $(\hat{b}X)^n = \hat{b}X^{n-1}$ denote the n-skeleta of C and $\hat{b}X$ respectively. Let

$$\mathit{DF}\hat{\mathit{M}}_{(f)}(n) \subset \mathit{DF}\hat{\mathit{M}}(n) \subset \mathit{DF}\hat{\mathit{M}}$$

be the full subcategories of $\mathit{DF}\hat{\mathit{M}}$ consisting of n-realizable objects (H,C) for which H is free as an R-module respectively. Let $\mathit{DFA}_{(f)}$ the full subcategory of DFA consisting of free chain algebras X for which H_0X is free as an R-module. Clearly, by $\hat{b}$ in (1.8) we have th functor

$$\hat{b}\colon \mathit{DFA}_{(f)} \longrightarrow \mathit{DF}\hat{\mathit{M}}_{(f)}(n)$$

for arbitrary n . //

The functor C in (3.5) has the following property:

(9.5) Proposition: *The functor*

$$C\colon H_{(f)}/\simeq \longrightarrow \mathit{DF}\hat{\mathit{M}}_{(f)}(2)/\simeq$$

is an equivalence of categories. Moreover, this functor satisfies the strong sufficiency condition.

We derive from (9.3):

(9.6) Corollary: *The functor*

$$\hat{b}\colon \mathit{DFA}_{(f)}/\simeq \longrightarrow \mathit{DF}\hat{\mathit{M}}_{(f)}(2)/\simeq$$

satisfies the strong sufficiency condition.

There are many corollaries of this result:

(9.7) Definition: We say, a free chain algebra $Y = (T(V),d)$ is finite, counta or of dimension $\leq n$ if V is finitely generated, countably generated or dimension $\leq n$ respectively.//

(9.8) Corollary: *Let* X *be a chain algebra for which* H_0X *is free as an* R*-module and suppose* $f: C \longrightarrow \hat{B}X$ *is a homotopy equivalence in* $\mathrm{DF}\hat{M}$ *where* C *is* 2*-realizable. Suppose* C *satisfies any combination of the conditions*

(i) C_i *is a finitely generated* R*-module for* $i \leq m_1$,

(ii) C_i *is countably generated for* $i \leq m_2$,

(iii) $C_i = 0$ *for* $i > m_3$.

Then X *is weakly equivalent to a free chain algebra* Y *satisfying the corresponding conditions*

(i) Y^{m_1} *is finite,*

(ii) Y^{m_2} *is countable,*

(iii) $\dim Y \leq m_3$.

We can follows Wall's proof for deleting the 2-realizability condition in (9.6), compare the remark on (A.8.11). Moreover, we obtain in the same way as in (A.8.12) the following obstruction to finiteness:

(9.9) Corollary: *Let* $n \geq 3$ *and let* X *be a free chain algebra for which* $H = H_0X$ *is a free* R*-module. Suppose the* n*-skeleton* X^{n-1} *is finite and suppose*

(i) $H_i(\hat{b}X) = 0$ *for* $i > n$ *and*

(ii) $d\hat{b}_{n+1}X$ *is a direct summand of* $\hat{b}_nX$ *over* $H \otimes H^{op}$ *, see (1.2).*

Let B_n *be a complement of this summand in* $\hat{b}_nX$ *and let*

$$\sigma(X) = (-1)^n\{B_n\} \in \tilde{K}_0(H \otimes H^{op})$$

be the element in the reduced projective class group of π *which is given by the finitely generated* $H\otimes H^{op}$*-module* B_n *, which is projective since* B_n *is a direct summand of* $\hat{b}_n X$ *. Then* $\sigma(X)$ *is an o struction, depending only on the homotopy type of* X *, which vanishe if* X *is finite, and whose vanishing is sufficient for* X *to be homotopy equivalent to a finite free chain algebra of dimension* $\leq n$.

__Addendum__: *If* X *is homotopy equivalent to an* n*-dimensional free chain algebra the conditions (i) and (ii) are satisfied.*

The projective class group $\tilde{K}_0(\Lambda)$ of a ring Λ is defined in (A.8. The proof of (9.9) is the same as the proof of (A.8.12).

If a chain algebra X is __connected__, $H_0X=R$, we can derive from (9 __small models__ as follows, compare (A.8.16):

(9.10) __Corollary__: *Let* X *be a chain algebra with* $H_0X=R$ *. Suppose for the ba construction* BX *a presentation*

$$0 \longrightarrow R^{r_k} \longrightarrow R^{b_k} \longrightarrow H_kBX \longrightarrow 0$$

of the R*-module* H_kBX *is given (* $r_1=b_1=0$ *), (recall that* R *is principal ideal domain). Then there is a weak equivalence*

$$K = (T(V),d) \longrightarrow X$$

in $\mathcal{DA}$ *where* K *is a free chain algebra for which*

$$\dim_R(V_{k-1}) = b_k + r_{k-1} \ .$$

__Proof__: For $\hat{B}X=BX$ there is a chain complex $C=(R\oplus sV,d)$ and a homot equivalence $C\simeq\hat{B}X$ in $\mathcal{DF}\hat{\mathcal{M}}$. Since $H_0X=0$ and $V_0=0$ we know that is 2-realizable. Thus the result follows from (9.6). □

(9.11) <u>Definition</u>: We call a free chain algebra $K = (T(V),d)$ <u>minimal</u> if $V_0 = 0$ and if d_Q on QK is trivial, $d_Q = 0$, or equivalently if dv, $v \in V$, is decomposable, $dV \subset \tilde{K} \cdot \tilde{K}$. We call K a <u>minimal model</u> of the chain algebra X if there is a weak equivalence

$$K = (T(V),d) \xrightarrow{\sim} X$$

where K is minimal.

(9.12) <u>Corollary</u>: *Let X be a chain algebra with $H_0X = 0$. Then there exists a minimal model of X iff HBX is free on an R-module. Two minimal models of X are isomorphic as chain algebras in DFA.*

In case R is a field, this result on minimal models is also proven in [2].

<u>Proof of (9.12)</u>: We can take $r_k = 0$ and $b_k = \dim H_k BX$ in (9.10). Clearly, the existence of a minimal model K implies that

$$\tilde{H}BX = \tilde{H}BK = \tilde{H}bK = sV$$

is a free R-module. $\square$

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(D) The chains on a loop space

§ 1 The universal cover of a classifying space

Let Δ^* be the simplicial category. Objects in Δ^* are the sets $\Delta(n) = \{0,\dots,n\}$, $n \geq 0$, of integers and morphisms are the monotone functions $\alpha: \Delta(n) \longrightarrow \Delta(m)$, that is $\alpha(i) \leq \alpha(j)$ if $i \leq j$. The morphisms of Δ^* are generated by the injective maps $d_i: \Delta(n-1) \longrightarrow \Delta(n) - \{i\} \subset \Delta(n)$, $i \in \Delta(n)$, and by the surjective maps $s_i: \Delta(n) \longrightarrow \Delta(n-1)$ with $s_i(i) = s_i(i+1) = i$ for $i \in \Delta(n-1)$.

Associated with Δ^* is the covariant functor

$$\rho: \Delta^* \longrightarrow \mathit{Top} , \Delta(n) \longmapsto \Delta^n \tag{1.1}$$

which carries the set $\Delta(n)$ to the standard simplex Δ^n . Δ^n is the convex hull of the unit vectors $v_i = (0,\dots,1,\dots,0)$ in $\mathbb{R}^{n+1}$ and is equal to

$$\Delta^n = \{ \sum_{i=0}^{n} t_i v_i : \sum_i t_i = 1 , 0 \leq t_i \leq 1 \} .$$

The mapping $\alpha_*: \Delta^n \longrightarrow \Delta^m$ induced by ρ is given by

$$\alpha_*(\sum_i t_i v_i) = \sum_i t_i v_{\alpha(i)} .$$

A simplicial space X is a contravariant functor

$$X: \Delta^* \longrightarrow \mathit{Top} , X_n = X(\Delta(n)) . \tag{1.2}$$

We define the realization, $|X|$, of the simplicial space X by

$$|X| = (\bigcup_{n \geq 0} \Delta^n \times X_n) /\sim . \tag{1.3}$$

Here the equivalence relation '$\sim$' is generated by

$$(\alpha_* t, x) \sim (t, \alpha^* x)$$

for $\alpha: \Delta(n) \longrightarrow \Delta(m)$, $t \in \Delta^n$, $x \in X_m$.

Let H be a topological monoid, that is H has an associative multiplication $\mu: H \times H \longrightarrow H$ with unit $* \in H$. We can form the contravariant functor

$$\underline{B}H: \Delta^* \longrightarrow \mathcal{T}op \tag{1.4}$$

which we call the <u>geometric bar construction</u> for H . This functor maps the set $\Delta(n)$ to the n-fold product $H^n = H \times \ldots \times H$ and is given on generating morphisms by

$$d_i^*: H^n \longrightarrow H^{n-1}$$

$$d_i^* = \begin{cases} pr_1 & , \quad i = 0 \\ \mu_i & , \quad i = 1, \ldots, n-1 \\ pr_n & , \quad i = n \end{cases}$$

$$s_i^* = j_{i+1}: H^{n-1} \longrightarrow H^n \quad (i = 0, \ldots, n-1) \quad .$$

Here $\mu_i(x_1,\ldots,x_n) = (x_1,\ldots,x_i \cdot x_{i+1},\ldots,x_n)$ and pr_i is the projection omitting the i-th coordinate. j_i is the inclusion filling in $*$ as the i-the coordinate of the tuple

$$j_i(x_1,\ldots,x_n) = (x_1,\ldots,x_{i-1},*,x_i,\ldots,x_n) \quad .$$

A space X is <u>wellpointed</u> if the inclusion $* \subset X$ of the basepoint is a closed cofibration in $\mathcal{T}op$. The following variant of the Dold-Lashof result [ ] is due to Milgram [ ], see also I.§ 1 in [ ].

(1.5) Proposition: *Let H be a topological group or let H be a path connected topological monoid, and let H be well-pointed. Then the realization $|\underline{B}H|$ is a classifying space for H .*

Proof: We define a contravariant functor

$$\underline{E}H: \Delta^* \longrightarrow Top\ ,\ \Delta(n) \longmapsto H^{n+1}\ ,$$

by

$$d_i^{\,*} = \begin{cases} pr_1: H^{n+1} \longrightarrow H^n & ,\ i=0 \\ \mu_i\ : H^{n+1} \longrightarrow H^n & ,\ i=1,\dots,n \end{cases}$$

and by

$$s_i^{\,*} = j_{i+1}: H^n \longrightarrow H^{n+1} \quad ,\ i=0,\dots,n-1\ .$$

Then $q = pr_{n+1}: \underline{E}H(\Delta(n)) \longrightarrow \underline{B}H(\Delta(n))$ is a natural transformation of functors. One can check that the realization of q ,

$$q: |\underline{E}H| \longrightarrow |\underline{B}H|$$

$$q(t,x) = (t,qx)$$

is a quasi-fibration with fiber H , see [ ]. Since $|\underline{E}H|$ is contractible, the result follows. ▯

(1.6) Example: The subspace $|\underline{B}^nH|$ of $|\underline{B}H|$ is given by points $(t,y) \in \Delta^i \times H^i$, $i \leq n$. If $H = \mathbb{Z}_n, S^1, S^3$ the space $\underline{B}^nH$ is the projective space RP_n , CP_n , HP_n respectively and the restriction of q above is the Hopf fibration.

