A NOTE ON FUNCTIONAL EQUATIONS OF POLYLOGARITHMS

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MPI/90-22

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0. Introduction

The function log z satisfies the functional equation

$$\log x + \log y = \log(xy) \ .$$

The dilogarithm $\mathrm{Li}_2(z) := \int_0^z \frac{-\log(1-z)}{z} dz$ satisfies the following functional equation

$$\mathrm{Li}_2\left(\frac{x}{1-x}\cdot\frac{y}{1-y}\right) = \mathrm{Li}_2\left(\frac{y}{1-x}\right) + \mathrm{Li}_2\left(\frac{x}{1-y}\right) - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \log(1-x)\log(1-y)$$

(see [1]).

Let us set $\mathrm{Li}_1(z) : -\log(1-z)$ and $\mathrm{Li}_n(z) := \int_0^z \frac{\mathrm{Li}_{n-1}(z)}{z} dz$. It was expected that functions $\mathrm{Li}_n(z)$ will satisfy functional equations similar to functional equations of log z and $\mathrm{Li}_2(z)$. In fact various functional equations of functions $\mathrm{Li}_n(z)$ for $n \leq 5$ were found. The basic reference is Lewin's book (see [5]).

Our aim is to find some new functional equations satisfied by these functions.

Definition 0.1. If $f : P^1(\mathbb{C}) \longrightarrow P^1(\mathbb{C})$ is a rational map then the divisor

$f^{-1}(1) = \sum\limits_{k=1}^{r} r_k \cdot c_k$, where $r_k \in Z$ and $c_k \in \mathbb{C}$, is the inverse image on $\mathbb{C}$ of 1 taken with multiplicities.

Now we shall formulate our main results.

Theorem A. Let $f(z) = \alpha \prod\limits_{i=1}^{n} (z-a_i)^{n_i} / \prod\limits_{j=1}^{m} (z-b_j)^{m_j}$ be a map from $P^1(\mathbb{C})$ to $P^1(\mathbb{C})$. Let $f^{-1}(1) = \sum\limits_{k=1}^{r} r_k \cdot c_k$. We have the following formula:

$$\mathrm{Li}_2\left[\alpha \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m_j}\right] - \mathrm{Li}_2\left(\alpha \prod_{i=1}^{n} (x-a_i)^{n_i} / \prod_{j=1}^{m} (x-b_j)^{m_j}\right] =$$

$$\sum_{i,k} n_i \cdot r_k \left[\mathrm{Li}_2\left[\frac{z-a_i}{c_k-a_i}\right]\right] - \mathrm{Li}_2\left[\frac{x-a_i}{c_k-a_i}\right] + \log\frac{x-c_k}{a_i-c_k} \log\frac{z-a_i}{x-a_i}\right] +$$

$$-\sum_{j,k} m_j \cdot r_k \left[\mathrm{Li}_2\left[\frac{z-b_j}{c_k-b_j}\right] - \mathrm{Li}_2\left[\frac{x-b_j}{c_k-b_j}\right] + \log\frac{x-c_k}{b_j-c_k} \log\frac{z-b_j}{x-b_j}\right] +$$

$$-\sum_{i,j} n_i \cdot m_j \left[\mathrm{Li}_2\left[\frac{z-a_i}{b_j-a_i}\right] - \mathrm{Li}_2\left[\frac{x-a_i}{b_j-a_i}\right] + \log\frac{x-b_j}{a_i-b_j} \log\frac{z-a_i}{x-a_i}\right] +$$

$$-\sum_{j<j'} m_j \cdot m_{j'} \left[\log\frac{z-b_j}{x-b_j}\right] \left[\log\frac{z-b_{j'}}{x-b_{j'}}\right] - \frac{1}{2}\sum_{j} m_j^2 \left[\log\frac{z-b_j}{x-b_j}\right]^2 .$$

The following summation convention is used in Theorem A and it will be used through the whole paper.

$$\sum_{i,k} = \sum_{i=1}^{n} \sum_{k=1}^{r} , \quad \sum_{j<j'} = \sum_{j=1}^{m-1} \sum_{j'=j+1}^{m} , \quad \sum_{j} = \sum_{j=1}^{m} \quad \text{and so on.}$$

We shall show that from the functional equation in Theorem A we can get all or almost all functional equations of the dilogarithm choosing suitably the function $f(z)$.

