Finite Generation of Canonical Rings and Flip Conjecture

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dedicated to Professor T. Nagano
on the occasion of his 60th birthday

Abstract

We prove the finite generation of canonical rings of projective varieties of general type and the flip conjecture in all dimension. As a consequence we prove the minimal model conjecture up to dimension 4 which is previously known to be true up to dimension 3 by Mori ([10]).

1 Introduction

The *classification theory of algebraic varieties* is the attempt to study all algebraic varieties by decomposing them into 3 kinds of particles :

1. varieties with negative K_X,
2. varieties with numerically trivial K_X,
3. varieties with positive K_X

and their fibre spaces. As for the particles of the 1-st kind, S. Mori invented his cone theorem ([11]) to single out these particles. The purpose of this article is to single out the particles of the 3-rd kind globally (the existence of canonical model) and locally (the flip conjecture). In comparison with Mori's theory, the method in the present paper is quite transcendental in nature. In my opinion it seems to be hopeless to obtain the results in this paper by a purely algebraic method because the canonical ring of an algebraic variety seems to be a quite transcendental object.

As for the 2nd particles, there are no essential ways to single out these particles at present. This problem is called *the abundance conjecture.* Our method does not work to single out the particles of 2nd kind.

The following conjecture is one of the central problem in the classification theory of algebraic varieties.

Conjecture 1.1 *(Minimal Model Conjecture) Let X be a normal projective variety. Assume that X is not uniruled. Then there exists a minimal projective variety X_{min} (cf. Definition 2.5) which is birational to X.*

This conjecture is trivial in the case of algebraic curve and is known to be true classically in the case of $\dim X = 2$. Recently S. Mori solved the conjecture in the case of $\dim X = 3$([10]). His method depends on the close study of 3-dimensional terminal singularities and it seems to be difficult to generalize his method to the case of higher dimensional varieties. I hope that the present paper will give a perspective of the conjecture in all dimension because our method is independent of the dimension of the variety. In fact, we prove *Flip Conjecture*(existence of flip) in all dimension in this paper. Hence to prove the minimal model conjecture , we only need to prove the termination of flips. In particular since the termination of flips is known in the case of $\dim X \leq 4$ ([8]), we have a solution of *Minimal Model Conjecture* in the case of $\dim X \leq 4$.

In this paper all varieties and morphisms are defined over **C**.

The the following theorems are main results in this paper.

Theorem 1.1 *Let X be a smooth projective variety of general type. Then the canonical ring*

$$R(X, K_X) = \oplus_{\nu\geq 0} H^0(X, \mathcal{O}_X(\nu K_X))$$

is finitely generated. Hence the canonical model

$$X_{can} = Proj\, R(X, K_X)$$

exists.

In the case of $\dim X = 2$, this theorem was proved by D. Mumford([17, appendix]) and recently S. Mori proved Theorm 1.1 in the case of $\dim X = 3$ in terms of the existence of minimal models of 3-folds ([10]).

The following conjecture is essential in the construction of minimal model in the case of dimension greater than 2.

Conjecture 1.2 *(Flip Conjecture) Let X be a projective variety with only terminal singularities. Let $\varphi : X \longrightarrow X'$ be a birational contraction of an extremal ray (cf.[7, 11]). Then*

$$R(X/X', K_X) = \oplus_{\nu\geq 0} \varphi_* \mathcal{O}_X([\nu K_X])$$

is finitely generated as an $\mathcal{O}_{X'}$-algebra.

As a corollary of Theorem 1.1, we have:

Theorem 1.2 *Flip conjecture holds in all dimension.*

This theorem implies the existence of minimal model in the case of $\dim X \leq 4$.

Theorem 1.3 *Let X be a normal projective variety of dimension ≤ 4. If X is not uniruled, then there exists a minimal algebraic variety X_{min} which is birational to X.*

The proof of Theorem 1.1 is closely related to the cone theorem of Mori and Kawamata([11, 6]) although it is purely analytic in nature. Mori proved his cone theorem by his method *bend and break curves.* Instead of curves we *bend and break Kähler forms* by Hamilton's heat flow.

2 Preliminaries

Let X be a normal projective vareity of dimension n. We denote by $Z_{n-1}(X)$ (resp. $Div(X)$), the group of Weil (resp. Cartier) divisor on X. The canonical divisor K_X is defined by

$$K_X = i_*\Omega^n_{X_{reg}},$$

where $i : X_{reg} \longrightarrow X$ is the canonical injection. K_X is an element of $Z_{n-1}(X)$. An $\mathbf{R}-$divisor D is an element of $Z_{n-1}(X)\otimes\mathbf{R}$, i.e. $D = \sum d_j D_j$ (finite sum). where $d_j \in \mathbf{R}$ and the D_j are mutually distinct prime divisor on X.

If $D \in Div(X) \otimes \mathbf{R}$, we say that D is $\mathbf{R}$-Cartier. We define round up $\lceil D\rceil$, the integral part $[D]$, the fractional part $\{D\}$ and the round off $\langle D\rangle$ by

$$\lceil D\rceil = \sum\lceil d_j\rceil D_j, [D] = \sum[d_j]D_j,$$
$$\{D\} = \sum\{d_j\}D_j, \langle D\rangle = \sum\langle d_j\rangle D_j,$$

where $\lceil r\rceil, [r]$ and $\langle r\rangle$ for $r \in \mathbf{R}$ are integers such that

$$r - 1 < [r] \leq r \leq \lceil r\rceil < r + 1$$

$$r - \frac{1}{2} \leq \langle r\rangle < r + \frac{1}{2}$$

and

$$\{r\} = r - [r].$$

Definition 2.1 *$D \in Div(X) \otimes \mathbf{R}$ is said to be nef if $D \cdot C \geq 0$ holds for every effective curve on X.*

Definition 2.2 *Let X be a normal projective variety. We say that X has only canonical (resp. terminal) singularity, if K_X is $\mathbf{Q}$-Cartier, i.e. $K_X \in Div(X) \otimes \mathbf{Q}$ and there is a resolution of singularity $\mu : Y \longrightarrow X$ such that the exceptional locus F of μ is a divisor with normal crossings and*

$$K_Y = \mu^*(K_X) + \sum a_j F_j,$$

where $a_j \geq 0$ (resp $a_j > 0$).

The following definition is more general.

Definition 2.3 *A pair (X, Δ) for $\Delta \in Z_{n-1}(X) \otimes \mathbf{Q}$ is said to be logcanonical (resp. logterminal) if the following conditions are satisfied.*

1. $[\Delta] = 0$ *and* $K_X + \Delta \in Div(X) \otimes \mathbf{Q}$.

2. *There is a resolution of singularity $\mu : Y \longrightarrow X$ such that the union F of the exceptional locus of μ and the inverse image opf the support of Δ is a divisor with normal crossings and*

$$K_Y = \mu^*(K_X + \Delta) + \sum a_j F_j, a_j \geq -1(resp. > -1).$$

Definition 2.4 *A normal projective variety X is said to be $\mathbf{Q}$-factorial, if every Weil divisor is $\mathbf{Q}$-Cartier.*

In this paper, we use the notion of minimal varieties in the following sense.

Definition 2.5 *Let X be a normal projective variety. X is said to be minimal, if the following condition is satisfied.*

1. *X has only terminal singularities.*

2. *K_X is nef.*

3. *X is $\mathbf{Q}$-factorial.*

Definition 2.6 *$D \in Div(X) \otimes \mathbf{Q}$ is said to be big, if $\kappa(X, D) = \dim X$*

Now we shall define Zariski decomposition.

Definition 2.7 *And expression $D = P + N$, $(D, P, N \in Div(X) \otimes \mathbf{R})$ is called a Zariski decomposition of D if the following conditions are satisfied.*

1. *D is big.*

2. *P is nef.*

3. *N is effective.*

4. *The natural homomorphisms*

$$H^0(X, \mathcal{O}_X([mP])) \longrightarrow H^0(X, \mathcal{O}_X([mD]))$$

are bijective for all positive integer m.