Let G be a topological group. We can define the universal cover $\hat{B}_G$ of a classifiying space B_G by identifications in the total space E_G . That is we identify in each fiber of

$$E_G \longrightarrow B_G$$

each path component to a point. This leads to the commutative diagram of fibrations

(1.7)
$$\begin{tikzcd}
\pi_o G \arrow[r] & \hat{B}_G \arrow[dr, "p"] & \\
 & & B_G \\
G \arrow[r] \arrow[uu, "\lambda"] & E_G \arrow[uu, twoheadrightarrow, "\lambda"] \arrow[ur, "q"'] &
\end{tikzcd}$$

where λ is the obvious map which maps a point in the fiber G to its path component in $\pi_o G$. From the construction in the proof of (1.5) we derive the simplicial space

(1.8)
$$\underline{\hat{B}G}\colon \Delta^* \longrightarrow \mathit{Top}$$

which carries $\Delta(n)$ to $G^n \times \pi$, $\pi = \pi_o G$, and which is defined on generating morphisms of Δ^* by

$$d_i^* = \begin{cases} pr_1\colon H^n \times \pi \longrightarrow H^{n-1} \times \pi , & i = 0 \\ \mu_i\ \colon H^n \times \pi \longrightarrow H^{n-1} \times \pi , & i = 1,\dots,n \end{cases}$$

with $\mu_n(x_1,\dots,x_n,\alpha) = (x_1,\dots,x_{n-1},(\lambda x_n)\cdot\alpha)$ and by

$$s_i^* = j_{i+1}\colon H^{i-1} \times \pi \longrightarrow H^n \times \pi$$

for $i = 0,\dots,n-1$.

(1.9) <u>Definition</u>: A topological group G is a <u>CW-group</u> if G is a CW-complex such that $* \in G$ is a zero cell and the multiplication μ: $G \times G \longrightarrow G$ is a cellular map. We use the CW-topology for $G \times G$. //

For a CW-group G we set

(1.10) $$B_G = |\underline{B}G| \ , \ \hat{B}_G = |\hat{\underline{B}}G| \ , \ E_G = |\underline{E}G| \ .$$

The projection q in the proof of (1.5) gives us

$$q: E_G \xrightarrow[\lambda]{} \hat{B}_G \xrightarrow[p]{} B_G$$

where λ is induced by $1 \times \lambda: G^n \times G \longrightarrow G^n \times \pi$.

Let $\dot{\Delta}^n = \Delta^n - \partial\Delta^n$ be the 'open' standard simplex.

(1.11) <u>Proposition:</u> *For a CW-group the spaces* $B_G, \hat{B}_G, E_G$ *are CW-complex* *the cells of which are*

(a) $$\dot{\Delta}^n \times e_1 \times \ldots \times e_n \subset B_G$$

(b) $$\dot{\Delta}^n \times e_1 \times \ldots \times e_n \times \alpha \subset \hat{B}_G$$

(c) $$\dot{\Delta}^n \times e_1 \times \ldots \times e_n \times a \subset E_G \quad .$$

Here e_i $(i = 1,\ldots,n)$ *are cells in* G *which are different from* $*$, α *is an element in* $\pi = \pi_o G$ *and* a *is any cell in* G . *If the cell* a *lies in the path component* α *of* G *we write* $\chi a = \alpha$. *In this case the map* λ *in (1.10) projects the cell (c) above to the cell (b). Moreover, the map* p *in (1.10) is the universal covering which projects the cell (b) above to the cell (a), compar* *(A .1.11).*

This is a consequence of (I.3.4) in [ ].

For the CW-group G we obtain the <u>chain algebra</u>

(1.12) $$A = C_*G$$

of cellular chains. Here C_*G is the cellular chain complex of the CW-complex G (over R). The multiplication of the chain algebra A is induced by the cellular map $\mu: G \times G \longrightarrow G$,

$$\mu_*: A \otimes A = C_*(G \times G) \longrightarrow C_*G = A \ .$$

In § 2 of C2 we defined the bar construction BA of A. Similarly as in (C2.2.12) we now define, $H = H_0 A$,

$$(BA) \otimes_\tau H \quad \text{and} \quad (BA) \otimes_\tau A \ . \tag{1.13}$$

Compare (C2.2.16). As modules these are tensor products $(BA) \otimes H$ and $(BA) \otimes A$ respectively. The differentials are given by the formulas

$$d([a_1|\dots|a_n] \otimes \alpha) =$$

$$(d_B[a_1|\dots|a_n]) \otimes \alpha + (-1)^n [a_1|\dots|a_{n-1}] \otimes (\lambda a_n \cdot \alpha)$$

$$d([a_1|\dots|a_n] \otimes a) =$$

$$d_\otimes([a_1|\dots|1_n] \otimes a) + (-1)^n [a_1|\dots|a_{n-i}] \otimes (a_n \cdot a) \ .$$

Here $\alpha \in H = H_0 A$ and $a \in A$ and $\lambda: A \longrightarrow H$ denotes the projection. We use the notation in (C2.2.7), $d_\otimes = d_B \otimes 1 + 1 \otimes d$ is the differential of the tensor product $BA \otimes A$ of chain complexes.

(1.14) **Theorem:** *For the CW-complexes in (1.10) we have canonical isomorphisms of chain complexes such that the diagram*

$$\begin{tikzcd}
C_*B_G \arrow[r, phantom, "\cong"] & BA \\
C_*\hat{B}_G \arrow[u, "p_*"'] \arrow[r, phantom, "\cong"] & BA\otimes_\tau H \arrow[u, "1\otimes\varepsilon"'] \\
C_*E_G \arrow[u, "\lambda_*"'] \arrow[r, phantom, "\overset{I}{\cong}"] & BA\otimes_\tau A \arrow[u, "1\otimes\lambda"']
\end{tikzcd}$$

commutes, $\varepsilon: H \longrightarrow R$ *is the augmentation. The diagram is natural for cellular homomorphisms* $f: G \longrightarrow G'$.

Since E_G is contractible we can derive that $BA\otimes_\tau A$ is contrac ible. This is a well known fact, see [ ].

<u>Proof of (1.14)</u>: We define the isomorphism I on generators of C_*E

(1) $$I(\dot{\Delta}^n \times e_1 \times \ldots \times e_n \times a) = [a_1|\ldots|a_n]\otimes a$$

where $a_i = e_i - \varepsilon(e_i) \in \tilde{A} = \text{kernel } \varepsilon$. Similarly, we define the isomor-phism on $C_*\hat{B}_G$ and C_*B_G . Compare the proof of theorem I.3.19 in [ ]. Clearly, I in (1) gives us an isomorphism of modules. We have to check $Id = dI$. We obtain the boundary in C_*E_G by

(2) $$d(\dot{\Delta}^n \times e_1 \times \ldots \times e_n \times a) =$$
$$(d\dot{\Delta}^n) \times e_1 \times \ldots \times e_n \times a + (-1)^n \dot{\Delta}^n \times d(e_1 \times .. \times e_n$$

Now it is well known that in $C_*\Delta^n$ we have

(3) $$d\dot{\Delta}^n = \sum_{i=0}^{n} (-1)^i d_{i*}(\dot{\Delta}^{n-1}) \quad .$$

From the identification in (1.3) we get

(4) $$(d_{i*}\dot{\Delta}^{n-1}) \times e_1 \times \dots \times e_n \times a = \dot{\Delta}^{n-1} \times d_i^*(e_1 \times \dots \times e_n \times a) \quad .$$

The definition of $\underline{EH}$ yields

(5) $$d_i^*(e_1 \times \dots \times e_n \times a) = \begin{cases} e_1 \times \dots \times (e_i \cdot e_{i+1}) \times \dots \times e_n \times a \ , & i=1,\dots,n- \\ e_2 \times \dots \times e_n \times a & , i=0 \\ e_1 \times \dots \times e_{n-1} \times (e_n \cdot a) & , i=n \ . \end{cases}$$

We conclude by (1) and by the formulas (2),...,(5) that on the cellular chain level we have the boundary with the following summands where we use the convention

(6) $$\bar{a}_i = (-1)^{|e_i|} a_i \quad , \quad a_i = e_i - \varepsilon(e_i) \quad ,$$

with $|e_i| = \text{dimension}(e_i)$.

(7) $$\text{Id}(\dot{\Delta}^n \times e_1 \times \dots \times e_n \times a) =$$

(8) $$+ (-1)^n \sum_{i=1}^{n} [\bar{a}_1 | \dots | de_j | \dots | \bar{a}_n] \otimes a$$

(9) $$+ (-1)^n [\bar{a}_1 | \dots | \bar{a}_n] \otimes da$$

(10) $$+ \sum_{i=1}^{n-1} (-1)^i [a_1 | \dots | e_i \cdot e_{i+1} - \varepsilon(e_i \cdot e_{i+1}) | \dots | a_n] \otimes a$$

(11) $$+ \varepsilon(e_1) \ [a_2 | \dots | a_n] \otimes a$$

(12) $$+ (-1)^n [a_1 | \dots | a_{n-1}] \otimes (e_n \cdot a) \quad .$$

Here (8) and (9) correspond to the second summand of (2) and (10) + (11) + (12) is the first summand of (2) by (3),(4),(5).

On the other hand we have by definition in (1) and (1.13)

(13) $$dI(\dot{\Delta}^n \times e_1 \times \ldots \times e_n \times a) =$$

(14) $$+ (d_B[a_1|\ldots|a_n])\otimes a$$

(15) $$+ (-1)^n[\bar{a}_1|\ldots|\bar{a}_n]\otimes da$$

(16) $$+ (-1)^n [a_1|\ldots|a_{n-1}]\otimes((e_n - \varepsilon e_n)\cdot a) \quad .$$

In (15) we have the factor $(-1)^n$ since the degree of $[a_1|\ldots|a_n]$ is $|e_1| + \ldots + |e_n| + n$, see (C2.2.7)(2). Clearly, (9) and (15) cancel out. Moreover, (12) and one summand of (16) cancel out. It remains to

(17) $$(14) = (8) + (10) + (11) + (-1)^n(\varepsilon e_n)[a_1|\ldots|a_{n-1}]\otimes a$$

This is a consequence of formula (C2.2.7)(3) since we insert there

(18) $$\begin{aligned} a_j\cdot a_{j+1} &= (e_j\cdot e_{j+1} - \varepsilon(e_j\cdot e_{j+1})) \\ &\quad - \varepsilon(e_j)a_{j+1} - \varepsilon(e_{j+1})a_j \quad . \end{aligned}$$ □

By (C2.2.12) we see that $\hat{B}A$ yields $BA\otimes_\tau H$ by the formula

(1.15) $$\begin{aligned} R\otimes_H \hat{B}A &= R\otimes_H H\otimes_\tau (BA\otimes_\tau H) \\ &= BA\otimes_\tau H \quad . \end{aligned}$$

Here we consider $\hat{B}A$ as a left H-module and we consider R as an H-module via the augmentation $\varepsilon: H \longrightarrow R$.