We have the similar formula for the trilogarithm $\mathrm{Li}_3(z)$. In the introduction we state only the special case when the function $f(z)$ is a polynomial function.

Theorem B. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i}$ and let $f^{-1}(1) = \sum_{k=1}^{r} r_k \cdot c_k$. We have the following formula

$$\mathrm{Li}_3\Big(\alpha \prod_{i=1}^{n} (z-a_i)^{n_i}\Big) - \mathrm{Li}_3\Big(\alpha \prod_{i=1}^{n} (x-a_i)^{n_i}\Big) =$$

$$\sum_{i<i',k} -n_i \cdot n_{i'} \cdot r_k \Big[\mathrm{Li}_3\Big[\frac{z-a_i}{z-a_{i'}} \cdot \frac{c_k-a_{i'}}{c_k-a_i}\Big] - \mathrm{Li}_3\Big[\frac{x-a_i}{x-a_{i'}} \cdot \frac{c_k-a_{i'}}{c_k-a_i}\Big]$$

$$-\mathrm{Li}_2\Big[\frac{x-a_i}{x-a_{i'}} \cdot \frac{c_k-a_{i'}}{c_k-a_i}\Big] \log\Big[\frac{z-a_i}{z-a_{i'}} \cdot \frac{x-a_{i'}}{x-a_i}\Big] - \frac{1}{2}\log\Big[\frac{a_{i'}-a_i}{x-a_{i'}} \cdot \frac{x-c_k}{c_k-a_i}\Big] \log^2\Big[\frac{z-a_i}{z-a_{i'}} \cdot \frac{x-a_{i'}}{x-a_i}\Big]\Big]$$

$$-\sum_{i',i,k} n_i \cdot n_{i'} r_k \Big[\mathrm{Li}_3\Big[\frac{z-a_i}{c_k-a_i}\Big] - \mathrm{Li}_3\Big[\frac{x-a_i}{c_k-a_i}\Big] - \mathrm{Li}_2\Big[\frac{x-a_i}{c_k-a_i}\Big] \cdot \log\Big[\frac{z-a_i}{x-a_i}\Big] +$$

$$-\frac{1}{2}\log\Big[\frac{c_k-x}{c_k-a_i}\Big] \log^2\Big[\frac{z-a_i}{x-a_i}\Big]\Big] +$$

$$+\sum_{i<i',k} n_i \cdot n_{i'} \cdot r_k \Big[\mathrm{Li}_3\Big[\frac{z-a_i}{z-a_{i'}}\Big] - \mathrm{Li}_3\Big[\frac{x-a_i}{x-a_{i'}}\Big] - \mathrm{Li}_2\Big[\frac{x-a_i}{x-a_{i'}}\Big] \log\Big[\frac{z-a_i}{z-a_{i'}} \cdot \frac{x-a_{i'}}{x-a_i}\Big] +$$

$$-\frac{1}{2}\log\left[\frac{a_i-a_{i'}}{x-a_i}\right]\log^2\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{x-a_{i'}}{x-a_i}\right]\Bigg] \quad .$$

We can observe that $\mathrm{Li}_2(f(z))$ we expressed as a sum of $\mathrm{Li}_2(g(z))'s$, where $g(z)$ are functions of degree one, logarithmic terms and constants. The same holds for $\mathrm{Li}_3(f(z))$. This is not a general phenomena as we shall see in the next theorem.

Definition 0.2. Let us assume that n_i $i=1,\dots,n$; m_j $j=1,\dots,m$ are positive integers. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m_j}$ be a rational function written in an irreducible form. We set $\deg f := \max\left[\sum_{i=1}^{n} n_i, \sum_{j=1}^{m} m_j\right]$ and we call this number the degree of f .

