More on Paperfolding

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It is a common knowledge that folding a sheet of paper yields a straight line. We start our discussion of paperfolding with a mathematical explanation of this phenomenon. The model for a paper sheet is a piece of the plane; folding is an isometry of the part of the plane on one side of the fold to another, the fold being the curve of fixed points of this isometry. The statement is that this curve is straight, that is, has zero curvature.

fig. 1

If not, consider an arc γ of the fold with nonvanishing curvature. Let γ_+ be the curve at (small) distance ε from γ on the concave side, and γ_- – that on the convex side. Then

$$\textit{length } \gamma_+ > \textit{length } \gamma > \textit{length } \gamma_-$$

(the difference being of order $\varepsilon \cdot \textit{length } \gamma \cdot \textit{curvature } \gamma$). On the other hand, the isometry takes γ_+ to γ_-, so $\textit{length } \gamma_+ = \textit{length } \gamma_-$. This is a contradiction.

fig. 2

In spite of what has just been said, *one can fold paper along an arbitrary smooth curve*! The reader is invited to perform an experiment: draw a curve on a sheet of paper and slightly fold the paper along the curve (a word of practical advice: press hard when drawing the curve. It also helps to cut a neighborhood of the curve not to mess with too large a sheet. A more serious reason for restricting to a neighborhood is that this way one avoids selfintersections of the sheets, unavoidable otherwise.) The result looks somewhat like Fig. 3(a):

fig. 3

One may even start with a closed curve drawn on paper. To be able to fold one has to cut a hole inside the curve; the result is shown below:

fig. 4

It goes without saying that the argument in the opening paragraphs of this article does not contradict the possibility to fold along a curve: the two sheets in Fig. 3(a) meet at a nonzero angle. To fix terminology call the curve drawn on paper the *fold* and denote it by Γ; call the curve in space obtained by folding along Γ the *ridge* and denote it by γ. The above described experiments and dozens of similar ones that kept us busy lately reveal the following:

1). It is possible to start with an arbitrary smooth fold and obtain an arbitrary ridge provided the ridge is "more curved" than the fold.

2) If the ridge is only slightly more curved than the fold, then the neighborhood of the fold to be folded should be taken very thin, at least from the side of the convex domain of the plane bounded by the fold.

3). If the fold has an inflection point (i.e. point of zero curvature) then the corresponding point of the ridge is also an inflection point (notice that unlike plane curves a generic space curve does not have inflection points at all).

4). If the fold is a closed strictly convex curve then the ridge has a nonzero torsion, that is, does not lie in one plane.

5). If the fold is a nonclosed arc the folded paper tends to occupy such a position that the ridge lies in a plane, and the angle made by the two sheets is constant along the ridge.

What follows is an attempt to explain these experimental observations. A surface obtained by bending, without folding, a paper sheet is a *developable* surface, that is, a surface locally isometric to the plane (one cannot stretch paper!). The theory of such surfaces is due to Euler; its main result is as follows. A developable surface is a ruled surface, i.e., it consists of a one-parameter family of straight lines called rulings. These lines are not arbitrary: they are tangent to a certain space curve called the edge of regression (this description does not include two special cases, cylinders and cones, which are also developable surfaces). The tangent planes to a developable surface along every ruling coincide: one can put not only a knitting needle on such a surface but also a ruler. Thus a developable surface is the envelop of a one-parameter family of planes.

fig. 5

Consider Fig. 3(b). One sees two developable surfaces intersecting along a space curve γ. Unfolding either of the surfaces to the plane transforms γ to the same plane curve Γ. Reverse the situation and pose the following question: given a plane curve Γ, a space curve γ and an isometry $f\colon \Gamma \to \gamma$, is it possible to extend f to an isometric embedding of a plane neighborhood of Γ to space? Said differently, can one bend a sheet of paper with a curve Γ drawn on it so that Γ bends to a given space curve γ?

THEOREM. *Assume that for every $x \in \Gamma$ the absolute value of the curvature of γ at point $f(x)$ is greater than that of Γ at x. Then there exist exactly two extensions of f to isometric embeddings of a plane neighborhood of Γ to space.*

These two embedded surfaces are the sheets intersecting along the ridge in Fig. 3(b). Extending the sheets beyond the ridge one obtains another configuration of sheets that meet along γ. Thus there are exactly two ways to fold paper along Γ to produce the curve γ. This explains and extends the first of the above made observations.