Conjecture 2.1 *Let X be a smooth projective variety of general type. Then there exists a modification*

$$f : \tilde{X} \longrightarrow X$$

such that f^*K_X has a Zariski decomposition.

By [6] to prove Theorem 1.1 it is sufficient to solve Conjecture 2.1. In this paper we shall prove Theorem 1.1 by solving Conjecture 2.1. To solve Conjecture 2.1, we shall use the theory of currents which is considered to be a generalization of the notion of subvarieties.

Let M be a complex manifold of dimension n.

Definition 2.8 *The current $\mathcal{D}^{p,q}(M)$ of type (p,q) are the continuous linear functional on the compactly supported C^∞ forms of type $(n-p, n-q)$, $A_c^{n-p,n-q}(M)$ with the C^∞-topology.*

$$\partial : \mathcal{D}^{p,q}(M) \longrightarrow \mathcal{D}^{p+1,q}(M),\ \bar{\partial} : \mathcal{D}^{p,q}(M) \longrightarrow \mathcal{D}^{p,q+1}(M)$$

are defined by

$$\partial T(\varphi) = (-1)^{p+q+1}T(\partial\varphi),\ \bar{\partial}T(\varphi) = (-1)^{p+q+1}T(\bar{\partial}\varphi)$$

for $T \in \mathcal{D}^{p,q}(M)$ and we set $d = \partial + \bar{\partial}$. A (p,p) current T is real in case $T = \bar{T}$ in the sense that $\overline{T(\varphi)} = T(\bar{\varphi})$v for all $\varphi \in A_c^{n-p,n-p}(M)$ and a real current T is positive in case

$$(\sqrt{-1})^{p(p-1)/2}T(\eta \wedge \bar{\eta}) \geq 0, \eta \in A_c^{n-p,0}(M).$$

Let V be a subvariety of codimension p in M. Then

$$V(\varphi) = \int_V \varphi, \quad \varphi \in A_c^{n-p,n-p}(M)$$

is a d-closed positive (p,p)-current. Hence we can consider subvarieties as d-closed positive currents. On the other hand, every $C^\infty(p,p)$-form ψ on M defines a (p,p)-current T_ψ by

$$T_\psi(\varphi) = \int_M \psi \wedge \varphi, \quad \varphi \in A_c^{n-p,n-p}(M).$$

The current of this type is called a smooth current. As we explain below, a general d-closed positive current is basically somewhere between the smooth currents and those supported by analytic varieties. Let T be a d-closed positive (p,p)-current on M. For each point $x \in M$ we define a number

$$\Theta(T, x)$$

defined as follows. Let (U, z) be a local coordinate around x $(z(x) = O)$. We set

$$B[r] = \{y \in U \,|\, \| z(y) \| < 1\},$$

$$\omega = \frac{\sqrt{-1}}{2}\sum_{i=1}^{n} dz_i \wedge d\bar{z}_i,$$

$\chi[r]$: the characteristic function of $B[r]$.

We define $\Theta(T, x)$ by

$$\Theta(T, x) = \lim_{r \downarrow 0} \frac{1}{\pi^{n-p}r^{2n-2p}} T(\chi[r]\omega^{n-p})$$

and call it the Lelong number of T at x. The Lelong number exists and finite for all d-closed positive (p,p)-current (cf. [4, pp.390-391]) and it is independent of the choice of the coordinate ([13]). It is easy to see that $\Theta(T, x) = 0$ for every $x \in M$, if T is a smooth current. On the other hand we have:

Theorem 2.1 *([4, p. 391]) Let $V \subset M$ be a subvariety of codimension p in M. Then we have*

$$\Theta(T_V, x) = mult_x V.$$

The following theorem is fundamental for our purpose.

Theorem 2.2 *([13]). Let T be a d-closed positive (p,p)-current on M. Then for every positive number ε*

$$S_\varepsilon(T) = \{x \in M \mid \Theta(T, x) \geq \varepsilon\}$$

is a subvariety of codimension $\geq p$.

For the later use, we need the following lemma.

Lemma 2.1 *Let T be a d-closed positive $(1,1)$-current on a smooth quasi-projective variety X such that*

1. *There exists a nonempty Zariski open subset Y of X such that $T \mid Y$ is smooth.*
2. *$\Theta(T, x) = 0$ for every $x \in X$.*

Then for every complete irreducible reduced curve C in X,

$$T(C) \geq 0.$$

Proof. Let $p \in C$ be a smooth point of C and let $(U, z_1, \ldots, z_n)$ be a local coordinate of X such that

1. $(z_1(p), \ldots, z_n(p)) = O$.
2. $V \cap C = \{q \in X \mid z_2(q) = \ldots = z_n(q) = 0\}$.

Then

$$T \mid V = (\sqrt{-1})^{n-1} \delta_{U \cap C} dz_2 \wedge d\bar{z}_2 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n,$$

where $\delta_{U \cap C}$ is a positive measure supported on $U \cap C$. Let $\{\delta_\varepsilon\}$ $(\varepsilon \in [0,1])$ be a smoothing of $\delta_{V \cap C}$ by positive smooth functions (for example molify $\delta_{U \cap C}$ by a Friedrichs molifier). Then we have

$$\begin{aligned}(T \mid U)(C) &= \lim_{\varepsilon \downarrow 0} (T \mid U)((\sqrt{-1})^{n-1} \delta_\varepsilon dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n) \\ &= \lim_{\varepsilon \downarrow 0} \int_U (\sqrt{-1})^{n-1} \delta_\varepsilon \cdot T \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n.\end{aligned}$$

Since $\Theta(T, x) = 0$ for every $x \in X$, we have

$$(T \mid U)(C) = \lim_{\varepsilon \downarrow 0} \int_{U \setminus C} (\sqrt{-1})^{n-1} \delta_\varepsilon \cdot T \wedge dz_2 \wedge d\bar{z}_2 \wedge \cdot \wedge dz_n \wedge d\bar{z}_n \geq 0.$$

If C is smooth this completes the proof of the lemma. If C is not smooth, we take an embedded resolution $\pi : \tilde{X} \longrightarrow X$ of C. Then by considering the pullback $\pi^* T$, the same argument is valid (note that $\Theta(\pi^* T, \tilde{x}) \geq \Theta(T, \pi(\tilde{x}))$ in general). Q.E.D.

3 Deformation of Kähler form I

Let X be a smooth projective variety of general type and let $n = \dim X$. Let ω_0 be a C^∞-Kähler form on X. We consider the initial value problem:

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}_\omega - \omega \quad \text{on } X \times [0, T) \tag{1}$$

$$\omega = \omega_0 \quad \text{on } X \times \{0\}, \tag{2}$$

where

$$\mathrm{Ric}_\omega = -\sqrt{-1}\partial\bar{\partial} \log \omega^n$$

and T is the maximal existence time for C^∞-solution.

Since

$$\frac{\partial}{\partial t}(d\omega) = -d\omega \quad \text{on } X \times [0, T)$$

$$d\omega_0 = 0 \quad \text{on } X \times \{0\},$$

we have that $d\omega = 0$ on $X \times [0, T)$, i.e. ,the equation preserves the Kähler condition. Let ω denote the de Rham cohomology class of ω in $H^2_{DR}(X, \mathbf{R})$. Since $-(2\pi)^{-1}\mathrm{Ric}_\omega$ is a first Chern form of K_X, we have

$$[\omega] = (1 - \exp(-t))2\pi c_1(K_X) + \exp(-t)[\omega_0]. \tag{3}$$

Let Ω be a C^∞-volume form on X and let

$$\omega_\infty = -\mathrm{Ric}\,\Omega = \sqrt{-1}\partial\bar{\partial}\log\Omega.$$

We set

$$\omega_t = (1 - \exp(-t))\omega_\infty + \exp(-t)\omega_0. \tag{4}$$

Since $[\omega] = [\omega_t]$ on $X \times \{t\}$ for every $t \in [0, T)$, there exists a C^∞-function u on $X \times [0, T)$ such that

$$\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u. \tag{5}$$

By (1), we have

$$\frac{\partial}{\partial t}(\omega_t + \sqrt{-1}\partial\bar{\partial}u) = \sqrt{-1}\partial\bar{\partial}\log(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n - (\omega_t + \sqrt{-1}\partial\bar{\partial}u).$$

