Category of Eilenberg - Mac Lane Fibrations and Cohomology of Grothendieck Constructions

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§1. Introduction

Let X be a path connected CW-space with basepoint and let $\pi = \pi_1(X)$. A $K(A,n)$-fibration over X is a fibration

$$K(A,n) \subset Y \longrightarrow X,$$

for which the fiber is an Eilenberg - MacLane space $K(A,n)$. It is well known that such a fibration yields for $n \geq 2$ the structure of a π-module on A and its k- invariant $k(Y)$ lies in cohomology with twisted coefficients $H^{n+1}(X,A)$. Let $n \geq 2$ and K_X^n be the full subcategory of the homotopy category of maps over X, consisting of the K(A,n) - fibration over X where A ranges over all π - modules.

Baues [2 VIII §2] or [3, 3.4.] has obtained the following (weak) linear extension of categories

(1.1). $$\mathbb{H}^n + \longrightarrow K^n_X \xrightarrow{\ p\ } K^{n+1}_X .$$

(A,k) where A is a π - module and $k \in H^{n+1}(X,A)$. Morphisms

$$\xi:(A',k') \longrightarrow (A,k)$$

are π-linear maps $\xi:A' \longrightarrow A$ with $\xi_*(k') = k$. Moreover

$$\mathbb{H}^n: K^{n+1}_X \longrightarrow Ab$$

and

$$p: K^n_X \longrightarrow K^{n+1}_X$$

are functors defined by

$$\mathbb{H}^n(A,k) = H^n(X,A),$$

$$p(K(A,n) \subset Y \longrightarrow X) = (A,\ k(Y)).$$

The linear extension (1.1) represents an element of the abelian group

(1.2). $$\mathcal{O}^n(X) \in H^2(\ K^{n+1}_X,\ \mathbb{H}^n),$$

see [3]. In the present work we compute the cohomology groups $H^*(K^{n+1}_X, \mathbb{H}^n)$ and the element $\mathcal{O}^n(X)$ in terms of homological invariants of X (see theorem 4.20 below). We prove for X = K(π,1), where π is finite group, that

$$H^i(\mathbb{K}_X^{n+1},\mathbb{H}^n) \cong \begin{cases} 0 \ , & i \neq 2, \\ \mathbb{Z}/|\pi| \ , & i = 2 \end{cases}$$

Here $\mathcal{O}^n(X)$ is a generator of the group $H^2(\mathbb{K}_X^{n+1},\mathbb{H}^n)$. We also consider the category $\mathbb{K}_{X,0}^n$ of orientable Eilenberg-MacLane fibrations over X and we shall construct a nice algebraic model for the category $\mathbb{K}_{X,0}^n$(see 4.26 below).

Note that the cohomology $H^*(\mathbb{K}_X^{n+1}, \mathbb{H}^n)$ in question is the cohomology of a small category with coefficients given by a functor but our method essentially uses the more general Baues & Wirsching cohomology with coefficients given by natural systems, see [3].

Note that the cohomologies and linear extensions above are defined only for small categories. But the categories $\mathbb{K}_X^n$, $\mathbb{K}_X^{n+1}$, $\mathbb{K}_{X,0}^n$ are not small. Here we mean that all objects of these categories belong to certain universe. Since the results do not depend on the choice of universum, we do not mention it. Similarly the categories in 4.14-4.19 are considered as small categories.

Main results of this paper were announced in [14]

§2. Preliminaries

We recall the definition of the cohomology of small categories with coefficients in a natural system (see [3]), and one result from [12].

Let I be a category. A *natural system of abelian groups* on I is a functor $D: FI \longrightarrow Ab$. Here FI is the *category of factorizations in* I : objects in FI are morphisms $\alpha: i \longrightarrow j$ in I , and morphisms $(\xi,\eta):\alpha \longrightarrow \alpha'$ are commutative diagrams

$$\begin{array}{ccc} j & \xrightarrow{\ \xi\ } & j' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\alpha'} \\ i & \xleftarrow{\ \eta\ } & i' \end{array}$$

i.e. $\alpha' = \xi\alpha\eta$. Composition is defined by

$$(\xi',\eta')(\xi,\eta) = (\xi'\xi,\ \eta\eta').$$

We clearly have $(\xi,\eta) = (\xi,1)\ (1,\eta) = (1,\eta)\ (\xi,1)$. We write $D(f) = D_f$ and $\xi_* = D(\xi,1)$, $\eta^* = D(1,\eta)$ for the induced maps of the functor D.

We have the functors

$$FI \xrightarrow{\ q_1\ } I^{op} \times I \xrightarrow{\ q_2\ } I$$

for which $q_1(\alpha: i \longrightarrow j) = (i,j)$ and $q_2(i,j) = j$. Hence any functor on I or any bifunctor $I^{op} \times I \longrightarrow Ab$ defines a natural system on I by composition with q_1 and q_2.

Let D be a natural system on I. The *cohomology of* I *with* *coefficients in* D is defined by

$$H^*(\underline{I},D) = H^*(C^*(\underline{I},D)).$$

Here $C^*(\underline{I},D)$ is the standard cochain complex defined in [3]. We recall that

$$C^n(\underline{I},D) = \prod D_{\alpha_1\alpha_2\ldots\alpha_n},$$

where product is taken over all composable n-tuples

$$i_n \xrightarrow{\alpha_n} \ldots \xrightarrow{\alpha_0} i_0 .$$

2.1.*PROPOSITION. Let* $\underline{I}$ *be a small category and* $Ab^{\underline{I}}$ *the category of functors from* $\underline{I}$ *to Ab. Let* T,F: $\underline{I} \longrightarrow$ *Ab be functors. Suppose that* T(i) *is a free abelian group for every* $i \in Ob(\underline{I})$. *Then*

$$H^*(\underline{I}, \mathcal{H}om(T,F)) \cong Ext^*_{Ab^{\underline{I}}}(T, F),$$

where the bifunctor

$$\mathcal{H}om(T, F): \underline{I}^{op} \times \underline{I} \longrightarrow Ab$$

is defined by

$$\mathcal{H}om(T, F)(i,j) = Hom(T(i), F(j)), \; i,j \in Ob(\underline{I}).$$

The proposition 2.1 follows from [12].

2.2.*Definition.* ([3]). *Let* D *be a natural system on* $\underline{I}$. *A linear extension of the category* $\underline{I}$ *by* D,

$$D + \longrightarrow \underline{C} \xrightarrow{p} \underline{I},$$

is a functor p *with the following properties*

i). $\underline{C}$ *and* $\underline{I}$ *have the same objects and* p *is a full functor which* *is the identity on objects.*

ii). For each $\alpha: i \longrightarrow j$ *the abelian group* D_α *acts transitively and effectively on the subset* $p^{-1}(\alpha)$ *of morphisms in* $\underline{C}$. *We write* $\alpha_o + a$ *for the action of* $a \in D_\alpha$ *on* $\alpha_o \in p^{-1}(\alpha)$.

iii). The action satisfies the linear distributivity law :

$$(\alpha_o + a)(\beta_o + b) = \alpha_o\beta_o + \alpha_* b + \beta^* a.$$

Two linear extensions $\underline{C}$ and $\underline{C}'$ are *equivalent* if there is an isomorphism of categories $r : \underline{C} \cong \underline{C}'$ with $p'r = p$ and $r(\alpha_0 + a) = r(\alpha_0) + \alpha$, $\alpha_o \in Mor(\underline{C})$, $a \in D_{p\alpha_o}$. The extension $\underline{C}$ is *split* if there is a functor $s: \underline{I} \longrightarrow \underline{C}$ with $ps = 1$.

2.3. *Proposition.*([3]) *Let* D *be a natural system on the category* $\underline{I}$ *and* $M(\underline{I}, D)$ *be the set of equivalence classes of linear extensions of* $\underline{I}$ *by* D. *Then there is a canonical bijection*

$$\Phi: M(\underline{I}, D) \cong H^2(\underline{I}, D)$$

which maps the split extension to the zero element.