In a particular case when γ lies in a plane one of the sheets is obtained from another by reflection in this plane. In the general case of a nonplanar curve γ the tangent planes of the two sheets are symmetric with respect to the osculating plane of γ at every its point.

PROOF OF THEOREM. Parametrize the curves γ and Γ by the arclength parameter t so that $\gamma(t) = f(\Gamma(t))$. Let the desired developable surface S make the angle $\alpha(t)$ with the osculating plane of the curve $\gamma(t)$ (well defined since, by assumption, the curvature of γ never vanishes). Denote by $k(t)$ the curvature of the space curve γ and by $K(t)$ that of the plane curve Γ. The geodesic curvature vector of γ in S is the projection of the curvature vector of γ in space onto S; thus the geodesic curvature of γ equals $k(t)\cos\alpha(t)$. Since an isometry preserves the geodesic curvature of curves, $k\cos\alpha = K$. This equation uniquely determines the nonvanishing function $\alpha(t)$ up to the substitution $\alpha \to \pi - \alpha$. To construct the developable surface S by the function $\alpha(t)$ consider the plane through point $\gamma(t)$ that makes the angle $\alpha(t)$ with the osculating plane of γ. Such planes constitute a one-parameter family, and according to the above described general theory, their envelop is a developable surface.

REMARKS. 1. The above theorem is hardly new – cf. L. Bianchi, Vorlesungen uber Differentialgeometrie, Leipzig 1910; W. Blaschke, Vorlesungen und Geometrische Grundlagen Einsteins Relativitatstheorie, Berlin 1930.

2. A direct computation involving the Frenet formulas for γ (which we omit) makes it possible to find the angle $\beta(t)$ made by the rulings $l(t)$ with the curve $\gamma(t)$ in terms of the torsion $\kappa(t)$ of γ:

$$\cot\beta(t) = \frac{\alpha'(t) - \kappa(t)}{k(t)\sin\alpha(t)}.$$

For the two developable surfaces corresponding to the angles $\alpha(t)$ and $\pi - \alpha(t)$ one has:

$$\cot\beta_1(t) + \cot\beta_2(t) = -\frac{2\kappa(t)}{k(t)\sin\alpha(t)}.$$

Therefore the ridge γ is a plane curve (i.e., $\kappa = 0$) iff $\beta_1 + \beta_2 = \pi$. In this case unfolding the two sheets on the plane yields the straight rulings that extend each other on both sides of the fold Γ.

fig. 6

The reader with a taste for further experimentation may find the following one of interest. Start with a fold Γ and paste (with scotch tape) a number of pins on its both sides. In this way one prescribes the angles $\beta_1(t)$ and $\beta_2(t)$. Then fold along Γ.

fig. 7

Remark 2 may be used for the explanation of the second of our experimental observations. Namely, since $\cos\alpha(t) = \dfrac{K(t)}{k(t)}$, the above formula for $\cot\beta(t)$ may be rewritten as

$$\cot\beta(t) = \frac{\alpha'(t) - \kappa(t)}{\sqrt{k(t)^2 - K(t)^2}}.$$

If $\kappa(t)$ is bounded from below, and $K(t)$ is C^1 close to $k(t)$, then $\alpha'(t)$ is small, and $\cot\beta(t)$ is large; hence the angle $\beta(t)$ is small. It is clear that straight lines, crossing the boundary of a convex domain in the plane under small angles cross each other near the boundary (actually, they cannot penetrate deep in the domain):

fig. 8

But the rulings of a non-selfintersecting developable surface do not cross each other. Hence, to avoid crossings we need to make the neighborhood of the fold thin. The limit case of the last observation is particularly interesting. Suppose that $K = k$. Then the isometry between the fold and the ridge cannot be extended into the convex domain bounded by the fold at all. It can be extended into the concave domain, and we get a developable surface, for which *the edge is the edge of regression.* Indeed, the above formula for $\cot\beta(t)$ gives $\cot\beta(t) = \infty$; hence $\beta(t) = 0$, and the rulings of the surface are all tangent to the ridge. Of course, in this way we get only one of the two pieces of the surface, cut along the edge of regression. The other piece may be made of another copy of the same concave domain. The difference between the two pieces is that for each tangent to the boundary of our concave domain, divided into two halves by the tangency point, one half is straight on one of the pieces and the other half is straight on the other piece. The image of the whole tangent on each piece is a curve, half of which is straight and half of which is curved:

fig. 9

Here is how the union of the two pieces with the images of the two tangents looks like:

fig.10

As a by-product of these observations we learn how to make a paper model of a developable surface with a prescribed edge of regression (without inflection points). To do this we should draw a planar curve whose curvature is precisely the same as that of the intended edge of regression; this drawing should be made in two copies on two separate sheets of paper. Then we cut the sheets along the curves and take the concave portions of both. After this we bend the two pieces to make their edges fit into the given spatial curve. This may be done in two different ways, and we must bend our (identical) pieces into different surfaces; this two surfaces compose the developable surface we are constructing. Since the "angle" between the two pieces should be 0, it may be useful to glue the two pieces before bending along a very thin neighborhood of the edges. But be aware, that this bending is not even a C^2 mapping (this is why we used quotation marks for the word "angle" above), and the paper will be resistant to this construction. Pins, attached to the two pieces tangentially to the edges as shown in the figure may help:

fig. 11

Back to the above formulated experimental observations. The first two have been explained, proceed to the third one. Let $\Gamma(t_0)$ be a nondegenerate inflection point, so the fold looks like a cubic parabola near this point.

fig. 12

Then $K(t_0) = 0$ and, according to the already familiar formula $k \cos \alpha = K$, either $\alpha(t_0) = \pi/2$ or $k(t_0) = 0$. We want to show that the latter possibility holds. Suppose not; then both sheets are perpendicular to the osculating plane of γ at point $\gamma(t_0)$ and, therefore, coincide. Moreover, if $k(t_0) \neq 0$ then the projection of the curvature vector of the space curve γ onto each sheet is the vector of the geodesic curvature therein. This vector lies on one side of γ on the surface at points $\gamma(t_0 - \varepsilon)$ just before the inflection point and on the other side at points $\gamma(t_0 + \varepsilon)$ just after it. Therefore the function $\alpha(t) - \pi/2$ changes sign at $t = t_0$. This means that the two sheets pass through each other at $t = t_0$. Possible in the class of immersions, this cannot happen with real paper. Thus $k(t_0) = 0$, that is, the ridge has an inflection point.

Next, consider the fourth observation. Assume that both γ and Γ are closed plane curves and Γ is strictly convex. The relation between the curvatures still holds: $k \cos \alpha = K$, and $K(t)$ does not vanish. Hence $k(t) \geq K(t)$ for all t and $\int k(t) > \int K(t)$ unless $\alpha(t)$ identically vanishes. On the other hand, the integral curvature of a simple closed plane curve equals 2π, so the above integrals must be equal. This is a contradiction. It is interesting that if Γ is closed *nonconvex* curve one can nontrivially bend paper keeping Γ in the plane.

fig. 13

Finally we turn to the fifth observation. This one takes us into dangerous waters because its explanation requires further assumptions concerning elasticity properties of paper. A strip of paper resists twisting: being relaxed it tends to become flat. Consider a space curve $\gamma(t)$ parametrized by the arclength. Let S be a thin strip along γ and $v(t)$ the unit normal vector field to γ in S. Define the *twist* of S to be the length of the projection of the vector $v'(t)$ to the normal plane of $\gamma(t)$. Our assumption is that a paper strip tends to minimize its twist. Let the strip make the angle $\alpha(t)$ with the osculating plane of the curve $\gamma(t)$. Then a computation, similar to the one mentioned in Remark 2, gives the value of the twist $|\kappa(t) - \alpha'(t)|$. Folding paper one produces two strips along the ridge $\gamma(t)$, the angles being $\alpha(t)$ and $\pi - \alpha(t)$. The twists of these strips are equal to $|\kappa - \alpha'|$ and $|\kappa + \alpha'|$. Both quantities attain minimum if $\kappa(t) = 0$ and $\alpha(t)$ is constant. This appears to explain the fifth experimental observation.