Hence

$$\begin{aligned} &\exp(-t)(\omega_\infty - \omega_0) + \sqrt{-1}\partial\bar{\partial}(\frac{\partial u}{\partial t}) \\ = &\sqrt{-1}\partial\bar{\partial}\log(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n - \omega_\infty + \exp(-t)(\omega_\infty - \omega_0). \end{aligned}$$

Then (1) is equivalent to the initial value problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u \quad \text{on } X \times [0, T) \\ u &= 0 \quad \text{on } X \times \{0\}. \end{aligned} \tag{6}$$

Let

$$A(X) = \{[\eta] \mid \eta : \text{Kähler form on } X\} \subset H^2_{DR}(X, \mathbf{R})$$

be the Kähler cone of X. Since $[\omega]$ moves on the segument connecting $[\omega_0]$ and $[\omega_\infty] = 2\pi c_1(K_X)$, we cannot expect T to be ∞, unless $2\pi c_1(K_X)$ is on the closure of $A(X)$ in $H^2_{DR}(X, \mathbf{R})$. We shall determine T. It is standard to see that $T > 0$ ([5]).

Theorem 3.1 *If $\omega_0 - \omega_\infty$ is a Kähler form, then T is equal to*

$$T_0 = \sup\{t > 0 \mid [\omega_t] \in A(X)\}.$$

The proof of Theorem 3.1 is almost parallel to that of [14].

Lemma 3.1 *If $\omega_0 - \omega_\infty$ is a Kähler form, then there exists a constant C_0 such that*

$$\frac{\partial u}{\partial t} \leq C_0 \exp(-t).$$

Proof.

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta_\omega \frac{\partial u}{\partial t} - \frac{\partial u}{\partial t} - tr_\omega(\omega_0 - \omega_\infty)$$

holds by defferentiating (5) by t. By the maximum principle, we have

$$\frac{\partial u}{\partial t} \leq (\max \log \frac{\omega_0^n}{\Omega}) \exp(-t).$$

Q.E.D.

To estimate u from below, we modify (6) as

$$\begin{aligned} \frac{\partial u}{\partial t} &= \log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_t^n} + f_t - u \quad \text{on } X \times [0, T_1) \\ u &= 0 \quad \text{on } X \times \{0\}, \end{aligned} \tag{7}$$

where

$$f_t = \log \frac{\omega_t^n}{\Omega} \tag{8}$$

and

$$T_1 = \min\{\sup\{t > 0 \mid \omega_t > 0\}, T\} \tag{9}$$

If $t \in [0, T_1)$, we have

$$\log \frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\omega_t^n} = \int_0^1 \frac{d}{ds} \log \frac{(\omega_t + \sqrt{-1}s\partial\bar{\partial}u)^n}{\omega_t^n} ds = \int_0^1 \Delta_s u ds,$$

where Δ_s is the Laplacian with respect to the Kähler form $\omega_t + \sqrt{-1}s\partial\bar{\partial}u$. Then by the minimum principle, (7) and Lemma 2.2, we have

Lemma 3.2

$$u \geq -C_0 \exp(-t) + \min_X f_t \quad \textit{on } X \times \{t\}, t \in [0, T_1)$$

We note that this estimate is depending on t and C_0 is independent of the choice of Ω.

For the next we shall obtain a C^2-estimate of u.

Lemma 3.3 *Let M be a compact Kähler manifold and let ω, $\tilde{\omega}$ be Kähler forms on M. Assume that there exists a C^∞-function φ such that*

$$\tilde{\omega} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi.$$

We set

$$f = \log \frac{\tilde{\omega}^n}{\omega^n},$$

R : curvature tensor of ω, Then for every positive constant C

$$\exp(C\varphi)\tilde{\Delta}(\exp(-C\varphi)(n+\Delta\varphi)) \geq$$

$$(\Delta f - n^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}} - Cn(n+\Delta\varphi)$$

$$+(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}})(n+\Delta\varphi)^{\frac{n}{n-1}} \exp(-\frac{f}{n-1})$$

holds

Applying this lemma to ω_t and $\omega = \omega_t + \sqrt{-1}\partial\bar{\partial}u$, we have:

Lemma 3.4 *For every $C > 0, t \in [0, T_1)$,*

$$\exp(Cu)(\Delta_\omega - \frac{\partial}{\partial t})(\exp(-Cu)tr_{\omega_t}\omega) \geq$$

$$-(\Delta_t \log \frac{\omega_t^n}{\Omega} + n^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}}(t) + n)$$

$$-C(n - \frac{1}{C} - \frac{\partial u}{\partial t})tr_{\omega_t}\omega - \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty)\cdot\sqrt{-1}\partial\bar{\partial}u$$

$$+(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}}(t))\exp(\frac{1}{n-1}(-\frac{\partial u}{\partial t} - u + \log\frac{\omega_t^n}{\Omega}))(tr_{\omega_t}\omega)^{\frac{n}{n-1}}$$

holds, where
Δ_t : Laplacian with respect to ω_t,
$R_{i\bar{i}l\bar{l}}(t)$: the bisectional curvature of ω_t.

Proof. Let

$$f = \log \frac{\omega^n}{\omega_t^n} = \frac{\partial u}{\partial t} + u - \log\frac{\omega_t^n}{\Omega}.$$

Then by Lemma 2.3, we have

$$\begin{aligned} &\exp(Cu)\Delta_\omega(\exp(-Cu)tr_{\omega_t}\omega) \\ &\geq (\Delta_t f - n^2 \inf_{i\neq l} R_{i\bar{i}l\bar{l}}(t)) - Cn(n+\Delta_t u) \\ &+(C + \inf_{i\neq l} R_{i\bar{i}l\bar{l}})(tr_{\omega_t}\omega)^{\frac{n}{n-1}} \exp(-\frac{f}{n-1}). \end{aligned}$$

Since

$$\begin{aligned} \Delta_t f &= \Delta_t(\frac{\partial u}{\partial t} + u - \log\frac{\omega_t^n}{\Omega}) \\ &= \Delta_t\frac{\partial u}{\partial t} + tr_{\omega_t}\omega - n - \Delta_t \log\frac{\omega_t^n}{\Omega} \end{aligned}$$

and

$$\exp(Cu)\frac{\partial}{\partial t}(\exp(-Cu)tr_{\omega_t}\omega)$$

$$= -C\frac{\partial u}{\partial t}tr_{\omega_t}\omega + tr_{\omega_t}\frac{\partial \omega}{\partial t} - tr_{\omega_t}\frac{\partial \omega_t}{\partial t}\cdot\omega$$

$$= -C\frac{\partial u}{\partial t}tr_{\omega_t}\omega + \Delta_t\frac{\partial u}{\partial t} - \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty) + \exp(-t)tr_{\omega_t}(\omega_0 - \omega_\infty)\cdot\omega,$$

we obtain the lemma. Q.E.D.

Let ε be an arbitrary small positive number. We set

$$T_1(\varepsilon) = \min\{\sup\{t > 0 \mid \omega_t > 0\} - \varepsilon, T\}$$

and let C be a positive number such that

$$C + \inf_{i \neq l} R_{i\bar{i}l\bar{l}}(t) > 0$$

for all $t \in [0, T_1(\varepsilon)]$. Then since the function $x\exp(-x)$ is bounded on $[0,\infty)$, by the maximum principle and Lemma 3.4, we have that if $(x_0, t_0) \in X \times [0, T_0(\varepsilon)]$ is a maximal point of $\exp(-Cu)tr_{\omega_t}\omega$, we have

$$tr_{\omega_t}\omega(x_0, t_0) < C_\varepsilon$$

for some $C_\varepsilon > 0$ depending only on ε. Then by the C^0-estimate of u ; Lemma 3.1 and Lemma 3.2, we have that there exists a positive constant $C'_{1,\varepsilon}$ such that

$$tr_{\omega_t}\omega < C'_{1,\varepsilon}.$$

Hence we obtain:

Lemma 3.5 *There exists a positive constant $C_{1,\varepsilon}$ depending only on $T_1(\varepsilon)$ such that*

$$\| u \|_{C^2(X)} \leq C_{2,\varepsilon}$$

for every $t \in [0, T_1(\varepsilon))$, where $\| \ \|_{C^r(X)}$ is the C^2-norm with repsect to ω_0.

