The Metric Aspect of Noncommutative Geometry

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THE METRIC ASPECT OF NONCOMMUTATIVE GEOMETRY

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Most of the previous work on "noncommutative geometry" could more accurately be labeled as noncommutative differential topology, in that it deals with the homology of differential forms on noncommutative spaces (cyclic homology) and vector bundles on noncommutative spaces (K-theory) [Co1]. However, the essence of geometry has to do with the metric properties of spaces.

In this paper we shall begin to investigate metric properties of noncommutative spaces. First, we shall show that the metric data of differential geometry can be reformulated in operator theoretic terms. This will be done using a familiar differential operator, the Dirac operator D. We shall see how the metric structure on a Riemannian manifold, namely the geodesic distance

(1) $d(p,q)$ = Infimum of the length of paths γ from p to q,

can be recovered from the selfadjoint operator D, acting in the Hilbert space $\mathfrak{h}$ of L^2 spinors, together with the representation in $\mathfrak{h}$ of the algebra of functions on the manifold.

Our data of a noncommutative metric space will consist of a triple $(\mathcal{A}, \mathfrak{h}, D)$ of a Hilbert space $\mathfrak{h}$, an involutive algebra $\mathcal{A}$ of operators on $\mathfrak{h}$ and a selfadjoint unbounded operator D on $\mathfrak{h}$. This new notion of a metric space includes as examples the following list of spaces, besides Riemannian manifolds:

α) Finite spaces.
β) Spaces of non integral Hausdorff dimension.
γ) Group rings of discrete subgroups of Lie groups.
δ) Configuration spaces in supersymmetric quantum field theory.
ε) "Quantum" tori.

Our task will be to show that this new class of spaces still deserves the name of geometry. For this, we shall replace the tools of the differential and integral calculus by operator theoretic tools. We shall develop a differential calculus on noncommutative spaces which, on a Riemannian manifold, reproduces the calculus of differential forms. The main tool of integration will be a nonstandard trace on operators, the Dixmier trace. It will allow us to develop the analogue of the Yang Mills action. (As an indication that we have the correct mathematical notion of the noncommutative Yang Mills action, we shall give a very general lower bound of the action in terms of a topological quantity.)

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The main example of a space to which all these considerations will be applied is an extended Euclidean space time. We shall give a geometric interpretation of the experimentally confirmed (at least at present energies) model of particle physics, namely the standard model. Our geometric interpretation of this model is as a pure gauge theory, but on a space time $E' = E \times F$, the product of ordinary Euclidean space time by a finite space. The geometry of the finite space is specified by a pair $(\mathfrak{h}, D)$ as above, where $\mathfrak{h}$ is finite dimensional and the self adjoint operator D encodes the fermion masses and the Kobayashi-Maskawa mixing parameters of the standard model.

Our analysis is limited to the classical level, and does not adress at the moment the questions related to renormalization. Nevertheless our geometric interpretation of the standard model gives an indication that particle physics is saying something about the small-scale geometric structure of space time.

Most of the results of this paper appeared in [CL], and will appear as a chapter in [Co2].

1. RIEMANNIAN MANIFOLDS AND THE DIRAC OPERATOR.

Let X be a compact Riemannian spin manifold and $D = \partial_X$ the corresponding Dirac operator (cf [LM]). Thus D is an unbounded self adjoint operator acting on the Hilbert space $\mathfrak{h}$ of L^2 spinors on the manifold X.

To fix notation, we can write $D = \Sigma\, \gamma^j\, \nabla_j$, where the matrices $\{\gamma^j\}$ are skew-Hermitian. Then if f is a function on X, also regarded as a multiplication operator on $\mathfrak{h}$, we have $[D, f] = \Sigma\gamma^j\, \partial_j f$, an operator of Clifford multiplication which we shall denote by $\gamma(df)$.

We shall give 4 formulae below which show how to reconstruct the ***metric space*** (X, d) with d being the geodesic distance, the ***volume measure*** dv on X, the space of ***gauge potentials*** and finally the ***Yang-Mills*** action functional, from the following purely operator theoretic data :

$(\mathcal{A}, \mathfrak{h}, D)$,

where D is the Dirac operator on the Hilbert space $\mathfrak{h}$ and $\mathcal{A}$ is the abelian algebra of continuous functions on X.

By Gelfand's theorem, we can recover the compact topological space X from $\mathcal{A}$. Namely, a point p of X gives a $*$ homomorphism $\rho : \mathcal{A} \to \mathbb{C}$ by setting $\rho(a) = a(p)$ for all $a \in \mathcal{A}$. And conversely, any such homomorphism ρ is given by evaluation at some point p, so X can be identified with the space of all such homomorphisms.
This is somewhat qualitative information, and we now come to the first interesting formula, giving us a natural distance function.

Formula 1.
For any pair of points $p, q \in X$, their geodesic distance is given by:

(2) $$d(p,q) = \operatorname{Sup} \{ \,|a(p) - a(q)|\, ; a \in \mathcal{A} , \| [D,a] \| \leq 1 \}$$

The proof is straightforward, but is worth going through. The operator $[D,a]$, which is bounded iff a is Lipschitz, is given by the Clifford multiplication $\gamma(da)$ by the differential of a. This differential is a section of the cotangent bundle T^*X of X and one has :

$$\| [D,a] \| = \text{Essential Sup } \| da \| = \text{Lipschitz norm of } a$$

So in (2), we are imposing that $|\text{grad } a|$ is everywhere ≤ 1, and then upon expressing $a(p) - a(q)$ as a line integral, it follows that the right hand side of (2) is $\leq$ the geodesic distance $d(p,q)$. But fixing the point p and considering the function

$$a(q) = d(q,p),$$

one checks that a is Lipschitz with constant one, so that $\| [D,a] \| \leq 1$ and one gets the desired equality. Note that (2) is in essence dual to the original formula (1) in that instead of involving arcs, namely copies of $\mathbb{R}$ inside the manifold X, it involves ***functions***, i.e. maps from X to $\mathbb{R}$ (or $\mathbb{C}$).

This is an essential point for us, since in the case of discrete spaces or of noncommutative spaces X there may be no interesting arcs in X, but nevertheless there are plenty of ***functions,*** namely the elements $a \in \mathcal{A}$ of the defining algebra. We note rightaway that the right hand side of (2) makes sense in that general context, and technically defines a distance on the space of ***states*** of the C^* algebra A:

$$d(\varphi, \psi) = \text{Sup} \, \{ \, | \, \varphi(a) - \psi(a) \, | \, ; \| [D,a] \| \leq 1 \, \}.$$

We have now recovered from our original data $(\mathcal{A}, \mathfrak{h}, D)$ the metric space (X,d), with d being the geodesic distance. We still need tools of Riemannian geometry which are not immediately implied by the metric structure, the first being the measure given by the volume form :

$$f \mapsto \int_X f \, dv,$$

where in local coordinates, one has :

$$dv = (\det (g_{\mu\nu}))^{1/2} \, | \, dx^1 \wedge \ldots \wedge dx^n \, |$$

This takes us to our second formula, which is nothing but a restatement of the H. Weyl theorem about the asymptotic behaviour of eigenvalues of elliptic differential operators. It does, however, involve a new tool, ***the Dixmier trace*** Tr_ω which, unlike asymptotic expansions, will make sense in full generality, and will be the correct operator theoretic replacement for integration.

Formula 2.
For any $f \in \mathcal{A}$ one has :

(3) $$\int_X f \, dv = \text{const.}(d) \; \mathrm{Tr}_\omega (f \, | \, D \, |^{-d}), \qquad d = \dim X.$$

For the detailed definition and properties of the Dixmier trace Tr_ω we refer to [Di, Co2]. For the time being we can interpret the right hand side as the limit of the sequence :

$$\frac{1}{\text{Log}\, N} \sum_0^N \lambda_j$$

where the λ_j 's are the eigenvalues of the compact operator $f \, | \, D \, |^{-d}$, or equivalently, as the residue of the function

$$\zeta(s) = \text{Trace}\,(f \, | \, D \, |^{-sd}) \qquad \text{Re}\, s > 1$$

at the point $s = 1$.

For us, the crucial fact is that the Dixmier trace is defined for operators on ***any*** Hilbert space, and that properties of the integral $\int_X f \, dv$ become corollaries of the general properties of the Dixmier trace :

A) Positivity : $\mathrm{Tr}_\omega(T) \geq 0$ if T is a ***positive*** operator.

B) Finiteness : $Tr_\omega(T) < \infty$ if the characteristic values of T satisfy $\sum_0^N \mu_n(T) = O(\text{Log } N)$.

C) Covariance : $Tr_\omega(U T U^*) = Tr_\omega(T)$ for any unitary U

D) Vanishing : $Tr_\omega(T)$ vanishes if T has finite trace in the usual sense.

Property D is the counterpart of locality in our framework. It shows that the Dixmier trace of an operator is unaffected by a finite rank perturbation.

One can see more clearly the locality of the Dixmier trace for a special class of operators, namely the pseudodifferential operators of order $-d$, acting on a vector bundle over X. If P is such an operator then, up to an overall constant which we shall neglect, the Dixmier trace is the integral, over the cosphere bundle of X, of the local trace of the symbol of P. Equation (3) is a special case of this last statement.

We shall now focus on defining the noncommutative Yang-Mills action. Before doing so, it may be worth making some general comments. First, the usual Yang-Mills action involves zeroth order information about the metric, i.e. no derivatives. We do not address the question of noncommutative ***Riemannian*** geometry, in the sense of Riemannian curvature, in this paper. Second, we shall shortly define a differential algebra on a noncommutative space, and use this to do gauge theory on the noncommutative space. If one is only interested in the differential calculus of gauge theory, it is enough to have a differential ***Lie*** algebra, such as, for example, the graded Lie algebra of g-valued differential forms for some Lie algebra g. (One still needs an inner product on the curvature forms, for which we shall use the Dixmier trace.) This fact was used to construct physical models in [CES]. However, one then loses the notion of an underlying space, which corresponds to an algebra. Given an algebra, one obtains a Lie algebra by putting $[x,y] = xy - yx$, but not every Lie algebra arises in this way. In this paper we base everything on ***algebras***, which, as we shall see, corresponds to a generalization of electrodynamics. However, we cannot say for *a priori* reasons that either approach is physically right or wrong.

Let us then clearly state our aim; it is to recover the Yang Mills functional making use only of the following data :

Definition 1.

A K cycle (h, D) *over an involutive algebra* $\mathcal{A}$ *is given by a representation of* $\mathcal{A}$ *on a Hilbert space* h *and a (possibly unbounded) selfadjoint operator* D *such that* $(1 + D^2)^{-1}$ *is compact and* $[D,a]$ *is bounded for all* $a \in \mathcal{A}$. ***We shall say that a K cycle is even if, in addition, there is a self-adjoint operator*** Γ ***on*** h, ***the*** $\mathbb{Z}/2$ ***grading-operator, such that*** $\Gamma^2 = 1$, $\Gamma D + D\Gamma = 0$ ***and*** $\Gamma a = a\Gamma$ ***for all*** $a \in \mathcal{A}$.

If the eigenvalues λ_n of $|D|$ are of the order of $n^{1/d}$ as $n \to \infty$, we say that the K cycle is (d, ∞) summable. On the algebra of smooth functions on a compact Riemannian spin manifold, the Dirac operator determines a K cycle which is (d,∞) summable where $d = \dim X$. If X is even-dimensional then the K cycle is even, with Γ being the chirality operator. We shall call this the Dirac K cycle. (The term K cycle comes from K-homology theory.)

The value of the following construction is that it will also apply when the $*$ algebra $\mathcal{A}$ is not commutative, or when D is no longer the Dirac operator. The reader can have in mind both the Riemannian case and the slightly more involved case where the algebra $\mathcal{A}$ is the $*$ algebra of matrix-valued functions on a Riemannian manifold, just in order to have in mind that the usual notion of exterior product does not make sense in the latter case.

We shall begin by the notion of a connection on the trivial bundle, i.e. the case of "electromagnetism", and define the vector potentials and Yang Mills action. We shall then treat the case of arbitrary hermitian bundles.

We want to define k-forms over $\mathcal{A}$ as operators on $\mathfrak{h}$ of the form

$$\omega = \Sigma \; a_0^j [D, a_1^j] \ldots [D, a_k^j]$$

where the a_i^j are elements of $\mathcal{A}$, represented as operators on $\mathfrak{h}$. This idea arises because although the operator D fails to be invariant under the representation on $\mathfrak{h}$ of the unitary group $\mathcal{U}$ of $\mathcal{A}$:

$$\mathcal{U} = \{ u \in \mathcal{A} \; ; u^* u = uu^* = 1 \}$$

the following equality shows that the failure of invariance is governed by a 1-form in the above sense :

$$u D u^* = D + \omega_u , \quad \omega_u = u [D, u^*].$$

Note that ω_u is self adjoint as an operator on $\mathfrak{h}$ and it is thus natural to adopt the following definition :

Definition 2.

A vector potential V *is a self adjoint element of the space of 1-forms :* $\Sigma \; a_0^j [D , a_1^j]$, $a_k^j \in \mathcal{A}$.

One can immediately check that in the basic example of the Dirac operator on a spin Riemannian manifold X, a vector potential in the above sense is exactly given by an imaginary 1-form v on X, the corresponding operator on spinors being $V = \gamma(v)$.

The action of the unitary group $\mathcal{U}$ on vector potentials is such that it replaces the operator $D + V$ by $u(D + V) u^*$. It is thus given by the algebraic formula :

$$\gamma_u(V) = u[D, u^*] + u V u^* \qquad u \in \mathcal{U}$$

Note that it is ***not true*** in general that $u V u^* = V$, as happens in the case of Riemannian manifolds.

We now just need to define the curvature or field strength θ for a vector potential, and use the analogue of equation (3) above to integrate the square of θ. So

$$YM(V) = Tr_\omega (\theta^2 |D|^{-d})$$

should give us the Yang Mills action.

The formula for θ should be of the form :

$$\theta = dV + V^2$$

and the only difficulty is to define properly the "differential" dV of a vector potential, as an operator on $\mathfrak{h}$. The naive formula is :

If $V = \Sigma \; a_0^j [D, a_1^j]$ then $dV = \Sigma \; [D, a_0^j] [D, a_1^j]$.

Before we point out what the difficulty is, let us check that if we replace V by $\gamma_u(V)$,

$$\gamma_u(V) = u [D, u^*] + \Sigma \; u a_0^j [D, a_1^j] u^*$$

then the curvature transforms in a covariant way :

$$d(\gamma_u(V)) + \gamma_u(V)^2 = u(dV + V^2)\, u^*$$

This computation is instructive so we shall do it in detail. First, to write $\gamma_u(V)$ in the same form as V, we use :

$$[D, a_1^j]\, u^* = [D, a_1^j u^*] - a_1^j [D, u^*]$$

Thus

$$\gamma_u(V) = u[D, u^*] + \Sigma\ u a_0^j\ [D, a_1^j u^*] - \Sigma\, u\, a_0^j a_1^j\, [D, u^*]$$

and one has :

$$d\gamma_u(V) = [D,u]\, [D,u^*] + \Sigma\, [D, u\, a_0^j]\, [D, a_1^j u^*]\ - \Sigma\, [D, u a_0^j a_1^j]\, [D, u^*].$$

Now we can see that the following operators on $\mathfrak{h}$ are equal :

α) $d\, \gamma_u (V) + \gamma_u(V)^2$

β) $u(dV + V^2)\, u^*$.