2.4.*Remark.* Let $\underline{I}$ be a small category and $T, F: \underline{I} \longrightarrow Ab$ be functors. Suppose that $T(i)$ is a free abelian group for any $i \in Ob(\underline{I})$. Let

$$\mathfrak{F} = (0 \longrightarrow F \xrightarrow{\mu} L_1 \xrightarrow{\tau} L_2 \xrightarrow{\sigma} T \longrightarrow 0)$$

be a two-fold extension in the category $Ab^{\underline{I}}$ and let $u(i)$ be a section of $\sigma(i)$ in the category *Ab*, $i \in Ob(\underline{I})$. Then by 2.3 the linear extension

$$\mathcal{H}om(T,F) \rightarrowtail E \xrightarrow{p} \underline{I}$$

represents an element in the group $H^2(\underline{I}, \mathcal{H}om(T,F))$, which corresponds to the class of $\mathcal{E}$ in $Ext^2_{Ab^{\underline{I}}}(T,F)$ via 2.1. Here $\underline{E}$ is the category whose objects are the same as of $\underline{I}$, and morphisms from i to j are pairs (α, x), where α is a morphism in $\underline{I}$ and $x: T(i) \longrightarrow L_1(j)$ is homomorphism of abelian groups satisfying the following equation

$$\tau(j)x = L_2(\alpha)u(i) - u(j)T(\alpha).$$

Composition in $\underline{E}$ and the functor p are defined by

$$(\beta,y)(\alpha,x) = (\beta\alpha, L_1(\beta)x + yT(\alpha)),$$

$$p(\alpha,x) = \alpha.$$

2.5.*Definition. Let* D *be a natural system on* $\underline{I}$. *A weak linear extension of the category* $\underline{I}$ *by* D *is a sequence*

$$(\mathcal{E}) \qquad D \rightarrowtail \underline{C} \xrightarrow{p} \underline{I}$$

where p is a functor with following properties:

1) p *is surjective on objects;*

2) *Let* $\underline{C}_0$ *be any full subcategory of* $\underline{C}$, *for which the restriction functor*

$$p_0 = p|_{\underline{C}_0}: \underline{C}_0 \longrightarrow \underline{I}$$

is bijective on objects, then the inclusion $\underline{C}_0 \subset \underline{C}$ *is an equiva-*

lence of categories and the sequence

$$(\mathfrak{F}_0) \qquad D \rightarrowtail \mathcal{C}_o \xrightarrow{p_o} \mathbf{I}$$

is a linear extension of categories. Moreover the corresponding cohomology class $\Psi(\mathfrak{F}_o) \in H^2(\mathbf{I}, D)$ *is independent of the choice ofthe subcategory* $\mathcal{C}_o$. *We call this class a characteristic class of the weak linear extension* $(\mathfrak{F})$.

For examples of weak linear extesions see §4 below.

§3. Calculations

The main theorem of this section is 3.7, we shall use it to calculate the cohomology of the categories in the introduction.

Let $\mathbf{I}$ be a small category and let

$$T: \mathbf{I} \longrightarrow \mathit{Sets}$$

be a functor. Consider the category $\mathbf{I}\int T$, whose objects are pairs (i,x), with $i \in Ob(\mathbf{I})$ and $x \in T(i)$, while a morphism from the pair (i,x) to (j,x) is represented by a morphism $\alpha: i \longrightarrow j$ in $\mathbf{I}$ such that $T(\alpha)(x) = y$. The category $\mathbf{I}\int T$ is commonly called the Grothendieck construction of T. By definition we have the following equality

$$\mathbf{k}_X^{n+1} = \pi\text{-}mod \int H^{n+1}(X,-).$$

3.1. *Proposition. Let* $\mathbf{I}$ *be a small category and let*

$$T:\mathbf{I} \longrightarrow Sets$$

be an arbitrary functor. For any natural system D *defined on the category* $\mathbf{I}\int T$, *there exists an isomorphism*

$$H^*(\mathbf{I}\int T;\ D) \cong H^*(\mathbf{I};\ D').$$

Here D' *is the natural system on* $\mathbf{I}$ *assigning to every arrow* $\alpha: i \longrightarrow j$ *the group*

$$\prod_{x\in T(i)} D(\alpha_x),$$

where α_x *denotes the*

$$\alpha:(i,x) \longrightarrow (j,\ T(\alpha)(x))$$

in $\mathbf{I}\int T$ *given by* α.

P r o o f. By definition we have

$$C^*(\mathbf{I}\int T;\ D) = \prod D_{\alpha_1\ldots\alpha_n},$$

where the product is taken over all composable n-tuples

$$(i_0,x_0)\xleftarrow{\alpha_1}(i_1,x_1) \longleftarrow \ldots\ldots \xleftarrow{\alpha_n} (i_n,\ x_n)$$

in the category $\mathbf{I}\int T$. Here we have

$$\alpha_k \in \mathbf{I}(i_k, i_{k-1}),$$
$$x_{k-1} = T(\alpha_k)(x_k),\ 1\leqslant k\leqslant n.$$

Hence

$$C^n(\mathbf{I}\int T;\ D) = \prod\prod D_{\alpha_1\ldots\alpha_n},$$

where the first product is taken over all composable n-tuples

$$i_o \xleftarrow{\alpha_1} \cdots \xleftarrow{\alpha_n} i_n$$

and second one over all $x_n \in T(i)$. But this double product is the same as $C^n(I;\ D')$. It is easily seen that the coboundary operator is compatible with the equality above, i.e.

$$C^*(I\int T;\ D) \cong C^*(I;\ D')\ ,$$

which proves the proposition.

3.2. *Remark.* Let $D+ \longrightarrow E \xrightarrow{p} I\int T$ be a linear extension of categories. We define the category X; objects are the same as those of I, and a morphism from i to j, $i,j \in Ob(I)$ is a pair (α, f). Here $\alpha: i \longrightarrow j$ is a morphism in I and f is a function assigning to each $x \in T(i)$ the morphism

$$f_x:\ (i,x) \longrightarrow (j,\ T(\alpha)(x))$$

in E, such that $pf_x = \alpha$. Let $q: X \longrightarrow I$ be the functor defined by $q(\alpha, f) = \alpha$. Then

$$D'+ \longrightarrow X \longrightarrow I$$

is the linear extension corresponding to

$$D + \longrightarrow E \xrightarrow{p} I\int T$$

via the isomorphism 3.1.

In the following $\mathbb{Z}[S]$ will denote the free abelian group with base S.

3.3. *Lemma. Let* $\mathcal{I}$ *be a small additive category and let* $Ab^{\mathcal{I}}$ *be the category whose objects are all functors from* $\mathcal{I}$ *to the category of abelian groups. Let* $\mathcal{I}$*- mod be the full subcategory of* $Ab^{\mathcal{I}}$ *whose objects are additive functors. Then the functor* $\mathbb{Z}:\mathcal{I}\text{-}mod \longrightarrow Ab^{\mathcal{I}}$ *defined by*

$$(\mathbb{Z}T)(i) = \mathbb{Z}[T(i)],$$

$T \in \mathrm{Ob}(\mathcal{I}\text{-}mod)$, $i \in \mathrm{Ob}(\mathcal{I})$, *carries projective objects to projective objects.*

P r o o f. It is well known that any projective object of the category $\mathcal{I}$-*mod* is a retract of a sum of objects of type $\mathrm{Hom}_{\mathcal{I}}(i,-)$ while every projective object of $Ab^{\mathcal{I}}$ is a retract of objects of type $\mathbb{Z}\mathrm{Hom}_{\mathcal{I}}(i,-)$. Hence it is sufficient to show that $\mathbb{Z}P$ is a projective object of $Ab^{\mathcal{I}}$, where

$$P = \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{I}}(i_\lambda,-).$$

The same remark can be applied if Λ is a singleton. Furthermore, if Λ is finite, then

$$P = \mathrm{Hom}_{\mathcal{I}}(\bigoplus_{\lambda\in\Lambda} i_\lambda, -),$$

and hence this reduces to the already considered case. Now suppose Λ is an arbitrary set. It is easily seen that the following holds

(3.4) $$\mathbb{Z}P = \bigoplus_{\{\lambda_1,\ldots,\lambda_k\}\subset\Lambda} \mathbb{Z}_k(\mathrm{Hom}_{\underline{I}}(i_{\lambda_1},-),\ldots.\mathrm{Hom}_{\underline{I}}(i_{\lambda_k},-).$$

Here $\mathbb{Z}_k$ denotes the k-th cross-effect of the functor $\mathbb{Z}$, in the sense of Eilenberg & MacLane [8], and the sum is taken over all finite subsets $\{\lambda_1,\ldots,\lambda_k\} \subset \Lambda$. Hence it is sufficient to show that

$$\mathbb{Z}_k(\mathrm{Hom}_{\underline{I}}(i_{\lambda_1},-),\ldots, \mathrm{Hom}_{\underline{I}}(i_{\lambda_k},-))$$

is projective in $Ab^{\underline{I}}$. To this end put $\Lambda = \{\lambda_1,\ldots\lambda_k\}$ in 3.4. Since in this case Λ is finite, $\mathbb{Z}P$ will be projective, hence it's direct summand $\mathbb{Z}_k(\mathrm{Hom}_{\underline{I}}(i_{\lambda_1},-),\ldots\mathrm{Hom}_{\underline{I}}(i_{\lambda_k},-))$ will be projective too.