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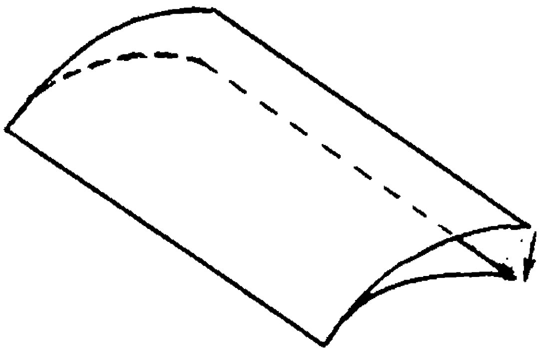

Fig. 1

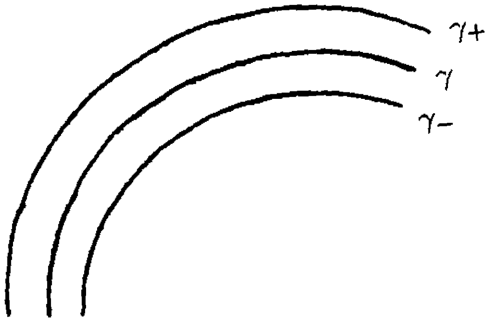

Fig. 2

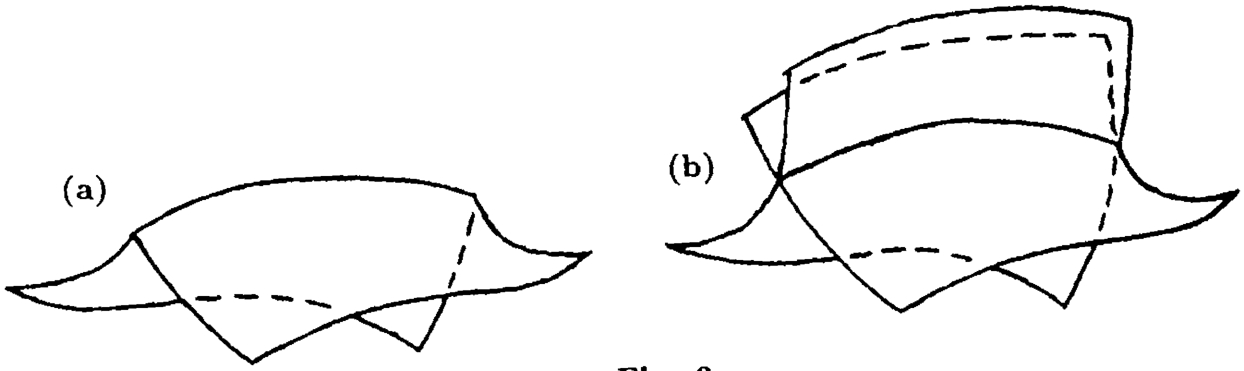

Fig. 3

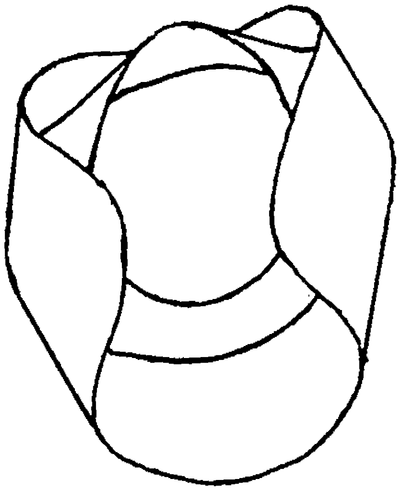