On the other hand we can derive from $BA\otimes_\tau H$ the chain complex $\hat{B}A$ in $\mathcal{DF}\Omega$:

For the chain algebra $A = C_*G$ we know that

$$H = H_0A = H_0G = R[\pi] \quad (\pi = \pi_0G)$$

is the group ring of π . We write the group $\pi = \pi_0 G$ multiplicatively. Let H^{op} be the opposite algebra of H and let

$$\xi: H \longrightarrow H^{op}$$

be the algebra homomorphism with $\xi[\alpha] = [\alpha^{-1}]$ for $\alpha \in \pi$. From ξ we derive the algebra homomorphism, see (C2.1.2),

(1.16) $$\bar{\xi}: H \longrightarrow H \otimes H^{op} \ , \ \bar{\xi}(x) = x \otimes \xi x \ .$$

Via $\bar{\xi}$ we consider $H \otimes H^{op}$ as being a left H-module which we denote by $\xi^*(H \otimes H^{op})$.

(1.17) **Theorem:** *For* $A = C_* G$ *there is an* H*-biequivariant isomorphism*

$$(BA \otimes_\tau H) \otimes_H \bar{\xi}^*(H \otimes H^{op}) \cong \hat{BA}$$

in $DF\hat{M}$. *Here* $\bar{\xi}^*(H \otimes H^{op})$ *is the left* H*-module given by the ring homomorphism (1.16). The isomorphism is natural with respect to a cellular homomorphism* $f: G \longrightarrow G'$. *Moreover, the isomorphism reduces to the identity of* $BA \otimes_\tau H$ *if we apply* $R \otimes_H$, *compare (1.15).*

Proof: For the cells e_i in $G - \{*\}$ the elements

(1) $$a_i = e_i - \varepsilon(e_i) \in \tilde{A}$$

form a basis of $\tilde{A} = \text{kernel}(\varepsilon: C_* G \longrightarrow R)$. We set

(2) $$\chi e_i = \alpha$$

if e_i is a cell in the path component $\alpha \in \pi_0(G)$. We define a map

(3) $$h: BA \otimes_\tau H \longrightarrow \hat{BA} = H \otimes_\tau BA \otimes_\tau H$$

on basis elements,

(4) $$x = [e_1 - \varepsilon e_1|\dots|e_n - \varepsilon e_n]\otimes\alpha \text{ , by}$$

(5) $$h(x) = \alpha^{-1}(\chi e_n)^{-1}\dots(\chi e_1)^{-1}\otimes[e_1 - \varepsilon e_1|\dots|e_n - \varepsilon e_n]$$

where e_i are cells in $G - \{*\}$ and where $\alpha\in\pi$. Thus h is well-defined as a map of R-modules. In fact, h is a map of right H-modu where H operates on $\hat{BA}$ via $\bar{\xi}$ in (1.16), compare (C2.1.2).

Below we check that h is a chain map, that is $\hat{d}h = hd$. Moreover, we show that the map

(6) $$\bar{h}\colon (BA\otimes_\tau H)\otimes_H\bar{\xi}^*(H\otimes H^{op}) \longrightarrow \hat{BA}$$

$$x\otimes(\alpha\otimes\beta) \longmapsto \beta\cdot(hx)\cdot\alpha$$

is an isomorphism in $\mathcal{DF}\hat{\mathcal{M}}$. Clearly, $\bar{h}$ is a chain map (if h is one since

$$\hat{d}\bar{h}(x\otimes(\alpha\otimes\beta)) = \hat{d}(\beta\cdot hx\cdot\alpha) = \beta\cdot\hat{d}hx\cdot\alpha$$

$$= \beta\cdot hdx\cdot\alpha = \bar{h}(dx\otimes(\alpha\otimes\beta)) \text{ .}$$

We define an inverse $\bar{\bar{h}}$ of $\bar{h}$ by

$$\bar{\bar{h}}(\beta\otimes[a_1|\dots|a_n]\otimes\alpha) = ([a_1|\dots|a_n]\otimes\alpha)\otimes_H(1\otimes\bar{\beta})$$

where $\bar{\beta} = \beta(\chi e_1)\dots(\chi e_n)\alpha$. Clearly, $\bar{h}\bar{\bar{h}}$ is the identity. Moreover, $\bar{\bar{h}}\bar{h}$ is the identity since we have for

$$x = [a_1|\dots|a_n]\otimes\eta\in BA\otimes_\tau H$$

the equations, $\chi = (\chi e_1)\dots(\chi e_n)$,

$$\bar{\bar{h}}\bar{h}(x\otimes_H(\alpha\otimes\beta)) = \bar{\bar{h}}(\alpha\cdot hx\cdot\beta).$$

$$= \bar{\bar{h}}(\beta\cdot\eta^{-1}\cdot\chi^{-1}\otimes[a_1|\ldots|a_n]\otimes\eta\alpha)$$

$$= ([a_1|\ldots|a_n]\otimes\eta\alpha)\otimes_H(1\otimes\beta\alpha)$$

$$= ([a_1|\ldots|a_n]\otimes\eta)\otimes_H(\bar{\xi}\alpha)\cdot(1\otimes\beta\alpha)$$

$$= x\otimes_H(\alpha\otimes\beta)$$

since in $H\otimes H^{op}$ we have

$$(\bar{\xi}\alpha)\circ(1\otimes\beta\alpha) = (\alpha\otimes\alpha^{-1})(1\otimes\beta\alpha)$$

$$= \alpha\otimes\beta\alpha\alpha^{-1} = \alpha\otimes\beta \quad .$$

For the proof of (1.17) it remains to check that h in (3) is in fact a chain map. To see this we use the notation in the proof of (1.14). For the basis element x in (4) we have with

$$\chi = (\chi e_1)\cdot\ \ldots\ \cdot(\chi e_n)\in\pi\subset R[\pi]$$

the equation:

(7) $$\hat{d}hx = \alpha^{-1}\chi^{-1}\cdot\hat{d}[a_1|\ldots|a_n]\cdot\alpha =$$

(8) $$+\ \alpha^{-1}\chi^{-1}\otimes d_B[a_1|\ldots|a_n]\otimes\alpha$$

(9) $$+\ \alpha^{-1}\chi^{-1}\lambda(a_1)\otimes[a_2|\ldots|a_n]\otimes\alpha$$

(10) $$+\ (-1)^n\ \alpha^{-1}\chi^{-1}\otimes[a_1|\ldots|a_{n-1}]\otimes\lambda(a_n)\cdot\alpha \quad .$$

Compare (C2.2.12). On the other hand we have by definition in (1.13)

(11) $$dx = d_B[a_1|\ldots|a_n]\otimes\alpha\ +$$

(12) $$+ (-1)^n [a_1|\dots|\varepsilon_{n-1}]\otimes\lambda(a_n)\cdot\alpha \quad .$$

For the computation of hdx we have to write the terms in (11) and (12) as a linear combination of basis elements in $BA\otimes H$. For $|e_n| > 0$ we have $\lambda(a_n)=0$ and (12) vanishes. For $|e_n|=0$ we have

(13) $$\lambda(a_n) = \chi e_n - \varepsilon e_n \in R[\pi] = H \quad .$$

Therefore we get for $|e_n|=0$ with $\chi_i = \chi(e_i)$

(14) $$h(12) = (-1)^n (\chi_n\alpha)^{-1}\chi_{n-1}^{-1} \dots \chi_1^{-1}\otimes[a_1|\dots|a_{n-1}]\otimes$$

(15) $$- (-1)^n (\varepsilon e_n)\alpha^{-1}\chi_{n-1}^{-1} \dots \chi_1^{-1}\otimes[a_1|\dots|a_{n-1}]\otimes\alpha$$

If we insert (13) in (10) we see that one summand of (10) and (14) cancel out. Moreover, we have to write (11) as a linear combination of basis elements in $BA\otimes H$. For this we only need the equation

$$d_B[a_1|\dots|a_n] = A+B+C+D$$

where

$$A = (-1)^n \sum_{j=1}^{n} [\bar{a}_1|\dots|de_j|\dots|a_n]$$

$$B = \sum_{j=1}^{n-1} (-1)^j [a_1|\dots|e_je_{j+1} - \varepsilon(e_je_{j+1})|\dots|\varepsilon_n]$$

$$C = \varepsilon(e_1)[a_2|\dots|a_n]$$

$$D = (-1)^n \varepsilon(e_n)[a_1|\dots|a_{n-1}] \quad .$$

Compare the proof of (1.14). For the boundary de_j in A we know th $de_j = \Sigma n_i e^i$ is a linear combination of cells e^i in the path comp of e_j . Since $\varepsilon de_j = 0$ we have

$$\begin{aligned} de_j &= \sum_i n_i(e^i - \varepsilon e^i) \\ \chi e^i &= \chi e_j \end{aligned} \tag{16}$$

Also $e_j e_{j+1} = \Sigma\, m_i e^{(i)}$ is a linear combination of cells $e^{(i)}$ in the path component of $e_j e_{j+1}$. Therefore

$$\begin{aligned} e_j e_{j+1} - \varepsilon(e_j e_{j+1}) &= \Sigma\, m_i(e^{(i)} - \varepsilon e^{(i)}) \\ \chi e^{(i)} &= \chi(e_j e_{j+1}) = (\chi e_j)(\chi e_{j+1}) \quad . \end{aligned} \tag{17}$$

Now (16) and (17) show that the summand $\alpha^{-1}\chi^{-1}\otimes(A+B)\otimes\alpha$ of (8) and the summand $h((A+B)\otimes\alpha)$ of hdx in h(11) cancel out. It remains to show that the following terms

$$\begin{aligned} &\alpha^{-1}\chi^{-1}\otimes(C+D)\otimes\alpha + (9) + \\ &-\ (-1)^n\alpha^{-1}\chi^{-1}\otimes[a_1|\ldots|a_{n-1}]\otimes(\varepsilon e_n)\cdot\alpha \quad \text{, and} \end{aligned} \tag{18}$$

$$h((C+D)\otimes\alpha) + (15) \tag{19}$$

are equal. In (18) the last summand and $\alpha^{-1}\chi^{-1}\otimes(D)\otimes\alpha$ cancel out. Also in (19) the summands $h(D\otimes\alpha)$ and (15) cancel out. Now we get for $|e_1| = 0$ the equations:

$$\begin{aligned} (18) &= \alpha^{-1}\chi^{-1}(\lambda(a_1) + \varepsilon(e_1))\otimes[a_2|\ldots|a_n]\otimes\alpha \\ &= \alpha^{-1}\chi_n^{-1}\ldots\chi_1^{-1}\chi_1\otimes[a_2|\ldots|a_n]\otimes\alpha \\ &= \alpha^{-1}\chi_n^{-1}\ldots\chi_2^{-1}\otimes[a_2|\ldots|a_n]\otimes\alpha \\ &= h(C\otimes\alpha) = (19) \end{aligned}$$

since $\varepsilon(e_1) = 1$ for $|e_1| = 0$. This proves the theorem. □

From (1.14) and (1.17) we derive the following result on the generalized Hochschild (co)homology of a chain algebra, see (C2.1.26) and (C2.2.15). R denotes a principal ideal domain, see (C1.§ 0).

(1.18) <u>Theorem</u>: *Let* G *be a topological group with classifying space* BG *and* $\pi = \pi_0 G = \pi_1 BG$. *Let* $C_* G$ *be the chain algebra of* G *over* R, *see (2.2), and let* Γ *be a* $R[\pi]$*-bimodule. Then there are the natural isomorphisms*

$$\hat{H}_*(C_*G,\Gamma) = \hat{H}_*(BG, {}_\xi\Gamma)$$

$$\hat{H}^*(C_*G,\Gamma) = \hat{H}^*(BG,\Gamma_\xi) \quad .$$

Here ${}_\xi\Gamma$ *and* Γ_ξ *are the left and right* $R[\pi]$*-modules* Γ *with* $\alpha\cdot x = \alpha x\alpha^{-1}$ *and* $x\cdot\alpha = \alpha^{-1}x\alpha$ *for* $\alpha\in\pi$, $x\in\Gamma$ *respectively.*

Thus, indeed, (co)homology with local coefficients of a space corre ponds to Hochschild (co)homology. For a discrete group G the theorem above is exactly the result of Eilenberg - Mac Lane in chapter X, theo 5.5 in [ ]. In this case $C_*G = R[\pi]$ is the group ring of $\pi = G$.