3.5. *Proposition. Let* $\underline{I}$ *be a small additive category and* $T,F \in \mathrm{Ob}(\underline{I}\text{-}mod)$. *Then, the composition*

$$\mathrm{Ext}^*_{\underline{I}\text{-mod}}(T,F) \longrightarrow \mathrm{Ext}^*_{Ab^{\underline{I}}}(T,F) \longrightarrow \mathrm{Ext}^*_{Ab^{\underline{I}}}(\mathbb{Z}T,F)$$

is an isomorphism, where the first map is induced by the exact inclusion $I\text{-}mod \longrightarrow Ab^{\underline{I}}$, *and the second by the augmentation* $\varepsilon\colon \mathbb{Z}T \longrightarrow T$. *In particular the first map is a split monomorphism.*

Before giving the proof we need to justify some notations. Let X_* be a simplicial object in an abelian category. Then $Ch(X_*)$ will denote the chain complex, whose n-th component is equal to X_n, while the boundary operator is $d = \Sigma(-1)^k\, \partial_k$. The

Moore normalization of X_* [13] is denoted by NX_*. So

$$\pi_* X_* = H_* NX_* \cong H_* Ch(X_*).$$

P r o o f of 3.5. Let P be a componentwise projective simplicial object of the category $\mathbf{I}$-*mod*, such that

$$\pi_o P_* = T, \ \pi_n P_* = 0, \ n>0.$$

Then

$$H_n(P_*(i),\mathbb{Z}) = 0, \ n>0$$

and

$$H_o(P_*(i),\mathbb{Z}) = \mathbb{Z}[T(i)], \ i \in Ob(\mathbf{I}).$$

Here on the left had side one has homology groups of simplicial sets $P_*(i)$ with integer coefficients [13]. By the definition of these we get

$$H_n(Ch(\mathbb{Z}P_*)) = 0, \ n>0,$$

$$H_o(Ch(\mathbb{Z}P_*)) = \mathbb{Z}T.$$

By 3.3, $\mathbb{Z}P_*$ is a componentwise projective simplicial object in $Ab^{\mathbf{I}}$, hence $Ch(\mathbb{Z}P_*) \longrightarrow \mathbb{Z}T$ is a projective resolution and

$$Ext^*_{Ab^{\mathbf{I}}}(\mathbb{Z}T,F) = H^*(Hom_{Ab^{\mathbf{I}}}(Ch(\mathbb{Z}P_*,F))) =$$

$$= H^*(Hom_{\mathbf{I}\text{-mod}}(Ch(P_*),F).$$

Here the second equality holds, since the 3.6 is valid in dimension zero, by direct checking. But $H_*(Ch\ P_*) = \pi_n P_*$, so that

$\mathrm{Ch}\,P_* \longrightarrow T$ is a projective resolution of T in $\underline{I}$-*mod*. Consequently $\varepsilon_*\colon \mathbb{Z}P_* \longrightarrow P_*$ induces the isomorphism

$$\mathrm{Ext}^*_{Ab^{\underline{I}}}(\mathbb{Z}T,\ F) \cong \mathrm{Ext}^*_{\underline{I}\text{-mod}}(T, F) \tag{3.6}$$

Let $Q_* \longrightarrow T$ be a projective resolution in the category $Ab^{\underline{I}}$. Then there exists morphisms $\alpha_*\colon \mathrm{Ch}\,\mathbb{Z}P_* \longrightarrow Q_*$, $\beta_*\colon Q_* \longrightarrow \mathrm{Ch}P_*$ such that the diagrams

$$\begin{array}{ccc} \mathrm{Ch}\mathbb{Z}P_* & \xrightarrow{\alpha_*} & Q_* \\ \downarrow & & \downarrow \\ \mathbb{Z}T & \xrightarrow{\varepsilon} & T \end{array}\quad , \qquad \begin{array}{ccc} Q_* & \xrightarrow{\beta_*} & \mathrm{Ch}P_* \\ \downarrow & & \downarrow \\ T & \xrightarrow{1_T} & T \end{array}$$

commute. In fact, the first and second vertical maps are projective resolutions in the category $Ab^{\underline{I}}$. Therefore $\beta_*\alpha_*$ and ε_* are homotopic and from this fact and 3.6 follows the proposition.

3.7. THEOREM. *Let $\underline{I}$ be a small additive category and let $T, F\colon \underline{I} \longrightarrow Ab$ be additive functors. Define the functor*

$$D\colon \underline{I}\textstyle\int T \longrightarrow Ab$$

by

$$D(i,x) = F(i),\ i \in \mathrm{Ob}(\underline{I}),\ x \in T(i).$$

Then there exists an isomorphism

$$H^*(\underline{I}\textstyle\int T; D) \cong \mathrm{Ext}^*_{\underline{I}\text{-mod}}(T,F).$$

P r o o f. By 3.1,

$$H^*(I\int T;\ D) \cong H^*(I;\ D').$$

Here D' is the natural system on I, assigning to the arrow $\alpha: i \longrightarrow j$ the group

$$\prod_{x\in T(i)} D(\alpha_x)\ ;\ \alpha_x \equiv \alpha:(i,x) \longrightarrow (j,T(\alpha)x).$$

By definition of D one has $D(\alpha_x) = F(j)$. Hence

$$D'(\alpha: i \longrightarrow j) = \prod_{x\in T(i)} F(j) = \mathrm{Hom}\,(\mathbb{Z}[T(i)],\ F(j)).$$

So D' turns out to be the bifunctor of type $\mathcal{H}om\,(\mathbb{Z}T,F)$. Thus by 2.1.

$$H^*(I\int T;\ D) \cong \mathrm{Ext}^*_{Ab^I}(\mathbb{Z}T,\ F)$$

and the theorem follows from 3.5.

3.8. *Remark.* Let

$$\mathcal{E} = (\ 0 \longrightarrow F \longrightarrow X \xrightarrow{\tau} Y \xrightarrow{p} T \longrightarrow 0\ )$$

be a twofold extension in the category *I-mod* and let $u(i)$ be a section of $p(i)$ in the category *Sets*. Let $D: I\int T \longrightarrow Ab$ be a functor for which $D(i,x) = F(i)$. From 2.4, 3.2, 3.5 follows that the extension

$$D \longrightarrow E \xrightarrow{q} I\int T,$$

represents (by 2.3) an element in the group $H^2(I,\ D)$ which corresponds to the class of $\mathcal{E}$ in $\mathrm{Ext}^2_{I\text{-mod}}(T,F)$ via 3.7. Here E is a category with the same objects as in $I\int T$. Morphisms in E from (i,x) to $(j,y), x \in T(i), y \in T(j)$, are pairs (α,h) where $\alpha:(i,x) \longrightarrow (j,y)$ is a morphism in $I\int T$ and $h \in X(j)$ satisfies the equation

$$\tau(j)h + u(j)(y) = Y(\alpha)(u(i)(x)).$$

The functor q is defined by

$$q(\alpha,h) = \alpha.$$

3.9. *Example*. Let $\mathbb{A}$ be an abelian category with enough projective objects. Let (C_*,d) be a componentwise projective chain complex in $\mathbb{A}$ and

$$K_n = \mathrm{Coker}(d:C_{n+1} \longrightarrow C_n).$$

Let I be a small additive full subcategory of $\mathbb{A}$ with $C_n, K_n \in \mathrm{Ob}(I)$, $n \geq 0$. For any $A \in \mathrm{Ob}(\mathbb{A})$, the cohomology of the cochain complex $\mathrm{Hom}_{\mathbb{A}}(C_*,A)$ is denoted by $H^*(C_*,A)$. It is clear that

3.10. $$0 \longrightarrow \mathrm{Hom}_{\mathbb{A}}(K_n,-) \longrightarrow \mathrm{Hom}_{\mathbb{A}}(C_n,-) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{\mathbb{A}}(K_{n+1},-) \longrightarrow H^{n+1}(C_*,-) \longrightarrow 0$$

is exact. Hence 3.10 is projective resolution of the object $H^{n+1}(C_*, -)$ in the category I-*mod*, because $\mathrm{Hom}_{\mathbb{A}}(X, -)$ is projective object in I- *mod*, if $X \in \mathrm{Ob}(I)$. Therefore 3.7 implies

(3.11) $$H^m(I\int H^{n+1}(C_*,-),D) \cong \begin{cases} 0, & \text{if } m \geq 3, \\ H^m(FK_{n+1} \longrightarrow FC_n \longrightarrow FK_n), & 0 \leq m \leq 2. \end{cases}$$

This holds for any additive functor $F:I \longrightarrow$ *Ab* where

$$D: \mathbb{A}\int H^{n+1}(C_*,-) \longrightarrow Ab$$

is functor for which $D(A,x) = F(A)$, $A \in Ob(\mathbb{I})$, $x \in H^{n+1}(C\ ,A)$.