Now by [15], for every $r \geq 2$ there exists a positive constant $C_{r,\varepsilon}$ depending only on $T_1(\varepsilon)$ such that

$$\| u \|_{C^r(X)} \leq C_{r,\varepsilon}.$$

Letting ε tend to 0, we have that

$$T \geq T_1$$

holds. Since $[\omega_{T_0}]$ is on the closure of the Kähler cone $A(X)$, by changing Ω properly, we can make $T_0 - T_1 > 0$ arbitarary small. Hence we conclude that $T = T_0$. This completes the proof of Theorem 3.1.

4 Deformation of Kähler form II

In this section we use the same notation as in the last section. In the laset section, we gave the maximal existence time of for the smooth solution of the initial value problem (1). In this section, we shall prove the long time existence of the current solution of (1) which is smooth on a Zariski open subset of X.

Theorem 4.1 *There exists a Zariski open subset U of X and a d-closed positive current solution ω of (1) such that*

1. ω is smooth on U.

2. $\omega_E = \lim_{t\to\infty}\omega$ exists as a d-closed positive $(1,1)$-current and ω_E is a smooth Kähler-Einstein form on U, i.e.,

$$Ric_{\omega_E} = -2\pi\omega_E$$

on U.

3. $[\omega_E] = 2\pi c_1(K_X)$.

The following lemma is fundamental in the proof of Theorem 4.1.

Lemma 4.1 *(Kodaira's lemma) Let D be a big divisor (cf. Definition 2.6) on a smooth projective variety M. Then there exists a effective $\mathbf{Q}$-divisor E such that $D - E$ is an ample $\mathbf{Q}$-divisor.*

Proof. Let H be a very ample divisor on M. Then

$$0 \to H^0(M, \mathcal{O}_M(mD - H)) \to H^0(M, \mathcal{O}_M(mD)) \to H^0(H, \mathcal{O}_H(mD \mid H))$$

is exact. Since D is big, for a sufficiently large m, $\mid mD - H \mid$ is nonempty. This completes the proof of the lemma. Q.E.D.

By Lemma 4.1, there exists an effective $\mathbf{Q}$-divisor F such that $K_X - F$ is ample. Let

$$F = \sum a_i F_i$$

be the decomoposition of E into the irreducible components. Let h_i be the hermitian metric of the line bundle $\mathcal{O}_X(F_i)$ and let σ_i be a nontrivial global section of $\mathcal{O}_X(F_i)$. We consider the degenerate volume form

$$\Omega_F = (\prod_i \parallel \sigma_i \parallel^{2a_i})\Omega,$$

where $\parallel \sigma_i \parallel$ denotes the norm of σ_i with respect to h_i respectively. By the definition of F, if we take h_i properly, we may assume that

$$-\mathrm{Ric}\,\Omega_F = \sqrt{-1}\partial\bar{\partial}\log\Omega_F$$

is a Kähler form on X. We set

$$\omega_{F,t} = \omega_t + \sqrt{-1}\sum a_i\partial\bar{\partial}\log \parallel \sigma_i \parallel^2 .$$

and

$$u_F = u - \sum a_i \log \parallel \sigma_i \parallel^2 .$$

Then u_F satisfies the partial differential eqation:

$$\frac{\partial u_F}{\partial t} = \log\frac{(\omega_{F,t} + \sqrt{-1}\partial\bar{\partial}u_F)^n}{\Omega_F} - u_F \quad \text{on } (X - F) \times [0, T_F)$$

$$u_F = -\sum a_i \log \parallel \sigma_i \parallel^2 \quad \text{on } (X - F) \times \{0\}, \qquad (10)$$

where

$T_F =$ the maximal existence time for the smooth solution $u_F \in C^\infty(X - F)$ of (10).

Proposition 4.1 $T_F = \infty$.

Because we consider the solution only on $X - F$, we would like to forget the boundary F. By a suitable blow up of X, we may assume that $SuppF$ is a divisor with simple normal crossings. Let A be a smooth ample divisor on X such that $A + F$ is a divisor with normal crossings and $A + F$ is also ample. Let ρ be a global holomorphic section of $\mathcal{O}_X(A)$ and let $\| \ \|$ be a hermitian norm on $\mathcal{O}_X(A)$ such that

$$-\sqrt{-1}\sum a_i\partial\bar{\partial}\log \| \sigma_i \|^2 - \sqrt{-1}\partial\bar{\partial}\log \| \rho \|^2$$

is a Kähler form on X. Then

$$v = -\sum a_i \log \| \sigma_i \|^2 - \log \| \rho \|^2$$

is a smooth strongly plurisubharmonic exhaustion function on $X - F - A$.

For every nonegative number ε, we set

$$\omega_{A,t} = \omega_t + \sqrt{-1}\varepsilon\sum_i \partial\bar{\partial}\log(\log \| \sigma_i \|^2)^2 + \sqrt{-1}\varepsilon\partial\bar{\partial}\log(\log \| \rho \|^2)^2$$

and

$$\omega_{F,A,t} = \omega_{A,t} + \sqrt{-1}\sum a_i\partial\bar{\partial}\log \| \sigma_i \|^2 .$$

If we take ε sufficiently small by direct computation, we see that $\omega_{A,0}$, $\omega_{F,A,t}$ are complete Kähler forms of logarithmic growth on $X - F - A$ and $\omega_{F,A,t}$ have uniformly bounded curvarure with respect to t as in [9]. We set

$$\Omega_{F,A} = (\log \| \rho \|^2)^{-2\varepsilon}\prod_i(\log \| \sigma_i \|^2)^{-2\varepsilon}\Omega_F$$

To prove Proposition 4.1, we consider the following perturbed equation:

$$\begin{aligned} \frac{\partial u_{F,A}}{\partial t} &= \log\frac{(\omega_{F,A,t} + \sqrt{-1}\partial\bar{\partial}u_{F,A})^n}{\Omega_{F,A}} - u_{F,A} \text{ on } (X - F - A) \times [0, T_{F,A}) \\ u_{F,A} &= -\sum_i a_i \log \| \sigma_i \|^2 - \varepsilon\log(\log \| \rho \|^2)^2 - \varepsilon\sum_i \log(\log \| \sigma_i \|^2)^2 \\ & \quad \text{on } (X - F - A) \times \{0\}, \end{aligned} \tag{11}$$

where $T_{F,A}$ is the maximal existence time of the smooth solution $u_{F,A}$. We note that this purturbed equation is essentially the same equation as (6) or (10) on $(X - F - A) \times [0, T)$.