(1.19) <u>Remark</u>: Let G be a topological group with classifying space BG an let $\pi = \pi_0 G = \pi_1 BG$. For left and right $R[\pi]$-modules M and N respectively we obtain the natural equations

$$\begin{aligned}\mathrm{Tor}_{C_*G}(\lambda^* N, \lambda^* M) &= \hat{H}_*(C_*G, M\otimes N)\\ &= \hat{H}_*(BG, {}_\xi(M\otimes N)) \quad .\end{aligned}$$

Here ${}_\xi(M\otimes N)$ is the left $\mathbb{Z}[\pi]$-module $M\otimes N$ defined by $\alpha(m\otimes n) = (\alpha m)\otimes(n\alpha^{-1})$. Moreover, $\lambda\colon C_*G \longrightarrow H_0G = \mathbb{Z}[\pi]$ is the canonical p jection. The first equation is a special case of (**) in (C2.2.16) and the second equation follows from (1.18) above. The equivalence (1) $\Leftrightarrow$ (4) in (C2.2.18) now follows for the special case $A = C_*G$, $X = C_*G'$ and $f\colon G \longrightarrow G'$, a homomorphism between topological groups. //

§ 2 The chain algebra of a loop space

We define the homology groups by use of the normalized cubical chain complex CS_* , see [ ]. To fix notation we give an explicit description:

Let $I^n = I \times \ldots \times I$ be the n-dimensional cube and $I^o = *$. A map $\sigma: I^n \longrightarrow X$ is called a singular n-cube in the space X . σ is degenerate if it factors over one of the projections $p_i: I^n \longrightarrow I^{n-1}$ which omites the i-th coordinate. Let (QX, ∂) be the chain complex which is the free R-module generated by all singular cubes in X . The boundary ∂ is defined by

$$\partial(\sigma) = \sum_{i=1}^{n} (-1)^i (\sigma \circ \partial_i^o - \sigma \circ \partial_i^1)$$

where $\partial_i^\varepsilon: I^{n-1} \subset I^n$ denotes a face of I^n which is defined by settin the i-th coordinate equal to $\varepsilon \in \{0,1\}$. The group DX generated by degenerate cubes is a subchain complex. We set

(2.1) $$CS_*X = QX / DX .$$

CS_*X is <u>augmented</u> by $\varepsilon: CS_*X \longrightarrow CS_*(\text{point}) = R$.

The (singular) homology of a pair (X,A) of spaces is defined by

$$H_n(X,A) = H_n(CS_*X / CS_*A) .$$

Let M be a topological monoid with multiplication $\mu: M \times M \longrightarrow M$ and unit $* \in M$. Then $CS_*(M)$ is the (singular) <u>chain algebra</u> of M the multiplication of which is

(2.2) $$CS_*M \otimes CS_*M \xrightarrow[\times]{} CS_*(M \times M) \xrightarrow[\mu_*]{} CS_*M \quad .$$

The crossproduct, $\sigma \times \tau$, for singular cubes $\sigma: I^n \longrightarrow M$ and $\tau: I^m \longrightarrow M$ is the singular cube

$$\sigma \times \tau: I^n \times I^m = I^{n+m} \longrightarrow M \times M \ .$$

The unit 1 of $CS_*(M)$ is the 0-cube $I^0 = * \longrightarrow * \in M$. The graded algebra $H(CS_*M) = H_*(M)$ is the <u>Pontryagin ring</u> of M . The multiplication in $H_*(M)$ is given by

$$H_*M \otimes H_*M \xrightarrow[\times]{} H_*(M \times M) \xrightarrow[\mu_*]{} H_*M \quad .$$

(2.3) <u>Remark</u>: If M is a CW-monoid, for example if M is a CW-group as in (1.9), then the <u>cellular</u> chain algebra C_*M is canonically equivalent to the <u>singular</u> chain algebra CS_*M in the category $Ho(\mathcal{D}A)$. From the context it will always be clear whether C_*M denotes the singular or the cellular chain algebra of M respectively. We obtain the equivalence

$$C_*M \sim CS_*M \quad \text{in} \quad Ho(\mathcal{D}A)$$

as follows. Let $\underline{M}$ be the cubical set of all singular cubes $I^n \longrightarrow M$ which are cellular maps. The realization $|\underline{M}|$ is a CW-monoid and we have a canonical map $|\underline{M}| \longrightarrow M$ which is cellular. Therefore we have equivalences

$$C_*M \xleftarrow{\sim} C_*|\underline{M}| \cong CS_*M$$

of chain algebras. //

For a space X with $x_0, x_1 \in X$ let $P(X, x_0, x_1)$ be the <u>Moore path</u>

space. An element $\omega \in P(X,x_o,x_1)$ is a pair $\omega = (f,r)$ where r is the length of the path $f\colon [0,r] \longrightarrow X$, $[0,r] = \{t \in \mathbb{R} : 0 \leq t \leq r\}$, with $f(0) = x_o$, $f(r) = x_1$. The topology is taken from $X^{\mathbb{R}} \times \mathbb{R}$, ($f(t) = x_1$ for $t \geq r$). The addition of paths

$$+\colon P(X,x_o,x_1) \times P(X,x_1,x_2) \longrightarrow P(X,x_o,x_2)$$

with

$$\begin{cases} (f,r) + (g,s) = (f+g,r+s) \ , \\ (f+g)(t) = f(t) \qquad \text{for} \quad 0 \leq t \leq r \ , \\ (f+g)(r+t) = g(t) \quad \text{for} \quad 0 \leq t \leq s \end{cases}$$

is associative. Therefore the *loop space*

$$X = P(X,*,*) \quad (* \in X) \tag{2.4}$$

is a topological monoid. Clearly, a base point preserving map $F\colon X \longrightarrow Y$ yields a map $\Omega F\colon \Omega X \longrightarrow \Omega Y$ between topological monoids, $(\Omega F)(f,r) = (Ff,r)$, which induces the map

$$(\Omega F)_*\colon C_*\Omega X \longrightarrow C_*\Omega Y$$

between chain algebras. This gives us the functor

$$C_*\Omega\colon \mathit{Top}_o \longrightarrow \mathit{DA} \tag{2.5}$$

which we already considered in § 0 of chapter C2. We obtain the following diagram of functors between homotopy categories:

(2.6)
$$\begin{tikzcd}
\mathrm{Ho}(Top_o) \arrow[r, "C_*\Omega"'] & \mathrm{Ho}(\mathcal{D}A) \\
CW_{/\simeq} \arrow[u, "\sim"] \arrow[r, dashed, "\alpha"'] \arrow[d, "\hat{C}_*"'] & \mathcal{D}FA_{/\simeq} \arrow[u, "\sim"'] \arrow[d, "\hat{b}"] \\
\mathrm{Chain}^{\wedge}_{R/\simeq} \arrow[r, "\hat{\alpha}"'] & \mathcal{D}F\hat{M}_{/\simeq}
\end{tikzcd}$$

Here the category $\mathrm{Chain}^{\wedge}_{R}$ is defined in the same way as $\mathrm{Chain}^{\wedge} = \mathrm{Chain}^{\wedge}_{\mathbb{Z}}$ in (A1.1.8) where we replace the group ring $\mathbb{Z}[\pi]$ by the group algebra $R[\pi]$. The functor $\hat{C}_*$ of cellular chains in the universal covering is defined in (A .§1). The functor

$$\hat{\alpha}\colon \mathrm{Chain}^{\wedge}_{R} \longrightarrow \mathcal{D}F\hat{M}$$

carries the object $(\pi,\hat{C})$ in $\mathrm{Chain}^{\wedge}_{R}$ to the object (H,C) in $\mathcal{D}F\hat{M}$ with

$$\begin{cases} H = R[\pi] \\ C = \hat{C}\otimes_H \bar{\xi}^*(H\otimes H^{op}) \end{cases} ,$$

compare (1.16) and (C2.1.2). Moreover, the functor $\hat{\alpha}$ carries the morphism (φ,f) to the φ_*-biequivariant map $f\otimes\varphi_*\otimes\varphi_*$.

We call any functor α for which the upper square in (2.6) commutes a <u>model functor</u> for $C_*\Omega$. We could take for α for example the functor $\Omega BC_*\Omega$, compare (C2.0.8).

(2.7) <u>Remark</u>: Let π be a group which we write additively and let $\hat{C}$ be a right H-module with $H = R[\pi]$. Then $H \otimes \hat{C}$ is an H-bimodule by setting

$$\alpha \cdot ([\beta] \otimes x) = [\alpha + \beta] \otimes x \quad ,$$

$$([\alpha] \otimes x) \cdot \beta = [\alpha + \beta] \otimes x^{\beta}$$

for $x \in \hat{C}$, $\alpha, \beta \in \pi$. Here x^{β} denotes the action of $\beta \in \pi$ on $x \in \hat{C}$ from the right. For the H-bimodule $H \otimes \hat{C}$ we have the canonical biequivariant isomorphism

$$H \otimes \hat{C} \cong \hat{C} \otimes_H \xi^*(H \otimes H^{op})$$

which maps $[\alpha] \otimes x$ to $x \otimes (1 \otimes [\alpha]^*)$, compare the notation in (C2.1.2).//

Clearly, we have

$$\varepsilon^* R \otimes_H (\hat{\alpha}\hat{C}) = \varepsilon^* R \otimes_H (H \otimes \hat{C}) = \hat{C} \quad .$$

This shows that $\hat{\alpha}$ is a faithful functor on $\mathit{Chain}_{\hat{R}}$. In fact, also the functor $\hat{\alpha}$ in (2.6) between homotopy categories is faithful since a homotopy $\Lambda: \hat{\alpha}f \simeq \hat{\alpha}g$ in $\mathcal{DF}\hat{M}$ gives us the homotopy $\varepsilon^* R \otimes_H \Lambda : f \simeq g$ in $\mathit{Chain}_{\hat{R}}$.

(2.8) <u>Theorem</u>: *The diagram of functors (2.6) commutes, that is, there is a canonical natural equivalence of functors* $\hat{b}\alpha \sim \hat{\alpha}\hat{C}_*$ *in the homotopy category* $\mathcal{DF}\hat{M}_{/\simeq}$.