§4. Categories of extensions and Eilenberg-MacLane fibrations

In this section we will be prove Theorem 4.20 which requires the consideration of various kinds of categories of extensions.

4.1. The category of extensions in abelian categories. Let $\mathbb{A}$ be a small abelian category with enough projective objects. Let $A \in Ob(\mathbb{A})$. We consider the category of extensions $\mathcal{E}xt^1_{\mathbb{A}}(A)$. Objects of this category are exact sequences in $\mathbb{A}$

$$\mathcal{E} = (0 \longrightarrow B \xrightarrow{\mu} X \xrightarrow{\sigma} A \longrightarrow 0),$$

and morphisms from $\mathcal{E}$ to $\mathcal{E}'$ are pairs $(f:B \longrightarrow B', g:X \longrightarrow X')$ with

$$\mu' f = g\mu, \ \sigma' g = \sigma.$$

Let (f,g) and (f',g') be two morphisms from $\mathcal{E}$ to $\mathcal{E}'$ and $f = f'$, then there exists a unique $h \in Hom_{\mathbb{A}}(A,B')$ with

$$g - g' = \mu' h\sigma.$$

This shows that the sequence

4.2. $$Hom \longrightarrow \mathcal{E}xt^1_{\mathbb{A}}(A) \xrightarrow{p} \mathbb{A}\int Ext^1_{\mathbb{A}}(A,-)$$

is weak linear extension of categories. Here p and

$$\mathrm{Hom}: \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(A,-) \longrightarrow Ab$$

are functors defined by

$$p(\xi) = (B, cl(\xi) \in \mathrm{Ext}^1_{\mathbb{A}}(A,B)),$$
$$\mathrm{Hom}(B,x) = \mathrm{Hom}_{\mathbb{A}}(A,B).$$

By 2.5 the weak linear extension 4.2 determines an element $\mathcal{O}(A) \in H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(A,-);\mathrm{Hom})$. We shall compute this element. For this we recall the following definition.

4.3. *Definition.* [10]. *Let* $f,g : A \longrightarrow B$ *be morphisms in an abe- lian category* $\mathbb{A}$. *We call* f *and* g *homotopic in the sense of Eckmann - Hilton and write* $f \simeq g$ *iff there exists a projective object* P *and morphisms* $s: A \longrightarrow P$, $t: P \longrightarrow B$ *such that* $f - g = ts$. *Let*

$$[A,B]_{\mathbb{A}} = \mathrm{Hom}_{\mathbb{A}}(A,B)/\simeq.$$

The natural epimorphism $\mathrm{Hom}_{\mathbb{A}}(A,B) \longrightarrow [A,B]_{\mathbb{A}}$ is denote by cl.

4.4. *Proposition. Let* $\mathcal{O}(A)$ *be the characteristic element of the weak linear extension 4.2. Then there exists an isomorphism*

$$H^m(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(A,-);\mathrm{Hom}) \cong \begin{cases} 0, & m \neq 2, \\ [A,A]_{\mathbb{A}}, & m = 2, \end{cases}$$

and $cl(1_A)$ *corresponds to* $\mathcal{O}(A)$ *under this isomorphism.*

P r o o f. Let $P_* \longrightarrow A$ be a projective resolution in $\mathbb{A}$. If we put in 3.9

$$C_* = P_*,\ \mathbb{I} = \mathbb{A},\ F = \mathrm{Hom}_{\mathbb{A}}(A,-),$$

then 3.11 implies the first part of the proposition.

Let

$$\mathcal{P} = 0 \longrightarrow K \xrightarrow{\mu} P \xrightarrow{\sigma} A \longrightarrow 0$$

be a short exact sequence with P projective. We denote its class in $Ext^1_{\mathbb{A}}(A,K)$ by u. It follows from the proof of 3.11 that the 2-fold extension

$$(4.5)\quad \begin{array}{l} 0 \longrightarrow Hom_{\mathbb{A}}(A,-) \longrightarrow Hom_{\mathbb{A}}(P,-) \longrightarrow \\ \longrightarrow Hom_{\mathbb{A}}(K,-) \longrightarrow Ext^1_{\mathbb{A}}(A,-) \longrightarrow 0. \end{array}$$

corresponds to $cl(1_A) \in [A,A]_{\mathbb{A}}$ via the isomorphism

$$Ext^2_{\mathbb{A}-mod}(Ext^1_{\mathbb{A}}(A,-);\ Hom_{\mathbb{A}}(A,-)) \cong [A,A]_{\mathbb{A}}.$$

For each $x \in Ext^1_{\mathbb{A}}(A,B)$ and

$$\mathfrak{X} = (\ 0 \longrightarrow B \longrightarrow X \longrightarrow A \longrightarrow 0)$$

we choose morphisms $f_{B,x}:(K,u) \longrightarrow (B,x)$ in $\mathbb{A}\int Ext^1_{\mathbb{A}}(A,-)$ and $(f_{B,cl\mathfrak{X}}, g_{\mathfrak{X}}): \mathcal{P} \longrightarrow \mathfrak{X}$ in $\mathfrak{X}xt^1_{\mathbb{A}}(A)$. Let $\mathfrak{X}x^1_{\mathbb{A}}(A)$ be a category with the same objects as in $\mathbb{A}\int Ext^1_{\mathbb{A}}(A,-)$. Morphisms from (B,x) to (B',x') are pairs (f,h), where $f:(B,x) \longrightarrow (B',x')$ is a morphism in $\mathbb{A}\int Ext^1_{\mathbb{A}}(A,-)$ and $h:P \longrightarrow B'$ is a morphism with

$$f_{B',x'} - h\mu = ff_{B,x}\ .$$

According to 3.8 the class of the linear extension

$$Hom + \longrightarrow \mathfrak{X}x^1_{\mathbb{A}}(A) \xrightarrow{q} \mathbb{A}\int Ext^1_{\mathbb{A}}(A,-)$$

in the group $H^2(\mathbb{A}\int Ext^1_{\mathbb{A}}(A,-), Hom)$ corresponds to 4.5 via the isomorphism 3.7. Here $q(f,h) = f$. Let $T:\mathcal{E}xt^1_{\mathbb{A}}(A) \longrightarrow \mathcal{E}x^1_{\mathbb{A}}(A)$ be a functor defined on objects by

$$T(\mathcal{E}) = (B, \mathrm{cl}\,\mathcal{E} \in \mathrm{Ext}^1_{\mathbb{A}}(A,B)).$$

Let $(f,g):\mathcal{E} \longrightarrow \mathcal{E}'$ be a morphism in $\mathcal{E}xt^1_{\mathbb{A}}(A)$. Then there exists a unique $h:F \longrightarrow B'$ such that

$$\mu' h = g g_{\mathcal{E}} - g_{\mathcal{E}'}.$$

Now we define the functor T on morphisms by $T(f,g) = (f,h)$. It is clear that the diagram

$$\begin{array}{ccccc}
\mathrm{Hom} & \rightarrowtail & \mathcal{E}xt^1_{\mathbb{A}}(A) & \longrightarrow & \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(A,-) \\
\| & & \downarrow T & & \| \\
\mathrm{Hom} & \rightarrowtail & \mathcal{E}x^1_{\mathbb{A}}(A) & \longrightarrow & \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(A,-)
\end{array}$$

commutes and hence the upper and lower extension determine both the same element in $H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(A,-),\mathrm{Hom})$. This complete the proof of proposition.