Fig. 4

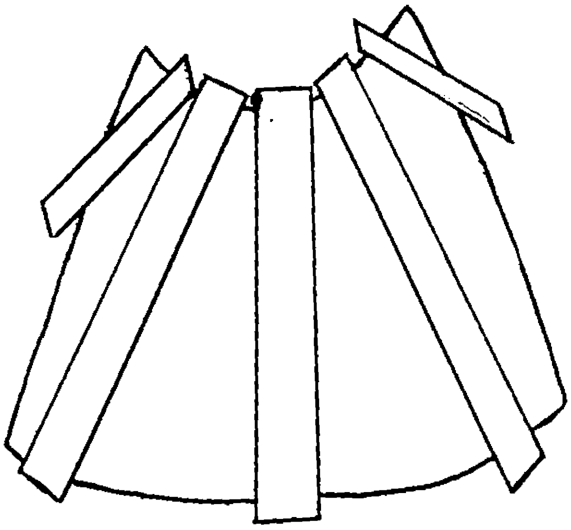

Fig. 5

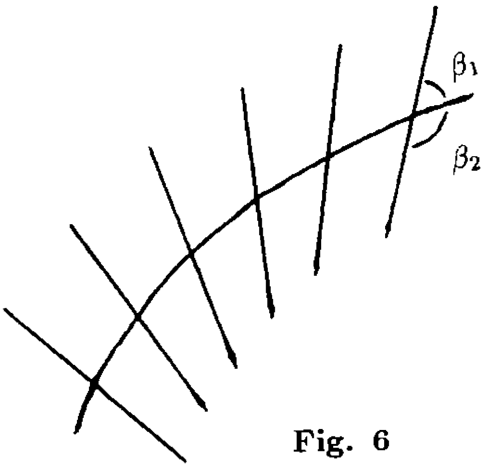

Fig. 6

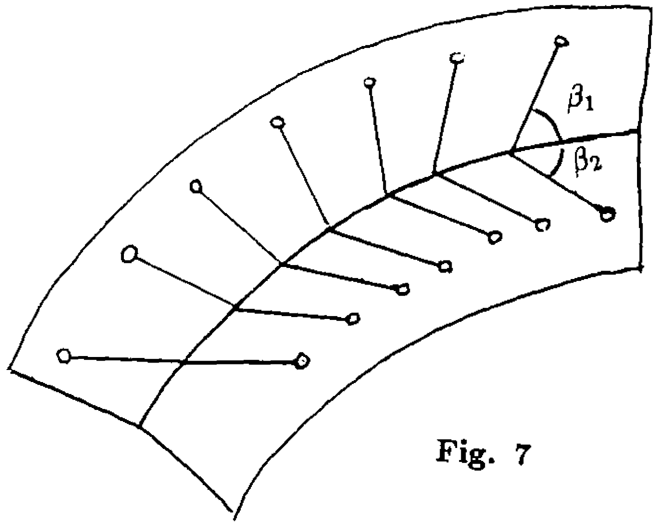

Fig. 7

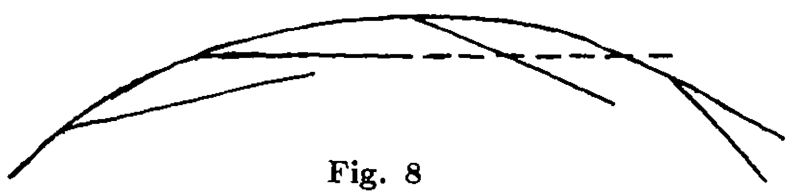

Fig. 8