<u>Proof</u>: This is a consequence of theorem (1.17). Let GX be the realization of the loop group $G\,\mathrm{Sing}\,X$ of Kan [ ]. Here $\mathrm{Sing}\,X$ is the reduced simplicial set of singular simplices $\sigma: \Delta^n \longrightarrow X$ with $\sigma(\Delta^{n,0}) = *$. Then GX is a CW-group and a map $f: X \longrightarrow Y$ in Top_o induces a cellular homomorphism $GX \longrightarrow GY$. Moreover, the functors $C_*\Omega$ and C_*G are canonically equivalent in $\mathrm{Ho}(\mathcal{DA})$. The

advantage of GX is, that GX is a topological group. We thus can apply (1.17) and obtain the natural equivalences in $\mathcal{DF}\hat{\mathcal{M}}_{/\simeq}$

$$
\begin{aligned}
\hat{b}\alpha(X) &= \hat{B}C_*X \qquad (\alpha = \Omega BC_*\Omega) \\
&\sim \hat{B}C_*GX \\
&\cong \hat{\alpha}\hat{C}_*(BGX) \qquad \text{, see (1.17), (1.14)} \\
&\sim \tilde{\hat{\alpha}}\tilde{\hat{C}}_*(X)
\end{aligned}
$$

In the last step we use the natural equivalence $BGX \simeq X$ in Ho(Top

§ 3 Cellular models of chain algebras

Let $FM(Z_1)$ be the free monoid generated by the set Z_1. We have the inclusion

(3.1) $$FM(Z_1) \subset \langle Z_1 \rangle$$

where $\langle Z_1 \rangle$ denotes the free group generated by Z_1. We say a presentation

$$Z_2 \xrightarrow{f} \langle Z_1 \rangle \xrightarrow{p} \pi = \langle Z_1 \rangle / NfZ_2$$

of the group π, see (A.1.40), is admissible if

(3.2) $$\begin{cases} fZ_2 \subset FM(Z_1) & \text{and} \\ pFM(Z_1) = \pi & . \end{cases}$$

If f is any presentation of π we obtain easily a presentation $\bar{f}$ of π which is admissible. Let $\bar{Z}_1$ be a further copy of the set Z_1. We define

(3.3) $$Z_2 \cup Z_1 \xrightarrow{\bar{f}} \langle Z_1 \cup \bar{Z}_1 \rangle$$

as follows: Let

$$\bar{f}(x) = x - \bar{x} \quad \text{if} \quad x \in Z_1 \quad ,$$

Here $\bar{x} \in Z_1$ is the element which corresponds to $x \in Z_1$. Moreover, assume $f(y) = y_1 + \ldots + y_n$ for $y \in Z_2$ is the reduced word in $\langle Z_1 \rangle$ with $y_i \in Z_1$ or $y_i \in -Z_1$. Then we set

$$\bar{f}(y) = x_1 + \ldots + x_n$$

where $x_i = y_i$ if $y_i \in Z_1$ and where $x_i = \overline{-y_i}$ if $y_i \in -Z_1$. The map

$$q: \langle Z_1 \cup \bar{Z}_1 \rangle \longrightarrow \langle Z_1 \rangle \quad ,$$

with $qx = x$, $q\bar{x} = -x$ for $x \in Z_1$, $\bar{x} \in \bar{Z}_1$ has the property

$$qFM(Z_1 \cup \bar{Z}_1) = \langle Z_1 \rangle \quad .$$

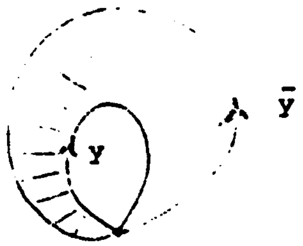
This shows that $\bar{f}$ is in fact an admissible presentation of $\pi = \langle Z_1 \rangle / NfZ_2$. //

Now let X be a strictly pointed CW-complex and let Z_n be the set of n-cells in X. The attaching map f of 2-cells in X gives us a presentation of the fundamental group $\pi = \pi_1 X$, see (A.1.40). We say X is admissible if this presentation π is admissible. For any complex X in *CW* it is easy to obtain an admissible complex $\bar{X}$ together with a homotopy equivalence

$$q: \bar{X} \xrightarrow{\simeq} X \quad . \tag{3.4}$$

Proof: We replace each 1-sphere S^1 in the 1-skeleton of X by the 2-dimensional complex

$\bar{S}_1 =$



with one 2-cell and two 1-cells, y and $\bar{y}$. The attaching map of t

2-cell is $y + \bar{y}$. There is an obvious homotopy equivalence

$$q\colon \bar{S}_1 \xrightarrow{\simeq} S_1$$

which is the identity, id, on y and which is $-\mathrm{id}$ on $\bar{y}$. From the attaching map of 2-cells in X, f, we derive the map

$$\bar{f}\colon \bigvee_{Z_2} S^1 \longrightarrow \bigvee_{Z_1} \bar{S}^1$$

which is defined for $y \in Z_2$ in the same way as $\bar{f}(y)$ in (3.3). Clearly, $q\bar{f} = f$, and therefore the mapping cone $\bar{X}^2 = C_{\bar{f}}$ is homotopy equivalent to the 2-skeleton $X^2 = C_f$. The attaching map of 2-cells in the complex $\bar{X}^2$ is $\bar{f}$ in (3.3). Now we define $\bar{X}$ by the push out diagram

$$\begin{tikzcd}
X \arrow[r, "\simeq"] & \bar{X} \\
\cup & \cup \\
X^2 \arrow[r, "\simeq", "q'"'] & \bar{X}^2
\end{tikzcd}$$

where q' is a homotopy inverse of q. Thus $\bar{X}$ is an admissible complex which satisfies (3.4). $\square$

For the cellular chains of $\bar{X}$ we have

$$C_n\bar{X} = \begin{cases} C_1X \oplus C_1X & , \quad n = 1 \\ C_2X \oplus sC_1X & , \quad n = 2 \\ C_nX & , \quad n \geq 3 \end{cases} \qquad (3.5)$$

The construction of $\bar{X}$ shows that the assumption of admissibility on a complex X is not restrictive if we want to deal with the homotopy classification problems. The great advantage of admissible complexes is the following property:

(3.6) **Proposition:** *Let* X *be an admissible complex. Then* $\hat{\alpha}\hat{C}_*X$ *is* 2-*realizable, see (C2.9.4).*

Proof: We consider $\hat{C}=\hat{C}_*X$ in degree ≤ 2 . With the convention in (A .1.16) we have

(1)
$$\begin{tikzcd}
\hat{C}_2 \arrow[r, "\hat{d}_2"] \arrow[d, equal] & \hat{C}_1 \arrow[r, "\hat{d}_1"] \arrow[d, equal] & \hat{C}_o \arrow[d, equal] \\
C_2\otimes R[\pi] & C_1\otimes R[\pi] & R[\pi]
\end{tikzcd} .$$

Here C_n is the free abelian group generated by the set Z_n of n-ce in X . For $y\in Z_1$ we have by (A .1.17):

(2) $$\hat{d}_1(y) = 1-[y] \quad .$$

Here $[y]$ denotes the element in $\pi\subset R[\pi]$ which is represented by the 1-cell y . We write $\pi=\pi_1X$ additively and the inclusion $\pi \subset R[\pi] = H$ is denoted by $\alpha \longmapsto [\alpha]$.

Now let $e\in Z_2$ be a 2-cell with attaching map

$$fe = y_1+\ldots+y_n \in FM(Z_1) \quad .$$

In (A.1.52) we have proven

(3) $$\hat{d}_2(e) = \sum_{i=1}^{n} y_i\otimes[y_{i+1}+\ldots+y_n] \quad .$$

By definition in (2.6) we obtain $\hat{\alpha}C_*X$ by the top row of the followin diagram:

(4)
$$\begin{tikzcd}
\hat{C}_2\otimes_H(H\otimes H^{op}) \arrow[r, "\hat{d}_2\otimes 1"] \arrow[d, "\tau"'] & \hat{C}_1\otimes_H(H\otimes H^{op}) \arrow[r, "\hat{d}_1\otimes 1"] \arrow[d, "\tau"'] & H\otimes_H(H\otimes H^{op}) \arrow[d, "\tau"'] \\
H\otimes C_2\otimes H \arrow[r, "\hat{\partial}_2"] & H\otimes C_1\otimes H \arrow[r, "\hat{\partial}_1"] & H\otimes H \ .
\end{tikzcd}$$

The isomorphisms τ are defined in the same way as in (2.7). That is

(5)
$$\tau((x\otimes[\alpha])\otimes_H([\beta]\otimes[\gamma]^*)) = [\gamma-\alpha]\otimes x\otimes[\alpha+\beta]$$

for $x\otimes[\alpha]\in\hat{C}_i = C_i\otimes H$, $\alpha,\beta,\gamma\in\pi$. Since for $e\in Z_2$ the word $f(e) = y_1+\ldots+y_n$ is a relation for π we obtain

(6)
$$[y_1+\ldots+y_i] = [-y_n-\ldots-y_{i+1}] \quad (1\leq i\leq n) \ .$$

For the H - biequivariant maps $\hat{\partial}_1$ and $\hat{\partial}_2$ in (4) (which make (4) commutative) we derive from (2),(3),(5) and (6) the equations

(7)
$$\hat{\partial}_1(x) = 1\otimes 1 - [-x]\otimes[x]$$

(8)
$$\begin{aligned}\hat{\partial}_2(y) &= \tau\hat{d}_2(y)\\ &= \sum_i [-y_n-\ldots-y_{i+1}]\otimes y_i\otimes[y_{i+1}+\ldots+y_n]\\ &= \sum_i ([y_1+\ldots+y_{i-1}]\cdot[y_i])\otimes y_i\otimes[y_{i+1}+\ldots+y_n] \ .\end{aligned}$$

We now show that $\hat{\alpha}\hat{C}_*X$ in (4) is in fact 2 - realizable. To see this we define a chain algebra

(9)
$$A = T(s^{-1}C_1\oplus s^{-1}C_2)$$

with $\bar{b}A$ being isomorphic to (4). The differential d in A is defined by

(10)
$$\begin{cases} d\colon s^{-1}C_2 \longrightarrow T(s^{-1}C_1) \\ d(s^{-1}e) = 1-(1+s^{-1}y_1)\otimes\ldots\otimes(1+s^{-1}y_n) \end{cases}$$

with $f_e = y_1+\ldots+y_n \in FM(Z_1)$ and $e \in Z_2$ as above.

There is an isomorphism ψ of algebras which makes the following diagram commutative

(11)
$$\begin{tikzcd} & R & \\ A \arrow[ur, dashed, "\varepsilon = p_o"] \arrow[r, "\lambda"] & H_oA \arrow[u, "\varepsilon"] \arrow[r, "\cong", "\psi"'] & H = R[\pi] \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

We set $\psi\{s^{-1}y\} = [y]-1$ for $y \in Z_1$.

By definition of $\hat{b}A$ we get

(12)
$$\begin{aligned} \hat{b}A &= (H_oA\otimes(R\oplus C_1\oplus C_2)\otimes H_oA,\hat{d}) \\ &\underset{\psi}{\cong} (\ H \otimes(R\oplus C_1\oplus C_2)\otimes\ H\ ,\partial) \end{aligned}$$

Here we identify H with H_oA by the isomorphism ψ in (11). The differential in $\hat{b}A$ is given by $\hat{d}e = -\bar{s}\Lambda d(s^{-1}e)$ for $e \in Z$. By definition of Λ and ψ we thus have for ∂ in (16)

(13)
$$\partial e = \sum_{i=1}^{n} [y_1+\ldots+y_{i-1}]\otimes y_i\otimes[y_{i+1}+\ldots+y_n] \ .$$

Similarly we get for $y \in Z_1$

(14)
$$\begin{aligned} \hat{d}y &= \lambda y\otimes 1 - 1\otimes\lambda y \qquad \text{and} \\ \partial y &= ([y]-1)\otimes 1 - 1\otimes([y]-1) \\ &= [y]\otimes 1 - 1\otimes[y] \ . \end{aligned}$$

Now we set up an isomorphism ∇ between the bottom row in (4) and the complex in (12). Let ∇ be the identity in degree $\neq 1$ and let in degree 1

(15) $$\nabla(y) = [y]\otimes y\otimes 1 \qquad \text{for} \quad y\in Z_1 .$$

Clearly, ∇ in degree 1 is an isomorphism of H-bimodules with inverse $\nabla^{-1}(y) = [-y]\otimes y$.