Theorem $\mathbb{C}$. Let $f(z)$ be a rational function of degree k greater than 1. Let us assume that $f(z)$ is not a k–th power. Let n be a natural number greater than 3. Then the function $L_n(f(z))$ cannot be expressed as a sum of $\pm L_n(f_i(z))$ with $\deg f_i = 1$, constants and products of $\pm \mathrm{Li}_j(g(z))$ with $j < n$ and g rational.

Theorem $\mathbb{C}$ follows immediately from Proposition 2.8 and Proposition 2.4 which we shall prove in section 2.

The functions $\mathrm{Li}_n(z)$ are special cases of Chen iterated integrals. We recall their definition. Let $\omega_1,\dots,\omega_n$ be one–forms on a manifold M , let x and z be two points of M and let $\gamma(t)$, $t \in [0,1]$ be a smooth path from x to z . Then we define by a recursive formula

$$\int_{x,\gamma}^{z} \omega_1,\dots,\omega_n := \int_{x,\gamma}^{z} \left[\int_{x,\gamma}^{\gamma(t)} \omega_1\right] \omega_2,\dots,\omega_n \ .$$

It is clear that $\mathrm{Li}_n(z) = \int_0^z \frac{-dz}{z-1}, \frac{dz}{z},\dots,\frac{dz}{z}$.

We have the following results.

Theorem D. Let a_1, a_2, a_3, a_4 be four different points in $\mathbb{C}$.

a) The function $N(z) = \int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \frac{dz}{z-a_3}$ can be expressed by classical polylogarithms.

b) Let $\L(z) = \int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \frac{dz}{z-a_3}, \frac{dz}{z-a_4}$. There is no polynomial $p(s,t_1,\dots,t_r)$ such that $p(\L(z),\mathrm{Li}_{n_1}(f_1(z)),\dots,\mathrm{Li}_{n_r}(f_r(z))) = 0$ where $\mathrm{Li}_{n_k}(z)$ are classical polylogarithms and $f_i(z)$ are rational functions. (see Propositions 2.10 and 2.12 in section 2.)

This note is an extended version of our preprint "A note on functional equation of the dilogarithm" CRM (Bellaterra) October 1984. We would like to thank very much P. Deligne for his comments on our manuscript under the same title, where he reinterpreted our results in terms of Lie algebras of fundamental groups. He also showed us the connection from section 1 in the special case of $\mathbb{C}\backslash\{0,1\}$. We acknowledge the influence of the lecture of D. Zagier (Bonn, April 1989). We acknowledge the influence of the paper of L.J. Rogers (see [6]), H.F. Sandham (see [7]) and R.F. Coleman (see [3]). We would like to thank very much J.L. Loday and Ch. Soulé who told us about functional equations of polylogarithms.

The principal tools in our investigations are two observations.

1. Functions of the type of polylogarithms are horizontal sections of the canonical unipotent connection on $P^1(\mathbb{C})\backslash\{a_1,\dots,a_n\}$.

2. The functional equations of functions of the type of polylogarithms are consequences of relations between maps induced by regular functions from $P^1(\mathbb{C})\backslash$several points to $P^1(\mathbb{C})\backslash$several points on Lie algebras of fundamental groups.

We illustrate the second principal with few examples.

Example 1. The maps $f(x) = x$ and $g(x) = 1-x$ from $X = P^1(\mathbb{C})\backslash\{0,1,\infty\}$ into itself induce opposite maps on $\Gamma^2\pi_1(X,x)/\Gamma^3\pi_1(X,x)$, therefore we have a functional equation

$$\mathrm{Li}_2(x) - \mathrm{Li}_2(1-x) = \text{l.d.t.} \; .$$

l.d.t. = lower degree terms.