Proposition 4.2 *$T_{F,A}$ is ∞.*

Proof of Proposition 4.2. Let K be a strongly psedoconvex subdomain in $X - F - A$ with smooth boundary ∂K. We consider the following initial value problem with the Dirichlet boundary condition.

$$\begin{aligned} \frac{\partial u_{K,A}}{\partial t} &= \log\frac{(\omega_{K,A,t} + \sqrt{-1}\partial\bar{\partial}u_{F,A})^n}{\Omega_{F,A}} - u_{K,A}\text{on } K \times [0, T_{K,A}) \\ u_{K,A} &= -\sum_i a_i \log \| \sigma_i \|^2 - \varepsilon\log(\log \| \rho \|^2)^2 - \sum_i \varepsilon\log(\log \| \sigma_i \|^2)^2 \text{ on } \partial K \times [0, T_{K,A}) \\ u_{K,A} &= -\sum_i a_i \log \| \sigma_i \|^2 - \varepsilon\log(\log \| \rho \|^2)^2 - \sum_i \varepsilon\log(\log \| \sigma_i \|^2)^2 \text{ on } K \times \{0\}, \end{aligned}$$

where $T_{K,A}$ is the maximal existence time for the smooth solution on $\bar{K}$(the topological closure of K). By the standard implicit funtion theorem and the boundary regularity of complex Monge-Ampere equations in [2], we see that $T_{K,A} > 0$. We note that for every sufficiently small positive number ε_0 , $K_X - F - \varepsilon_0 A$ is ample. Then we have the following C^0-estimate

$$\frac{\partial u_{K,A}}{\partial t} \leq C_0 \exp(-t)$$

and

$$u_{K,A} \geq -C_0 + \varepsilon_0 \log \| \rho \|^2$$

by the maximum principle as Lemma 3.1, where C_0 is a constant independent of K. Let $\{K_\nu\}_{\nu=1}^{\infty}$ be an exhaustion of $X - F - A$ by a sequence of strongly pseudoconvex subdomains with smooth boundary. We would like to show that $\{u_{K_\nu,A}\}$ converges uniformly on every compact subset of $X - F - A$. We set

$$H = \| \rho \|^{2\varepsilon_0} (-\log \| \rho \|^2)^{2\varepsilon} \prod_i \| \sigma_i \|^{2a_i} (-\log \| \sigma_i \|^2)^{2\varepsilon}.$$

We need the following lemma.

Lemma 4.2 *([14, Lemma 3.2])*

$$\begin{aligned} H^{-C} \exp(C u_{K,A})(\Delta_{\omega_{K,A}} - \frac{\partial}{\partial t})(\exp(-C u_{K,A}) H^C tr_{\omega_{F,t}} \omega_{K,A}) \\ \geq (-\Delta_{\omega_{F,A,t}} \log \frac{\omega_{F,A,t}^n}{\Omega_{F,A}} - n^2 \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(F, A, t) - n) \\ +C(n - \frac{1}{C} - \frac{\partial u_{K,A}}{\partial t}) tr_{\omega_{F,A,t}} \omega_{K,A} - \exp(-t) tr_{\omega_{F,A,t}} (\omega_0 - \omega_\infty) \cdot \omega_{K,A} \\ +(C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(F, A, t)) \exp(\frac{1}{n-1}(-\frac{\partial u_{K,A}}{\partial t} - u_{K,A} + \log \frac{\omega_{F,A,t}^n}{\Omega}))(tr_{\omega_{F,A,t}} \omega_{K,A})^{\frac{n}{n-1}}, \end{aligned}$$

where $\inf_{i \neq j} R_{i\bar{i}j\bar{j}}(F, A, t)$ denotes the infimum of the bisectional curvature of th Kähler form $\omega_{F,A,t}$ and C is a positive constant such that $C + \inf_{i \neq j} R_{i\bar{i}j\bar{j}}(F, A, t) >$ 0 for all t.

The proof of Lemma 4.2 is the same as the proof of Lemma 3.4. Hence we omit it. Let us take K_ν as

$$K_\nu = \{x \in X \mid v(x) \leq 2^\nu\}.$$

Without loss of generality, we may assume that K_ν has a smooth boundary. We note that the second fundamental form of ∂K_ν is of polynomial growth with respect to H^{-1}. Then Lemma 4.3 and the C^2-estimate on the boundary by [2], if we take C sufficiently large, by the maximum principle, we obtain that there exists a positive constant C_2 independent of ν such that

$$H^C tr_{\omega_{F,A,t}} \omega_{K_\nu,A} \leq C_2$$

holds. Hence by the C^0-estimate of $u_{K,A}$ above, we have

$$tr_{\omega_{F,A,t}} \omega_{K_\nu,A} \leq C_2 H^{-3C}. \qquad (12)$$

This implies that on every compact subset of $X - F$ the C^2-norms of $\{u_{K_\nu A}\}$ is uniformly bounded. Again by [15], we obtain the uniform estimate of the higher derivatives of $\{u_{K_\nu,A}\}$ on every compact subset of $X - F - A$. Then

$$u_{F,A} = \lim_{\nu \to \infty} u_{K_\nu,A}.$$

is a solution of (11) on $(X - F - A) \times [0, \infty)$. This completes the proof of Proposition 4.2. Q.E.D.

Since the equation (11) is essentially the same as (10) or (6) on $(X - F - A) \times [0, T)$, we may assume that

$$\omega_{F,A,t} + \sqrt{-1}\partial\bar{\partial}u_{F,A} = \omega \text{ on } (X - F - A) \times [0, T).$$

Then by the real analyticity of the solution $u_{F,A}$, by moving A, we can define u_F by

$$u_F \mid (X - F - A) \times [0, \infty) = u_{F,A}.$$

This completes the proof of Proposition 4.1.

By the construction of u_F we have

Lemma 4.3 *There exists a constant C_0 such that*

$$\frac{\partial u_F}{\partial t} \leq C_0 \exp(-t)$$

and

$$u_F \geq -C_0.$$

By Lemma 4.4,

$$\omega = \omega_{F,t} + \sqrt{-1}\partial\bar{\partial}u_F$$

defines a d-closed positive $(1,1)$-current on X (where $\partial\bar{\partial}$ is the derivation as a current). Since ω satisfies the equation

$$\frac{\partial \omega}{\partial t} = -\mathrm{Ric}_\omega - \omega$$

on $X \times [0, \infty)$ as a d-closed positive $(1,1)$-current, we see that

$$[\omega] = \exp(-t)[\omega_0] + (1 - \exp(-t))2\pi c_1(K_X).$$

Moereover by the above estimate

$$\omega_E = \lim_{t \to \infty} \omega$$

exists as a d-closed positive $(1,1)$-current and ω_E is smooth on $X - F$.

Definition 4.1 *Let M be a compact complex manifold and let ω be a d-closed positive $(1,1)$-current. ω is said to be a Kähler-Einstein current if there exists a constant c and a nonempty Zariski open subset U of M such that*

1. *ω is smooth on U.*
2. *$Ric_\omega = c\omega$ on U.*

Definition 4.2 *Let $D \in Div(X)\otimes\mathbf{R}$ be an $\mathbf{R}$ Cartier divisor on a projective variety Y. Then the stable base locus of D is defined by :*

$$SBs(D) = \cap_{\nu>0} SuppBs \mid [\nu D] \mid .$$

Definition 4.3 *Let D be a a Cartier divisor on a projective variety V and let $\Phi_{|\nu D|} : V - \cdots \rightarrow \mathbf{P}^{N(\nu)}$ be the rational map associated with $\mid \nu D \mid$. Let $f_\nu : \tilde{X} \longrightarrow X$ be a resolution of the base locus of $\mid \nu D \mid$ and let $\tilde{\Phi}_{|\nu D|} : \tilde{X} \longrightarrow \mathbf{P}^{N(\nu)}$ be the associated morphism. Let $\tilde{E}(\nu D)$ denote the exceptional locus of $\tilde{\Phi}_{|\nu D|}$. We set*

$$E(\nu D) = f_\nu(\tilde{E}(\nu D))$$

and call it the exceptional locus of $\mid \nu D \mid$. It is easily be seen that $E(\nu D)$ is independent of the choice of the resolution of the base locus. We set

$$SE(D) = \cap_{\nu>0} E(\nu D)$$

and call it the stable exceptional locus of D.

Lemma 4.4 *We set $S = SBs(K_X) \cup SE(K_X)$. Then there exists a modification $\mu : Y \longrightarrow X$ such that*

1. *$\mu \mid Y - \mu^{-1}(S) : Y - \mu^{-1}(S) \longrightarrow X - S$ is biregular.*
2. *The exceptional locus E of μ is a divisor with normal crossings.*
3. *There exists an effective $\mathbf{Q}$-divisor D such that $SuppD \subseteq E$ and $\mu^* K_X - D$ is ample on Y.*

Proof. Elementary. Q.E.D.

Hence by the proof of Theorem 4.1, we obtain :

Theorem 4.2 *Let X be a smooth projective variety of general type. Let $S = SBs(K_X) \cup SE(K_X)$. Then there exists a Kähler-Einstein current ω_E on X which is smooth on $X - S$.*

5 L^2-vanishing theorem

In the last section we constructed a Kähler-Einstein current ω_E on X. In this section we shall prove the following theorem.