In case when $\mathbb{A} = \mathcal{Ab}$ we denote $\mathcal{E}xt^1_{\mathbb{A}}(A)$ by $\mathcal{E}xt(A)$. If A is a finitely generated abelian group, then $[A,A]_{\mathcal{Ab}} = \mathrm{End}(t(A))$, where $t(A)$ is the torsion part of A.

4.6. <u>The category of relative extensions.</u> Let

$$\mathbb{A} \xrightarrow{\ U\ } \mathbb{B} \xrightarrow{\ F_0\ } \mathbb{A}$$

be functors between categories, satisfying the following conditions:

a) $\mathbb{A}$ and $\mathbb{B}$ are small abelian categories with enough projectives;

b) U is exact, full and faithful;

c) F_0 is the left adjoint of U.

Let $F_*:\mathbb{B}\longrightarrow\mathbb{A}$ be the collection of left derived functors of the functor F_0, $n\geq 0$. The conditions b),c) imply, that F_0 carries projective objects to projective objects. Hence c) and Grothendiecks spectral sequence for derived functors of compositions shows that for $A\in Ob(\mathbb{A})$, $B\in Ob(\mathbb{B})$ there exists a spectral sequence

$$E_2^{pq} = Ext^p_{\mathbb{A}}(F_q(B),A) \Longrightarrow Ext^{p+q}_{\mathbb{B}}(B,UA).$$

In particular one gets an exact sequence

4.7. $$0 \longrightarrow Ext^1_{\mathbb{A}}(F_o(B),A) \xrightarrow{\phi} Ext^1_{\mathbb{B}}(B,UA) \xrightarrow{\psi} Hom_{\mathbb{A}}(F_1(B),A) \longrightarrow$$
$$\longrightarrow Ext^2_{\mathbb{A}}(F_o(B),A) \longrightarrow Ext^2_{\mathbb{B}}(B,A).$$

Let $B \in Ob(\mathbb{B})$ and $\mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B)$ be the corresponding full subcategory of $\mathcal{E}xt^1_{\mathbb{B}}(B)$ whose objects are exact sequences

4.8. $$\mathcal{E} = (\ 0 \longrightarrow UA \longrightarrow X \longrightarrow B \longrightarrow 0)$$

where $A \in Ob(\mathbb{A})$. We define functors

$$\Phi:\mathcal{E}xt^1_{\mathbb{A}}(F_o(B)) \longrightarrow \mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B),$$
$$\Psi:\mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B) \longrightarrow H_1(B)\downarrow\mathbb{A},$$

by

$$\Phi(\mathcal{E}) = \psi(cl(\mathcal{E})),$$

$$\Phi(0 \longrightarrow A \longrightarrow A_1 \longrightarrow F_0(B) \longrightarrow 0) =$$

$$= (0 \longrightarrow UA \longrightarrow B_1 \longrightarrow B \longrightarrow 0),$$

Here $\mathcal{E}$ is the same as in 4.8, $cl(\mathcal{E}) \in Ext^1_{\mathbb{B}}(B, UA)$ is the characteristic element of $\mathcal{E}$, B_1 is defined by the pullback diagram

$$\begin{array}{ccc} B_1 & \longrightarrow & B \\ \downarrow & & \downarrow \\ UA_1 & \longrightarrow & UF_0(B), \end{array}$$

and $B \longrightarrow UF_0(B)$ is the counit of the adjunction c).

Let

$$\phi: Ext^1_{\mathbb{A}}(F_0B, -) \longrightarrow Ext^1_{\mathbb{B}}(B, U(-)),$$

$$\psi: Ext^1_{\mathbb{B}}(B, U(-)) \longrightarrow Hom_{\mathbb{A}}(F_1B, -)$$

be the natural transformations in 4.7. Then ϕ and ψ induce the natural transformations

$$\phi_*: \mathbb{A}\int Ext^1_{\mathbb{A}}(F_0B, -) \longrightarrow \mathbb{A}\int Ext^1_{\mathbb{B}}(B, U(-)),$$

$$\psi_*: \mathbb{A}\int Ext^1_{\mathbb{B}}(B, U(-)) \longrightarrow \mathbb{A}\int Hom_{\mathbb{A}}(F_1B, -).$$

It is clear that there exists a commutative diagram of weak linear extensions

4.9.
$$\begin{array}{ccccc} \mathrm{Hom} & \rightarrowtail & \mathcal{E}xt^1_{\mathbb{A}}(F_0B) & \longrightarrow & \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(F_0B,-) \\ \| & & \Phi \downarrow & & \phi_* \downarrow \\ \mathrm{Hom} & \rightarrowtail & \mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B) & \longrightarrow & \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)). \end{array}$$

The upper and lower extension determine the element

$$\mathcal{O}_{\mathbb{A}}(F_0B) \in H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(F_0B,-),\mathrm{Hom})$$

and

$$\mathcal{O}_{\mathbb{A},\mathbb{B}}(B) \in H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)),\mathrm{Hom}).$$

respectively. From commutativity of 4.9 follows that

4.10. $$\phi_{**}(\mathcal{O}_{\mathbb{A},\mathbb{B}}(B)) = \mathcal{O}_{\mathbb{A}}(F_0B),$$

Here $\phi_{**}: H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)),\mathrm{Hom}) \longrightarrow H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(F_0B,U(-)),\mathrm{Hom})$ is induced by ϕ_*. Let P_* be a projective resolution. If we take in 3.9

$$C_* = F_0\ P_*\ ,\ F = \mathrm{Hom}_{\mathbb{A}}(F_0B,-),$$

then 3.11 yulds

$$H^m(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-));\mathrm{Hom}\ ) \cong 0,\ \text{if } m \neq 2,$$
$$\cong [F_0B,F_0B]_{\mathbb{A}},\ \text{if } m = 2.$$

Now we compare this equation with 4.4. Then we conclude that ϕ_{**} is isomorphism and therefore we proved the following proposition

4.11. *Proposition. The characteristic element* $\theta_{\mathbb{A},\mathbb{B}}$ in

$$H^2(\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-));\mathrm{Hom}) \cong [F_0B,F_0B]_{\mathbb{A}}$$

corresponds via this isomorphism to $\mathrm{cl}(1_{F_0B})$.

4.12. *Proposition. Suppose that the projective homological dimension* p.d.$(F_0B) \leq 1$. *Then there exists the functor*

$$\Omega: \mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B) \longrightarrow \mathcal{E}xt^1_{\mathbb{A}}(F_0(B))$$

for which $\Omega\Phi = 1$. *Moreover the diagram*

$$\begin{array}{ccc} \mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B) & \xrightarrow{\ \Phi\ } & F_1(B) \downarrow \mathbb{A} \\ {\scriptstyle\Omega}\downarrow & & \downarrow{\scriptstyle p} \\ \mathcal{E}xt^1_{\mathbb{A}}(F_0(B)) & \xrightarrow{\ q\ } & \mathbb{A} \end{array}$$

is essentially a pullback with

$$p(F_1(B) \longrightarrow A) = A$$

and

$$q(0 \longrightarrow A \longrightarrow A_1 \longrightarrow F_0(B) \longrightarrow 0) = A.$$

4.13. *Remark.* We call a commutative square

$$\begin{array}{ccc} \mathbb{I} & \longrightarrow & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{D} & \longrightarrow & \mathbb{E} \end{array}$$

of categories and functors *essentially a pullback* iff the natural functor $\mathbb{I} \longrightarrow \mathbb{D}\times_{\mathbb{E}}\mathbb{C}$ is an equivalence of categories.