Example 2. The maps $f(x) = x^2$, $g(x) = x$ and $h(x) = -x$ from $X = P^1(\mathbb{C})\backslash\{0,1,-1,\infty\}$ to $P^1(\mathbb{C})\backslash\{0,1,\infty\}$ satisfies

$$f_* - 2g_* - 2h_* = 0$$

on $\Gamma^2\pi_1(X,x)/\Gamma^3\pi_1(X,x)$, therefore there is a functional equation

$$\mathrm{Li}_2(x^2) - 2\,\mathrm{Li}_2(x) - 2\,\mathrm{Li}_2(-x) = \text{l.d.t.} \; .$$

Example 3. Let $f_1(x) = x$, $f_2(x) = \frac{1}{1-x}$, $f_3(x) = \frac{x}{x-1}$, $f_4(x) = \frac{1}{x}$ be maps from

$X = P^1(\mathbb{C})\setminus\{0,1,\infty\}$ into itself. In $\mathrm{Hom}\left[\Gamma^3\pi_1(X,x)/\Gamma^4\pi_1(X,x);\ \Gamma^3\pi_1(X,x)/\Gamma^4\pi_1(X,x) + [V[U,V]]\right]$, where U is a loop around 0 and V is a loop around 1 we have

$$f_{1*} = f_{4*} \text{ and } f_{1*} + f_{2*} + f_{3*} = 0 \ .$$

Hence there are functional equations

$$\mathrm{Li}_3(x) = \mathrm{Li}_3(\tfrac{1}{x}) + \text{l.d.t.}$$

and

$$\mathrm{Li}_3(x) + \mathrm{Li}_3(\tfrac{1}{1-x}) + \mathrm{Li}_3(\tfrac{x}{x-1}) = \text{l.d.t.} \ .$$

Example 4. Let $X = P^1(\mathbb{C})\setminus\{0,1,\infty\}$, $f(x) = x$ and $g(x) = 1/x$. Let U be a loop around 0 and let V be a loop around 1. On the quotient $\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)+L$, where L is a subgroup of $\Gamma^n\pi_1(X,x)$ generated by all these commutators which contain V at least twice, we have

$$f_* = (-1)^{n-1}g_* \ .$$

Therefore we have a functional equation

$$\mathrm{Li}_n(z) = (-1)^{n-1}\mathrm{Li}_n(1/z) + \text{l.d.t.} \ .$$

All these examples follow easily from the following theorem:

Theorem E. Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_r,\infty\}$ and $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$. Let U (resp. V) be a loop around 0 (resp. 1) in Y . Let $f_1,\dots,f_N : X \longrightarrow Y$ be regular maps from X to Y and let $n_1,\dots,n_N$ be integers. There is a functional equation

$$n_1 Li_n(f_1(z)) + \dots + n_N Li_n(f_N(z)) + \text{l.d.t.} = 0$$

if and only if

$$n_1 f_{1*} + \dots + n_N f_{N*} = 0$$

in the Z–module $\mathrm{Hom}(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x), \Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y)+L)$, where L is a subgroup of $\Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y)$ generated by all commutators which contain V at least twice and f_{i*} is the map induced by f_i on fundamental groups.

This theorem will follow from Theorem 2.1, Proposition 2.4 and Lemma 2.7 from section 2.

1. The universal unipotent connection on $P^1(\mathbb{C})\setminus$several points and functional equations.

Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_n,\infty\}$. Let $H = H_1(X) = Zx_1 + Zx_2 + \dots + Zx_n$ where x_i is the class of a loop around a_i . Let $\mathbb{C}[[H]] = \mathbb{C}[[x_1,\dots,x_n]]$ be an algebra of formal power series in non–commutative variables $x_1,\dots,x_n$. Let I be an augmentation ideal. Then $C[[H]]/I^n$ is a finite dimensional vector space over $\mathbb{C}$, so it has the standard, complex topology, $C[[H]] = \varprojlim_n C[[H]]/I^n$ and we equipped $C[[H]]$ with the topology of the inverse limit.