By comparing (13) with (8) and (14) with (7) respectively, we see that the diagram

(16)
$$\begin{tikzcd}
H\otimes C_2\otimes H \arrow[r, "\tilde{\partial}_2"] & H\otimes C_1\otimes H \arrow[r, "\tilde{\partial}_1"] & H\otimes H \\
H\otimes C_2\otimes H \arrow[r, "\partial_2"'] \arrow[u, "\mathrm{id}"'] & H\otimes C_1\otimes H \arrow[r, "\partial_1"'] \arrow[u, "\nabla"'] & H\otimes H \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

commutes. This proves the proposition since we have the isomorphism

(17) $$t = \psi^{-1}\nabla^{-1}\tau : \hat{\alpha}\hat{C}_*X^2 \cong \hat{b}A$$

in $\hat{\mathcal{DFM}}$. $\square$

For a CW-complex X with $X^0 = *$ we consider the free algebra $T(s^{-1}\tilde{C}_*X)$ the generators, $s^{-1}e$, of which are in 1-1 correspondence with the cells, e , in $X - *$. In the following theorem we introduce a differential on this algebra:

(3.7) **Theorem:** *Let X be an admissible CW-complex. Then there is a differential d on $T(s^{-1}\tilde{C}_*X)$ together with a weak equivalence*

$$\vartheta : A(X) = (T(s^{-1}\tilde{C}_*X),d) \longrightarrow C_*\Omega X$$

in $\mathcal{D}A$ *. Moreover, the chain algebra* $A(X)$ *has the property that the chain complexes* $\hat{b}(AX)$ *and* $\hat{\alpha}\hat{C}_*X$ *are canonicall isomorphic in* $\mathcal{D}F\hat{M}$ *by the isomorphism* t *below. For a 2-cell* e *in* X *the differential* $ds^{-1}e$ *in* $A(X)$ *is given by the formula*

$$ds^{-1}e = 1-(1+s^{-1}y_1)\otimes\ldots\otimes(1+s^{-1}y_n) \quad .$$

Here $f_e = y_1+\ldots+y_n$ *is the attaching map of* e *for which all* y_i *are 1-cells in* X *, (compare the definition of admissibility above).*

The isomorphism t is defined by $t = \psi^{-1}\nabla^{-1}\tau$ as in the proof of (

For convenience of the reader we recall the definition of t as follo

We have the isomorphism

$$\psi\colon\ H_o = H_oAX \cong H = R[\pi]$$

of algebras with $\psi\{s^{-1}y\} = 1-[y]$ for each 1-cell y in X . The isomorphism t is obtained by the commutative diagram

$$\begin{tikzcd}
\hat{\alpha}\hat{C}_*X \arrow[ddd, "t"'] & = & (C_*X\otimes H)\otimes_H\bar{\xi}^*(H\otimes H^{op}) \arrow[d, "\tau"', "\cong"] \\
 & & H\otimes C_*X\otimes H \\
 & & H\otimes C_*X\otimes H \arrow[u, "\nabla", "\cong"'] \\
\hat{b}A(X) & = & H_o\otimes(R\bullet\tilde{C}_*X)\otimes H_o \arrow[u, "\bar{\psi}", "\cong"']
\end{tikzcd}$$

Here we use the identification $\tilde{C}_*X = C_*X\otimes H$ in (A .1 16) and the identification $C_*X = R \oplus \tilde{C}_*X$. We set $\bar{\psi} = \psi \otimes 1 \otimes \psi$. The isomorphism ∇ is the identity in degree $\neq 1$ and is defined in degree 1 by

$$\nabla(y) = [y] \otimes y \otimes 1$$

for each 1 - cell y in X . Moreover, the isomorphism τ is defined by

$$\tau(x \otimes [\alpha]) \otimes_H ([\beta] \otimes [\gamma]^*) = [\gamma - \alpha] \otimes x \otimes [\alpha + \beta]$$

for $x \in C_*X$ and $\alpha, \beta, \gamma \in \pi$.

(3.8) <u>Remark</u>: Let X be any CW - complex with trivial 0 - skeleton, $X^0 = *$. Then there is a differential d on

$$A = T(s^{-1}\tilde{C}_*X \oplus C_1X \oplus s^{-1}C_1X)$$

such that (A,d) is a chain algebra which is equivalent to $C_*\Omega X$. We set $(A,d) = A(\bar{X})$ with $\bar{X}$ constructed in (3.4). //

<u>Proof of (3.7)</u>: We know by (C2.9.6) that the functor

$$\hat{b}: \mathcal{DFA}_{(f)/\simeq} \longrightarrow \mathcal{DF}\hat{\mathcal{M}}_{(f)}(2)_{/\simeq}$$

satisfies the <u>strong sufficiency</u> condition. Let

$$K = (H_0 \otimes (R \oplus \tilde{C}_*X) \otimes H_0, \partial) \tag{1}$$

be the chain complex which is isomorphic to $\hat{\alpha}\hat{C}_*X$ via t above. In the proof of (3.6) we constructed the algebra $A = A(X^2)$ with

$$\hat{b}A = K^2 \tag{2}$$

This shows that the homotopy equivalence

(3) $$K \overset{\sim}{=} \tilde{\alpha}C_*X \simeq \tilde{b}\alpha X \text{ , see (2.8),}$$

is a map in $\mathcal{DF}\tilde{M}_{(f)}(2)$. By strong sufficiency of $\hat{b}$, see (A.8.1), we thus obtain inductively an object $A(X)$ in $\mathcal{DFA}$ equal to A and with

(4) $$\tilde{b}A(X) = K \quad .$$

Moreover, the homotopy equivalence (3) can be realized by a homotopy equivalence $\vartheta\colon AX \simeq \alpha X$. By (4) and (1) we see that the underlying algebra of $A(X)$ is the free tensor algebra $T(s^{-1}\tilde{C}_*X)$. This proves t theorem. ▯

We call a chain algebra $A(X)$ as in (3.7) together with a weak equivalence

(3.9) $$\vartheta\colon AX \simeq \alpha X \xrightarrow{\sim} C_*\Omega X$$

as in the proof of (3.7) a <u>cellular model</u> of $C_*\Omega X$. By choosing a ce ular model $(A(X),\vartheta)$ for each admissible CW - complex X we obtain a functor A which makes the following diagram of functors commutative

(3.10) $$\begin{tikzcd}
Ho(\mathit{Top}_o) \arrow[r, "C_*\Omega"'] & Ho(\mathcal{DA}_f) \\
\mathit{CW}_{a/\simeq} \arrow[u, "\sim"'] \arrow[r, "A"'] \arrow[d, "\tilde{C}_*"'] & \mathcal{DFA}_{(f)/\simeq} \arrow[u, "\sim"'] \arrow[d, "\tilde{b}"'] \\
\tilde{\mathit{Chain}}_{R/\simeq} \arrow[r, "\tilde{\alpha}"] & \mathcal{DF}\tilde{M}_{(f)/\simeq}
\end{tikzcd}$$

Here $\mathcal{CW}_a$ denotes the full subcategory of $\mathcal{CW}$ consisting of admissible complexes. The functor A carries the CW-complex X to the chosen cellular model A(X) and carries the homotopy class of $F: X \longrightarrow Y$ to the unique class $\{AF\}$ which makes the diagram

$$\begin{tikzcd}
AX \arrow[r, dashed, "AF"] \arrow[d, "\vartheta"] & AY \arrow[d, dotted, "\vartheta"] \\
C_*\Omega X \arrow[r, "C_*\Omega F"'] & C_*\Omega Y
\end{tikzcd}$$

homotopy commute in $\mathcal{DA}$. Compared with (2.6) the advantage of diagram (3.10) is the fact that the lower square of this diagram actually commutes on objects if we identify $\hat{b}AX$ and $\hat{\alpha}\hat{C}_*X$ by the isomorphism t in (3.7).

A cellular model $(A(X),\vartheta)$ can be characterized in a different way by the properties which we describe in the addendum (3.14) below.

To this end we introduce the <u>Hurewicz-map</u>

(3.11) $$h: \pi_n(X,A) \longrightarrow H_n(X,A) \qquad (n \geqq 1)$$

as follows: We set $S^1 = I/\{0,1\}$. The quotient map $I^n \longrightarrow E^n = I \wedge S^1 \wedge \ldots \wedge S^1$ (where 0 is the base point of I) represents a canonical generator in cubical homology:

$$e^n \in H_n(E^n, S^{n-1}) \cong \mathbb{Z} \quad .$$

An element $\alpha \in \pi_n(X,A)$ is the homotopy class of a pointed pair map

$$\alpha: (E^n, S^{n-1}, *) \longrightarrow (X, A, *) \quad .$$

We define h in (3.11) by $h(\alpha) = a_*(e^n)$. If $A = *$ is a point we o tain $h\colon \pi_n(X) \longrightarrow \tilde{H}_n(X)$, $n \geq 0$. The boundary maps, ∂ , satisfy

$$\partial h(\alpha) = h(\partial\alpha)$$

as follows from the definition of ∂ in (2.1). (Here it is important that 0 is the basepoint in I .) Clearly, $\partial\alpha$ is represented by th restriction of a to S^{n-1} .

We define the *adjunction isomorphism*

(3.12) $$\sigma\colon \pi_{n-1}(\Omega X,\Omega A) \xrightarrow{\cong} \pi_n(X,A)$$

by $\sigma\{a\} = \{\bar{a}\circ T\}$. Here $\bar{a}\colon S^1 \wedge E^{n-1} \longrightarrow X$ is the adjoint of $a\colon E$ $\longrightarrow \Omega X$ and $T\colon (E^n,S^{n-1}) \longrightarrow (S^1 \wedge E^{n-1},S^{n-1})$ is a map of degree This shows that

$$\partial\sigma = -\sigma\partial \quad ,$$

(as it should be by the Koszul convention). The homomorphism

(3.13) $$\tau = h\sigma^{-1}\colon \pi_n(X,A) \longrightarrow H_{n-1}(\Omega X,\Omega A)$$

satisfies $\partial\tau = -\tau\partial$. For an $(n+1)$-cell e in a strictly pointed CW complex X let the element

$$f_e \in \pi_{n+1}(X^{n+1},X^n)$$

be determined by the *characteristic* map of e .

(3.14) *Addendum*: Assume that X is an admissible complex. We can choose AX (3.9) together with a weak equivalence

$$U = U_X\colon AX \xrightarrow{\sim} C_*\Omega X$$

such that the following properties hold:

(a) υ_X preserves the filtration of subcomplexes. That is, for any subcomplex Y in X we have an inclusion of chain algebras

$$AY = (T(s^{-1}\tilde{C}_*Y),d) \subset (T(s^{-1}\tilde{C}_*X,d) = AX$$

which is given by the inclusion $C_*Y \subset C_*X$ and we have a commutative diagram

$$\begin{tikzcd} AX \arrow[r, "\upsilon_X"'] & C_*\Omega X \\ AY \arrow[u, phantom, "\cup"] \arrow[r, "\upsilon_Y"'] & C_*\Omega Y \arrow[u, phantom, "\cup"] \end{tikzcd} .$$

(b) By (a) the chain map υ induces for all $n \geq 1$

$$\upsilon_*: H_n(AX^{n+1},AX^n) \longrightarrow H_n(\Omega X^{n+1},\Omega X^n) .$$

This map satisfies $\upsilon_*\{s^{-1}e\} = \tau(f_e)$ for each $(n+1)$-cell e in $X^{n+1} - X^n$, see (3.13).