P r o o f of 4.12. It follows from 4.7 that there exists a natural transformation

$$\omega\colon \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)) \longrightarrow \mathrm{Ext}^1_{\mathbb{A}}(F_0B,-)$$

for which $\omega\phi = 1$ since $\mathrm{Ext}^2_{\mathbb{A}}(F_0B,-)= 0$ and since $\mathrm{Hom}_{\mathbb{A}}(F_0B,-)$ is a projective object in $\mathbb{A}$-*mod*. Then ω induces the natural transformation

$$\omega_*\colon \mathbb{A}\textstyle\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)) \longrightarrow \mathbb{A}\textstyle\int \mathrm{Ext}^1_{\mathbb{A}}(F_0B,-).$$

We showed above, that ϕ_{**} is an isomorphism. Therefore ω_{**} is an isomorphism too since $\omega_{**}\phi_{**} = 1$. Hence 4.10 implies that

$$\omega_*(\theta_{\mathbb{A}}(F_0(B)) = \theta_{\mathbb{A},\mathbb{B}}(B).$$

Therefore there exists $\Omega\colon \mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B) \longrightarrow \mathcal{E}xt^1_{\mathbb{A}}(F_0B)$ such that the diagram

$$\begin{array}{ccccc}
\mathrm{Hom} & \longrightarrow & \mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B) & \longrightarrow & \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)) \\
\Vert & & \Big\downarrow \Omega & & \Big\downarrow \omega \\
\mathrm{Hom} & \longrightarrow & \mathcal{E}xt^1_{\mathbb{A}}(F_0(B)) & \longrightarrow & \mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(F_0B,-)).
\end{array}$$

commutes and hence the second square in this diagram is essentially a pullback. This completes the proof of the proposition since in fact the diagram

$$\begin{array}{ccc}
\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{B}}(B,U(-)) & \xrightarrow{\ \psi_*\ } & \mathbb{A}\int \mathrm{Hom}_{\mathbb{A}}(F_1B,-) \\
\omega_* \downarrow & & \downarrow p \\
\mathbb{A}\int \mathrm{Ext}^1_{\mathbb{A}}(F_0B,-) & \xrightarrow{\ q\ } & \mathbb{A}
\end{array}$$

is pullback and $\mathbb{A}\int \mathrm{Hom}_{\mathbb{A}}(F_1B,-) = H_1B\downarrow\mathbb{A}$.

4.14. *Remark*. The conditions of proposition 4.12 are satisfied if

$$\mathbb{A} = Ab\ ,\ \ \mathbb{B} = G\text{-}mod,\ \ F_0 = H_0(G,-),$$

where G is an arbitrary group. In this case objects of the category $\mathcal{E}xt^1_{\mathbb{A},\mathbb{B}}(B)$ are called G-*central extensions* [6].

Let G be a group and $\mathcal{C}ext(G)$ be the category of central extensions of G. The study of this category is analogous to 4.6. The role of 4.7 is played by the well known universal coefficient formula

$$0 \longrightarrow \mathrm{Ext}(G_{ab},-) \xrightarrow{\ \phi\ } H^2(G,-) \xrightarrow{\ \psi\ } \mathrm{Hom}(H_2G,-) \longrightarrow 0.$$

The arguments in the proof of 4.12 are still true and hence we obtain the following proposition.

4.15. *Proposition. Let*

$$\Psi: \mathcal{C}ext(G) \longrightarrow H_2G \downarrow Ab,$$

$$p:\ H_2G \downarrow Ab \longrightarrow Ab,$$

$$q: \mathcal{E}xt(G_{ab}) \longrightarrow Ab\ ,$$

$$\Phi: \mathcal{E}xt(G_{ab}) \longrightarrow \mathcal{C}ext(G),$$

be functors defined as follows

$$\Psi(\mathfrak{X}) = \psi(cl(\mathfrak{X})),$$

$$p(H_2G \longrightarrow A) = A,$$

$$q(0 \longrightarrow A \longrightarrow X \longrightarrow G_{ab} \longrightarrow 0) = A,$$

and Φ *is given by the pullback along the natural morphism*

$$G \longrightarrow G_{ab}.$$

Then there exist a functor $\Omega: \mathcal{C}ext(G) \longrightarrow \mathcal{E}xt(G_{ab})$, *such that* $\Phi\Omega = 1$ *and such that the following diagram is essentially a pullback*

$$\begin{array}{ccc} \mathcal{C}ext(G) & \xrightarrow{\Psi} & H_2G{\downarrow}Ab \\ \Omega \downarrow & & \downarrow p \\ \mathcal{E}xt\ (G_{ab}) & \xrightarrow{q} & Ab. \end{array}$$

For a different description of the category $\mathcal{C}ext(G)$ see [11].

4.16. <u>The category of abelian extensions of groups</u>. Let G be a group and $\mathcal{A}ext(G)$ be the category of abelian extensions of G. Objects of this category are exact sequences of groups

$$\mathfrak{X} = 0 \longrightarrow A \longrightarrow B \longrightarrow G \longrightarrow 1,$$

where A is abelian. As in 4.6 we have the weak linear extension

4.17. $$\mathbf{Der} \longmapsto \mathcal{A}ext(G) \longrightarrow G\text{-}mod\textstyle\int H^2(G,-),$$

where

$$\mathbf{Der}:\ G\text{-}mod\textstyle\int H^2(G,-) \longrightarrow Ab$$

is defined by

$$\mathbf{Der}(A,x) = \mathrm{Der}(G,A).$$

4.18. *Proposition. If* G *is a finite group then*

$$H^m(G\text{-}\mathit{mod}\int H^2(G,-);\mathbf{Der}) \cong \mathbb{Z}/\,|G|, \text{ if } m = 2,$$
$$\cong 0, \text{ if } m \neq 2$$

and the characteristic element $\theta^a(G)$ *of* 4.17 *is a generator of this group.*

P r o o f. Let $I(G) = \mathrm{Ker}(\mathbb{Z}[G] \longrightarrow \mathbb{Z})$ be the augmentation ideal. It is well known that

$$H^2(G,-) \cong \mathrm{Ext}^1_G(I(G),-),\ \mathrm{Der}(G,-) \cong \mathrm{Hom}_G(I(G),-),$$
$$\phi: \mathcal{E}xt^1_G(I(G)) \cong \mathcal{L}ext(G),$$

where

$$\phi(0 \longrightarrow M \longrightarrow N \xrightarrow{\sigma} I(G) \longrightarrow 0) =$$
$$= (0 \longrightarrow M \longrightarrow X \longrightarrow G \longrightarrow 1),$$

Here $X = \{(x,g) \in X \times G,\ \sigma x = g - 1\}$ is a group with theoperation $(x,g)(y,s) = (x + gy,\ gs)$, $x,y \in X$, $g,s \in G$. Hence the weak linear extensions 4.17 is isomorphic to 4.2 in case $B = I(G)$, $\mathbb{A} = G\text{-}\mathit{mod}$. Therefore 4.4 implies that

$$H^m(G\text{-}\mathit{mod}\int H^2(G,-),\mathbf{Der}) \cong [I(G),I(G)]_G, \text{ if } m = 2,$$
$$\cong 0, \text{ if } m \neq 2$$

and $\theta^a(G) = \mathrm{cl}(1_{I(G)})$. Let

$$0 \longrightarrow Q \longrightarrow P \longrightarrow I(G) \longrightarrow 0$$

be an exact sequence of G-modules with projective P. Then

$$\mathrm{Hom}_G(I(G),P) \longrightarrow \mathrm{Hom}_G(I(G),I(G)) \longrightarrow \mathrm{Ext}^1_G(I(G),Q) \longrightarrow \mathrm{Ext}^1_G(I(G)(P)$$

is an exact sequence. Since $|G|<\infty$ we get

$$0 = H^m(G,P) = \mathrm{Ext}^m_G(I(G),P),$$

$m>0$ [5]. Therefore

$$[I(G),I(G)]_G \cong \mathrm{Ext}^1_G(I(G),Q) \cong H^2(G,Q) \cong$$
$$\cong H^1(G,I(G)) \cong \mathrm{Ext}^1_G(\mathbb{Z},I(G)).$$

From the exact sequence

$$0 \longrightarrow I(G) \longrightarrow \mathbb{Z}[G] \longrightarrow \mathbb{Z} \longrightarrow 0$$

follows that the sequences

$$\mathrm{Hom}_G(\mathbb{Z}[G],I(G)) \longrightarrow \mathrm{Hom}_G(I(G),I(G)) \xrightarrow{\delta} \mathrm{Ext}^1_G(\mathbb{Z},I(G)) \longrightarrow \ldots,$$

$$\begin{array}{ccccccc} \mathrm{Hom}_G(\mathbb{Z},\mathbb{Z}[G]) & \longrightarrow & \mathrm{Hom}_G(\mathbb{Z},\mathbb{Z}) & \xrightarrow{\partial} & \mathrm{Ext}^1_G(\mathbb{Z},I(G)) & \longrightarrow & 0 \\ \| & & \| & & \| & & \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}/|G| & & \end{array}$$

are exact. By [4, Cor.1 of prop.5 in §6.7] we have

$$\delta(1_{I(G)}) = -\partial(1_{\mathbb{Z}}).,$$

Hence $\mathcal{O}^a(G) = cl(1_{I(G)}) = \delta(1_{I(G)})$ is a generator of $[I(G),I(G)]_G$.