Theorem 5.1 *$\{\Theta(\omega_E, x) \mid x \in X\}$ is a finite set.*

The following lemma is the starting point of our argument.

Lemma 5.1 *Let $S = SBs(K_X) \cup SE(K_X)$. Then $X - S$ is pseudoconcave in the sense of Andreotti([1]).*

Proof. Unless $SBs(K_X)$ contains a divisor, we have nothing to prove. Let $D \subseteq SBs(K_X)$ be an irreducible divisor. We clain that D is contained in $SE(K_X)$. Let $\Phi_\nu : X - \cdots \rightarrow \mathbf{P}^{N(\nu)}$ be the rational map associated with $\mid \nu K_X \mid$ and let $\pi_\nu : \tilde{X} \longrightarrow X$ be the resolution of $Bs \mid \nu K_X \mid$ and let $\tilde{\Phi}_\nu : \tilde{X} \rightarrow \mathbf{P}^{N(\nu)}$be the associated morphism. Let $\tilde{D}$ be the strict transform of D. If we take ν properly, we may assume that $\tilde{\Phi}_\nu : \tilde{X} \longrightarrow \tilde{\Phi}_\nu(\tilde{X})$ is birational and $\tilde{\Phi}_\nu(\tilde{D})$ is a divisor. If we take some positive multiple of ν if necessary, we may assume that $\tilde{\Phi}_\nu(\tilde{X})$ is normal. Then $\tilde{\Phi}_\nu(\tilde{X})_{reg} \cap \tilde{\Phi}_\nu(\tilde{D})$ is nonempty. On the other hand $\tilde{D}$ is in the branched locus of $\tilde{\Phi}_\nu$ because D is in $Bs \mid \nu K_X \mid$. Since $\tilde{\Phi}_\nu$ is birational, we see that codim $\tilde{\Phi}_\nu(\tilde{D}) \geq 2$. This is a contradiction. Q.E.D.

Now we shall briefly review the L^2-estimate on a complete Kähler manifold.

Let (M, ω) be a complete Kähler manifold of dimension m and let (L, h) be a hermitian line bundle on M. Let $A_c^{0,p}(M, L)(0 \leq p \leq m)$ denote the space of L-valued smooth $(0,p)$ form on M with compact support. Let

$$\bar{\partial} : A_c^{0,p}(M, L) \longrightarrow A_c^{0,p+1}(M, L)$$

be the natural $\bar{\partial}$ operator and let

$$\vartheta : A_c^{0,p}(M, L) \longrightarrow A_c^{0,p-1}(M, L)$$

be the formal adjoint of $\bar{\partial}$ Let $\mathcal{L}^{0,p}(M, L)$ denote the space obtained by taking the form closure with respect to the graph norm

$$A_c^{0,p}(M, L) \ni f \mapsto \| f \|^2 + \| \bar{\partial} f \|^2 + \| \vartheta f \|^2 .$$

We define the L^2-cohomology group $H_2^p(M, \mathcal{O}_M(L))$ by

$$H_{(2)}^p(M.\mathcal{O}_M(L)) = \frac{\ker \bar{\partial} \mid \mathcal{L}^{0,p}(M, L)}{\overline{\bar{\partial} A_c^{0,p-1}(M.L)}},$$

where the closure is taken with respect to the graph norm. By Hörmander's L^2-estimate, we obtain:

Theorem 5.2 *Assume that there exists a volume form Ω in M and a positive constant c such that*

$$Ric\,\Omega - \sqrt{-1}\partial\bar{\partial} \log h \geq c\omega.$$

Then we have

$$H_{(2)}^p(M, \mathcal{O}_M(L)) = 0 \textit{ for } p > 0,$$

where the L^2-cohomology is taken with respect to the volume form Ω and h.

First we shall prove the following proposition.

Proposition 5.1

$$\{x \in X \mid \Theta(\omega_E, x) = 0\} \cap SBs(K_X) = \phi.$$

Proof of Proposition 5.1 . The following lemma is well known.

Lemma 5.2 *([13, p.95, Lemma 7.5]). Let T be a d-closed positive $(1,1)$ current on $B(r) = \{z \in \mathbf{C}^n \mid \| z \| < r\}$ for some $r > 0$. Let us consider $\mathbf{P}^{n-1}$ as a parameter space which parametrizes complex line through the origin O. Then for almost all $L \in \mathbf{P}^{n-1}$,*

$$\Theta(T, O) = \Theta(T \mid L \cap B(r))$$

holds. And for every $L \in \mathbf{P}^{n-1}$ such that $T \mid L \cap B(r)$ is well defined,

$$\Theta(T, O) \leq \Theta(T \mid L \cap B(r))$$

holds.

Lemma 5.3 *Let p be a point on X such that $\Theta(\omega_E, x) = 0$ and let $B_p : X_p \longrightarrow X$ be the blowing up with centre p. Let E_p denote the exceptional divisor of B_p. Then for every $x \in E_p$,*

$$\Theta(B_p^*\omega_E, x) = 0.$$

holds.

Proof of Lemma 5.3. By Lemma 5.2, we have that

$$\Theta(B_p^*\omega_E, x) = 0$$

holds for almost all $x \in E$. We note that $B_p^*(K_X)$ is numerically trivial on E_p. Let C be an irreducible reduced curve in $E_p \simeq \mathbf{P}^{n-1}$ then by the same augument as in the proof of Lemma 2.1, we see that $\Theta(\omega_E, x) = 0$ for all $x \in C$ since $(B_p^*K_X)\cdot C = 0$. This completes the proof of the lemma. Q.E.D.

Let p be a point on X such that $\Theta(\omega_E, p) = 0$ and let $B_p : X_p \longrightarrow X$ be as in Lemma 5.3. Let B denote the strict transform of $SBs(K_X)$. Let $\tau : \bar{W} \longrightarrow B_p(X)$ be a modification such that $\tau^{-1}(B)$ is a divisor with normal crossings. Let F be a reduced divisor such that $Supp F = \tau^{-1}(B)$ and let $W = \bar{W} - F$. Let E be the strict transform of E_p in $\bar{W}$. We set $\pi = \tau \circ B_p$ and $D = \pi^*(K_X)$. We consider the exact sequence :

$$H^0(W, \mathcal{O}_W(\nu D \mid W)) \to H^0(E\cap W, \mathcal{O}_{E\cap W}(\nu D \mid E\cap W)) \to H^1(W, \mathcal{O}_W(\nu D - E \mid W)).$$

Since $D \mid E$ is trivial, $H^0(E, \mathcal{O}_E(\nu D)) \simeq \mathbf{C}$. Let us consider the homomorphism

$$u : H^0(E, \mathcal{O}_E(\nu D)) \to H^0(E\cap W, \mathcal{O}_{E\cap W}(\nu D \mid E\cap D)) \to H^1(W, \mathcal{O}_W(\nu D - E \mid W)).$$

If Im $u = 0$, we are done by the above exact sequence since because of the pseudoconcavity and the property of canonical singularity, we have the isomorphism:

$$H^0(X_p, \mathcal{O}_{X_p}(\nu D)) \simeq H^0(W, \mathcal{O}_W(\nu D \mid W)).$$

To prove that Im $u = 0$, we shall use the L^2-estimate. First we construct a complete Kähler metric on W. Let $F = \sum_i F_i$ be the decomposition of F into the irreducible components. Let σ_i be a nontrivial global holomorphic section of $\mathcal{O}_{\bar{W}}(F_i)$ with divisor F_i. Let $\omega_{\bar{W}}$ be a smooth Kähler form and let $\| \ \|$ be hermitian metrics on $\mathcal{O}_{\bar{W}}(F_i)$ We set

$$\omega_W = \omega_{\bar{W}} + \sqrt{-1}\partial\bar{\partial}(\sum_i \log(\log \| \sigma_i \|)^2).$$

Then by a direct calculation as in [3, p.565, Proposition 2.1], if we multiply a sufficiently large positive number to $\omega_{\bar{W}}$, ω_W is a complete Kähler form on W. Let L denote $\mathcal{O}_W(\nu D \mid W)$. Since $\Theta(\pi^*\omega_E)$ is 0 on W, there exists a singular hermitian metric h on L such that

1. $\nu\pi^*\omega_E \mid W = -\sqrt{-1}\partial\bar{\partial}\log h$.
2. Let h_0 be any smooth hermitian metric on $\mathcal{O}_{\bar{W}}(\nu D)$, then $\varphi_c = (h/h_0)^c$ is locally integrable on W for all $c > 0$(cf. Lemma 5.3 and Lemma 5.5 below).