(c) Each 1-cell y in X yields the loop $y \in \Omega X \subset C_o\Omega X$ with

$$\upsilon(s^{-1}y) = y - 1 \in C_o\Omega X .$$

(d) For a 2-cell e in X the element $\upsilon(s^{-1}e) \in C_1\Omega X$ is given by the singular 1-cube

$$\upsilon(s^{-1}e) = \bar{f}_e : I \longrightarrow \Omega X .$$

Here $\bar{f}_e$ is the adjoint of the characteristic map f_e of e, see (3.12).//

(3.15) <u>Remark</u>: $\bar{f}_e$ in (3.14)(d) satisfies

$$\bar{f}_e(1) = * \quad \text{and} \quad \bar{f}_e(0) = y_1 + \dots + y_n \in \Omega X \ .$$

Here $\bar{f}_e(0)$ is the attaching map of e for which the loops y_i (i= are 1-cells in X. From the definition of ∂ in (2.1) follows:

$$\partial \bar{f}_e = 1 - y_1 \cdot \dots \cdot y_n \in C_0 \Omega X \ .$$

This shows that with the definition of $ds^{-1}e$ in (3.9) we have

$$\begin{aligned} \upsilon d(s^{-1}e) &= \upsilon(1 - (1+s^{-1}y_1) \otimes \dots \otimes (1+s^{-1}y_n)) \\ &= 1 - (1+y_1-1) \dots (1+y_n-1) \\ &= 1 - y_1 \cdot \dots \cdot y_n \\ &= \partial \bar{f}_e = d\upsilon(s^{-1}e) \ . \end{aligned}$$

Moreover, by (3.14)(c) the map υ induces in degree 0 the map

$$\psi = \upsilon_*: H_0 = H_0 AX \xrightarrow{\cong} H_0 \Omega X = H$$

which is the isomorphism in (3.7).//

(3.16) <u>Proposition</u>: *Let X be an admissible CW-complex and suppose*

$$\upsilon: A(X) = (T(s^{-1}\tilde{C}_*X), d) \longrightarrow C_*\Omega X$$

is a map in $\mathcal{D}A$ which satisfies the properties in (3.14). Then υ is a weak equivalence and the isomorphism t in (3.7) is an isomo phism $\hat{b}A(X) \cong \hat{\hat{\alpha}}C_*X$. Thus $(A(X),\upsilon)$ is a cellular model. This sh that we can give an <u>inductive construction</u> of a cellular model: As we have constructed

$$\upsilon_n: A(X^n) \longrightarrow C_*\Omega X^n$$

satisfying the properties in (3.14), $n \geq 2$. *Then* υ_n *is a weak equivalence which gives us for each (n+1) - cell,* e , *the homology class*

$$\{z_e\} = (\upsilon_{n*})^{-1}\tau(\partial f_e) \in H_{n-1}(AX^n) \quad .$$

We choose a cycle z_e *in this class and we define* $ds^{-1}e = z_e$. *Then there exists a map* υ_{n+1} *which extends* υ_n *and which satisfies the properties in (3.14) as well. Thus* υ_{n+1} *is again a weak equivalence and we can proceed.*

(3.17) Remark: For a complex X with trivial 1 - skeleton, $X^1 = *$, the inductive construction (3.16) is due to Adams and Hilton [ ]. Their method of proof relies on the Moore comparison theorem and cannot be generalized to the case $X^1 \neq *$. Our proof of (3.16) is essentially a consequence of the fact that the functor $\hat{b}$ satisfies the strong sufficiency condition.//

(3.18) Remark: As an example Adams in [ ] computed $A(|X|)$ for a simplicial set X with $|X|^1 = *$. He showed that

$$A(|X|) = \Omega(C_*X,\Delta)$$

is the cobar construction. Here C_*X is the coalgebra with the Alexander - Whitney diagonal. Compare also [ ]. It would be of interest to compute $A(\overline{|X|})$ where $|X|^0 = *$ and where $\overline{|X|}$ is the construction in (3.4).//

§ 4 The homology suspension

We fist recall some well known operators for homology. In this sec

$R = \mathbb{Z}$ is the ring of coefficients. The suspension ΣX of a pointed s

X is the quotient space

$$\Sigma X = (I \times X) / (I \times * \cup \partial I \times X) \tag{4.1}$$

where $\partial I = \{0,1\}$. Let $p\colon I \times X \longrightarrow \Sigma X$ be the quotient map. For th

cubical chains, $CS_*(X)$, on X we define a chain map of degree +1

$$\Sigma\colon \widetilde{CS}_* X \longrightarrow \widetilde{CS}_* \Sigma X \tag{4.2}$$

by

$$\Sigma(\sigma) = p \circ (1_I \times \sigma) \ , \ \sigma\colon I^n \longrightarrow X \ .$$

By the definition of ∂ in (2.1) it is clear that Σ in (4.2) is a

map of degree +1, that is $\partial\Sigma = -\Sigma\partial$. This chain map induces the suspe

isomorphism in homology, Σ , for which the following diagram commute

$$\begin{tikzcd}
\tilde{H}_{n-1}(X) \arrow[d, dashed, "\Sigma"'] & & H_n(I \wedge X, X) \arrow[ll, "\partial"', "\cong"] \arrow[d, dashed, "p_*"', "\cong"] \\
\tilde{H}_n(\Sigma X) \arrow[rr, equal] & & H_n(\Sigma X, *)
\end{tikzcd} \tag{4.3}$$

Here $0 \in I$ is the basepoint of I and $p\colon I \wedge X \longrightarrow \Sigma X$ is the quo

map. Moreover, we obtain the following commutative diagram for any po

pair (X,A) , $* \in A$,

(4.4)
$$\begin{tikzcd}
\pi_n(X,A) \arrow[r, "h"] \arrow[d, "\Sigma"] & H_n(X,A) \arrow[d, "\cong\ \Sigma"] \\
\pi_{n+1}(\Sigma X,\Sigma A) \arrow[r, "h"'] & H_{n+1}(\Sigma X,\Sigma A)
\end{tikzcd} .$$

Here h is the Hurewicz map defined in (3.8). Since we want this diagram to be commutative we define the suspension homomorphism on homotopy groups as follows: For a class $\{f\} \in \pi_n(X,A)$ let $\Sigma\{f\}$ be represented by

$$\Sigma f\colon I \wedge S^1 \wedge S^{n-1} \xrightarrow{T} S^1 \wedge I \wedge S^{n-1} \xrightarrow{(-1)\wedge f} S^1 \wedge X = \Sigma X$$

where T is the interchange map for $I \wedge S^1$ and where $(-1)\colon S^1 \longrightarrow S^1$ is given by $(-1)(t) = 1-t$, $t \in I$. This definition satisfies the formula

(4.5)
$$\partial\Sigma = -\Sigma\partial$$

which is valid for homotopy groups and for homology groups.

The adjoint map of $f\colon \Sigma X \longrightarrow Y$ is

$$\bar{f}\colon X \longrightarrow \Omega Y \quad \text{with} \quad \bar{f}(x) = (\omega,1) \text{ , } \omega(t) = f(t,x) \text{ .}$$

Compare the definition of the Moore loop space in (2.4). Vice versa, the adjoint of $g\colon X \longrightarrow \Omega Y$ is

$$\bar{g}\colon \Sigma X \longrightarrow Y \quad \text{with} \quad \bar{g}(t,x) = \omega(t\cdot r) \quad \text{if} \quad gx = (\omega,r) \text{,}$$

This gives us the adjunction isomorphism

(4.6)
$$\sigma\colon [\Sigma X,Y] \cong [X,\Omega Y]$$

of groups, $\sigma\{f\} = \{\bar{f}\}$. The group structure is induced by addition of loops on ΩY . Taking the adjoints of the identities of ΣX and ΩX respectively we obtain

(4.7)
$$\begin{cases} i_X \colon X \longrightarrow \Omega\Sigma X & \text{(suspension map)} \quad , \\ p_X \colon \Sigma\Omega X \longrightarrow X & \text{(evaluation map)} \quad . \end{cases}$$

We now define the homology suspension σ by the composition

(4.8)
$$\sigma\colon \tilde{H}_{n-1}(\Omega X) \xrightarrow[\Sigma]{} \tilde{H}_n(\Sigma\Omega X) \xrightarrow[p_{X*}]{} \tilde{H}_n(X) \quad .$$

Here we can replace X by a pointed pair (X,A) with $\Omega(X,A) = (\Omega X,$

In analogy to diagram (4.3) the diagram

(4.9)
$$\begin{tikzcd} \tilde{H}_{n-1}(\Omega X) \arrow[d, "\sigma"'] & & H_n(PX,\Omega X) \arrow[ll, "\partial", "\cong"'] \arrow[d, dashed, "q_*"] \\ \tilde{H}_n(X) \arrow[rr, phantom, "="] & & H_n(X,*) \end{tikzcd}$$

commutes. Here $q\colon PX \longrightarrow X$ is the path space fibration, $PX = \bigcup_{x \in X} P(X,*,x)$, compare (2.4). Since PX is contractible the boundar in (4.9) is an isomorphism. Thus we can define the homology sus sion σ by $\sigma = q_*\partial^{-1}$ as well.

(4.10) Lemma: *Diagram (4.9) commutes.*

Proof: There is a mapping Q such that the diagram of pairs

$$\begin{tikzcd} (I \wedge \Omega X, \Omega X) \arrow[r, "Q"'] \arrow[d, "p"] & (PX,\Omega X) \arrow[d, dashed, "q"] \\ (\Sigma\Omega X, *) \arrow[r, "p_X"'] & (X,*) \end{tikzcd}$$

is commutative. We can define Q by

$$Q(t,(\omega,r)) = (\omega_t, tr)$$

where $\omega_t(\tau) = \omega(\tau)$. Commutativity of (4.9) now is a consequence of (4.3). [

In analogy to (4.4) we get the commutative diagram

(4.11)
$$\begin{tikzcd}
\pi_n(X,A) \arrow[r, "h"] \arrow[rd, dashed, "\tau"] & H_n(X,A) \\
\pi_{n-1}(\Omega X,\Omega A) \arrow[u, "\sigma", "\cong"'] \arrow[r, "h"] & H_{n-1}(\Omega X,\Omega A) \arrow[u, "\sigma"']
\end{tikzcd}$$

where h is the Hurewicz homomorphism. Here σ is defined on either side by $\sigma = (p_X)_* \circ \Sigma$. Thus commutativity follows from the naturality of the Hurewicz map and from (4.4). By (4.7) we see that σ for $\pi_n(X,A)$ is the adjunction isomorphism. Therefore we can define τ with

(4.12)
$$\tau = h\sigma^{-1} \ , \ \sigma\tau = h \ ,$$

compare (3.12). From (4.5) we derive

(4.13)
$$\partial\tau = -\tau\partial \ , \ \partial\sigma = -\sigma\partial \ ,$$

compare (3.11) and (3.12).