Indeed, the operator α) is equal to :

$$d\gamma_u(V) + (u[D,u^*] + u\, V\, u^*)^2 =$$

$$d\, \gamma_u(V) + u[D, u^*]\, u[D,u^*] + u[D,u^*]\, u\, V\, u^* + u\, V\, u^*\, u[D,u^*] + u\, V^2\, u^* =$$

$$d\gamma_u(V) - [D,u]\, [D,u^*] - [D,u]\, Vu^* + u\, V[D,u^*] + u\, V^2\, u^* =$$

$$\Sigma\, [D, u a_0^j]\, [D, a_1^j\, u^*] - \Sigma\, [D, u\, a_0^j a_1^j]\, [D,u^*] - [D,u]\, Vu^* + uV\, [D,u^*] + u\, V^2 u^* =$$

$$u\, (d\, V)\, u^* + u\, V^2\, u^*,$$

where the last equality follows from :

$$\Sigma\, [D,u]\, a_0^j\, [D, a_1^j\, u^*] - \Sigma\, [D, u]\, a_0^j a_1^j [D, u^*] = [D,u]\, Vu^*,$$

$$\Sigma\, u\, [D, a_0^j]\, [D, a_1^j\, u^*] - \Sigma\, u\, [D, a_0^j]\, a_1^j\, [D, u^*] = u\, d\, V\, u^*,$$

$$\Sigma\, u\, a_0^j\, [D, a_1^j]\, [D, u^*] = u\, V\, [D, u^*].$$

However, there is a big difficulty that we overlooked, namely that the same vector potential V might be written in several ways as $V = \Sigma\ a_j^0\, [D,a_j^1]$, and so the definition of dV as $\Sigma\ [D,a_j^0]\, [D,a_j^1]$ is ambiguous.

To understand the nature of the problem, let us introduce some algebraic notation. We let $\Omega^*(\mathcal{A})$ be the universal differential graded algebra over $\mathcal{A}$. It is a formal object equal to $\mathcal{A}$ in degree 0 and generated by symbols da, $a \in \mathcal{A}$, of degree 1 with the following relations :

α) $d(ab) = (da)\, b + a\, db \quad \forall\ a, b \in \mathcal{A}$

β) $d1 = 0$

The involution $*$ of $\mathcal{A}$ extends uniquely to an involution on $\Omega^*(\mathcal{A})$ by the rule :

γ) $(da)^* = -\, da^*$

The differential d on $\Omega^*(\mathcal{A})$ is defined *unambiguously* by

$$d^0(a^0 da^1 \dots da^n) = da^0\, da^1 \dots da^n \quad , \quad \text{for all} \quad a^j \in \mathcal{A}$$

and it satisfies :

δ) $d^2\omega = 0$ for all $\omega \in \Omega^*(\mathcal{A})$

ε) $d(\omega_1 \omega_2) = (d\omega_1)\, \omega_2 + (-1)^{\deg \omega_1}\, \omega_1\, d\omega_2$ for all $\omega_j \in \Omega^*(\mathcal{A})$.

We will abbreviate $\Omega^*(\mathcal{A})$ by Ω^*, and $\Omega^k(\mathcal{A})$ by Ω^k.

Proposition 3.

1) ***The following equality defines an involutive representation*** π ***of the algebra*** Ω^* ***on*** h ***:***

$$\pi(a^0\, da^1 \dots da^n) = a^0\, [D,a^1] \dots [D,a^n] \qquad \text{for all} \quad a^j \in \mathcal{A}.$$

2) ***Let*** $J_0 = \operatorname{Ker} \pi \subset \Omega^*$ ***be the two sided ideal of*** Ω^* ***given by*** $J_0^{(k)} = \{ \omega \in \Omega^k \, , \; \pi(\omega) = 0 \}$. ***Then*** $J = J_0 + dJ_0$ ***is a two sided ideal of*** Ω^****, invariant under d.***

The first statement is easy to see. Using it, we can define the Yang-Mills action unambiguously for any self-adjoint element of Ω^1. Let us discuss 2). By construction, J_0 is a two sided ideal, but it is not in general a ***differential*** ideal i.e. if $\omega \in \Omega^k$ and $\pi(\omega) = 0$ one does not have in general $\pi(d\omega) = 0$. This is exactly the reason why the above definition of $\Sigma\, [D, a_j^0]\, [D, a_j^1]$ as the differential of the 1-form $\Sigma\, a_j^0\, [D, a_j^1]$ was ambiguous.

Let us show, however, that $J = J_0 + dJ_0$ is still a two sided ideal. Since we have $d^2 = 0$, it is obvious that J is then a differential ideal. Let $\omega \in J^{(k)}$ be a homogenous element of J. Then ω can be written in the form $\omega = \omega_1 + d\omega_2$, where $\omega_1 \in J_0 \cap \Omega^k$, $\omega_2 \in J_0 \cap \Omega^{k-1}$. Choose $\omega' \in \Omega^{k'}$ and let us show that $\omega\omega' \in J^{(k+k')}$. One has

$$\omega\omega' = \omega_1\omega' + (d\omega_2)\, \omega' = \omega_1\, \omega' + d(\omega_2\, \omega') - (-1)^{k-1}\, \omega_2\, d\omega' =$$
$$(\omega_1\omega' + (-1)^k\, \omega_2\, d\omega') + d(\omega_2\, \omega').$$

But the first term belongs to $J_0 \cap \Omega^{k+k'}$, and $\omega_2\, \omega' \in J_0 \cap \Omega^{k+k'-1}$. Similarly, one shows that $\omega'\, \omega$ is in $J^{(k+k')}$. QED

Now using proposition 3, we can introduce the following graded differential algebra :

$$\Omega_D^* = \Omega^*/J.$$

(Quotienting out by J has the effect of eliminating the spurious fields of [CL], and makes the comparison easier with other treatments, such as [CES].)

Let us look at Ω_D^0 , Ω_D^1 and Ω_D^2 .

We have $J \cap \Omega^0 = J_0 \cap \Omega^0 = \{0\}$ if we assume, as we shall, that $\mathcal{A}$ is embedded in $\mathcal{L}(\mathfrak{h})$, the bounded operators on $\mathfrak{h}$. Thus $\Omega_D^0 = \mathcal{A}$. Next,

$$J \cap \Omega^1 = J_0 \cap \Omega^1 + d(J_0 \cap \Omega^0) = J_0 \cap \Omega^1.$$

Thus Ω_D^1 is the quotient of Ω^1 by the kernel of π, and so it is exactly the space $\pi(\Omega^1)$ of operators ω of the form :

$$\omega = \Sigma\ a_j^0 [D, a_j^1] \quad ; \quad a_j^k \in \mathcal{A}.$$

Now $J \cap \Omega^2 = J_0 \cap \Omega^2 + d(J_0 \cap \Omega^1)$ and the representation π gives us an isomorphism :

(4) $$\Omega_D^2 \simeq \pi(\Omega^2) / \pi(d(J_0 \cap \Omega^1)).$$

More precisely, this means that we can view an element ω of Ω_D^2 as a class of elements ρ of the form :

$$\rho = \Sigma\ a_j^0 [D, a_j^1] [D, a_j^2] \quad ; \quad a_j^k \in \mathcal{A}$$

modulo the subspace of elements of the form :

$$\rho_0 = \Sigma [D, b_j^0] [D, b_j^1] \quad ; \quad b_j^k \in \mathcal{A} \quad , \quad \Sigma\ b_j^0 [D, b_j^1] = 0.$$

It is clear now that because we work ***modulo the subspace*** $\pi(d(J_0 \cap \Omega^1))$, the question of the ambiguity in the definition of $d\omega$ for $\omega \in \pi(\Omega^1)$ no longer arises.

Note that equality (4) makes sense for all k :

(5) $$\Omega_D^k \simeq \pi(\Omega^k) / \pi(d(J_0 \cap \Omega^{k-1}))$$

and allows us to define an inner product on Ω_D^k: for each k, let $\mathfrak{h}_k$ be the Hilbert space completion of $\pi(\Omega^k)$ with respect to the inner product

$$< T_1, T_2 >_k = \mathrm{Tr}_\omega (T_2^* T_1 |D|^{-d}) \ , \ \forall\ T_j \in \pi(\Omega^k).$$

Let P be the orthogonal projection from $\mathfrak{h}_k$ onto the orthogonal of the subspace $\pi(d(J_0 \cap \Omega^{k-1}))$. By construction, for $\omega_j \in \pi(\Omega^k)$, the inner product $< P\omega_1, P\omega_2 >$ only depends upon the classes of the ω_j in Ω_D^k. We let Λ^k be the Hilbert space completion of Ω_D^k for this inner product; it is of course equal to $P\mathfrak{h}_k$.

Proposition 4.

1) *The actions of* $\mathcal{A}$ *on* Λ^k *by left and right multiplication define commuting unitary representations of* $\mathcal{A}$ *on* Λ^k.

2) *The functional* $YM(V) = < dV + V^2 \ , dV + V^2 >$ *is positive, quartic and invariant under gauge transformations,*

$$\gamma_u(V) = udu^* + uVu^* \qquad \forall\ u \in \mathcal{U}(\mathcal{A})$$

3) *The functional* $I(\alpha) = \mathrm{Tr}_\omega(\theta^2 |D|^{-d})$, $\theta = \pi(d\alpha + \alpha^2)$ *is positive,*

quartic and gauge invariant on $\{ \alpha \in \Omega^1 : \alpha = \alpha^* \}$.

4) ***One has*** $YM(V) = \mathrm{Inf}\{ I(\alpha);\ \pi(\alpha) = V \}$

Let us say a few words about the easy proof. Since $\pi(d(J_0 \cap \Omega^{k-1})) \subset \pi(\Omega^k)$ is invariant under left and right multiplication by $\mathcal{A}$, and since the left and right actions of $\mathcal{A}$ on h_k are unitary, it follows that $P(a\xi b) = aP(\xi)b$ for all $a, b \in \mathcal{A}$ and $\xi \in h_k$. So 1) follows. For 2), one just notes that by the above calculation, with dV unambiguous now, $\theta = dV + V^2$ is covariant under gauge transformations. For 3), one uses again the above calculation to show that $d\alpha + \alpha^2$ transforms covariantly under gauge transformations. Finally, to see 4), note that if $\pi(\alpha) = V$ then $dV + V^2$, as an element of Λ^2, is equal to $P(\pi(d\alpha + \alpha^2))$. Thus $I(\alpha) \geqslant YM(V)$. To see that the infimum is attained, fix V and take any α such that $\pi(\alpha) = V$. We have that $\pi(d\alpha + \alpha^2) - (dV + V^2)$ is an element of $\pi(d(J_0 \cap \Omega^1))$, say $d\sigma$. Put $\beta = \alpha - \sigma$. Then $\pi(\beta) = V$ and $\pi(d\beta + \beta^2) = dV + V^2$, so $I(\beta) = YM(V)$. QED

Stated in simpler terms the meaning of proposition 4 is that the ambiguity that we met above in the definition of the Yang-Mills action can be resolved by taking the infimum over all possibilities. The obtained action is nevertheless quartic by 4.2.

Example 1: We shall now check that in the case of Riemannian manifolds, with the Dirac K cycle, the graded differential algebra Ω_D^* is the same as the de Rham algebra of differential forms on X, with its usual prehilbert space structure. We now specialize to the Riemannian case, where $\mathcal{A}$ is the algebra of functions (with some regularity) on an even-dimensional compact spin manifold X and $D = \partial_X$ is the Dirac operator on the Hilbert space $L^2(X,S)$ of spinors. We let C be the bundle over X whose fiber at each $p \in X$ is the complexified Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(T_p^*(X))$ of the cotangent space at $p \in X$. Any measurable bounded section ρ of C defines a bounded operator $\gamma(\rho)$ on $h = L^2(X,S)$. For any $f^0, \dots, f^n \in \mathcal{A}$, $f_0\,df_1 \dots df_n$ is an element of $\Omega^*(A)$ and one has $\pi(f_0\,df_1 \dots df_n) = f_0\,df_1 \cdot \dots \cdot df_n$, where on the right hand side, the usual differential df is considered as a section $[D, f]$ of C, and $\cdot$ denotes the product in C.

For each $p \in X$, the Clifford algebra C_p has a filtration by $\{ C_p^{(k)} \}$, where $C_p^{(k)}$ is the subspace spanned by products of $\leqslant k$ elements of $T_p^*(X)$. The associated graded algebra $\{ C_p^{(k)} / C_p^{(k-1)} \}$ is isomorphic to the complexified exterior algebra $\Lambda_{\mathbb{C}}(T_p^*(X))$ and we shall let $\sigma_k : C^{(k)} \to \Lambda_{\mathbb{C}}^k(T^*)$ be the quotient map.

Using the inner product on C given by the trace in the spinor representation, one can also identify $\Lambda_{\mathbb{C}}^k$ with the orthogonal complement of $C^{(k-1)}$ in $C^{(k)}$, or equivalently, if we let C^k be the subspace of $C^{(k)}$ of elements of the same parity as k then

$$\Lambda_{\mathbb{C}}^k = C^k \ominus C^{k-2}.$$

Lemma 5.
Let (h,D) *be the Dirac K cycle on the algebra* $\mathcal{A}$ *of functions on* X. *For* $k \in \mathbb{N}$, *a pair* T_1, T_2 *of operators on* h *is of the form* $T_1 = \pi(\omega)$, $T_2 = \pi(d\omega)$ *for some* $\omega \in \Omega^k$ *iff there exist sections* ϱ_1, ϱ_2 of C^k *and* C^{k+1} *such that :*

$$T_j = \gamma(\varrho_j),\ j = 1, 2, \quad d\sigma_k(\varrho_1) = \sigma_{k+1}(\varrho_2).$$

Here $\sigma_k(\varrho_1)$ is an ordinary k-form on X and d is its ordinary differential. Note that for $k > d = \dim X$, one has $\sigma_k(\varrho) = 0$. We shall omit the proof of Lemma 5.

We can now easily determine the graded differential algebra Ω_D^*. First, let us identify $\pi(\Omega^k)$ with the space of sections of $C^{(k)}$. Then lemma 5 shows that :

$$\pi(d(J_0 \cap \Omega^{k-1})) = \operatorname{Ker} \sigma_k.$$

(If ϱ is a section of C^k with $\sigma_k(\varrho) = 0$ then the pair $\varrho_1 = 0$, $\varrho_2 = \varrho$ in C^{k-1} and C^k fulfills the condition of lemma 5, and so $\varrho = \pi(d\omega)$ for some ω with $\pi(\omega) = 0$.)

Thus σ_k is an isomorphism : $\Omega_D^k \simeq$ Sections of $\wedge_{\mathbb{C}}^k(T^*)$, which again by Lemma 5 commutes with the differential. We can then state :

Formula 3.
The map $a^0 da^1 \dots da^n \to a^0 da^1 \wedge \dots \wedge da^n$ *from* Ω^* to $\wedge_{\mathbb{C}}^k(X)$ *extends to an isomorphism of the differential graded algebra* Ω_D^* *with the de Rham algebra of differential forms on* X *. Under this isomorphism, the inner product on* Ω_D^k *is the Riemannian inner product of k-forms :*

$$\langle \omega, \omega' \rangle = \int_X \omega \wedge * \omega'.$$

The last equality follows from the computation of the Dixmier trace for the operator on $h = L^2(X,S)$ associated to a section ϱ of the bundle C of Clifford algebras:

$$\operatorname{Tr}_\omega(\varrho |D|^{-d}) = \int_X \operatorname{trace}(\varrho(p))\, dv(p)$$

As an immediate corollary of formula 3 we get :

$$YM(V) = \int_X \| dV \|^2 \, dv$$

for any vector potential V on X. **End of Example 1.**

Let us now consider the generalized fermionic action. A fermion field will simply be an element of the Hilbert space h. The operator $D + V$ is self-adjoint.

Definition 6.
The fermionic action is $\langle \psi, (D + V)\psi \rangle$, *for* $\psi \in h$ *and* $V \in \Omega_D^1$.