Let $\mathcal{L}(X)$ be a Lie algebra of Lie elements in $C[[H]]$. This is a free Lie algebra on H. Let $L(X)$ be a completion of $\mathscr{L}(X)$ with respect to the lower central series of $\mathscr{L}(X)$. We equipped $L(X)$ with a group law given by the Baker–Hausdorff formula and we denote this new group by $\pi(X)$. The group $\pi(X)$ is a topological group. The topology is induced from $C[[H(X)]]$. This topology coincides with the topology of the inverse limit given by $\pi(X) = \varprojlim_n \pi(X)/\Gamma^n\pi(X)$, where $(\Gamma^n\pi(X))_{n\geq 2}$ is the lower central series and $\pi(X)/\Gamma^n\pi(X)$ is a complex Lie group.

Let $C[[H(X)]]^*$ be a group of invertible elements in $C[[H(X)]]$. This is also a topological group, an inverse limit of finite dimensional Lie groups.

The map

$$\exp : \pi(X) \longrightarrow C[[H(X)]]^*$$

given by $w \longrightarrow e^w = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \dots$ is a continous homomorphism of topological groups. In fact this map is a monomorphism whose image is a closed subgroup of $C[[H(X)]]^*$.

The Lie algebra of $\pi(X)$ is $L(X)$. We shall consider $L(X)$ as a Lie subalgebra of the Lie algebra of $C[[H]]^*$. The tangent vector at $0 \in \pi(X)$ given by $t \longrightarrow t \cdot x_i$ we denote therefore by x_i. The tangent vector at $1 \in \mathbb{C}[[H]]$ given by $t \longrightarrow 1+t\cdot x_i$ we denote also by x_i.

We consider on X two one–forms ω_X and $\bar{\omega}_X$ with values in the Lie algebra of $\pi(X)$ and $C[[H]]^*$.

We set

$$\omega_X = \frac{dz}{z-a_1} \otimes x_1 + \dots + \frac{dz}{z-a_n} \otimes x_n \in \Omega^1(X) \otimes L(X)$$

and

$$\overline{\omega}_X = \frac{dz}{z-a_1} \otimes x_1 + \dots + \frac{dz}{z-a_n} \otimes x_n \in \Omega^1(X) \otimes \mathrm{Lie}\, C[[H]]^*$$

(Lie $C[[H]]^*$ is the Lie algebra of $C[[H]]^*$).

The monomorphism $\exp : \pi(X) \longrightarrow C[[H]]^*$ maps ω_X into $\overline{\omega}_X$ and in the sequel we shall denote both forms by ω_X.

The principal fibre bundles

$$X \times \pi(X) \longrightarrow X$$

and

$$X \times C[[H]] \longrightarrow X$$

we equipped with the connections given by the one–form ω_X (see [4]).

Let us set $\omega_i = -\frac{dz}{z-a_i}$ $i = 1,\dots,n$. Let γ be a path from x to z . Let us define the following functions of z .

$$\Lambda_x(\epsilon_1,\dots,\epsilon_k)(z) := \int_\gamma \omega_{\epsilon_k}, \omega_{\epsilon_{k-1}},\dots,\omega_{\epsilon_1} .$$

Theorem 1.1. The application

$$X \ni z \longrightarrow (z, 1 + \sum \Lambda_x(\epsilon_1,\dots,\epsilon_k)(z) x_{\epsilon_1},\dots,x_{\epsilon_k})$$

is horizontal with respect to the connection ω_X on $X \times \mathbb{C}[[H_1(X)]]^*$.

The proof of this result is a straightford calculation of horizontal liftings.

Let $X = P^1(\mathbb{C})\setminus\{x_1,\dots,x_n,\infty\}$ and $Y = P^1(\mathbb{C})\setminus\{y_1,\dots,y_m,\infty\}$. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m}$ be a rational function. Let us assume that f restricts to a regular map $f : X \longrightarrow Y$. The map $f_* : H_1(X) \longrightarrow H_1(Y)$ induces homomorphisms of groups

$$f_* : \mathbb{C}[[H_1(X)]]^* \longrightarrow \mathbb{C}[[H_1(Y)]]^*$$

and

$$f_* : \pi(X) \longrightarrow \pi(Y) .$$

In the sequel $G(X)$ is $\mathbb{C}[[H_1(X)]]^*$ or $\pi(X)$ and $G(Y)$ is $\mathbb{C}[[H_1(Y)]]^*$ or $\pi(Y)$.