We note that h is unique up to positive constant multiple because of the pseudoconcavity of W (cf. Lemma 5.1).

By the upper semicontinuity of Lelong numbers and Theorem 2.3, Lemma 5.3, 5.5, we have

Lemma 5.4 *For every $c > 0$, φ_c is locally integrable on a neighbourhood of E.*

Let τ be a nontrivial global section of $\mathcal{O}_W(E)$. Then since $\mathcal{O}_E(-E)(\simeq \mathcal{O}_{\mathbf{P}^{n-1}}(1))$ is ample and in $Div(\bar{W})$

$$\nu D - E - K_{\bar{W}} = (\nu - 1)D - nE - \sum_i b_i F_i$$

for some $b_i > 0$, if we take positive numbers d_i, hermitian metrics $\| \ \|$ of $\mathcal{O}_{\bar{W}}(F_i)$ and $\mathcal{O}_E$ properly, we may assume that there exists a positive number c_0 and a smooth volume form $\Omega_{\bar{W}}$ such that

$$\mathrm{Ric}\,\Omega_{\bar{W}} + \nu\pi^*\omega_E + \sum_i \sqrt{-1}d_i\partial\bar{\partial}\log \| \sigma_i \|^2 + \sqrt{-1}\partial\bar{\partial}\log \| \tau \|^2 \geq c_0\omega_{\bar{W}}$$

holds for every sufficiently large ν. Hence if we replace $\Omega_{\bar{W}}$ by

$$\Omega_W = (\prod_i (\log \| \sigma_i \|^2)^2)^{-\varepsilon}\Omega_{\bar{W}}$$

for a sufficiently small positive number ε, we see that there exists a singular hermitian metric h_ν on $L \otimes \mathcal{O}_W(-E \cap W)$ and a positive constant c_1 such that

$$\mathrm{Ric}\,\Omega_W + \sqrt{-1}\partial\bar{\partial}\log h_\nu \geq c_1\omega_W$$

holds for every sufficiently large ν. Then by a minor modification of Hörmander's L^2-estimate for $\bar{\partial}$, we obtain

$$H^p_{(2)}(W, \mathcal{O}_W(\nu L \otimes \mathcal{O}(-E)) = 0 (p > 0),$$

where L^2-cohomologies are taken with respect to the volume form Ω_W and the hermitian metric h_ν. By Lemma 5.5 and the construction of h_ν we see that if we take d_is' sufficiently small

$$\mathrm{Im}\, u \subseteq H^1_{(2)}(W, \mathcal{O}_W(\nu D - E \mid W)) = 0.$$

Hence we completes the proof of Proposition 5.1. Q.E.D.

Now we shall relate the Lelong number of ω_E and the multiplicity of the base scheme $Bs \mid \nu K_X \mid$. The following lemma is essential for the purpose.

Lemma 5.5 *([13, p.85, Lemma 5.3]) Let φ be an arbitrary plurisubharmonic function on an open subset U of $\mathbf{C}^n$ and let x be a point in U. Let $T = \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi$. Then the followings are true.*

1. *If $\Theta(T, x) < 2$, then $e^{-\varphi}$ is locally integrable at x.*

2. *If $\Theta(T, x) \geq 2n$, then $e^{-\varphi}$ is not locally integrable at x.*

Lemma 5.6 *There exists a positive integer ν_0 such that for every positive integer ν and $x \in X$,*

$$mult_x Bs \mid \nu K_X \mid \geq \nu\Theta(\omega_E, x)$$

holds.

Proof. Let $\Phi_\nu : X - \cdots \rightarrow \mathbf{P}^{N(\nu)}$ denote the rational map associated with $\mid \nu K_X \mid$. Let ν_0 be a positive integer such that Φ_ν is birational for every $\nu \geq \nu_0$. We fix ν such that $\nu \geq \nu_0$. Let $\pi_\nu : \tilde{X}_\nu \longrightarrow X$ be the resolution of $Bs \mid \nu K_X \mid$ such that $\pi_\nu^{-1}(Bs \mid \nu K_X \mid)$ is a divisor with normal crossings. Let $F_\nu = \sum c_i F_i$ be the fixed component of $\mid \nu\pi_\nu^* D \mid$. Then there exists an effective **Q**-divisor $R_\nu = \sum_i r_i F_i$ such that

$$\nu\pi_\nu^* K_X - F_\nu - R_\nu$$

is ample. We note that for every $\varepsilon > 0$, we can take R_ν such that $0 \leq r_i \leq \varepsilon$. Then by the proof of Theorem 4.2 (cf. Lemma 4.3) and Lemma 5.5, we have

$$\nu\Theta(\pi_\nu^*\omega_E, x) \leq na_i \;\; (x \in (F_{\nu,red})_{reg} \cap F_i).$$

This is not enough for our purpose, but by considering (local) holomorphic curves through x, by slicing the current $\pi_\nu^*\omega_E$ (cf. Lemma 5.3), we have

$$\nu\Theta(\pi_\nu^*\omega_E, x) \leq a_i \;\; (x \in (F_{\nu,red})_{reg} \cap F_i).$$

Since

$$\Theta(\omega_E, \pi_\nu(x)) \leq \Theta(\pi_\nu^*\omega_E, x)$$

holds for every $x \in \tilde{X}_\nu$ again by Lemma 5.3, we have

$$\nu\Theta(\omega_E, x) \leq \text{mult}_x Bs \mid \nu K_X \mid$$

for every $x \in X$ and $\nu \geq \nu_0$. Q.E.D.

Proposition 5.2 *We set*

$$S(\omega_E) = \{x \in X \mid \Theta(\omega_E, x) > 0\}.$$

Then $S(\omega_E) = SBs(K_X)$.

Proof of Proposition 5.2. By Lemma 5.6, we have that $S(\omega_E) \subseteq SBs(K_X)$. On the other hand, by Proposition 5.1, we have that $SBs(K_X) \subseteq S(\omega_E)$. Q.E.D. Let

$$S(\omega_E) = \sum_\alpha B_\alpha$$

be the decomposition of $S(\omega_E)$ into the irreducible components. Let

$$e_\alpha = \inf_{x \in B_\alpha} \Theta(\omega_E, x).$$

By Proposition 5.1, we see that $e_\alpha > 0$ for every α.

Lemma 5.7 $\Theta(\omega_E, x) = e_\alpha$ *for almost all $x \in B_\alpha$.*

Proof. Let $\pi_\alpha : X_\alpha \longrightarrow X$ be the blowing up with center B_α and let E_α be the excepsional divisor. By slicing lemma (Lemma 5.3), we have

$$\Theta(\pi_\alpha^* \omega_E, x) \geq \Theta(\omega_E, \pi_\alpha(x))$$

for all $x \in X_{\alpha,reg}$ and the equality holds for almost all $x \in E_\alpha \cap X_{\alpha,reg}$. Then the lemma follows from [13, p.89 Lemma 6.3]. Q.E.D.