By construction, the fermionic action is gauge invariant in that for any $u \in \mathcal{U}(\mathcal{A})$, it is invariant under the transformations

$$\psi \to u\psi, \quad V \to \gamma_u(V).$$

Example 2: *Massless chiral electrodynamics.*

We can call the following the action for a generalized "massless chiral electrodynamics":

$$L(V, \psi) = g^{-2}\ YM(V) + \langle\psi, (D + V)\psi\rangle \qquad \text{for } \psi \in \mathfrak{h} \text{ and } V \in \Omega_D^1.$$

The reason that the name is appropriate is that if we use the Dirac K cycle on a 4-dimensional spin manifold, then we obtain exactly the usual Euclidean action of massless electrodynamics with one fermion field, and with g being the physical coupling constant. We should note that one cannot really write a Euclidean action for chiral fermions, and the ψ field above is a 4-component spinor field. The reason to call it "chiral" electrodynamics is that one can Wick-rotate to Minkowski space, and then impose $\Gamma\psi = \psi$.

Similarly, to obtain the action of massless electrodynamics with N_+ right-handed fermions and N_- left-handed fermions, we would take the Hilbert space to be $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$, where $\mathfrak{h}_+$ consists of N_+ copies of $L^2(X,S)$ and $\mathfrak{h}_-$ consists of N_- copies. With the grading operator given by $\Gamma = (\gamma_5 \otimes I_{N+}) \oplus (-\gamma_5 \otimes I_{N-})$, we can then write out $L(V, \psi)$, Wick-rotate and impose $\Gamma\psi = \psi$.

End of Example 2.

Let us now extend the definition of the Yang Mills action to connections on arbitrary hermitian vector bundles.

First of all, we need to express in algebraic terms, i.e. using only the involutive algebra $\mathcal{A} = C(X)$, the notion of a hermitian vector bundle over X. A vector bundle E is entirely characterized by the vector space $\mathcal{E}$ of its sections. Furthermore, this vector space has an $\mathcal{A}$-action, which we shall take to be on the right. In other words, $\mathcal{E}$ is a right $\mathcal{A}$-module. The local triviality of E and the finite dimensionality of its fibers translate algebraically to saying that there is an $\mathcal{E}'$ such that $\mathcal{E} \oplus \mathcal{E}'$ is $\mathcal{A}^N$ for some finite N, or in more fancy terms, that $\mathcal{E}$ is a finite projective module over $\mathcal{A}$.

The Hermitian structure on E, i.e. the inner product $(\xi, \eta)_p$ on each fiber E_p, allows us to construct a sesquilinear map :

$$(\ ,\ ) : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$$

given by $(\xi, \eta)(p) = (\xi(p), \eta(p))_p$.

This map (,) satisfies the following conditions :

1) $(\xi a, \eta b) = a^* (\xi, \eta)\, b$ for all $\xi, \eta \in \mathcal{E}$, $a, b \in \mathcal{A}$

2) $(\xi, \xi) \geq 0$ for all $\xi \in \mathcal{E}$

3) $\mathcal{E}$ is self dual for (,).

Thus the hermitian vector bundles over X correspond to the hermitian finite projective modules over $\mathcal{A}$ in the following sense :

Definition 7.
*Let $\mathcal{A}$ be an algebra with an involution * and a unit. Then a hermitian structure on a finite projective module $\mathcal{E}$ over $\mathcal{A}$ is given by a sesquilinear map (,) : $\mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying* 1, 2 *and* 3 .

One can show that all hermitian structures on a given finite projective module $\mathcal{E}$ over $\mathcal{A}$ can be obtained as follows: one writes $\mathcal{E}$ as $e\mathcal{A}^N$ for appropriate e and N, where e, an NxN matrix with entries in $\mathcal{A}$, is selfadjoint and satisfies $e^2 = e$. One then restricts to $\mathcal{E}$ the hermitian structure on $\mathcal{A}^N$ given by :

$$(\xi, \eta) = \Sigma\, \xi_i^* \eta_i \in \mathcal{A} \quad \text{for all } \xi = (\xi_i)\ , \ \eta = (\eta_i).$$

The algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ of endomorphisms of $\mathcal{E}$, that is, linear maps T from $\mathcal{E}$ to $\mathcal{E}$ which commute with the $\mathcal{A}$-action, has a natural involution, given by :

$$(T^*\xi, \eta) = (\xi , T\eta) \quad \text{for all } \xi , \eta \in \mathcal{E}.$$

With this involution, $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is isomorphic to the algebra of matrices $eM_N(\mathcal{A})e$.

As before, we let (h , D) be a K cycle over $\mathcal{A}$.

Definition 8.
The Hilbert space of "gauged spinors" is $\mathcal{E} \otimes_{\mathcal{A}} h$. Its inner product is given by

$$<\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2> = <\eta_1, (\xi_1, \xi_2)\eta_2>.$$

If h has a grading operator Γ, it extends to a grading operator on $\mathcal{E} \otimes_{\mathcal{A}} h$.

Definition 9.
Let $\mathcal{E}$ be a hermitian finite projective module over $\mathcal{A}$. Then a connection on $\mathcal{E}$ is given by a linear map $\nabla: \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ such that

$$\nabla(\xi a) = (\nabla\xi)\, a + \xi \otimes da \quad \text{for all } \xi \in \mathcal{E}, a \in \mathcal{A}.$$

A connection ∇ is compatible (with the metric) iff :

$$< \xi , \nabla\eta > - < \nabla\xi , \eta > = d < \xi , \eta > \quad \text{for all } \xi , \eta \in \mathcal{E}.$$

Both sides of the last equation lie in Ω_D^1. (In computations, one should remember that $(da)^* = - da^*$ for all $a \in \mathcal{A}$, and if $\nabla\xi = \Sigma\, \xi_i \otimes \omega_i$ with $\omega_i \in \Omega_D^1$ then $< \nabla\xi , \eta > = \Sigma\, \omega_i^* < \xi_i , \eta >$.)

Such connections always exist, for with $\mathcal{E}$ expressed as $e\,\mathcal{A}^N$, one may take ∇ to be :

$$\nabla_0\xi = e\, d\xi.$$

Two connections ∇ and ∇' on ε differ by an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1)$. Any compatible connection can be written as $\nabla\xi = e\, d\xi + \rho\, \xi$, where ρ is a self-adjoint NxN matrix of 1-forms satisfying $e\rho = \rho\, e = \rho$.

As in proposition 4 we shall now give two equivalent definitions of the action functional $YM(\nabla)$ on the affine space $C(\mathcal{E})$ of compatible connections.

The group $\mathcal{U}(\mathcal{E})$ of unitary automorphisms of $\mathcal{E}$,

$$\mathcal{U}(\mathcal{E}) = \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})\ ; uu^* = u^*u = 1\}$$

will be the unitary gauge group, in that it acts by gauge-transformations (explicitly, $\gamma_u(\nabla) = u\, \nabla\, u^*$) on the space $C(\mathcal{E})$. To define the curvature θ of a connection ∇, we first need to define the covariant derivative of a vector-valued form. Put

$$\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^*,$$

the space of vector-valued forms. Extend ∇ to a unique linear map from $\mathcal{E}'$ to $\mathcal{E}'$, which we shall also denote by ∇, by

$$\nabla(\xi \otimes \omega) = (\nabla\xi)\, \omega + \xi \otimes d\omega \qquad \text{for all } \xi \in \mathcal{E}\ ,\ \omega \in \Omega_D^*$$

One finds that this linear map ∇ satisfies :

$$\nabla(\eta\omega) = (\nabla\eta)\,\omega + (-1)^{\deg \eta}\,\eta\, d\omega$$

for any homogeneous $\eta \in \mathcal{E}'$ and $\omega \in \Omega_D^*$.

It follows that $\nabla^2(\eta\omega) = (\nabla^2\eta)\,\omega$, i.e. ∇^2 is an endomorphism of the right Ω_D^* module $\mathcal{E}'$. It is determined by its restriction to $\mathcal{E}$, which we shall denote by θ :

$$\theta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2).$$

If we use the representation $\mathcal{E} = e\mathcal{A}^N$ then θ will be a self-adjoint NxN matrix of 2-forms such that $e\theta = \theta e = \theta$.

Next, using the inner product on Ω_D^2 and the hermitian structure on $\mathcal{E}$, one gets a natural inner product on

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^2).$$

Using this we define

Definition 10. $\mathrm{YM}(\nabla) = \langle \theta, \theta \rangle$.

By construction, this action is gauge invariant, positive and quartic.

Formula 4.
Let X be a Riemannian spin manifold with its Dirac K cycle $(\mathfrak{h}, D)$. Then the notion of connection (Def. 9) is the usual one, and one has :

$$\mathrm{YM}(\nabla) = \int_M \|\theta\|^2\, dv,$$

where θ is the usual curvature of ∇.

This follows immediately from Formula 3.

Thus we recover in this case the usual Yang Mills action. For the fermionic action, we define the gauged Dirac operator on $\mathcal{E} \otimes_{\mathcal{A}} \mathfrak{h}$ by

$$D_\nabla(\xi \otimes \eta) = \xi \otimes D\eta + (\nabla\xi)\,\eta \qquad \text{for all } \xi \in \mathcal{E} \text{ and } \eta \in \mathfrak{h}.$$

Definition 11.
The fermionic action is $\langle\psi, D_\nabla \psi\rangle$, for $\psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{h}$ and ∇ a compatible connection.

Example 3: *U(N) gauge theory with chiral fermions.*

Given an even $(4,\infty)$-summable K cycle, take $\mathcal{E}$ to be $\mathcal{A}^N$. Then $\mathcal{E} \otimes_{\mathcal{A}} \mathfrak{h}$ is simply N copies of $\mathfrak{h}$. Let ∇ be a compatible connection on $\mathcal{E}$. Consider the action

$$\mathcal{L}(\nabla, \psi) = g^{-2}\, \mathrm{YM}(\nabla) + \langle\psi, D_\nabla \psi\rangle \qquad \text{for } \psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{h}.$$

Its group of gauge invariances consists of the unitary NxN matrices over $\mathcal{A}$.

In the case of the Dirac K cycle on a 4-dimensional spin manifold, we obtain the usual Euclidean action of a U(N)-gauge theory with one N-tuple of fermion fields in the fundamental representation of U(N). **End of Example 3.**

We shall now mention the easy adaptation of proposition 4.4 to the general case. First of all, any compatible connection in the sense of Definition 9 is the composition with π of a ***universal compatible connection,*** i.e. a linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1$$

fulfilling exactly the conditions of Definition 9.

To see the surjectivity of the map :

$$\pi : CC(\mathcal{E}) \to C(\mathcal{E})$$

(where $CC(\mathcal{E})$ is the space of universal compatible connections), it is enough to note that the special "Grassmannian" connection ∇_0 is of this form, and that π is a surjection of Ω^1 onto Ω^1_D. Next, a universal compatible connection extends uniquely to a linear map :

$$\nabla : \mathcal{E} \otimes_{\mathcal{A}} \Omega^* \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^*$$

such that

$$\nabla (\eta\omega) = (\nabla\eta)\, \omega + (-1)^{\deg \eta}\, \eta\, d\omega$$

for any homogeneous $\eta \in \mathcal{E} \otimes_{\mathcal{A}} \Omega^*$ and $\omega \in \Omega^*$.

The curvature $\theta = \nabla^2$ is then an endomorphism of the module $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{A}} \Omega^*$ over Ω^*, and $\pi(\theta)$ makes sense as a bounded operator on the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} h$. Then the analogue of the action I of proposition 4 is given by :

$$I(\nabla) = \mathrm{Tr}_\omega (\pi(\theta)^2 |D_\nabla|^{-d})$$

One proves in the same way as before that for a given compatible connection $\nabla \in C(\mathcal{E})$, one has :

$$YM(\nabla) = \mathrm{Inf}\{ I(\nabla_1) ; \pi(\nabla_1) = \nabla\}.$$

Let us briefly state the inequality between the Yang-Mills action and a topological quantity. For the background notions and notations, we refer to [Co1]. Suppose that we have an even $(4,\infty)$-summable K cycle. Define a Hochschild cocycle by

$$\Phi(a^0 da^1 \dots da^4) = \mathrm{Tr}_\omega (\Gamma a^0 [D,a^1] \dots [D,a^4] D^{-4}) \quad \text{for all } a^j \in \mathcal{A}.$$

Let B be the operator :

$$B : H^4(\mathcal{A}, \mathcal{A}^*) \to HC^3(\mathcal{A}).$$

Then one can show that $B\Phi = 0$. We shall, however, need :

Hypothesis 12 $B\Phi = 0$ as a cochain.

(In the case of the Dirac operator on a 4-dimensional manifold one has :

$$\Phi(f^0, \dots, f^4) = \int f^0 df^1 \wedge df^2 \wedge \dots \wedge df^4,$$

which satisfies the hypothesis.)

Since $B_0\Phi$ is already cyclic :

$$B_0\Phi(a^0, a^1, a^2, a^3) = \Phi(1, a^0, a^1, a^2, a^3) = \mathrm{Tr}_\omega (\Gamma [D, a^0] \dots [D, a^3] D^{-4})$$

the condition $B\Phi = 0$ means that in fact, $B_0\Phi = 0$. This, together with $b\Phi = 0$, implies that Φ is a cyclic cocycle.

Theorem 13.
For any hermitian finite projective module $\mathcal{E}$ over $\mathcal{A}$ one has :

$$YM(\nabla) \geq \langle [\mathcal{E}], \Phi \rangle \qquad \text{for all } \nabla \in C(\mathcal{E})$$

The right hand side is the pairing between K theory and cyclic cohomology. In the case of the Dirac operator on a compact 4-dimensional spin manifold X, one recovers the usual lower bound for the Yang-Mills action in terms of the topological charge of the vector bundle.

2. PRODUCT OF CONTINUUM BY DISCRETE AND THE SYMMETRY BREAKING MECHANISM.

We have shown how to extend the notions of gauge potentials and Yang Mills action to finitely summable K cycles $(\mathfrak{h}, D)$ over an algebra $\mathcal{A}$, and we have also defined the fermion action.

In this section we shall give two examples of computations of these actions:
a) The case of a discrete 2 pt space.
b) The case of a product of a 4 dimensional manifold by case a).

We first need a brief discussion of product spaces. Suppose that we have two triples :

$$(\mathcal{A}_1, \mathfrak{h}_1, D_1) \quad , \quad (\mathcal{A}_2, \mathfrak{h}_2, D_2).$$

We assume that one of them is *even* , i.e. we are given a $\mathbb{Z}/2$ grading, say Γ_1, on $\mathfrak{h}_1$. We define the product to be the triple $(\mathcal{A}, \mathfrak{h}, D)$:

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \quad , \quad \mathfrak{h} = \mathfrak{h}_1 \otimes \mathfrak{h}_2 \ ,$$

$$D = D_1 \otimes 1 + \Gamma_1 \otimes D_2 .$$

One can check that if our two triples are Dirac K cycles coming from two Riemannian manifolds, then the product K-cycle corresponds to the Dirac K cycle of the product manifold. If we have finite hermitian projective modules $\mathcal{E}_j$ over the $\mathcal{A}_j$, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a finite hermitian projective module over $\mathcal{A}$.

Next, the formula $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$, which follows from the anticommutation of D_1 with Γ_1, shows that dimensions add up, that is, if the D_j are (p_j, ∞) summable then D is $(p_1 + p_2, \infty)$ summable. Moreover, one can show that

$$\mathrm{Tr}_\omega((T_1 \otimes T_2)\,|D|^{-(p_1+p_2)}) = \mathrm{Tr}_\omega\,(T_1 |D_1|^{-p_1})\ \mathrm{Tr}_\omega(T_2 |D_2|^{-p_2}),$$

$$\text{for all } T_j \in \mathcal{L}(\mathfrak{h}_j).$$

More precisely, this is true provided that $p_j \geqslant 1$, but in the case of interest (Example 5), we have $p_1 = 4$ and $p_2 = 0$. The corresponding formula turns out to be :

$$\mathrm{Tr}_\omega((T_1 \otimes T_2)\,|D|^{-p}) = \mathrm{Tr}_\omega\,(T_1 |D_1|^{-p})\ \mathrm{Tr}(T_2)$$

Thus in the 0-dimensional case, we should replace the Dixmier trace by the ordinary trace, and the Yang-Mills action $YM(\nabla)$ is just given by $\mathrm{Tr}\,(\theta^2)$.

