Wedge Sobolev Spaces

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Abstract

The pseudodifferential calculus on a wedge in the form developed by the first author provides a general framework for the analysis on manifolds with edges. Its components are (1) wedge Sobolev spaces; (2) wedge algebra of pseudodifferential operators; and (3) the corresponding space of residual operators. One of the drawbacks of the theory so far has been that all constructions are performed in a fixed set of coordinates. While it could, of course, be conjectured that the actual choice of coordinates was irrelevant, this has never been shown. The present paper is first in the series of papers intended to examine the invariance. We show that there are diffeomorphisms of a coordinate wedge which don't leave weighted Sobolev spaces invariant, even if they keep the edge. We also indicate a reasonable class of diffeomorphisms of a coordinate wedge under which the weighted Sobolev are invariant. Consequently, our main result just amounts to saying that these weighted Sobolev spaces make sense on a manifold with edge-like singularities, provided that its transition diffeomorphisms belong to the above class.

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Introduction

This work presents a first step in a series of forthcoming papers in which we are going to establish the invariance of the calculus of pseudodifferential operators on manifolds with edges in the form developed by Schulze [15, 16]. It is devoted to examining the weighted Sobolev spaces on manifolds with edges which enter the calculus.

Let L be a Banach space with a fixed $\mathbb{R}_+$ action $(\kappa_\lambda)_{\lambda\in\mathbb{R}_+}$ on L. Given any $s \in \mathbb{R}$, the Sobolev space $\mathcal{W}^s(\mathbb{R}^q, L)$ is defined to consists of all L-valued distributions u on $\mathbb{R}^q$ such that $\mathcal{F}_{y\to\eta}u \in L^1_{loc}(\mathbb{R}^q, L)$ and

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,L)} = \left(\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \left\|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta}u\right\|^2_L d\eta\right)^{\frac{1}{2}} < \infty.$$

The function $\eta \mapsto \langle\eta\rangle$ entering this definition is a so-called *smooth norm function.* This means, it is positive, smooth and equal to $|\eta|$ for η large enough. The particular choice of this function does not affect the definition up to an equivalent norm.

This spaces were introduced by Schulze (see [15] and the references given there) for constructing algebras of pseudodifferential operators on manifolds with edges.

In order to localize the spaces $\mathcal{W}^s(\mathbb{R}^q, L)$ to open subsets of $\mathbb{R}^q$ we invoke the standard construction of "loc" and "comp" spaces. Namely, given an open set $U \subset \mathbb{R}^q$, we denote by $\mathcal{W}^s_{loc}(\mathbb{R}^q, L)$ the space of all distributions $u \in \mathcal{D}'(U, L)$ such that $\varphi u \in \mathcal{W}^s(\mathbb{R}^q, L)$ for any $\varphi \in \mathcal{D}(U)$. And by $\mathcal{W}^s_{comp}(U, L)$ we denote the space of all L-valued distributions on U of the form φu, where $\varphi \in \mathcal{D}(U)$ and $u \in \mathcal{W}^s(\mathbb{R}^q, L)$. Since $\mathcal{W}^s(\mathbb{R}^q, L)$ is a module over $\mathcal{D}(\mathbb{R}^q)$, it follows that $\mathcal{W}^s_{comp}(U, L)$ consists of all $u \in \mathcal{W}^s(\mathbb{R}^q, L)$ supported on U.

The spaces $\mathcal{W}^s(\mathbb{R}^q, L)$ are natural generalizations of the usual Sobolev spaces and possess many common properties. In particular, it is a simple matter to see that if $\delta : U_1 \to U_2$ is a diffeomorphism of open sets in $\mathbb{R}^q$, then the pull-back operator δ^* on distributions induces the isomorphisms

$$\begin{aligned} \delta^* : \mathcal{W}^s_{loc}(U_2, L) &\to \mathcal{W}^s_{loc}(U_1, L), \\ \delta^* : \mathcal{W}^s_{comp}(U_2, L) &\to \mathcal{W}^s_{comp}(U_1, L) \end{aligned}$$

(see, for instance, Behm [2, p.57]).

In the wedge theory, however, L is considered to be a weighted Sobolev space on a stretched cone $X^\square = \overline{\mathbb{R}_+} \times X$, where X is either $\mathbb{R}^n$ or a compact manifold of dimension n. If it is the case, then $\mathcal{W}^s(\mathbb{R}^q, L)$ is in fact a function space on the product $X^{\square} \times \mathbb{R}^q$. (In this way we obtain what we shall call the *model wedge* with edge $Y = \mathbb{R}^q$ and basic cone $X^{\square}$.) Hence one may ask whether $\mathcal{W}^s(\mathbb{R}^q, L)$ is still locally invariant under diffeomorphisms of $X^{\square} \times \mathbb{R}^q$. The answer is negative in general because the diffeomorphisms of the model wedge "mix" up the variables. On the other hand, it might be expected that $\mathcal{W}^s(\mathbb{R}^q, L)$ is locally invariant if we restrict our attention to the diffeomorphisms of $X^{\square} \times \mathbb{R}^q$ which preserve the edge $\mathbb{R}^q$.

In this paper we first construct a diffeomorphism δ of the model wedge such that δ preserves the edge $\mathbb{R}^q$ while δ^* does not preserve the Sobolev spaces on $X^{\square} \times \mathbb{R}^q$. The idea consists of "mixing" up the variables of X and Y, namely $\delta(t, x, y) = (t, y, x)$. In spite of this explicit form of δ, the proof of the non-invariance of Sobolev spaces is not obvious because the group action occuring in the definition of $\mathcal{W}^s(\mathbb{R}^q, L)$ evokes many technical troubles.

On the other hand, there is a reasonable class of diffeomorphisms of the model wedge consisting of those which preserve the typical differential operators on the wedge (so-called *edge-degenerate operators*). Roughly speaking, these diffeomorphisms are close to those acting separately in (t, x) and y. More precisely, they are of the form

$$(t, x, y) \mapsto (\tau(t, x, y), \chi(t, x, y), v(t, x, y)),$$

where

$$\begin{aligned} &\tau(0, x, y) \equiv 0, \\ &v(0, x, y) \textit{ does not depend on } x. \end{aligned}$$

For such diffeomorphisms, we prove the local invariance of Sobolev spaces on the wedge under changing the variables.

It is worth pointing out that the wedge theory contains the theory of boundary value problems as a *special case* corresponding to $n = 0$. Namely, in the case of boundary value problems the edge is just the boundary, and the model cone $\mathbb{R}_+$ is the "inner normal" to the boundary. By the above, the Sobolev spaces entering this special case are locally invariant under *all* diffeomorphisms of the model cone. So, these spaces make sense on *any* manifold with boundary.

Chapter 1

Cone Sobolev Spaces

1.1 Sobolev Spaces on a Coordinate Cone

1.1.1 *Model cone*

Let X be an open subset of the unit sphere S^n in $\mathbb{R}^{1+n}$. We tacitly assume that X is different from the whole sphere S^n, and write $x = (x_1, \ldots, x_n)$ for local coordinates in X.

By a *model cone* is meant the geometrical cone in $\mathbb{R}^{1+n}$ given by

$$X^\Delta = \{\lambda p : \ \lambda \geq 0, \ p \in X\}.$$

(Here λp denotes the λ multiple of the vector p under the standard vector structure on $\mathbb{R}^{1+n}$.)

In the sequel, $int\, X^\Delta$ stands for the set of the interior points of X^Δ in $\mathbb{R}^{1+n}$, i.e., $int\, X^\Delta = \{\lambda p : \ \lambda > 0, \ p \in X\}$.

The natural volume form on X^Δ is that induced by the Lebesgue measure $dv = dz_1 \wedge \ldots \wedge dz_n$ on $\mathbb{R}^{1+n}$.

1.1.2 *Polar coordinates*

Given any point $z \in X^\Delta$, the pair (t, p), where $t = |z|$ and $p = \frac{z}{|z|}$, can be considered as *polar coordinates* of z.

Under the polar coordinates, the cone X^Δ can be identified with the cylinder

$$X^\Box = \overline{\mathbb{R}_+} \times X.$$

(One should however have kept in mind that the vertex of the cone is blown up to the base $\{0\} \times X$ of the cylinder.) This is a local description of what we call the passage to the "*stretched object.*"

We write $int\, X^{\square}$ for the set of the interior points of $X^{\square}$ on $\mathbb{R}\times S^n$, i.e., $int\, X^{\square} = \mathbb{R}_+ \times X$. Then

$$\pi(t,x) = t\, p(x)$$

is a diffeomorphism of $int\, X^{\square} \to int\, X^{\triangle}$.

For a distribution u in the interior of $X^{\triangle}$, we denote by $\pi^* u \in \mathcal{D}'(int\, X^{\square})$ the *pull-back* of u under π.

Thus, instead of analyzing functions in Euclidean coordinates of the cone $X^{\triangle}$ whose boundary is

$$\{0\} \cup \{\lambda p : \ \lambda > 0,\ p \in \partial X\},$$

we may analyze those in the polar coordinates of the cylinder $X^{\square}$ whose boundary is

$$(\{0\} \times X) \cup (\mathbb{R}_+ \times \partial X).$$

To have inherited the Riemannian structure on $X^{\square}$ under this passage, we need the following elementary result.

Lemma 1.1.1 *We have $dv = t^n\, dt\, dx$, where dx is the area form on X induced by the standard area form on the unit sphere.*

Proof. Indeed,

$$\begin{aligned} & dz_1 \wedge \ldots \wedge dz_{1+n} \\ &= d\left(|z|\,\frac{z_1}{|z|}\right) \wedge \ldots \wedge d\left(|z|\,\frac{z_{1+n}}{|z|}\right) \\ &= \left(\frac{z_1}{|z|}\, d|z| + |z|\, d\frac{z_1}{|z|}\right) \wedge \ldots \wedge \left(\frac{z_{1+n}}{|z|}\, d|z| + |z|\, d\frac{z_{1+n}}{|z|}\right) \\ &= |z|^n\, d|z| \sum_{j=1}^{1+n} (-1)^{j-1}\, \frac{z_j}{|z|}\, d\frac{z}{|z|}[j], \end{aligned}$$

where $d\frac{z}{|z|}[j]$ is the wedge product of the differentials $d\frac{z_1}{|z|}, \ldots, d\frac{z_{1+n}}{|z|}$ one after another excepting $d\frac{z_j}{|z|}$. Hence our statement follows.

□

1.1.3 *Fuchs-type operators*

Given a differential operator $M(z,D) = \sum_{|\alpha|\le m} M_\alpha(z) D^\alpha$ of order m with C^∞ coefficients on $X^{\triangle}$, we are going to write it in the coordinates (t,x).

Lemma 1.1.2 *For every $j = 1, \ldots, 1+n$, we have*

$$\frac{\partial}{\partial z_j} = p_j\, \frac{\partial}{\partial t} + \frac{1}{t}\, \frac{\partial}{\partial t_j}, \tag{1.1.1}$$

where $\frac{\partial}{\partial t_j}$ is a tangential vector field on S^n with C^∞ coefficients independent of t.

Proof. Let $p = p(x_1, \ldots, x_n)$ be local coordinates on the sphere S^n. Here, p is a smooth function from an open set U in $\mathbb{R}^n$ to S^n.

We have $z = t\,p(x)$, whence

$$\frac{\partial z}{\partial(t,x)} = \begin{pmatrix} p & t\,\frac{\partial p}{\partial x} \end{pmatrix}.$$

Moreover, it follows from $p_1^2 + \ldots p_{1+n}^2 \equiv 1$ that $p^t\,\frac{\partial p}{\partial x} \equiv 0$, where p^t is the transpose line to the row vector p. By Cramer's rule, the inverse matrix for $\frac{\partial w}{\partial(t,x)}$ is of the form

$$\frac{\partial(t,x)}{\partial z} = \begin{pmatrix} p^t \\ \frac{1}{t}\left(\frac{\partial p}{\partial x}\right)^{-1} \end{pmatrix},$$

where $\left(\frac{\partial p}{\partial x}\right)^{-1}$ is the left inverse for $\frac{\partial p}{\partial x}$.

Thus, the chain rule yields

$$\frac{\partial}{\partial z_j} = p_j\,\frac{\partial}{\partial t} + \frac{1}{t}\,t_j(x, D_x), \qquad j = 1, \ldots, 1+n,$$

$t_j(x, D_x)$ being a first order differential operator on U whose coefficients are independent of t.

The proof above gives more, namely t_j is actually independent of which local coordinates on S^n we choose to define it. Hence the lemma follows.

□

Lemma 1.1.2 shows that the differential operator $M(z, D)$ transforms into an operator of the form

$$\pi^\sharp M\left((t,x),(D_t,D_x)\right) = \sum_{\alpha_0+|\alpha|\le m} \widetilde{M}_{\alpha_0,\alpha}(t,x)\,D_t^{\alpha_0}\left(\frac{1}{t}D_x\right)^{\alpha}$$

on the cylinder $X^\square$. (This operator $\pi^\sharp M$ is called the pull-back of M under π.) The coefficients $\widetilde{M}_{\alpha_0,\alpha}$ can be computed from (1.1.1) and M_α. Notice that they are smooth up to $t = 0$, provided that M_α are.

Lemma 1.1.3 *For every non-negative integer N, there are (unique) constants c'_{Nj} and c''_{Nj} such that*

$$\begin{aligned} (tD_t)^N &= \textstyle\sum_{j=1}^N c'_{Nj}\,t^j D_t^j, \\ t^N D_t^N &= \textstyle\sum_{j=1}^N c''_{Nj}\,(tD_t)^j. \end{aligned}$$

Proof. Use induction on N and the obvious equality

$$(tD_t)\,(t^j D_t^j) = j\,t^j D_t^j + t^{j+1} D_t^{j+1}.$$

□

Using this lemma we may also write

$$\pi^{\sharp} M\left((t,x),(D_t,D_x)\right) = \sum_{\alpha_0+|\alpha|\le m} \widetilde{M}_{\alpha_0,\alpha}(t,x)\,\frac{1}{t^{\alpha_0+|\alpha|}}(tD_t)^{\alpha_0} D_x^{\alpha},$$

the coefficients $\widetilde{M}_{\alpha_0,\alpha}$ having been slightly modified.

Definition 1.1.4 *By Fuchs-type operators on $X^{\square}$ are meant differential operators of the form $\frac{1}{t^m}\sum_{j=0}^{m} M_j(t)\,(-t\partial_t)^j$ with operator-valued coefficients $M_j \in C^{\infty}_{loc}(\overline{\mathbb{R}_+}, Diff^{m-j}(X))$.*

(Here $Diff^{m-j}(X)$ is the Fréchet space of all differential operators of order $m-j$ on X.)

As above, the analysis of differential operators in polar coordinates leads to Fuchs-type operators.

1.1.4 *Transformation of Sobolev spaces*

In a similar way we may look at the function spaces. On the whole, we are interested in Sobolev spaces $H^s(X^{\Delta})$ on the cone X^{Δ}.

We assume that X lies in a coordinate patch on S^n with local coordinates $p = p(x)$. We begin with an example.

Example 1.1.5 Let $n = 1$. Use the standard polar coordinates

$$\begin{cases} p_1(\phi) = \cos\phi, \\ p_2(\phi) = \sin\phi \end{cases} \quad (\phi \in [0, 2\pi))$$

on the unit circle. Then

$$\begin{cases} \frac{\partial}{\partial z_1} = p_1\frac{\partial}{\partial t} - \frac{1}{r}p_2\frac{\partial}{\partial \phi}, \\ \frac{\partial}{\partial z_2} = p_2\frac{\partial}{\partial r} + \frac{1}{r}p_1\frac{\partial}{\partial \phi} \end{cases} \quad \text{while} \quad \begin{cases} r\frac{\partial}{\partial r} &= z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}, \\ \frac{\partial}{\partial \phi} &= -z_2\frac{\partial}{\partial z_1} + z_1\frac{\partial}{\partial z_2}. \end{cases}$$

Given any $u \in H^2(X^{\Delta})$, we denote by $\tilde{u}(r,\phi) = u(rp(\phi))$ the corresponding *pull-back* function on the cylinder $X^{\square}$. An easy calculation shows that

$$\begin{aligned} \|u\|^2_{H^2(X^{\Delta})} &:= \int_{X^{\Delta}} \left(\sum_{|\alpha|\le 2} |D^{\alpha}u|^2\right) dv \\ &= \int_{X^{\square}} \Big(|\tilde{u}|^2 \\ &\quad + \frac{1}{r^2}\left(|(rD_r)\tilde{u}|^2 + |D_{\phi}\tilde{u}|^2\right) \\ &\quad + \frac{1}{r^4}\left(|(rD_r)^2\tilde{u} - (rD_r)\tilde{u}|^2 + 2\,|(rD_r)D_{\phi}\tilde{u} - D_{\phi}\tilde{u}|^2 + |D_{\phi}^2\tilde{u} + (rD_r)\tilde{u}|^2\right)\Big)\, r\,dr\,d\phi \end{aligned}$$

while

$$\|\tilde{u}\|^2_{\mathcal{H}^2(X^\square)} := \int_{X^\square} \left(\sum_{\alpha_0+\alpha\le 2} \frac{1}{r^{2(\alpha_0+\alpha)}} \left|(rD_r)^{\alpha_0} D_\phi^\alpha \tilde{u}\right|^2 \right) r dr d\phi$$
$$= \int_{X^\triangle} \Big(|u|^2$$
$$+ \left(\left|\frac{\partial u}{\partial z_1}\right|^2 + \left|\frac{\partial u}{\partial z_2}\right|^2 \right)$$
$$+ \left(\left|\frac{\partial^2 u}{\partial z_1^2} + \frac{z_1}{|z|^2}\frac{\partial u}{\partial z_1} - \frac{z_2}{|z|^2}\frac{\partial u}{\partial z_2}\right|^2 + 2\left|\frac{\partial^2 u}{\partial z_1 \partial z_2} + \frac{z_2}{|z|^2}\frac{\partial u}{\partial z_1} + \frac{z_1}{|z|^2}\frac{\partial u}{\partial z_2}\right|^2 \right.$$
$$\left. + \left|\frac{\partial^2 u}{\partial z_2^2} - \frac{z_1}{|z|^2}\frac{\partial u}{\partial z_1} + \frac{z_2}{|z|^2}\frac{\partial u}{\partial z_2}\right|^2 \right) \Big) dv.$$

Hence it follows that for any $\varepsilon > 0$ there are positive constants c_1 and c_2 depending on ε, such that

$$c_1 \|\tilde{u}\|_{\mathcal{H}^2(X^\square)} \le \|u\|_{H^2(X^\triangle)} \le c_2 \|\tilde{u}\|_{\mathcal{H}^2(X^\square)}$$

whenever $u \in H^2(X^\triangle)$ has its support away from $B(0,\varepsilon) \cap X^\triangle$, where $B(0,\varepsilon)$ is the ball of center 0 and radius ε in $\mathbb{R}^2$.

□

(Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space L, we say that these norms are *equivalent* (written $\|\cdot\|_1 \sim \|\cdot\|_2$), if the ratio $\frac{\|u\|_1}{\|u\|_2}$ is bounded uniformly in $u \in L$ both from below and above by positive constants .)

We conclude therefore that the norm $\|\pi^\star u\|_{\mathcal{H}^2(X^\square)}$ sharply catches the behavior of Sobolev functions $u \in H^2(X^\triangle)$ away from any neighborhood of the vertex $t = 0$. It is clear that this example is of general character.

Definition 1.1.6 *Given any $s \in \mathbb{Z}_+$, denote by $\mathcal{H}^s(X^\square)$ the set of all distributions u in the interior of $X^\square$ whose derivatives up to the order s are locally integrable with respect to the measure $t^n dtdx$ and satisfy*

$$\|u\|^2_{\mathcal{H}^s(X^\square)} := \int_{X^\square} \left(\sum_{\alpha_0+|\alpha|\le s} \frac{1}{t^{2(\alpha_0+|\alpha|)}} |(tD_t)^{\alpha_0} D_x^\alpha u|^2 \right) t^n dtdx < \infty.$$

For integer $s < 0$, we define the space $\mathcal{H}^s(X^\square)$ by duality as $\mathcal{H}^s(X^\square) = \mathcal{H}^{-s}_{comp}(X^\square)'$ where the subscript "*comp*" refers to functions of compact support in the cylinder. Further, for fractional s, the space $\mathcal{H}^s(X^\square)$ is defined by (complex) interpolation.

Lemma 1.1.7 *Let $s \in \mathbb{R}$. To any $\varepsilon > 0$, there correspond positive constants c_1 and c_2 depending on ε, such that*

$$c_1 \|\pi^* u\|_{\mathcal{H}^s(X^\square)} \le \|u\|_{H^s(X^\triangle)} \le c_2 \|\pi^* u\|_{\mathcal{H}^s(X^\square)}$$

whenever $u \in H^s(X^\triangle)$ satisfies supp $u \subset X^\triangle \setminus B(0,\varepsilon)$.

Proof. For non-negative integer s, the proof is similar to that given in Example 1.1.5. All of we need in addition is Lemma 1.1.2.

If we proof the statement for negative integers s, the lemma follows by interpolation properties.

Let $f \in H^s(X^\triangle)$ vanish for $t < \varepsilon$. Pick a function $\chi \in C^\infty_{loc}(\mathbb{R})$ with $\chi(t) = 0$ for $t < \frac{1}{3}\varepsilon$ and $\chi(t) = 1$ for $t > \frac{2}{3}\varepsilon$. Then $f = \chi(|z|)f$ in $X^\triangle$.

Given any function $u \in H^{-s}_{comp}(X^\triangle)$, the product $\chi(|z|)u$ is supported away from the ball $B(0, \frac{\varepsilon}{3})$ in $X^\triangle$. From what has already been proved, it follows that there are constants c_1, $c_2 > 0$ independent of u, such that

$$c_1 \, \|\chi \, \pi^* u\|_{\mathcal{H}^s(X^\square)} \le \|\chi(|z|)u\|_{H^s(X^\triangle)} \le c_2 \, \|\chi \, \pi^* u\|_{\mathcal{H}^s(X^\square)}.$$

By definition,

$$\begin{aligned} \|f\|_{H^s(X^\triangle)} &= \sup_{u \in H^{-s}_{comp}(X^\triangle)} \frac{|\langle f, u\rangle|}{\|u\|_{H^{-s}(X^\triangle)}} \\ &= \sup_{u \in H^{-s}_{comp}(X^\triangle)} \frac{|\langle f, \chi(|z|)u\rangle|}{\|u\|_{H^{-s}(X^\triangle)}}. \end{aligned}$$

Since $\|\chi(|z|)u\|_{H^{-s}(X^\triangle)} \le c' \, \|u\|_{H^{-s}(X^\triangle)}$ with some constant c' depending only on χ, we conclude that

$$\begin{aligned} \|f\|_{H^s(X^\triangle)} &\le c' \sup_{u \in H^{-s}_{comp}(X^\triangle)} \frac{|\langle f, \chi(|z|)u\rangle|}{\|\chi(|z|)u\|_{H^{-s}(X^\triangle)}} \\ &\le \frac{c'}{c_1} \sup_{u \in \mathcal{H}^{-s}_{comp}(X^\triangle)} \frac{|\langle \pi^* f, \chi \, \pi^* u\rangle|}{\|\chi \, \pi^* u\|_{\mathcal{H}^{-s}(X^\square)}} \\ &\le \frac{c'}{c_1} \, \|\pi^* f\|_{\mathcal{H}^s(X^\square)}, \end{aligned}$$

$\pi^* f$ being the pull-back of f in $int\, X^\square$ under the polar coordinates.

Interchanging $H^s(X^\triangle)$ and $\mathcal{H}^s(X^\square)$, we can see in the same manner that

$$\|\pi^* f\|_{\mathcal{H}^s(X^\square)} \le c'' c_2 \, \|f\|_{H^s(X^\triangle)},$$

where c'' depends only on χ. This is the desired conclusion.

□

1.1.5 *Mellin transform*

We leave it to the reader to verify to what extent the norm $\|\pi^* u\|_{\mathcal{H}^s(X^\square)}$ controls the behavior of Sobolev functions $u \in H^s(X^\triangle)$ near the vertex $t = 0$. Our next goal is to present a general technique for testing the behavior of functions on $X^\square$ close to $t = 0$.

On the real axis $\mathbb{R}$, the local regularity of a function $u(t)$ near $t = 0$ can be characterized by the order of growth of the Fourier transformation of ωu at infinity, where $\omega(t)$ is a cut-off function near zero. However, the Fourier transform is no longer applicable if u is given on the half-axis $\mathbb{R}_+$ only.

Nevertheless, we could pass to the new coordinate $t = e^{-r}$ to obtain a function $u(e^{-r})$ that is already defined on the whole line $\mathbb{R}$. Our task consists then of testing the regularity of $u(e^{-r})$ at infinity $r = +\infty$. To this end, we may again invoke the Fourier transform $\mathcal{F}_{r\mapsto\rho}$ on $\mathbb{R}$.

The change of variables $e^{-r} \mapsto t$ implies

$$\begin{aligned}\mathcal{F}_{r\mapsto\rho}(u(e^{-r})) &= \int_{-\infty}^{\infty} e^{-\sqrt{-1}\rho r} u(e^{-r})dr \\ &= \int_0^{\infty} t^{\sqrt{-1}\rho} u(t)\frac{dt}{t}.\end{aligned}$$

In this way we obtain what is known as the *Mellin transform*. Namely,

$$\mathcal{M}_{t\mapsto z}(u(t)) = \int_0^{\infty} t^z u(t)\frac{dt}{t}, \qquad z \in \mathbb{C},$$

where we first assume $u \in C^{\infty}_{comp}(\mathbb{R}_+)$. Obviously, $\mathcal{M}u$ is an entire function in the complex plane.

Fortunately, the Mellin transform is related to Fuchs-type operators in the same manner as the Fourier transform is to the usual differentiation operator.

Lemma 1.1.8 *For any $u \in C^{\infty}_{comp}(\mathbb{R}_+)$, it follows that*

$$\mathcal{M}\,(-t\partial_t)u = z\,\mathcal{M}u.$$

Proof. Indeed, integrating by parts yields

$$\begin{aligned}\mathcal{M}_{t\mapsto z}((-t\partial_t)u) &= \int_0^{\infty} t^z\,(-t\partial_t)u\,\frac{dt}{t} \\ &= -\,t^z u\,|_0^{\infty} + \int_0^{\infty} (\partial_t t^z)\,u\,dt \\ &= z\,\mathcal{M}_{t\mapsto z}(u),\end{aligned}$$

as desired.

□

The second basic property is that multiplication of u by the weight function t^{γ} is interpreted under the Mellin transform as the displacement of the reference line by γ.

Lemma 1.1.9 *For any $\gamma \in \mathbb{R}$, we have*

$$\mathcal{M}(t^{\gamma}u)\,(\sqrt{-1}\rho) = \mathcal{M}(u)\,(\gamma + \sqrt{-1}\rho), \qquad \rho \in \mathbb{R}.$$

Proof. Indeed,

$$\mathcal{M}(t^\gamma u)(z) = \int_0^\infty t^z (t^\gamma u) \frac{dt}{t} = \mathcal{M}(u)(z+\gamma),$$

as desired.