Let $\pi_\alpha : X_\alpha \longrightarrow X$ be as in the proof of Lemma 5.7 above and let $\mu_\alpha : \tilde{X}_\alpha \longrightarrow X_\alpha$ be a modifaication such that $(\mu_\alpha \circ \pi_\alpha)^{-1}(B)$ is a divisor with normal crossings. Let $\tilde{E}_\alpha$ be the strict transform of $E_\alpha (= \pi_\alpha^{-1}(B_\alpha))$ and let $\varpi = \mu_\alpha \circ \pi_\alpha$. Let p be a point on $\tilde{E}_\alpha$ such that $\Theta(\varpi^* \omega_E, p) = e_\alpha$. Let $B_p : \bar{Y} \longrightarrow \tilde{X}_\alpha$ be the blowing up at p and let E be the exceptional divisor. Let F be the strict transform of $\varpi^{-1}(B)$ We set $Y = \bar{Y} - F$. Then by the same argument as in the proof of Proposition 5.1, we obtain:

Proposition 5.3

$$\{p \in \tilde{E}_\alpha \mid \Theta(\varpi^* \omega_E, p) = e_\alpha\} \cap SBs(\varpi^* K_X - e_\alpha \tilde{E}_\alpha) = \phi.$$

Proposition 5.3 implies that there exists a zariski open subset U_α of $\tilde{E}_\alpha$ such that

$$\Theta(\varpi^* \omega_E, x) = e_\alpha$$

for all $x \in U_\alpha$. Hence we have :

$$\Theta(\omega_E, x) = e_\alpha \text{ for } x \in \varpi(U_\alpha).$$

Then by Noetherian induction, we conclude that

$$\{\Theta(\omega_E, x) \mid x \in X\}$$

is a finite set. This completes the proof of Theorem 5.1.

6 Zariski Decomposition of Canonical Divisor

In this section we shall prove Conjecture 2.1. Let X, ω_E be as in Thoerem 5.1. Let

$$f : \tilde{X} = X_m \rightarrow X_{m-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 = X$$

be a successive blowing up such that

1. Let Z_i denote the centre of the blowing up

$$\mu_i : X_i \longrightarrow X_{i-1}.$$

   Then Z_i is smooth for each $1 \leq i \leq m$. We set $f_i : X_i \rightarrow X$ be the composition of the blowing ups.

2. $\Theta(f_{i-1}^* \omega_E)$ is constant and positive on Z_i.

3. $N_{red} = f^{-1}(S(\omega_E)_{red}$ is a divisor with normal crossings.

4. Let $N_{red} = \sum_j N_j$ be the decomposition of N_{red} into the irreducible components. Then $\Theta(f^*\omega_E)$ is constant on $N_j \cap (N_{red})_{reg}$ for every j. We denote the constant by n_j.

5. Let $p \in X_m$. Then
$$\Theta(f^*\omega_E, p) = \sum_{p \in N_j} n_j.$$

The existence of such successive blowing ups follows from Theorem 5.1 and the fact that ω_E has finite order along S by Lemma 4.3 and the estimate (12).

Now we set
$$N = \sum_j n_j N_j.$$

By the definition n_j are nonnegative real number. And let
$$P = f^*K_X - N.$$

Then we have :

Theorem 6.1 *The expression*
$$f^*K_X = P + N \, (P, N \in Div(\tilde{X}) \otimes \mathbf{R})$$
is a Zariski decomposition of f^*K_X. Hence Conjecture 2.1 is true.

Proof of Theorem 6.1. By the definition of N, N is effective. For the next, we shall prove that P is numerically effective.

Lemma 6.1 *([13, p. 87,Lemma 6.2]). Suppose T is a d-closed positive $(1,1)$-current on an open subset U of $\mathbf{C}^n$ and V is a a complex submanifold of condimension 1 in U. Suppose $c > 0$ and $\Theta(T, x) \geq c$ for every $x \in V$. Then $T - cV$ is positive.*

By Lemma 6.1, we see that $f^*\omega_E - N$ is a d-closed positive $(1,1)$-current on $\tilde{X}$. Hence P is numerically effective by Lemma 2.1.

On the other hand by Lemma 5.6, we have a natural inclusion:
$$0 \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu f^*K_X)) \to H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}([\nu P]))$$
for all $\nu \geq 0$. Because of the converse inclusion is trivial, we see that
$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}([\nu P])) \simeq H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\nu f^*(K_X)))$$
holds for all positive integer ν. This completes the proof of Theorem 6.1. Q.E.D.

Now Theorem 1.1 follows from the following theorem.

Theorem 6.2 *([6, p.425, Theorem 1]) Let $f : X \longrightarrow S$ be a proper surjective morphism of normal algebraic varieties, let Δ be a $\mathbf{Q}$-divisor on X such that the pair (X, Δ) is log-terminal. Assume that $K_X + \Delta$ is f-big, i.e. $\kappa(X_\eta, K_{X_\eta} + \Delta_\eta) = \dim X_\eta$, where X_η is the generic fibre of f and $\Delta_\eta = \Delta \mid X_\eta$, and that there exists the Zariski decomposition*
$$K_X + \Delta = P + N \quad \textit{in } Div(X) \otimes \mathbf{R}$$

of $K_X + \Delta$ relative to f. Then the positive part P is f-semiample. i.e., $mP \in Div(X)$ and the natural homomorphism

$$f^* f_* \mathcal{O}_X(mP) \to \mathcal{O}_X(mP)$$

is surjective for some positive integer m. Thus the relative log-canonical ring

$$R(X/S, K_X + \Delta) = \sum_{m \geq 0} f_* \mathcal{O}_X([m(K_X + \Delta)])$$

is finitely generated as an $\mathcal{O}_S$-algebra.

Theorem 1.2 follows from Theorem 1.1 easily because the problem is completely local (for the proof see [12, p.479,Proposition 4.4]). Since the termination of flips is known up to dimension 4 ([8, p.337,Theorem 5.15]), we have Theorem 1.3.

References

[1] Andreotti, A.- Tomassini, G., Some remarks on pseudoconcave manifolds, Essays on Topology and Related Topics, dedicated to G. de Rham, ed. by Haefliger and R. Narasimhan, 85-104, springer-Verlag (1970).

[2] Caffarelli, L.-Kohn J.J.-Nirenberg L.-Spruck J., The Dirichlet problem for nonlinear second-order elliptic equations. II. Complex Monge-Ampere, and uniformly elliptic equations, Comm. Pure and Appl. Math. 38, 209-252 (1985).

[3] Carlson, J.-Griffith, P.H., A defect relation for equidimensional holomorphic mappings between algebraic varieties, Ann. of Math. 95 (1972), 557-584.

[4] Griffith, P.A.-Harris, J, Priciples of Algebraic Geoemetry, John Wiley and Sons (1980).

[5] Hamilton, R.S., Three manifolds with positive Ricci curvature, J. Diff. Geom. 17, 255-306 (1982).

[6] Kawamata, Y., The Zariski Decomposition of Log Canonical Divisors, Proc. of Symp. in Pure Math. 46 (1987), 425-433.

[7] Kawamata, Y., The cone of curves of algebraic varieties, Ann. of Math. 119, 603-633 (1984).

[8] Kawamata, Y.-Matsuda, K.-Matsuki, K., Introduction to the Minimal Model Problem, Adv. Studies in Pure Math. 10, Algebraic Geometry Sedai 1985, 283-360 (1987).

[9] Kobayashi R., Einstein Kähler metrics on an open algebraic manifold, Osaka J of Math. 21 (1984), 399-418.

[10] Mori, S., Flip conjecture and the existence of minimal models for 3-folds, Jour. of Amer. Math. Soc. 1 (1988), 117-253.

[11] Mori, S., Three folds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982),133-176.

[12] Moriwaki, A., Semi-ampleness of the numerically effective part of Zariski decomposition, J. Math. Kyoto Univ. 26 (1986), 465-481.

[13] Siu, Y.-T., Analyticity of sets associocited to Lelong numbers and the extension of closed positive currents, Invent. Math. 26 (1976).

[14] Tsuji. H.: Existence and Degeneration of Kähler-Einstein Metrics on Minimal Algebraic Varieties of General Type, Math. Ann. 281, 123-133 (1988).

[15] Trudinger, N.S.: Fully nonlinear elliptic equation under natural structure conditions, Trans. Am. Math. Soc. 272, 751-769 (1983).

[16] Yau, S.-T., On the Ricci curvature of a compact Kähler manifold an the complex Monge Ampere equation I. Comm. Pure Appl. Math. 31, 339-411 (1978).

[17] Zariski, O. The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface, Ann. of Math. 76, 560-615 (1962).