Example 4.
The space we are dealing with has *two points* a and b . Thus the algebra $\mathcal{A}$ is just $\mathbb{C} \oplus \mathbb{C}$, the direct sum of two copies of $\mathbb{C}$. An element $f \in \mathcal{A}$ is given by two complex numbers $f(a)$, $f(b) \in \mathbb{C}$. Let $(\mathfrak{h}, D, \Gamma)$ be a 0-dimensional K cycle over $\mathcal{A}$. Then $\mathfrak{h}$ is *finite dimensional* and the representation of $\mathcal{A}$ on $\mathfrak{h}$ corresponds to a decomposition of $\mathfrak{h}$ as a direct sum $\mathfrak{h} = \mathfrak{h}_a \oplus \mathfrak{h}_b$, with the action of $\mathcal{A}$ given by :

$$f \in \mathcal{A} \to \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix}$$

If we write D as a 2×2 matrix in this decomposition :

$$D = \begin{pmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{pmatrix}$$

we can ignore the diagonal elements since they commute with the action of $\mathcal{A}$. We shall thus take D to be of the form :

$$D = \begin{pmatrix} 0 & D_{ab} \\ D_{ba} & 0 \end{pmatrix}$$

where $D_{ba} = D_{ab}^*$ and D_{ab} is a linear map from h_b to h_a. We shall denote by M this linear map. As will become clear, a "standard" geometry on our 2-point space corresponds to $M = 0$. Thus, although our algebra in this example is ***commutative***, our more general notion of geometry allows us to consider nonstandard geometries on this "commutative" space.

We shall take for Γ the $\mathbb{Z}/2$ grading given by the matrix

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So the geometry of our 2-point space is given by :

$$\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \quad , \quad h = h_a \oplus h_b \quad , \quad D = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \ \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Let us first compute the metric on the space $F = \{a , b\}$, using Formula 1. Given $f \in \mathcal{A}$, one has :

$$[D , f] = \left[\begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} , \begin{pmatrix} f(a) & 0 \\ 0 & f(b) \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & M(f(b)-f(a)) \\ -M^*(f(b)-f(a)) & 0 \end{pmatrix} = (f(b)-f(a)) \begin{pmatrix} 0 & M \\ -M^* & 0 \end{pmatrix}$$

Thus the norm of this commutator is $|f(b) - f(a)| \, \|M\|$, where $\|M\|$ is the largest characteristic value of M. Hence :

$$d(a, b) = \text{Sup} \ \{ |f(a) - f(b)| \ , \ \|[D,f]\| \leqslant 1 \} = 1/\|M\|$$

Let us now determine the space of gauge potentials, the curvature and the action in two cases :

α) $\quad \mathcal{E} = \mathcal{A}$ (i.e. the trivial bundle over F) .

First, let e be the idempotent $e \in \mathcal{A}$ given by $e(a) = 1$ and $e(b) = 0$. For notational simplicity, let us write e' for the idempotent $1 - e$. Then $\mathcal{A}$ is spanned by e and e', and

$de = - de'$. Thus the space Ω^1 of universal 1-forms over $\mathcal{A}$ is a 2 dimensional space, with the basis {e de, e' de}. So every element of Ω^1 can be written in the form δ_a e de + δ_b e'de for some complex numbers δ_a and δ_b. We shall denote this 1-form by the pair (δ_a, δ_b). Using the identities

$$e\, de = (de)\, e', \quad e'\, de = de\, (e),$$

one finds that the action of $\mathcal{A}$ on Ω^1 is given by

$$f\,(\delta_a, \delta_b) = (f(a)\, \delta_a, f(b)\, \delta_b); \qquad (\delta_a, \delta_b)\, f = (\,\delta_a\, f(b), \delta_b\, f(a)).$$

We see that although the algebra $\mathcal{A}$ is commutative, the algebra Ω^* is ***not*** graded-commutative.

The differential $d : \mathcal{A} \to \Omega^1$ is essentially a ***finite difference*** operator:

$$df = (\Delta f,\ \Delta f) \quad , \quad \Delta f = f(a) - f(b).$$

One computes:

$$\pi((\delta_a, \delta_b)) = \begin{pmatrix} 0 & -\delta_a M \\ \delta_b M^* & 0 \end{pmatrix} \qquad \in \mathcal{L}(h).$$

So provided $M \neq 0$, the representation $\pi : \Omega^* \to \mathcal{L}(h)$ is 1-1 on Ω^1, and $\Omega^1 = \Omega^1_D$.

Next, let us find what Ω^2_D is. Recall that $\Omega^2_D = \pi(\Omega^2)/\pi(d(J_0 \cap \Omega^1)$. Now any element of Ω^2 can be written as h_a e de de + h_b e' de de, which we shall denote by the pair of complex numbers (h_a , h_b). One computes:

$$\pi(h_a , h_b) = \begin{pmatrix} -h_a MM^* & 0 \\ 0 & -h_b M^*M \end{pmatrix} \qquad \in \mathcal{L}(h).$$

So if $M \neq 0$ then the representation $\pi : \Omega^* \to \mathcal{L}(h)$ is 1-1 on Ω^2. And since π is 1-1 on Ω^1, $J_0 \cap \Omega^1 = 0$. Thus $\Omega^2_D = \Omega^2$. The differential

$d : \Omega^1 \to \Omega^2$ is given by $d(\delta_a, \delta_b) = (\delta_a - \delta_b)$ de de $= (\delta_a - \delta_b, \delta_a - \delta_b)$.

And the multiplication

$\Omega^1 \times \Omega^1 \to \Omega^2$ is given by $(\delta_a, \delta_b)\, (\delta_a', \delta_b') = (\delta_a\, \delta_b', \delta_b\, \delta_a')$.

We now have enough of the differential calculus on the 2-point space to compute the gauge theory. A vector potential is given by a self adjoint element of Ω^1_D , or in our case, by $V = (-\Phi, \Phi^*)$, with Φ a complex number. Its curvature is :

$$\theta = dV + V^2 = -\,(\,|\Phi + 1|^2 - 1,\ |\Phi + 1|^2 - 1\,).$$

This gives the following formula for the Yang Mills action :

$$YM(V) = 2(\,|\Phi + 1|^2 - 1\,)^2 \quad \mathrm{Trace}\ ((M^*M)^2\,)$$

The action of the gauge group $\mathcal{U} = U(1) \times U(1)$ on the space of vector potentials, i.e. on Φ, is given by $\gamma_u(V) = u\, du^* + u\, V\, u^*$, which, with $u = u_a\, e + u_b\, e'$, gives:

$$\gamma_u(V) = (1 - u_a u_b^* (\Phi + 1), -1 + u_b u_a^* (\Phi^* + 1)).$$

On the variable $\Phi + 1$, this just means multiplication by $u_a u_b^*$.

Thus in this very simple case, our action $YM(V)$ reproduces the symmetry-breaking Higgs potential. It has nonunique minima, given by $|\Phi + 1| = 1$, which are acted upon non trivially by the gauge group.

The fermionic action is given by $<\psi, (D + V)\psi>$, where the operator $D + V$ is equal to :

$$\begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Phi M \\ \Phi^* M^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & (1+\Phi)M \\ (1+\Phi)^* M^* & 0 \end{pmatrix}$$

Writing ψ as $(\psi_+, \psi_-)^T$, we see that the fermionic action is

$$<\psi, (D+V)\psi> = \psi_+^* (1+\Phi) M \psi_- + \text{(complex conjugate)},$$

which is a sort of a primitive Yukawa coupling. Let us note that for the "standard" geometry with $M = 0$, the two points are infinitely far apart, and the Yang-Mills action vanishes.

β) Let us take for $\mathcal{E}$ the non trivial bundle over $F = \{a, b\}$ with fibers of dimension n_a and n_b on a and b respectively. (This does not affect the differential calculus that we worked out before.) The bundle is nontrivial iff $n_a \neq n_b$ and we shall consider the simple case of $n_a = 1$ and $n_b = 2$. The finite projective module $\mathcal{E}$ is of the form :

$$\mathcal{E} = f \mathcal{A}^2$$

where the idempotent $f \in M_2(\mathcal{A})$ is given in terms of the notation of α by the formula

$$f = \begin{pmatrix} (1,1) & 0 \\ 0 & (0,1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix}$$

To the idempotent f corresponds a particular compatible connection on $\mathcal{E}$, given by $\nabla_0 \xi = f d\xi$ with obvious notations. An arbitrary compatible connection on $\mathcal{E}$ has the form :

$$\nabla \xi = \nabla_0 \xi + \rho \xi$$

where ρ is a self adjoint element of $M_2(\Omega_D^1)$ such that $f\rho = \rho f = \rho$. If we write ρ as a matrix,

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix},$$

these conditions are:

(6) $$e'\rho_{21} = \rho_{21}, \quad e'\rho_{22} = \rho_{22} = \rho_{22} e', \quad \rho_{12} e' = \rho_{12}.$$

Thus we get :

$$\rho_{11} = -\Phi_1 \, ede + \Phi_1^* \, e'de, \quad \rho_{21} = \Phi_2^* \, e'de, \quad \rho_{12} = -\Phi_2 \, ede, \quad \rho_{22} = 0,$$

where Φ_1 and Φ_2 are arbitrary complex numbers.

The curvature θ is given by $\theta = fdfdf + f(d\rho)f + \rho^2$

$$= \begin{pmatrix} 0 & 0 \\ 0 & e'dede \end{pmatrix} + \begin{pmatrix} d\rho_{11} & (d\rho_{12})e \\ e'd\rho_{21} & 0 \end{pmatrix} + \begin{pmatrix} \rho_{11}\rho_{11} + \rho_{12}\rho_{21} & \rho_{11}\rho_{12} \\ \rho_{21}\rho_{11} & \rho_{21}\rho_{12} \end{pmatrix}$$

Explicitly, the components of θ are:

$$\theta_{11} = (1 - |\Phi_1 + 1|^2 - |\Phi_2|^2)\, edede + (1 - |\Phi_1 + 1|^2)\, e'dede,$$

$$\theta_{12} = -\Phi_2(\Phi_1 + 1)^*\, e'dede,$$

$$\theta_{21} = -\Phi_2^*(\Phi_1 + 1)\, e'dede,$$

$$\theta_{22} = (1 - |\Phi_2|^2)\, e'dede.$$

An easy calculation then gives the action $YM(\nabla)$ in terms of the variables Φ_1 and Φ_2 :

$$YM(\nabla) = (1 + 2(1 - (|\Phi_1 + 1|^2 + |\Phi_2|^2))^2) \quad Tr((M^*M)^2).$$

It is by construction invariant under the gauge group $U(1) \times U(2)$. We see that the minimum of $YM(\nabla)$ is strictly positive, and so the bundle $\mathcal{E}$ does not admit any compatible connection with vanishing curvature. We also see, after the fact, that there is nothing special about ∇_0 ; any connection with $|\Phi_1 + 1|^2 + |\Phi_2|^2 = 1$ also minimizes the Yang-Mills action.

To write the fermionic action, note that $\mathcal{E} \otimes_{\mathcal{A}} h$ is $f\mathcal{A}^2 \otimes_{\mathcal{A}} h = fh^2$. Let us write a typical element of $\mathcal{E} \otimes_{\mathcal{A}} h$ as $\psi = ((e_R, e_L), (0, \nu_L))^T$. Then the fermionic action is

$$<\psi, D_\nabla \psi> = <\psi, (D \otimes 1_2 + \rho)\psi> = e_R^*(1 + \Phi_1) M e_L + e_R^* \Phi_2 M \nu_L$$
$$+ \text{(complex conjugate)},$$

which is like the leptonic Yukawa coupling. **End of Example 4.**

Example 5. 4 dim. Riemannian manifold x (2-point space)

To fix notations, we let X be a compact Riemannian Spin 4 manifold, $\mathcal{A}_1$ the algebra of functions on X and (h_1, D_1, Γ_1) the Dirac K-cycle on $\mathcal{A}_1$. We shall let the triple $(\mathcal{A}_2, h_2, D_2)$ to be as in example 4 above, i.e.

$$\mathcal{A}_2 = \mathbb{C} \oplus \mathbb{C}, \quad h_2 = h_{2,a} \oplus h_{2,b}, \quad D_2 = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}$$

We put $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $h = h_1 \otimes h_2$, $D = D_1 \otimes 1 + \Gamma_1 \otimes D_2$.

The algebra $\mathcal{A}$ is commutative. It is the algebra of complex valued functions on the space $Y = X \times F$, which is the union of two copies of the manifold X: $Y = X_a \cup X_b$.

Let us first compute the metric on Y associated to the K-cycle (h, D) :

$$d(p,q) = \sup_{f \in \mathcal{A}} \{ |f(p) - f(q)| ; \| [D, f] \| \leq 1 \}$$

Every $f \in \mathcal{A}$ is a pair (f_a, f_b) of functions on X. Also, to the decomposition of

$\mathfrak{h}_2$ as :

$$\mathfrak{h}_2 = \mathfrak{h}_{2,a} \oplus \mathfrak{h}_{2,b}$$

corresponds a decomposition $\mathfrak{h} = \mathfrak{h}_a \oplus \mathfrak{h}_b$, on which the action of $f = (f_a , f_b) \in \mathcal{A}$ is diagonal :

$$f \to \begin{pmatrix} f_a & 0 \\ 0 & f_b \end{pmatrix} \in \mathcal{L}(\mathfrak{h})$$

In this decomposition, the operator D becomes :

$$D = \begin{pmatrix} \partial_X \otimes 1 & \gamma_5 \otimes M \\ \gamma_5 \otimes M^* & \partial_X \otimes 1 \end{pmatrix}$$

where ∂_X is the Dirac operator on X and γ_5 the $\mathbb{Z}/2$ grading of its spinor bundle.

This gives us the following formula for the "differential" $[D,f]$ of a function f :

$$[D,f] = \begin{pmatrix} \gamma(df_a) \otimes 1 & (f_b - f_a)\, \gamma_5 \otimes M \\ (f_a - f_b)\, \gamma_5 \otimes M^* & \gamma(df_b) \otimes 1 \end{pmatrix}$$

Thus the differential $[D,f]$ contains three parts :

α) The usual differential df_a of the restriction of f to the copy X_a of X .

β) The usual differential df_b of the restriction of f to the copy X_b of X .

γ) The finite difference $\Delta f = f(p_a) - f(p_b)$ where p_a and p_b are the points of X_a and X_b above a given point p of X.

One can then show:

Proposition 14.

1) The restriction of the metric d *on* $X_a \cup X_b$ *to each copy* (X_a *or* X_b) *of* X *is the Riemannian geodesic distance of* X.

2) For each point $p = p_a$ *of* X_a*, the distance* $d(p_a , X_b)$ *equals* $\|M\|^{-1}$ *and is attained at the unique point* p_b.

Let us now pass to the computation of the space Ω_D^1 of 1-forms on Y. As a 1-form is a sum of terms of the form $\pi(f_0\, df_1)$, the above computation of $[D,f] = \pi(df)$ shows that an element α of Ω_D^1 is given by :

α) A complex 1-form ω_a on X_a

β) A complex 1-form ω_b on X_b

γ) A pair of complex valued functions δ_a , δ_b on X.

The corresponding operator on $\mathfrak{h}$ is given by :

$$\begin{pmatrix} \gamma(\omega_a) \otimes 1 & -\delta_a \gamma_5 \otimes M \\ \delta_b \gamma_5 \otimes M^* & \gamma(\omega_b) \otimes 1 \end{pmatrix}$$

The action of $\mathcal{A}$ on Ω_D^1 is given, with obvious notations, by :

$$(f_a, f_b)(\omega_a, \omega_b, \delta_a, \delta_b) = (f_a\omega_a, f_b\omega_b, f_a\delta_a, f_b\delta_b)$$

$$(\omega_a, \omega_b, \delta_a, \delta_b)(f_a, f_b) = (f_a\omega_a, f_b\omega_b, f_b\delta_a, f_a\delta_b)$$

The involution on Ω_D^1 is given by $(\omega_a, \omega_b, \delta_a, \delta_b)^* = -(\omega_a^*, \omega_b^*, \delta_b^*, \delta_a^*)$.