□

Given a $\gamma \in \mathbb{R}$, set $\Gamma_\gamma = \{z \in \mathbb{C} : \operatorname{Re} z = \gamma\}$ and consider the "weighted Mellin transform" $\mathcal{M}_\gamma u = \mathcal{M}u\,|_{\Gamma_{\frac{1}{2}-\gamma}}$.

Lemma 1.1.10 *For any $\gamma \in \mathbb{R}$, the $\mathcal{M}_\gamma$ extends by continuity to a unitary isomorphism*

$$\mathcal{M}_\gamma : \ L^2(\mathbb{R}_+, t^{-2\gamma}dt) \xrightarrow{\cong} L^2(\Gamma_{\frac{1}{2}-\gamma}). \qquad (1.1.2)$$

Note that by the *unitary property* of $\mathcal{M}_\gamma$ is meant that

$$(\mathcal{M}_\gamma u, \mathcal{M}_\gamma v)_{L^2(\Gamma_{\frac{1}{2}-\gamma})} = 2\pi\,(u,v)_{L^2(\mathbb{R}_+, t^{-2\gamma}dt)}$$

for all u, $v \in L^2(\mathbb{R}_+, t^{-2\gamma}dt)$.

Proof. From $u \in C^\infty_{comp}(\mathbb{R}_+)$ it follows that

$$\sup_{z \in \Gamma_{\frac{1}{2}-\gamma}} (1+|z|)^\nu\,|\mathcal{M}_\gamma u\,(z)| < \infty$$

for all $\nu \in \mathbb{Z}_+$. Hence (1.1.2) is an easy consequence of the relation between $\mathcal{M}$ and the Fourier transform on $\mathbb{R}_+$ via the substitution $t \mapsto e^{-r}$, and of Parseval's formula.

□

The following is actually an equivalent formulation of the *Fourier inversion formula.*

Lemma 1.1.11 *The inverse of (1.1.2) is given by the formula*

$$\mathcal{M}_\gamma^{-1} f\,(t) = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{\frac{1}{2}-\gamma}} t^{-z} f(z)\,dz.$$

Proof. Indeed, given any $u \in L^2(\mathbb{R}_+, t^{-2\gamma}dt)$, we have

$$\begin{aligned} \mathcal{M}_\gamma u\,(\rho) &= \int_0^\infty t^{(\frac{1}{2}-\gamma)+\sqrt{-1}\rho}\, u(t)\,\frac{dt}{t} \\ &= \mathcal{F}_{r\mapsto\rho}\left(e^{-(\frac{1}{2}-\gamma)r} u(e^{-r})\right), \end{aligned}$$

whence by Fourier's inversion formula

$$\begin{aligned} &\mathcal{M}_\gamma^{-1}(\mathcal{M}_\gamma u)(t) \\ &= t^{-(\frac{1}{2}-\gamma)} \frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^\infty e^{\sqrt{-1}(-\ln t)\rho}\, \mathcal{F}_{r\mapsto\rho}\left(e^{-(\frac{1}{2}-\gamma)r} u(e^{-r})\right) d\rho \\ &= u. \end{aligned}$$

This proves the lemma.

□

Lemmas 1.1.8 and 1.1.10 imply that $\mathcal{M}(-t\partial_t) = z\mathcal{M}$, defined, for instance, on all $u \in L^2(\mathbb{R}_+)$ with $t\partial_t u \in L^2(\mathbb{R}_+)$. This leads to the notion of *Mellin pseudodifferential operators*

$$\mathrm{op}_{\mathcal{M}}(p)u\,(t) = \mathcal{M}^{-1}_{z\mapsto t}p(t,z)\mathcal{M}_{t\mapsto z}u(t)$$

(see Schulze [15, 16]).

1.1.6 *Sobolev spaces based on the Mellin transform*

We want to define function spaces on $X^\square$ based on the Mellin transform along $\mathbb{R}_+$ and the Fourier transform locally along X.

Definition 1.1.12 *Let $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$. Denote by $\mathcal{H}^{s,\gamma}(X^\square)$ the set of all distributions u on int $X^\square$ whose derivatives up to order s are locally integrable with respect to the measure $t^n dtdx$ and satisfy*

$$\|u\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} := \int_{X^\square} \frac{1}{t^{2\gamma}} \left(\sum_{\alpha_0+|\alpha|\le s} |(tD_t)^{\alpha_0} D_x^\alpha u|^2 \right) t^n dtdx < \infty.$$

For integer $s < 0$, we define the space $\mathcal{H}^{s,\gamma}(X^\square)$ by duality as $\mathcal{H}^{s,\gamma}(X^\square) = \mathcal{H}^{-s,-\gamma}_{comp}(X^\square)'$ where the subscript "*comp*" refers to functions of compact support in the cylinder. Further, for fractional s, the space $\mathcal{H}^{s,\gamma}(X^\square)$ is defined by (complex) interpolation.

Lemma 1.1.13 *For any $u \in \mathcal{H}^{s,\gamma}(X^\square)$ whose support projects onto a compact subset of X, we have*

$$\|u\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} = \frac{1}{\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} \int_{\mathbb{R}^n} (1+|z|^2+|\xi|^2)^s \, |\mathcal{M}_{t\mapsto z}\mathcal{F}_{x\mapsto\xi}\tilde{u}|^2 \, dzd\xi$$

up to a factor in the range $\left[\frac{1}{s!}\frac{1}{(2\pi)^{1+n}}, \frac{1}{(2\pi)^{1+n}}\right]$.

Proof. A familiar argument shows that it is sufficient to prove the statement for integer $s \ge 0$ only.

If $s \in \mathbb{Z}_+$, we transform the norm $\|\cdot\|^2_{\mathcal{H}^{s,\gamma}(X^\square)}$ by using the Mellin transform. Namely, Lemmas 1.1.10 and 1.1.8 imply

$$\begin{aligned}
\|u\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} &= \sum_{\alpha_0+|\alpha|\le s} \int_{\mathbb{R}^n} dx \int_0^\infty |(tD_t)^{\alpha_0} D_x^\alpha u|^2 \, t^{-2(\gamma-\frac{n}{2})} dt \\
&= \sum_{\alpha_0+|\alpha|\le s} \int_{\mathbb{R}^n} dx \, \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} |\mathcal{M}_{t\mapsto z}(tD_t)^{\alpha_0} D_x^\alpha u|^2 \, dz \\
&= \sum_{\alpha_0+|\alpha|\le s} \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} dz \int_{\mathbb{R}^n} |z^{\alpha_0}\, \mathcal{M}_{t\mapsto z} D_x^\alpha u|^2 \, dx.
\end{aligned}$$

Thus, applying Parseval's formula yields

$$\|u\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} = \frac{1}{(2\pi)^{1+n}} \frac{1}{\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} \int_{\mathbb{R}^n} \left(\sum_{\alpha_0+|\alpha|\le s} |z^{\alpha_0}\xi^\alpha|^2 \right) |\mathcal{M}_{t\mapsto z}\mathcal{F}_{x\mapsto\xi}\tilde{u}|^2 \, dz d\xi.$$

Since

$$\frac{1}{s!}(1+|z|^2+|\xi|^2)^s \le \sum_{\alpha_0+|\alpha|\le s} |z^{\alpha_0}\xi^\alpha|^2 \le (1+|z|^2+|\xi|^2)^s,$$

we conclude that

$$\begin{aligned} &\frac{1}{s!} \frac{1}{(2\pi)^{1+n}} \frac{1}{\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} \int_{\mathbb{R}^n} (1+|z|^2+|\xi|^2)^s |\mathcal{M}_{t\mapsto z}\mathcal{F}_{x\mapsto\xi}u|^2 \, dz d\xi \\ &\le \|u\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} \\ &\le \frac{1}{(2\pi)^{1+n}} \frac{1}{\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} \int_{\mathbb{R}^n} (1+|z|^2+|\xi|^2)^s |\mathcal{M}_{t\mapsto z}\mathcal{F}_{x\mapsto\xi}u|^2 \, dz d\xi. \end{aligned}$$

This is precisely the assertion of the lemma.

□

1.1.7 *The space* $\mathcal{K}^{s,\gamma}(X^\square)$

The analysis on the cone employs the Mellin transform and weights only near the vertex that corresponds to a neighborhood of $t = 0$. The weight factor $t^{-2\gamma}$ in Definition 1.1.12 affects the space also for $t \to \infty$. It will be advantageous to introduce another variant of spaces on $X^\square$ that refers to the Mellin transform and to weight factors only near $t = 0$.

The idea is to multiply $\mathcal{H}^{s,\gamma}(X^\square)$ by a cut-off function ω and then to add $(1-\omega)\,\mathcal{H}^s(X^\square)$.

Here, by a *cut-off function* is meant any $\omega \in C^\infty(\overline{\mathbb{R}_+})$ with $\omega(t) = 1$ close to $t = 0$ and $\omega(t) = 0$ away from a neighborhood of $t = 0$.

Lemma 1.1.14 *Let* $\omega \in C^\infty_{comp}(\overline{\mathbb{R}_+})$. *Then the multiplication operator* $u \mapsto \omega u$ *is a continuous mapping of* $\mathcal{H}^{s,\gamma}(X^\square) \to \mathcal{H}^{s,\gamma}(X^\square)$.

Proof. The proof is straightforward.

□

If ω is a cut-off function, then the difference $\chi = 1 - \omega$ belongs to $C^\infty(\overline{\mathbb{R}_+})$, vanishes near $t = 0$ and is equal to 1 away from a neighborhood of $t = 0$. In this way we obtain what will be referred to as the *excision function.*

Lemma 1.1.15 *Let* $\chi \in C^\infty(\overline{\mathbb{R}_+})$ *vanish near* $t = 0$, *and let the derivatives of* χ *be bounded at infinity. Then the multiplication operator* $u \mapsto \chi u$ *is a continuous mapping of* $\mathcal{H}^s(X^\square) \to \mathcal{H}^s(X^\square)$.

Proof. The proof is straightforward.

□

Having disposed of this preliminary step, we can now define our main Sobolev space $\mathcal{K}^{s,\gamma}(X^{\square})$.

Definition 1.1.16 *For $s, \gamma \in \mathbb{R}$ and $\omega \in C^{\infty}_{comp}(\overline{\mathbb{R}_+})$ a cut-off function, let*

$$\mathcal{K}^{s,\gamma}(X^{\square}) = \omega\, \mathcal{H}^{s,\gamma}(X^{\square}) + (1-\omega)\, \mathcal{H}^{s}(X^{\square}).$$

We topologize $\mathcal{K}^{s,\gamma}(X^{\square})$ by the norm

$$\|u\|_{\mathcal{K}^{s,\gamma}(X^{\square})} = \inf_{u=\omega u_1+(1-\omega)u_2} \left(\|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^{\square})} + \|(1-\omega)u_2\|_{\mathcal{H}^{s}(X^{\square})} \right).$$

Lemma 1.1.17 *The space $\mathcal{K}^{s,\gamma}(X^{\square})$ is independent of the particular choice of the cut-off function ω up to an equivalent norm.*

Proof. By the above, it is sufficient to prove the statement for integer $s \geq 0$ only.

To this end, let ω_1 and ω_2 be two cut-off functions. Let $\omega_i(t) = 1$ for $t \leq a_i$ and $\omega_i(t) = 0$ for $t \geq A_i$, with $0 < a_i < A_i < \infty$.

We assume that $u = \omega u_1 + (1-\omega)u_2$ for some functions $u_1 \in \mathcal{H}^{s,\gamma}(X^{\square})$ and $u_2 \in \mathcal{H}^{s}(X^{\square})$. Fix a cut-off function ω which is equal to 1 in a neighborhood of $[0, A_2]$, and an excision function χ which is equal to 1 in a neighborhood of $[a_2, \infty)$. Then

$$\begin{aligned} u &= \omega_2 u + (1-\omega_2)u \\ &= \omega_2(\omega u) + (1-\omega_2)(\chi u). \end{aligned}$$

We have

$$\|\omega u\|_{\mathcal{H}^{s,\gamma}(X^{\square})} \leq \|\omega\, \omega_1 u_1\|_{\mathcal{H}^{s,\gamma}(X^{\square})} + \|\omega\,(1-\omega_1)u_2\|_{\mathcal{H}^{s,\gamma}(X^{\square})}.$$

By Lemma 1.1.14,

$$\begin{aligned} \|\omega\, \omega_1 u_1\|_{\mathcal{H}^{s,\gamma}(X^{\square})} &\leq c_1' \ \|\omega_1 u_1\|_{\mathcal{H}^{s,\gamma}(X^{\square})}, \\ \|\omega\,(1-\omega_1)u_2\|_{\mathcal{H}^{s,\gamma}(X^{\square})} &\leq c_2' \ \|(1-\omega_1)u_2\|_{\mathcal{H}^{s}(X^{\square})}, \end{aligned}$$

where c_1' depends only on ω, while c_2' depends on ω and a_1 (but not on u).

On the other hand,

$$\|\chi u\|_{\mathcal{H}^{s}(X^{\square})} \leq \|\chi\, \omega_1 u_1\|_{\mathcal{H}^{s}(X^{\square})} + \|\chi\,(1-\omega_1)u_2\|_{\mathcal{H}^{s}(X^{\square})},$$

and Lemma 1.1.15 shows that

$$\begin{aligned} \|\chi\, \omega_1 u_1\|_{\mathcal{H}^{s}(X^{\square})} &\leq c_1'' \ \|\omega_1 u_1\|_{\mathcal{H}^{s,\gamma}(X^{\square})}, \\ \|\chi\,(1-\omega_1)u_2\|_{\mathcal{H}^{s}(X^{\square})} &\leq c_2'' \ \|(1-\omega_1)u_2\|_{\mathcal{H}^{s}(X^{\square})}, \end{aligned}$$

where c_1'' depends on χ and A_1, while c_2'' depends only on χ.

We have thus proved that $\omega u \in \mathcal{H}^{s,\gamma}(X^\square)$ and $\chi u \in \mathcal{H}^s(X^\square)$. Moreover, we get

$$\begin{aligned}
&\inf_{u=\omega_2 v_1+(1-\omega_2)v_2} \left(\|\omega_2 v_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega_2)v_2\|_{\mathcal{H}^s(X^\square)}\right) \\
&\leq \|\omega_2(\omega u)\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega_2)(\chi u)\|_{\mathcal{H}^s(X^\square)} \\
&\leq const(\omega_2)\left((c_1'+c_1'')\|\omega_1 u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + (c_2'+c_2'')\|(1-\omega_2)u_2\|_{\mathcal{H}^s(X^\square)}\right),
\end{aligned}$$

the constant c being independent on u.

Interchanging ω_1 and ω_2 we obtain a reverse estimate, and the proof is complete. □

It is worth pointing out that the topology of $\mathcal{K}^{s,\gamma}(X^\square)$ is again generated by the *Hilbert inner product*

$$(u,v)_{\mathcal{K}^{s,\gamma}(X^\square)} = (\omega u, \omega v)_{\mathcal{H}^{s,\gamma}(X^\square)} + ((1-\omega)u,(1-\omega)v)_{\mathcal{H}^s(X^\square)}.$$

1.1.8 *Weight shift operators*

By a *weight shift operator* of order $\rho \in \mathbb{R}$, is meant any positive function on $\mathbb{R}_+$ of the form $s_\rho(t) = t^\rho\omega(t) + \chi(t)$, where ω is a cut-off function, and χ is an excision function.

In this subsection we show that the weight shift operators form an algebra, i.e., they are kept under multiplication and division.

Lemma 1.1.18 *If s_{ρ_i} is a weight shift operator of order ρ_i $(i=1,2)$, then $s_{\rho_1}s_{\rho_2}$ is a weight shift operator of order $\rho_1+\rho_2$.*

Proof. For $i=1,2$, let $s_{\rho_i} = t^{\rho_i}\omega_i + \chi_i$. Then $s_{\rho_1}s_{\rho_2} = t^{\rho_1+\rho_2}\omega + \chi$, where

$$\begin{aligned}
\omega &= \omega_1\omega_2, \\
\chi &= t^{\rho_1}\omega_1\chi_2 + t^{\rho_2}\omega_2\chi_1 + \chi_1\chi_2.
\end{aligned}$$

It is evident that ω is a cut-off function, and χ is an excision function. This completes the proof. □

Lemma 1.1.19 *If s_ρ is a weight shift operator of order $\rho \in \mathbb{R}$, then $\frac{1}{s_\rho}$ is a weight shift operator of order $-\rho$.*

Proof. By definition, there are $a, A > 0$ with $a < A$, such that $s_\rho(t) = t^\rho$ for $t \in [0,a]$, and $s_\rho(t) = 1$ for $t \in [A,\infty)$. Pick a cut-off function ω with a support on $[0,A)$, such that $\omega(t) = 1$ for $t \leq a$. Let χ be defined by

$$\frac{1}{s_\rho(t)} = t^{-\rho}\omega(t) + \chi(t), \quad t \in \mathbb{R}_+.$$

We check at once that χ is infinitely differentiable on $\mathbb{R}_+$, vanishes on the interval $(0,a]$, and is equal to 1 on the interval $[A,\infty)$. In other words, χ is an excision function. This completes the proof.

□

1.1.9 *Properties of* $\mathcal{K}^{s,\gamma}(X^\square)$

Here are some elementary properties of the space $\mathcal{K}^{s,\gamma}(X^\square)$. They go back to the books of Schulze [15, 16].

Proposition 1.1.20 *Let $s,\gamma \in \mathbb{R}$. For any $\varepsilon > 0$, there are positive constants c_1 and c_2 depending on ε, such that*

$$c_1 \|u\|_{\mathcal{H}^s(X^\square)} \le \|u\|_{\mathcal{K}^{s,\gamma}(X^\square)} \le c_2 \|u\|_{\mathcal{H}^s(X^\square)}$$

whenever $u \in \mathcal{H}^s(X^\square)$ is supported away from $[0,\varepsilon) \times X$.

Proof. Pick a cut-off function ω with a support on $[0,\varepsilon)$. Then

$$u = \omega u + (1-\omega)u = (1-\omega)u \qquad \text{on } X^\square.$$

To complete the proof, it suffices to use Lemma 1.1.17.

□

Our next result justifies the term "*weight shift operator*" to some degree.

Proposition 1.1.21 *Let s_ρ be a weight shift operator of order $\rho \in \mathbb{R}$. Given any $s,\gamma \in \mathbb{R}$, the operator s_ρ acts continuously in the following mapping:*

$$s_\rho : \mathcal{K}^{s,\gamma}(X^\square) \xrightarrow{\cong} \mathcal{K}^{s,\gamma+\rho}(X^\square). \tag{1.1.3}$$

Proof. By interpolation and duality arguments, we need only consider the case when $s \in \mathbb{Z}_+$. For such s's, however, the proof of the continuity of the mapping (1.1.3) is straightforward by Lemma 1.1.17. The only point remaining concerns the invertibility of this mapping, which is a consequence of Lemma 1.1.19 and what has already been proved.

□

The following result shows that the norm $\|\cdot\|_{\mathcal{H}^{s,\gamma}(X^\square)}$ controls the behavior of functions $u \in \mathcal{K}^{s,\gamma}(X^\square)$ near $t=0$ only to a limited extent.

Proposition 1.1.22 *For any $u \in \mathcal{K}^{s,\gamma}(X^\square)$ whose support projects onto a compact subset of X, there is a sequence $\{u_\nu\}$ in $C^\infty_{comp}(\text{int}\, X^\square)$ such that*

$$\|u - u_\nu\|_{\mathcal{K}^{s,\gamma}(X^\square)} \to 0 \qquad \text{as } \nu \to \infty.$$

Proof. Pick a cut-off function ω which is equal to 1 on the interval $[0,1)$. Write $u = u' + u''$, where $u' = \omega u$ and $u'' = (1-\omega)u$.

We first approximate the function u'' in $\mathcal{K}^{s,\gamma}(X^{\square})$ by elements of $C^{\infty}_{comp}(int\, X^{\square})$. Since u'' is supported on the set $[1,\infty)\times K$, where K is a compact subset of X, we can invoke the fact that C^{∞} functions of compact support are dense in the usual Sobolev space $H^{s}(\mathbb{R}^{1+n})$. Using Lemma 1.1.7 we find a sequence $\{u''_{\nu}\}$ in $C^{\infty}_{comp}(int\, X^{\square})$ such that each u''_{ν} vanishes for $t \le \frac{1}{2}$ and $\|u'' - u''_{\nu}\|_{\mathcal{H}^{s}(X^{\square})} \to 0$ as $\nu \to \infty$. Proposition 1.1.20 now shows that $\{u''_{\nu}\}$ converges to u'' in the norm of $\mathcal{K}^{s,\gamma}(X^{\square})$, as desired.

It is a more delicate problem to approximate the function u' in the norm of $\mathcal{K}^{s,\gamma}(X^{\square})$ by C^{∞} functions of compact support. Proposition 1.1.21 enables us to assume without loss of generality that $\gamma = 0$.

By Lemma 1.1.13,

$$\|u'\|^2_{\mathcal{H}^{s,0}(X^{\square})} \sim \int_{\mathbb{R}^1_{\rho}}\int_{\mathbb{R}^n_{\xi}} (1+\rho^2+|\xi|^2)^s \left|\mathcal{F}_{(r,x)\mapsto(\rho,\xi)}\left(e^{-\frac{1+n}{2}r}\, u'(e^{-r},x)\right)\right|^2 d\rho d\xi, \quad (1.1.4)$$

with the symbol $\sim$ denoting here the equivalence of norms. Hence it follows that

$$e^{-\frac{1+n}{2}r}\, u'(e^{-r},x) \in H^{s}(\mathbb{R}^1_r \times \mathbb{R}^n_x).$$

As $C^{\infty}_{comp}(\mathbb{R}^{1+n})$ is dense in $H^{s}(\mathbb{R}^{1+n})$, there is a sequence $\{v_{\nu}\}$ in $C^{\infty}_{comp}(\mathbb{R}\times X)$ converging to $e^{-\frac{1+n}{2}r}\, u'(e^{-r},x)$ in $H^{s}(\mathbb{R}^1_t \times \mathbb{R}^n_x)$. We invoke equality (1.1.4) again to see that the sequence

$$u'_{\nu}(t,x) = t^{-\frac{1+n}{2}}\, v_{\nu}(-\ln t, x), \quad \nu = 1,2,\ldots,$$

converges to u' in the norm of $\mathcal{H}^{s,0}(X^{\square})$ as $\nu \to \infty$.

As $t \mapsto -\ln t$ is a proper mapping of $\mathbb{R}_+ \to \mathbb{R}$, it follows that $u'_{\nu} \in C^{\infty}_{comp}(int\, X^{\square})$ for all ν. This completes the proof.

□

1.2 Invariance under Automorphisms of the Cone

What we want to treat in this section is whether the cone Sobolev spaces $\mathcal{K}^{s,\gamma}(X^{\square})$ are invariant under automorphisms of $X^{\triangle}$, which preserve the vertex. To this end, we first look carefully at the properties of such automorphisms.

1.2.1 *Properties of the automorphisms of the cone*

Let $X^{\triangle}$ and $Y^{\triangle}$ be two model cones in $\mathbb{R}^{1+n}$, with X and Y open subsets of the unite sphere.

By a *diffeomorphism* of these cones is meant any diffeomorphism $w = \delta(z)$ of a neighborhood of $X^{\triangle}$ to a neighborhood of $Y^{\triangle}$ in $\mathbb{R}^{1+n}$, such that $\delta(X^{\triangle}) = Y^{\triangle}$.

If $w = \delta(z)$ is a diffeomorphism of $X^\triangle \to Y^\triangle$, then δ acts smoothly near the vertex of $X^\triangle$. Moreover, a familiar topological argument shows that δ maps the vertex of $X^\triangle$ to that of $Y^\triangle$, i.e., $\delta(0) = 0$.

Composing δ with polar coordinates $z = t\,p(x)$ in $X^\triangle$ and $w = \tau\,q(\chi)$ in $Y^\triangle$, we obtain a diffeomorphism

$$\begin{cases} \tau = \tau(t,x), \\ \chi = \chi(t,x), \end{cases}$$

of $int\,X^\square \to int\,Y^\square$, where

$$\begin{aligned} \tau(t,x) &= |\delta(t\,p(x))|, \\ q(\chi(t,x)) &= \frac{\delta(t\,p(x))}{|\delta(t\,p(x))|}. \end{aligned}$$

Since the polar coordinates are "singular" at the origin, it may happen (?) that this diffeomorphism (denoted by $\tilde{\delta}$) does not extend to a diffeomorphism of neighborhoods of the stretched cones on $\mathbb{R} \times S^n$. However, it is clear from the construction that $\tilde{\delta}$ preserves the base of the cylinder, i.e., $\tau(0,x) = 0$ for all $p(x) \in X$.

By a *diffeomorphism* of the stretched cones, we shall now mean any diffeomorphism $\tilde{\delta}$ of a neighborhood of $X^\square$ to a neighborhood of $Y^\square$ in $\mathbb{R} \times S^n$, such that $\tilde{\delta}(X^\square) = Y^\square$.

Fix a diffeomorphism $\tilde{\delta} = (\tau, \chi)$ of $X^\square \to Y^\square$. By definition, $\tilde{\delta}$ acts smoothly in a neighborhood of the base of $X^\square$. Moreover, it is clear by topological reasons that $\tilde{\delta}$ maps the base of $X^\square$ to the base of $Y^\square$.

Lemma 1.2.1 *Given any $R > 0$ and compact set $K \subset X$, there exists a constant $c > 0$ such that*

$$c \le |J(t,x)| \le \frac{1}{c} \qquad \textit{for all} \ \ (t, p(x)) \in [0,R] \times K,$$

where $J = \det \frac{\partial(\tau,\chi)}{\partial(t,x)}$ is the Jacobian of $\tilde{\delta}$.

Proof. By condition, $\frac{\partial(\tau,\chi)}{\partial(t,x)}$ extends continuously to a non-singular matrix in a neighborhood of the compact set $[0,R] \times K$ on $\mathbb{R} \times S^n$. Hence the lemma follows by a familiar argument.

□

We now invoke the condition "$\tau(0,x) = 0$ *for all* $p(x) \in X$" in deriving the following main property of $\tilde{\delta}$.

Lemma 1.2.2 *Assume that K is a compact subset of X. There are an $R > 0$ and a C^∞ function F with bounded derivatives on $[0,R] \times K$ such that*

$$\tau(t,x) = t\,e^{F(t,x)} \qquad \textit{for all} \ \ (t, p(x)) \in [0,R] \times K.$$

Proof. Indeed, $\tau(0,x)=0$ implies

$$\begin{aligned}\tau(t,x) &= \int_0^1 \frac{\partial}{\partial\theta}(\tau(\theta t,x))\,d\theta \\ &= t\int_0^1 \frac{\partial\tau}{\partial t}(\theta t,x)\,d\theta.\end{aligned}$$

We are going to set

$$F(t,x)=\ln\int_0^1 \frac{\partial\tau}{\partial t}(\theta t,x)\,d\theta. \tag{1.2.1}$$

The only point remaining concerns the behavior of the derivative $\frac{\partial\tau}{\partial t}$ for $t>0$ small enough. The task is to show that this derivative is bounded from below by a positive constant.

