Max-Planck-Institut für Mathematik
Bonn

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Preprint Series 2018 (26)

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MPIM 18-26

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Abstract

Using the exact formula about the A-twisted correlation functions of two dimensional $\mathcal{N} = (2, 2)$ gauged linear sigma model, we reconsidered the computation of the B-model Yukawa couplings of the local toric Calabi-Yau varieties. Our analysis is based on the exact result which has been evaluated from the supersymmetric localization technique and a careful treatment of its application. We provide a detailed description of a procedure to investigate the local B-model Yukawa couplings and also test our prescription by comparing the results with known expressions evaluated from the local mirror symmetry approach. Especially, we find that the ambiguities of classical intersection numbers of certain class of local toric Calabi-Yau varieties discovered before can be interpreted as degrees of freedom of the twisted mass deformations.

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1 Introduction

Two dimensional $\mathcal{N} = (2, 2)$ gauged linear sigma model (GLSM) [1] has played significant roles in the study of the topological quantum field theories and the mirror symmetry. A physical proof of mirror symmetry in [2] makes use of the duality properties of this model. Shortly after the work of [1], the topological A-twisted version of the GLSM was studied in [3] and the genus zero correlation functions were exactly computed for the A-models with target space a toric variety or a toric Calabi-Yau hypersurface. These developments led to a conjecture called toric residue mirror conjecture [4, 5] (later proven in [6, 7, 8, 9]), which states the existence of an equality between an integration over the toric compactification[1] of the moduli space of genus zero maps in the A-model and the toric residue [12] over the space of a holomorphic section in a toric variety in the B-model (see also [13]). The toric residue mirror conjecture provides an efficient method to compute the B-model Yukawa couplings exactly for the mirror varieties of toric Calabi-Yau complete intersections.

On the other hand, by using the supersymmetric localization technique [14], the exact formula about the A-twisted $\mathcal{N} = (2, 2)$ GLSM correlation functions on a supersymmetric two-sphere background has been clarified in [15, 16].[2] This approach does not depend on a particular choice of gauge group and enables us to calculate the Yukawa couplings for the GLSMs with non-abelian gauge groups efficiently [15, 19] (see also [20]). Meanwhile, the application for the models with non-compact target spaces has not been thoroughly investigated. The aim of this work is to fill this gap and establish the explicit formalism to investigate the local B-model Yukawa couplings from the A-twisted GLSM correlation functions on local toric Calabi-Yau varieties.

In this paper, we will explicitly demonstrate how to apply the exact formula about A-twisted GLSM correlators in [15] for the explicit computation of the B-model Yukawa couplings of local toric Calabi-Yau varieties. We remind that the application of the formula to the local toric Calabi-Yau varieties requires a careful treatment of the mass parameters of the chiral multiplets. Then we propose how to determine the proper mass deformations to conduct the exact calculation of the A-twisted GLSM correlators and check that the resulting A-twisted correlators coincide with known results for the B-model Yukawa couplings evaluated by the local mirror symmetry approach [21, 22, 23]. To our best knowledge, this relationship has not been thoroughly investigated before. Moreover, as a by-product, we also find that an ambiguity of the classical intersection numbers of a certain class of local toric Calabi-Yau varieties argued before in [22] can admit a new interpretation.

This paper is organized as follows. First we will take a brief look at the exact results about the GLSM correlation functions in Section 2. In Section 3, we explain the details about the topological properties of local toric Calabi-Yau varieties and propose the rules of mass assignment for applying the formula into the non-compact backgrounds. Then in Section 4 we will evaluate

[1] Toric compactifications can be generalized to stable toric quasimaps [10] allowing some degeneracies of moduli space of maps and further to stable quasimaps [11] which also include non-abelian varieties.

[2] It is worth noting that the supersymmetric localization of $\mathcal{N} = (2, 2)$ gauge theories has been also performed on a different two-sphere background, which corresponds to a fusion of the A- and $\bar{A}$- twists on two hemispheres [17, 18].

the GLSM correlators for several examples of local toric Calabi-Yau threefolds and fourfolds, and demonstrate that the resulting expressions are completely consistent with known results obtained by the local mirror symmetry approach. Finally we will conclude and propose several future directions. In Appendix A, we collect the building blocks of the localization formula, which are useful to understand the mirror transformation. The list of the local B-model Yukawa couplings for the local del Pezzo surface K_{dP_2} and the local A_2 geometry are summarized in Appendix B and C, respectively.

2 $\mathcal{N} = (2,2)$ GLSM on the Ω-deformed two-sphere

Here we will take a brief look at the two dimensional $\mathcal{N} = (2,2)$ GLSM on the Ω-deformed two-sphere $S^2_\hbar$ which is a one-parameter deformation of the A-twisted sphere. As performed in [15], supersymmetric localization formula considerably simplifies the exact calculations of the A-twisted correlation functions and realizes the quantum cohomology relations appropriately.

2.1 Exact formula about GLSM correlation functions

Following the approach to curved space rigid supersymmetry advocated in [24], supersymmetric backgrounds in two dimensions were studied in detail in [25]. A remarkable result is that the topological A-twist on the two-sphere admits an interesting $U(1)$ equivariant deformation called the Ω-deformation [26]. This deformation can be characterized by an equivariant parameter $\hbar$ which corresponds to a non-trivial expectation value for the graviphoton field. The ordinary topological A-twist is obtained by setting $\hbar = 0$.

Let G be a rank $\mathrm{rk}(G)$ gauge group with Lie algebra $\mathfrak{g}$ and Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In [15], an exact formula about correlation functions of $\mathcal{N} = (2,2)$ GLSM with gauge group G on the Ω-deformed two-sphere $S^2_\hbar$ was clarified as

$$\langle \mathcal{O}_1(\mathbf{u})|_N \, \mathcal{O}_2(\mathbf{u})|_S \rangle_\hbar = \frac{1}{|\mathcal{W}|} \sum_{\mathbf{d} \in \mathbb{Z}^{\mathrm{rk}(G)}} \mathbf{z}^{\mathbf{d}} \sum_{\mathbf{u}_*} \underset{\mathbf{u}=\mathbf{u}_*}{\text{JK-Res}} \left[\mathsf{Q}_*, \eta\right] \mathbf{I}_{\mathbf{d}} \left(\mathcal{O}_1 \mathcal{O}_2\right), \tag{2.1}$$

by using supersymmetric localization technique (see also [16]). Here $\mathcal{O}_1(\mathbf{u})|_N$ and $\mathcal{O}_2(\mathbf{u})|_S$ are gauge invariant operators constructed from the complex scalars $\mathbf{u} = \left(u_1, \ldots, u_{\mathrm{rk}(G)}\right) \in \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ in the vector multiplet inserted at the north and south poles, respectively. The order of the Weyl group of G is denoted by $|\mathcal{W}|$, and

$$\mathbf{z}^{\mathbf{d}} = \prod_{i=1}^{r} z_i^{d_i} = \prod_{i=1}^{r} \mathrm{e}^{2\pi\sqrt{-1}\left(\frac{\theta_i}{2\pi} + \sqrt{-1}\xi_i\right) d_i} \qquad (i = 1, \ldots r), \tag{2.2}$$

where θ_i and ξ_i are theta angles and Fayet-Iliopoulos parameters for the central $U(1)^r \subset G$. The parameters $\mathbf{d} = (d_1, \ldots, d_r)$ represent magnetic charges for the central $U(1)^r$ called GNO charges [27].

The factor $\mathbf{I}_{\mathbf{d}}\left(\mathcal{O}_1 \mathcal{O}_2\right)$ is a differential form given by

$$\mathbf{I}_{\mathbf{d}}\left(\mathcal{O}_1 \mathcal{O}_2\right) = \mathcal{O}_1\Big(\mathbf{u} - \frac{\mathbf{d}}{2}\hbar\Big) \, \mathcal{O}_2\Big(\mathbf{u} + \frac{\mathbf{d}}{2}\hbar\Big) Z^{\mathrm{vec}}_{\mathbf{d}}(\mathbf{u};\hbar) \prod_a Z^{\Phi_a}_{\mathbf{d}}(\mathbf{u};\hbar)\, du_1 \wedge \cdots \wedge du_{\mathrm{rk}(G)}, \tag{2.3}$$

which consists of the one-loop determinants

$$Z_{\mathbf{d}}^{\mathrm{vec}}(\mathbf{u};\hbar) = (-1)^{\sum_{\alpha\in\Delta_+}(\alpha(\mathbf{d})+1)} \prod_{\alpha\in\Delta_+}\Big(\alpha(\mathbf{u})^2 - \frac{\alpha(\mathbf{d})^2}{4}\hbar^2\Big), \tag{2.4}$$

for the vector multiplet and

$$Z_{\mathbf{d}}^{\Phi_a}(\mathbf{u};\hbar) = (-1)^{\delta_{r_a,2}} \prod_{\rho_a\in R_a} \hbar^{r_a-\rho_a(\mathbf{d})-1} \frac{\Gamma\Big(\frac{\rho_a(\mathbf{u})+\lambda_a}{\hbar} + \frac{r_a-\rho_a(\mathbf{d})}{2}\Big)}{\Gamma\Big(\frac{\rho_a(\mathbf{u})+\lambda_a}{\hbar} - \frac{r_a-\rho_a(\mathbf{d})}{2} + 1\Big)}, \tag{2.5}$$

for the chiral matter multiplets Φ_a in a representation R_a with R-charge r_a and the twisted mass λ_a. Here Δ_+ is the set of positive roots and ρ_a denote the weights of R_a. The products $\alpha(*)$ and $\rho_a(*)$ are defined by the canonical pairing.

The one-loop determinant for chiral multiplets (2.5) has poles along hyperplanes defined from the Γ-function in the denominators. For later convenience, we will use a collective form for the gauge charges as $\mathsf{Q} = \big\{Q_i \in \mathbb{Z}^{\mathrm{rk}(G)} \subset \mathfrak{h}^*\big\}$ where i labels all the components of the multiplets of the GLSM. The hyperplanes can intersect at a point $\mathbf{u} = \mathbf{u}_* = (u_1^*, \ldots, u_{\mathrm{rk}G}^*)$ and realize a codimension $\mathrm{rk}(G)$ pole. Such intersecting hyperplanes simultaneously specify a subset of charge vectors with at least $\mathrm{rk}(G)$ elements and we denote it by $\mathsf{Q}_* \subset \mathsf{Q}$.

A crucial ingredient of the exact formula (2.1), whose treatment will be described in the next section, is the Jeffrey-Kirwan residue operation [28][3] $\text{JK-Res}\,[\mathsf{Q}_*, \eta]$ at $\mathbf{u} = \mathbf{u}_*$ depending on a choice of a covector $\eta \in \mathfrak{h}^*$. As adopted in [15], we always choose η to be parallel and pointing in the same direction with ξ_i and therefore η specifies the phase of the model.

2.2 The Jeffrey-Kirwan residue operation

In this section, we will briefly review the Jeffrey-Kirwan residue operation. For more details, we refer the reader to [7, 31].

Definition 2.1. *When $\mathsf{Q}_* \subset \mathsf{Q}$ lies within an open half-space of $\mathfrak{h}^*$, the associated intersection point $\mathbf{u} = \mathbf{u}_*$ is called projective point. Such an arrangement of hyperplanes is called projective arrangement.*

Definition 2.2. *Consider an integrand $\frac{du_1\wedge\cdots\wedge du_{\mathrm{rk}(G)}}{Q_1(\mathbf{u})\cdots Q_{\mathrm{rk}(G)}(\mathbf{u})}$ with a projective point $\mathbf{u}_* = \mathbf{0}$ and $\mathsf{Q}_* = \{Q_1, \ldots, Q_{\mathrm{rk}(G)}\}$. The cases with generic $\mathbf{u}_*$ can be realized by shifting the coordinates appropriately. Then the Jeffrey-Kirwan residue $\text{JK-Res}_{\mathbf{u}=\mathbf{u}_*}[\mathsf{Q}_*, \eta]$ is given by*

$$\underset{\mathbf{u}=\mathbf{0}}{\text{JK-Res}}\,[\mathsf{Q}_*, \eta]\,\frac{du_1\wedge\cdots\wedge du_{\mathrm{rk}(G)}}{Q_1(\mathbf{u})\cdots Q_{\mathrm{rk}(G)}(\mathbf{u})} = \begin{cases} \frac{1}{|\det(Q_1,\ldots,Q_{\mathrm{rk}(G)})|} & \text{if } \eta \in \mathrm{Cone}(Q_1,\ldots,Q_{\mathrm{rk}(G)}), \\ 0 & \text{if } \eta \notin \mathrm{Cone}(Q_1,\ldots,Q_{\mathrm{rk}(G)}), \end{cases}$$

where $\mathrm{Cone}(Q_1, \ldots, Q_{\mathrm{rk}(G)})$ is the closed cone spanned by $Q_1, \ldots, Q_{\mathrm{rk}(G)}$.