The differential $f : \mathcal{A} \to \Omega_D^1$ is given by:

$$f = (f_a, f_b) \to (df_a, df_b, f_a - f_b, f_a - f_b) \in \Omega_D^1.$$

When we write things in terms only of X, we can view Ω_D^1 as a 10 dimensional bundle over X, given by two copies of the complexified cotangent bundle and a trivial 2 dimensional bundle, so that over a point p in X, the fiber consists of

$$T_p^*(X)_{\mathbb{C}} \oplus T_p^*(X)_{\mathbb{C}} \oplus \mathbb{C} \oplus \mathbb{C}$$

But we have to keep in mind the nontrivial multiplication structure in the last two terms.

As in the case of the Dirac operator on Riemannian manifolds (Lemma 6), let us compute the pairs of operators of the form $\pi(\rho) = T_1$, $\pi(d\rho) = T_2$ for $\rho \in \Omega^1(\mathcal{A})$.

Given $\rho = \Sigma\, f_j\, dg_j \in \Omega^1(\mathcal{A})$, with $f_j, g_j \in \mathcal{A}$, one has :

$$\pi(\rho) = \begin{pmatrix} \gamma(\omega_a) \otimes 1 & -\delta_a \gamma_5 \otimes M \\ \delta_b \gamma_5 \otimes M^* & \gamma(\omega_b) \otimes 1 \end{pmatrix}$$

with $\omega_a = \Sigma\, f_{ja}\, dg_{ja}$, $\omega_b = \Sigma\, f_{jb}\, dg_{jb}$ and :

$$\delta_a = \Sigma\, f_{ja}(g_{ja} - g_{jb}), \quad \delta_b = \Sigma\, f_{jb}(g_{ja} - g_{jb}).$$

One has $\pi(d\rho) = \Sigma\, \pi(df_j)\, \pi(dg_j)$, which gives the 2×2 matrix :

$$\pi(d\rho) = \begin{pmatrix} \gamma(\xi_a) \otimes 1 + (\delta_b - \delta_a) \otimes MM^* & -\gamma(\eta_a)\gamma_5 \otimes M \\ \gamma(\eta_b)\gamma_5 \otimes M^* & \gamma(\xi_b) \otimes 1 + (\delta_b - \delta_a) \otimes M^*M \end{pmatrix}$$

where $\xi_a = \Sigma\, df_{ja} \cdot dg_{ja}$ and $\xi_b = \Sigma\, df_{jb} \cdot dg_{jb}$ are sections of the Clifford algebra bundle C^2 over X, while

$$\eta_a = \Sigma\, ((f_{jb} - f_{ja})\, dg_{jb} - (g_{jb} - g_{ja})\, df_{ja}) \quad \text{and}$$

$$\eta_b = \Sigma\, ((g_{ja} - g_{jb})\, df_{jb} - (f_{ja} - f_{jb})\, dg_{ja}).$$

Using the equalities :

$$d\delta_a = \Sigma\, f_{ja}(dg_{ja} - dg_{jb}) + (g_{ja} - g_{jb})\, df_{ja}$$

$$d\delta_b = \Sigma\, f_{jb}(dg_{ja} - dg_{jb}) + (g_{ja} - g_{jb})\, df_{jb}$$

$$\omega_a = \Sigma\, f_{ja}\, dg_{ja}, \qquad \omega_b = \Sigma\, f_{jb}\, dg_{jb}$$

we can rewrite η_a and η_b as follows :

$$\eta_a = \omega_b - \omega_a + d\delta_a$$

$$\eta_b = \omega_b - \omega_a + d\delta_b.$$

Thus knowing T_1 fixes δ_a, δ_b, η_a and η_b. As in the Riemannian case (Lemma 5), the sections ξ_a, ξ_b of C^2 are arbitrary except for $\sigma_2(\xi_a) = d\omega_a$ and $\sigma_2(\xi_b) = d\omega_b$.

This shows that the subspace $\pi\,(d(J_0 \cap \Omega^1))$ of $\pi(\Omega^2)$ is the space of 2×2 matrices of operators of the form :

$$T = \begin{pmatrix} \gamma(\xi_a) \otimes 1 & 0 \\ 0 & \gamma(\xi_b) \otimes 1 \end{pmatrix}$$

where ξ_a and ξ_b are sections of C^0 , i.e. are just arbitrary scalar valued functions on X.

A general element of $\pi(\Omega^2)$ is a 2×2 matrix of operators of the form :

$$T = \begin{pmatrix} \gamma(\alpha_a) \otimes 1 - h_a \otimes MM^* & -\gamma(\beta_a)\,\gamma_5 \otimes M \\ \gamma(\beta_b)\,\gamma_5 \otimes M^* & \gamma(\alpha_b) \otimes 1 - h_b \otimes M^*M \end{pmatrix}$$

where α_a and α_b are arbitrary sections of C^2, h_a and h_b are arbitrary functions on X and β_a and β_b are arbitrary sections of C^1 (i.e. 1-forms). We thus get :

Lemma 15.
Assume that M^*M *is not a scalar multiple of the identity matrix. Then an element of* Ω_D^2 *is given by*

1) a pair of complex 2-forms σ_a, σ_b on X
2) a pair of complex 1-forms β_a, β_b on X
3) a pair of complex functions h_a, h_b on X

The hypothesis $M^*M \neq$ const. Id enters because otherwise the functions h_a and h_b are eliminated when we quotient out by $\pi(d(J_0 \cap \Omega^1))$.

Using the above computation of $\pi(d\rho)$ we can compute the differential $d\omega$ of an element $\omega = (\omega_a, \omega_b, \delta_a, \delta_b)$ of Ω_D^1. We get :

1) $\sigma_a = d\omega_a$, $\sigma_b = d\omega_b$

2) $\beta_a = \omega_b - \omega_a + d\delta_a$, $\beta_b = \omega_b - \omega_a + d\delta_b$

3) $h_a = \delta_a - \delta_b$, $h_b = \delta_a - \delta_b$.

So we see that the differential $d\,\omega \in \Omega_D^2$ involves both the differential terms $d\omega_a$, $d\omega_b$, $d\delta_a$ and $d\delta_b$, and the finite difference terms $\omega_a - \omega_b$ and $\delta_a - \delta_b$, but in combinations imposed by $d(df) = 0$.

Next, let us compute the product $\omega\,\omega' \in \Omega_D^2$ of elements $\omega = (\omega_a, \omega_b, \delta_a, \delta_b)$ and $\omega' = (\omega'_a, \omega'_b, \delta'_a, \delta'_b)$ of Ω_D^1. We get :

1) $\sigma_a = \omega_a \wedge \omega'_a$ $\qquad$ $\sigma_b = \omega_b \wedge \omega'_b$
2) $\beta_a = -\delta_a\,\omega'_b + \delta'_a\,\omega_a$ $\qquad$ $\beta_b = -\delta_b\,\omega'_a + \delta'_b\,\omega_b$
3) $h_a = \delta_a\,\delta'_b$ $\qquad$ $h_b = \delta_b\,\delta'_a$.

Comparing these formulae with those of Example 4, one can summarize the results by saying that the differential algebra on Y is simply the ***graded tensor product*** of the differential algebras on X and F.

The next step is to determine the inner product on the space Ω_D^2 of 2-forms given in section 1. By definition we take the orthogonal of $\pi(d(J_0 \cap \Omega^1))$ in $\pi(\Omega^2)$, gifted with the inner product $< T_1 , T_2 > = \mathrm{Tr}_\omega (T_1^* T_2 | D |^{-4})$.

An easy calculation then gives :

Lemma 16.

Let $P(M^*M)$ *be the orthogonal projection of the matrix* M^*M *on the scalar matrices* const. Id. *Then the square norm of an element* $(\sigma_a , \sigma_b , \beta_a , \beta_b , h_a , h_b)$ *of* Ω_D^2 *is given by*

$$\int_X (N_a \| \sigma_a \|^2 + N_b \| \sigma_b \|^2)\, dv + \mathrm{tr}\,(M^*M) \int_X (\| \beta_a \|^2 + \| \beta_b \|^2)\, dv +$$
$$\mathrm{tr}\,((M^*M - P(M^*M))^2) \int_X (\| h_a \|^2 + \| h_b \|^2)\, dv$$

where $N_a = \dim \mathfrak{h}_a$, $N_b = \dim \mathfrak{h}_b$.

We are now ready to compute the action $YM(\nabla)$. For $\mathcal{E}$, we shall take the space of sections of the hermitian vector bundle E on $Y = X_a \cup X_b$ which has fiber $\mathbb{C}$ on the copy X_a of X, and fiber $\mathbb{C}^2$ on the copy X_b of X. In other words we consider the product of Example 2 and Example 4 β. We can say immediately that the gauge group $\mathcal{U} = \mathrm{End}_{\mathcal{A}} (\mathcal{E})$ of our gauge theory is the unitary gauge group of the vector bundle E over $Y = X_a \cup X_b$, or equivalently the group :

$$\mathcal{U} = \mathrm{Map}\,(X , U(1) \times U(2)).$$

As in Example 4 β, we can write $\mathcal{E}$ as $f\mathcal{A}^2$, where $f \in M_2(\mathcal{A})$ is the idempotent

$$f = \begin{pmatrix} 1 & 0 \\ 0 & e' \end{pmatrix} \quad \text{and} \quad e' = (0, 1) \in \mathcal{A}.$$

Then a compatible connection ∇ has the form

$$\nabla \xi = f d\xi + \rho\xi \quad \in \quad \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1 \quad , \quad \text{for all } \xi \in \mathcal{E},$$

where ρ is a self adjoint element of $M_2(\Omega_D^1)$ which satisfies the conditions (6). Using the description of an element ω of Ω_D^1 as a quadruple $(\omega_a, \omega_b, \delta_a, \delta_b)$, we find that the entries of the 2 x 2 matrix ρ have the form:

$$\rho_{11} = (\omega_{11}^a \; , \; \omega_{11}^b \; , \; -\Phi_1 \; , \; \Phi_1^*)$$
$$\rho_{12} = (\; 0 \; , \; \omega_{12}^b \; , \; -\Phi_2 \; , \; 0 \;)$$
$$\rho_{21} = (\; 0 \; , \; \omega_{21}^b \; , \; 0 \; , \; \Phi_2^*)$$
$$\rho_{22} = (\; 0 \; , \; \omega_{22}^b \; , \; 0 \; , \; 0 \;),$$

where ω^a is a u(1)-valued 1-form on X, ω^b is a u(2)-valued 1-form on X and where (Φ_1 , Φ_2) is a pair of complex-valued functions on X. In other words, a compatible connection on $\mathcal{E}$ consists of

α) A u(1)- connection ∇_a on the restriction of E to X_a

β) A u(2)– connection ∇_b on the restriction of E to X_b
γ) A linear map (Φ_1, Φ_2) from E over X_b, to E over X_a.

The action of the gauge group on ∇_a and ∇_b is the obvious one, and the action on (Φ_1, Φ_2) is given by composition.

Next, the curvature θ is the following element of $f\, M_2(\Omega_D^2)\, f$:

$$\theta = f\, df\, df + f\, d\rho\, f + \rho^2,$$

which is easily computed using the above formulae for

$$d : \Omega_D^1 \to \Omega_D^2 \quad \text{and} \quad \wedge : \Omega_D^1 \times \Omega_D^1 \to \Omega_D^2.$$

Let us write an element of Ω_D^2 as a sextuple $(\sigma_a, \sigma_b, \beta_a, \beta_b, h_a, h_b)$. Then one finds that the components of θ are

$$\theta_{11} = (F^a_{11}, F^b_{11}, -D(1+\Phi_1), D(1+\Phi_1)^*, 1 - |\Phi_1 + 1|^2 - |\Phi_2|^2, 1 - |\Phi_1 + 1|^2)$$

$$\theta_{12} = (0 \;, F^b_{12}, \; -D\Phi_2 \;, \; 0 \;, \; 0 \;, -\Phi_2(\Phi_1 + 1)^*)$$

$$\theta_{21} = (0 \;, F^b_{21}, \; 0 \;, \; D\Phi_2 \;, \; 0 \;, -\Phi_2^*(\Phi_1 + 1))$$

$$\theta_{22} = (0 \;, F^b_{22}, \; 0 \;, \; 0 \;, \; 0 \;, \; 1 - |\Phi_2|^2).$$

Here F^a and F^b are the curvatures of ω^a and ω^b respectively, and

$$D(1+\Phi_1, \Phi_2) = d(1+\Phi_1, \Phi_2) - (1+\Phi_1, \Phi_2) \begin{pmatrix} \omega^b_{11} - \omega^a_{11} & \omega^b_{12} \\ \omega^b_{21} & \omega^b_{22} - \omega^a_{11} \end{pmatrix}$$

(Note that the calculation of the (h_a, h_b)'s is exactly the same as in Example 4 β.)

Applying Lemma 16 gives that the Yang-Mills action is the integral of a Lagrangian density $\mathcal{L}_B$ over X, with

$$\mathcal{L}_B = c_1 \,\|F^a\|^2 + c_2 \,\|F^b\|^2 + c_3 \,\mathrm{Tr}(M^*M)\,\|D(1+\Phi_1, \Phi_2)\|^2 +$$
$$c_4 \,\mathrm{Tr}((M^*M - P(M^*M))^2)\,(1 + 2(1 - (|\Phi_1 + 1|^2 + |\Phi_2|^2))^2).$$

The c_i's are various positive constants; we shall come back to their meaning in the next section.

The fermionic action is even easier to compute. Note that $\mathcal{E} \otimes_{\mathcal{A}} h$ is $f\,h^2$. Let us write a typical element of $\mathcal{E} \otimes_{\mathcal{A}} h$ as $\psi = ((e_R, e_L), (0, \nu_L))^T$. Then the fermionic action is

$$<\psi, D_\nabla \psi> = <\psi, (D \otimes I_2 + \rho)\psi>,$$

which is the integral of a Lagrangian density $\mathcal{L}_F$ over X, with

$$\mathcal{L}_F = e_R^* \,\partial_a\, e_R + (e_L, \nu_L)^* \,\partial_b\, (e_L, \nu_L) +$$
$$[e_R^*(1+\Phi_1)\, M\, \gamma_5\, e_L + e_R^* \Phi_2\, M\, \gamma_5\, \nu_L + (\text{complex conjugate})].$$

Here ∂_a is the Dirac operator on X, when coupled to the u(1)–gauge field ω^a, and similarly ∂_b is the Dirac operator on X, when coupled to the u(2)–gauge field ω^b.

It should now be clear that the total Lagrangian density $\mathcal{L}_B + \mathcal{L}_F$ of our noncommutative gauge theory is almost the same as that of the

Glashow-Weinberg-Salam (GWS) model of leptons [GWS]. In fact, there are only two differences. First, the global gauge group of the GWS model is not U(1) x U(2), but U(1) x SU(2). In order to reduce our gauge group, we impose the

Ad Hoc Condition : $\mathrm{tr}(\omega^a) = \omega^b$.

We shall give a less ad hoc formulation of this condition in Section 4 . The second difference is that we need for the fermions to be chiral. To achieve this, we simply Wick-rotate to Minkowski space and impose the condition $\Gamma\psi = \psi$.

End of Example 5.