Proposition 1.2. The map $(f, f\times f_*)$ of principal fibre bundles

$$\begin{tikzcd}
X\times G(X) \arrow[r, "f\times f_*"] \arrow[d, "(1)"'] & Y\times G(Y) \arrow[d, "(2)"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is such that

$$(\mathrm{id} \otimes f_*)\omega_X = (f^* \otimes \mathrm{id})\omega_Y .$$

This is the direct verification.

The staightford consequence of the theorem is the following result.

Corollary 1.3. The map $(f, f\times f_*)$ of the principal fibre bundles maps horizontal sections of the first bundle into horizontal sections along f of the second bundle.

Let γ be a path in X. We shall denote by $(z, \ell_X(x,z;\gamma))$ or shortly by $(z, \ell_X(z))$ the value at z of the horizontal section of the bundle $X \times G(X) \longrightarrow X$ along the path γ with the initial condition $\ell_X(x,x,\gamma) = (x,0)$ if $G(X) = \pi(X)$, and $\ell_X(x,x,\gamma) = (x,1)$ if $G(X) = \mathbb{C}[[H_1(X)]]^*$. Corollary 2.2 implies that we have an equality

$$f_* \ell_X(z;x,\gamma) = \ell_Y(f(z);f(x),f(\gamma)) \ .$$

The group $G(Y)$ is an affine group. Let $\mathrm{Alg}(G(Y))$ be an algebra of polynomial, complex valued functons on $G(Y)$.

Theorem 1.4 (General functional equation). Let $f_1,\dots,f_n : X \longrightarrow Y$ be regular functions. Let $\mathscr{C}_1,\dots,\mathscr{C}_n$ be elements of $\mathrm{Alg}(G(Y))$ and let $p(t_1,\dots,t_n)$ be a polynomial in variables $t_1,\dots,t_n$.

i) Let $G(\ ) = \pi(\ )$. There is a functional equation

$$(1) \qquad p(\mathscr{C}_1(\ell_Y(f_1(z);f_1(x),f_1(\gamma))),\dots, \mathscr{C}_n(\ell_Y(f_n(z);f_n(x),f_n(\gamma))) = 0$$

if and only if

$$(2) \qquad p(\mathscr{C}_1 \mathrm{o} f_{1*},\dots, \mathscr{C}_n \mathrm{o} f_{n*}) = 0 \ .$$

ii) Let $G(\ ) = \mathbb{C}[[H_1(\ )]]^*$. If

$$p(\mathscr{C}_1 \mathrm{o} f_*,\dots, \mathscr{C}_n \mathrm{o} f_*) = 0$$

then there is a functional equation

$$p(\ \mathscr{C}_1(\ell_Y(f_1(z)),\dots,\ \mathscr{C}_n(\ell_Y(f_n(z))) = 0\ .$$

Proof. Let us assume that we have (2). Corollary 1.3 implies that

$$\mathscr{C}_i(f_{i*}(\ell_X(z))) =\ \mathscr{C}_i(\ell_Y(f(z))\ .$$

Replacing $\mathscr{C}_i(f_{i*}(\ell_X(z))$ by $\mathscr{C}_i(\ell_Y(f(z))$ in the formula (2) we get the functional equation (1).

Let us assume that we have a functional equation (1). The set of values $\ell_X(x,x,\gamma)$, for all closed loops γ, is Zariski dense in $\pi(X)$. The vanishing of the regular function (2) on the Zariski dense set implies that this regular function is zero.