More constructive definition can be represented as follows. Let $\mathcal{FL}(\mathsf{Q}_*)$ be a finite set of flags

$$F = \big[F_0 = \{0\} \subset F_1 \subset \cdots \subset F_{\mathrm{rk}(G)} = \mathfrak{h}^*\big]\,, \quad \dim F_j = j,$$

[3] See also [29, 30] for the early developments.

such that Q_* contains a basis of each of the flags F_j, $j = 1, \dots, \mathrm{rk}(G)$. Then one can choose an ordered set $\mathsf{Q}_*^F = (Q_{i_1}, \dots, Q_{i_{\mathrm{rk}(G)}})$ such that the first j elements $\{Q_{i_m}\}_{m=1}^j$ give a basis of F_j.

Definition 2.3. *For the above defined ordered basis Q_*^F of a flag $F \in \mathcal{FL}(\mathsf{Q}_*)$, the iterated residue Res_F of $\omega = \omega_{1\dots\mathrm{rk}(G)} dQ_{i_1}(\mathbf{u}) \wedge \cdots \wedge dQ_{i_{\mathrm{rk}(G)}}(\mathbf{u})$ is defined by*

$$\operatorname*{Res}_F \omega = \operatorname*{Res}_{Q_{i_{\mathrm{rk}(G)}}(\mathbf{u})=0} \cdots \operatorname*{Res}_{Q_{i_1}(\mathbf{u})=0} \omega_{1\dots\mathrm{rk}(G)}.$$

Here in each step of the residue operations on the right hand side, the higher variables are remained to be free parameters. Note that the iterated residue only depends on the flag F, and does not depend on the choice of the ordered basis.

Let us define the closed simplicial cone for a flag $F \in \mathcal{FL}(\mathsf{Q}_*)$ as

$$\mathfrak{s}^+(F, \mathsf{Q}_*) = \sum_{j=1}^{\mathrm{rk}(G)} \mathbb{R}_{\geq 0} \kappa_j^F, \qquad \kappa_j^F := \sum_{Q_i \in F_j} Q_i, \quad j = 1, \dots, \mathrm{rk}(G), \tag{2.6}$$

and denote by $\mathcal{FL}^+(\mathsf{Q}_*, \eta)$ a set of flags such that the corresponding cone $\mathfrak{s}^+(F, \mathsf{Q}_*)$ contains η.

Definition 2.4. *By the partial sums of the elements of $\mathsf{Q}_* = \{Q_1, \dots, Q_n\}$, $n \geq \mathrm{rk}(G)$, define a set*

$$\Sigma\mathsf{Q}_* = \left\{ \sum_{i \in \pi} Q_i, \ \pi \subset \{1, \dots, n\} \right\}.$$

An element $v \in \mathrm{Cone}(\mathsf{Q}_*)$ is called regular with respect to $\Sigma\mathsf{Q}_*$ if $v \notin \mathrm{Cone}_{sing}(\Sigma\mathsf{Q}_*)$, where $\mathrm{Cone}_{\mathrm{sing}}(\Sigma\mathsf{Q}_*)$ is the union of the closed cones spanned by $\mathrm{rk}(G) - 1$ independent elements of $\Sigma\mathsf{Q}_*$. A regular $v \notin \mathrm{Cone}_{sing}(\Sigma\mathsf{Q}_*)$ identifies one chamber in $\mathfrak{h}^*$.

Then the following theorem can be proved [7].

Theorem 2.5. *If $\eta \in \mathrm{Cone}(\mathsf{Q}_*)$ is regular with respect to $\Sigma\mathsf{Q}_*$, the Jeffrey-Kirwan residue at a projective point $\mathbf{u} = \mathbf{u}_* = \mathbf{0}$ can be written in terms of the iterated residue as*

$$\operatorname*{JK\text{-}Res}_{\mathbf{u}=\mathbf{0}} [\mathsf{Q}_*, \eta] = \sum_{F \in \mathcal{FL}^+(\mathsf{Q}_*, \eta)} \nu(F) \operatorname*{Res}_F, \tag{2.7}$$

where $\nu(F) = 0$ if κ_j^F, $j = 1, \dots, \mathrm{rk}(G)$ are linearly dependent, and $\nu(F) = 1$ (resp. -1) if κ_j^F are linearly independent and the ordered basis $\kappa^F := (\kappa_1^F, \dots, \kappa_{\mathrm{rk}(G)}^F) \in \mathfrak{h}^*$ is positively (resp. negatively) oriented, i.e. $\mathrm{sign}(\det \kappa^F) = 1$ (resp. -1).

3 GLSM correlators and local B-model Yukawa couplings

It has been clarified in [15] that the A-twisted correlators given by (2.1) with vanishing Ω-deformation precisely give the B-model Yukawa couplings of the mirror of compact Calabi-Yau manifolds and non-compact orbifolds.[4] Here, we consider local toric Calabi-Yau varieties. In

[4]See also [19, 20] for the treatment of the manifolds with non-abelian GLSM descriptions.

order to perform the Jeffrey-Kirwan residue operation for these backgrounds appropriately, it is required to introduce the twisted masses to matter fields and realize a projective hyperplane arrangement of Definition 2.1. The main purpose of this paper is to provide a detailed description of this procedure to investigate the local B-model Yukawa couplings.

3.1 Local toric Calabi-Yau varieties and correlation functions

Consider a local toric Calabi-Yau m-fold X described by a symplectic quotient $X = \mathbb{C}^n /\!/_{\xi} (\mathbb{C}^*)^r$ with $m = n - r,\ m \geq 3$.[5] In terms of the GLSM, this background can be specified by n chiral matter multiples Φ_i, $i = 1, \ldots, n$ with R-charge 0 and $U(1)^r$ charge vectors $Q_i \in \mathbb{Z}^r$ satisfying the Calabi-Yau condition $\sum_{i=1}^n Q_i = \mathbf{0}$. One can also assign the twisted masses $-\lambda_i$ for the chiral multiplets Φ_i. Then a model can be specified by a set of charge vectors and twisted masses as

$$\left(\begin{array}{cccc} Q_1 & Q_2 & \cdots & Q_n \\ \hline \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{array} \right). \tag{3.1}$$

In the limit of the vanishing Ω-deformation $\hbar = 0$, the exact formula about correlation functions (2.1) in the geometric phase reduces to a simple expression

$$\langle \mathcal{O}(\mathbf{u}) \rangle_{\hbar=0} = \sum_{\mathbf{d} \in (\mathbb{Z}_{\geq 0})^r} \mathbf{z}^{\mathbf{d}} \sum_{\mathbf{u}_*} \underset{\mathbf{u}=\mathbf{u}_*}{\text{JK-Res}} \left[\mathsf{Q}_*, \eta \right] \mathcal{O}(\mathbf{u}) \prod_{i=1}^{n} Z_{\mathbf{d}}^{\Phi_i}(\mathbf{u}) \, du_1 \wedge \cdots \wedge du_r, \tag{3.2}$$

where the moduli parameters $\mathbf{z} = (z_1, \ldots, z_r)$ are defined in (2.2) and the one-loop determinant for chiral multiplets in (2.5) or (A.5) takes a reduced form

$$Z_{\mathbf{d}}^{\Phi_i}(\mathbf{u}) := Z_{\mathbf{d}}^{\Phi_i}(\mathbf{u}; 0) = (Q_i(\mathbf{u}) - \lambda_i)^{-Q_i(\mathbf{d})-1}. \tag{3.3}$$

Choosing the monomials of $\mathbf{u}$ as the operators $\mathcal{O}(\mathbf{u})$, we focus on the following GLSM correlators:

$$Y_{z_{i_1} \cdots z_{i_m}}(\mathbf{z}) := \langle u_{i_1} \cdots u_{i_m} \rangle_{\hbar=0}, \quad 1 \leq i_1 \leq \cdots \leq i_m \leq r. \tag{3.4}$$

Remark 3.1. The phase structure of the Calabi-Yau variety X is described by the secondary fan which consists of the set of the charge vectors $\{Q_1, \ldots, Q_n\}$ in $\mathfrak{h}^*$ (see e.g. [32]). Although we will consider a particular choice of $\eta \in \mathfrak{h}^*$ inside the geometric phase of X for actual computation of the GLSM correlators (3.4), the result does not depend on the choice of η as long as the Jeffrey-Kirwan residue operation is carried out appropriately.

3.2 Criteria for the twisted mass deformations

Computations of the GLSM correlators (3.4) for local toric Calabi-Yau varieties require a careful treatment. This is because generically the associated hyperplane arrangement without twisted mass parameters λ_i becomes non-projective and one should not use the Jeffrey-Kirwan residue operation directly. Therefore it is required to introduce twisted masses for chiral multiplets in a proper way, such that the hyperplane arrangement of the model becomes projective. Although

[5] We refer the reader to [32] for a detailed introduction on this subject.

the necessity of this kind of prescription has been argued before, up to the authors knowledge, there has been no satisfactory attempt to clarify the proper way of the twisted mass insertions. Here we propose the following rules as a proper determination of the twisted mass deformations:

1. The twisted masses should not be inserted for chiral multiplets with $Q_i \neq \mathbf{0}$ describing a non-compact fiber coordinate, or a blow up coordinate of a singularity.

2. Suppose a GLSM of a local toric Calabi-Yau variety of interest has a chiral multiplet Φ_0 which is neutral under the j-th $U(1)$ gauge symmetry and describes a non-compact fiber coordinate X_0 by its scalar component.[6] Let λ_i, $i = 1, \ldots, s$ be twisted masses for other chiral multiplets Φ_i with non-zero charges with respect to the j-th $U(1)$ gauge symmetry. If a divisor defined by $X_0 = 0$ contains a blow up coordinate of a singularity, one need to impose

$$\lambda_0 + \sum_{i=1}^{s} \lambda_i = 0,$$

for the twisted mass λ_0 of Φ_0 as a "Calabi-Yau condition on the divisor".

3. As long as the above requirements are fulfilled, one can turn on generic twisted masses for the remaining chiral multiplets while respecting the symmetries of the model such as the permutation of the homogeneous coordinates of the toric variety.

In Section 4, by using the above prescription for various examples, we will explicitly check that the GLSM correlators (3.4) give the same results predicted by the mirror symmetry approach. Moreover, we find that the ambiguities of the intersection numbers for a certain class of varieties argued in [22] can be reinterpreted as degrees of freedom of the proper twisted mass deformations.

Remark 3.2. In this paper, we do not consider the so-called non-nef toric varieties such as $\mathcal{O}(k) \oplus \mathcal{O}(-2-k) \to \mathbb{P}^1$ with $k \geq 1$ [33, 34, 35]. It would be interesting to investigate such kind of varieties and try to extend our prescription.

3.3 Local Yukawa couplings and the mirror map

The topological B-model m-point Yukawa couplings of a Calabi-Yau m-fold $X^\vee$ are defined by

$$Y_{z_{i_1} \cdots z_{i_m}}(\mathbf{z}) = \int_{X^\vee} \Omega(\mathbf{z}) \wedge \nabla_{z_{i_1} \partial_{z_{i_1}}} \cdots \nabla_{z_{i_m} \partial_{z_{i_m}}} \Omega(\mathbf{z}) \ \in \ \mathrm{Sym}^m \left(T^*\mathcal{M}\right) \otimes \mathcal{L}^{-2}, \tag{3.5}$$

where ∇ is a flat connection called Gauss-Manin connection (see e.g. [32]) and $T^*\mathcal{M}$ is the cotangent bundle of the complex structure moduli space $\mathcal{M}$ of the Calabi-Yau m-fold $X^\vee$. $\mathcal{L}$ is a holomorphic line bundle over $\mathcal{M}$ whose section is given by the nowhere vanishing holomorphic m-form $\Omega(\mathbf{z})$ on $X^\vee$. The topological A-model m-point Yukawa couplings for the mirror X of $X^\vee$ can be obtained from (3.5) by changing the complex structure moduli parameters $\{\log z_i\}$ into the flat coordinates $\{\log q_i\}$ parametrizing $h_i \in H^{1,1}(X)$ as

$$\widetilde{Y}_{h_{i_1} \cdots h_{i_m}}(\mathbf{q}) = \frac{1}{X_0(\mathbf{z}(\mathbf{q}))^2} \sum_{j_1, \ldots, j_m = 1}^{m} Y_{z_{j_1} \cdots z_{j_m}}(\mathbf{z}(\mathbf{q})) \frac{\partial \log z_{j_1}(\mathbf{q})}{\partial \log q_{i_1}} \cdots \frac{\partial \log z_{j_m}(\mathbf{q})}{\partial \log q_{i_m}}. \tag{3.6}$$

[6] A typical example with this property will be shown in (4.5).

This transformation map is called mirror map. Note that the quantity $X_0(\mathbf{z})^2$ is a square of the monodromy invariant fundamental period of $X^\vee$ and is equal to 1 for local toric Calabi-Yau varieties, while for compact Calabi-Yau m-folds it is a non-trivial function of the moduli parameters $\mathbf{z}$.