4.19. <u>The category of Eilenberg-MacLane fibrations.</u> Now we consider the category K^n_X and the element $\mathcal{O}^n(X)\in H^2(K^{n+1}_X,\mathbb{H}^n)$ (see §1). We assume that $X = BG_*$, where G_* is a component wise free simplicial group. Let $\pi = \pi_0(G_*) = \pi_1(BG_*)$. For every G_*-module A_* we denote $Ch(A_* \otimes_{G_*} \mathbb{Z}\pi)$ by $\mathcal{F}(A_*)$. Let $I[G_*]\in Ob(G_*\text{-}mod)$ be the augmentation ideal of G_* and let $\hat{C}_* = \mathcal{F}(I[G_*])$. We recall that

$$H^m(BG_*,M) = H^{m-1}\mathrm{Hom}_\pi(\hat{C}_*,M),\ M\in Ob(\pi\text{-}mod),\ m\geq 2,$$

(see appendix). Hence from 3.11 follows

$$\begin{aligned} H^m(K_X^{n+1},\mathbb{H}^n) &= 0,\ m\neq 0,2, \\ &= \mathrm{Coker}(H^n(BG_*,\hat{C}_{n-1})\longrightarrow H^n(BG_*,\hat{C}_{n-1}/d\hat{C}_n),\ m=2. \end{aligned}$$

It is clear that the natural projection $\hat{C}_{n-1}\longrightarrow \hat{C}_{n-1}/d\hat{C}_n$ is cocycle in $\mathrm{Hom}_\pi(\hat{C}_*,\hat{C}_{n-1}/d\hat{C}_n)$. We denote by i_n its class in the cohomology $H^n(BG_*,\hat{C}_{n-1}/d\hat{C}_n)$.

4.20. **Theorem.** *There exists an exact sequence*

$$H^n(BG_*,\hat{C}_{n-1})\longrightarrow H^n(BG_*,\hat{C}_{n-1}/d\hat{C}_n)\xrightarrow{\ \xi\ } H^2(K_X^{n+1},\mathbb{H}^n)\longrightarrow 0,$$

and in the group $H^2(K_X^{n+1},\mathbb{H}^n)$ *holds* $\xi(i_n) = \theta^n(BG_*)$.

4.21. *Remark.* Let $\Sigma\hat{C}_*$ be the chain complex with

$$(\Sigma\hat{C}_*)_0 = \mathbb{Z},\ (\Sigma\hat{C}_*)_{n+1} = \hat{C}_n,\ d_0^\Sigma = 0,\ d_{n+1}^\Sigma = (-1)^n d^{\hat{C}},$$

then there exists a π-equivariant homotopy equivalence between $\Sigma\hat{C}_*$ and $\hat{C}_*(X)$. Here $\hat{C}_*(X)$ is the cellular chain complex of universal covering of X.

P r o o f of 4.20. The first part of the theorem is proved above. For any abelian group A we denote by K(A,n) the "standard Eilenberg-MacLane space", this is the simplicial abelian group whose normalization satisfied

$$(NK(A,n))_i = 0, \ i \neq n,$$
$$= A, \ i = n.$$

If A is a π-module then $K(A,n)$ is considered as a G_*-module via the natural augmentation $G_* \longrightarrow \pi$. Then we have the natural functor

$$K(-,n): \pi\text{-}mod \longrightarrow G_*\text{-}mod.$$

This functor has a left adjoint which carries $A_* \in G_*\text{-}mod$ to

$$\mathcal{F}_n(A_*)/d\mathcal{F}_{n+1}(A_*).$$

If we take in 4.6

$$\mathbb{A} = \pi\text{-}mod, \ \mathbb{B} = G_*\text{-}mod, \ U = K(-,n-1), \ B = I[G_*],$$

we obtain the weak linear extension

4.22. $\mathbf{Hom} + \longrightarrow \mathcal{E}xt^1_{\pi,G_*}(I[G_*]) \longrightarrow \pi\text{-}mod \int Ext^1_{G_*\text{-}mod}(I[G_*], K(-n-1)).$

By A1(see appendix below) $\mathcal{E}xt^1_{\pi,G_*}(I[G_*])$ is equivalent to the category of short exact sequences of simplicial groups of type

$$0 \longrightarrow K(A,n-1) \longrightarrow X \longrightarrow G_* \longrightarrow 1,$$

A ranges in the category of π-modules and

$$\mathbf{Hom} : \pi\text{-}mod \int Ext^1_{G_*\text{-}mod}(I[G_*], K(-,n-1)) \longrightarrow Ab,$$

is the functor defined by $\mathbf{Hom}(A,x) = Hom_\pi(\hat{C}_{n-1}/d\hat{C}_n, A)$.

By 4.11 follows that the characteristic element $\theta_{\pi,G_*}(I[G_*]$ of the weak linear extension 4.22 in

$$H^2(\pi\text{-}mod \int \mathrm{Ext}^1_{G_*\text{-mod}}(I[G_*], K(-,n-1)) \cong$$

$$\cong [\ \hat{C}_{n-1}/d\hat{C}_n,\ \hat{C}_{n-1}/\ d\hat{C}_n]_\pi$$

corresponds to $cl(1_{\hat{C}_{n-1}/d\hat{C}_n})$. By theorem A2 below we have a canonical isomorphism

$$\mathrm{Ext}^1_{G_*\text{-mod}}(I[G_*],\ K(-,n-1) \cong H^{n+1}(BG_*,-).$$

Let $B: \mathcal{E}xt^1_{\pi,G_*}(I[G_*]) \longrightarrow K^n_{BG_*}$ be the functor induced by the classifying space functor. Then the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}\ + \longrightarrow & \mathcal{E}xt^1_{\pi,G_*}(I[G_*]) & \longrightarrow & \pi\text{-}mod \int \mathrm{Ext}^1_{G_*\text{-mod}}(IG_*,K(-,n-1)) \\
\downarrow \beta & \downarrow B & & \| \\
H^n\ + \longrightarrow & K^n_{BG_*} & \longrightarrow & K^{n+1}_{BG_*} ,
\end{array}$$

is commutative. Here β is induced by B. In particular $\mathcal{O}^n(BG_*) = \beta_*(\mathcal{O}_{\pi,G_*}(I[G_*]))$. From 3.11 follows that the following diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_\pi(\hat{C}_{n-1}/d\hat{C}_n, \hat{C}_{n-1}/d\hat{C}_n) & \xrightarrow{cl} & H^2(K^{n+1}_{BG_*}, \mathrm{Hom}) & \longrightarrow & 0 \\
\downarrow & & \downarrow \beta_* & & \\
H^n(BG_*, \hat{C}_{n-1}/d\hat{C}_n) & \xrightarrow{\xi} & H^2(K^{n+1}_{BG_*}, H^n) & \longrightarrow & 0
\end{array}$$

is commutative. Hence

$$\mathcal{O}^n(B) = \beta_*(\mathcal{O}_{\pi,G_*}(IG_*)) = \beta_*(cl(1_{\hat{C}_{n-1}/d\hat{C}_n}) = \xi(1_n).$$

4.23. *Corollary. If* $X = K(\pi,1)$ *with* π *finite, then*

$$H^2(\mathbb{K}_X^{n+1},\mathbb{H}^n) \cong \mathbb{Z}/|\pi|$$

and $O^n(X)$ *is generator of this group.*

In this case $\hat{C}_* \longrightarrow I[\pi]$ is a projective resolution and $H^*(X,-) = H^*(\pi,-)$. It is well known that for finite π one has $\hat{H}^*(\pi,\hat{C}_*) = 0$, where $\hat{H}^*(\pi,-)$ denotes the Tate cohomology [5]. Therefore

$$H^2(\mathbb{K}_X^{n+1}, \mathbb{H}^n) \cong H^n(\pi,\hat{C}_{n-1}/d\hat{C}_n) \cong$$

$$\cong H^{n-1}(\pi,\hat{C}_{n-2}/d\hat{C}_{n-1}) \cong \ldots \hat{H}^0(\pi,\mathbb{Z}) \cong \mathbb{Z}/|\pi|.$$

4.24. The category of orientable Eilenberg-MacLane fibrations. Let $n \geq 1$. A fibration $K(A,n) \longrightarrow E \longrightarrow X$ is called orientable iff A is abelian and if $\pi = \pi_1 X$ acts trivially on A. Let $\mathbb{K}_{X,o}^n$ be the full subcategory of $\mathbb{K}_X^n$ whose objects are orientable fibrations. The restriction of 1.1 gives the following weak linear extension

4.25. $$\mathbb{H}^n + \longrightarrow \mathbb{K}_{X,0}^n \longrightarrow Ab\int H^{n+1}(X,-).$$

The study of this weak linear extensions is similar to 4.6 and the proof of the following proposition is analogous to 4.12.