Corollary 1.5. Let $f_1,\dots,f_n : X \longrightarrow Y$ be regular functions. Let $\mathscr{C}_1,\dots,\ \mathscr{C}_n$ be elements of $\mathrm{Alg}(\pi(Y))$ and let $\psi_1,\dots,\psi_m$ be elements of $\mathrm{Alg}(\pi(X))$. Let $p(s_1,\dots,s_m,t_1,\dots,t_n)$ be a polynomial in $s_1,\dots,s_m,\ t_1,\dots,t_n$. Let γ be a path in X from x to z.

There is a functonal equation

$$p(\psi_1(\ell_X(z;x,\gamma)),\dots,\psi_m(\ell_X(z;x,\gamma)),\ \mathscr{C}_1(\ell_Y(f_1(z);f_1(x),f_1(\gamma)),\dots) = 0$$

if and only if

$$p(\psi_1,\dots,\psi_m,\ \mathscr{C}_1 o f_{1*},\dots,\ \mathscr{C}_n o f_{n*}) = 0\ .$$

Proof. First we replace Y by Y' such that the inclusion $i : X \longrightarrow Y'$ $i(z) = z$ is regular. Let B' be a base of $\mathscr{L}(Y')$ which extends some base B of $\mathscr{L}(Y)$. We extend $\mathscr{C}_i$ to $\mathscr{C}'_i \in \mathrm{Alg}(\pi(Y'))$ in such a way that for any $b \in B'$, $\mathscr{C}'_i(b) = 0$. Then we have an equality

$$p(\psi_1,\ldots,\psi_m, \mathscr{C}'_1 \circ f_{1*},\ldots, \mathscr{C}'_n \circ f_{n*}) = 0$$

where $f_{i*} : \pi(X) \longrightarrow \pi(Y')$. The corollary now follows from Theorem 1.4.

We finish this section with an easy lemma about the monodromy of the function $\ell_X(z)$. We recall that we have a natural identification $\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X) \approx (\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)) \otimes \mathbb{C}$.

Lemma 1.6. Let $\mathscr{C} \in (\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X))^*$. The monodromy of the function $\mathscr{C}(\ell_X(z))$ on $\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x)$ is given by $(-2\pi i)^n\ \mathscr{C}$.

Proof. The monodromy of $\ell_X(z)$ around a_i is given by $\ell_X(z) \longrightarrow \ell_X(z) + (-2\pi i)x_i$ + terms of degree ≥ 2. This implies the lemma.

2. General functional equation of polylogarithms.

Let $X = P^1(\mathbb{C})\setminus\{x_1,\ldots,x_n,\infty\}$ and let $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$. $\mathscr{L}(X)$ and $\mathscr{L}(Y)$ are free Lie algebras on $H_1(X)$ and $H_1(Y)$ respectively. Let U, V be a base of $H_1(Y)$ given by loops in clock–wise direction around points 0 and 1 respectively. For a free Lie algebra L, let us set $L' = [L,L]$ and $L'' = [L',L']$. Let L_n be the vector space of degree n elements in L/L''. In $\mathscr{L}(Y)$ we fix a base given by elementary basic elements. This base determines a base of $\mathscr{L}(Y)_n$. Let us set $e_0 := U$, $e_1 := V$, $e_2 := [U,V]$, $e_n := [U,e_{n-1}]$. Let $\hat{e}_n$ (resp. e^*_n) be the linear form on $\mathscr{L}(Y)$ (resp. $\mathscr{L}(Y)_n$) dual to e_n .

Theorem 2.1 (General functional equation of polylogarithms). Let $f_1,f_2,\dots,f_k : X \longrightarrow Y$ be regular functions and let $n_1,n_2,\dots,n_k$ be integers. Let $G(\ ) = \pi(\ )$. There is a functional equation

$$(3) \qquad n_1\hat{e}_n(\ell_Y(f_1(z);f_1(x),f_1(\gamma)) + \dots + n_k\hat{e}_n(\ell_Y(f_k(z);f_k(x),f_k(\gamma)) = 0$$

if and only if

$$(4) \qquad n_1(\overset{*}{e}_n \circ f_{1*}) + n_2(\overset{*}{e}_n \circ f_{2*}) + \dots + n_k(\overset{*}{e}_n \circ f_{k*}) = 0$$

in $(\mathscr{L}(X)_n)^*$.
(f_{i*} are maps f_{i*} restricted to $\mathscr{L}(X)_n$).