Generically the A-model three-point Yukawa coupling of a Calabi-Yau threefold, say X_3, takes the following form [36, 37, 38, 39, 21]:

$$\widetilde{Y}_{h_i h_j h_k}(\mathbf{q}) = \kappa_{h_i h_j h_k} + \sum_{\mathbf{d}\in H_2(X_3,\mathbb{Z})\backslash\{\mathbf{0}\}} n_{\mathbf{d}} \frac{d_i d_j d_k \mathbf{q}^{\mathbf{d}}}{1-\mathbf{q}^{\mathbf{d}}}, \qquad \mathbf{q}^{\mathbf{d}} = q_1^{d_1} q_2^{d_2} \cdots . \tag{3.7}$$

This provides a generating function of integer invariants $n_{\mathbf{d}}$ which give the Gromov-Witten invariants and enumerates the number of holomorphic maps $\phi : \mathbb{P}^1 \to X_3$ of class $\mathbf{d} \in H_2(X_3, \mathbb{Z})$ intersecting with cycles dual to h_i, h_j, and h_k. Here $\kappa_{h_i h_j h_k}$ is called classical triple intersection number and the summation with respect to $\mathbf{d}$ is taken only over non-negative elements.

For Calabi-Yau m-folds, the m-point Yukawa couplings can be factorized into three-point Yukawa couplings in accordance with the fusion rules of the underlying Frobenius algebra (see [40, 41, 42] for details). The so obtained A-model three-point Yukawa couplings similarly provide generating functions of genus zero Gromov-Witten invariants for m-folds. In the case of a Calabi-Yau fourfold X_4, the A-model four-point Yukawa couplings are factorized into three-point Yukawa couplings as

$$\widetilde{Y}_{h_i h_j h_k h_l}(\mathbf{q}) = \eta^{\alpha\beta}\, \widetilde{Y}_{H_\alpha h_i h_j}(\mathbf{q})\, \widetilde{Y}_{H_\beta h_k h_l}(\mathbf{q}), \tag{3.8}$$

where $H_\alpha, H_\beta \in H^{2,2}(X_4)$ represent the so-called primary elements generated by the wedge product of the elements in $H^{1,1}(X_4)$ and $\eta^{\alpha\beta}$ is the inverse matrix of the intersection matrix associated with H_α and H_β. The A-model three-point Yukawa couplings $\widetilde{Y}_{h_i h_j H}(\mathbf{q})$ have a generic form [40, 41, 42, 43]:

$$\widetilde{Y}_{h_i h_j H}(\mathbf{q}) = \kappa_{h_i h_j H} + \sum_{\mathbf{d}\in H_2(X_4,\mathbb{Z})\backslash\{\mathbf{0}\}} n_{\mathbf{d}}(H) \frac{d_i d_j \mathbf{q}^{\mathbf{d}}}{1-\mathbf{q}^{\mathbf{d}}}, \qquad \mathbf{q}^{\mathbf{d}} = q_1^{d_1} q_2^{d_2} \cdots , \tag{3.9}$$

where $\kappa_{h_i h_j H}$ is the classical intersection number associated with cycles dual to h_i, h_j and H.

Mirror map from the localization formula

Starting from the factors (2.4) and (2.5) in the localization formula, one can find building blocks of the Givental I-function [44, 45, 46] from which the mirror map can be properly derived. Here we go straight to the point and some technical details are relegated to Appendix A.

For a local toric Calabi-Yau m-fold X with GLSM description (3.1), one can construct the Givental I-function for X with the following form

$$I_X^{\{\lambda_i\}}(\mathbf{z};\mathbf{x};\hbar) = \mathbf{z}^{\mathbf{x}/\hbar} \sum_{\mathbf{d}\in\left(\mathbb{Z}_{\geq 0}\right)^r} \left(\prod_{i=1}^{n} I_{\mathbf{d}}^{\Phi_i}(\mathbf{x},\lambda_i;\hbar) \right) \mathbf{z}^{\mathbf{d}}, \qquad \mathbf{z}^{\mathbf{x}/\hbar} = \prod_{j=1}^{r} z_j^{x_j/\hbar}, \tag{3.10}$$

which consists of the building blocks defined in (A.8). To find the mirror map, it is sufficient to consider the case without twisted mass deformations. Expanding the I-function with $\lambda_i = 0$ around $\hbar = \infty$, we obtain

$$I_X^{\{\mathbf{0}\}}(\mathbf{z};\mathbf{x};\hbar) = 1 + \left(I_{1,1}(\mathbf{z})x_1 + \cdots + I_{1,r}(\mathbf{z})x_r\right)\hbar^{-1} + O(\hbar^{-2}). \tag{3.11}$$

Then the inverse mirror map $\log q_i$, $i = 1, \ldots, r$ can be obtained by $\log q_i = I_{1,i}(\mathbf{z})$ [21, 47] and the result is

$$\log \mathbf{q} = \log \mathbf{z} - \sum_{\mathbf{d}\in(\mathbb{Z}_{\geq 0})^r\setminus\{\mathbf{0}\}} \sum_I{}' (-1)^{Q_I(\mathbf{d})} Q_I \frac{(-Q_I(\mathbf{d})-1)!}{\prod_{i\neq I} Q_i(\mathbf{d})!}\mathbf{z}^{\mathbf{d}}, \tag{3.12}$$

where we put $1/n! = 0$ for $n < 0$ and $\sum_I'$ means that the summation is taken over all I satisfying $Q_I(\mathbf{d}) < 0$.

Remark 3.3. For a compact Calabi-Yau manifold, the I-function is generically annihilated by the Picard-Fuchs equation for the mirror.[7] Then the higher order terms of the expansion in (3.11) are expected to possess the information about the genus zero Gromov-Witten invariants (see e.g. [49]). But for a local toric Calabi-Yau variety, say X_{local}, the information obtained from the I-function without twisted mass parameters (i.e. $\lambda_i = 0$ for all i) is sometimes not enough to get Gromov-Witten invariants. For instance, if $\dim H_4(X_{local}, \mathbb{Z}) = 0$, the higher order terms $O(\hbar^{-2})$ in (3.11) vanish and cannot say anything about the Gromov-Witten invariants.

It is worth noting that by using the I-function with twisted mass parameters and the Birkhoff factorization [46], the equivariant local mirror symmetry for local toric Calabi-Yau threefolds was developed in [33, 34, 35] and the Gromov-Witten invariants have been computed.

4 Examples

Based on the criteria for the twisted mass deformations mentioned in Section 3.2, here we will compute the A-twisted GLSM correlators (3.4) for various local nef toric Calabi-Yau varieties in the geometric phase and demonstrate that the corresponding local B-model Yukawa couplings can be calculated appropriately. A novel aspect of the previously observed ambiguities for certain class of varieties [22] will be also discussed.

4.1 Local toric Calabi-Yau threefolds

First we will focus on the local nef toric Calabi-Yau threefolds whose exact properties have been studied by using the local mirror symmetry [21, 22, 33, 34, 35, 50]. We confirm that the exact formula about the A-twisted GLSM correlation functions match the previous results of the local B-model Yukawa couplings with the aid of our prescriptions. We also see that the ambiguities of the intersection numbers in [22] can be interpreted as degrees of freedom of the twisted mass deformations.

[7] See [48] for a recent development.

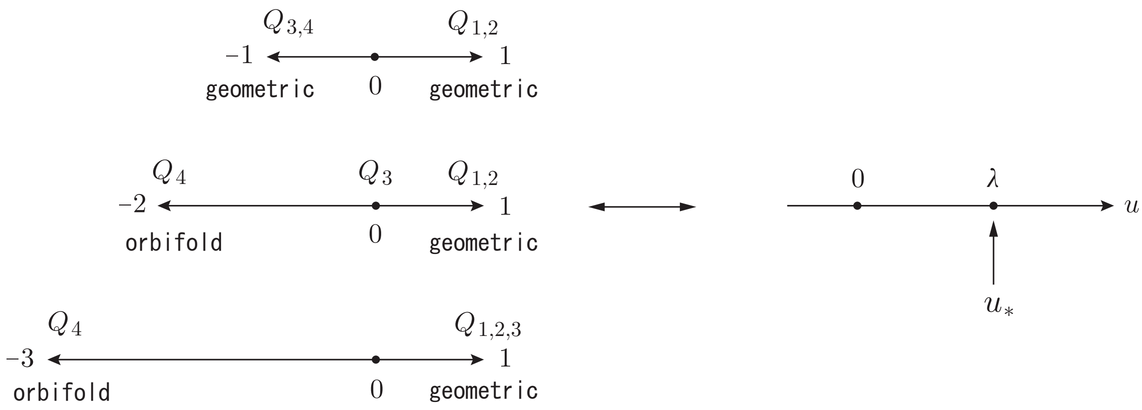

Figure 1: **1)** The upper left diagram describes the secondary fan of the resolved conifold and the right one represents the corresponding hyperplanes $\{u = \lambda, u = 0\}$ corresponding to the charge vectors $\{Q_{1,2} = 1, Q_{3,4} = -1\}$. The point $u_* = \lambda$ is a projective point associated with $\mathsf{Q}_* = Q_{1,2}$. **2)** The middle (resp. lower) left diagram describes the secondary fan of $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1$ (resp. $\mathcal{O}(-3) \to \mathbb{P}^2$) and the right figure again describes the hyperplanes $\{u = \lambda, u = 0\}$ associated with $\{Q_{1,2} = 1, Q_4 = -2\}$ (resp. $\{Q_{1,2,3} = 1, Q_4 = -3\}$). The point $u_* = \lambda$ is a projective point associated with $\mathsf{Q}_* = Q_{1,2}$ (resp. $\mathsf{Q}_* = Q_{1,2,3}$). The orbifold phase of the middle (resp. lower) left diagram is given by the orbifold geometry $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ (resp. $\mathbb{C}^3/\mathbb{Z}_3$).

Resolved conifold: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$

Let us consider the resolved conifold described by a $U(1)$ GLSM with the following charge vectors and the twisted masses:

$$\left(\begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{array}\right) = \left(\begin{array}{cccc} 1 & 1 & -1 & -1 \\ \hline \lambda & \lambda & 0 & 0 \end{array}\right), \tag{4.1}$$

where the inclusion of the twisted mass parameters $\lambda \neq 0$ has been determined in accordance with the rules in Section 3.2, such that the pole $u_* = \lambda$ of the exact formula associated with $\mathsf{Q}_* = \{Q_1, Q_2\}$ in Figure 1 becomes a projective point. In other words, without turning on the twisted masses, the hyperplane arrangement of the model remains to be non-projective and the usage of the Jeffrey-Kirwan residue operation cannot be justified. By taking $\eta \in \mathrm{Cone}(\mathsf{Q}_*)$, namely in the geometric phase, the projective point $u_* = \lambda$ contributes to the Jeffrey-Kirwan residue in (3.4) and we obtain

$$Y_{zzz}(z) = \sum_{d=0}^{\infty} z^d \operatorname*{Res}_{u=\lambda} \frac{u^3\,(-u)^{2(d-1)}}{(u-\lambda)^{2(d+1)}} = \frac{1}{1-z}. \tag{4.2}$$

In this example the mirror map (3.12) becomes trivial, i.e. $\log q = \log z$, and the A-model Yukawa coupling (3.6) is given by

$$\widetilde{Y}_{hhh}(q) = \frac{1}{1-q}. \tag{4.3}$$

Remark 4.1. Turning on the twisted masses in a symmetric manner for the base and fiber directions as $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$ to maintain the flop symmetry $z \leftrightarrow z^{-1}$ of the resolved conifold interchanging the compact and the non-compact directions, we obtain

$$Y'_{zzz}(z) = \sum_{d=0}^{\infty} z^d \operatorname*{Res}_{u=\lambda} \frac{u^3 \left(-u-\lambda\right)^{2(d-1)}}{(u-\lambda)^{2(d+1)}} = \frac{1}{2} + \frac{z}{1-z}. \tag{4.4}$$

Interestingly, this expression has a different classical triple intersection number and consistent with the result argued in [22].