As the Jacobian matrix

$$\frac{\partial(\tau,\chi)}{\partial(t,x)}=\begin{pmatrix}\frac{\partial\tau}{\partial t} & \frac{\partial\tau}{\partial x}\\ \frac{\partial\chi}{\partial t} & \frac{\partial\chi}{\partial x}\end{pmatrix}$$

is non-singular in a neighborhood of $X^\square$ on $\mathbb{R}\times S^n$ and $\frac{\partial\tau}{\partial x}(0,x)\equiv 0$, we deduce that $\frac{\partial\tau}{\partial t}\neq 0$ for all $p(x)$ in a neighborhood of K on S^n. Even $\frac{\partial\tau}{\partial x}(0,x)>0$, since the $\mathbb{R}_+$-direction is preserved. Hence it follows by the compactness of K that there are positive constants R and ε such that

$$\varepsilon\le\frac{\partial\tau}{\partial t}(t,x)\le\frac{1}{\varepsilon}\qquad\text{for all }\ (t,p(x))\in[0,R]\times K.$$

Thus, the function F given by (1.2.1) possesses all the desired properties, and the proof is complete.

□

1.2.2 *The invariance*

Every function $u\in\mathcal{K}^{s,\gamma}(Y^\square)$ defines a continuous linear functional on $C^\infty_{comp}(int\,Y^\square)$, and so is a distribution in the interior of $Y^\square$. For this reason, given any diffeomorphism $\tilde\delta: X^\square\to Y^\square$, we may define the *pull-back* $\tilde\delta^*u=u\circ\tilde\delta$ of u under $\tilde\delta$ in the usual way. The $\tilde\delta^*u$ is a distribution on $int\,X^\square$, and the question arises whether it still belongs to $\mathcal{K}^{s,\gamma}(X^\square)$.

The following simple result seems to be first proved in Schrohe [14].

Theorem 1.2.3 *Let $\tilde\delta$ be a diffeomorphism of $X^\square\to Y^\square$. Then $\tilde\delta^*u\in\mathcal{K}^{s,\gamma}(X^\square)$ for any $u\in\mathcal{K}^{s,\gamma}(Y^\square)$ vanishing away from a compact subset of $Y^\square$.*

Proof. In view of interpolation and duality we may assume that $s\in\mathbb{Z}_+$.

Given a function $u\in\mathcal{K}^{s,\gamma}(Y^\square)$ of compact support, we pick a compact set $K\subset X$ such that $\tilde\delta^{-1}(supp\,u)\subset\overline{\mathbb{R}_+}\times K$. Hence it follows that the pull-back $\tilde\delta^*u$ of u is compactly supported in $\overline{\mathbb{R}_+}\times K$.

Let $\tilde{\delta} = (\tau, \chi)$, and let $R > 0$ be the number of Lemma 1.2.2. There is an $A > 0$ such that from $(t, p(x)) \in \overline{\mathbb{R}_+} \times K$ and $\tau(t, p(x)) \in [0, A]$ it follows that $t \in [0, R]$.

Indeed, since $\tilde{\delta}$ is continuous on $X^{\square}$, there is an $A' > 0$ such that $\tau(t, p(x)) \in [0, A']$ for all $(t, p(x)) \in [0, R] \times K$. As $\tilde{\delta}^{-1}$ is also continuous, we deduce that there is an $R' \geq R$ such that from $(t, p(x)) \in \overline{\mathbb{R}_+} \times K$ and $\tau(t, p(x)) \in [0, A']$ it follows that $t \in [0, R']$. Denote by A'' the infimum of the function $\tau(t, p(x))$ on $[R, R'] \times K$. Clearly, $A'' > 0$, since otherwise we had $\tau(t, p(x)) = 0$ for some $t \neq 0$ and $p(x) \in K$. If $A'' > A'$, we take $A = A'$; otherwise we let A be any positive number less than A''. Now it is a simple matter to see that A possesses the desired property.

Pick a cut-off function ω which is equal to 1 for $\tau \leq a$ and to 0 for $\tau \geq A$, where $0 < a < A$. Write $u = u_1 + u_2$ with $u_1 = \omega u$ and $u_2 = (1 - \omega)u$.

The function $u_2 \in \mathcal{H}^s(Y^{\square})$ is supported away from $[0, a) \times Y$. Since the invariance of the usual Sobolev spaces is well-known, it follows from Lemma 1.1.7 that $\tilde{\delta}^* u_2 \in \mathcal{H}^s(X^{\square})$, provided that either u vanishes outside of a compact subset of $Y^{\square}$ or $\tilde{\delta}$ behaves well at the infinity of $X^{\square}$. If it is the case, then Lemma 1.1.17 shows that in fact $\tilde{\delta}^* u_2 \in \mathcal{K}^{s,\gamma}(X^{\square})$ because $\tilde{\delta}^* u_2$ vanishes for $t > 0$ small enough.

It remains to prove that $\tilde{\delta}^* u_1 \in \mathcal{K}^{s,\gamma}(X^{\square})$. For this purpose, we shall write u instead of u_1 and assume that u is supported on $[0, A] \times Y$.

The change of variables $(t, x) = \tilde{\delta}^{-1}(\tau, \chi)$ enables us to write

$$
\begin{aligned}
&\|u \circ \tilde{\delta}\|^2_{\mathcal{H}^{s,\gamma}(X^{\square})} \\
&= \int_{(\tilde{\delta}^{-1})_* Y^{\square}} \frac{1}{t^{2\gamma}} \left(\sum_{\alpha_0 + |\alpha| \leq s} \left| (tD_t)^{\alpha_0} D_x^{\alpha} \, (u \circ \tilde{\delta}) \right|^2 \right) t^n dt dx \\
&= \int_{[0,A] \times Y} \frac{1}{\tau^{2\gamma}} (\tilde{\delta}^{-1})^* \left(|J|^{-1} e^{(2\gamma - n)F} \sum_{\alpha_0 + |\alpha| \leq s} \left| (tD_t)^{\alpha_0} D_x^{\alpha} \, (u \circ \tilde{\delta}) \right|^2 \right) \tau^n d\tau d\chi,
\end{aligned}
$$

the last equality being a consequence of Lemma 1.2.2.

On the other hand, applying the chain rule and Lemma 1.2.2 yields

$$
\begin{aligned}
t \frac{\partial}{\partial t} (u \circ \tilde{\delta}) &= \left(1 + t \frac{\partial F}{\partial t} \right) \left(\tau \frac{\partial}{\partial \tau} u \right) \circ \tilde{\delta} + \sum_{j=1}^{n} t \frac{\partial \chi_j}{\partial t} \left(\frac{\partial u}{\partial \chi_j} \right) \circ \tilde{\delta}, \\
\frac{\partial}{\partial x_i} (u \circ \tilde{\delta}) &= \frac{\partial F}{\partial x_i} \left(\tau \frac{\partial}{\partial \tau} u \right) \circ \tilde{\delta} + \sum_{j=1}^{n} \frac{\partial \chi_j}{\partial x_i} \left(\frac{\partial u}{\partial \chi_j} \right) \circ \tilde{\delta},
\end{aligned}
$$

for $i = 1, \ldots, n$. Therefore,

$$
(tD_t)^{\alpha_0} D_x^{\alpha} \, (u \circ \tilde{\delta}) = \sum_{\beta_0 + |\beta| \leq \alpha_0 + |\alpha|} c^{\alpha_0, \alpha}_{\beta_0, \beta}(t, x) \left((\tau D_\tau)^{\beta_0} D_\chi^{\beta} u \right) \circ \tilde{\delta},
$$

the coefficients $c^{\alpha_0, \alpha}_{\beta_0, \beta}$ being bounded on $[0, R] \times K$.

Summarizing we see that

$$
\begin{aligned}
\|u \circ \tilde{\delta}\|^2_{\mathcal{H}^{s,\gamma}(X^{\square})} &\leq c \int_{[0,A] \times Y} \frac{1}{\tau^{2\gamma}} \left(\sum_{\beta_0 + |\beta| \leq s} \left| (\tau D_\tau)^{\beta_0} D_\chi^{\beta} u \right|^2 \right) \tau^n d\tau d\chi \\
&= c \, \|u\|^2_{\mathcal{H}^{s,\gamma}(Y^{\square})},
\end{aligned}
$$

where the constant c can be estimated via the *sup*-norms of $|J|^{-1}$ and the derivatives of $\tilde{\delta}$ up to order $s+1$ on $[0,R] \times K$.

We have thus proved that $\tilde{\delta}^* u \in \mathcal{H}^{s,\gamma}(X^{\square})$. Since this function is supported on $[0,R] \times K$, Lemma 1.1.17 gives $\tilde{\delta}^* u \in \mathcal{K}^{s,\gamma}(X^{\square})$. This is the desired conclusion.

□

1.3 Sobolev Spaces on a Manifold with Conical Singularities

Now let us pass to the weighted Sobolev spaces on arbitrary manifolds with conical singularities.

1.3.1 *The topological cone over a space*

Given a topological space T and a closed subset Σ of T, we denote by $\frac{T}{\Sigma}$ the quotient space of T over the equivalence relation corresponding to the decomposition of T into the set Σ and the singletons formed from the points of the complement $T \setminus \Sigma$.

Example 1.3.1 Let X be any topological space. Consider the topological product $T = [0,1) \times X$ and the closed subset $\Sigma = \{0\} \times X$. The quotient space

$$C_t(X) = \frac{[0,1) \times X}{\{0\} \times X}$$

is known as the *topological cone over the space* X.

□

1.3.2 *Manifolds with conical singularities*

An $(1+n)$-dimensional manifold is a topological (second countable) Hausdorff space M such that each point $p \in M$ has a neighborhood which is diffeomorphic to $\mathbb{R}^{1+n}$.

Definition 1.3.2 *By a manifold with conical singularities of dimension* $1+n$, *we mean a topological (second countable) Hausdorff space* K *with a finite subset* $V \subset K$ *("singularities") such that*

- $K \setminus V$ *is an* $(1+n)$*-dimensional manifold;*
- *for each point* $v \in V$ *there exist a neighborhood* $\mathcal{N}$ *of* v *in* K, *a compact manifold* X *of dimension* n *and a homeomorphism* $x : \mathcal{N} \to \frac{[0,1)\times X}{\{0\}\times X}$ *such that* $x(v) = \frac{\{0\}\times X}{\{0\}\times X}$ *and the restriction* $x : \mathcal{N} \setminus \{v\} \to (0,1) \times X$ *is a diffeomorphism.*

We may always think of X embedded to the unit sphere S^N for N large enough, even for $N = n$ if we consider local embeddings. If it is the case, the quotient $\frac{[0,1)\times X}{\{0\}\times X}$ is identified with the model cone $\{\lambda p : \lambda \in [0,1), \; p \in X\}$ in the obvious way.

Our next goal is to determine a "*cone structure*" on K close to each point $v \in V$.

Given any two diffeomorphisms

$$\phi_1 : \mathcal{N} \setminus \{v\} \to (0,1) \times X,$$
$$\phi_2 : \mathcal{N} \setminus \{v\} \to (0,1) \times X,$$

the composition $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism of $(0,1) \times X \to (0,1) \times X$. We say that ϕ_1 and ϕ_2 are *equivalent* if $\phi_2 \circ \phi_1^{-1}$ is the restriction of some diffeomorphism $(-1,1) \times X \to (-1,1) \times X$.

The equivalence class of such diffeomorphisms is regarded as a part of the structure of $K \setminus V$ in a neighborhood of v. It is kept fixed and determines the "*cone structure*" on K close to $v \in V$ via the local $\mathbb{R}_+$ action on $(t,x) \in (0,1) \times X$, i.e., $\lambda(t,x) = (\lambda t, x)$ for all $\lambda \in \mathbb{R}_+$ with $\lambda t \in (0,1)$.

Example 1.3.3 If X is a submanifold of S^N of dimension $n \le N$, then the geometric cone $X^\Delta = \{\lambda p : \lambda \ge 0, \; p \in X\}$ is a manifold with conical singularity $v = 0$.

□

1.3.3 *Stretched manifolds*

The analysis on a manifold with conical singularity has always referred to the corresponding "*stretched manifold.*"

Proposition 1.3.4 *For any manifold K with conical singularities V there is a smooth manifold with boundary $\mathbb{K}$ such that*

1) $K \setminus V$ is diffeomorphic to $\mathbb{K} \setminus \partial\mathbb{K}$; and

2) there is a neighborhood $\mathcal{N}$ of V in K and a collar neighborhood $\mathbb{N} \simeq \partial\mathbb{K} \times [0,1)$ of $\partial\mathbb{K}$ in $\mathbb{K}$ such that $\mathcal{N} \setminus V$ is diffeomorphic to $\partial\mathbb{K} \times (0,1)$.

Proof. We construct $\mathbb{K}$ by replacing, for every singularity v, the neighborhood $\mathcal{N}$ in Definition 1.3.2 by $(0,1) \times X$ via gluing with any one of the diffeomorphisms ϕ. We even get $\partial\mathbb{K} = \cup_{v \in V} X_v$, the subscript v pointing to the dependence of X on v.

□

This manifold $\mathbb{K}$ is called the "*stretched object*" associated with K. It is worth pointing out that the idea of invoking stretched objects is of general character for the analysis on manifolds with singularities.

1.3.4 *Definition of Sobolev spaces*

Assume that K is a compact manifold with conical singularities of dimension $1+n$, and $\mathbb{K}$ is the corresponding "stretched object."

The definition of weighted Sobolev spaces (or Mellin-Sobolev spaces) $\mathcal{K}^{s,\gamma}(\mathbb{K})$ is based, as usually, on the standard localization procedure.

By the definition of $\mathbb{K}$, each interior point of $\mathbb{K}$ has a neighborhood which is diffeomorphic to $\mathbb{R}^{1+n}$. And every point of $\partial\mathbb{K}$ possesses a neighborhood which is diffeomorphic to $\overline{\mathbb{R}_+} \times \mathbb{R}^n$.

Since $\mathbb{K}$ is compact, there exists a finite covering $\{\mathcal{N}_\nu\}$ of $\mathbb{K}$ by open subsets each of them lies in a coordinate patch on $\mathbb{K}$. It follows that, for $\mathcal{N}_\nu \cap \partial\mathbb{K} = \emptyset$, we have a diffeomorphism $\phi_\nu : \mathcal{N}_\nu \to \mathbb{R}^{1+n}$. If $\mathcal{N}_\nu \cap \partial\mathbb{K} \neq \emptyset$, then we have a diffeomorphism $\phi_\nu : \mathcal{N}_\nu \to \overline{\mathbb{R}_+} \times \mathbb{R}^n$.

Let $\{\varphi_\nu\}$ be a C^∞ partition of unity on $\mathbb{K}$ subordinated to the covering $\{\mathcal{N}_\nu\}$.

As expected, given any $s, \gamma \in \mathbb{R}$, the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ is defined to consist of all distributions u in the interior of $\mathbb{K}$ such that, for every ν, the product $(\varphi_\nu u) \circ \phi_\nu^{-1}$ belongs to either $H^s(\mathbb{R}^{1+n})$, if $\mathcal{N}_\nu \cap \partial\mathbb{K} = \emptyset$, or $\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n)$, if $\mathcal{N}_\nu \cap \partial\mathbb{K} \neq \emptyset$.

Proposition 1.3.5 *As defined above, the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ is independent of the particular choice of the covering $\{\mathcal{N}_\nu\}$, diffeomorphisms $\{\phi_\nu\}$, and the partition of unity $\{\varphi_\nu\}$.*

Proof. Let

$$T' = \{\mathcal{N}'_\mu,\ \phi'_\mu,\ \varphi'_\mu\},$$
$$T'' = \{\mathcal{N}''_\nu,\ \phi''_\nu,\ \varphi''_\nu\}$$

be two triples as above.

We introduce the temporary notation $\mathcal{K}^{s,\gamma}(\mathbb{K}, T')$ or $\mathcal{K}^{s,\gamma}(\mathbb{K}, T'')$ for the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ corresponding to either triple. We shall have established the proposition if we prove the following: if $u \in \mathcal{K}^{s,\gamma}(\mathbb{K}, T')$, then $u \in \mathcal{K}^{s,\gamma}(\mathbb{K}, T'')$.

To this end, pick a $u \in \mathcal{K}^{s,\gamma}(\mathbb{K}, T')$. Given any number ν, write

$$\begin{aligned}(\varphi''_\nu u) \circ (\phi''_\nu)^{-1} &= \left(\varphi''_\nu \left(\sum_\mu \varphi'_\mu\right) u\right) \circ (\phi''_\nu)^{-1} \\ &= \sum_\mu \left(\phi'_\mu \circ (\phi''_\nu)^{-1}\right)^* \left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}.\end{aligned}$$

The task is now to show that every summand on the right-hand side here is in $H^s(\mathbb{R}^{1+n})$, if $\mathcal{N}''_\nu \cap \partial\mathbb{K} = \emptyset$, or in $\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n)$, if $\mathcal{N}''_\nu \cap \partial\mathbb{K} \neq \emptyset$. We give the proof only for the latter case, i.e., when $\mathcal{N}''_\nu \cap \partial\mathbb{K} \neq \emptyset$. Similar considerations apply to the first case, and will only refer to the invariance of the usual Sobolev spaces.

We can certainly assume that $\mathcal{N}'_\mu \cap \mathcal{N}''_\nu \neq \emptyset$, since otherwise $\varphi''_\nu \varphi'_\mu \equiv 0$ and so the corresponding (zero!) summand is obviously in $\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n)$.

Suppose that $\mathcal{N}'_\mu \cap \partial\mathbb{K} = \emptyset$. By condition, $\left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}$ belongs to $H^s_{comp}(\mathbb{R}^{1+n})$. As $\phi'_\mu \circ (\phi''_\nu)^{-1}$ is a diffeomorphism of

$$\left(\overline{\mathbb{R}_+} \times \mathbb{R}^n\right) \cap \phi''_\nu \left(\mathcal{N}'_\mu \cap \mathcal{N}''_\nu\right) \to \mathbb{R}^{1+n},$$

it follows that the pull-back of $\left(\left(\varphi''_\nu\varphi'_\mu\right)u\right)\circ\left(\phi'_\mu\right)^{-1}$ under this diffeomorphism belongs to $H^s_{comp}(\mathbb{R}_+\times\mathbf{R}^n)$. Applying Lemmas 1.1.7 and 1.1.17 we can assert that

$$\left(\phi'_\mu\circ(\phi''_\nu)^{-1}\right)^*\left(\left(\varphi''_\nu\varphi'_\mu\right)u\right)\circ\left(\phi'_\mu\right)^{-1}\in\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbf{R}^n),$$

as desired.

It remains to consider the case when $\mathcal{N}'_\mu\cap\partial\mathbb{K}\neq\emptyset$. By condition, the function $\left(\left(\varphi''_\nu\varphi'_\mu\right)u\right)\circ\left(\phi'_\mu\right)^{-1}$ belongs to $\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n)$. Moreover, it vanishes away from a compact subset of $\overline{\mathbb{R}_+}\times\mathbb{R}^n$. Since $\phi'_\mu\circ(\phi''_\nu)^{-1}$ is a diffeomorphism of

$$\left(\overline{\mathbb{R}_+}\times\mathbf{R}^n\right)\cap\phi''_\nu\left(\mathcal{N}'_\mu\cap\mathcal{N}''_\nu\right)\to\overline{\mathbb{R}_+}\times\mathbb{R}^n,$$

it follows from Theorem 1.2.3 that the pull-back of $\left(\left(\varphi''_\nu\varphi'_\mu\right)u\right)\circ\left(\phi'_\mu\right)^{-1}$ under this diffeomorphism belongs to $\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n)$. This completes the proof.

□

Another way of stating Proposition 1.3.5 is to say that these weighted Sobolev spaces make sense on a manifold with conical singularities.

The space $\mathcal{K}^{s,\gamma}(\mathbb{K})$, when endowed with the norm

$$\|u\|_{\mathcal{K}^{s,\gamma}(\mathbf{K})} = \left(\sum_{\nu:\,\mathcal{N}_\nu\cap\partial\mathbf{K}\neq\emptyset}\|(\varphi_\nu u)\circ\phi_\nu^{-1}\|^2_{\mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n)}+\sum_{\nu:\,\mathcal{N}_\nu\cap\partial\mathbf{K}=\emptyset}\|(\varphi_\nu u)\circ\phi_\nu^{-1}\|^2_{H^s(\mathbb{R}^{1+n})}\right)^{\frac{1}{2}},$$

is a Banach space and even a Hilbert space.

Moreover, analysis similar to that in the proof of Proposition 1.3.5 shows that the norm $\|\cdot\|_{\mathcal{K}^{s,\gamma}(\mathbf{K})}$ is independent, up to an equivalent norm, of the particular choice of the covering $\{\mathcal{N}_\nu\}$, diffeomorphisms $\{\phi_\nu\}$, and the partition of unity $\{\varphi_\nu\}$.

1.3.5 *Order reduction*

In order to give a global description of the norm in $\mathcal{K}^{s,\gamma}(\mathbb{K})$, we shall need $\mathcal{L}^m_{cl}(X;\Lambda)$, for $\Lambda=\mathbb{R}^l$ with $l\in\mathbb{Z}_+$, which is the space of all λ-dependent classical pseudodifferential operators on a compact manifold X.

A family $M(\lambda)$ of operators in $\mathcal{L}^m(X)$ is called *dependent on the parameter* $\lambda\in\Lambda$ if its symbol depends on (x,ξ,λ), where λ is treated as an additional covariable. We write $M(\lambda)\in\mathcal{L}^m(X;\Lambda)$. The family $M(\lambda)$ is said to be *classical*, and we write $M(\lambda)\in\mathcal{L}^m_{cl}(X;\Lambda)$, if its symbol is classical in (ξ,λ).

Generally speaking, a parameter-dependent theory of operators gives more information about the corresponding theory of operators not depending on a parameter. For example, it is an important tool in spectral theory.

The *principal symbol* of $M(\lambda)\in\mathcal{L}^m_{cl}(X;\Lambda)$ is defined to be the component of its symbol which is homogeneous in (ξ,λ) of order m (written $\sigma^m(M)$).

An operator $M(\lambda) \in \mathcal{L}^m_{cl}(X;\Lambda)$ is called *parameter-dependent elliptic* if

$$\sigma^m(M)(x,\xi,\lambda) \neq 0 \quad \text{on } (T^*(X) \times \Lambda) \setminus \{0\}.$$

If it is the case, then there exists a $\Pi(\lambda) \in \mathcal{L}^{-m}_{cl}(X;\Lambda)$ with both $1 - \Pi(\lambda)M(\lambda)$ and $1 - M(\lambda)\Pi(\lambda)$ in $\mathcal{L}^{-\infty}_{cl}(X;\Lambda)$. (Any such $\Pi(\lambda)$ is called a *parameter-dependent parametrix* of $M(\lambda)$.)

Let us denote by $\{H^s(X)\}_{s\in\mathbb{R}}$ the scale of classical Sobolev spaces on X. If $M(\lambda)$ is parameter-dependent elliptic, then there exists an $R > 0$ such that $M(\lambda) : H^s(X) \to H^{s-m}(X)$ is an isomorphism for all $s \in \mathbb{R}$ and all $\lambda \in \Lambda$ with $|\lambda| \geq R$ (cf. Shubin [20, 11.9.2]).

Lemma 1.3.6 *Given any $m \in \mathbb{R}$ and $l \in \mathbb{Z}_+$, there is a parameter-dependent elliptic operator $\Lambda^m \in \mathcal{L}^m_{cl}(X;\mathbb{R}^l)$ such that*

$$\Lambda^m(\lambda) : H^s(X) \to H^{s-m}(X)$$

is an isomorphism for all $s \in \mathbb{R}$ and $\lambda \in \mathbb{R}^l$.

Proof. See, for instance, Schulze [16].

□

Pseudodifferential operators of this kind will also be called *order reductions* (for the scale $\{H^s(X)\}_{s\in\mathbb{R}}$ of Sobolev spaces over X).

For many purposes the algebra $\mathcal{L}^0(X)$ of pseudodifferential operators of order 0 is more accessible. Order reductions "reduce" the analysis of $\mathcal{L}^m(X)$ to that of $\mathcal{L}^0(X)$. They induce isomorphisms $\Lambda^m\, \mathcal{L}^0(X) = \mathcal{L}^m(X)$ and $\mathcal{L}^0(X)\, \Lambda^m = \mathcal{L}^m(X)$, which preserve elliptic operators.

1.3.6 *Globalization of the norm*

Let K be a manifold with conical singularities, and let $\mathbb{K}$ be the corresponding stretched object. For simplicity, we assume that there is only one singularity with cross-section X.

In the sequel, Λ^s ($s \in \mathbb{R}$) stands for a fixed family of order reductions on X, which depends on a parameter $\lambda \in \mathbb{R}$. For every such Λ^s, we have the equivalence of norms

$$\|u\|_{H^s(X)} \sim \|\Lambda^s u\|_{L^2(X)}, \quad u \in H^s(X).$$

By Proposition 1.3.4, there is a collar neighborhood $\mathcal{N}$ of $\partial\mathbb{K}$ which is diffeomorphic to $[0,1) \times X$. Denote by $\phi : \mathcal{N} \to [0,1) \times X$ one of such diffeomorphisms.

Given a distribution u in the interior of $\mathbb{K}$, we say that u is supported close to the boundary of $\mathbb{K}$ if it vanishes away from a compact subset of $\mathcal{N}$.

If ω is a cut-off function with a support in $[0,1)$, then the pull-back $\phi^*\omega$ yields a C^∞ function on $\mathbb{K}$ which is supported close to the boundary of $\mathbb{K}$ and equal to 1 near $\partial\mathbb{K}$. We still call $\phi^*\omega$ the *cut-off* function with respect to $\partial\mathbb{K}$.

Fix a cut-off function $\omega \in C^\infty(\mathbb{K})$ with respect to the boundary of $\mathbb{K}$.

Proposition 1.3.7 *For any $s, \gamma \in \mathbb{R}$, we have the equivalence of norms*

$$\|u\|_{\mathcal{K}^{s,\gamma}(\mathbb{K})} \sim \left(\frac{1}{\sqrt{-1}} \int_{\Gamma_{\frac{1+n}{2}-\gamma}} \|\Lambda^s(Im\, z)\ \mathcal{M}(\omega u \circ \phi^{-1})(z)\|^2_{L^2(X)} dz + \|(1-\omega)u\|^2_{H^s(\mathbb{K})} \right)^{\frac{1}{2}}.$$

(Here $H^s(\mathbb{K})$ stands for the usual Sobolev space on the compact manifold with boundary $\mathbb{K}$. The norm in $H^s(\mathbb{K})$ is defined by using some finite coordinate covering of $\mathbb{K}$. It is independent of the covering up to an equivalent norm.)

Proof. This follows from Lemma 1.1.13 by the same method as in the proof of Lemma 1.1.17.

□

1.3.7 *Properties*

In this subsection we briefly sketch only those new properties of the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ which differ from the corresponding properties of $\mathcal{K}^{s,\gamma}(X^\square)$.