3. BIMODULES

In the discussion so far, we have had a single algebra $\mathcal{A}$ acting on the Hilbert space $\mathfrak{h}$. In fact, it turns out to be natural to extend this to having two algebras $\mathcal{A}$ and $\mathfrak{B}$ acting on $\mathfrak{h}$, whose actions commute. We can express this by saying that $\mathcal{A}$ acts on $\mathfrak{h}$ on the left, and $\mathfrak{B}$ acts on $\mathfrak{h}$ on the right. Alternatively, we can say that $\mathcal{A} \otimes \mathfrak{B}$ acts on $\mathfrak{h}$ on the left. Given the first description, we get the second description by putting

$$(a \otimes b)\eta = a\eta b \quad \text{for all } a \in \mathcal{A},\ b \in \mathfrak{B} \text{ and } \eta \in \mathfrak{h}.$$

This situation of having two algebras acting arises when one want to extend Poincaré duality to an algebraic setting. We shall not need the details of this, for which we refer to [CS,Ka,Co2], and will only broadly state the ideas . Recall that if X is a closed oriented manifold then Poincaré duality gives an isomorphism between the cohomology and homology of X. Similarly, if X is a spin^c manifold then X is K-oriented and there is an isomorphism between the K-theory $K^*(X)$ and K-homology $K_*(X)$ of X.

Let us consider what the analogous statement would be for general algebras. On the level of K-groups, it would be an isomorphism between $K_*(\mathcal{A})$ and $K^*(\mathfrak{B})$. (In the special case that X is a spin^c manifold, we can take both $\mathcal{A}$ and $\mathfrak{B}$ to be C(X).) Of course, one needs additional structure to have such an isomorphism. The essential piece of information needed is a K cycle $(\mathcal{A} \otimes \mathfrak{B}, \mathfrak{h}, D)$ for the algebra $\mathcal{A} \otimes \mathfrak{B}$.

On the level of homology, we want a map from the homology of the complex $\Omega_D^*(\mathcal{A})$ to the periodic cyclic cohomology of $\mathfrak{B}$. It turns out that such a map can defined provided that one has certain relations, one of which is

$$(7) \qquad [[D,a],b] = 0 \quad \text{for all } a \in \mathcal{A} \text{ and } b \in \mathfrak{B}.$$

We note that this relation is actually symmetric in a and b, as they commute.

The point of this general discussion is that it is natural to look at a K cycle for the tensor product $\mathcal{A} \otimes \mathfrak{B}$ of two algebras, which satisfies (7). We can apply the constructions of section 1 to such a K cycle, and in particular the notions of a vector potential V and its Yang Mills action YM(V). The gauge group of such a gauge theory would be the group $\mathcal{U}(\mathcal{A} \otimes \mathfrak{B})$ of unitaries of $\mathcal{A} \otimes \mathfrak{B}$. However, we can use the fact that we have two algebras to single out a class of vector potentials which is invariant under the action of the subroup $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathfrak{B}}$.

Proposition 17.

Let $\mathcal{V} = \mathcal{V}_{\mathcal{A}} + \mathcal{V}_{\mathfrak{B}}$ be the subspace of the vector potentials $\mathcal{V}_{\mathcal{A}\otimes\mathfrak{B}}$ which consists of sums of vector potentials relative to $\mathcal{A}$ and $\mathfrak{B}$. Then $\mathcal{V}$ is invariant under the action of $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathfrak{B}}$, and for every $V \in \mathcal{V}$, the operator $D + V$ still satifies equation (7).

To see this, recall that the action of the unitary group of $\mathcal{A} \otimes \mathfrak{B}$ on vector potentials is determined by

$$g(D + V)g^* = D + \gamma_g(V).$$

Let us specialize this equation to elements $g = u v \in \mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{B}}$. Take $V = V^a + V^b \in \mathcal{V}_{\mathcal{A}} + \mathcal{V}_{\mathcal{B}}$. Then

$$uv(D + V^a + V^b)\, v^* u^* = u\, v\, D\, v^* u^* + u\, V^a u^* + v\, V^b v^*,$$

since by (7), every element V^a of $\mathcal{V}^a$ (resp. V^b of $\mathcal{V}^b$) commutes with $\mathcal{B}$ (resp. $\mathcal{A}$). Next,

$$u\, v\, D\, v^* u^* = u(D + v\,[D, v^*])\, u^* = D + u\,[D, u^*] + v[D, v^*],$$

again using (7). So we get :

$$\gamma_{uv}(V^a + V^b) = \gamma_u(V^a) + \gamma_v(V^b),$$

which shows that the space $\mathcal{V}$ is invariant under the action of $\mathcal{U}_{\mathcal{A}} \times \mathcal{U}_{\mathcal{B}}$. Finally, we compute:

$$[[D + V^a + V^b , a] , b] = [[V^a, a], b] = [[V^a, b], a] = 0.$$

4. The standard U(1) × SU(2) × SU(3) model

In this section we shall build on the computation of the action functional in Example 5 i.e. in the case of the product of a continuum by a discrete 2-point space. We saw that almost by accident, we recovered the GWS model for leptons from a simple modification of the 4-D continuum. The question which we address in this section is: can one by a similar procedure incorporate the quarks as well as strong interactions?

Let us make some preliminary remarks. First, there is presently (1991) no doubt that the standard model of electroweak and strong interactions gives a remarkably accurate description of the known elementary particles. We refer to other works, such as [El], for a survey of the standard model, and will only give a skeleton description in order to fix notation.

The goal is to find a modification of the continuum spacetime geometry such that the bosonic part of the standard model becomes a pure gauge theory on this modified spacetime. (The fermionic part will be straightforward.) That is, we wish to find a new geometry such that the gauge fields and the Higgs fields of the continuum geometry become unified into a gauge field on the new geometry. In itself this is not a new idea, and most previous attempts to do such a unification used a new geometry consisting of $\mathbb{R}^4$ x F, where F is a compact homogeneous space [CJ]. However, none of these attempts were able to succesfully reproduce realistic particle models, partly because of problems in producing <u>chiral</u> fermions on $\mathbb{R}^4$ from fermions on $\mathbb{R}^4$ x F [Wi]. What is new in our approach is to take F to be a finite set, albeit with a nonstandard geometry. Then the problems with producing chiral fermions immediately go away.

We wish, then, to find a finite space F such that when one computes the analog of the classical Lagrangian of electrodynamics, but instead on $\mathbb{R}^4$ x F, one finds the classical

Lagrangian of the standard model. Once the structure of this finite space F is given, we just apply our general method of computing the Yang-Mills action to $\mathbb{R}^4$ x F, and will find the bosonic terms of the standard model action. The Higgs boson will be part of a gauge field, but coming from a finite difference, rather than a differential. The fermionic action will be straightforward to derive, and will give the fermionic terms of the standard model action.

a) *The standard model*

The Lagrangian density of the standard model contains five different terms:

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_\Psi + \mathcal{L}_\phi + \mathcal{L}_Y + \mathcal{L}_V$$

which we now describe in a Euclidean version of the model.

1) *The pure gauge boson part* $\mathcal{L}_G$

$$\mathcal{L}_G = \frac{1}{4} g^{-2} \left(F_{\mu\nu} F^{\mu\nu}\right) + \frac{1}{4} g'^{-2} \left(G_{\mu\nu a} G_a^{\mu\nu}\right) + \frac{1}{4} g''^{-2} \left(H_{\mu\nu b} H_b^{\mu\nu}\right)$$

where $F_{\mu\nu}$ is the field strength tensor of a U(1)-gauge field A_μ, $G_{\mu\nu}$ is the field strength tensor of an SU(2)-gauge field W_μ, and $H_{\mu\nu}$ is the field strength tensor of an SU(3)-gauge field V_μ, the gluon field.

2) *The Fermion kinetic term* $\mathcal{L}_\Psi$

$$\mathcal{L}_\Psi = \overline{\Psi} \left(\gamma^\mu D_\mu\right) \Psi,$$

where Ψ is a spinorial field consisting of N copies, or generations, of a certain $\mathbb{C}^{15}$ representation of U(1) x SU(2) x SU(3). Here D_μ is the covariant derivative of the spinor field:

$$D_\mu \Psi = [\partial_\mu + \pi(A_\mu) + \pi'(W_\mu) + \pi''(V_\mu)]\ \Psi,$$

and π, π' and π'' are the respective representations of the Lie algebras of U(1), SU(2) and SU(3) on Ψ. The decomposition of the $\mathbb{C}^{15}$-representation into its irreducible components, listed by the particles of the first generation, is as follows:

Particle	$\pi \otimes \pi' \otimes \pi''$	Y
e_R	$\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}$	-2
(e_L, ν_L)	$\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}$	-1
d_R	$\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}^3$	-2/3

u_R	$\mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C}^3$	4/3
(d_L, u_L)	$\mathbb{C} \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$	1/3

The hypercharge Y, when multiplied by 3, labels the U(1) representation π. Hereafter we will write the fermion fields as N-vectors, labelled by their first-generation particle. For example, with the three known generations, using the standard particle names we have

$$\vec{e} = (e, \mu, \tau), \quad \vec{v} = (v_e, v_\mu, v_\tau), \quad \vec{u} = (u, c, t), \quad \vec{d} = (d, s, b).$$

3) *The kinetic terms for the Higgs fields*

$$\mathcal{L}_\phi = (D_\mu \phi)^* \; D^\mu \phi ,$$

where $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is an SU(2) doublet of complex scalar fields with hypercharge Y = - 1.

4) *The Yukawa coupling of Higgs fields with Fermions*

$$\mathcal{L}_Y = \overline{e_R} \, M_e \phi_1 e_L + \overline{e_R} \, M_e \phi_2 v_L + \overline{d_R} \, M_d \phi_1 d_L + \overline{d_R} \, M_d \phi_2 u_L$$

$$+ \overline{u_R} \, M_u (-\overline{\phi}_2) d_L + \overline{u_R} \, M_u \overline{\phi}_1 u_L + \text{complex conjugate.}$$

Here M_e, M_d and M_u are N x N matrices whose singular values are, up to a constant, the masses of the fermions. Let us note that while $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ is involved in the Yukawa couplings to the electron e_R and down quark d_R, its conjugate

(8) $$\tilde{\phi} = \begin{pmatrix} -\overline{\phi}_2 \\ \overline{\phi}_1 \end{pmatrix}$$

is in the Yukawa coupling to the up quark u_R.

5) *The Higgs self interaction*

$$\mathcal{L}_V = -\mu^2 \phi^* \phi + \frac{1}{2} \lambda (\phi^* \phi)^2$$

has exactly the same form as in the GWS model.

Thus we see that there are essentially three new features of the complete standard model as compared to the GWS model:

A. The new SU(3) gauge symmetry, whose gauge fields are responsible for the strong interaction.

B. The new fermions, the quarks, with their new hypercharges.

C. The new Yukawa coupling terms involving the quarks.

We shall now briefly explain how these new features motivate a modification of Example 5, which lead us above to the GWS model for leptons. First, our model will still be a *product* of an ordinary Euclidean continuum by a finite space.

In example 4 β, our algebra $\mathcal{A}$ of functions on the finite space,was $\mathbb{C} \oplus \mathbb{C}$. But since we then considered a bundle on $\{a,b\}$ with fiber $\mathbb{C}$ on a and $\mathbb{C}^2$ on b, we could have equally well used $\mathcal{A} = \mathbb{C} \oplus M_2(\mathbb{C})$, and taken the module $\mathcal{E}$ to be the same as $\mathcal{A}$. Let us see how point C. leads us to replace $\mathbb{C} \oplus M_2(\mathbb{C})$ by $\mathcal{A} = \mathbb{C} \oplus \mathbb{H}$ where $\mathbb{H}$ is the Hamilton algebra of quaternions. The point is simply that the equation (8) which relates ϕ and $\tilde{\phi}$ is the same as the unitary equivalence $2 \sim \bar{2}$ between the fundamental representation 2 of SU(2) and its complex conjugate or contragredient representation, i.e. one has :

$$g \in U(2)\ ,\ J\, g\, J^{-1} = \bar{g} \Leftrightarrow g \in SU(2),$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We remark that :

$$\{\ x \in M_2(\mathbb{C})\ ,\ J\, x\, J^{-1} = \bar{x}\ \}$$

defines an algebra, the quaternion algebra $\mathbb{H}$.

Next let us see how point A. leads us to the formalism of bimodules of Section 3. Indeed, look at any isodoublet of the form $\begin{pmatrix} u_L \\ d_L \end{pmatrix}$ of left handed quarks. It appears in 3 colors:

$$\begin{pmatrix} u_L^r & u_L^y & u_L^b \\ d_L^r & d_L^y & d_L^b \end{pmatrix}$$

which makes it clear that the corresponding representation of $SU(2) \times SU(3)$ is the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^3$ of their fundamental representations. It is easy to convince oneself that even if one neglects the difference between U(n) and SU(n), there is no way to obtain such groups and representations from a single algebra and its unitary group. The solution that we found is to take $(\mathcal{A}, \mathcal{B})$ bimodules, with $\mathcal{A} = \mathbb{C} \oplus \mathbb{H}$ and $\mathcal{B} = \mathbb{C} \oplus M_3(\mathbb{C})$.

We are now ready to describe the geometric structure of a finite space F which, when crossed by $\mathbb{R}^4$, gives the standard model.

b) *Geometric structure of the finite space* F

The structure is given by an $\mathcal{A}$ - $\mathcal{B}$ bimodule $(\mathfrak{h}, D, \Gamma)$ where $\mathcal{A}$ is the involutive algebra $\mathbb{C} \oplus \mathbb{H}$ while $\mathcal{B}$ is the involutive algebra $\mathbb{C} \oplus M_3(\mathbb{C})$. Unlike $\mathcal{B}$, the algebra $\mathcal{A}$ is only an algebra over $\mathbb{R}$. The involutive representations π of $\mathcal{A}$ in a finite dimensional Hilbert space are characterized (up to unitary equivalence) by three multiplicities: n_+, n_-, m, where

$$\mathfrak{h}_\pi = \mathbb{C}^{n_+} \oplus \mathbb{C}^{n_-} \oplus \mathbb{C}^{2m},$$

and if $a = (\lambda, q) \in \mathbb{C} \oplus \mathbb{H}$, $\pi(a)$ is the block diagonal matrix:

$$\pi(a) = \left(\lambda \otimes I_{n_+}\right) \oplus \left(\bar{\lambda} \otimes I_{n_-}\right) \oplus \left(\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \otimes I_m\right).$$

Here we are writing the quaternion q as $q = \alpha + \beta j$, with $\alpha, \beta \in \mathbb{C} \subset \mathbb{H}$.

The representation of the complex involutive algebra $\mathcal{B}$ in $\mathfrak{h}$ gives a decomposition:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus (\mathfrak{h}_1 \otimes \mathbb{C}^3)$$

in which $b = (b_0, b_1) \in \mathbb{C} \oplus M_3(\mathbb{C})$ acts by $\pi(b) = b_0 \oplus (1 \otimes b_1)$. It follows that the representation of $\mathcal{A}$ (which commutes with the representation of $\mathcal{B}$) is given by *a pair* π_0, π_1 of representations of $\mathcal{A}$ in Hilbert spaces $\mathfrak{h}_0$ and $\mathfrak{h}_1$. The $\mathcal{A}$ - $\mathcal{B}$ bimodule $\mathfrak{h}$ is thus completely described by the six multiplicities, namely (n_+^0, n_-^0, m^0) for π_0 and (n_+^1, n_-^1, m^1) for π_1. We shall take them to be of the form:

$$(n_0^+, n_0^-, m_0) = N(1,0,1)$$

$$(n_1^+, n_1^-, m_1) = N(1,1,1),$$

where N will eventually be the number of generators. That is,

$$\mathfrak{h} = [(\mathbb{C} \oplus \mathbb{H}) \oplus ((\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}) \otimes \mathbb{C}^3)] \otimes \mathbb{R}^N.$$

We shall take the $\mathbb{Z}/2$ grading Γ in $\mathfrak{h}$ to be given by the element $\Gamma = (1, -1)$ of the center of $\mathfrak{A}$. Finally, for D we shall take the most general selfadjoint operator in $\mathfrak{h}$ which anticommutes with Γ and commutes with $\mathbb{C} \oplus \mathfrak{B}$, where $\mathbb{C} \subset \mathfrak{A}$ is the diagonal subalgebra: $\{(\lambda, \lambda), \lambda \in \mathbb{C}\}$. (As we shall see, D encodes both the fermion masses and the Kobayaski-Maskawa mixing parameters.) It follows that the action of $\mathfrak{A}$ and the operator D in $\mathfrak{h}_0$ (resp. $\mathfrak{h}_1$) have the following general form: (with $q = \alpha + \beta j \in \mathbb{H}$)

$$\pi_0(f,q) = \begin{bmatrix} f & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\bar{\beta} & \bar{\alpha} \end{bmatrix} \qquad D_0 = \begin{bmatrix} 0 & M_e & 0 \\ M_e^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\pi_1(f,q) = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & \bar{f} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & 0 & M_d & 0 \\ 0 & 0 & 0 & M_u \\ M_d^* & 0 & 0 & 0 \\ 0 & M_u^* & 0 & 0 \end{bmatrix}$$

where M_e, M_u, M_d are arbitrary complex $N \times N$ matrices.

c) *Gauge Theory on the finite space* F

We shall take the modules for $\mathfrak{A}$ and $\mathfrak{B}$ to be $\mathcal{E} = \mathfrak{A}$ and $\mathcal{F} = \mathfrak{B}$ respectively. Then the Hilbert space $\mathcal{E} \otimes_{\mathfrak{A}} \mathfrak{h} \otimes_{\mathfrak{B}} \mathcal{F}$ of gauged fermions is the same as $\mathfrak{h}$. The unitary gauge groups are $\mathcal{U}_{\mathfrak{A}} = U(1) \times SU(2)$ and $\mathcal{U}_{\mathfrak{B}} = U(1) \times U(3)$. The gauge fields are simply given by vector potentials, i.e. self-adjoint elements of $\Omega^1_D(\mathfrak{A})$ and $\Omega^1_D(\mathfrak{B})$. As $\mathfrak{B}$ commutes with D, $\Omega^1_D(\mathfrak{B})$ vanishes, and so $\mathfrak{B}$ will play no role in the finite geometry.