<u>$\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1$</u>

Next, we consider the local toric Calabi-Yau threefold $\mathcal{O}(0) \oplus \mathcal{O}(-2) \to \mathbb{P}^1$ described by a $U(1)$ GLSM with

$$\left(\begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{array}\right) = \left(\begin{array}{cccc} 1 & 1 & 0 & -2 \\ \hline \lambda & \lambda & -2\lambda & 0 \end{array}\right). \tag{4.5}$$

Following the criteria in Section 3.2, a twisted mass parameter $\lambda \neq 0$ has been introduced such that $u_* = \lambda$ associated with $\mathsf{Q}_* = \{Q_1, Q_2\} = 1$ gives a projective point (see Figure 1). Especially, according to the rule **2**, we have assigned a twisted mass $\lambda_3 = -2\lambda$ for the neutral chiral multiplet Φ_3 corresponding to $\mathcal{O}(0)$ in order to satisfy the "Calabi-Yau condition on the divisor" $\lambda_1 + \lambda_2 + \lambda_3 = 0$. For $\eta \in \mathrm{Cone}(1)$ in the geometric phase, the projective point $u_* = \lambda$ contributes to the Jeffrey-Kirwan residue in (3.4) and we obtain

$$Y_{zzz}(z) = \sum_{d=0}^{\infty} z^d \operatorname*{Res}_{u=\lambda} \frac{u^3 \left(-2u\right)^{2d-1}}{2\lambda(u-\lambda)^{2(d+1)}} = -\frac{1}{2(1-4z)^2}. \tag{4.6}$$

This agrees with the Yukawa coupling evaluated in [22]. The mirror map (3.12) becomes

$$\log q = \log z + 2\sum_{d=1}^{\infty} \frac{(2d-1)!}{(d!)^2} z^d = \log z - 2\log\frac{1+\sqrt{1-4z}}{2}, \quad \longleftrightarrow \quad z = \frac{q}{(1+q)^2}, \tag{4.7}$$

and the A-model Yukawa coupling (3.6) is given by

$$\widetilde{Y}_{hhh}(q) = -\frac{1}{2} + \frac{q}{1+q}. \tag{4.8}$$

<u>**Local $\mathbb{P}^2$: $\mathcal{O}(-3) \to \mathbb{P}^2$**</u>

As an explicit example with $\dim H_4(X, \mathbb{Z}) = 1$, we consider the local Calabi-Yau threefold $K_{\mathbb{P}^2}$ described by a $U(1)$ GLSM with

$$\left(\begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{array}\right) = \left(\begin{array}{cccc} 1 & 1 & 1 & -3 \\ \hline \lambda & \lambda & \lambda & 0 \end{array}\right). \tag{4.9}$$

Here we have introduced a twisted mass parameter $\lambda \neq 0$ for the base space $\mathbb{P}^2$ such that $u_* = \lambda$ associated with $\mathsf{Q}_* = \{Q_1, Q_2, Q_3\} = 1$ provides a projective point (see Figure 1). For

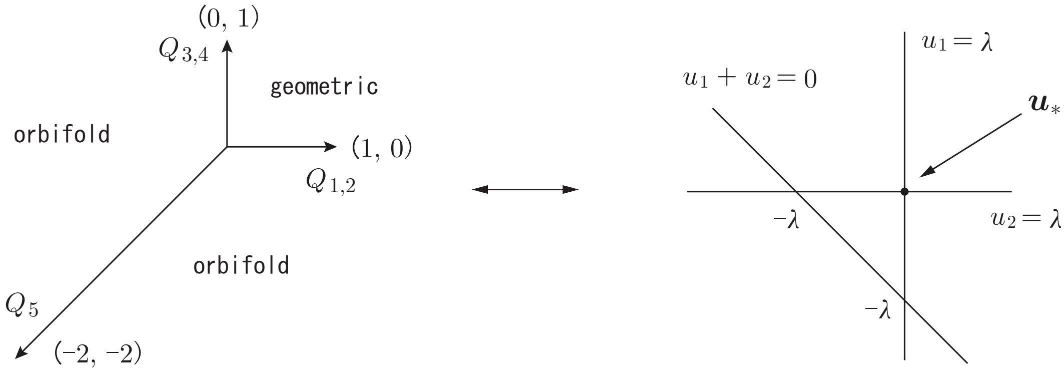

Figure 2: The left figure describes the secondary fan of the local F_0, and the right figure describes the hyperplanes $\{u_1 = \lambda, u_2 = \lambda, u_1 + u_2 = 0\}$ corresponding to the charge vectors $\{Q_{1,2}, Q_{3,4}, Q_5\}$. The point $\mathbf{u}_* = (\lambda, \lambda)$ is a projective point associated with $\mathsf{Q}_* = \{Q_{1,2}, Q_{3,4}\}$. The orbifold phases in the left figure are described by the orbifold geometry $T^*S^3/\mathbb{Z}_2$.

$\eta \in \mathrm{Cone}(1)$ in the geometric phase, the projective point $u_* = \lambda$ gives a contribution to the Jeffrey-Kirwan residue in the GLSM correlator (3.4) and we obtain

$$Y_{zzz}(z) = \sum_{d=0}^{\infty} z^d \operatorname*{Res}_{u=\lambda} \frac{u^3 \left(-3u\right)^{3d-1}}{(u-\lambda)^{3(d+1)}} = -\frac{1}{3(1+27z)}. \tag{4.10}$$

The mirror map (3.12) is given by

$$\log q = \log z + 3\sum_{d=1}^{\infty} (-z)^d \frac{(3d-1)!}{(d!)^3}, \tag{4.11}$$

and by using (3.7), the Gromov-Witten invariants computed in [21] can be precisely reproduced.

Local F_0: $\mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$

As a second example with $\dim H_4(X,\mathbb{Z}) = 1$, let us consider the local Hirzebruch surface $K_{F_0} = K_{\mathbb{P}^1\times\mathbb{P}^1}$ described by a $U(1)^2$ GLSM with

$$\left(\begin{array}{ccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{array}\right) = \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ \hline \lambda & \lambda & \lambda & \lambda & 0 \end{array}\right). \tag{4.12}$$

Here, based on the rules in Section 3.2, we have introduced a twisted mass parameter $\lambda \neq 0$ in a symmetric way for $z_1 \leftrightarrow z_2$ in the base space $\mathbb{P}^1 \times \mathbb{P}^1$, such that $\mathbf{u}_* = (\lambda, \lambda)$ associated with $\mathsf{Q}_* = \{Q_1, Q_2, Q_3, Q_4\}$ provides a projective point (see Figure 2). For $\eta \in \mathrm{Cone}(Q_1, Q_1 + Q_3)$ inside the geometric phase, the projective point $\mathbf{u}_* = (\lambda, \lambda)$ gives a contribution to the Jeffrey-Kirwan residue in the GLSM correlation functions in (3.4) as

$$Y_{z_i z_j z_k}(z_1, z_2) = \sum_{d_1, d_2 = 0}^{\infty} z_1^{d_1} z_2^{d_2} \operatorname*{Res}_{u_2=\lambda} \operatorname*{Res}_{u_1=\lambda} \frac{u_i u_j u_k \left(-2u_1 - 2u_2\right)^{2d_1+2d_2-1}}{(u_1-\lambda)^{2(d_1+1)}(u_2-\lambda)^{2(d_2+1)}}, \tag{4.13}$$

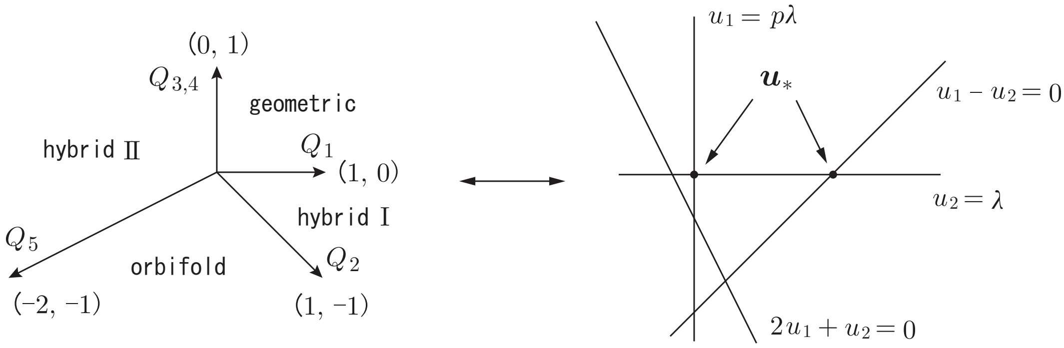

Figure 3: The left figure describes the secondary fan of the local F_1, and the right figure describes the hyperplanes $\{u_1 = p\lambda, u_1 - u_2 = 0, u_2 = \lambda, 2u_1 + u_2 = 0\}$ corresponding to the charge vectors $\{Q_1, Q_2, Q_{3,4}, Q_5\}$. The points $\mathbf{u}_* = (p\lambda, \lambda)$, (λ, λ) are projective points associated with $\mathsf{Q}_* = \{Q_1, Q_{3,4}\}$, $\{Q_2, Q_{3,4}\}$.

and each component has the following exact form:

$$\begin{aligned} Y_{z_1z_1z_1}(z_1,z_2) &= \frac{(1-4z_2)^2 - 16z_1(1+z_1)}{4\Delta_{F_0}(z_1,z_2)}, \quad Y_{z_1z_1z_2}(z_1,z_2) = \frac{16z_1^2 - (1-4z_2)^2}{4\Delta_{F_0}(z_1,z_2)}, \\ Y_{z_1z_2z_2}(z_1,z_2) &= \frac{16z_2^2 - (1-4z_1)^2}{4\Delta_{F_0}(z_1,z_2)}, \quad Y_{z_2z_2z_2}(z_1,z_2) = \frac{(1-4z_1)^2 - 16z_2(1+z_2)}{4\Delta_{F_0}(z_1,z_2)}, \end{aligned} \tag{4.14}$$

where

$$\Delta_{F_0}(z_1,z_2) = 1 - 8(z_1+z_2) + 16(z_1-z_2)^2. \tag{4.15}$$

The above result is in agreement with the Yukawa couplings evaluated from the local mirror symmetry approach in [22, 23]. The mirror map for K_{F_0} is obtained by (3.12) as

$$\log q_i = \log z_i + 2 \sum_{\substack{d_1,d_2=0 \\ (d_1,d_2)\neq(0,0)}}^{\infty} \frac{(2d_1+2d_2-1)!}{(d_1!)^2(d_2!)^2} z_1^{d_1} z_2^{d_2}, \quad i = 1,2, \tag{4.16}$$

and then the formula (3.7) correctly reproduces the Gromov-Witten invariants studied in [21].

Local F_1

As a third example with $\dim H_4(X,\mathbb{Z}) = 1$, we consider the local Hirzebruch surface K_{F_1} obtained by the one point blow up of $K_{\mathbb{P}^2}$, which can be described by a $U(1)^2$ GLSM with

$$\left(\frac{Q_1 \quad Q_2 \quad Q_3 \quad Q_4 \quad Q_5}{\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5}\right) = \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & -2 \\ 0 & -1 & 1 & 1 & -1 \\ \hline p\lambda & 0 & \lambda & \lambda & 0 \end{array}\right). \tag{4.17}$$

Here we have introduced a twisted mass parameter $\lambda \neq 0$ and a free parameter $p \neq -1/2$ for the base space F_1, such that $\mathbf{u}_* = (p\lambda, \lambda)$ and (λ, λ) associated with $\mathsf{Q}_* = \{Q_1, Q_3, Q_4\}$ and $\{Q_2, Q_3, Q_4\}$ provide projective points respectively (see Figure 3). Note that if we choose

$p = -1/2$, the point $\mathbf{u}_* = (p\lambda, \lambda)$ becomes non-projective. For $\eta \in \mathrm{Cone}(Q_1, Q_1 + Q_3)$ in the geometric phase, the projective points $\mathbf{u}_* = (p\lambda, \lambda)$, (λ, λ) contribute to the Jeffrey-Kirwan residue in (3.4) as

$$Y_{z_i z_j z_k}(z_1, z_2) = \sum_{d_1, d_2 = 0}^{\infty} z_1^{d_1} z_2^{d_2} \left(\operatorname*{Res}_{u_2=\lambda} \operatorname*{Res}_{u_1 = p\lambda} + \operatorname*{Res}_{u_1=\lambda} \operatorname*{Res}_{u_2=\lambda} \right) \times \frac{u_i u_j u_k \left(-2u_1 - u_2\right)^{2d_1 + d_2 - 1}}{(u_1 - p\lambda)^{d_1+1} (u_2 - \lambda)^{2(d_2+1)} (u_1 - u_2)^{d_1 - d_2 + 1}}, \tag{4.18}$$

and we obtain

$$\begin{aligned} Y_{z_1 z_1 z_1}(z_1, z_2) &= -6x + \frac{1}{3} + \frac{-1 - 4z_1^2 + z_2 - z_1(7 - 6z_2)}{3\Delta_{F_1}(z_1, z_2)}, \\ Y_{z_1 z_1 z_2}(z_1, z_2) &= 12x - \frac{2}{3} + \frac{-1 + 8z_1^2 + z_2 + z_1(2 - 3z_2)}{3\Delta_{F_1}(z_1, z_2)}, \\ Y_{z_1 z_2 z_2}(z_1, z_2) &= -24x + \frac{4}{3} + \frac{z_2(1 - 12z_1) - (1 - 4z_1)^2}{3\Delta_{F_1}(z_1, z_2)}, \\ Y_{z_2 z_2 z_2}(z_1, z_2) &= 48x - \frac{8}{3} + \frac{2(1 - 4z_1)^2 + z_2(1 + 60z_1)}{3\Delta_{F_1}(z_1, z_2)}, \end{aligned} \tag{4.19}$$

where

$$\Delta_{F_1}(z_1, z_2) = (1 - 4z_1)^2 - z_2(1 - 36z_1 + 27z_1 z_2), \tag{4.20}$$

and

$$x = \frac{(p+1)(3p+1)}{18(2p+1)^2}. \tag{4.21}$$

The above expressions completely agree with the result in [22] (and [23] for $p = 0$ or equivalently $x = 1/18$) and we also see that the free parameter x arising from a degree of freedom of the twisted mass of a chiral matter multiplet reproduces the "moduli parameter" discussed in [22].