Proposition 1.3.8 *For any $s, \gamma \in \mathbb{R}$, the subspace $C^\infty_{comp}(int\, \mathbb{K})$ is dense in $\mathcal{K}^{s,\gamma}(\mathbb{K})$.*

Proof. Use the standard localization procedure and Proposition 1.1.22.

□

Thus, the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ could be also defined as the completion of $C^\infty_{comp}(int\, \mathbb{K})$ in the norm $\|\cdot\|_{\mathcal{K}^{s,\gamma}(\mathbb{K})}$.

Proposition 1.3.9 *Assume that $s, \gamma \in \mathbb{R}$. The $\mathcal{K}^{0,0}(\mathbb{K})$-scalar product (u, v), for $u, v \in C^\infty_{comp}(int\, \mathbb{K})$, extends to a non-degenerate sesqui-linear pairing*

$$\mathcal{K}^{s,\gamma}(\mathbb{K}) \times \mathcal{K}^{-s,-\gamma}(\mathbb{K}) \to \mathbb{C},$$

under which $\mathcal{K}^{-s,-\gamma}(\mathbb{K})$ is topologically isomorphic to the dual of $\mathcal{K}^{s,\gamma}(\mathbb{K})$.

Proof. It is sufficient to prove that

$$\mathcal{K}^{-s,-\gamma}(\mathbb{K}) \overset{top.}{\cong} \mathcal{K}^{s,\gamma}(\mathbb{K})'$$

for $s \in \mathbb{Z}_+$. The general case follows from here by interpolation and duality because the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ is reflexive.

That every function $v \in \mathcal{K}^{-s,-\gamma}(\mathbb{K})$ defines a continuous linear functional F_v on $\mathcal{K}^{s,\gamma}(\mathbb{K})$ by $F_v(u) = (u, v)$, is a direct consequence of the definition of the space $\mathcal{K}^{s,\gamma}(\mathbb{K})$ and the equality $\mathcal{K}^{-s,-\gamma}(X^\square) = \mathcal{K}^{s,\gamma}_{comp}(X^\square)'$ for the corresponding local spaces.

Conversely, let F be a continuous linear functional on $\mathcal{K}^{s,\gamma}(\mathbb{K})$. As $C^\infty_{comp}(int\,\mathbb{K})$ is dense in this space, there is a unique distribution $v \in \mathcal{D}'(int\,\mathbb{K})$ such that

$$F(u) = \langle \overline{v}, u \rangle \quad \text{for all } u \in C^\infty_{comp}(int\,\mathbb{K}), \tag{1.3.1}$$

$\overline{v}$ being the complex conjugate to v.

We claim that $v \in \mathcal{K}^{-s,-\gamma}(\mathbb{K})$. To see this, we have to verify, by Proposition 1.3.5, that the restrictions of v to coordinate patches in $\mathbb{K}$ belong the the corresponding local spaces $\mathcal{K}^{-s,-\gamma}(X^\square)$. But this is just the case, as follows from (1.3.1) and the equality $\mathcal{K}^{-s,-\gamma}(X^\square) = \mathcal{K}^{s,\gamma}_{comp}(X^\square)'$ having already been mentioned.

This finishes the proof, the detailed verification being left to the reader.

□

1.4 Group Action

Group actions are necessary auxiliary tools for defining weighted Sobolev spaces on manifolds with edges.

1.4.1 *Definition*

Let L be a Banach space, and $\mathcal{L}(L)$ the space of all continuous linear operators in L. Unless otherwise stated we assume that $\mathcal{L}(L)$ is equipped with the topology of uniform convergence on bounded subsets of L (i.e., with the norm $\|\cdot\|_{\mathcal{L}(L)}$).

By $\mathcal{L}_\sigma(L)$ we denote the space $\mathcal{L}(L)$ endowed with the topology of pointwise convergence (i.e., with the system of seminorms $\mathcal{L}(L) \ni M \mapsto \|Mu\|_L$, where u varies over L).

Definition 1.4.1 *By an $\mathbb{R}_+$-action on L, is meant any continuous mapping $\kappa : \mathbb{R}_+ \ni \lambda \mapsto \kappa_\lambda \in \mathcal{L}_\sigma(L)$ satisfying*

$$\begin{aligned} \kappa_\lambda \kappa_\mu &= \kappa_{\lambda\mu}, \\ \kappa_\lambda{}^{-1} &= \kappa_{\lambda^{-1}} \end{aligned} \tag{1.4.1}$$

for all $\lambda, \mu \in \mathbb{R}_+$.

This definition just amounts to saying that $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a commutative group of operators in $\mathcal{L}(E)$, and $\lim_{\lambda \to 1} \kappa_\lambda u = u$ for all $u \in L$. For this reason, we also write $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ instead of κ and call it a *group action* on L.

The simplest example of an $\mathbb{R}_+$-action on L is the so-called *trivial $\mathbb{R}_+$-action*, i.e., $\kappa_\lambda := id_L$ for all $\lambda \in \mathbb{R}_+$. Let us mention another example to be paradigmatic for the sequel.

Example 1.4.2 Let $L = L^2(\mathbb{R}_+)$. Then one can take $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}} u(\lambda t)$ for $\lambda \in \mathbb{R}_+$. Here κ_λ is unitary for every λ.

□

The following simple result is often of use.

Proposition 1.4.3 *For any group action $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ on L, there are constants $c \geq 0$, $R > 0$ such that*

$$\|\kappa_\lambda\|_{\mathcal{L}(L)} \leq \begin{cases} c\,\lambda^R & \text{for } \lambda \geq 1, \\ c\,\lambda^{-R} & \text{for } \lambda \leq 1. \end{cases}$$

Proof. See Hirschmann [9].

□

1.4.2 *Continuity*

In the previous sections we defined the spaces $\mathcal{H}^s(X^\square)$, $\mathcal{H}^{s,\gamma}(X^\square)$ and $\mathcal{K}^{s,\gamma}(X^\square)$ for the case when X was an open subset of the unit sphere S^n in $\mathbb{R}^{1+n}$.

Now let X be an arbitrary compact manifold of dimension n. Without loss of generality we can assume that X is a submanifold of S^N for N large enough.

As above, we denote by $X^\triangle$ the geometrical cone in $\mathbb{R}^{1+n}$ which consists of the points λp, with $\lambda \geq 0$ and $x \in X$.

There is a finite open covering of X by coordinate patches $\{X_\nu\}$ each of which is diffeomorphic to an open subset of S^n. Then the family $\{X_\nu^\triangle\}$ covers $X^\triangle$, and every $X_\nu^\triangle$ can be identified with a model cone in $\mathbb{R}^{1+n}$.

Using such coverings we can define, in the standard manner, the spaces $\mathcal{H}^s(X^\square)$, $\mathcal{H}^{s,\gamma}(X^\square)$ and $\mathcal{K}^{s,\gamma}(X^\square)$ on the cylinder $X^\square = \overline{\mathbb{R}_+} \times X$ for any compact manifold X, too.

In the sequel, (t, x) stands for the coordinates in the cylinder $X^\square$, where t is the global coordinate in $\overline{\mathbb{R}_+}$ and $x = (x_1, \ldots, x_n)$ are local coordinates in X.

Given any $s \in \mathbb{Z}_+$ and $\gamma \in \mathbb{R}$, we have

$$\begin{aligned} \|u\|^2_{\mathcal{H}^s(X^\square)} &= \sum_{\alpha_0 + j \leq s} \int_0^\infty \tfrac{1}{t^{2(\alpha_0 + j)}} \|(tD_t)^{\alpha_0} u\|^2_{H^j(X)}\, t^n dt, \\ \|u\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} &= \sum_{\alpha_0 + j \leq s} \int_0^\infty \tfrac{1}{t^{2\gamma}} \|(tD_t)^{\alpha_0} u\|^2_{H^j(X)}\, t^n dt \end{aligned} \tag{1.4.2}$$

up to equivalent norms.

In $\mathcal{K}^{s,\gamma}(X^\square)$ the norm is defined just in the same way as in Subsection 1.1.7.

If $n = 0$, then $\mathcal{H}^0(X^\square) = L^2(\mathbb{R}_+)$ and $\mathcal{H}^{0,\gamma}(X^\square) = L^2(\mathbb{R}_+, t^{-2\gamma} dt)$. We are going to generalize the $\mathbb{R}_+$-action of Example 1.4.2 to the spaces $\mathcal{K}^{s,\gamma}(X^\square)$.

To this end, we consider the $\mathbb{R}_+$-action κ on $\mathcal{K}^{s,\gamma}(X^\square)$ given by

$$\kappa_\lambda u\,(t, x) = \lambda^{\frac{1+n}{2}}\, u(\lambda t, x) \quad \text{for } u \in \mathcal{K}^{s,\gamma}(X^\square). \tag{1.4.3}$$

Lemma 1.4.4 *For any $s, \gamma \in \mathbb{R}$, it follows that*

$$\|\kappa_\lambda\|_{\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\square))} \le c \ \max(1, \lambda^s, \lambda^\gamma) \quad \textit{for all } \lambda > 0,$$

the constant c being independent of λ.

Proof. By interpolation and duality, it is sufficient to prove the lemma for $s \in \mathbf{Z}_+$.

Let ω be a fixed cut-off function, and let $\omega(t) = 1$ for $t \le a$ and $\omega(t) = 0$ for $t \ge A$.

Given a function $u \in \mathcal{K}^{s,\gamma}(X^\square)$, assume that $u = \omega u_1 + (1-\omega)u_2$ with some $u_1 \in \mathcal{H}^{s,\gamma}(X^\square)$ and $u_2 \in \mathcal{H}^s(X^\square)$. By definition,

$$\begin{aligned}\|\kappa_\lambda u\|_{\mathcal{K}^{s,\gamma}(X^\square)} &\le \|\omega\kappa_\lambda u\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)\kappa_\lambda u\|_{\mathcal{H}^s(X^\square)} \\ &\le \|\omega\kappa_\lambda(\omega u_1)\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|\omega\kappa_\lambda((1-\omega)u_2)\|_{\mathcal{H}^{s,\gamma}(X^\square)} \\ &\quad + \|(1-\omega)\kappa_\lambda(\omega u_1)\|_{\mathcal{H}^s(X^\square)} + \|(1-\omega)\kappa_\lambda((1-\omega)u_2)\|_{\mathcal{H}^s(X^\square)}.\end{aligned}$$

Having disposed of this preliminary step, we can now change the variables $\tau = \lambda t$ and observe that $tD_t = \tau D_\tau$. This leads to

$$\begin{aligned}&\|\omega\kappa_\lambda(\omega u_1)\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} \\ &= \sum_{\alpha_0+j\le s} \int_0^\infty \frac{1}{t^{2\gamma}} \left\|(tD_t)^{\alpha_0}\left(\omega(t)\,\lambda^{\frac{1+n}{2}}(\omega u_1)(\lambda t,\cdot)\right)\right\|^2_{H^j(X)} t^n dt \\ &= \lambda^{2\gamma} \sum_{\alpha_0+j\le s} \int_0^\infty \frac{1}{\tau^{2\gamma}} \left\|(\tau D_\tau)^{\alpha_0}\left(\omega(\frac{\rho}{\lambda})\,(\omega u_1)(\tau,\cdot)\right)\right\|^2_{H^j(X)} \tau^n d\tau \\ &\le (c')^2\,\lambda^{2\gamma}\,\|\omega u_1\|^2_{\mathcal{H}^{s,\gamma}(X^\square)},\end{aligned} \tag{1.4.4}$$

the constant c' being estimated by $\max_{\alpha_0\le s} |(tD_t)^{\alpha_0}\omega|$.

Similarly,

$$\begin{aligned}&\|\omega\kappa_\lambda((1-\omega)u_2)\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} \\ &= \lambda^{2\gamma} \sum_{\alpha_0+j\le s} \int_0^\infty \tau^{2(\alpha_0+j-\gamma)} \frac{1}{\tau^{2(\alpha_0+j)}} \left\|(\tau D_\tau)^{\alpha_0}\left(\omega(\frac{\tau}{\lambda})\,((1-\omega)u_2)(\tau,\cdot)\right)\right\|^2_{H^j(X)} \tau^n d\tau \\ &\le (c')^2\,\lambda^{2\gamma} \max_{a\le\tau\le\lambda A}(\tau^{-2\gamma}, \tau^{2(1-\gamma)}, \ldots, \tau^{2(s-\gamma)})\,\|(1-\omega)u_2\|^2_{\mathcal{H}^s(X^\square)}\end{aligned} \tag{1.4.5}$$

with the same constant c' as in (1.4.4).

In the same manner we can see that

$$\begin{aligned}&\|(1-\omega)\kappa_\lambda(\omega u_1)\|^2_{\mathcal{H}^s(X^\square)} \\ &= \sum_{\alpha_0+j\le s} \lambda^{2(\alpha_0+j)} \int_0^\infty \tau^{2(\gamma-\alpha_0-j)} \frac{1}{\tau^{2\gamma}} \left\|(\tau D_\tau)^{\alpha_0}\left((1-\omega(\frac{\tau}{\lambda}))\,(\omega u_1)(\tau,\cdot)\right)\right\|^2_{H^j(X)} \tau^n d\tau \\ &\le (c'')^2 \max_{\lambda a\le\tau\le A}(\tau^{2\gamma}, \lambda^2\tau^{2(\gamma-1)}, \ldots, \lambda^{2s}\tau^{2(\gamma-s)})\,\|\omega u_1\|^2_{\mathcal{H}^{s,\gamma}(X^\square)},\end{aligned} \tag{1.4.6}$$

where the constant c'' can be estimated by the *sup*-norm of the derivatives $(tD_t)^{\alpha_0}\omega$ up to order s on $\mathbb{R}_+$.

Likewise,

$$\begin{aligned}
&\|(1-\omega)\kappa_\lambda((1-\omega)u_2)\|^2_{\mathcal{H}^s(X^\square)} \\
&= \sum_{\alpha_0+j\le s} \lambda^{2(\alpha_0+j)} \int_0^\infty \frac{1}{\tau^{2(\alpha_0+j)}} \left\|(\tau D_\tau)^{\alpha_0}\Big((1-\omega(\frac{\tau}{\lambda}))\,((1-\omega)u_2)(\tau,\cdot)\Big)\right\|^2_{H^j(X)} \tau^n d\tau \\
&\le (c'')^2 \max(1,\lambda^2,\ldots,\lambda^{2s})\, \|(1-\omega)u_2\|^2_{\mathcal{H}^s(X^\square)}, \qquad (1.4.7)
\end{aligned}$$

the constant c'' being the same as in (1.4.6).

Combining (1.4.4) - (1.4.7) we can assert that

$$\|\kappa_\lambda u\|_{\mathcal{K}^{s,\gamma}(X^\square)} \le c\ \max(1,\lambda^s,\lambda^\gamma)\left(\|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u_2\|_{\mathcal{H}^s(X^\square)}\right),$$

with c a constant independent of u and λ. Hence

$$\|\kappa_\lambda u\|_{\mathcal{K}^{s,\gamma}(X^\square)} \le c\ \max(1,\lambda^s,\lambda^\gamma)\ \|u\|_{\mathcal{K}^{s,\gamma}(X^\square)},$$

which is our assertion.

□

It is also a simple matter to see that κ is a continuous mapping of $\mathbb{R}_+ \to \mathcal{L}_\sigma(\mathcal{K}^{s,\gamma}(X^\square))$.

1.4.3 *Invariance under Fuchs-type operators*

The following result is basic in the theory of pseudodifferential operators on manifolds with conical singularities (see Schulze [15]).

Lemma 1.4.5 *Let M be a Fuchs-type operator of order m, with coefficients independent of t for $t \ge c$ where $c > 0$. Given any $s, \gamma \in \mathbb{R}$, the operator M induces a continuous linear mapping*

$$M:\ \mathcal{K}^{s,\gamma}(X^\square) \to \mathcal{K}^{s-m,\gamma-m}(X^\square).$$

Proof. We give the proof only for the case when s is an integer $\ge m$; the other cases are left to the reader.

Pick a cut-off function ω on $\overline{\mathbb{R}_+}$. Given any $u \in \mathcal{K}^{s,\gamma}(X^\square)$, we assume that $u = \omega u_1 + (1-\omega)u_2$ where $u_1 \in \mathcal{H}^{s,\gamma}(X^\square)$ and $u_2 \in \mathcal{H}^s(X^\square)$. Then

$$\begin{aligned}
\|Mu\|_{\mathcal{K}^{s-m,\gamma-m}(X^\square)} &\le \|\omega Mu\|_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} + \|(1-\omega)Mu\|_{\mathcal{H}^{s-m}(X^\square)} \\
&\le \|\omega M(\omega u_1)\|_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} + \|\omega M((1-\omega)u_2)\|_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} \\
&\quad + \|(1-\omega)M(\omega u_1)\|_{\mathcal{H}^{s-m}(X^\square)} + \|(1-\omega)M((1-\omega)u_2)\|_{\mathcal{H}^{s-m}(X^\square)}.
\end{aligned}$$

Analysis similar to that in the proof of Lemma 1.4.4 shows that

$$\begin{aligned} \|\omega M(\omega u_1)\|_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} &\le c_1' \, \|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)}, \\ \|\omega M((1-\omega)u_2)\|_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} &\le c_2' \, \|(1-\omega)u_2\|_{\mathcal{H}^{s}(X^\square)} \end{aligned}$$

as well as

$$\begin{aligned} \|(1-\omega)M(\omega u_1)\|_{\mathcal{H}^{s-m}(X^\square)} &\le c_1'' \, \|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)}, \\ \|(1-\omega)M((1-\omega)u_2)\|_{\mathcal{H}^{s-m}(X^\square)} &\le c_2'' \, \|(1-\omega)u_2\|_{\mathcal{H}^{s}(X^\square)}, \end{aligned}$$

where the constants c_1', c_2' and c_2', c_2'' depend only on M and ω.

Let us check, for instance, the first of these estimates. Since

$$M = \frac{1}{t^m} \sum_{k=0}^{m} M_k(t)\,(-t\partial_t)^k$$

with some $M_k \in C^\infty_{loc}(\overline{\mathbb{R}_+}, Diff^{m-k}(X))$, it follows that

$$\begin{aligned} &\|\omega M(\omega u_1)\|^2_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} \\ &= \sum_{\alpha_0+j\le s-m} \int_0^\infty \frac{1}{t^{2(\gamma-m)}} \left\| \frac{1}{t^m} \sum_{k=0}^{m} \sum_{\alpha=0}^{\alpha_0} M_{k,\alpha}(t)\,(tD_t)^{k+\alpha}(\omega u_1) \right\|^2_{H^j(X)} t^n dt, \end{aligned}$$

where $M_{k,\alpha} \in C^\infty_{comp}(\overline{\mathbb{R}_+}, Diff^{m-k}(X))$. Therefore,

$$\begin{aligned} &\|\omega M(\omega u_1)\|^2_{\mathcal{H}^{s-m,\gamma-m}(X^\square)} \\ &\le c_1^2 \sum_{\alpha_0+j\le s-m} \sum_{k=0}^{m} \sum_{\alpha=0}^{\alpha_0} \int_0^\infty \frac{1}{t^{2\gamma}} \left\|(tD_t)^{k+\alpha}(\omega u_1)\right\|^2_{H^{j+(m-k)}(X)} t^n dt \\ &\le (c_1')^2 \, \|\omega u_1\|^2_{\mathcal{H}^{s,\gamma}(X^\square)}, \end{aligned}$$

as desired.

Summarizing we get

$$\|Mu\|_{\mathcal{K}^{s-m,\gamma-m}(X^\square)} \le c \left(\|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u_2\|_{\mathcal{H}^{s}(X^\square)} \right),$$

with c a constant independent of u. From this the lemma follows.

□

As follows, the Fuchs-type operators are invariant under the $\mathbf{R}_+$-action κ on $\mathcal{K}^{s,\gamma}(X^\square)$ given by (1.4.3).

Proposition 1.4.6 *For any Fuchs-type operator M with coefficients independent of t, we have*

$$\kappa_\lambda^{-1}\, M\, \kappa_\lambda = M, \quad \lambda > 0.$$

Proof. Use the equality $tD_t = \rho D_\rho$, where $\rho = \lambda r$.

□

1.4.4 *Unitary property*

Fix a density dx on X based on some Riemannian metric. It follows immediately that the normed spaces $\mathcal{K}^{0,0}(X^\square)$ and $L^2(X^\square, t^{\frac{n}{2}}\,dt\,dx)$ are isomorphic. We shall fix once and for all the scalar product on $\mathcal{K}^{0,0}(X^\square)$ induced under this identification.

Proposition 1.4.7 *For any $\lambda > 0$, the operator κ_λ on $\mathcal{K}^{0,0}(X^\square)$ is unitary.*

Proof. The proof is straightforward.

□

Chapter 2

Wedge Sobolev Spaces

The role of conical singularities in the present discussion is that manifolds with edges are locally close to an edge of dimension q of the form of a wedge $X^{\Delta} \times U$ with open $U \subset \mathbb{R}^q$.

2.1 Sobolev Spaces on a Coordinate Wedge

2.1.1 *Model wedge*

By a *model wedge* is meant a direct product $X^{\Delta} \times U$, where X^{Δ} is a model cone in $\mathbb{R}^{1+n}$ and U is an open subset of $\mathbb{R}^q$.

More generally, we consider the products $X^{\Delta} \times U$, with X an n-dimensional submanifold of the unit sphere S^N in $\mathbb{R}^{1+N}$. They will also be referred to as model wedges.

The natural volume form on $X^{\Delta} \times U$ is induced by embedding this wedge (locally) to $\mathbb{R}^{1+n+q}$.

If $q = 0$, then $U = \{0\}$, and so $X^{\Delta} \times U$ is identified with X^{Δ}. This is precisely how the *wedge theory* encompasses the *cone theory.*

Another extreme case is obtained by setting $n = 0$. Then the cone X^{Δ} degenerates into the semiaxis $\overline{\mathbb{R}_+}$ (or into the disjoint union of two copies of $\overline{\mathbb{R}_+}$), and so the wedge $X^{\Delta} \times U$ is identified with the cylinder $\overline{\mathbb{R}_+} \times U$ (or with the disjoint union of two copies of $\overline{\mathbb{R}_+} \times U$).

2.1.2 *Stretched object*

Write $z = (z_1, z_2)$ for the variable in $\mathbb{R}^{1+n+q}$, where $z_1 \in \mathbb{R}^{1+n}$ stands for the "cone variable," and $z_2 \in \mathbb{R}^q$ runs over the edge.

Set

$$\begin{cases} z_1 = t\, p(x), \\ z_2 = y, \end{cases}$$

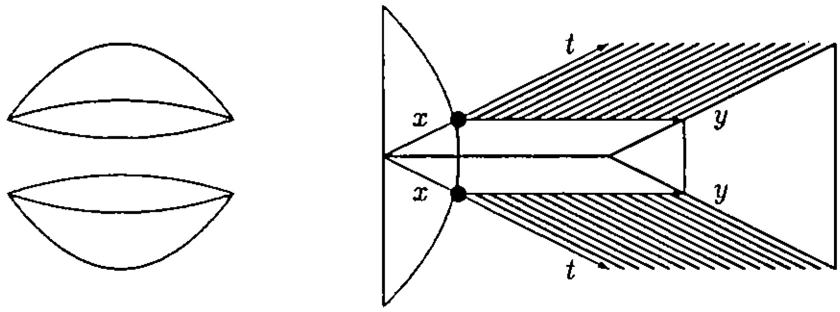

Fig. 2.1: In this way we obtain what we shall call the "*stretched wedge.*"

which gives a diffeomorphism $\pi : \, int\, X^{\square} \times U \to int\, X^{\Delta} \times U$.

For a distribution u in the interior of $X^{\Delta} \times U$, we denote by $\pi^{*}u$ the *pull-back* of u under π. This is a distribution in the interior of $X^{\square} \times U$.

Thus, the analysis on a model wedge $X^{\Delta} \times U$ will refer to the open *stretched wedge*

$$X^{\square} \times U = \overline{\mathbb{R}_{+}} \times X \times U.$$

In the sequel, (t, x, y) stands for the coordinates in $X^{\square} \times U$, where t is the global coordinate in $\overline{\mathbb{R}_{+}}$, $x = (x_1, \ldots, x_n)$ are local coordinates in X, and $y = (y_1, \ldots, y_q)$ are coordinates in U.

From what has already been said, it follows that the natural volume form on $X^{\square} \times U$ is $t^{n} dt\, dx\, dy$, where dx is a volume form on X related to some Riemannian metric on this manifold, and dy is that on U.

2.1.3 *Typical differential operators*

Let $(t, x) \in X^{\square}$ be polar coordinates in the cone X^{Δ}, i.e., $t = |p|$ and x are the coordinates of the point $\frac{p}{|p|}$ in X, for $p \in X^{\Delta}$.

Given a differential operator M of order m on $X^{\Delta} \times U$, we are going to write it in the coordinates (t, x, y).

Using Lemma 1.1.2 we can assert that M transforms into an operator of the form

$$\pi^{\sharp} M \left((t, x, y), (D_t, D_x, D_y) \right) = \sum_{\alpha_0 + |\alpha| + |\beta| \leq m} \widetilde{M}_{\alpha_0, \alpha, \beta}(t, x, y)\, D_t^{\alpha_0} \left(\frac{1}{t} D_x \right)^{\alpha} D_y^{\beta}$$

on the stretched wedge $X^{\square} \times U$. The coefficients $\widetilde{M}_{\alpha_0, \alpha, \beta}$ can be computed from (1.1.1) and the coefficients of M. Notice that they are smooth up to $t = 0$, provided that the coefficients of M are.

Definition 2.1.1 *By edge-degenerate operators on* $X^{\square} \times U$ *are meant differential operators of the form*

$$\frac{1}{t^m} \sum_{j+|\beta| \le m} M_{j,\beta}(t,y)\,(-t\partial_t)^j (tD_y)^\beta$$

with operator-valued coefficients $M_{j,\beta} \in C^\infty_{loc}(\overline{\mathbb{R}_+} \times U, Diff^{m-(j+|\beta|)}(X))$.

As above, the analysis of differential operators in polar coordinates on a wedge leads to edge-degenerate operators.

Let us mention yet another important example motivating the introduction of the edge-degenerate operators.

Example 2.1.2 Let $h(t,y)$ $(t \ge 0,\ y \in U)$ be a (t,y)-dependent Riemannian metric on X which is C^∞ up to $t = 0$. Then the Laplace-Beltrami operator on $X^{\square} \times U$ associated to the metric of the "geometric wedge" $dt^2 + t^2 h(t,y) + dy^2$ is just edge-degenerate, of order $m = 2$.

□

2.1.4 *Transformation of Sobolev spaces*

As in Example 1.1.5, we can see in what way Sobolev spaces

$$H^s(X^{\triangle} \times U) = H^s(\mathbb{R}^{1+n} \times \mathbb{R}^q)\,|_{X^{\triangle} \times U}$$

on the model wedge $X^{\triangle} \times U \subset \mathbb{R}^{1+n} \times \mathbb{R}^q$ transforms under polar coordinates in the cone $X^{\triangle}$.