Let us look at $\Omega^1_D(\mathfrak{A})$. Write an element ρ of $\Omega^1(\mathfrak{A})$ as $\rho = \sum a_j da'_j$, with $a_j, a'_j \in \mathfrak{A}$; $a_j = (\lambda_j, q_j)$, $a'_j = (\lambda'_j, q'_j)$; $q_j = \alpha_j + \beta_j j$, $q'_j = \alpha'_j + \beta'_j j$. (One can simplify the calculations by noting that (π_0, D_0) is essentially the degenerate case $M_u = 0$ of (π_1, D_1)).

One finds

$$\pi_0(\rho) = \begin{pmatrix} 0 & \phi_1 M_e & \phi_2 M_e \\ \phi'_1 M_e^* & 0 & 0 \\ -\bar{\phi}'_2 M_e^* & 0 & 0 \end{pmatrix} ,$$

$$\pi_1(\rho) = \begin{pmatrix} 0 & 0 & \phi_1 M_d & \phi_2 M_d \\ 0 & 0 & -\bar{\phi}_2 M_u & \bar{\phi}_1 M_u \\ \phi'_1 M_d^* & \phi'_2 M_u^* & 0 & 0 \\ -\bar{\phi}'_2 M_d^* & \bar{\phi}'_1 M_u^* & 0 & 0 \end{pmatrix}$$

where

$$\phi_1 = \Sigma \lambda_j (\alpha'_j - \lambda'_j) \qquad \phi_2 = \Sigma \lambda_j \beta'_j$$

$$\phi'_1 = \Sigma \alpha_j (\lambda'_j - \alpha'_j) + \beta_j \overline{\beta'_j} \qquad \phi'_2 = \Sigma\ \beta_j (\bar{\lambda}'_j - \bar{\alpha}'_j) - \alpha_j \beta'_j$$

Thus $\Omega^1_D(\mathcal{A}) = \{(\phi_1, \phi_2, \phi_1', \phi_2') \in \mathbb{C}^4\}$, the differential $d: \mathcal{A} \to \Omega^1_D(\mathcal{A})$ being given by

$$d(\lambda, \alpha, \beta) = (\alpha - \lambda, \beta, \lambda - \alpha, -\beta).$$

If ρ is a vector potential then $\phi'_1 = \bar{\phi}_1$ and $\phi'_2 = -\phi_2$. Similarly, one computes that

$$\pi_0(d\rho) = \begin{pmatrix} (\phi_1 + \phi'_1)\ M_e M_e^* & 0 \\ 0 & P \end{pmatrix}$$

with $$P = 1/2 \begin{pmatrix} \phi_1 + \phi'_1 + Y & \phi_2 + \phi'_2 + Z \\ -\bar{\phi}_2 - \bar{\phi}'_2 + \bar{Z} & \bar{\phi}_1 + \bar{\phi}'_1 - \bar{Y} \end{pmatrix} \otimes M_e^* M_e,$$

and $$\pi_1(d\rho) = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$$

with $$Q = \begin{pmatrix} (\phi_1 + \phi'_1) M_d M_d^* & (\phi_2 + \phi'_2)\ M_d M_u^* \\ (-\bar{\phi}_2 - \bar{\phi}'_2) M_u M_d^* & (\bar{\phi}_1 + \bar{\phi}'_1)\ M_u M_u^* \end{pmatrix}$$

and $$R = 1/2 \begin{pmatrix} \phi_1 + \phi'_1 & \phi_2 + \phi'_2 \\ -\bar{\phi}_2 - \bar{\phi}'_2 & \bar{\phi}_1 + \bar{\phi}'_1 \end{pmatrix} \otimes (M_d^* M_d + M_u^* M_u)$$

$$+ 1/2 \begin{pmatrix} Y & Z \\ \bar{Z} & -\bar{Y} \end{pmatrix} \otimes (M_d^* M_d - M_u^* M_u).$$

Here Y and Z are new fields given by

$$Y = \Sigma (\alpha_j - \lambda_j)(\lambda'_j - \alpha'_j) - \beta_j \overline{\beta'_j} \quad \text{and} \quad Z = \Sigma - (\alpha_j - \lambda_j)\beta'_j + \beta_j (\overline{\alpha'_j} - \overline{\lambda'_j}).$$

We see that $\{\pi(d\rho): \rho \in J_0 \cap \Omega^1_D(\mathcal{A})\} = \{\pi(d\rho): (\phi_1, \phi_2, \phi_1', \phi_2') = 0\}$ consists of the Y and Z fields. Then the quotienting used to define $\Omega^2_D(\mathcal{A})$ amounts to quotienting out the Y and Z fields. Considering the products $\pi(\rho_1)\,\pi(\rho_2)$, we see that $\Omega^2_D(\mathcal{A}) \cong \mathbb{C}^4$, with the product $\Omega^1_D(\mathcal{A}) \times \Omega^1_D(\mathcal{A}) \to \Omega^2_D(\mathcal{A})$ given by

$$(\phi_1, \phi_2, \phi'_1, \phi'_2) \times (\eta_1, \eta_2, \eta'_1, \eta'_2) =$$
$$(\phi_1\eta_1' - \phi_2\overline{\eta_2}', \phi_1\eta_2' + \phi_2\overline{\eta_1}', \phi'_1\eta_1 - \phi'_2\overline{\eta_2}, \phi'_1\eta_2 + \phi'_2\overline{\eta_1})$$

and the differential $d\colon \Omega^1_D(\mathcal{A}) \to \Omega^2_D(\mathcal{A})$ given by

$$d(\phi_1, \phi_2, \phi_1', \phi_2') = (\phi_1 + \phi_1', \phi_2 + \phi_2', \phi_1 + \phi_1', \phi_2 + \phi_2').$$

It follows that the curvature $\theta = dV + V^2$ of $V = (\phi_1, \phi_2, \overline{\phi_1}, -\phi_2)$ has image in $\Omega^2_D(\mathcal{A})$ given by

$$\pi_D(\theta) = \left(|1 + \phi_1|^2 + |\phi_2|^2 - 1 \right) (1, 0, 1, 0) \in \Omega^2_D(\mathcal{A}).$$

Then the Yang-Mills action is

$$YM = \langle\theta, \theta\rangle = \text{const.} \left(|1 + \phi_1|^2 + |\phi_2|^2 - 1 \right)^2.$$

Up to a shift of the ϕ_1 variable, this is the symmetry-breaking potential for the Higgs field, with $(\phi_1, \phi_2) = 0$ being a minimum.

Writing a fermion field $\Psi \in \mathfrak{h}$ as

$$\Psi = (e_R, e_L, \nu_L) \oplus (d_R, u_R, d_L, u_L),$$

the fermionic action

$$\mathcal{L}_\Psi = \overline{\Psi}\,(D + V)\,\Psi$$

gives exactly the Yukawa couplings of the standard model, after a shift of the ϕ_1 field. When (ϕ_1, ϕ_2) is frozen at its minimum $(0, 0)$, $\mathcal{L}_\Psi$ becomes

$$\overline{e_R}\, M_e\, e_L + \overline{d_R}\, M_d\, d_L + \overline{u_R}\, M_u\, u_L + \text{complex conjugate}$$

The "normal modes" of Ψ are given by the eigenstates of the matrices

$$(M_eM_e^*, M_e^*M_e, 0),\ (M_dM_d^*, M_uM_u^*, M_d^*M_d, M_u^*M_u),$$

and the corresponding fermion masses are the square roots of the eigenvalues. In this finite geometry, the fermion masses are the only physical information in the matrices M_e, M_d and M_u, but in the full standard model, to be described next, there is also a physically relevant N x N unitary matrix, the mixing matrix U. This matrix comes from the discrepancy between the eigenstates of the mass matrices and the weak curent interaction [El]. Explicitly, suppose that $M_u^*M_u$ and $M_d^*M_d$ are diagonalized by unitary matrices V_u and V_d:

$$M_u^*M_u = V_u\,\mathrm{Diag}_u\,V_u^{-1} \text{ and } M_d^*M_d = V_d\,\mathrm{Diag}_d\,V_d^{-1}.$$

Then $U = V_u^{-1}V_d$. We can easily describe U in terms of our finite-geometry. There are orthonormal bases for the vector spaces of d_L's and u_L's given by the eigenstates of the matrices $M_d^*M_d$ and $M_u^*M_u$. As $d_L + u_L j$ lies in $\mathbb{H}$, multiplication by the unit quaternion j maps the vector space of d_L's to the vector space of u_L's. Then U is simply the writing of this multiplication operator in terms of the preferred bases. As the eigenstates are only defined up to a phase, U is only defined up to right and left multiplication by $U(1)^N$.

Before leaving the finite-point geometry, we remark that there is a compact way to write its differential algebra. First, $\Omega_D^0(\mathcal{Q}) = \mathcal{Q} = \mathbb{C} \oplus \mathbb{H} \subset \mathbb{H} \oplus \mathbb{H}$. Next,

$$\Omega_D^1(\mathcal{Q}) = \{(\phi_1, \phi_2), (\phi_1', \phi_2') \in \mathbb{C}^2 \oplus \mathbb{C}^2\} \cong \{(q_1, q_2) \in \mathbb{H} \oplus \mathbb{H}\}.$$

With the identification $q_1 = \phi_1 + \phi_2 j$ and $q_2 = \phi'_1 + \phi'_2 j$, the bimodule structure on $\Omega_D^1(\mathcal{Q})$ is given by

$$(\lambda, q)\,(q_1, q_2) = (\lambda q_1, q q_2) \qquad \forall q_1, q_2 \in \mathbb{H}$$

$$(q_1, q_2)\,(\lambda, q) = (q_1\,q, q_2\,\lambda) \qquad \lambda \in \mathbb{C}, q \in \mathbb{H}$$

and the differential d being again the *finite difference:*

$$d(\lambda, q) = (q - \lambda, \lambda - q) \in \mathbb{H} \oplus \mathbb{H}.$$

The involution on $\Omega_D^1(\mathcal{Q})$ is given by:

$$(q_1, q_2)^* = (q_2^*, q_1^*) \quad \forall q_1, q_2 \in \mathbb{H}.$$

The space $\mathcal{V}$ of vector potentials is thus naturally isomorphic to $\mathbb{H}$.

Finally, $\Omega^2_D(\mathcal{A}) \cong \mathbb{H} \oplus \mathbb{H}$ with an $\mathcal{A}$-bimodule structure given by:

$$(\lambda, q)\,(q_1, q_2)\,(\lambda', q') = (\lambda q_1 \lambda', q q_2 q') \qquad \forall \lambda, \lambda' \in \mathbb{C}\ , q\text{'s} \in \mathbb{H}.$$

The product: $\Omega^1_D \times \Omega^1_D \to \Omega^2_D$ is given by:

$$(q_1, q_2)\ \ (q'_1, q'_2) = (q_1\, q'_2\, , q_2\, q'_1)$$

and the differential $d : \Omega^1_D \to \Omega^2_D$ is given by

$$d(q_1, q_2) = (q_1 + q_2, q_1 + q_2)\,.$$

The curvature θ of a vector potential $V = (\,q, q^*\,)$ is then

$$\theta = dV + V^2 = (q + q^* + qq^*\, , q + q^* + q^*q) = (|1 + q|^2 - 1)\ \ (1,1).$$

d) *Geometric structure of the standard model*

For the full standard model, we take the product geometry of the finite space (b) and the Riemannian geometry of a spin 4-manifold X, where the product is in the sense of Section 2. So we have an $\mathcal{A}$ - $\mathcal{B}$ bimodule $(\mathfrak{H}, D, \Gamma)$ with

$$\mathcal{A} = C^\infty(M, \mathbb{R}) \otimes (\mathbb{C} \oplus \mathbb{H}),\ \mathcal{B} = C^\infty(M, \mathbb{R}) \otimes (\mathbb{C} \oplus M_3(\mathbb{C})).$$

The corresponding unitary gauge groups are

$$\mathcal{U}_{\mathcal{A}} = \mathrm{Map}(X, U(1) \times SU(2)),\ \ \mathcal{U}_{\mathcal{B}} = \mathrm{Map}(X, U(1) \times U(3))$$

The Hilbert space is $\mathfrak{H} = \mathfrak{H}_0 \oplus (\mathfrak{H}_1 \otimes \mathbb{C}^3)$, with

$$\mathfrak{H}_0 = L^2(X,S) \otimes (\mathbb{C} \oplus \mathbb{H}) \otimes \mathbb{R}^N,\ \mathfrak{H}_1 = L^2(X,S) \otimes (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}) \otimes \mathbb{R}^N.$$

The representations π_0 and π_1 of $\mathcal{A}$ are the same as in (b). Letting z denote the element (1, -1) of the center of $\mathcal{A}$, the grading operator on $\mathfrak{H}$ is $\Gamma = \gamma_5 \otimes \pi(z)$. The self-adjoint operator $D = D_0 \oplus D_1$ is given by

$$D_0 = \begin{bmatrix} \partial_X \otimes I_N & \gamma_5 \otimes M_e & 0 \\ \gamma_5 \otimes M_e^* & \partial_X \otimes I_N & 0 \\ 0 & 0 & \partial_X \otimes I_N \end{bmatrix}$$

$$D_1 = \begin{bmatrix} \partial_X \otimes I_N & 0 & \gamma_5 \otimes M_d & 0 \\ 0 & \partial_X \otimes I_N & 0 & \gamma_5 \otimes M_u \\ \gamma_5 \otimes M_d^* & 0 & \partial_X \otimes I_N & 0 \\ 0 & \gamma_5 \otimes M_u^* & 0 & \partial_X \otimes I_N \end{bmatrix}$$

Here M_e, M_d and M_u are complex N x N matrices.