4.26. *Proposition. There exists an isomorphism*

$$H^2(Ab\int H^{n+1}(X,-),\mathbb{H}^n) \cong [H_nX, H_nX]_{Ab}$$

which carries the class of 4.27 *to* $cl(1_{H_nX})$ *Let*

$$\mathcal{E}xt(H_nX) \oplus Ext(H_{n-1}X,-)$$

be the following category. Objects are the same as in $\mathcal{E}xt(H_nX)$ *and morphisms from*

$$\mathfrak{E} = 0 \longrightarrow A \longrightarrow X \longrightarrow H_nX \longrightarrow 0$$

to $\mathfrak{E}'$ *are triples* (f,g,h) *where* $(f,g):\mathfrak{E}\longrightarrow\mathfrak{E}'$ *is a morphism in* $\mathcal{E}xt(H_nX)$ *and* $a\in Ext(H_{n-1}X,A')$; *the composition is defined by*

$$(f',g',a')(f,g,a) = (f'f,g'g,a'+f'_*(a)).$$

Then the category $K^n_{X,0}$ *is equivalent to the pullback of the following diagram*

$$\begin{array}{ccc} & & H_{n+1}X\!\downarrow Ab \\ & & \downarrow p \\ \mathcal{E}xt(H_nX)\oplus Ext(H_{n-1}X,-) & \xrightarrow{\quad q \quad} & Ab \end{array}$$

where $p(H_{n+1}X \longrightarrow A) = A, q(\mathfrak{E}) = A, q(f,g,a) = f.$

Appendix.

In this appendix we prove theorem A2 below, which plays a crucial role in the proof of 4.20.

Let $G:I \longrightarrow$ *Groups* be a functor from a small category I

to the category of groups and let A be a G-module. A crossed homomorphism from G to A is a natural transformation of set valued functors $f:G \longrightarrow A$ such that $f(i)$ is a usual crossed homomorphism for each $i \in Ob(\mathbb{I})$. Let $\mathbb{D}er(G,A)$ be the set of all crossed homomorphism from G to A. The following theorem isproved by Basistov [1] (see also [9] for analogous facts about algebra valued functors).

A1. **Theorem.** *Let* $G:\mathbb{I} \longrightarrow$ *Groups be an arbitrary functor and A be a G-module. Let* $P_* \longrightarrow G$ *be an augmented simplicial object in* $(Groups)^{\mathbb{I}}$ *such that* P_n *is a projective object in* $(Groups)^{\mathbb{I}}$ *and*

$$\pi_m(P_*(i)) \quad = 0, \quad m>0, \qquad = G(i), \quad m=0.$$

Then there exists a natural bijection

$$Ext^*_{G\text{-mod}}(I[G],A) \cong \pi^* \mathbb{D}er(P_*,A),$$

where $I[G]$ *denotes the augmentaation ideal of* G. *The left hand side denotes the cohomology groups of a cosimplicial abelian group with*

$$\mathbb{D}er(P_*,A)_m = \mathbb{D}er(P_m,A).$$

Moreover in dimension one has a bijection between these groups and the set of equivalence classes of all short exact sequences in $(Groups)^{\mathbb{I}}$:

$$0 \longrightarrow A \xrightarrow{\mu} X \xrightarrow{\sigma} G \longrightarrow 1,$$

such that for each $x \in X(1)$, $a \in A(1)$ *the following equality holds,*

$$\mu(\sigma(x)a) = x\mu(a)x^{-1}.$$

The main theorem of this appendix is

A2. Theorem. *Let* G_* *be a componentwise free simplicial group,* $\pi = \pi_0 G_*$ *and* A *be a* π*-module. Then there exists a natural isomorphisms*

$$\mathrm{Ext}^m_{G_*\text{-mod}}(I[G_*], K(A,n)) \cong H^{m+1+n}(BG_*,A), \ m>0,$$

where $K(A,n)$ *is a "standard Eilenberg - MacLane space" (see the proof of* 4.20).

Before we prove the theorem we recall some basic facts about $H^*(BG_*,A)$. It follows from the classical Eilenberg-Zilber-Cartier theorem [7] that $H^*(BG_*,A)$ is isomorphic to the homology of the total complex of the bicomplex $C^*(G_*,A)$. Its (m,k) component is $\mathrm{Maps}(G^m_k,A)$ and its horizontal coboundaries are defined from the theory of cohomology of groups and its vertical coboundaries are induced from G_*. In case G_* is component wise free the spectral sequence of this bicomplex is degenerat and we obtain the isomorphism

A3. $$H^{m+1}(BG_*,A) \cong \pi^m \mathrm{Der}(G_*,A), \ m>0.$$

We recall that the projective objects in the category of simplicial groups are retracts of sums of objects of the type $Fr\Delta[n]$, $n\geq 0$, whose m component is a free group generated by m simplexes of $\Delta[n]$. Therefore for any projective object P_* (in the category of simplicial groups) $\pi_0 P_*$ is a free group and $P_* \longrightarrow \pi_0 P_*$ is a contractible augmented simplicial group.

Also recall that the simplicial maps from G_* to $K(A,n)$ are isomorphic to n-dimensional cocycles [13]. Therefore one gets

A4. $\mathbb{D}er(G_*, K(A,n)) \cong Ker(Der(G_n, A) \xrightarrow{d} Der(G_{n+1}, A))$,

where $d = \sum(-1)^i \partial_i$.

P r o o f of A2. Let $P_{**} \longrightarrow G_*$ be a simplicial projective resolution in the category of simplicial groups. This means that P_{**} is a bisimplicial group such that

a) for every $m\geq 0$ P_{*m} is a projective object in the category of simplicial groups and

b) for each $m\geq 0$ one has $\pi_o P_{m*} \cong G_m$, $\pi_k P_{m*} = 0$.

Since P_{*m} is a projective object $\pi_0 P_{*m}$ is a free group and $\pi_k P_{*m} = 0$, $k>0$. It follows from Quillens spectral sequences for bisimplicial groups [15] that simplicial groups G_* and $m \longmapsto \pi_0 P_{*m}$ are weak homotopy equivalent.

Since $P_{*m} \longrightarrow \pi_0 P_{*m}$ is a weak homotopy equivalence with free

$\pi_0 P_{*m}$ we have (by A3)

$$\pi^k \mathrm{Der}(P_{*m}, A) \cong H^{k+1}(BP_{*m}, A) \cong H^{k+1}(B\pi_0 P_{*m}, A) = 0, \ k>0.$$

Therefore A4 implies that there exists an isomorphism

$$\mathbb{D}\mathrm{er}(P_{*m}, K(A,0)) \cong \mathrm{Der}(\pi_0 P_{*m}, A), \ m>0,$$

and an exact sequences

A5. $$0 \longrightarrow \mathbb{D}\mathrm{er}(P_{*m}, K(A,n-1)) \longrightarrow \mathrm{Der}(P_{n-1,m}, A) \longrightarrow$$
$$\longrightarrow \mathbb{D}\mathrm{er}(P_{*m}, K(A,n)) \longrightarrow 0.$$

It follows from A3 and b) that

$$\pi^k \mathrm{Der}(P_{m*}, A) \cong H^{k+1}(BP_{m*}, A) \cong H^{k+1}(BG_m, A) = 0, \ k>0,$$

since G_m is free. Therefore A5 shows that for k>0 we have natural isomorphisms:

$$\pi^k \mathbb{D}\mathrm{er}((P_*)_*, K(A,n)) \cong \pi^{k+1} \mathbb{D}\mathrm{er}((P_*)_*, K(A,n-1)) \cong$$
$$\cong \ldots \cong \pi^{k+n} \mathbb{D}\mathrm{er}((P_*)_*, K(A,0)) \cong \pi^{k+n} \mathrm{Der}(m \longmapsto \pi_0 P_{*m}, A) \cong$$
$$\cong H^{k+n+1}(B(m \longmapsto \pi_0 P_{*m}), A) \cong H^{k+n+1}(BG_*, A).$$

Hence theorem A2 is an immediate consequence of A1.

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