Definition 2.1.3 *Given any* $s \in \mathbb{Z}_+$*, denote by* $\mathcal{H}^s(X^{\square} \times U)$ *the set of all distributions* u *in the interior of* $X^{\square} \times U$ *whose derivatives up to the order* s *are locally integrable with respect to the measure* $t^n dt dx dy$ *and satisfy*

$$\|u\|^2_{\mathcal{H}^s(X^{\square} \times U)} := \sum_{\alpha_0 + j + |\beta| \le s} \int_0^\infty \int_U \frac{1}{t^{2(\alpha_0 + j + |\beta|)}} \left\| (tD_t)^{\alpha_0} (tD_y)^\beta \tilde{u} \right\|^2_{H^j(X)} t^n dt dx dy < \infty.$$

For integer $s < 0$, we define the space $\mathcal{H}^s(X^{\square} \times U)$ by duality as $\mathcal{H}^s(X^{\square} \times U) = \mathcal{H}^{-s}_{comp}(X^{\square} \times U)'$. Further, for fractional s, the space $\mathcal{H}^s(X^{\square} \times U)$ is defined by (complex) interpolation.

As follows, the norm $\|\pi^* u\|_{\mathcal{H}^s(X^{\square} \times U)}$ sharply catches the behavior of Sobolev functions $u \in H^s(X^{\triangle} \times U)$ away from any neighborhood of the wedge $t = 0$ under the correspondence $\pi^* u\ (t,x,y) = u(t\,p(x), y)$.

Lemma 2.1.4 *Let $s \in \mathbb{R}$. To any $\varepsilon > 0$, there correspond positive constants c_1 and c_2 depending on ε, such that*

$$c_1 \|\pi^* u\|_{\mathcal{H}^s(X^\square \times U)} \leq \|u\|_{H^s(X^\triangle \times U)} \leq c_2 \|\pi^* u\|_{\mathcal{H}^s(X^\square \times U)}$$

whenever $u \in H^s(X^\triangle \times U)$ satisfies supp $u \subset (X^\triangle \setminus B(0,\varepsilon)) \times U$.

Proof. The proof is similar to that in Lemma 1.1.7.

□

Our next concern will be to find a suitable abstract framework for the spaces $\mathcal{H}^s(X^\square \times U)$.

2.1.5 *Vector-valued distributions*

Given a locally convex space $\mathcal{F}$ of distributions on $\mathbb{R}^q$, we denote by $\mathcal{F}'_\tau$ the dual space to $\mathcal{F}$ with Mackey's topology.

Let L be an arbitrary locally convex $\mathbb{C}$-vector space. Denote by $\mathcal{L}_\varepsilon(\mathcal{F}'_\tau \to L)$ the space of all continuous linear mappings of $\mathcal{F}'_\tau$ to L equipped with the topology of uniform convergence on equicontinuous sets of functionals in $\mathcal{F}'_\tau$. After Schwartz [18, 19], the elements of $\mathcal{F}(\mathbb{R}^q, L) := \mathcal{L}_\varepsilon(\mathcal{F}'_\tau \to L)$ are said to be *distributions of class $\mathcal{F}$ on $\mathbb{R}^q$ with values in L.*

The following result was proved by Grothendieck [8, Ch.2,Theorem 6].

Proposition 2.1.5 *If $\mathcal{F}$ and L are complete locally convex spaces and one of them is nuclear, then the canonical mapping $L \otimes \mathcal{F} \to \mathcal{F}(\mathbb{R}^q, L)$ may be extended to a topological isomorphism $\mathcal{F}(\mathbb{R}^q, L) \cong L \otimes_\pi \mathcal{F}$.*

(Here $L \otimes_\pi \mathcal{F}$ stands for the completed projective tensor product of L and $\mathcal{F}$.)

In particular, we have the Schwartz space $\mathcal{S}(\mathbb{R}^q, L)$ of rapidly decreasing L-valued functions on $\mathbb{R}^q$, the space $\mathcal{S}'(\mathbb{R}^q, L)$ of temperate L-valued distributions on $\mathbb{R}^q$, the space $\mathcal{A}(\mathcal{O}, L)$ of L-valued holomorphic functions on an open set $\mathcal{O} \subset \mathbb{C}$, and so on.

2.1.6 *Abstract Sobolev spaces*

Assume that L is a Banach space, and $(\kappa_\lambda)_{\lambda \in \mathbb{R}_+}$ is an $\mathbb{R}_+$-action on L.

Fix a strictly positive C^∞ function $\eta \mapsto \langle \eta \rangle$ on R^q such that $\langle \eta \rangle = |\eta|$ for all $|\eta| \geq c$ with some $c > 0$. (In this way we obtain what will be referred to as the *smoothed norm function.*)

The following property of the smoothed norm functions proves extremely useful in the sequel.

Lemma 2.1.6 (Peetre's inequality) *There is a constant $C > 0$ such that, given any $s \in \mathbb{R}$, we have*

$$\langle\eta\rangle^{s} \leq C^{|s|}\, \langle\eta-\theta\rangle^{|s|}\, \langle\theta\rangle^{s} \quad \textit{for all } \eta,\theta \in \mathbb{R}^{q}.$$

Proof. The proof is elementary.

□

A motivation to introducing the *abstract Sobolev spaces* in Schulze [15] was that $H^s(\mathbb{R}^q, H^s(\mathbb{R}^{1+n}) \neq H^s(\mathbb{R}^{1+n+q})$, unless $s = 0$.

Definition 2.1.7 *Given any $s \in \mathbb{R}$, the space $\mathcal{W}^s(\mathbb{R}^q, L)$ is defined to be the set of all $u \in \mathcal{S}'(\mathbb{R}^n, L)$ such that $\mathcal{F}u \in L^1_{loc}(\mathbb{R}^q, L)$ and*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^q,L)} = \left(\int \langle\eta\rangle^{2s} \left\|\kappa^{-1}_{\langle\eta\rangle}\, \mathcal{F}_{y\mapsto\eta} u\right\|^2_L d\eta\right)^{\frac{1}{2}} \leq \infty.$$

If we replace $\|\cdot\|_L$ by another equivalent norm on L, then we get an equivalent norm on $\mathcal{W}^s(\mathbb{R}^q, L)$. Furthermore, this norm is independent on the concrete choice of $\langle\eta\rangle$ modulo equivalence of norms.

Here are some basic properties of this concept. They may be found in Schulze [15].

The space $\mathcal{W}^s(\mathbb{R}^q, L)$ does depend on the particular choice of the group action κ.

Example 2.1.8 Let $\kappa_\lambda = 1$ be the trivial $\mathbb{R}_+$-action on L. Then $\mathcal{W}^s(\mathbb{R}^q, L)$ coincides with the usual Sobolev space $H^s(\mathbb{R}^q, L)$ of L-valued distributions of smoothness s on $\mathbb{R}^q$.

□

For general κ, the mapping

$$u \mapsto \mathcal{F}^{-1}_{\eta\mapsto y}\, \kappa^{-1}_{\langle\eta\rangle}\, \mathcal{F}_{y\mapsto\eta} u$$

is easily verified to be an isometrical isomorphism of $\mathcal{W}^s(\mathbb{R}^q, L) \to H^s(\mathbb{R}^q, L)$.

Proposition 2.1.9 *The space $\mathcal{W}^s(\mathbb{R}^q, L)$ is a Banach space.*

Proof. See Proposition 2.4 in Hirschmann [9].

□

If L is a Hilbert space, then $\mathcal{W}^s(\mathbb{R}^q, L)$ is even a Hilbert space with respect to the inner product

$$(u,v)_{\mathcal{W}^s(\mathbb{R}^q,L)} := \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \left(\kappa^{-1}_{\langle\eta\rangle}\mathcal{F}_{y\mapsto\eta} u, \kappa^{-1}_{\langle\eta\rangle}\mathcal{F}_{y\mapsto\eta} v\right)_L d\eta, \quad u,v \in \mathcal{W}^s(\mathbb{R}^q, L). \quad (2.1.1)$$

Proposition 2.1.10 *For any $s \in \mathbb{R}$, the bilinear form*

$$\langle v, u\rangle := \int_{\mathbb{R}^q} \left\langle \kappa_{\langle\eta\rangle}^{-1}\mathcal{F}_{y\mapsto\eta}v, \kappa_{\langle\eta\rangle}^{-1}\mathcal{F}_{y\mapsto\eta}u\right\rangle d\eta, \quad v \in \mathcal{W}^{-s}(\mathbb{R}^q, L'),\ u \in \mathcal{W}^s(\mathbb{R}^q, L),$$

defines a pairing of $\mathcal{W}^{-s}(\mathbb{R}^q, L') \times \mathcal{W}^s(\mathbb{R}^q, L) \to \mathbb{C}$, under which $\mathcal{W}^{-s}(\mathbb{R}^q, L')$ is topologically isomorphic to the dual of $\mathcal{W}^s(\mathbb{R}^q, L)$.

Proof. This repeats the proof of an analogous result for the usual Sobolev spaces.

□

Rapidly decreasing functions on $\mathbb{R}^q$ are multipliers of $\mathcal{W}^s(\mathbb{R}^q, L)$. It is the point at which there originates pseudodifferential calculus in the spaces $\mathcal{W}^s(\mathbb{R}^q, L)$ (see [15], [9]).

Proposition 2.1.11 *Let $\varphi \in \mathcal{S}(\mathbb{R}^q)$. For any $s \in \mathbb{R}$, the multiplication operator $M_\varphi : u \mapsto \varphi u$ is a continuous mapping of $\mathcal{W}^s(\mathbb{R}^q, L) \to \mathcal{W}^s(\mathbb{R}^q, L)$. Moreover, the operator $\varphi \mapsto M_\varphi$ is a continuous mapping of $\mathcal{S}(\mathbb{R}^q) \to \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, L))$.*

Proof. See Hirschmann [9, Theorem 3.2]. Behm [2] observed that even

$$\|M_\varphi\| \le \|\langle\eta\rangle^{R+s}\,\mathcal{F}_{y\mapsto\eta}\varphi\|_{L^1(\mathbb{R}^q)},$$

where R is the number of Proposition 1.4.3.

□

Proposition 2.1.12 *Given any $s \in \mathbb{R}$, the subspace $C^\infty_{comp}(\mathbb{R}^q, L)$ is dense in $\mathcal{W}^s(\mathbb{R}^q, L)$.*

Proof. If $u \in \mathcal{W}^s(\mathbb{R}^q, L)$, then the function

$$\eta \mapsto \langle\eta\rangle^s\,\kappa_{\langle\eta\rangle}^{-1}\mathcal{F}_{y\mapsto\eta}u$$

belongs to $L^2(\mathbb{R}^q, L)$. Hence there is a sequence $\{f_\nu\}$ of functions in $C^\infty_{comp}(\mathbb{R}^q, L)$ such that

$$\langle\eta\rangle^s\,\kappa_{\langle\eta\rangle}^{-1}f_\nu \to \langle\eta\rangle^s\,\kappa_{\langle\eta\rangle}^{-1}\mathcal{F}_{y\mapsto\eta}u \quad \text{in } L^2(\mathbb{R}^q, L).$$

It is evident that $f_\nu \in \mathcal{S}(\mathbb{R}^q, L)$ are the Fourier transforms of functions $u_\nu \in \mathcal{S}(\mathbb{R}^q, L)$, and $\|u - u_\nu\|_{\mathcal{W}^s(\mathbb{R}^q,L)} \to 0$ as $\nu \to \infty$.

It remains to verify that $C^\infty_{comp}(\mathbb{R}^q, L)$ is dense in $\mathcal{S}(\mathbb{R}^q, L)$ in the norm of $\mathcal{W}^s(\mathbb{R}^q, L)$.

To this end, pick a function $\omega \in C^\infty_{comp}(\mathbb{R}^q)$ with $\omega(y) = 1$ for $|y| \le 1$. If $u \in \mathcal{S}(\mathbb{R}^q, L)$, then $\omega(\varepsilon\,\cdot)u \in C^\infty_{comp}(\mathbb{R}^q, L)$. Moreover, an easy verification shows that $\omega(\varepsilon\,\cdot)u \to u$ in $\mathcal{W}^s(\mathbb{R}^q, L)$ as $\varepsilon \to 0$, because $\omega(\varepsilon\, y)u(y) \ne u(y)$ only for $|y| > \frac{1}{\varepsilon}$.

This completes the proof.

□

The following embedding theorem is a direct generalization of the *Rellich Theorem*.

Proposition 2.1.13 *Let $L_2 \subset L_1$ be Banach spaces with a group action κ on L_1 which restricts to a group action on L_2. If the embedding $L_2 \hookrightarrow L_1$ is compact and if $s_2 > s_1$, then the embedding*

$$\mathcal{W}^{s_2}_{loc}(\mathbb{R}^q, L_2) \hookrightarrow \mathcal{W}^{s_1}_{loc}(\mathbb{R}^q, L_1)$$

is compact.

Proof. See Behm [2, p.57].

□

2.1.7 *Interpolation*

A couple of Banach spaces (L_0, L_1) is called a *Banach couple* if there is a Hausdorff topological vector space L such that both L_0 and L_1 are continuously embedded subspaces of L.

If (L_0, L_1) is a Banach couple and $\theta \in [0,1]$, then we denote by $[L_0, L_1]_\theta$ the corresponding interpolation space defined by the complex interpolation method (see Bergh and Löfström [3]).

Theorem 2.1.14 *Let (L_0, L_1) be a Banach couple with a group action on $L_0 \oplus L_1$ which restricts to group actions on both the subspaces. Then*

$$[\mathcal{W}^{s_0}(\mathbb{R}^q, L_0), \mathcal{W}^{s_1}(\mathbb{R}^q, L_1)]_\theta = \mathcal{W}^{(1-\theta)s_0+\theta s_1}(\mathbb{R}^q, [L_0, L_1]_\theta)$$

holds isometrically for all $s_0, s_1 \in BbbR$ and $0 < \theta < 1$.

.

Proof. See Hirschmann [9].

□

2.1.8 *Sobolev spaces on the wedge*

In the present calculus it is sufficient to deal with Hilbert spaces L. Such a case is $L = \mathcal{K}^{s,\gamma}(X^\square)$ with (κ_λ) given in (1.4.3).

Definition 2.1.15 *Given any $s, \gamma \in \mathbb{R}$, the space*

$$\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q) := \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\square))$$

is said to be the weighted Sobolev space of smoothness s and weight γ on the stretched wedge $X^\square \times \mathbb{R}^q$.

The remainder of this section will be devoted to some specific properties of this concept, which are different of those listed in the previous subsection. Until further notice we assume that X is either $\mathbb{R}^n$ or a compact manifold of dimension n.

Proposition 2.1.16 *The subspace $C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$ is dense in $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$ for each $s, \gamma \in \mathbb{R}$.*

Proof. Pick a $u \in \mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$. By Proposition 2.1.12, there is a sequence $u'_\nu \in C^\infty_{comp}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\square))$ converging to u in the norm of $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$.

On the other hand, $C^\infty_{comp}(int\, X^\square)$ is dense in $\mathcal{K}^{s,\gamma}(X^\square)$ by Proposition 1.3.8, which remains still valid with $\mathbb{K}$ replaced by $X^\square$. Proposition 2.1.5 now shows that

$$C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q) = C^\infty_{comp}(int\, X^\square) \otimes_\pi C^\infty_{comp}(\mathbb{R}^q)$$

is dense in

$$C^\infty_{comp}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\square)) = \mathcal{K}^{s,\gamma}(X^\square) \otimes_\pi C^\infty_{comp}(\mathbb{R}^q).$$

Thus, for any ν, there exists a function $u_\nu \in C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$ such that

$$\|u'_\nu - u_\nu\|_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} < \frac{1}{\nu}.$$

Obviously, $u_\nu \to u$ in the norm of $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$, and the proof is complete.

□

Our next result extends Proposition 1.3.9 to the case of wedge Sobolev spaces.

Proposition 2.1.17 *Assume that $s, \gamma \in \mathbb{R}$. The $\mathcal{W}^{0,0}(X^\square \times \mathbb{R}^q)$-product (u, v), first taken for $u, v \in C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$, extends to a non-degenerate sesqui-linear pairing $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q) \times \mathcal{W}^{-s,-\gamma}(X^\square \times \mathbb{R}^q) \to \mathbb{C}$, under which $\mathcal{W}^{-s,-\gamma}(X^\square \times \mathbb{R}^q)$ is topologically isomorphic to the dual of $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$.*

Proof. Use Propositions 2.1.10 and 1.3.9, the latter being still true with $\mathbb{K}$ replaced by the stretched cone $X^\square$.

□

Note that

$$(u, v)_{\mathcal{W}^{0,0}(X^\square \times \mathbb{R}^q)} = \int_{\mathbb{R}^q} (\mathcal{F}_{y\mapsto\eta} u, \mathcal{F}_{y\mapsto\eta} v)_{\mathcal{K}^{0,0}(X^\square)}\, d\eta$$

because κ is unitary on $\mathcal{K}^{0,0}(X^\square)$.

Proposition 2.1.18 *Let $s_0, s_1 \in \mathbb{R}$ and $\gamma_0, \gamma_1 \in \mathbb{R}$ satisfy $s_0 \le s_1$ and $\gamma_0 \le \gamma_1$. Then*

$$[\mathcal{W}^{s_0,\gamma_0}(X^\square \times \mathbb{R}^q), \mathcal{W}^{s_1,\gamma_1}(X^\square \times \mathbb{R}^q)]_\theta = \mathcal{W}^{(1-\theta)s_0+\theta s_1,(1-\theta)\gamma_0+\theta\gamma_1}(X^\square \times \mathbb{R}^q)$$

holds up to an equivalent norm for all $0 < \theta < 1$.

Proof. It follows from Proposition 1.3.8 that $\mathcal{K}^{s_1,\gamma_1}(X^\square)$ is embedded to $\mathcal{K}^{s_0,\gamma_0}(X^\square)$ continuously and densely. Consequently, Theorem 2.1.14 shows that

$$\begin{aligned}&[\mathcal{W}^{s_0,\gamma_0}(X^\square \times \mathbb{R}^q), \mathcal{W}^{s_1,\gamma_1}(X^\square \times \mathbb{R}^q)]_\theta \\ &\quad = \mathcal{W}^{(1-\theta)s_0+\theta s_1}(\mathbb{R}^q, [\mathcal{K}^{s_0,\gamma_0}(X^\square), \mathcal{K}^{s_1,\gamma_1}(X^\square)]_\theta)\end{aligned}$$

is fulfilled isometrically for all $0 < \theta < 1$.

To complete the proof it remains to note that

$$[\mathcal{K}^{s_0,\gamma_0}(X^\square), \mathcal{K}^{s_1,\gamma_1}(X^\square)]_\theta = \mathcal{K}^{(1-\theta)s_0+\theta s_1,(1-\theta)\gamma_0+\theta\gamma_1}(X^\square)$$

holds up to an equivalent norm for all $0 < \theta < 1$.

□

Proposition 2.1.19 *Let $s \in \mathbb{R}$. For each $\varepsilon > 0$, there are positive constants c_1 and c_2 depending on ε, such that*

$$c_1 \|u\|_{\mathcal{H}^s(X^\square \times \mathbb{R}^q)} \le \|u\|_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} \le c_2 \|u\|_{\mathcal{H}^s(X^\square \times \mathbb{R}^q)} \tag{2.1.2}$$

whenever $u \in \mathcal{H}^s(X^\square \times \mathbb{R}^q)$ is supported away from the band $[0,\varepsilon) \times X \times \mathbb{R}^q$.

Proof. By Propositions 2.1.17 and 2.1.18, we may assume without loss of generality that s is a non-negative integer. Moreover, Proposition 2.1.16 shows that it suffices to prove estimate (2.1.2) for $u \in C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$ only. Since X is compact, the standard localization argument reduces the matter to the case $X = \mathbb{R}^n$.

Pick a $u \in C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$ which vanishes for $t < \varepsilon$. By definition,

$$\|u\|_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} = \left(\int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \left\| \kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u \right\|^2_{\mathcal{K}^{s,\gamma}(X^\square)} d\eta \right)^{\frac{1}{2}}.$$

Set $\delta = \min_{\eta\in\mathbb{R}^q} \langle\eta\rangle$. By definition, $\delta > 0$. As $u(t,x,y) = 0$ for $t < \varepsilon$, we conclude that

$$\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\ (t,x,\eta) = \langle\eta\rangle^{-\frac{1+n}{2}} \mathcal{F}_{y\mapsto\eta} u\ (\frac{t}{\langle\eta\rangle}, x, \eta)$$

vanishes for $t < \delta\varepsilon$.

Applying Proposition 1.1.20 we can assert that there exist positive constants c_1 and c_2 depending on $\delta\varepsilon$, such that

$$c_1 \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\|_{\mathcal{H}^s(X^\square)} \le \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\|_{\mathcal{K}^{s,\gamma}(X^\square)} \le c_2 \|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\|_{\mathcal{H}^s(X^\square)}$$

for all $\eta \in \mathbb{R}^q$. Multiplying this estimate by $\langle\eta\rangle^s$ and integrating in $\eta \in \mathbb{R}^q$ therefore yield

$$c_1 \|u\|_{\mathcal{W}^s(\mathbb{R}^q, \mathcal{H}^s(X^\square))} \le \|u\|_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} \le c_2 \|u\|_{\mathcal{W}^s(\mathbb{R}^q, \mathcal{H}^s(X^\square))} \tag{2.1.3}$$

for all $u \in C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$ supported away from $[0,\varepsilon) \times X \times \mathbb{R}^q$.

It remains to evaluate the norm $\|u\|_{\mathcal{W}^s(\mathbb{R}^q,\mathcal{H}^s(X^\square))}$. To do this, we may invoke the first equality of (1.4.2) because $s \in \mathbb{Z}_+$. Changing the variables $t = \langle\eta\rangle\tau$ and taking the equality $tD_t = \tau D_\tau$ into account, we deduce that

$$\begin{aligned}
\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,\mathcal{H}^s(X^\square))} &= \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \left\|\kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u\right\|^2_{\mathcal{H}^s(X^\square)} d\eta \\
&= \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \sum_{\alpha_0+j\le s} \int_0^\infty \frac{1}{t^{2(\alpha_0+j)}} \left\|(tD_t)^{\alpha_0} \left(\langle\eta\rangle^{-\frac{1+n}{2}} \mathcal{F}_{y\mapsto\eta} u\left(\frac{t}{\langle\eta\rangle}, x, \eta\right)\right)\right\|^2_{H^j(X)} t^n dt\, d\eta \\
&= \sum_{\alpha_0+j\le s} \int_0^\infty \frac{1}{\tau^{2(\alpha_0+j)}} \left(\int_{\mathbb{R}^q} \langle\eta\rangle^{2(s-\alpha_0-j)} \|(\tau D_\tau)^{\alpha_0} \mathcal{F}_{y\mapsto\eta} u\|^2_{H^j(X)} d\eta\right) \tau^n d\tau.
\end{aligned}$$

Since $(\tau D_\tau)\tau^{|\beta|} = |\beta|\tau^{|\beta|}$, Parseval's formula gives, as in the proof of Lemma 1.1.13,

$$\begin{aligned}
&\|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,\mathcal{H}^s(X^\square))} \\
&\sim \sum_{\alpha_0+j\le s} \int_0^\infty \frac{1}{\tau^{2(\alpha_0+j)}} \left(\sum_{|\beta|\le s-\alpha_0-j} \int_{\mathbb{R}^q} \left\|(\tau D_\tau)^{\alpha_0} D_y^\beta u\right\|^2_{H^j(X)} dy\right) \tau^n d\tau \\
&\sim \|u\|^2_{\mathcal{H}^s(X^\square\times\mathbb{R}^q)}. \qquad (2.1.4)
\end{aligned}$$

Combining (2.1.3) with (2.1.4) yields (2.1.2), as desired.

□

2.2 Invariance under Automorphisms of the Wedge

The analytical framework above allows one also to distinguish a reasonable class of automorphisms of the wedge, which keep invariant the wedge Sobolev spaces.

2.2.1 *An auxiliary result*

Proposition 2.1.19 gives a satisfactory description of the norm $\|\cdot\|_{\mathcal{W}^{s,\gamma}(X^\square\times\mathbb{R}^q)}$ for functions with supports away from the edge. What is still lacking is an explicit description of this norm on functions supported close to the singularity.

In this subsection we are aimed in removing the cut-off function ω which enters the definition of $\|\cdot\|_{\mathcal{W}^{s,\gamma}(X^\square\times\mathbb{R}^q)}$ through the norm $\|\cdot\|_{\mathcal{K}^{s,\gamma}(X^\square)}$. More precisely, it is shown that ω can be replaced by the characteristic function of any interval $[0, a]$, where $a > 0$.

Lemma 2.2.1 *Let $s \in \mathbb{Z}_+$, $\gamma \in \mathbb{R}$. Given any $a > 0$, it follows that*

$$\|u\|^2_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} \sim \sum_{\alpha_0+j\le s} \int_0^\infty \left(\int_{\langle\eta\rangle \le \frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2\gamma}} \|(tD_t)^{\alpha_0} \mathcal{F}_{y\mapsto\eta} u\|^2_{H^j(X)} \, d\eta + \int_{\langle\eta\rangle \ge \frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2(\alpha_0+j)}} \|(tD_t)^{\alpha_0} \mathcal{F}_{y\mapsto\eta} u\|^2_{H^j(X)} \, d\eta \right) t^n dt$$

for $u \in \mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$.

Proof. By Proposition 2.1.16, it is sufficient to prove the desired estimate for $u \in C^\infty_{comp}(int\, X^\square \times \mathbb{R}^q)$ only.

To define a norm in $\mathcal{K}^{s,\gamma}(X^\square)$, we fix a cut-off function ω on $\overline{\mathbb{R}_+}$ with $\omega(t) = 1$ for $t \in [0, a]$ and $\omega(t) = 0$ for $t \in [A, \infty)$. We claim that

$$\|u\|_{\mathcal{K}^{s,\gamma}(X^\square)} \sim \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u\|_{\mathcal{H}^s(X^\square)} \quad \text{for } u \in \mathcal{K}^{s,\gamma}(X^\square). \tag{2.2.1}$$

Indeed, by definition,

$$\|u\|_{\mathcal{K}^{s,\gamma}(X^\square)} \le \|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u\|_{\mathcal{H}^s(X^\square)} \quad \text{for all } u \in \mathcal{K}^{s,\gamma}(X^\square).$$

On the other hand, if $u = \omega u_1 + (1-\omega)u_2$ for some $u_1 \in \mathcal{H}^{s,\gamma}(X^\square)$ and $u_2 \in \mathcal{H}^s(X^\square)$, then

$$\begin{aligned} &\|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u\|_{\mathcal{H}^s(X^\square)} \\ &\le \|\omega\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|\omega(1-\omega)u_2\|_{\mathcal{H}^{s,\gamma}(X^\square)} \\ &\quad + \|(1-\omega)\omega u_1\|_{\mathcal{H}^s(X^\square)} + \|(1-\omega)(1-\omega)u_2\|_{\mathcal{H}^s(X^\square)}. \end{aligned}$$

Applying Lemmas 1.1.14 and 1.1.15 thus yields

$$\begin{aligned} &\|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u\|_{\mathcal{H}^s(X^\square)} \\ &\le c_1' \, \|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + c_2' \, \|(1-\omega)u_2\|_{\mathcal{H}^s(X^\square)} \\ &\quad + c_1'' \, \|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + c_2'' \, \|(1-\omega)u_2\|_{\mathcal{H}^s(X^\square)} \\ &\le c \left(\|\omega u_1\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u_2\|_{\mathcal{H}^s(X^\square)} \right), \end{aligned}$$

the constant c depending on ω only. Hence

$$\|\omega u\|_{\mathcal{H}^{s,\gamma}(X^\square)} + \|(1-\omega)u\|_{\mathcal{H}^s(X^\square)} \le c \, \|u\|_{\mathcal{W}^{s,\gamma}(X^\square)},$$

which implies (2.2.1).