The computation of the gauge theory on this space is similar to that done in Example 5, so we shall only state the results. First, consider the $\mathfrak{A}$ algebra. One finds that a universal 1-form ρ is represented by

$$\pi_0(\rho) = \begin{pmatrix} \gamma(A) \otimes I_N & \phi_1 \gamma_5 \otimes M_e & \phi_2 \gamma_5 \otimes M_e \\ \phi'_1 \gamma_5 \otimes M_e^* & \gamma(W_1) \otimes I_N & \gamma(W_2) \otimes I_N \\ -\overline{\phi}'_2 \gamma_5 \otimes M_e^* & -\gamma(\overline{W}_2) \otimes I_N & \gamma(\overline{W}_1) \otimes I_N \end{pmatrix},$$

$$\pi_1(\rho) = \begin{pmatrix} \gamma(A) \otimes I_N & 0 & \phi_1 \gamma_5 \otimes M_d & \phi_2 \gamma_5 \otimes M_d \\ 0 & \gamma(\overline{A}) \otimes I_N & -\overline{\phi}_2 \gamma_5 \otimes M_u & \overline{\phi}_1 \gamma_5 \otimes M_u \\ \phi'_1 \gamma_5 \otimes M_d^* & \phi'_2 \gamma_5 \otimes M_u^* & \gamma(W_1) \otimes I_N & \gamma(W_2) \otimes I_N \\ -\overline{\phi}'_2 \gamma_5 \otimes M_d^* & \overline{\phi}'_1 \gamma_5 \otimes M_u^* & \gamma(-\overline{W}_2) \otimes I_N & \gamma(\overline{W}_1) \otimes I_N \end{pmatrix}$$

Here (A, W_1, W_2) are complex-valued 1-forms on X, and $(\phi_1, \phi_2, \phi'_1, \phi'_2)$ are complex-valued functions on X. So $\Omega^1_D(\mathfrak{A}) \cong (\Lambda^1(X, \mathbb{C}))^3 \oplus (\Lambda^0(X, \mathbb{C}))^4$. The differential map $d: \mathfrak{A} \to \Omega^1_D(\mathfrak{A})$ is given by

$$d(\lambda, \alpha + \beta j) = (d\lambda, d\alpha, d\beta) \oplus (\alpha - \lambda, \beta, \lambda - \alpha, -\beta) \in \Omega^1_D(\mathfrak{A}).$$

If ρ is a vector potential then A is u(1)-valued, W is su(2)-valued and $\phi'_1 = \bar{\phi}_1$, $\phi'_2 = -\phi_2$. Thus a vector potential consists of a u(1) gauge field A, an su(2) gauge field W and a Higgs doublet ϕ.

In order to compute $\Omega^2_D(\mathcal{C})$, it is enough to just consider $\pi_1(d\rho)$, as $\pi_0(d\rho)$ is then obtained by taking $M_u = 0$. Separating the various terms with respect to their differential-form grading on M, one finds

$$\pi_1(d\rho) = \begin{pmatrix} \gamma(dA)\otimes I_N & 0 & 0 & 0 \\ 0 & \gamma(d\bar{A})\otimes I_N & 0 & 0 \\ 0 & 0 & \gamma(dW_1)\otimes I_N & \gamma(dW_2)\otimes I_N \\ 0 & 0 & \gamma(-d\bar{W}_2)\otimes I_N & \gamma(d\bar{W}_1)\otimes I_N \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & D\phi_1\gamma_5\otimes M_d & D\phi_2\gamma_5\otimes M_d \\ 0 & 0 & -\overline{D\phi_2}\gamma_5\otimes M_u & \overline{D\phi_1}\gamma_5\otimes M_u \\ D\phi'_1\gamma_5\otimes M_d^* & D\phi'_2\gamma_5\otimes M_u^* & 0 & 0 \\ -\overline{D\phi'_2}\gamma_5\otimes M_d^* & \overline{D\phi'_1}\gamma_5\otimes M_u^* & 0 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} \tilde{A}\otimes I_N & 0 & 0 & 0 \\ 0 & \bar{\tilde{A}}\otimes I_N & 0 & 0 \\ 0 & 0 & \tilde{W}_1\otimes I_N & \tilde{W}_2\otimes I_N \\ 0 & 0 & -\bar{\tilde{W}}_2\otimes I_N & \bar{\tilde{W}}_1\otimes I_N \end{pmatrix}$$

$$+ \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix}$$

with

$$Q = \begin{pmatrix} (\phi_1 + \phi'_1) M_d M_d^* & (\phi_2 + \phi'_2) M_d M_u^* \\ (-\bar{\phi}_2 - \bar{\phi}'_2) M_u M_d^* & (\bar{\phi}_1 + \bar{\phi}'_1) M_u M_u^* \end{pmatrix}$$

and

$$R = 1/2 \begin{pmatrix} \phi_1 + \phi'_1 & \phi_2 + \phi'_2 \\ -\bar{\phi}_2 - \bar{\phi}'_2 & \bar{\phi}_1 + \bar{\phi}'_1 \end{pmatrix} \otimes (M_d^* M_d + M_u^* M_u)$$

$$+ 1/2 \begin{pmatrix} Y & Z \\ \bar{Z} & -\bar{Y} \end{pmatrix} \otimes (M_d^* M_d - M_u^* M_u).$$

Here $D(\phi_1, \phi_2) = d(\phi_1, \phi_2) + A(\phi_1, \phi_2) - (\phi_1, \phi_2)\begin{pmatrix} W_1 & W_2 \\ -\overline{W}_2 & \overline{W}_1 \end{pmatrix}$, and $\tilde{A}$, $\widetilde{W}$, Y and Z are new scalar fields. We see that

$$\{\pi(d\rho): \rho \in J_0 \cap \Omega^1(\mathfrak{A})\} = \{\pi(d\rho): A = W = (\phi_1, \phi_2, \phi_1', \phi_2') = 0\}$$

consists of the $\tilde{A}$, $\widetilde{W}$, Y and Z fields. Then the quotienting used to define $\Omega^2_D(\mathfrak{A})$ amounts to quotienting out the new scalar fields. Let us assume, for example, that the matrix $M_d^* M_d + M_u^* M_u$ is not a multiple of the N x N identity matrix; then we find that $\Omega^2_D(\mathfrak{A})$ comes from the tensor product of the differential algebra of the finite-point space by the exterior algebra of X. Namely, an element of $\Omega^2_D(\mathfrak{A})$ consists of

a. A $\mathbb{C}$-valued 2-form on X and an $\mathbb{H}$-valued 2-form on X.

b. Two $\mathbb{H}$-valued 1-forms on X.

c. Two $\mathbb{H}$-valued 0-forms on X.

If $V_{\mathfrak{A}}$ is a vector potential then its curvature $\theta_{\mathfrak{A}} = dV_{\mathfrak{A}} + V_{\mathfrak{A}}^2 \in \Omega^2_D(\mathfrak{A})$, a self-adjoint element, consists of the following components:

a. The curvature F_A of the u(1)-gauge field A and the curvature F_W of the su(2) gauge field W.

b. The covariant derivative $D\phi$ of the Higgs field ϕ, and its conjugate.

c. The function $\left(|1 + \phi_1|^2 + |\phi_2|^2 - 1\right)$ times (1,1).

The story with the $\mathfrak{B}$ algebra is much simpler. As $\mathfrak{B}$ commutes with the off-diagonal terms of D, it is easy to see that $\Omega^*_D(\mathfrak{B})$ is just $\mathfrak{B} \otimes \Lambda^*(X, \mathbb{C})$, with the obvious multiplication and differentiation. Then a vector potential $V_{\mathfrak{B}}$ is the sum of a u(1) gauge field K and a u(3) gauge field V, and its curvature $\theta_{\mathfrak{B}} \in \Omega^2_D(\mathfrak{B})$ is same as the usual field strength $(dK, dV + V^2) \in (u(1) \oplus u(3)) \otimes \Lambda^2(X)$.

The gauge group of our theory is $Map(X, U(1)_{\mathfrak{A}} \times SU(2)_{\mathfrak{A}} \times U(1)_{\mathfrak{B}} \times U(3)_{\mathfrak{B}})$. In order to correctly reduce the gauge fields to take values in $u(1) \oplus su(2) \oplus su(3)$, we must impose following condition on the gauge fields :

(9) $\quad A = K = -\mathrm{Tr}\, V$.

Then the contributions to the net hypercharges of the fermions are as shown:

	A	K	V	Y
e_R	-1	-1	0	-2
(e_L, ν_L)	0	-1	0	-1
d_R	-1	0	1/3	-2/3
u_R	1	0	1/3	4/3
(d_L, u_L)	0	0	1/3	1/3.

We show in the Appendix that (9) has a natural interpretation as a ***unimodularity*** condition on the gauge fields, of the same general type as the reduction from a U(N) gauge theory to an SU(N) gauge theory. In particular, equation (9) is an infinitesimal version of equation (12) of the Appendix.

We shall now compute the action. Let us start with the fermionic part. If we write a fermion field $\Psi \in \mathfrak{h}$ as

$$\Psi = (e_R, e_L, \nu_L) \oplus (d_R, u_R, d_L, u_L),$$

then $\overline{\Psi}\,(D + V_{\mathfrak{A}} + V_{\mathfrak{B}})\,\Psi$ becomes the terms $\mathcal{L}_\Psi$ and $\mathcal{L}_Y$ of the standard model, after a shift of the ϕ_1 field.

We now must compute the Yang-Mills action. If we were to follow the previous discussion, we would simply take

$$\text{(10)} \qquad YM = \langle\theta_{\mathfrak{A}}, \theta_{\mathfrak{A}}\rangle + \langle\theta_{\mathfrak{B}}, \theta_{\mathfrak{B}}\rangle.$$

However, this would be physically wrong, as our Hilbert spaces of fermions are not irreducible under the action of the gauge group. Consequently, using (10) would have the effect of artificially imposing relations among coupling constants. A more general gauge invariant bosonic action is given by

$$\text{(11)} \qquad \mathcal{L} = \mathrm{Tr}_\omega\,(z_1\, \theta_{\mathfrak{A}}{}^2\, D_V{}^{-4}) + \mathrm{Tr}_\omega\,(z_2\, \theta_{\mathfrak{B}}{}^2\, D_V{}^{-4}),$$

where z_1 and z_2 are arbitrary positive operators on $\mathfrak{h}$ which commute with the actions of $\mathfrak{A}$ and $\mathfrak{B}$, and with the operator D. With this freedom, the Lagrangian (11) reproduces the terms $\mathcal{L}_G + \mathcal{L}_\phi + \mathcal{L}_V$ of the standard model, with arbitrary constants in $\mathcal{L}_G$ and $\mathcal{L}_V$ (after a rescaling of the Higgs field). Thus we recover the standard model on the nose, with the same number of arbitrary coupling constants. On the other hand, one could require in addition that the operators z_1 and z_2 lie in the center of $\mathcal{A} \oplus \mathcal{B}$. In this case we find one relationship among the coupling constants of the standard model. We will not write out this relationship here, but will simply note that it gives the Higgs mass in terms of the W mass and the fermion masses. In particular, if the top quark mass is of the same order of magnitude as the W mass, then the relationship implies that the Higgs mass would be, also. However, this relationship is not preserved by the usual renormalization flow, and we do not know if it is physically meaningful.

Let us summarize some of the improvements of the present paper over our previous paper. In [CL] we had the following :

1. The complex conjugate of the up quark in the Hilbert space, and a charge conjugation in the operator D.

2. An $(\mathcal{A}, \mathcal{B})$ bimodule structure.

3. Equation (9) relating the U(1) factors.

In the present paper, we simplified the first point by changing the action of the $\mathcal{A}$ algebra. (This simplification was noticed independently by D. Kastler.) We again have the bimodule structure, but Section 3 of the present paper puts this into a more general context. And equation (9) is now interpreted in the Appendix as a special case of a unimodularity condition which makes sense in noncommutative geometry.

5. Appendix

We discuss a notion of unimodularity which makes sense in a general algebraic setting. First, suppose that one has a C^*-algebra C and a self-adjoint trace τ on C. That is, $\tau(x^*) = \overline{\tau(x)}$ for all $x \in C$. Then one can define the phase of a unitary element of C by

$$\mathrm{Phase}_\tau(u) = \frac{1}{2\pi i} \int_0^1 \tau(u'(t)\, u(t)^{-1})\, dt,$$

where u(t) is a smooth path of unitaries joining 1 to u. So this phase is only defined in the connected component of the identity in the group $\mathcal{U}(C)$ of unitaries, and is ambiguous up to a countable subgroup of $\mathbb{R}$, namely the image $\langle \tau, K_0(C) \rangle$ of $K_0(C)$ by the trace [CK].

The condition $\mathrm{Phase}_\tau(u) = 0$ defines a normal subgroup of the connected component of the identity, which we will denote by $S_\tau(C)$.

Now let $\mathcal{A}$ and $\mathcal{B}$ be involutive algebras, and $(\mathfrak{h}, D)$ a (d, ∞)-summable bimodule over $\mathcal{A}$ and $\mathcal{B}$. We shall apply the above considerations to the C^*-algebra C generated by $\mathcal{A}$ and $\mathcal{B}$ in $\mathfrak{h}$, with a family of traces τ_ρ on C constructed from self-adjoint elements $\rho = \rho^*$ of the center of $\mathcal{A}$:

$$\tau_\rho(x) = \mathrm{Tr}_\omega(\rho\, x\, |D|^{-d}) \qquad \text{for all } x \in C.$$

We thus get a normal subgroup $S_{\mathcal{A}}(C)$ of the unitary group of C by intersecting all of the $S_{\tau_\rho}(C)$'s. Since $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$ is a subgroup of $\mathcal{U}(C)$, its intersection with $S_{\mathcal{A}}(C)$ gives a normal subgroup $S(\mathcal{A}, \mathcal{B})$ of $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$.

Example 6: Let X be a Riemannian spin manifold. Take $\mathcal{A} = M_N(C^\infty(X))$, $\mathcal{B} = \mathbb{C}$, $\mathfrak{h} = (L^2(X,S))^N$ and D to be the Dirac operator. Then the space of self-adjoint elements of the center of $\mathcal{A}$ is $\{f\, I_N : f \in C^\infty(X) \text{ real}\}$, and one finds $S(\mathcal{A}, \mathcal{B}) = \mathrm{Map}(X, SU(N))$. This is why in general, one can think of $S(\mathcal{A}, \mathcal{B})$ as a sort of unimodular unitary group.

Example 7: Take $\mathcal{A}$, $\mathcal{B}$, $\mathfrak{h}$ and D as in Section 4b above. A self-adjoint element of the center of $\mathcal{A}$ can be written as $\lambda_1 e + \lambda_2(1 - e)$ for some real numbers λ_1 and λ_2, with $e = (1, 0) \in \mathbb{C} \oplus \mathbb{H}$ and $1 - e = (0, 1) \in \mathbb{C} \oplus \mathbb{H}$. It follows that

$$S(\mathcal{A}, \mathcal{B}) = (\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})) \cap (SU(e\mathfrak{h}) \times SU((1 - e)\mathfrak{h})).$$

Let then U be an element of $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$. It is given by a quadruple:

$$U = (\lambda, q), (u, v)); \qquad \lambda \in U(1),\ q \in SU(2),\ u \in U(1),\ v \in U(3).$$

We have $\mathfrak{h} = \mathfrak{h}_0 \oplus (\mathfrak{h}_1 \otimes \mathbb{C}^3)$, with the action of U given by

$$(\pi_0(\lambda, q) \otimes u) \oplus (\pi_1(\lambda, q) \otimes v).$$

This operator restricts to both $e\mathfrak{h}$ and $(1 - e)\mathfrak{h}$, and we must compute the determinants of these restrictions. We get

$$\det(U_e) = (u^2 (\det(v))^2)^N, \qquad \det(U_{1-e}) = (\lambda\, u\, (\det(v))^2)^N.$$

So the unimodularity condition is

$$(12) \qquad \lambda = u = (\det(v))^{-1},$$

and $S(\mathcal{A}, \mathcal{B}) = U(1) \times SU(2) \times SU(3)$.

Example 8: Take $\mathcal{A}$, $\mathcal{B}$, $\mathfrak{h}$ and D as in Section 4d above. Then it is easy to see that $S(\mathcal{A}, \mathcal{B})$ consists of maps from X to the unimodular unitary group of Example 7, that is $\mathrm{Map}(X, U(1) \times SU(2) \times SU(3))$.

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