Remark 4.2. In addition to the twisted masses represented in (4.17), we can also turn on a non-zero value for λ_2 as $\lambda_1 = p\lambda$, $\lambda_2 = q\lambda$, $\lambda_3 = \lambda_4 = \lambda$, $\lambda_5 = 0$ to make the poles associated with $\mathsf{Q}_* = \{Q_1, Q_3, Q_4\}$ and $\{Q_2, Q_3, Q_4\}$ projective. Then we obtain the same result with (4.19) while x is defined by

$$x = \frac{(2q+3)^2 p^2 + (4q^2 + 15q + 12)p + (q+1)(q+3)}{6(2p+1)^2(2q+3)^2}. \tag{4.22}$$

This means that the additional mass deformation is possible in this case, but does not introduce an independent moduli parameter.

From (3.12), one can easily show that the mirror map for K_{F_1} is given by

$$\log q_i = \log z_i + c_i \sum_{\substack{d_1, d_2 = 0 \\ (d_1, d_2) \neq (0,0)}}^{\infty} \frac{(-1)^{d_2} (2d_1 + 3d_2 - 1)!}{d_1! (d_2!)^2 (d_1 + d_2)!} z_1^{d_1 + d_2} z_2^{d_2}, \quad i = 1, 2, \tag{4.23}$$

where $c_1 = 2$ and $c_2 = 1$. Combining with the formula in (3.7), the Gromov-Witten invariants studied in [21] can be appropriately reproduced.

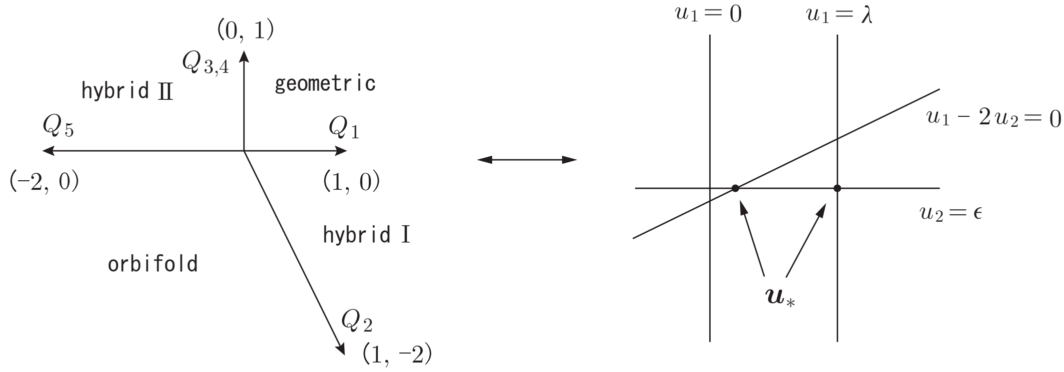

Figure 4: The left figure describes the secondary fan of the local F_2, and the right figure describes the hyperplanes $\{u_1 = \lambda, u_1 - 2u_2 = 0, u_2 = \epsilon, u_1 = 0\}$ corresponding to the charge vectors $\{Q_1, Q_2, Q_{3,4}, Q_5\}$. The points $\mathbf{u}_* = (\lambda, \epsilon)$, $(2\epsilon, \epsilon)$ are projective points associated with $\mathsf{Q}_* = \{Q_1, Q_{3,4}\}$, $\{Q_2, Q_{3,4}\}$. The orbifold phase in the left figure is described by the orbifold geometry $\mathbb{C}^3/\mathbb{Z}_4$.

Local F_2

As a fourth example with $\dim H_4(X,\mathbb{Z}) = 1$, let us study the local Hirzebruch surface K_{F_2} (a local A_1 geometry) described by a $U(1)^2$ GLSM with

$$\left(\begin{array}{ccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{array}\right) = \left(\begin{array}{ccccc} 1 & 1 & 0 & 0 & -2 \\ 0 & -2 & 1 & 1 & 0 \\ \hline \lambda & 0 & \epsilon & \epsilon & 0 \end{array}\right). \tag{4.24}$$

Here we have introduced twisted mass parameters $\epsilon, \lambda \neq 0$ with $\epsilon \neq \lambda$ for compact directions such that $\mathbf{u}_* = (\lambda, \epsilon)$ and $(2\epsilon, \epsilon)$ associated with $\mathsf{Q}_* = \{Q_1, Q_3, Q_4\}$ and $\{Q_2, Q_3, Q_4\}$ provide the projective points respectively (see Figure 4).

In this model, the chiral multiplet Φ_5 which is neutral under the second $U(1)$ gauge symmetry describes the non-compact fiber coordinate X_5, and the compact divisor $\{X_5 = 0\}$ (i.e. F_2) contains the blow up mode of the $\mathbb{Z}_2$ singularity. Therefore, following the rule **2** in Section 3.2, we impose the "Calabi-Yau condition on the divisor" as $\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$, which implies to take a limit $\epsilon \to 0$ in the final step. For $\eta \in \mathrm{Cone}(Q_1, Q_1 + Q_3)$ in the geometric phase, the projective points $\mathbf{u}_* = (\lambda, \epsilon)$ and $(2\epsilon, \epsilon)$ contribute to the Jeffrey-Kirwan residue in (3.4) as

$$\begin{aligned} Y_{z_i z_j z_k}(z_1, z_2) = \lim_{\epsilon \to 0} \sum_{d_1, d_2 = 0}^{\infty} z_1^{d_1} z_2^{d_2} \left(\operatorname*{Res}_{u_2 = \epsilon} \operatorname*{Res}_{u_1 = \lambda} + \operatorname*{Res}_{u_1 = 2\epsilon} \operatorname*{Res}_{u_2 = \epsilon} \right) \\ \times \frac{u_i u_j u_k \, (-2u_1)^{2d_1 - 1}}{(u_1 - \lambda)^{d_1 + 1} (u_2 - \epsilon)^{2(d_2 + 1)} (u_1 - 2u_2)^{d_1 - 2d_2 + 1}}, \end{aligned} \tag{4.25}$$

and we obtain

$$\begin{aligned} Y_{z_1 z_1 z_1}(z_1, z_2) &= \frac{-1}{\Delta_{F_2}(z_1, z_2)}, \quad Y_{z_1 z_1 z_2}(z_1, z_2) = \frac{2z_1 - \frac{1}{2}}{\Delta_{F_2}(z_1, z_2)}, \\ Y_{z_1 z_2 z_2}(z_1, z_2) &= \frac{z_2(1 - 8z_1)}{(1 - 4z_2)\Delta_{F_2}(z_1, z_2)}, \quad Y_{z_2 z_2 z_2}(z_1, z_2) = \frac{z_2(24z_1 z_2 + 2z_1 - 2z_2 - \frac{1}{2})}{(1 - 4z_2)^2 \Delta_{F_2}(z_1, z_2)}, \end{aligned} \tag{4.26}$$

where

$$\Delta_{F_2}(z_1,z_2) = (1-4z_1)^2 - 64z_1^2 z_2. \tag{4.27}$$

The above expressions indeed agree with the result obtained in [22]. The mirror map (3.12) for K_{F_2} is given by

$$\log q_i = \log z_i + c_i \log \frac{1+\sqrt{1-4z_2}}{2} + 2\delta_{i,1} \sum_{\substack{d_1,d_2=0 \\ (d_1,d_2)\neq(0,0)}}^{\infty} \frac{(2d_1+4d_2-1)!}{d_1!(d_2!)^2(d_1+2d_2)!} z_1^{d_1+2d_2} z_2^{d_2}, \tag{4.28}$$

where $i=1,2$, and $(c_1,c_2)=(1,-2)$. The Gromov-Witten invariants computed in [21] from the local mirror symmetry approach can be also reproduced by using (3.7).

Local dP_2

As a last example with $\dim H_4(X,\mathbb{Z})=1$, we consider the local del Pezzo surface K_{dP_2} which can be obtained by a one point blow up of K_{F_0} or K_{F_1}. This local toric variety can be described by a $U(1)^3$ GLSM with

$$\left(\begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{array}\right) = \left(\begin{array}{cccccc} 1 & -1 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ \hline \lambda & q\lambda & \lambda & p\lambda & p\lambda & 0 \end{array}\right). \tag{4.29}$$

Here we have introduced a twisted mass parameter $\lambda \neq 0$ and two free parameters p and q respecting the interchanging symmetry $z_2 \leftrightarrow z_3$ for the base space dP_2, such that $\mathbf{u}_* = ((p+1)\lambda, p\lambda, p\lambda)$, $((p+1)\lambda, p\lambda, (q+1)\lambda)$, $((p+1)\lambda, (q+1)\lambda, p\lambda)$ and $((q+2)\lambda, (q+1)\lambda, (q+1)\lambda)$ associated with $\mathsf{Q}_* = \{Q_1,Q_3,Q_4,Q_5\}$, $\{Q_1,Q_2,Q_5\}$, $\{Q_2,Q_3,Q_4\}$ and $\{Q_1,Q_2,Q_3\}$ provide the projective points, respectively. Note that the particular values for p and q satisfying $3p+1=0$, $2p+q+2=0$, $3q+4=0$ should be excluded in order to maintain the projective condition.

For $\eta=(5,3,2)$ in the geometric phase, we find that the above projective points contribute to the Jeffrey-Kirwan residue in (3.4) as

$$\begin{aligned} Y_{z_i z_j z_k}(z_1,z_2,z_3) = \sum_{d_1,d_2,d_3=0}^{\infty} & z_1^{d_1} z_2^{d_2} z_3^{d_3} \Big(\operatorname*{Res}_{u_1=(p+1)\lambda} \operatorname*{Res}_{u_3=p\lambda} \operatorname*{Res}_{u_2=p\lambda} + \operatorname*{Res}_{u_3=(q+1)\lambda} \operatorname*{Res}_{u_1=(p+1)\lambda} \operatorname*{Res}_{u_2=p\lambda} \\ & + \operatorname*{Res}_{u_2=(q+1)\lambda} \operatorname*{Res}_{u_3=p\lambda} \operatorname*{Res}_{u_1=u_3+\lambda} + \operatorname*{Res}_{u_3=(q+1)\lambda} \operatorname*{Res}_{u_1=u_3+\lambda} \operatorname*{Res}_{u_2=u_1-u_3+q\lambda} \Big) \\ & \times \frac{u_i u_j u_k \left(-u_1-u_2-u_3\right)^{d_1+d_2+d_3-1}}{(u_1-u_2-\lambda)^{d_1-d_2+1}(-u_1+u_2+u_3-q\lambda)^{-d_1+d_2+d_3+1}} \\ & \times \frac{1}{(u_1-u_3-\lambda)^{d_1-d_3+1}(u_3-p\lambda)^{d_3+1}(u_2-p\lambda)^{d_2+1}}. \end{aligned} \tag{4.30}$$

Note that the number of intersecting hyperplanes at the point $\mathbf{u}_* = ((p+1)\lambda, p\lambda, p\lambda)$ is larger than $r=3$. To deal with this "degenerated point" which requires a careful treatment of the order of the iterated residue, we have applied Theorem 2.5 for a flag F with $\kappa^F = (Q_5, Q_4+Q_5, Q_1+Q_3+Q_4+Q_5)$ and $\nu(F)=1$ as the only constituent in $\mathcal{FL}^+(\mathsf{Q}_*,\eta)$ for $\mathsf{Q}_* = \{Q_1,Q_3,Q_4,Q_5\}$.

As a result, we finally obtain

$$
\begin{aligned}
Y_{z_1z_1z_1}(z_1,z_2,z_3) &= 6x+2y-1+Y_{z_1z_1z_1}(z_1,z_2,z_3)\big|_{p=q=0},\\
Y_{z_1z_1z_2}(z_1,z_2,z_3) &= -3x-y+\frac{1}{2}+Y_{z_1z_1z_2}(z_1,z_2,z_3)\big|_{p=q=0},\\
Y_{z_1z_2z_2}(z_1,z_2,z_3) &= x+y+Y_{z_1z_2z_2}(z_1,z_2,z_3)\big|_{p=q=0},\\
Y_{z_1z_2z_3}(z_1,z_2,z_3) &= 2x-\frac{1}{2}+Y_{z_1z_2z_3}(z_1,z_2,z_3)\big|_{p=q=0},\\
Y_{z_2z_2z_2}(z_1,z_2,z_3) &= -y-\frac{1}{4}+Y_{z_2z_2z_2}(z_1,z_2,z_3)\big|_{p=q=0},\\
Y_{z_2z_2z_3}(z_1,z_2,z_3) &= -x+\frac{1}{4}+Y_{z_2z_2z_3}(z_1,z_2,z_3)\big|_{p=q=0},
\end{aligned}
\tag{4.31}
$$

and

$$
\begin{aligned}
&Y_{z_1z_1z_3}(z_1,z_2,z_3)=Y_{z_1z_1z_2}(z_1,z_3,z_2), \quad Y_{z_1z_3z_3}(z_1,z_2,z_3)=Y_{z_1z_2z_2}(z_1,z_3,z_2),\\
&Y_{z_2z_2z_2}(z_1,z_2,z_3)=Y_{z_3z_3z_3}(z_1,z_3,z_2), \quad Y_{z_2z_2z_3}(z_1,z_2,z_3)=Y_{z_2z_3z_3}(z_1,z_3,z_2),
\end{aligned}
\tag{4.32}
$$

with $z_2 \leftrightarrow z_3$ symmetry. Here

$$
\begin{aligned}
x &= \frac{\left(12p^3+33p^2+19p+3\right)q+2\left(8p^3+14p^2+7p+1\right)+(3pq+q)^2}{(3p+1)^2(3q+4)(2p+q+2)},\\
y &= -\frac{5p^3(3q+4)+2p^2(3q+4)^2+2p\left(6q^2+13q+7\right)+2(q+1)^2}{(3p+1)^2(3q+4)(2p+q+2)},
\end{aligned}
\tag{4.33}
$$

and the analytic expressions of $Y_{z_iz_jz_k}(z_1,z_2,z_3)\big|_{p=q=0}$ are summarized in Appendix B. Note that $p=q=0$ corresponds to $(x,y)=(1/4,-1/4)$.