Thus,

$$\begin{aligned} \|u\|^2_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} &= \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \left\| \kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u \right\|^2_{\mathcal{K}^{s,\gamma}(X^\square)} d\eta \\ &\sim \int_{\mathbb{R}^q} \langle\eta\rangle^{2s} \left(\left\| \omega\, \kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u \right\|^2_{\mathcal{H}^{s,\gamma}(X^\square)} + \left\| (1-\omega)\, \kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta} u \right\|^2_{\mathcal{H}^s(X^\square)} \right) d\eta, \end{aligned} \tag{2.2.2}$$

the last relation being a consequence of the elementary inequality $a^2+b^2 \le (a+b)^2 \le 2(a^2+b^2)$ for $a, b \ge 0$.

We now substitute expressions (1.4.2) into (2.2.2) and make the change of variables $t = \langle\eta\rangle\tau$. As above, $tD_t = \tau D_\tau$, so a direct calculation shows that

$$\begin{aligned}
&\|u\|^2_{\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)} \\
&\sim \sum_{\alpha_0+j\le s} \int_0^\infty \Big(\frac{1}{\tau^{2\gamma}} \int \langle\eta\rangle^{2(s-\gamma)} \|(\tau D_\tau)^{\alpha_0}(\omega(\langle\eta\rangle\tau)\mathcal{F}_{y\mapsto\eta}u)\|^2_{H^j(X)}\, d\eta \\
&\qquad + \frac{1}{\tau^{2(\alpha_0+j)}} \int \langle\eta\rangle^{2(s-\alpha_0-j)} \|(\tau D_\tau)^{\alpha_0}((1-\omega(\langle\eta\rangle\tau))\mathcal{F}_{y\mapsto\eta}u)\|^2_{H^j(X)}\, d\eta\Big) \tau^n d\tau.
\end{aligned}$$

To deal with the expressions on the right side, let us introduce the temporary notations $n_1^{(a)}(u)$ and $n_2^{(a)}(u)$ for

$$\begin{aligned}
n_1^{(a)}(u) &= \left(\sum_{\alpha_0+j\le s} \int_0^\infty \left(\int_{\langle\eta\rangle \le \frac{a}{\tau}} \frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2\gamma}} \|(\tau D_\tau)^{\alpha_0}\mathcal{F}_{y\mapsto\eta}u\|^2_{H^j(X)}\, d\eta \right) \tau^n d\tau \right)^{\frac{1}{2}}, \\
n_2^{(a)}(u) &= \left(\sum_{\alpha_0+j\le s} \int_0^\infty \left(\int_{\langle\eta\rangle \ge \frac{a}{\tau}} \frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2(\alpha_0+j)}} \|(\tau D_\tau)^{\alpha_0}\mathcal{F}_{y\mapsto\eta}u\|^2_{H^j(X)}\, d\eta \right) \tau^n d\tau \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.2.3}$$

It is easy to see that both $n_1^{(a)}(\cdot)$ and $n_2^{(a)}(\cdot)$ are norms on $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$. As $\omega(\langle\eta\rangle\tau) = 1$ for $\langle\eta\rangle \le \frac{a}{\tau}$ and $\omega(\langle\eta\rangle\tau) = 0$ for $\langle\eta\rangle \ge \frac{A}{\tau}$, it follows from what has already been proved that

$$\|u\|^2_{\mathcal{W}^{s,\gamma}(X^\square\times\mathbb{R}^q)} \le c_1 \left(n_1^{(A)}(u)\right)^2 + c_2\left(n_2^{(a)}(u)\right)^2 \quad \text{for all } u \in \mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q), \tag{2.2.4}$$

the constants c_1, c_2 being independent of u.

The rest of the proof consists of analyzing the properties of the norms $n_i^{(a)}$, $i = 1, 2$.

If $0 < a < A$, then $n_1^{(a)}(u) \le n_1^{(A)}(u)$. Moreover, $n_1^{(A)}(u) \to \|u\|_{\mathcal{W}^s(\mathbb{R}^q, \mathcal{H}^{s,\gamma}(X^\square))}$ as $A \to \infty$.

Similarly, if $0 < a < A$, then $n_2^{(a)}(u) \ge n_2^{(A)}(u)$. And $n_2^{(a)}(u) \to \|u\|_{\mathcal{W}^s(\mathbb{R}^q, \mathcal{H}^s(X^\square))}$ as $a \to 0$.

Our next claim is that

$$\min_{i=0,\dots,s} (a^{2(i-\gamma)}, A^{2(i-\gamma)}) \le \frac{(n_1^{(A)}(u))^2 - (n_1^{(a)}(u))^2}{(n_2^{(a)}(u))^2 - (n_2^{(A)}(u))^2} \le \max_{i=0,\dots,s} (a^{2(i-\gamma)}, A^{2(i-\gamma)}), \tag{2.2.5}$$

whenever $0 < a < A$. Indeed,

$$\begin{aligned}
&(n_1^{(A)}(u))^2 - (n_1^{(a)}(u))^2 \\
&= \sum_{\alpha_0+j\le s} \int_0^\infty \left(\int_{\frac{a}{\tau} < \langle\eta\rangle \le \frac{A}{\tau}} \frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2\gamma}} \|(\tau D_\tau)^{\alpha_0}\mathcal{F}_{y\mapsto\eta}u\|^2_{H^j(X)}\, d\eta \right) \tau^n d\tau \\
&= \sum_{\alpha_0+j\le s} \int_0^\infty \left(\int_{\frac{a}{\tau} \le \langle\eta\rangle \le \frac{A}{\tau}} (\tau\langle\eta\rangle)^{2(\alpha_0+j-\gamma)} \frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2(\alpha_0+j)}} \|(\tau D_\tau)^{\alpha_0}\mathcal{F}_{y\mapsto\eta}u\|^2_{H^j(X)}\, d\eta \right) \tau^n d\tau,
\end{aligned}$$

whence (2.2.5) follows.

Combining (2.2.4) and (2.2.5) gives

$$\begin{aligned}\|u\|^2_{\mathcal{W}^{s,\gamma}(X^\Box\times\mathbb{R}^q)} &\le c_1\left(n_1^{(a)}(u)\right)^2 + c_1\left(\left(n_1^{(A)}(u)\right)^2 - \left(n_1^{(a)}(u)\right)^2\right) + c_2\left(n_2^{(a)}(u)\right)^2\\ &\le c_1\left(n_1^{(a)}(u)\right)^2 + c_1\max_{i=0,\dots,s}\left(a^{2(i-\gamma)}, A^{2(i-\gamma)}\right)\left(\left(n_2^{(a)}(u)\right)^2 - \left(n_2^{(A)}(u)\right)^2\right) + c_2\left(n_2^{(a)}(u)\right)^2\\ &\le C\left(\left(n_1^{(a)}(u)\right)^2 + \left(n_2^{(a)}(u)\right)^2\right),\end{aligned} \tag{2.2.6}$$

with C a constant independent of u. This is just one part of the lemma.

We shall have established the other part of the lemma if we prove that also both $n_1^{(a)}(u)$ and $n_2^{(a)}(u)$ are majorized by $\|u\|_{\mathcal{W}^{s,\gamma}(X^\Box\times\mathbb{R}^q)}$. We give the proof only for $n_2^{(a)}(u)$; the norm $n_1^{(a)}(u)$ can be handled in much the same way.

Substituting

$$\mathcal{F}_{y\mapsto\eta}u = \omega(\langle\eta\rangle\tau)\,\mathcal{F}_{y\mapsto\eta}u + (1-\omega(\langle\eta\rangle\tau))\mathcal{F}_{y\mapsto\eta}u$$

into (2.2.3), using the *triangle inequality* and taking into account that $\omega(\langle\eta\rangle\tau) = 0$ for $\langle\eta\rangle \ge \frac{A}{\tau}$, we get

$$\begin{aligned}&\left(n_2^{(a)}(u)\right)^2\\ &\le 2\sum_{\alpha_0+j\le s}\int_0^\infty\Bigg(\int_{\frac{a}{\tau}\le\langle\eta\rangle\le\frac{A}{\tau}}\frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2(\alpha_0+j)}}\,\|(\tau D_\tau)^{\alpha_0}(\omega(\langle\eta\rangle\tau)\mathcal{F}_{y\mapsto\eta}u)\|^2_{H^j(X)}\,d\eta\\ &\qquad + \int_{\langle\eta\rangle\ge\frac{a}{\tau}}\frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2(\alpha_0+j)}}\,\|(\tau D_\tau)^{\alpha_0}((1-\omega(\langle\eta\rangle\tau))\mathcal{F}_{y\mapsto\eta}u)\|^2_{H^j(X)}\,d\eta\Bigg)\tau^n d\tau.\end{aligned}$$

Since

$$\frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2(\alpha_0+j)}} \le \max_{i=0,\dots,s}\left(a^{2(\gamma-i)}, A^{2(\gamma-i)}\right)\frac{\langle\eta\rangle^{2s}}{(\tau\langle\eta\rangle)^{2\gamma}} \quad \text{for all } \frac{a}{\tau}\le\langle\eta\rangle\le\frac{A}{\tau},$$

we get

$$\begin{aligned}&\left(n_2^{(a)}(u)\right)^2\\ &\le 2\max_{i=0,\dots,s}\left(a^{2(\gamma-i)}, A^{2(\gamma-i)}\right)\int_{\mathbb{R}^q}\langle\eta\rangle^{2s}\left\|\omega\kappa^{-1}_{\langle\eta\rangle}\,\mathcal{F}_{y\mapsto\eta}u\right\|^2_{\mathcal{H}^{s,\gamma}(X^\Box)}d\eta\\ &\quad +2\int_{\mathbb{R}^q}\langle\eta\rangle^{2s}\left\|(1-\omega)\kappa^{-1}_{\langle\eta\rangle}\,\mathcal{F}_{y\mapsto\eta}u\right\|^2_{\mathcal{H}^{s}(X^\Box)}d\eta\\ &\le c\int_{\mathbb{R}^q}\langle\eta\rangle^{2s}\left(\left\|\omega\kappa^{-1}_{\langle\eta\rangle}\,\mathcal{F}_{y\mapsto\eta}u\right\|^2_{\mathcal{H}^{s,\gamma}(X^\Box)} + \left\|(1-\omega)\kappa^{-1}_{\langle\eta\rangle}\,\mathcal{F}_{y\mapsto\eta}u\right\|^2_{\mathcal{H}^{s}(X^\Box)}\right)d\eta,\end{aligned}$$

where c is independent of u.

As $\frac{1}{2}(a+b)^2 \le a^2+b^2 \le (a+b)^2$ for $a, b\ge 0$, we can assert by (2.2.1) that the right-hand side here is equivalent to $\|u\|^2_{\mathcal{W}^{s,\gamma}(X^\Box\times\mathbb{R}^q)}$, which is the desired conclusion. □

2.2.2 *More on tensor products*

No one has yet given a direct proof of the fact that, given any $\varphi \in C^{\infty}_{comp}(\overline{\mathbb{R}_+} \times X \times \mathbb{R}^q)$, the multiplication operator $u \mapsto \varphi u$ is a continuous mapping of $\mathcal{W}^{s,\gamma}(X^{\square} \times \mathbb{R}^q) \to \mathcal{W}^{s,\gamma}(X^{\square} \times \mathbb{R}^q)$. This can be easily verified for those φ which are of the form $\varphi_1(t,x)\varphi_2(y)$, where $\varphi_1 \in C^{\infty}_{comp}(\overline{\mathbb{R}_+} \times X)$ and $\varphi_2 \in C^{\infty}_{comp}(\mathbb{R}^q)$. To derive the desired conclusion for general φ from this, one then uses a *representation theorem* on projective tensor products of two Fréchet spaces to be formulated.

Let l^1 be the usual Banach space of all summable sequences of complex numbers.

Given a Fréchet space L, we denote by $s_0(L)$ the space of all zero sequences $(l^{(\nu)})$ in L with the topology given by the family of seminorms $(l^{(\nu)}) \mapsto \sup_\nu p(l^{(\nu)})$, where p runs over a system of seminorms defining the topology of L.

If $(c_\nu) \in l^1$ and $(l^{(\nu)}) \in s_0(L)$, $(f^{(\nu)}) \in s_0(\mathcal{F})$, then the series $\sum_\nu c_\nu\, l^{(\nu)} \otimes f^{(\nu)}$ is absolutely convergent in the (completed) projective tensor product $L \otimes_\pi \mathcal{F}$. Conversely, if both L and $\mathcal{F}$ are Fréchet, then we can expand every element of $L \otimes_\pi \mathcal{F}$ into a sum of the above type.

Theorem 2.2.2 *Let L, $\mathcal{F}$ be Fréchet spaces. Suppose that $\mathcal{N}_1$ and $\mathcal{N}_2$ are absolutely convex neighborhoods of zero in L and $\mathcal{F}$ respectively, and $\mathcal{N}$ is the neighborhood of zero in $L \otimes_\pi \mathcal{F}$ corresponding to the absolutely convex hull of $\mathcal{N}_1 \otimes \mathcal{N}_2$ in $L \otimes_\pi \mathcal{F}$. Then for any compact set $K \subset \mathcal{N}$ there are a compact subset k of the unit ball in l^1 and sequences $(l^{(\nu)}) \in s_0(\mathcal{N}_1)$, $(f^{(\nu)}) \in s_0(\mathcal{N}_2)$ such that every element $u \in K$ can be written as $u = \sum_\nu c_\nu\, l^{(\nu)} \otimes f^{(\nu)}$ with some $(c_\nu) \in k$.*

Proof. See Treves [21, Theorem 45.2].

□

The theorem gains in interest if we realize that the sequence (c_ν) can be chosen of small norm in l^1, provided u is close to zero in $L \otimes_\pi \mathcal{F}$.

Corollary 2.2.3 *Let L, $\mathcal{F}$ be Fréchet spaces. Then for any sequence $(u^{(i)}) \subset L \otimes_\pi \mathcal{F}$ converging to zero there are sequences $(l^{(\nu)}) \in s_0(L)$, $(f^{(\nu)}) \in s_0(\mathcal{F})$ and $(c^{(i)}) = \left((c^{(i)}_\nu)\right) \in s_0(l^1)$ such that*

$$u^{(i)} = \sum_\nu c^{(i)}_\nu\, l^{(\nu)} \otimes f^{(\nu)} \qquad \textit{for } i = 1, 2, \ldots.$$

Proof. See Hirschmann [9].

□

2.2.3 *Boundedness of edge-degenerate operators*

The result to be proved here extends to the statement that the space $\mathcal{W}^{s,\gamma}(X^{\square} \times \mathbb{R}^q)$ is locally invariant under pseudodifferential operators of order 0 on $\mathbb{R}^q$ whose symbols take values in $\mathcal{L}(\mathcal{K}^{s,\gamma}(X^{\square}))$ (see Schulze [17]).

Theorem 2.2.4 *Let $\varphi \in \mathcal{S}(\mathbb{R}^{1+n+q})$. Then the multiplication operator $M_\varphi : u \mapsto \varphi u$ is a continuous mapping of $\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q) \to \mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$. Moreover, the operator $\varphi \mapsto M_\varphi$ is a continuous mapping of $\mathcal{S}(\mathbb{R}^{1+n+q}) \to \mathcal{L}(\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q))$.*

Proof. By duality and interpolation arguments, it is sufficient to prove the theorem for $s \in \mathbb{Z}_+$.

We first prove a reduced form of the theorem. Namely, let us assume that $\varphi(t,x,y) = \varphi_1(t,x)\varphi_2(y)$, where $\varphi_1 \in \mathcal{S}(\mathbb{R}^{1+n})$ and $\varphi_2 \in \mathcal{S}(\mathbb{R}^q)$.

If $u \in \mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$, then

$$\begin{aligned}
&\|\varphi u\|^2_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)} \\
&\quad = \int \langle\eta\rangle^{2s} \left\| \varphi_1\left(\frac{\cdot}{\langle\eta\rangle}, \cdot\right) \kappa^{-1}_{\langle\eta\rangle} \mathcal{F}_{y\mapsto\eta}(\varphi_2 u) \right\|^2_{\mathcal{K}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n)} d\eta \\
&\quad \le c \sup_{\substack{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n \\ \alpha_0'+|\alpha'|\le s}} \left|(tD_t)^{\alpha_0'} D_x^{\alpha'} \varphi_1\right|^2 \|\varphi_2 u\|^2_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)},
\end{aligned}$$

the constant c being independent of u. Since φ_2 depends only on y, Proposition 2.1.11 gives

$$\begin{aligned}
&\|\varphi u\|^2_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)} \\
&\quad \le c \sup_{\substack{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n \\ \alpha_0'+|\alpha'|\le s}} \left|(tD_t)^{\alpha_0'} D_x^{\alpha'} \varphi_1\right| \left\| \langle\eta\rangle^{s+\max(s,|\gamma|)} \mathcal{F}_{y\mapsto\eta}\varphi_2 \right\|_{L^1(\mathbb{R}^q)} \|u\|_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)},
\end{aligned} \tag{2.2.7}$$

as desired.

For the general case, given any $\varphi \in \mathcal{S}(\mathbb{R}^{1+n+q})$, we can find, by Theorem 2.2.2, sequences $(\varphi_1^{(\nu)}) \in s_0(\mathcal{S}(\mathbb{R}^{1+n}))$, $(\varphi_2^{(\nu)}) \in s_0(\mathcal{S}(\mathbb{R}^q))$ such that

$$\varphi(t,x,y) = \sum_\nu c_\nu \varphi_1^{(\nu)}(t,x) \otimes \varphi_2^{(\nu)}(y)$$

with some $(c_\nu) \in l^1$. Applying (2.2.7) we get

$$\begin{aligned}
&\|\varphi u\|_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)} \\
&\quad \le \sum_\nu |c_\nu| \left\| \varphi_1^{(\nu)} \varphi_2^{(\nu)} u \right\|_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)} \\
&\quad \le c \left(\sum_\nu |c_\nu| \right) \sup_{\substack{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n \\ \alpha_0'+|\alpha'|\le s \\ \nu=1,2,\dots}} \left|(tD_t)^{\alpha_0'} D_x^{\alpha'} \varphi_1^{(\nu)}\right| \sup_{\nu=1,2,\dots} \left\| \langle\eta\rangle^{s+\max(s,|\gamma|)} \mathcal{F}_{y\mapsto\eta}\varphi_2^{(\nu)} \right\|_{L^1(\mathbb{R}^q)} \\
&\qquad \times \|u\|_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)},
\end{aligned}$$

which proves the first part of the theorem.

The second part of the theorem follows from Corollary 2.2.3 because were φ close to zero, we would choose (c_ν) of small $\sum_\nu |c_\nu|$.

□

If X is a compact manifold of dimension n, then the theorem shows that $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$ is a module over $C^\infty_{comp}(\overline{\mathbb{R}_+} \times X \times \mathbb{R}^q)$.

We now invoke a standard procedure to localize the spaces $\mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$ to open subsets of $\overline{\mathbb{R}_+} \times X \times \mathbb{R}^q$. Namely, given any open set $\mathcal{O} \subset \overline{\mathbb{R}_+} \times X \times \mathbb{R}^q$, we denote by $\mathcal{W}^{s,\gamma}_{loc}(\mathcal{O})$ the space of all distributions u on $\mathcal{O} \cap (int\, X^\square \times \mathbb{R}^q)$ such that $\varphi u \in \mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$ for any $\varphi \in C^\infty_{comp}(\overline{\mathbb{R}_+} \times X \times \mathbb{R}^q)$. And by $\mathcal{W}^{s,\gamma}_{comp}(\mathcal{O})$ we denote the space of all distributions on $\mathcal{O} \cap (int\, X^\square \times \mathbb{R}^q)$ of the form φu, where $\varphi \in C^\infty_{comp}(\overline{\mathbb{R}_+} \times X \times \mathbb{R}^q)$. By the above, $\mathcal{W}^{s,\gamma}_{comp}(\mathcal{O})$ consists of all $u \in \mathcal{W}^{s,\gamma}(X^\square \times \mathbb{R}^q)$ supported on $\mathcal{O}$.

Corollary 2.2.5 *Assume that M is an edge-degenerate differential operator of order m on an open set $\mathcal{O} \subset \overline{\mathbb{R}_+} \times X \times \mathbb{R}^q$. Then M induces continuous mappings*

$$M : \mathcal{W}^{s,\gamma}_{loc}(\mathcal{O}) \to \mathcal{W}^{s-m,\gamma-m}_{loc}(\mathcal{O})$$

for all $s, \gamma \in \mathbb{R}$.

Proof. By Theorem 2.2.4, we are left with the task of showing that the operator $\frac{1}{t^m}(-t\partial_t)^{\alpha_0'} D_x^{\alpha'} (tD_y)^\beta$ is a continuous mapping of

$$\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q) \to \mathcal{W}^{s-m,\gamma-m}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q),$$

provided that $\alpha_0' + |\alpha'| + |\beta| \le m$.

We give the proof only for the case of integer $s \ge m$; the other cases are left to the reader. For any $u \in \mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$, Lemmas 2.2.1 and 1.1.3 yield

$$\begin{aligned}
&\left\| \frac{1}{t^m}(-t\partial_t)^{\alpha_0'} D_x^{\alpha'} (tD_y)^\beta u \right\|^2_{\mathcal{W}^{s-m,\gamma-m}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)} \\
&\sim \sum_{\alpha_0 + j \le s-m} \int_0^\infty \\
&\quad \left(\int_{\langle\eta\rangle \le \frac{a}{t}} \frac{\langle\eta\rangle^{2(s-m)}}{(t\langle\eta\rangle)^{2(\gamma-m)}} \left\| (tD_t)^{\alpha_0} \mathcal{F}_{y\mapsto\eta} \left(\frac{1}{t^m}(-t\partial_t)^{\alpha_0'} D_x^{\alpha'} (tD_y)^\beta u \right) \right\|^2_{H^j(\mathbb{R}^n)} d\eta \right. \\
&\quad \left. + \int_{\langle\eta\rangle \ge \frac{a}{t}} \frac{\langle\eta\rangle^{2(s-m)}}{(t\langle\eta\rangle)^{2(\alpha_0+j)}} \left\| (tD_t)^{\alpha_0} \mathcal{F}_{y\mapsto\eta} \left(\frac{1}{t^m}(-t\partial_t)^{\alpha_0'} D_x^{\alpha'} (tD_y)^\beta u \right) \right\|^2_{H^j(\mathbb{R}^n)} d\eta \right) t^n dt \\
&\le c \sum_{\alpha_0 + j \le s-m} \int_0^\infty \\
&\quad \left(\int_{\langle\eta\rangle \le \frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2\gamma}} \left\| (tD_t)^{\alpha_0+\alpha_0'} \mathcal{F}_{y\mapsto\eta} \left((tD_y)^\beta u \right) \right\|^2_{H^{j+|\alpha'|}(\mathbb{R}^n)} d\eta \right. \\
&\quad \left. + \int_{\langle\eta\rangle \ge \frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2(\alpha_0+j+m)}} \left\| (tD_t)^{\alpha_0+\alpha_0'} \mathcal{F}_{y\mapsto\eta} \left((tD_y)^\beta u \right) \right\|^2_{H^{j+|\alpha'|}(\mathbb{R}^n)} d\eta \right) t^n dt,
\end{aligned}$$

with c a constant independent of u.

As $\mathcal{F}_{y\mapsto\eta}\left((tD_y)^\beta u\right) = t^{|\beta|}\eta^\beta\,\mathcal{F}_{y\mapsto\eta}u$ and $|\eta^\beta| \le const\,\langle\eta\rangle^{|\beta|}$ for all $\eta\in\mathbb{R}^q$, it follows that

$$\left\|\frac{1}{t^m}(-t\partial_t)^{\alpha_0'}D_x^{\alpha'}(tD_y)^\beta u\right\|^2_{\mathcal{W}^{s-m,\gamma-m}(\overline{\mathbb{R}_+}\times\mathbb{R}^n\times\mathbb{R}^q)}$$
$$\le c'\sum_{\alpha_0+j\le s}\int_0^\infty\left(a^{2|\beta|}\int_{\langle\eta\rangle\le\frac{a}{t}}\frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2\gamma}}\|(tD_t)^{\alpha_0}\mathcal{F}_{y\mapsto\eta}u\|^2_{H^j(\mathbb{R}^n)}\,d\eta\right.$$
$$\left.+\frac{1}{a^{2(m-\alpha_0'-|\alpha'|-|\beta|)}}\int_{\langle\eta\rangle\ge\frac{a}{t}}\frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2(\alpha_0+j)}}\|(tD_t)^{\alpha_0}\mathcal{F}_{y\mapsto\eta}u\|^2_{H^j(\mathbb{R}^n)}\,d\eta\right)t^n dt,$$

which completes the proof.

□

2.2.4 *An example of non-invariance*

In this subsection we will look more closely at the norm of $\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)$, with $\gamma\le 0$. More precisely, we examine this norm on functions of separated variables, thus showing that the variables x and y enter the norm in an essentially asymmetric way. This naturally leads to a simple diffeomorphism of the wedge, under which the space $\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)$ is not locally invariant.

Given any $u(t,x,y) = u_0(t)u_1(x)u_2(y)$ in $\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)$, we have, by Lemma 2.2.1,

$$\|u\|^2_{\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)}$$
$$\sim\int_{\mathbb{R}^q}\left(\langle\eta\rangle^{-2\gamma}\int_0^{\frac{a}{\langle\eta\rangle}}t^{-2\gamma}|u_0|^2t^ndt+\int_{\frac{a}{\langle\eta\rangle}}^\infty|u_0|^2t^ndt\right)|\mathcal{F}u_2|^2d\eta\;\|u_1\|^2_{L^2(X)}.$$

Assume that u_0 vanishes away from the line segment $[0,T]$. Then an easy computation shows that

$$\|u\|^2_{\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)}$$
$$\sim\left(\int_0^T t^{-2\gamma}|u_0|^2t^ndt\right)\left(\int_{\langle\eta\rangle\le\frac{a}{T}}\langle\eta\rangle^{-2\gamma}|\mathcal{F}u_2|^2d\eta\right)\|u_1\|^2_{L^2(X)}$$
$$+\int_{\langle\eta\rangle\ge\frac{a}{T}}\left(\langle\eta\rangle^{-2\gamma}\int_0^{\frac{a}{\langle\eta\rangle}}t^{-2\gamma}|u_0|^2t^ndt+\int_{\frac{a}{\langle\eta\rangle}}^T|u_0|^2t^ndt\right)|\mathcal{F}u_2|^2d\eta\;\|u_1\|^2_{L^2(X)}.$$
$$(2.2.8)$$

Hence it follows that u_0 has to be square-integrable on $\mathbb{R}_+$ with respect to the measure $t^{-2\gamma+n}dt$. So the integral $\int_0^{\frac{a}{\langle\eta\rangle}}t^{-2\gamma}|u_0|^2t^ndt$ is infinitesimal when $\eta\to\infty$.