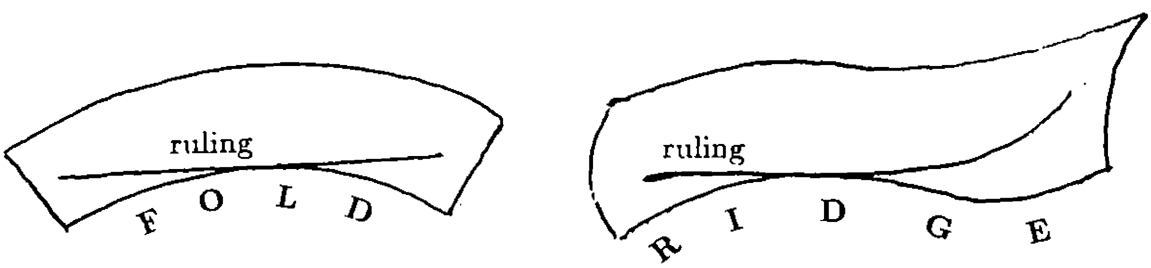

Fig. 9

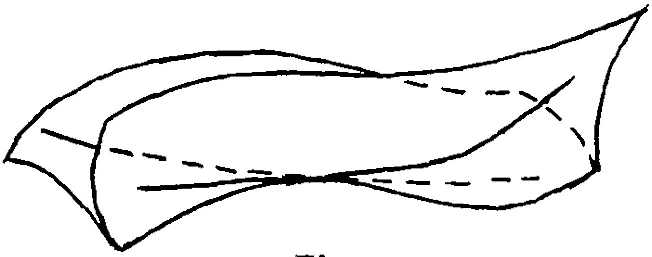

Fig. 10

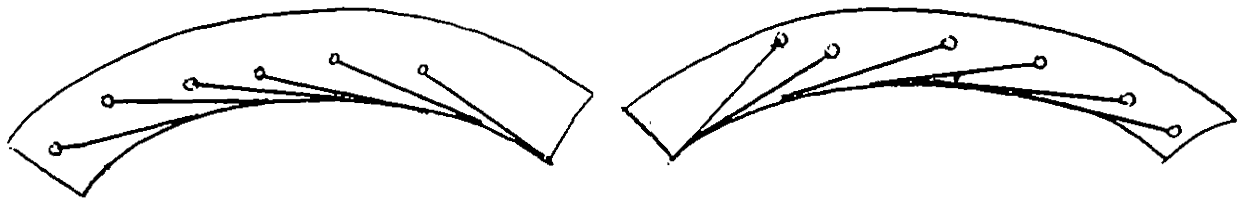

Fig. 11

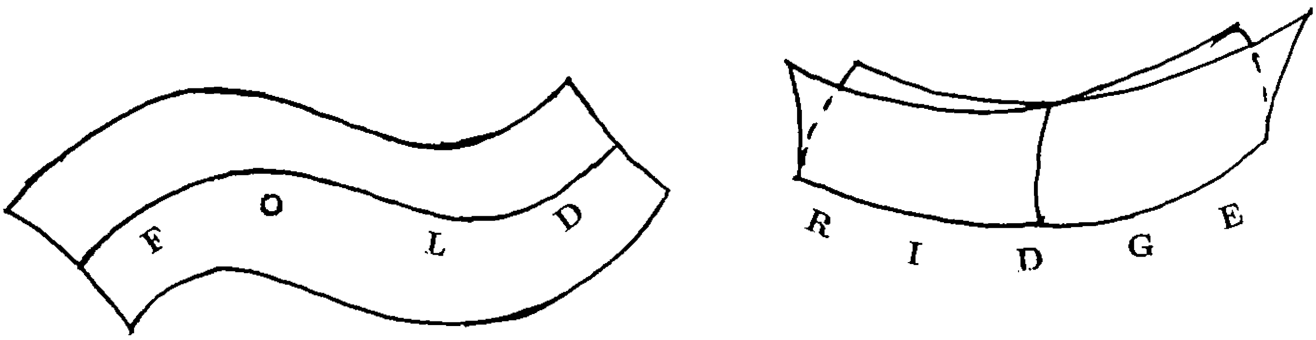

Fig. 12

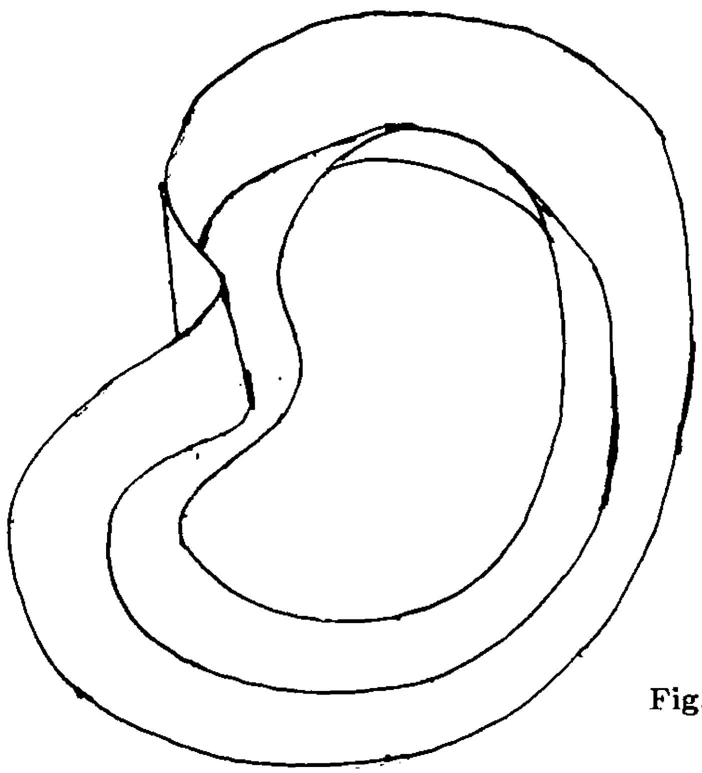

Fig. 13