The above results completely agree with the Yukawa couplings with the "moduli parameters" x and y given in [22]. The mirror map (3.12) for K_{dP_2} is given by

$$
\log q_i = \log z_i + \sum_{\substack{d_1,d_2,d_3=0\\(d_1,d_2,d_3)\neq(0,0,0)}}^{\infty} \frac{(-1)^{d_1}(3d_1+2d_2+2d_3-1)!}{d_1!d_2!d_3!(d_1+d_2)!(d_1+d_3)!} z_1^{d_1+d_2+d_3} z_2^{d_1+d_2} z_3^{d_1+d_3}, \tag{4.34}
$$

where $i=1,2,3$, and the associated Gromov-Witten invariants studied in [21] are correctly reproduced by using (3.7).

Local A_2 geometry

Let us consider a local A_2 geometry with $\dim H_4(X,\mathbb{Z})=2$, which is a fibered A_2 geometry over $\mathbb{P}^1$ represented in Figure 5. This geometry engineers the four dimensional pure $SU(3)$ gauge theory and can be described by a $U(1)^3$ GLSM with [51, 52, 21]:

$$
\left(\begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{array}\right) = \left(\begin{array}{cccccc} 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ \hline \epsilon & \epsilon & 0 & 0 & 0 & \lambda \end{array}\right). \tag{4.35}
$$

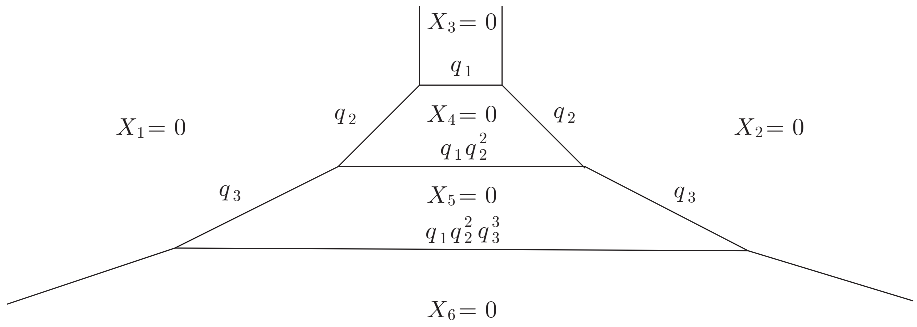

Figure 5: The toric web diagram for the local A_2 geometry. X_i, $i = 1, \ldots, 6$ are homogeneous local coordinates corresponding to the chiral multiplets Φ_i, and $\{X_4 = 0\}$ or $\{X_5 = 0\}$ gives a compact divisor. The parameters q_1, q_2 and q_3 are exponentiated Kähler moduli associated with the $U(1)^3$ gauge group.

Here we have introduced the twisted mass parameters $\epsilon, \lambda \neq 0$ with $\epsilon \neq \lambda$ for a projective hyperplane arrangement in the geometric phase with $\mathsf{Q}_* = \{Q_1, Q_2, Q_3, Q_6\}$, $\{Q_1, Q_2, Q_5, Q_6\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$.

Following the rule **1** in Section 3.2, we have not included the mass parameters for the chiral multiplets Φ_3, Φ_4 and Φ_5 with charge vectors Q_3, Q_4 and Q_5 which blow up the singularities. Furthermore, the chiral multiplet Φ_4 which is neutral with respect to the first $U(1)$ gauge symmetry describes a non-compact fiber coordinate X_4 and the compact divisor $\{X_4 = 0\}$ contains the blow up mode of the $\mathbb{Z}_2$ singularity. Therefore, by following the rule **2**, we impose the "Calabi-Yau condition on the divisor" as $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, which implies to take a limit $\epsilon \to 0$ in the final step.

By taking e.g. $\eta = (2, 1, 2)$ in the geometric phase, the projective points $\mathbf{u}_* = (\epsilon, 2\epsilon, \lambda)$, $(\epsilon, 2\lambda, \lambda)$ and $(\epsilon, 2\epsilon, 4\epsilon)$ contribute to the Jeffrey-Kirwan residue in the GLSM correlators (3.4) as

$$Y_{z_i z_j z_k}(z_1, z_2, z_3) = \lim_{\epsilon \to 0} \sum_{d_1, d_2, d_3 = 0}^{\infty} z_1^{d_1} z_2^{d_2} z_3^{d_3} \left(\operatorname*{Res}_{u_2 = 2\epsilon} \operatorname*{Res}_{u_3 = \lambda} \operatorname*{Res}_{u_1 = \epsilon} + \operatorname*{Res}_{u_2 = 2\lambda} \operatorname*{Res}_{u_1 = \epsilon} \operatorname*{Res}_{u_3 = \lambda} + \operatorname*{Res}_{u_3 = 4\epsilon} \operatorname*{Res}_{u_2 = 2\epsilon} \operatorname*{Res}_{u_1 = \epsilon} \right)$$
$$\times \frac{u_i u_j u_k \, (-2u_2 + u_3)^{2d_2 - d_3 - 1} \, (u_2 - 2u_3)^{-d_2 + 2d_3 - 1}}{(u_1 - \epsilon)^{2(d_1+1)} (-2u_1 + u_2)^{-2d_1 + d_2 + 1} (u_3 - \lambda)^{d_3 + 1}}, \tag{4.36}$$

and we obtain exact expressions represented in Appendix C. The mirror map (3.12) is given by

$$\begin{aligned}
&\log q_1 = \log z_1 - 2 \log \frac{1 + \sqrt{1 - 4z_1}}{2}, \quad \longleftrightarrow \quad z_1 = \frac{q_1}{(1 + q_1)^2}, \\
&\log q_2 = \log z_2 + \log \frac{1 + \sqrt{1 - 4z_1}}{2} + 2M_1(z_1, z_2, z_3) - M_2(z_1, z_2, z_3), \\
&\log q_3 = \log z_3 - M_1(z_1, z_2, z_3) + 2M_2(z_1, z_2, z_3),
\end{aligned} \tag{4.37}$$

where

$$
\begin{aligned}
M_1(z_1,z_2,z_3) &= \sum_{\substack{d_1,d_2,d_3=0\\(d_1,d_2,d_3)\neq(0,0,0)}}^{\infty} \frac{(-1)^{d_3}(2d_2+3d_3-1)!}{(d_1!)^2 d_2! d_3!(-2d_1+d_2+2d_3)!} z_1^{d_1} z_2^{d_2+2d_3} z_3^{d_3}, \\
M_2(z_1,z_2,z_3) &= \sum_{\substack{d_1,d_2,d_3=0\\(d_1,d_2,d_3)\neq(0,0,0)}}^{\infty} \frac{(-1)^{d_2}(6d_1+3d_2+2d_3-1)!}{(d_1!)^2 d_2! d_3!(4d_1+2d_2+d_3)!} z_1^{d_1} z_2^{2d_1+d_2} z_3^{4d_1+2d_2+d_3}.
\end{aligned}
\tag{4.38}
$$

From (3.7) we can reproduce the Gromov-Witten invariants studied in Table 4 of [21], except a one coefficient $n_{1,0,0} = -2/3$. A similar fractional number also appears at the degree $(0,1)$ invariant $n_{0,1} = -1/2$ for the local F_2 that we have investigated in this section (see also Table 11 in [21]).

4.2 Local toric Calabi-Yau fourfolds

Finally we focus on the local nef toric Calabi-Yau fourfolds whose exact properties have been studied in [43]. We again confirm that the localization formula about the A-twisted GLSM correlation functions provides the exact expressions for local B-model Yukawa couplings appropriately, with the aid of our formalism.

$\mathcal{O}(-1)\oplus\mathcal{O}(-2)\to\mathbb{P}^2$

Let us consider the local toric Calabi-Yau fourfold $\mathcal{O}(-1)\oplus\mathcal{O}(-2)\to\mathbb{P}^2$ described by a $U(1)$ GLSM with

$$
\left(\begin{array}{ccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{array}\right) = \left(\begin{array}{ccccc} 1 & 1 & 1 & -1 & -2 \\ \hline \lambda & \lambda & \lambda & 0 & 0 \end{array}\right), \tag{4.39}
$$

where, following the rules in Section 3.2, we have introduced a twisted mass parameter $\lambda \neq 0$ for the base space $\mathbb{P}^2$ such that $u_* = \lambda$ associated with $\mathsf{Q}_* = \{Q_1, Q_2, Q_3\} = 1$ gives a projective point (see Figure 6). For $\eta \in \mathrm{Cone}(1)$ in the geometric phase, the projective point $u_* = \lambda$ contributes to the Jeffrey-Kirwan residue in the GLSM correlation function (3.4) as

$$
Y_{zzzz}(z) = \sum_{d=0}^{\infty} z^d \operatorname*{Res}_{u=\lambda} \frac{u^4\,(-u)^{d-1}\,(-2u)^{2d-1}}{(u-\lambda)^{3(d+1)}} = \frac{1}{2(1+4z)}. \tag{4.40}
$$

In this example, the mirror map (3.12) is trivial, i.e. $\log q = \log z$, and the A-model Yukawa coupling (3.6) takes a form

$$
\widetilde{Y}_{hhhh}(q) = \frac{1}{2(1+4q)}. \tag{4.41}
$$

One finds that the identity

$$
\frac{1}{1-4q} = \left(\sum_{d=0}^{\infty} \binom{2d}{d} q^d\right)^2, \tag{4.42}
$$

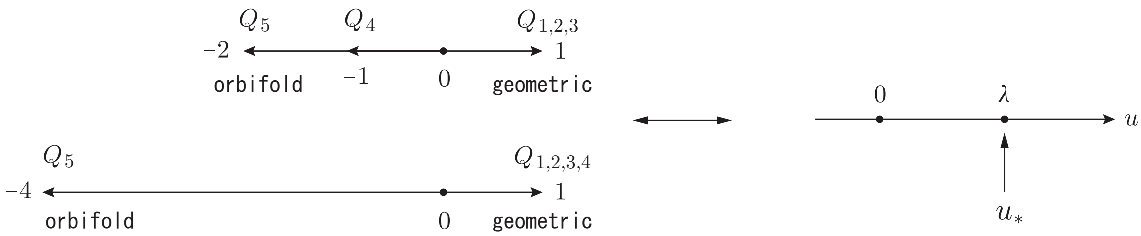

Figure 6: The first (resp. second) figure in the left describes the secondary fan of $\mathcal{O}(-1) \oplus \mathcal{O}(-2) \to \mathbb{P}^2$ (resp. $\mathcal{O}(-4) \to \mathbb{P}^3$), and the right figure describes the hyperplanes $\{u = \lambda, u = 0\}$ corresponding to the charge vectors $\{Q_{1,2,3}, Q_{4,5}\}$ (resp. $\{Q_{1,2,3,4}, Q_5\}$). The point $u_* = \lambda$ is a projective point associated with $\mathsf{Q}_* = Q_{1,2,3} = 1$ (resp. $\mathsf{Q}_* = Q_{1,2,3,4} = 1$).

is equivariant to the factorization (3.8) of the four-point Yukawa coupling $\widetilde{Y}_{hhhh}(q)$ into the three-point Yukawa coupling $\widetilde{Y}_{hhh^2}(q)$ in [43]:

$$\widetilde{Y}_{hhhh}(q) = 2\widetilde{Y}_{hhh^2}(q)^2, \qquad \widetilde{Y}_{hhh^2}(q) = \sum_{d=0}^{\infty} \frac{1}{2} \binom{2d}{d} (-q)^d. \tag{4.43}$$

Local $\mathbb{P}^3$: $\mathcal{O}(-4) \to \mathbb{P}^3$

Next, let us consider the local toric Calabi-Yau fourfold $K_{\mathbb{P}^3}$ described by a $U(1)$ GLSM with

$$\left(\begin{array}{ccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \end{array} \right) = \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & -4 \\ \hline \lambda & \lambda & \lambda & \lambda & 0 \end{array} \right), \tag{4.44}$$

where a twisted mass parameter $\lambda \neq 0$ has been introduced such that $u_* = \lambda$ associated with $\mathsf{Q}_* = \{Q_1, Q_2, Q_3, Q_4\} = 1$ provides a projective point (see Figure 6). For $\eta \in \mathrm{Cone}(1)$ in the geometric phase, the projective point $u_* = \lambda$ contributes to the Jeffrey-Kirwan residue in (3.4) and we obtain

$$Y_{zzzz}(z) = \sum_{d=0}^{\infty} z^d \operatorname*{Res}_{u=\lambda} \frac{u^4 \, (-4u)^{4d-1}}{(u-\lambda)^{4(d+1)}} = -\frac{1}{4\,(1-4^4 z)}. \tag{4.45}$$

The mirror map (3.12) for $K_{\mathbb{P}^3}$ is given by

$$\log q = \log z + 4 \sum_{d=1}^{\infty} \frac{(4d-1)!}{(d!)^4} \, z^d, \tag{4.46}$$

and one can check that the factorization (3.8) into the three-point Yukawa couplings correctly reproduces the previous results about the Gromov-Witten invariants in [43].