On the other hand, the integral $\int_{\frac{a}{\langle\eta\rangle}}^T|u_0|^2t^ndt$ may be unbounded as $\eta\to\infty$. However, its growth is controlled by a multiple of $\langle\eta\rangle^{-2\gamma}$ because

$$\int_{\frac{a}{\langle\eta\rangle}}^T|u_0|^2t^ndt\le\left(\frac{\langle\eta\rangle}{a}\right)^{-2\gamma}\int_0^T t^{-2\gamma}|u_0|^2t^ndt.$$

For this reason, the norm of u being finite requires the smoothness of u_2 in y of degree close to $-\gamma$ in accordance with u_0.

Example 2.2.6 Take $u_0(t) = t^p \chi_{[0,1]}(t)$, where $\chi_{[0,T]}$ is the characteristic function of the line segment $[0,T]$. This u_0 is square-integrable with respect to the measure $t^{-2\gamma+n}dt$ if and only if $-2\gamma+2p+n+1>0$. Moreover,

$$\langle\eta\rangle^{-2\gamma}\int_0^{\frac{a}{\langle\eta\rangle}} t^{-2\gamma}|u_0|^2 t^n dt + \int_{\frac{a}{\langle\eta\rangle}}^T |u_0|^2 t^n dt$$
$$= \frac{a^{-2\gamma+2p+1+n}}{-2\gamma+2p+1+n}\langle\eta\rangle^{-2p-n-1} + \left(\frac{T^{2p+1+n}}{2p+1+n} - \frac{a^{2p+1+n}}{2p+1+n}\langle\eta\rangle^{-2p-n-1}\right).$$

In the merely interesting case $2p+1+n<0$, formula (2.2.8) thus becomes

$$\|t^p\chi_{[0,T]}\,u_1u_2\|_{\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)} \sim \|u_1\|_{L^2(X)}\,\|u_2\|_{H^{-p-\frac{1+n}{2}}(\mathbb{R}^q)}. \tag{2.2.9}$$

Therefore, a function $u(t,x,y) = t^p\chi_{[0,T]}(t)\,u_1(x)u_2(y)$ belongs to $\mathcal{W}^{0,\gamma}(X^\square\times\mathbb{R}^q)$ if and only if $u_1\in L^2(X)$ and $u_2\in H^{-p-\frac{1+n}{2}}(\mathbb{R}^q)$. In particular, we conclude that the space $\mathcal{W}^{0,\gamma}(\overline{\mathbb{R}_+}\times\mathbb{R}^q\times\mathbb{R}^q)$ is not locally invariant under the diffeomorphism

$$(t,x,y)\mapsto(\tau(t,x,y),\chi(t,x,y),v(t,x,y))$$

of the wedge $\overline{\mathbb{R}_+}\times\mathbb{R}^q\times\mathbb{R}^q$, where

$$\begin{aligned}\tau(t,x,y)&=t,\\ \chi(t,x,y)&=y,\\ v(t,x,y)&=x\end{aligned} \tag{2.2.10}$$

rearranges x and y.

□

2.2.5 *Properties of the automorphisms of the wedge*

Example 2.2.6 demonstrates rather strikingly that it is important to pay attention to admissible diffeomorphisms of the stretched wedges when discussing the invariance of weighted Sobolev spaces on them. In fact, it is to interest of wedge calculus that such diffeomorphisms have to preserve edge-degenerate differential operators. The chain rule shows that, for this to happen for a diffeomorphism $(t,x,y)\mapsto(\tau(t,x,y),\chi(t,x,y),v(t,x,y))$, it must satisfy

- $\frac{1}{t}\frac{\partial\tau}{\partial t}\in C^\infty$ up to $t=0$; and
- $\frac{1}{t}\frac{\partial v}{\partial x}\in C^\infty$ up to $t=0$.

As we shall see, both the conditions make also sense under the geometric setting. Namely, the first condition is an analytical interpretation on the fact that the edge of the wedge must be preserved under the diffeomorphism. And the second condition just amounts to saying that the diffeomorphism of the wedge is a family of diffeomorphisms of the cone bases, which is parameterized by the variable y running over the edge.

Note that the diffeomorphism given in (2.2.10) satisfies the first condition, while the second condition fails to be fulfilled for it.

Let Z_1^Δ and W_1^Δ be two model cones in $\mathbb{R}^{1+n}$, with Z_1 and W_1 open subsets of the unit sphere S^n.

Given open subsets Z_2 and W_2 of $\mathbb{R}^q$, we consider the geometric wedges $Z = Z_1^\Delta \times Z_2$ and $W = W_1^\Delta \times W_2$ in $\mathbb{R}^{1+n+q}$.

Accordingly, we write $z = (z_1, z_2)$ and $w = (w_1, w_2)$ for the variables in $\mathbb{R}^{1+n+q}$, where $z_1, w_1 \in \mathbb{R}^{1+n}$ stand for the "cone variables," and $z_2, w_2 \in \mathbb{R}^q$ run over the edge.

By a *diffeomorphism* of the wedges Z and W is meant any diffeomorphism $w = \delta(z)$ of a neighborhood of Z to a neighborhood of W in $\mathbb{R}^{1+n+q}$, such that $\delta(Z) = W$.

Given a diffeomorphism $w = \delta(z)$ of $Z \to W$, it follows that δ acts smoothly near the edge $\{0\} \times Z_2$ of Z. Moreover, a familiar topological argument shows that δ maps the edge of Z to that of W, i.e., $\delta(\{0\} \times Z_2) = \{0\} \times W_2$.

Composing δ with polar coordinates $z_1 = tp(x)$ in Z_1^Δ and $w_1 = \tau q(\chi)$ in W_1^Δ yields a diffeomorphism

$$\begin{cases} \tau = \tau(t, x, y), \\ \chi = \chi(t, x, y), \\ v = v(t, x, y) \end{cases}$$

of the stretched wedges $int\,\mathbb{Z} \to int\,\mathbf{W}$, where

$$\begin{cases} \tau(t, x, y) = |w_1(tp(x), y)|, \\ \chi(t, x, y) = \frac{w_1(tp(x), y)}{|w_1(tp(x), y)|}, \\ v(t, x, y) = w_2(tp(x), y) \end{cases} \tag{2.2.11}$$

and

$$\begin{aligned} \mathbb{Z} &= Z_1^\square \times Z_2, \\ \mathbf{W} &= W_1^\square \times W_2. \end{aligned}$$

As polar coordinates are singular at the origin, this diffeomorphism (denoted by $\tilde{\delta}$) may fail to extend to a diffeomorphism of neighborhoods of $\mathbb{Z}$ and $\mathbf{W}$ in $\mathbb{R} \times S^n \times \mathbb{R}^q$. However, the following properties of $\tilde{\delta}$ are immediate from the construction of this diffeomorphism:

$$\begin{aligned} &\tau(0, x, y) \equiv 0 \text{ for all } x \in Z_1,\ y \in Z_2, \\ &v(0, x, y) = w_2(0, y) \textit{ is independent of } x \in Z_1. \end{aligned} \tag{2.2.12}$$

The first condition (2.2.12) means that the base $(\{0\} \times Z_1) \times Z_2$ of the wedge $\mathbb{Z}$ transforms to the base $(\{0\} \times W_1) \times W_2$ of the wedge $\mathbf{W}$ by $\tilde{\delta}$. And the second

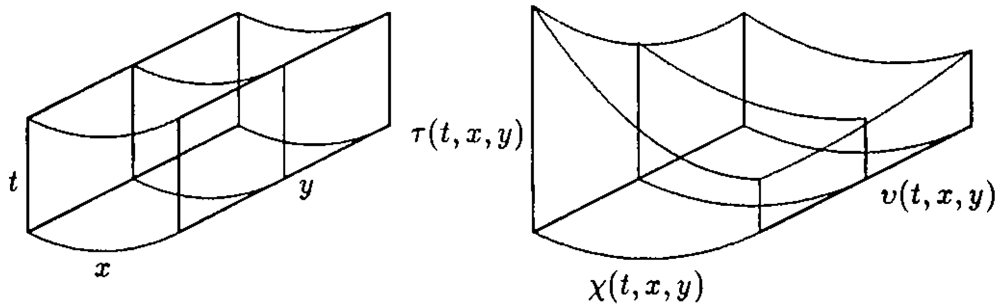

Fig. 2.2: In this way the stretched cone $([0,1) \times Z_1) \times \{y\}$ transforms to the stretched cone $\{(\tau, \chi) : \tau < \tau(1, \chi, y),\ \chi \in W_1\} \times \{\upsilon(t, x, y)\}$ under $\tilde{\delta}$.

condition (2.2.12) means that, for any fixed $y \in Z_2$, the base $(\{0\} \times Z_1) \times \{y\}$ of the stretched cone $Z_1^{\square}$ transforms to the base $(\{0\} \times W_1) \times \{\upsilon(0, x, y)\}$ of the stretched cone $W_1^{\square}$.

It is worth pointing out that if $\tilde{\delta}$ keeps invariant the cone structure of $\mathbb{Z}$ close to $t = 0$, i.e., given any $y \in Z_2$, the stretched cone $([0,1) \times Z_1) \times \{y\}$ transforms to the stretched cone $([0, \tau(t, x, y)) \times W_1) \times \{\upsilon(t, x, y)\}$, then the $\upsilon(t, x, y)$ will be independent of both t and x for $t \in [0, 1)$.

By a *diffeomorphism* of the stretched wedges, we shall now mean any diffeomorphism $\tilde{\delta} : (t, x, y) \mapsto (\tau, \chi, \upsilon)$ of a neighborhood of $\mathbb{Z}$ to a neighborhood of $\mathbb{W}$ in $\mathbb{R} \times S^n \times \mathbb{R}^q$, such that $\tilde{\delta}(\mathbb{Z}) = \mathbb{W}$ and the restriction of $\tilde{\delta}$ to the base of $\mathbb{Z}$ preserves the cone bases.

Fix a diffeomorphism $\tilde{\delta}$ of $\mathbb{Z} \to \mathbb{W}$. By the above, both conditions (2.2.12) are fulfilled.

Lemma 2.2.7 *Given any $R > 0$ and compact sets $K_1 \subset Z_1$ and $K_2 \subset Z_2$, there is a constant $c > 0$ such that*

$$c \le |J(t, x, y)| \le \frac{1}{c} \qquad \text{for all} \;\; (t, x, y) \in [0, R] \times K_1 \times K_2,$$

where $J = \det \frac{\partial(\tau, \chi, \upsilon)}{\partial(t, x, y)}$ is the Jacobian of $\tilde{\delta}$.

Proof. By assumption, the Jacobian matrix $\frac{\partial(\tau, \chi, \upsilon)}{\partial(t, x, y)}$ is non-singular near the compact set $[0, R] \times K_1 \times K_2$. Hence our statement follows by a familiar argument. □

We now invoke the first condition of (2.2.12) to derive one of two main properties of $\tilde{\delta}$.

Lemma 2.2.8 *Assume that K_i is a compact subset of Z_i, for $i = 1, 2$. Then there are an $R > 0$ and a C^∞ function F_0 in a neighborhood of $[0, R] \times K_1 \times K_2$, such that*

$$\tau(t, x, y) = t\, e^{F_0(t, x, y)} \qquad \text{for all} \;\; (t, x, y) \in [0, R] \times K_1 \times K_2.$$

Proof. Indeed, $\tau(0,x,y)=0$ implies

$$\begin{aligned}\tau(t,x,y) &= \int_0^1 \frac{\partial}{\partial\theta}(\tau(\theta t,x,y))\,d\theta \\ &= t\int_0^1 \frac{\partial\tau}{\partial t}(\theta t,x,y)\,d\theta.\end{aligned}$$

We are going to set

$$F_0(t,x,y)=\ln\int_0^1 \frac{\partial\tau}{\partial t}(\theta t,x,y)\,d\theta. \tag{2.2.13}$$

The only point remaining concerns the behavior of the derivative $\frac{\partial\tau}{\partial t}$ for $t>0$ small enough. More precisely, our objective is to show that this derivative is bounded from below by a positive constant.

As the Jacobian matrix

$$\frac{\partial(\tau,\chi,v)}{\partial(t,x,y)}=\begin{pmatrix}\frac{\partial\tau}{\partial t} & \frac{\partial\tau}{\partial x} & \frac{\partial\tau}{\partial y}\\ \frac{\partial\chi}{\partial t} & \frac{\partial\chi}{\partial x} & \frac{\partial\chi}{\partial y}\\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}$$

is non-singular on $\overline{\mathbb{R}_+}\times Z_1\times Z_2$ and

$$\begin{aligned}\tfrac{\partial\tau}{\partial x}(0,x,y)&=0,\\ \tfrac{\partial\tau}{\partial y}(0,x,y)&=0\end{aligned}$$

for all $x\in Z_1$, $y\in Z_2$, we deduce that $\frac{\partial\tau}{\partial t}(0,x,y)\neq 0$ for those x and y. Even $\frac{\partial\tau}{\partial t}(0,x,y)>0$, because the $\mathbb{R}_+$-direction is preserved. Hence it follows by the compactness of $K_1\times K_2$ that there are positive constants R and ε such that

$$\varepsilon\le\frac{\partial\tau}{\partial t}(t,x,y)\le\frac{1}{\varepsilon}\qquad\text{for all }\ (t,x,y)\in[0,R]\times K_1\times K_2.$$

Thus, the function F_0 given by (2.2.13) possesses all the desired properties, and the proof is complete.

□

The same reasoning applies to the component $v(t,x,y)$ of $\tilde{\delta}$, when using the second condition (2.2.12).

Lemma 2.2.9 *There exist a diffeomorphism $v=F_1(y)$ of $Z_2\to W_2$ and a C^{infty} function F_2 on a neighborhood of $\mathbb{Z}$ with values in $\mathbb{R}^q$, such that*

$$v(t,x,y)=F_1(y)+tF_2(t,x,y)\qquad\textit{for all }\ (t,x,y)\in\mathbb{Z}.$$

Proof. Indeed,

$$\begin{aligned}v(t,x,y) &= v(0,x,y)+\int_0^1\frac{\partial}{\partial\theta}(v(\theta t,x,y))\,d\theta\\ &= v(0,x,y)+t\int_0^1\frac{\partial v}{\partial t}(\theta t,x,y)\,d\theta.\end{aligned}$$

Set

$$\begin{array}{rcl} F_1(y) & = & v(0,x,y), \\ F_2(t,x,y) & = & \int_0^1 \frac{\partial v}{\partial t}(\theta t,x,y)\,d\theta \end{array}$$

(that F_1 does not depend on x follows from (2.2.12)).

Then both F_1 and F_2 are C^∞ functions on a neighborhood of $\mathbb{Z}$ with values in $\mathbb{R}^q$. Moreover, since

$$\begin{aligned} &\tilde{\delta}((\{0\} \times Z_1) \times Z_2) \\ &\quad = \{(\tau(0,x,y), \chi(0,x,y), v(0,x,y)) : \ (x,y) \in Z_1 \times Z_2\} \\ &\quad = \{(0, \chi(0,x,y), F_1(y)) : \ (x,y) \in Z_1 \times Z_2\} \\ &\quad = (\{0\} \times W_1) \times W_2 \end{aligned}$$

and

$$\frac{\partial(\tau,\chi,v)}{\partial(t,x,y)}(0,x,y) = \begin{pmatrix} e^{F_0(0,x,y)} & 0 & 0 \\ \frac{\partial \chi}{\partial t}(0,x,y) & \frac{\partial \chi}{\partial x}(0,x,y) & \frac{\partial \chi}{\partial y}(0,x,y) \\ F_2(0,x,y) & 0 & \frac{\partial F_1}{\partial y}(y) \end{pmatrix}, \tag{2.2.14}$$

we can assert that F_1 is a diffeomorphism of $Z_2 \to W_2$, as desired.

□

The class of diffeomorphisms we obtained in the course of the proof seems to be of independent interest.

Lemma 2.2.10 *As defined above, the wedge diffeomorphisms form a group, i.e., the composition of any two diffemorphisms is a diffeomorphism and the inverse to any diffeomorphism is a diffeomorphism.*

Proof. This is evident by a purely geometric reasoning. The analytical proof is also straightforward.

□

2.2.6 *The invariance*

Every function $u \in \mathcal{W}^{s,\gamma}_{loc}(\mathbf{W})$ defines a continuous linear functional on $C^\infty_{comp}(int\,\mathbf{W})$, and so is a distribution in the interior of $\mathbf{W}$. For this reason, given any diffeomorphism $\tilde{\delta} : \mathbb{Z} \to \mathbf{W}$, we may define the *pull-back* $\tilde{\delta}^* u = u \circ \tilde{\delta}$ of u under $\tilde{\delta}$ in the usual way. The $\tilde{\delta}^* u$ is a distribution on $int\,\mathbb{Z}$, and the question arises whether it still belongs to $\mathcal{W}^{s,\gamma}_{loc}(\mathbb{Z})$.

The following result answers this question.

Theorem 2.2.11 *Let $\tilde{\delta}$ be a diffeomorphism of $\mathbb{Z} \to \mathbf{W}$. Then $\tilde{\delta}^* u \in \mathcal{W}^{s,\gamma}_{loc}(\mathbb{Z})$ for any $u \in \mathcal{W}^{s,\gamma}_{loc}(\mathbf{W})$.*

Proof. By a purely formal argument, it suffices to prove that if $u \in \mathcal{W}^{s,\gamma}(\mathbf{W})$ vanishes away from a compact subset of $\mathbf{W}$, then $\tilde{\delta}^* u \in \mathcal{W}^{s,\gamma}(\mathbf{Z})$. Moreover, because of interpolation and duality we may assume that s is a non-negative integer.

Fix a $u \in \mathcal{W}^{s,\gamma}_{comp}(\mathbf{W})$. As $\tilde{\delta} : \mathbf{Z} \to \mathbf{W}$ is a proper mapping, there are compact sets $K_1 \subset Z_1$ and $K_2 \subset Z_2$ such that $\tilde{\delta}^{-1}(supp\, u) \subset \overline{\mathbb{R}_+} \times K_1 \times K_2$. Hence it follows that the pull-back $\tilde{\delta}^* u$ is supported in $\overline{\mathbb{R}_+} \times K_1 \times K_2$.

Let $\tilde{\delta} = (\tau, \chi, v)$, and let R be the number of Lemma 2.2.8. Analysis similar to that in the proof of Theorem 1.2.3 shows that there is an $A > 0$ such that from $(t, x, y) \in \mathbf{Z}$ and $\tau(t, x, y) \in [0, A)$ it follows that $t \in [0, R)$.

Pick a cut-off function ω with a support on $[0, A)$, which is equal to 1 for $\tau \le a$, where $0 < a < A$. Write $u = u_1 + u_2$, with $u_1 = \omega u$ and $u_2 = (1-\omega)u$.

The function $u_2 \in \mathcal{H}^s(\mathbf{W})$ is supported away from $([0,a) \times W_1) \times W_2$. Since the invariance of the usual Sobolev spaces is well-known, it follows from Lemma 2.1.4 that $\tilde{\delta}^* u_2 \in \mathcal{H}^s(\mathbf{Z})$, provided that either u vanishes outside of a compact subset of $\mathbf{W}$ (which is the case) or $\tilde{\delta}$ behaves well at the "infinity" of $\mathbf{Z}$. Thus, Proposition 2.1.19 shows that in fact $\tilde{\delta}^* u_2 \in \mathcal{W}^{s,\gamma}(\mathbf{Z})$ because $\tilde{\delta}^* u_2$ vanishes for $t > 0$ small enough.

It remains to prove that $\tilde{\delta}^* u_1 \in \mathcal{W}^{s,\gamma}(\mathbf{Z})$. To do this, we shall write u instead of u_1 and assume that u is supported on $([0,A) \times W_1) \times W_2$.

As $\mathcal{F}_{y\mapsto\eta}(tD_y)^\beta = (t\eta)^\beta \mathcal{F}_{y\mapsto\eta}$ and $|(t\eta)^\beta| \le const\,(\beta)$ for $\langle\eta\rangle \le \frac{a}{t}$, we can assert that

$$
\begin{aligned}
&\|\tilde{\delta}^* u\|^2_{\mathcal{W}^{s,\gamma}(\mathbf{Z})}\\
&= \sum_{\alpha_0+|\alpha|\le s} \int_0^\infty \int_{\mathbb{R}^n_x} \left(\int_{\langle\eta\rangle\le\frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2\gamma}} \left|(tD_t)^{\alpha_0} D_x^\alpha \mathcal{F}_{y\mapsto\eta}\tilde{\delta}^* u\right|^2 d\eta \right.\\
&\qquad \left. + \int_{\langle\eta\rangle\ge\frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2(\alpha_0+|\alpha|)}} \left|(tD_t)^{\alpha_0} D_x^\alpha \mathcal{F}_{y\mapsto\eta}\tilde{\delta}^* u\right|^2 d\eta \right) t^n dt dx\\
&\sim \sum_{\alpha_0+|\alpha|+|\beta|\le s} \int_0^\infty \int_{\mathbb{R}^n_x} \left(\int_{\langle\eta\rangle\le\frac{a}{t}} \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2\gamma}} \left|(tD_t)^{\alpha_0} D_x^\alpha \mathcal{F}_{y\mapsto\eta}((tD_y)^\beta\tilde{\delta}^* u)\right|^2 d\eta \right.\\
&\qquad \left. + \int_{\langle\eta\rangle\ge\frac{a}{t}} \frac{1}{t^{2(\alpha_0+|\alpha|+|\beta|)}} \left|(tD_t)^{\alpha_0} D_x^\alpha \mathcal{F}_{y\mapsto\eta}((tD_y)^\beta\tilde{\delta}^* u)\right|^2 d\eta \right) t^n dt dx.
\end{aligned}
$$

Denote by $\chi_{\langle\eta\rangle\le\frac{a}{t}}(\eta)$ and $\chi_{\langle\eta\rangle\ge\frac{a}{t}}(\eta)$ the characteristic functions of the sets $\langle\eta\rangle \le \frac{a}{t}$ and $\langle\eta\rangle \ge \frac{a}{t}$ in $\mathbb{R}^q$ respectively. Given any $N = 0, 1, \ldots$, let

$$
\omega^{(N)}(t,\eta) = \frac{\langle\eta\rangle^{2s}}{(t\langle\eta\rangle)^{2\gamma}} \chi_{\langle\eta\rangle\le\frac{a}{t}}(\eta) + \frac{1}{t^{2N}} \chi_{\langle\eta\rangle\ge\frac{a}{t}}(\eta), \tag{2.2.15}
$$

then from the above it follows that

$$
\begin{aligned}
&\|\tilde{\delta}^* u\|^2_{\mathcal{W}^{s,\gamma}(\mathbf{Z})} \sim \sum_{\alpha_0+|\alpha|+|\beta|\le s}\\
&\int_0^\infty \int_{\mathbb{R}^n_x} \left(\int_{\mathbb{R}^q_\eta} \left|\mathcal{F}_{y\mapsto\eta}\left((tD_t)^{\alpha_0} D_x^\alpha (tD_y)^\beta \tilde{\delta}^* u\right)\right|^2 \omega^{(\alpha_0+|\alpha|+|\beta|)}(t,\eta) d\eta \right) t^n dt dx.
\end{aligned}
\tag{2.2.16}
$$

On the other hand, applying the chain rule and Lemmas 2.2.8 and 2.2.9 gives

$$\begin{aligned} t\frac{\partial}{\partial t}(u\circ\tilde\delta) &= \Big(1+t\frac{\partial F_0}{\partial t}\Big)\Big(\tau\frac{\partial u}{\partial \tau}\Big)\circ\tilde\delta+\sum_{j=1}^{n} t\frac{\partial \chi_j}{\partial t}\Big(\frac{\partial u}{\partial \chi_j}\Big)\circ\tilde\delta+\sum_{j=1}^{q} e^{-F_0}\frac{\partial v_j}{\partial t}\Big(\tau\frac{\partial u}{\partial v_j}\Big)\circ\tilde\delta,\\ \frac{\partial}{\partial x_i}(u\circ\tilde\delta) &= \frac{\partial F_0}{\partial x_i}\Big(\tau\frac{\partial u}{\partial \tau}\Big)\circ\tilde\delta+\sum_{j=1}^{n} \frac{\partial \chi_j}{\partial x_i}\Big(\frac{\partial u}{\partial \chi_j}\Big)\circ\tilde\delta+\sum_{j=1}^{q} e^{-F_0}\frac{\partial F_{2j}}{\partial x_i}\Big(\tau\frac{\partial u}{\partial v_j}\Big)\circ\tilde\delta,\\ t\frac{\partial}{\partial y_k}(u\circ\tilde\delta) &= t\frac{\partial F_0}{\partial y_k}\Big(\tau\frac{\partial u}{\partial \tau}\Big)\circ\tilde\delta+\sum_{j=1}^{n} \tau\frac{\partial \chi_j}{\partial y_k}\Big(\frac{\partial u}{\partial \chi_j}\Big)\circ\tilde\delta+\sum_{j=1}^{q} e^{-F_0}\frac{\partial v_j}{\partial y_k}\Big(\tau\frac{\partial u}{\partial v_j}\Big)\circ\tilde\delta, \end{aligned}$$

for $i=1,\dots,n$ and $k=1,\dots,q$. Therefore,

$$\begin{aligned} &(tD_t)^{\alpha_0}D_x^{\alpha}(tD_y)^{\beta}(u\circ\tilde\delta)\\ &= \sum_{\gamma_0+|\gamma|+|\vartheta|\le\alpha_0+|\alpha|+|\beta|} c_{\gamma_0,\gamma,\vartheta}^{\alpha_0,\alpha,\beta}(t,x,y)\left((\tau D_\tau)^{\gamma_0}D_\chi^{\gamma}(\tau D_v)^{\vartheta}u\right)\circ\tilde\delta, \end{aligned}$$

the coefficients $c_{\gamma_0,\gamma,\vartheta}^{\alpha_0,\alpha,\beta}$ being infinitely differentiable near $[0,R]\times K_1\times K_2$.

We now invoke the elementary equality

$$v\,\partial^{\alpha}u=\sum_{\beta\le\alpha}(-1)^{|\beta|}\binom{\alpha}{\beta}\partial^{\beta}(\partial^{\alpha-\beta}v\;u),$$

to rewrite this in the form

$$\begin{aligned} &(tD_t)^{\alpha_0}D_x^{\alpha}(tD_y)^{\beta}(u\circ\tilde\delta)\\ &= \sum_{\gamma_0+|\gamma|+|\vartheta|\le\alpha_0+|\alpha|+|\beta|} \left((\tau D_\tau)^{\gamma_0}D_\chi^{\gamma}(\tau D_v)^{\vartheta}\left(c_{\gamma_0,\gamma,\vartheta}^{\alpha_0,\alpha,\beta}\,u\right)\right)\circ\tilde\delta \end{aligned} \tag{2.2.17}$$

with some new C^∞ functions $c_{\gamma_0,\gamma,\vartheta}^{\alpha_0,\alpha,\beta}(\tau,\chi,v)$ defined near the support of u. There is no loss of generality in assuming that $\left(c_{\gamma_0,\gamma,\vartheta}^{\alpha_0,\alpha,\beta}\right)$ are C^∞ functions of compact support on the whole wedge $\mathbf{W}$, for if not, we modify them away from a neighborhood of *supp* u.