The exact sequence of J.H.C. Whitehead in (A.§ 3) embeds the Hurewicz map $h: \pi_n(\hat{X}) \longrightarrow H_n(\hat{X})$ into a long exact sequence. We now do the same for the homology suspension $\sigma: H_{n-1}(\Omega\hat{X}) \longrightarrow H_n(\hat{X})$. To this end we intrc duce the functor Γ^T_{n-1} as follows: Let X be a connected CW-complex. By use of the inclusion, $i: X^{n-1} \subset X^n$, of skeleta we define for $n \geq 1$

(4.14)
$$\Gamma^T_{n-1}(X) = \mathrm{image}(\Omega i)_*: H_{n-1}\Omega X^{n-1} \longrightarrow H_{n-1}\Omega X^n \quad .$$

For $n=2$ we know $H_0\Omega X^1 = 0$ since X^1 is a wedge of 1-spheres. Th $\Gamma_1^T = 0$. While (4.14) is defined by skeleta of X this is in fact a hom invariant of X . Clearly, a cellular map $F: X \longrightarrow Y$ induces

$$F_*: \Gamma_{n-1}^T X \longrightarrow \Gamma_{n-1}^T Y$$

and if G is a cellular map homotopic to F then $F_* = G_*$. Thus Γ_{n-1}^T is a functor defined on the homotopy category $CW/\simeq$. In fact, we have

(4.15) **Lemma:** *Let X be an admissible complex in CW and let AX be a cellu model of $C_*\Omega X$. Then we have the natural isomorphism*

$$\Gamma_{n-1}^T(X) = \Gamma_{n-1}^T(AX)$$

where Γ_{n-1}^T is the homotopy functor defined in (C2.4.3).

Proof: By definition of Γ_{n-1}^T this is a corollary of (3.14). □

The functor Γ_{n-1}^T fits into the following commutative diagram (n ≥ 2 ($\hat{X}$ denotes the universal cover of X):

(4.16)
$$\begin{tikzcd}
H_{n+1}\hat{X} \arrow[r, "b"] \arrow[d, "id"] & \Gamma_n\hat{X} \arrow[r, "i"] \arrow[d, "\tau"] & \Gamma_n\hat{X} \arrow[r, "h"] \arrow[d, "\tau"] \arrow[r, phantom, shift right=6, "(*)"] & H_n\hat{X} \arrow[r, "b"] \arrow[d, "id"] & \Gamma_{n-1}\hat{X} \arrow[d, "\tau"] \\
H_{n+1}\hat{X} \arrow[r, "b"] & \Gamma_{n-1}^T\hat{X} \arrow[r, "i"] & H_{n-1}\Omega\hat{X} \arrow[r, "\sigma"] & H_n\hat{X} \arrow[r, "b"] & \Gamma_{n-2}^T\hat{X}
\end{tikzcd}$$

In (4.11) we have seen that the square (*) of this diagram is in fact commutative. By definition of $\Gamma_n\hat{X}$ in (A.§ 3) and of $\Gamma_{n-1}^T\hat{X}$ the natur transformation τ in (4.11) induces the natural map $\tau: \Gamma_n\hat{X} \longrightarrow \Gamma_{n-1}^T$ as well.

(4.17) **Theorem**: *Diagram (4.16) is welldefined and is natural with respect to basepoint preserving maps* $\hat{X} \longrightarrow \hat{Y}$. *The rows of this diagram are exact.*

Proof: The upper row is the exact sequence of J.H.C. Whitehead for $\hat{X}$, se (A.§ 3) . The lower row is the exact sequence in (C2.4.5) which is defined for a free model of $C_*\Omega\hat{X}$. We can derive the result directly from (A.3.13 as well. In this case we have to compute the homology of the chain complex (C_n, ∂) with

$$C_n = H_{n-1}(\Omega\hat{X}^n, \Omega\hat{X}^{n-1}) \quad , \quad n \geq 3 .$$

Since $\hat{X}$ is 1-connected, the homology suspension yields the isomorphism

$$\sigma: H_{n-1}(\Omega\hat{X}^n, \Omega\hat{X}^{n-1}) \cong H_n(\hat{X}^n, \hat{X}^{n-1}) \quad .$$

Therefore $H_n(C_*, \partial) = H_n(\hat{X})$. ▯

By covering transformations the group $\pi_1 X = \pi$ acts from the right on For $\alpha \in \pi$ we then have a map $\hat{X} \longrightarrow \hat{X}$, $x \mapsto x \cdot \alpha$, which, however, is no basepoint preserving for $\alpha \neq 0$. Since $\hat{X}$ is 1-connected there is up to homotopy a unique basepoint preserving map $\alpha^{\#} : \hat{X} \longrightarrow \hat{X}$ which is freely homotopic to $x \mapsto x \cdot \alpha$ with $(\alpha + \beta)^{\#} \simeq \beta^{\#} \circ \alpha^{\#}$. Therefore the functors in diagram (4.16) have the additional structure that

$$(4.18) \quad \begin{cases} \Gamma_k\hat{X} \ , \ \Gamma_k^T\Omega\hat{X} \ , \ H_k\hat{X} \ , \ H_k\Omega\hat{X} \text{ and } \pi_k\hat{X} \text{ are right } \mathbb{Z}[\pi]\text{-modules by} \\ \text{the action } x^{\alpha} = (\alpha^{\#})_*(x) \text{ for } \alpha \in \pi . \end{cases}$$

Clearly, since diagram (4.16) is natural with respect to maps all homomorphisms of this diagram are actually homomorphisms of right $\mathbb{Z}[\pi]$-modules.

The covering projection $\hat{X} \longrightarrow X$ gives us the isomorphism of $\mathbb{Z}[\pi]$-modules $(n \geq 2)$:

(4.19) $$\pi_n\hat{X} = \pi_n X \quad , \quad \Gamma_n\hat{X} = \Gamma_n X \quad .$$

Therefore the upper row of diagram (4.16) is isomorphic to the exact quence of J.H.C. Whitehead for the space X, see (A.§ 3). In fact, i non simply connected case we have the following commutative diagram w $H = H_o\Omega X = \mathbb{Z}[\pi]$ and $n \geq 2$.

(4.20)
$$\begin{tikzcd}
\Gamma_n\hat{X} \arrow[r, "i"] \arrow[d, "p", "\cong"'] & \pi_n\hat{X} \arrow[r, "h"] \arrow[d, dashed, "p", "\cong"'] & H_n(\hat{X}) \arrow[r, "b"] \arrow[d, dashed, "q", "="'] & \Gamma_{n-1} \arrow[d, "\cong"'] \\
\Gamma_n X \arrow[r, "i"] \arrow[d, dashed, "\tau"] & \pi_n X \arrow[r, "j"] \arrow[d, dashed, "\tau"] & \hat{H}_n(X;H) \arrow[r, "b"] \arrow[d, dashed, "\bar{\xi}_*"] & \Gamma_{n-1} \arrow[d, dashed] \\
\Gamma^T_{n-1}X \arrow[r, "i"] & H_{n-1}\Omega X \arrow[r, "j"] & \hat{H}_n(X;\bar{\xi}_* H\otimes H^{op}) \arrow[r, "b"] & \Gamma^T_{n-1} \\
H\otimes\Gamma^T_{n-1}\hat{X} \arrow[r, "1\otimes i"'] \arrow[u, "\bar{p}"', "\cong"] & H\otimes H_{n-1}\Omega\hat{X} \arrow[r, "1\otimes\sigma"'] \arrow[u, "\bar{p}"', "\cong"] & H\otimes H_n(\hat{X}) \arrow[r, "1\otimes b"'] \arrow[u, "\bar{q}"', "\cong"] & H\otimes\Gamma^T_n \arrow[u, "\cong"]
\end{tikzcd}$$

The <u>top row</u> of this diagram is the same as the one in diagram (4.16). isomorphisms p of right H-modules are given by (4.19). Moreover, denotes homology with local coefficients in a left H-module and q the identity, compare the definition of $\hat{H}_n$.

The <u>second row</u> is the exact sequence of J.H.C. Whitehead for X, we described in (A.§ 3). This is an exact sequence of right H-modul

The homomorphisms τ are defined by τ in (4.11). Moreover, $\bar{\xi}_*$ the map induced by the homomorphism between coefficient modules $\bar{\xi}$: H $H\otimes H^{op}$, see (1.16).

The third row is the second sequence in (C2.4.5) for the free chain algebra $A(X)$, (we can assume that X is an admissible complex). Here we use (4.15) and (3.14)(a). This row is an exact sequence of H - bimodules and via $\bar{\xi}$ the maps τ are maps of right H-modules, that is $\tau(x^{\alpha}) = [-\alpha]\cdot x\cdot[\alpha]$

The bottom row of diagram (4.20) is the bottom row of diagram (4.16) tensored with H from the left. All modules in the bottom row are H - bimodules by (4.18) where we use the convention in (2.7).

The maps $\bar{p}$ on the bottom row of (4.20) are defined by

(4.21) $$\bar{p}([\alpha]\otimes x) = \alpha\cdot(p_* x)$$

where $p\colon \hat{X} \longrightarrow X$ is the covering projection. These maps, $\bar{p}$, are isomorphisms of H - bimodules. The H - biequivariant map $\bar{q}$ is defined by

(4.22) $$\bar{q}([\alpha]\otimes y) = \alpha\cdot\bar{\xi}_*(qx) \quad .$$

This is an isomorphism of H - bimodules as well, compare the proof of the following theorem.

(4.23) **Theorem:** *Diagram (4.20) commutes and is natural with respect to maps* $X \longrightarrow Y$ *in* CW . *All rows of this diagram are exact sequences and the composition of maps in a column of (4.20) satisfies:*

$$\bar{p}^{-1}\tau p(x) = 1\otimes(\tau x) \qquad \text{and}$$

$$\bar{q}^{-1}\bar{\xi}_* q(y) = 1\otimes y \quad .$$

This shows that diagram (4.20) up to isomorphism is totally determined by diagram (4.16) and by (4.18).

Proof: The homeomorphism $\Omega\hat{X} \approx \Omega_o X$, $\sigma \mapsto p\sigma$, shows that the map $\Omega\alpha^{\#}\colon \Omega\hat{X} \longrightarrow \Omega\hat{X}$ is homotopic to the map $\Omega_o X \longrightarrow \Omega_o X$ which carries the loop

$\sigma \in \Omega_o X$ to the loop $-a_o + \sigma + a_o$ where $a_o \in \alpha$. Thus we see as in (6. the maps $\bar{p}$ are isomorphisms of H - bimodules. By naturality of the th sequence the squares between the third and the bottom row commute. Now can deduce that all squares in (4.20) commute. ▯

Since the maps $\bar{p}$ are isomorphsims, by the five lemma, also $\bar{q}$ is isomorphism.

Since the map j in the third row of (4.20) is induced by the cha map $\bar{s}\Lambda$ of degree +1 in (C2.1.20) we see that the following diagram c mutes $(n \geq 1)$:

(4.24)
$$\begin{tikzcd}
H_n AX \arrow[r, "\cong"', "\vartheta_*"] \arrow[d, "\bar{s}\Lambda_*"] & H_n \Omega X \arrow[r, "\bar{p}", "\cong"'] \arrow[d, dashed, "j"] & H \otimes H_n \hat{\Omega X} \arrow[d, dashed, "1 \otimes \sigma"] \\
H_{n+1} \hat{b}AX \arrow[r, "\cong", "t_*"'] & \hat{H}_{n+1}(X, \xi^* H \otimes H^{op}) \arrow[r, "\bar{q}", "\cong"'] & H \otimes H_{n+1} \hat{X}
\end{tikzcd}$$

If the cellular model AX is known this diagram can be used for t effective computation of the map $1 \otimes \sigma$.