$\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$

Finally, we consider the local toric Calabi-Yau fourfold $\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$ described by a $U(1)^2$ GLSM with

$$\left(\begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \\ \hline \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{array} \right) = \left(\begin{array}{cccccc} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ \hline \lambda & \lambda & \lambda & \lambda & 0 & 0 \end{array} \right), \tag{4.47}$$

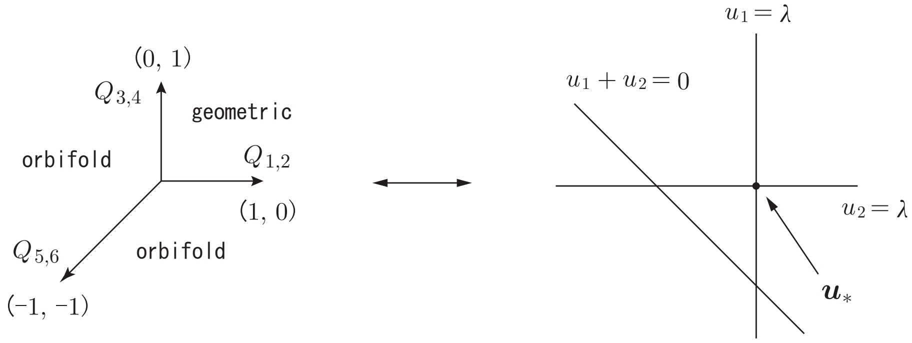

Figure 7: The left figure describes the secondary fan of $\mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$, and the right figure describes the hyperplanes $\{u_1 = \lambda, u_2 = \lambda, u_1 + u_2 = 0\}$ corresponding to the charge vectors $\{Q_{1,2}, Q_{3,4}, Q_{5,6}\}$. The point $\mathbf{u}_* = (\lambda, \lambda)$ is a projective point associated with $\mathsf{Q}_* = \{Q_{1,2}, Q_{3,4}\}$.

where we have introduced a twisted mass parameter $\lambda \neq 0$ in a symmetric way for $z_1 \leftrightarrow z_2$ in the base space $\mathbb{P}^1 \times \mathbb{P}^1$, such that $\mathbf{u}_* = (\lambda, \lambda)$ associated with $\mathsf{Q}_* = \{Q_1, Q_2, Q_3, Q_4\}$ gives a projective point (see Figure 7). For $\eta \in \mathrm{Cone}(Q_1, Q_1 + Q_3)$ in the geometric phase, the projective point $\mathbf{u}_* = (\lambda, \lambda)$ gives a contribution to the Jeffrey-Kirwan residue in the GLSM correlators (3.4) as

$$Y_{z_i z_j z_k z_l}(z_1, z_2) = \sum_{d_1, d_2 = 0}^{\infty} z_1^{d_1} z_2^{d_2} \operatorname*{Res}_{u_2 = \lambda} \operatorname*{Res}_{u_1 = \lambda} \frac{u_i u_j u_k u_l \, (-u_1 - u_2)^{2(d_1 + d_2 - 1)}}{(u_1 - \lambda)^{2(d_1+1)} (u_2 - \lambda)^{2(d_2+1)}}. \tag{4.48}$$

As the result, we obtain

$$\begin{aligned}
Y_{z_1 z_1 z_1 z_1}(z_1, z_2) &= \frac{-5(1 - z_2)^2 + z_1(10 + 11 z_1 + 10 z_2)}{8 \Delta'_{F_0}(z_1, z_2)}, \\
Y_{z_1 z_1 z_1 z_2}(z_1, z_2) &= \frac{(1 - z_2)^2 + z_1(6 - 7 z_1 - 10 z_2)}{8 \Delta'_{F_0}(z_1, z_2)}, \\
Y_{z_1 z_1 z_2 z_2}(z_1, z_2) &= \frac{3 - 6(z_1 + z_2) + 3 z_1^2 + 10 z_1 z_2 + 3 z_2^2}{8 \Delta'_{F_0}(z_1, z_2)}, \\
Y_{z_1 z_2 z_2 z_2}(z_1, z_2) &= \frac{(1 - z_1)^2 + z_2(6 - 10 z_1 - 7 z_2)}{8 \Delta'_{F_0}(z_1, z_2)}, \\
Y_{z_2 z_2 z_2 z_2}(z_1, z_2) &= \frac{-5(1 - z_1)^2 + z_2(10 + 10 z_1 + 11 z_2)}{8 \Delta'_{F_0}(z_1, z_2)},
\end{aligned} \tag{4.49}$$

where

$$\Delta'_{F_0}(z_1, z_2) = 1 - 2(z_1 + z_2) + (z_1 - z_2)^2. \tag{4.50}$$

In this example, the mirror map (3.12) becomes trivial, i.e. $\log q_i = \log z_i$, $i = 1, 2$, and we see that the factorization into the three-point Yukawa couplings (3.8) reproduces the previous results in [43] appropriately.

5 Conclusions

In this paper, we thoroughly investigated the relationship between the topological B-model Yukawa couplings for the backgrounds with non-compact directions and the exact localization formula about the A-twisted correlation functions of the two dimensional $\mathcal{N} = (2,2)$ gauged linear sigma models. Starting from the exact results for the A-twisted correlators of $\mathcal{N} = (2,2)$ gauged linear sigma models [15], we demonstrated how to extract appropriately the B-model Yukawa couplings for the local nef toric Calabi-Yau varieties. First we explained that the inclusion of the twisted masses for the chiral matter multiplets is indispensable to conduct the explicit calculations about the Jeffrey-Kirwan residue formalism. Although it has been argued in the literatures that the twisted mass deformation is required to deal with the backgrounds with non-compact directions, a comprehensive study has not been conducted before. We addressed this important issue and proposed an algorithm to compute the GLSM correlation functions for local toric Calabi-Yau varieties appropriately.

We have also checked that our prescription for the twisted mass deformations of the GLSM correlation functions is totally consistent with known results for the Yukawa couplings evaluated from the local mirror symmetry approach. Moreover, we found that the ambiguities of classical intersection numbers of certain class of local toric Calabi-Yau varieties argued before are identified with the degrees of freedom of the assignment of the proper twisted mass parameters. Combining the exact localization formula about the A-twisted GLSM correlation functions, our prescription would provide an alternative efficient formalism to compute the B-model Yukawa couplings for generic local nef toric Calabi-Yau varieties.

Finally we would like to comment on the possible future research directions. Throughout this paper, we have not considered the models whose target spaces are the non-nef varieties [33, 34, 35]. It would be interesting to extend our analysis for the GLSM correlation functions and twisted mass deformations into such intriguing examples.

In our framework, the twisted masses are introduced only for the compact directions of the target space. It would be interesting to clarify a physical meaning of this requirement. One possible explanation is the following. From the viewpoint of the SUSY algebra, inclusion of the twisted masses modify the central charges of the model and implies the existence of additional charged BPS particles in a specific vacuum. The appearance of such extra massless states can be naturally interpreted as the D-branes wrapping on compact directions as discussed in [53, 54], and is known to be indispensable to regularize a singularity of the model appropriately.

Acknowledgements

We would like to thank Masazumi Honda, Masao Jinzenji, and Yutaka Yoshida for useful discussions and comments. The work of YH was supported in part by the grant MOST-105-2119-M-007-018 and 106-2119-M-007-019 from the Ministry of Science and Technology of Taiwan. The work of MM was supported by the ERC Starting Grant no. 335739 "Quantum fields and knot homologies" funded by the European Research Council under the European Union's Seventh Framework Programme and currently by the Max-Planck-Institut für Mathematik in Bonn.

A Building blocks of I-functions from GLSM correlators

Let us consider a GLSM which flows in the IR to two dimensional non-linear sigma model with a Fano or a Calabi-Yau variety X as a target space. Following [19] (see also [55]), here we will show that the building blocks of the Givental I-function [44, 45, 46] for X can be derived from the factors (2.4) and (2.5) in the localization formula.

First, let us reparametrize $\mathbf{u}$ and $\mathbf{d}$ as

$$\mathbf{u} = \mathbf{x} + \frac{\mathbf{d}}{2}\hbar - \mathfrak{q}'\hbar, \qquad \mathbf{d} = \mathfrak{q} + \mathfrak{q}'. \tag{A.1}$$

Then the factor (2.4) can be decomposed as

$$Z^{\text{vec}}_{\mathbf{d}}(\mathbf{u};\hbar) = I^{\text{vec}}_{\text{pert}}(\mathbf{x})\, I^{\text{vec}}_{\mathfrak{q}}(\mathbf{x};\hbar)\, I^{\text{vec}}_{\mathfrak{q}'}(\mathbf{x};-\hbar), \tag{A.2}$$

where

$$I^{\text{vec}}_{\text{pert}}(\mathbf{x}) = \prod_{\alpha\in\Delta_+} (-1)\,\alpha(\mathbf{x})^2, \tag{A.3}$$

$$I^{\text{vec}}_{\mathfrak{q}}(\mathbf{x};\hbar) = \prod_{\alpha\in\Delta_+} (-1)^{\alpha(\mathfrak{q})}\,\frac{\alpha(\mathbf{x})+\alpha(\mathfrak{q})\hbar}{\alpha(\mathbf{x})}. \tag{A.4}$$

The factor (2.5) for $\Phi_a = \Phi$ with representation $R_a = R$, R-charge $r_a = 0$ and twisted mass $\lambda_a = \lambda$ can be decomposed as

$$Z^{\Phi}_{\mathbf{d}}(\mathbf{u};\hbar) = \begin{cases} \prod_{\rho\in R}\prod_{p=0}^{\rho(\mathbf{d})}\left(\rho(\mathbf{u})+\lambda+p\hbar-\frac{\rho(\mathbf{d})}{2}\hbar\right)^{-1}, & \text{if } \rho(\mathbf{d})\geq 0,\\ \prod_{\rho\in R}\prod_{p=1}^{-\rho(\mathbf{d})-1}\left(\rho(\mathbf{u})+\lambda-p\hbar-\frac{\rho(\mathbf{d})}{2}\hbar\right), & \text{if } \rho(\mathbf{d})\leq -1,\end{cases} \tag{A.5}$$

$$= I^{\Phi}_{\text{pert}}(\mathbf{x},\lambda)\, I^{\Phi}_{\mathfrak{q}}(\mathbf{x},\lambda;\hbar)\, I^{\Phi}_{\mathfrak{q}'}(\mathbf{x},\lambda;-\hbar), \tag{A.6}$$

where

$$I^{\Phi}_{\text{pert}}(\mathbf{x},\lambda) = \prod_{\rho\in R}\frac{1}{\rho(\mathbf{x})+\lambda}, \tag{A.7}$$

$$I^{\Phi}_{\mathfrak{q}}(\mathbf{x},\lambda;\hbar) = \begin{cases} \prod_{\rho\in R}\prod_{p=1}^{\rho(\mathfrak{q})}\left(\rho(\mathbf{x})+\lambda+p\hbar\right)^{-1}, & \text{for } \rho(\mathfrak{q})\geq 0,\\ \prod_{\rho\in R}\prod_{p=0}^{-\rho(\mathfrak{q})-1}\left(\rho(\mathbf{x})+\lambda-p\hbar\right), & \text{for } \rho(\mathfrak{q})\leq -1.\end{cases} \tag{A.8}$$

Similarly, the factor (2.5) for $\Phi_a = P$ with representation $R_a = R$, R-charge $r_a = 2$, and twisted mass $\lambda_a = \lambda$ can be decomposed as

$$Z^{P}_{\mathbf{d}}(\mathbf{u};\hbar) = \begin{cases} -\prod_{\rho\in R}\prod_{p=0}^{-\rho(\mathbf{d})}\left(\rho(\mathbf{u})+\lambda-p\hbar-\frac{\rho(\mathbf{d})}{2}\hbar\right), & \text{if } \rho(\mathbf{d})\leq 0,\\ -\prod_{\rho\in R}\prod_{p=1}^{\rho(\mathbf{d})-1}\left(\rho(\mathbf{u})+\lambda+p\hbar-\frac{\rho(\mathbf{d})}{2}\hbar\right)^{-1}, & \text{if } \rho(\mathbf{d})\geq 1,\end{cases} \tag{A.9}$$

$$= I^{P}_{\text{pert}}(\mathbf{x},\lambda)\, I^{P}_{\mathfrak{q}}(\mathbf{x},\lambda;\hbar)\, I^{P}_{\mathfrak{q}'}(\mathbf{x},\lambda;-\hbar)\prod_{\rho\in R}(-1)^{\rho(\mathbf{d})}, \tag{A.10}$$

where

$$I^P_{\text{pert}}(\mathbf{x},\lambda) = -\prod_{\rho\in R}(\rho(\mathbf{x})+\lambda)\,, \tag{A.11}$$

$$I^P_{\mathfrak{q}}(\mathbf{x},\lambda;\hbar) = \begin{cases}\prod_{\rho\in R}\prod_{p=1}^{-\rho(\mathfrak{q})}(-\rho(\mathbf{x})-\lambda+p\hbar)\,, & \text{for} \quad \rho(\mathfrak{q})\leq 0,\\ \prod_{\rho\in R}\prod_{p=0}^{\rho(\mathfrak{q})-1}(-\rho(\mathbf{x})-\lambda+p\hbar)^{-1}\,, & \text{for} \quad \rho(\mathfrak{q})\geq 1.\end{cases} \tag{A.12}$$

After taking the above decomposition, the ingredients $I^{\text{vec}}_{\mathfrak{q}}(\mathbf{x};\hbar)$ in (A.4), $I^{\Phi}_{\mathfrak{q}}(\mathbf{x},\lambda;\hbar)$ in (A.8) and $I^P_{\mathfrak{q}}(\mathbf{x},\lambda;\hbar)$ in (A.12) are known to provide the building blocks of the Givental I-function for the variety X. There $\mathbf{x}$ and λ are identified with the equivariant cohomology elements or the Chern roots of X and the equivariant parameter acting on X, respectively [55, 19, 56, 20, 48]. Starting from the I-function, one can find an associated quantum differential equation called Picard-Fuchs equation from which the mirror map and genus zero Gromov-Witten invariants of X can be evaluated [44, 45, 46].