If $\langle\eta\rangle\ge\frac{a}{t}$, then

$$\begin{aligned} \frac{1}{t^{2(\alpha_0+|\alpha|+|\beta|}} &= \frac{1}{t^{2(\gamma_0+|\gamma|+|\vartheta|}}\,\frac{\langle\eta\rangle^{2((\alpha_0+|\alpha|+|\beta|)-(\gamma_0+|\alpha|+|\vartheta|)}}{(t\langle\eta\rangle)^{2((\alpha_0+|\alpha|+|\beta|)-(\gamma_0+|\gamma|+|\vartheta|)}}\\ &\le \frac{1}{a^{2((\alpha_0+|\alpha|+|\beta|)-(\gamma_0+|\gamma|+|\vartheta|)}}\,\frac{1}{t^{2(\gamma_0+|\gamma|+|\vartheta|}}\,\langle\eta\rangle^{2((\alpha_0+|\alpha|+|\beta|)-(\gamma_0+|\alpha|+|\vartheta|)}. \end{aligned}$$

Substituting (2.2.17) to (2.2.16) we thus get

$$\begin{aligned} &\|\tilde\delta^{*}u\|^2_{\mathcal{W}^{s,\gamma}(\mathbf{Z})}\le const\sum_{\gamma_0+|\gamma|+|\vartheta|\le s}\\ &\int_0^\infty\int_{\mathbb{R}^n_x}\left(\int_{\mathbb{R}^q_\eta}\left|\mathcal{F}_{y\mapsto\eta}\tilde\delta^{*}\left((\tau D_\tau)^{\gamma_0}D_\chi^{\gamma}(\tau D_v)^{\vartheta}(c_{\gamma_0,\gamma,\vartheta}u)\right)\right|^2\omega^{(\gamma_0+|\gamma|+|\vartheta|)}(t,\eta)d\eta\right)t^n dt dx, \end{aligned} \tag{2.2.18}$$

$(c_{\gamma_0,\gamma,\vartheta})$ being C^∞ functions of compact support in $\mathbf{W}$.

Having disposed of this preliminary step, we can now highlight the main technical idea of the proof.

Lemma 2.2.12 *For any compact set $K \subset [0, A) \times W_1 \times W_2$ there is a constant $c > 0$ such that*

$$\int_0^\infty \int_{\mathbb{R}^n_x} \left(\int_{\mathbb{R}^q_\eta} \left| \mathcal{F}_{y \mapsto \eta}(v \circ \tilde{\delta}) \right|^2 \omega^{(N)}(t, \eta) d\eta \right) t^n dt dx$$
$$\leq c \int_0^\infty \int_{\mathbb{R}^n_\chi} \left(\int_{\mathbb{R}^q_\eta} |\mathcal{F}_{\upsilon \mapsto \eta} v|^2 \omega^{(N)}(\tau, \eta) d\eta \right) \tau^n d\tau d\chi \tag{2.2.19}$$

for all $v \in C^\infty_{comp}(\mathbf{W})$ with support in K.

Assuming this estimate, we show how to complete the proof of the theorem. To this end, apply Lemma 2.2.12 to each function

$$v = (\tau D_\tau)^{\gamma_0} D_\chi^\gamma (\tau D_\upsilon)^\vartheta (c_{\gamma_0,\gamma,\vartheta} u)$$

on the right-hand side of (2.2.18) and $N = \gamma_0 + |\gamma| + |\vartheta|$. (That (2.2.19) is valid for v of finite smoothness, can be seen by a passage to the limit.) This yields

$$\|\tilde{\delta}^* u\|^2_{\mathcal{W}^{s,\gamma}(\mathbf{Z})} \leq c \sum_{\gamma_0 + |\gamma| + |\vartheta| \leq s}$$
$$\int_0^\infty \int_{\mathbb{R}^n_\chi} \left(\int_{\mathbb{R}^q_\eta} \left| \mathcal{F}_{\upsilon \mapsto \eta} (\tau D_\tau)^{\gamma_0} D_\chi^\gamma (\tau D_\upsilon)^\vartheta (c_{\gamma_0,\gamma,\vartheta} u) \right|^2 \omega^{(\gamma_0 + |\gamma| + |\vartheta|)}(\tau, \eta) d\eta \right) \tau^n d\tau d\chi.$$

By Theorem 2.2.4, every summand on the right-hand side here is majorized by a multiple of $\|u\|^2_{\mathcal{W}^{s,\gamma}(\mathbf{W})}$, with a constant depending only on $(c_{\gamma_0,\gamma,\vartheta})$. Summarizing we see that

$$\|\tilde{\delta}^* u\|_{\mathcal{W}^{s,\gamma}(\mathbf{Z})} \leq c \, \|u\|_{\mathcal{W}^{s,\gamma}(\mathbf{W})},$$

the constant c being independent of u provided u is supported in a fixed compact subset of $[0, A) \times W_1 \times W_2$. This is the desired conclusion.

□

Thus, the theorem will be proved once we prove Lemma 2.2.12.

Proof of Lemma 2.2.12. The proof lies beyond the range of the paper. This is a standard result on the L^2-boundedness of pseudodifferential operators. We refer the reader to Calderon and Vaillancourt [4], Coifman and Meyer [6], Kato [12], Kumano-go [13], Beals [1], Cordes [7], Hwang [11], Hunt, Muckenhoupt and Wheeden [10], Coifman and Fefferman [5] and others.

□

2.3 Sobolev Spaces on a Manifold with Edges

We now turn to the weighted Sobolev spaces on arbitrary manifolds with edge-like singularities.

2.3.1 *Manifolds with edges*

To be short, we begin with a definition.

Definition 2.3.1 *By a manifold with edges, we mean a topological (second countable) Hausdorff space W with an exceptional set $Y \subset W$ ("singularities") such that*

- *Y is a compact manifold of dimension $q \geq 1$;*
- *$W \setminus Y$ is an $(1+n+q)$-dimensional manifold;*
- *for each point $y \in Y$ there exist a neighborhood $\mathcal{N}$ in W and a homeomorphism $\phi : \mathcal{N} \to \frac{[0,1)\times X}{\{0\}\times X} \times U$, for X a compact manifold of dimension n and U an open subset of $\mathbb{R}^q$, such that ϕ restricts to diffeomorphisms of $\mathcal{N} \setminus Y \to (0,1) \times X \times U$ and $\mathcal{N} \cap Y \to \frac{\{0\}\times X}{\{0\}\times X} \times U$.*

We may easily generalize the notion of manifolds with edges by allowing Y to be the disjoint union of components of different dimensions. This causes trivial modifications of our considerations. For similar reasons, there is no loss of generality in assuming that Y is connected. Then the cone bases $X(v_1)$ and $X(v_2)$ for different $v_1, v_2 \in Y$ are diffeomorphic. Hence we may simply talk about X.

We may always think of X embedded to the unit sphere S^N for N large enough, even for $N = n$ if we consider local embeddings. If it is the case, the product $\frac{[0,1)\times X}{\{0\}\times X} \times U$ is identified with the model wedge $\{\lambda p : \lambda \in [0,1), p \in X\} \times U$ in the obvious way.

Our next goal is to determine a "*wedge structure*" on W close to each point $v \in Y$.

Given any two diffeomorphisms

$$\begin{aligned} \phi_1 &: \mathcal{N} \setminus Y \to (0,1) \times X \times U_1, \\ \phi_2 &: \mathcal{N} \setminus Y \to (0,1) \times X \times U_2, \end{aligned}$$

the composition $\phi_2 \circ \phi_1^{-1}$ is a diffeomorphism of $(0,1) \times X \to (0,1) \times X \times U_1 \to (0,1) \times X \to (0,1) \times X \times U_2$. We say that ϕ_1 and ϕ_2 are *equivalent* if $\phi_2 \circ \phi_1^{-1}$ is the restriction of some diffeomorphism $(-1,1) \times X \times U_1 \to (-1,1) \times X \times U_2$.

The system of the equivalence classes of such diffeomorphisms, when v varies over Y, is regarded as a part of the structure of $W \setminus Y$ near Y. It is kept fixed and determines the "*wedge structure*" on W close to $v \in Y$ via the local $\mathbb{R}_+$ action on $(t,x,y) \in (0,1) \times X \times U$, i.e., $\lambda(t,x,y) = (\lambda t, x, y)$ for all $\lambda \in \mathbb{R}_+$ with $\lambda t \in (0,1)$.

Note that W is no manifold near Y, unless the base of the model cone is the sphere S^n. Nevertheless we will talk about manifolds with edges because the analysis takes place on $W \setminus Y$.

2.3.2 *Stretched manifolds*

The analysis on a manifold with edges has always referred to the corresponding *"stretched manifold."*

Proposition 2.3.2 *For any manifold W with edge Y there is a smooth manifold with boundary $\mathbf{W}$ such that*

1) $W \setminus Y$ is diffeomorphic to $\mathbf{W} \setminus \partial\mathbf{W}$; and

2) there is a neighborhood $\mathcal{N}$ of Y in W and a collar neighborhood $\mathbb{N} \simeq \partial\mathbf{W} \times [0,1)$ of $\partial\mathbf{W}$ in $\mathbf{W}$ such that $\mathcal{N} \setminus Y$ is diffeomorphic to $\partial\mathbf{W} \times (0,1)$.

Proof. We construct $\mathbf{W}$ by replacing, for every singularity v, the neighborhood $\mathcal{N}$ in Definition 2.3.1 by $(0,1) \times X \times U$ via gluing with any one of the diffeomorphisms ϕ. We even get $\partial\mathbf{W} = \cup_{v \in Y} X_v$, the subscript v pointing to the dependence of X on v.

□

This manifold $\mathbf{W}$ is called the *"stretched object"* associated with W.

Example 2.3.3 Assume that K is a manifold with conical singularities, and U is an open subset of $\mathbb{R}^q$. Then $W = K \times U$ is a manifold with edges, and we have $\mathbf{W} = \mathbb{K} \times U$.

□

The important point to note here is the form of the *transition diffeomorphisms* close to the boundary of the manifold $\mathbf{W}$. By definition, every point of $\partial\mathbf{W}$ has a neighborhood $\mathcal{N}$ in $\mathbf{W}$, which is diffeomorphic to $\overline{\mathbb{R}_+} \times \mathbb{R}^n \times R^q$. If

$$\begin{aligned} \phi' :\ & \mathcal{N}' \to \overline{\mathbb{R}_+} \times \mathbb{R}^n \times R^q, \\ \phi'' :\ & \mathcal{N}'' \to \overline{\mathbb{R}_+} \times \mathbb{R}^n \times R^q \end{aligned}$$

are two such diffeomorphisms, then the composition $\phi' \circ (\phi'')^{-1}$ is a diffeomorphism of $\phi''(\mathcal{N}' \cap \mathcal{N}'') \to \phi'(\mathcal{N}' \cap \mathcal{N}'')$ of the form

$$(t, x, y) \mapsto (\tau(t,x,y), \chi(t,x,y), v(t,x,y)),$$

where

$$\begin{aligned} &\tau(0,x,y) \equiv 0, \\ &v(0,x,y) \textit{ does not depend on } x. \end{aligned}$$

To see this, we recall that t and x enter the function v only through the aggregate $t\,p(x)$, where $p(x) \in S^N$.

2.3.3 *Definition of Sobolev spaces*

Assume that W is a compact manifold of dimension $1+n+q$ with edges, and $\mathbf{W}$ is the corresponding "stretched object."

The definition of weighted Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbf{W})$ is based, as usually, on the standard localization procedure.

By the definition of $\mathbf{W}$, each interior point of $\mathbf{W}$ has a neighborhood which is diffeomorphic to $\mathbb{R}^{1+n+q}$. And every point of $\partial\mathbf{W}$ possesses a neighborhood which is diffeomorphic to $\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q$.

Since $\mathbf{W}$ is compact, there exists a finite covering $\{\mathcal{N}_\nu\}$ of $\mathbf{W}$ by open subsets each of them lies in a coordinate patch on $\mathbf{W}$. It follows that, for $\mathcal{N}_\nu \cap \partial\mathbf{W} = \emptyset$, we have a diffeomorphism $\phi_\nu : \mathcal{N}_\nu \to \mathbb{R}^{1+n+q}$. If $\mathcal{N}_\nu \cap \partial\mathbf{W} \neq \emptyset$, then we have a diffeomorphism $\phi_\nu : \mathcal{N}_\nu \to \overline{\mathbb{R}_+} \times \mathbb{R}^n \times R^q$.

Let $\{\varphi_\nu\}$ be a C^∞ partition of unity on $\mathbf{W}$ subordinated to the covering $\{\mathcal{N}_\nu\}$.

As expected, given any $s, \gamma \in \mathbb{R}$, the space $\mathcal{W}^{s,\gamma}(\mathbf{W})$ is defined to consist of all distributions u in the interior of $\mathbf{W}$ such that, for every ν, the product $(\varphi_\nu u) \circ \phi_\nu^{-1}$ belongs to either $H^s(\mathbb{R}^{1+n+q})$, if $\mathcal{N}_\nu \cap \partial\mathbb{K} = \emptyset$, or $\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$, if $\mathcal{N}_\nu \cap \partial\mathbb{K} \neq \emptyset$.

Proposition 2.3.4 *As defined above, the space $\mathcal{W}^{s,\gamma}(\mathbf{W})$ is independent of the particular choice of the covering $\{\mathcal{N}_\nu\}$, diffeomorphisms $\{\phi_\nu\}$, and the partition of unity $\{\varphi_\nu\}$.*

Proof. Let

$$T' = \{\mathcal{N}'_\mu,\ \phi'_\mu,\ \varphi'_\mu\},$$
$$T'' = \{\mathcal{N}''_\nu,\ \phi''_\nu,\ \varphi''_\nu\}$$

be two triples as above.

We introduce the temporary notation $\mathcal{W}^{s,\gamma}(\mathbf{W}, T')$ or $\mathcal{W}^{s,\gamma}(\mathbf{W}, T'')$ for the space $\mathcal{W}^{s,\gamma}(\mathbf{W})$ corresponding to either triple. We shall have established the proposition if we prove the following: if $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, T')$, then $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, T'')$.

To this end, pick a $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, T')$. Given any number ν, write

$$\begin{aligned} (\varphi''_\nu u) \circ (\phi''_\nu)^{-1} &= \left(\varphi''_\nu \left(\sum_\mu \varphi'_\mu\right) u\right) \circ (\phi''_\nu)^{-1} \\ &= \sum_\mu \left(\phi'_\mu \circ (\phi''_\nu)^{-1}\right)^* \left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}. \end{aligned}$$

The task is now to show that every summand on the right-hand side here is in $H^s(\mathbb{R}^{1+n+q})$, if $\mathcal{N}''_\nu \cap \partial\mathbb{K} = \emptyset$, or in $\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$, if $\mathcal{N}''_\nu \cap \partial\mathbb{K} \neq \emptyset$. We give the proof only for the latter case, i.e., when $\mathcal{N}''_\nu \cap \partial\mathbb{K} \neq \emptyset$. Similar considerations apply to the first case, and will only refer to the invariance of the usual Sobolev spaces.

We can certainly assume that $\mathcal{N}'_\mu \cap \mathcal{N}''_\nu \neq \emptyset$, because otherwise $\varphi''_\nu \varphi'_\mu \equiv 0$ and so the corresponding (zero!) summand is obviously in $\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$.

Suppose that $\mathcal{N}'_\mu \cap \partial\mathbb{K} = \emptyset$. By condition, $\left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}$ belongs to $H^s_{comp}(\mathbb{R}^{1+n+q})$. As $\phi'_\mu \circ (\phi''_\nu)^{-1}$ is a diffeomorphism of

$$\left(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q\right) \cap \phi''_\nu \left(\mathcal{N}'_\mu \cap \mathcal{N}''_\nu\right) \to \mathbb{R}^{1+n+q},$$

it follows that the pull-back of $\left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}$ under this diffeomorphism belongs to $H^s_{comp}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q)$. Applying Lemma 2.1.4 and Proposition 2.1.19 we can assert that

$$\left(\phi'_\mu \circ (\phi''_\nu)^{-1}\right)^* \left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1} \in \mathcal{H}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q),$$

as desired.

It remains to consider the case when $\mathcal{N}'_\mu \cap \partial \mathbf{W} \neq \emptyset$. By condition, the function $\left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}$ belongs to $\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$. Moreover, it vanishes away from a compact subset of $\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q$. Since $\phi'_\mu \circ (\phi''_\nu)^{-1}$ is a diffeomorphism of

$$\left(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q\right) \cap \phi''_\nu \left(\mathcal{N}'_\mu \cap \mathcal{N}''_\nu\right) \to \overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q,$$

it follows from Theorem 2.2.11 that the pull-back of $\left(\left(\varphi''_\nu \varphi'_\mu\right) u\right) \circ \left(\phi'_\mu\right)^{-1}$ under this diffeomorphism belongs to $\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)$. This completes the proof.

□

Another way of stating Proposition 2.3.4 is to say that these weighted Sobolev spaces make sense on a manifold with conical singularities.

The space $\mathcal{W}^{s,\gamma}(\mathbf{W})$, when endowed with the norm

$$\|u\|_{\mathcal{W}^{s,\gamma}(\mathbf{W})} = \left(\sum_{\nu:\, \mathcal{N}_\nu \cap \partial \mathbf{W} \neq \emptyset} \|(\varphi_\nu u) \circ \phi_\nu^{-1}\|^2_{\mathcal{W}^{s,\gamma}(\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q)} + \sum_{\nu:\, \mathcal{N}_\nu \cap \partial \mathbf{W} = \emptyset} \|(\varphi_\nu u) \circ \phi_\nu^{-1}\|^2_{H^s(\mathbb{R}^{1+n+q})} \right)^{\frac{1}{2}},$$

is a Banach space and even a Hilbert space.

Moreover, analysis similar to that in the proof of Proposition 2.3.4 shows that the norm $\|\cdot\|_{\mathcal{W}^{s,\gamma}(\mathbf{W})}$ is independent, up to an equivalent norm, of the particular choice of the covering $\{\mathcal{N}_\nu\}$, diffeomorphisms $\{\phi_\nu\}$, and the partition of unity $\{\varphi_\nu\}$.

2.3.4 *Properties*

The spaces $\mathcal{W}^{s,\gamma}(\mathbf{W})$ have actually the same properties as the weighted Sobolev spaces on a wedge of Subsection 2.1.8.

2.4 Sobolev Sections of Vector Bundles

From now on we assume that W is a compact manifold of dimension $1+n+q$ with an q-dimensional edge Y. As above, we denote by $\mathbf{W}$ the corresponding stretched manifold.

2.4.1 *Sections of vector bundles*

Let B be a differentiable $\mathbb{C}$-vector bundle of rank k over $\mathbf{W}$, or, more concisely, $\pi : B \to \mathbf{W}$.

By $\sigma(B)$ we denote the vector space of all sections of B, that is, all mappings $u : \mathbf{W} \to B$ such that the composition πu gives the identity mapping of $\mathbf{W}$. In other words u maps a point $p \in \mathbf{W}$ into the fiber B_p of the bundle B over this point.

Locally each section of B is represented by a vector-function on an open subset of either the space $\mathbb{R}^{1+n+q}$ or the subspace $\overline{\mathbb{R}_+} \times \mathbb{R}^n \times \mathbb{R}^q$ with values in $\mathbb{C}^k$. How does this happen?

We suppose that $\mathcal{N}$ is some coordinate neighborhood on $\mathbf{W}$ over which B is trivial, and let $t : \ B|_{\mathcal{N}} \to \mathcal{N} \times \mathbb{C}^k$ be this trivialization. For $u \in \sigma(B)$, the composition

$$\mathcal{N} \xrightarrow{u} B|_{\mathcal{N}} \xrightarrow{t} \mathcal{N} \times \mathbb{C}^k \xrightarrow{proj} \mathbb{C}^k$$

defines a vector-function $u_{\mathcal{N}} : \mathcal{N} \to \mathbb{C}^k$ that is called the representation of u in $\mathcal{N}$.

Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be some coordinate neighborhoods on $\mathbf{W}$ with $\mathcal{N}_1 \cap \mathcal{N}_2 \neq \emptyset$, and let B be trivial over these neighborhoods with trivializations t_1 and t_2 respectively. Then the representations u_1 and u_2 of the section u in $\mathcal{N}_1$ and $\mathcal{N}_2$ are connected in the intersection $\mathcal{N}_1 \cap \mathcal{N}_2$ by means of the equality $u_2 = t_{21} u_1$. Here $t_{21} : \ \mathcal{N}_1 \cap \mathcal{N}_2 \to GL(k, \mathbb{C})$ are the transition matrices for the bundle B induced by the mapping $t_2 t_1^{-1} : (\mathcal{N}_1 \cap \mathcal{N}_2) \times \mathbb{C}^k \to (\mathcal{N}_1 \cap \mathcal{N}_2) \times \mathbb{C}^k$.

The smoothness of the transition matrices is determined by the smoothness of the bundle B so an analysis of differentiability properties of sections of vector bundles is reduced to an investigation of such properties for their local representations. Moreover, in each neighborhood it is sufficient to limit oneself to some one representation.

The use of local representations also gives a way to define a topology in spaces of differentiable sections of B. We shall illustrate this by the example of weighted Sobolev sections of B.

2.4.2 *Definition*

The definition of weighted Sobolev spaces $\mathcal{W}^{s,\gamma}(\mathbf{W}, B)$ is based, as usually, on the standard localization procedure.

Fix a finite coordinate covering $\{\mathcal{N}_\nu\}$ of the manifold $\mathbf{W}$, such that B is trivial over each $\mathcal{N}_\nu$. Pick also some trivializations $t_\nu : \ B|_{\mathcal{N}_\nu} \to \mathcal{N}_\nu \times \mathbb{C}^k$ for the restrictions of B.

Let $\{\varphi_\nu\}$ be a C^∞ partition of unity on $\mathbf{W}$ subordinated to the covering $\{\mathcal{N}_\nu\}$.

Given any section $u \in \sigma(B)$, denote by u_ν the representation of u in $\mathcal{N}_\nu$. For every ν, the product $\varphi_\nu u_\nu$ is a well-defined function on $\mathbf{W}$ with values in $\mathbb{C}^k$.

Let $s, \gamma \in \mathbb{R}$. The space $\mathcal{W}^{s,\gamma}(\mathbf{W}, B)$ is defined to consist of all sections $u \in \sigma(B)$ such that, for every ν, the product $\varphi_\nu u_\nu$ belongs to $\mathcal{W}^{s,\gamma}(\mathbf{W})^k$ (the direct sum of k copies of $\mathcal{W}^{s,\gamma}(\mathbf{W})$).

Proposition 2.4.1 *As defined above, the space $\mathcal{W}^{s,\gamma}(\mathbf{W}, B)$ is independent of the particular choice of the covering $\{\mathcal{N}_\nu\}$, trivializations $\{t_\nu\}$, and the partition of unity $\{\varphi_\nu\}$.*

Proof. Let

$$T' = \{\mathcal{N}'_\mu,\ t'_\mu,\ \varphi'_\mu\},$$
$$T'' = \{\mathcal{N}''_\nu,\ t''_\nu,\ \varphi''_\nu\}$$

be two triples as above.

We introduce the temporary notation $\mathcal{W}^{s,\gamma}(\mathbf{W}, B, T')$ or $\mathcal{W}^{s,\gamma}(\mathbf{W}, B, T'')$ for the space $\mathcal{W}^{s,\gamma}(\mathbf{W}, B)$ corresponding to either triple. We shall have established the proposition if we prove that $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, B, T')$ implies $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, B, T'')$.

To this end, pick a $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, B, T')$. Given any number ν, write

$$\begin{aligned} \varphi''_\nu u''_\nu &= \varphi''_\nu \left(\sum_\mu \varphi'_\mu\right) u''_\nu \\ &= \varphi''_\nu \left(\sum_{\mu:\, \mathcal{N}'_\mu \cap \mathcal{N}''_\nu \neq \emptyset} \varphi'_\mu\right) u''_\nu. \end{aligned}$$

Since $u''_\nu = t''_\nu \left(t'_\mu\right)^{-1} u'_\mu$ in a neighborhood of $supp\, \varphi'_\mu \cap supp\, \varphi''_\nu$, we get

$$\varphi''_\nu u''_\nu = \sum_{\mu:\, \mathcal{N}'_\mu \cap \mathcal{N}''_\nu \neq \emptyset} \left(\varphi''_\nu\, t''_\nu \left(t'_\mu\right)^{-1}\right) \varphi'_\mu u'_\mu. \tag{2.4.1}$$

By assumption, every product $\varphi'_\mu u'_\mu$ belongs to $\mathcal{W}^{s,\gamma}(\mathbf{W})$. As multiplication by functions of $C^\infty_{loc}(\mathbf{W})$ is a continuous operator in $\mathcal{W}^{s,\gamma}(\mathbf{W})$ (see Subsection 2.3.4), it follows that every summand on the right-hand side of (2.4.1) is in $\mathcal{W}^{s,\gamma}(\mathbf{W})$.

Hence $u \in \mathcal{W}^{s,\gamma}(\mathbf{W}, B, T'')$, which is the desired conclusion.

□

The space $\mathcal{W}^{s,\gamma}(\mathbf{W}, B)$, when endowed with the scalar product

$$(u, v) = \sum_\nu (\varphi_\nu u_\nu, \varphi_\nu v_\nu)_{\mathcal{W}^{s,\gamma}(\mathbf{W})^k}, \tag{2.4.2}$$

is a Hilbert space. Again this Hilbert structure doesn't depend, up to an equivalent norm, on the concrete choice of the covering $\{\mathcal{N}_\nu\}$, trivializations $\{t_\nu\}$, and the partition of unity $\{\varphi_\nu\}$.

Our next theorem yields information about the dual of $\mathcal{W}^{s,\gamma}(\mathbf{W}, B)$.

Proposition 2.4.2 *For any $s, \gamma \in \mathbb{R}$, it follows that*

$$\mathcal{W}^{s,\gamma}(\mathbf{W}, B)' \overset{top.}{\cong} \mathcal{W}^{-s,-\gamma}(\mathbf{W}, B'),$$

B' being the dual vector bundle of B.

Proof. This is an immediate consequence of the corresponding local result given in Proposition 2.1.17.

□

2.4.3 *Rellich Theorem*

For weighted Sobolev spaces of sections of a vector bundle B over $\mathbf{W}$ the *Rellich Theorem* remains valid in full generality.

Theorem 2.4.3 *If W is compact, then the embedding*

$$\mathcal{W}^{s_2,\gamma_2}(\mathbf{W}, B) \hookrightarrow \mathcal{W}^{s_1,\gamma_1}(\mathbf{W}, B)$$

is compact provided that $s_1 < s_2$ and $\gamma_1 < \gamma_2$.

Proof. See Behm [2, Theorem 3.3.2.4]. □

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