B Local B-model Yukawa couplings of local dP_2

The exact B-model Yukawa couplings $Y^{(0)}_{z_iz_jz_k}(\mathbf{z}) = Y_{z_iz_jz_k}(z_1,z_2,z_3)\big|_{p=q=0}$ for $p=q=0$ of the local del Pezzo surface K_{dP_2} defined in (4.30) have the following expressions:

$$\begin{aligned}
Y^{(0)}_{z_1z_1z_1}(\mathbf{z}) &= \big(4z_1^2\,(z_2-z_3)^2\,(z_2+z_3-2)+z_1\,(z_2-1)\,(z_3-1)\,(5z_3-z_2\,(9z_3-5)-1)\,\big)/\Delta_{dP_2}(\mathbf{z}),\\
Y^{(0)}_{z_1z_1z_2}(\mathbf{z}) &= \big(-4\left(2z_2^3-(3z_3+2)\,z_2^2+(z_3+1)^2\,z_2-z_3\right)z_1^2+\big(\left(9z_3^2-8z_3-2\right)z_2^2-2\left(7z_3^2-7z_3-1\right)z_2\\
&\quad +4z_3^2-4z_3-1\big)z_1+(z_2-1)\,(z_3-1)\,\big)/2\Delta_{dP_2}(\mathbf{z}),\\
Y^{(0)}_{z_1z_2z_2}(\mathbf{z}) &= \big(4\,(z_2-1)\,z_2\,(z_2-2z_3+1)\,z_1^2+z_2\,(z_3-1)\,(8z_3-z_2\,(9z_3-5)-4)\,z_1\big)/\Delta_{dP_2}(\mathbf{z}),\\
Y^{(0)}_{z_1z_2z_3}(\mathbf{z}) &= \big(4\left(z_3z_2^2+\left(z_3^2-4z_3+1\right)z_2+z_3\right)z_1^2+\big(\left(9z_3^2-14z_3+4\right)z_2^2-2\left(7z_3^2-10z_3+2\right)z_2\\
&\quad +4z_3^2-4z_3-1\big)z_1+(z_2-1)\,(z_3-1)\,\big)/2\Delta_{dP_2}(\mathbf{z}),\\
Y^{(0)}_{z_2z_2z_2}(\mathbf{z}) &= \big(-16\left(z_2^2-z_3^2\right)z_1^3-8\left(4z_2^3-4z_3z_2^2-\left(z_3^2+2z_3-2\right)z_2-2z_3^3+2z_3^2+z_3\right)z_1^2\\
&\quad +\big(\left(45z_3^2-52z_3+4\right)z_2^2+\left(20z_3^2-38z_3+20\right)z_2-8z_3^2+8z_3+1\big)\,z_1\\
&\quad +(3z_2+1)\,(z_3-1)\,\big)/4\Delta_{dP_2}(\mathbf{z}),\\
Y^{(0)}_{z_2z_2z_3}(\mathbf{z}) &= \big(16\left(z_2^2-z_3^2\right)z_1^3+8\left(2z_2^3-2z_2^2-z_3^2z_2-2z_3^3+2z_3^2+z_3\right)z_1^2+\big(\left(-9z_2^2-4z_2+8\right)z_3^2\\
&\quad +2\left(4z_2^2+3z_2-4\right)z_3-1\big)z_1+(z_2-1)\,(z_3-1)\,\big)/4\Delta_{dP_2}(\mathbf{z}),
\end{aligned}$$

and

$$\begin{aligned}
Y^{(0)}_{z_1z_1z_3}(z_1,z_2,z_3) &= Y^{(0)}_{z_1z_1z_2}(z_1,z_3,z_2), \quad & Y^{(0)}_{z_1z_3z_3}(z_1,z_2,z_3) &= Y^{(0)}_{z_1z_2z_2}(z_1,z_3,z_2),\\
Y^{(0)}_{z_2z_2z_2}(z_1,z_2,z_3) &= Y^{(0)}_{z_3z_3z_3}(z_1,z_3,z_2), \quad & Y^{(0)}_{z_2z_2z_3}(z_1,z_2,z_3) &= Y^{(0)}_{z_2z_3z_3}(z_1,z_3,z_2),
\end{aligned}$$

where

$$\begin{aligned}
\Delta_{dP_2}(\mathbf{z}) &= 16(z_2-z_3)^2z_1^3+8\left((2z_2+2z_3-3)(z_2-z_3)^2-(z_2+z_3)(1-z_2)(1-z_3)\right)z_1^2\\
&\quad +\left(4(9z_2z_3-2z_2-2z_3+2)(z_2+z_3)+1-30z_3z_2-27z_2^2z_3^2\right)z_1-(1-z_2)(1-z_3).
\end{aligned}$$

These expressions indeed agree with the local B-model Yukawa couplings in [22] with $x=1/4$ and $y=-1/4$.

C B-model Yukawa couplings of local A_2 geometry

The exact B-model Yukawa couplings of the local A_2 geometry defined in (4.36) have the following expressions:

$$
\begin{aligned}
Y_{z_1z_1z_1}(\mathbf{z}) = {} & 2z_1\big(-16z_1^2z_2^2(3z_3(9z_3(3z_2(6z_3-1)+4z_3-5)+14)-4)\\
& +4z_1\big(324(z_2-2)z_2^2z_3^3+2(3z_2(32-9(z_2-3)z_2)-8)z_3^2+(8-92z_2)z_3+11z_2-1\big)\\
& +(z_2(6z_3-1)-4z_3+1)(z_2(9z_3(3z_2z_3-2)+4)+4z_3-1)\big)/3(1-4z_1)^2\Delta_{A_2}(\mathbf{z}),\\
Y_{z_1z_1z_2}(\mathbf{z}) = {} & -4z_1\big(-108z_2^2z_3^3(4z_1(3z_2+1)-3z_2+3)+z_3^2(9z_2(3z_2(4z_1(2z_2+5)-2z_2-1)+16)-16)\\
& -2z_2z_3(21(4z_1-1)z_2+34)+4z_2((4z_1-1)z_2+2)+8z_3-1\big)/3(1-4z_1)\Delta_{A_2}(\mathbf{z}),\\
Y_{z_1z_1z_3}(\mathbf{z}) = {} & -2z_1\big(-54z_2^2z_3^3(z_1(60z_2+8)-15z_2+6)+z_3^2(9z_2(3z_2(4z_1(8z_2+11)-8z_2-7)+16)-16)\\
& -4z_2z_3(33(4z_1-1)z_2+17)+8z_2((8z_1-2)z_2+1)+8z_3-1\big)/3(1-4z_1)\Delta_{A_2}(\mathbf{z}),\\
Y_{z_1z_2z_2}(\mathbf{z}) = {} & 2\big(-54z_2^2z_3^3(4z_1(3z_2+2)-3z_2+2)+z_3^2(3z_2(9z_2(4z_1(z_2+5)-z_2-3)+32)-16)\\
& -2z_2z_3(21(4z_1-1)z_2+22)+z_2(4(4z_1-1)z_2+5)+8z_3-1\big)/3\Delta_{A_2}(\mathbf{z}),\\
Y_{z_1z_2z_3}(\mathbf{z}) = {} & \big(-27z_2^2z_3^3(4z_1(15z_2+4)-15z_2+4)+z_3^2(3z_2(9z_2(4z_1(4z_2+11)-4z_2-9)+44)-16)\\
& +z_2z_3(132(1-4z_1)z_2-65)+8z_2((8z_1-2)z_2+1)+8z_3-1\big)/3\Delta_{A_2}(\mathbf{z}),\\
Y_{z_1z_3z_3}(\mathbf{z}) = {} & 2\big(-54(4z_1-1)z_2^3z_3^2(3z_3-2)-z_2^2(4z_1(3z_3(9(z_3-5)z_3+32)-16)\\
& +3z_3(9z_3(z_3+3)-32)+16)+z_2\big(42z_3^2-44z_3+8\big)+z_3(5-4z_3)-1\big)/3\Delta_{A_2}(\mathbf{z}),\\
Y_{z_2z_2z_2}(\mathbf{z}) = {} & 4\big(-(4z_1-1)z_2^2(3z_3(3z_3-2)(12z_3-7)-4)+z_2(4z_3(12z_3-5)+2)-(1-4z_3)^2\big)/3\Delta_{A_2}(\mathbf{z}),\\
Y_{z_2z_2z_3}(\mathbf{z}) = {} & 2\big(-(4z_1-1)z_2^2(3z_3(9z_3(4z_3-11)+44)-16)\\
& +2z_2(z_3(60z_3-31)+4)-(1-4z_3)^2\big)/3\Delta_{A_2}(\mathbf{z}),\\
Y_{z_2z_3z_3}(\mathbf{z}) = {} & 4\big(-(4z_1-1)z_2^2(3z_3(9(z_3-5)z_3+32)-16)\\
& +z_2\big(48z_3^2-44z_3+8\big)+z_3(5-4z_3)-1\big)/3\Delta_{A_2}(\mathbf{z}),\\
Y_{z_3z_3z_3}(\mathbf{z}) = {} & \big(-2(4z_1-1)z_2^2(3z_3(9(z_3-8)z_3+80)-64)\\
& +4z_2(z_3(33z_3-52)+16)-8z_3^2+22z_3-8\big)/3\Delta_{A_2}(\mathbf{z}),
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{A_2}(\mathbf{z}) = {} & 729(1-4z_1)^2z_2^4z_3^4+108(4z_1-1)z_2^3(9z_3-2)z_3^2+2z_2^2(4z_1(9z_3(3z_3(4z_3-7)+8)-8)\\
& +9z_3(3z_3(4z_3+5)-8)+8)-4z_2(4z_3-1)(9z_3-2)+(1-4z_3)^2.
\end{aligned}
$$

Derivation of $\Delta_{A_2}(\mathbf{z})$

The factor $\Delta_{A_2}(\mathbf{z})$ appeared in the above expressions can be regarded as a discriminant which describes the degenerate points of the mirror curve [2, 57]. To find a mirror curve explicitly, let us first consider the mirror local Calabi-Yau threefold of the local A_2 geometry defined by

$$
\begin{aligned}
X^\vee = \Big\{(\omega_+,\omega_-,x_1,x_2,x_3,x_4,x_5,x_6)\in\mathbb{C}^2\times(\mathbb{C}^*)^6\ \Big|\ & \omega_+\omega_- = \sum_{i=1}^{6}x_i,\ x_1x_2=z_1x_3^2\\
& x_3x_5=z_2x_4^2,\ x_4x_6=z_3x_5^2\Big\},
\end{aligned} \tag{C.1}
$$

where each of x_i, $i=1,\ldots,6$ is the mirror coordinate corresponding to X_i in Figure 5. Let us take a local coordinate $x_1=y$, $x_3=0$, $x_4=x$ which corresponds to the mirror of a brane wrapping on a Lagrangian submanifold located at the external leg of the local atlas around

$X_1 = X_3 = X_4 = 0$. Then x becomes the open string moduli parameter in the B-model. From (C.1), we obtain a mirror curve in $X^\vee$ at $\omega_+ = 0$ or $\omega_+ = 0$ as

$$y^2 + \left(1 + x + z_2 x^2 + z_2^2 z_3 x^3\right) y + z_1 = 0. \tag{C.2}$$

The branch points can be obtained from

$$\left(1 + x + z_2 x^2 + z_2^2 z_3 x^3\right)^2 - 4 z_1 = 0,$$

and the discriminant for x of the above equation given by

$$2^{12} z_1^3 z_2^{16} z_3^6 \, \Delta_{A_2}(\mathbf{z})$$

describes the degenerate points of the mirror curve (C.2).

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