Proof. Let us notice that the formula (4) is equivalent to the equality

$$(4') \qquad n_1(\hat{e}_n \circ f_{1*}) + n_2(\hat{e}_n \circ f_{2*}) + \dots + n_k(\hat{e}_n \circ f_{k*}) = 0$$

in $(\mathscr{L}(X))^*$. Theorem 1.4 implies that (3) and (4′) are equivalent.

Definition 2.2 (Higher Rogers functions). Let γ be a path from x to z in $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$. We set

$$\mathscr{L}_n(z;x,\gamma) := \hat{e}_n(\ell_Y(z;x,\gamma))$$

and

$$\mathcal{L}_n(z;\gamma) := \mathcal{L}_n(z;0,\gamma) \ .$$

(If the integration path γ is obvious we shall write $\mathcal{L}_n(z)$ instead of $\mathcal{L}_n(z;x,\gamma)$.)

We point out that functions $\mathcal{L}_n(z)$ are in some sense higher analogs of the Rogers function $L(z) = -\frac{1}{2}\int_0^z \left[\frac{\log(1-z)}{z}+\frac{\log z}{1-z}\right] dz$. The Rogers function $L(z)$ satisfies functional equations which usually have less lower degree terms than the analogous equations for $Li_2(z) = \int_0^z \frac{-\log(1-z)}{z} dz$. For example for the Rogers function we have

$$L(z) + L(1-z) = L(1)$$

while for the classical dilogarithm we have

$$Li_2(z) + Li_2(1-z) = \pi^2/6 - \log z \log(1-z) \ .$$

We shall see later that in general the functions $\mathcal{L}_n(z)$ satisfy functional equations with less lower degree terms than analogous functional equations for classical polylogarithms $Li_n(z)$. We also have the following proposition which justifies the name "higher Rogers functions".

Proposition 2.3. We have

$$\mathcal{L}_2(z) = \frac{1}{2}\int_0^z \left[\frac{\log(1-z)}{z} + \frac{\log z}{1-z}\right] dz$$

and

$$\mathscr{L}_3(z) = \int_0^z \left[\left[\frac{1}{12}\log z \log(1-z) + \frac{1}{2}\mathscr{L}_2(z)\right]\frac{1}{z} + \frac{1}{12}\log^2 z \frac{1}{1-z}\right] dz \ .$$

The next proposition gives the relation between higher Rogers functions $\mathscr{L}_n(z)$ and classical polylogarithms $\mathrm{Li}_n(z)$.

Proposition 2.4. i) The difference $\mathscr{L}_n(z) - \mathrm{Li}_n(z)$ can be expressed as a sum of products of lower degree polylogarithms $\mathrm{Li}_k(z)$ for $k < n$ and $\log z$.

ii) The difference $\mathscr{L}_n(z) - \mathrm{Li}_n(z)$ can be expressed as a sum of products of lower degree functions $\mathscr{L}_k(z)$ for $k < n$ and $\log z$ and $\log(1-z)$.

Proof. It follows from Theorem 1.1 applied to $\mathbb{C}[[H_1(Y)]]^* = \mathbb{C}[[U,V]]$ that the coefficient at U^nV of the horizontal section starting from 0 is equal to $\mathrm{Li}_n(z)$. The homomorphism $\exp : \pi(Y) \longrightarrow \mathbb{C}[[U,V]]$ maps horizontal sections into horizontal sections. Comparing coefficients at U^nV for $\exp(\ell_Y(z))$ ($\ell_Y(z)$ is the horizontal section in $\pi(Y)$) and for the horizontal section in $\mathbb{C}[[U,V]]$ we get the part ii). The point i) follows then easily by induction.

We shall consider the question whether $\mathscr{L}_n(f(z))$ can be expressed as a sum of $\mathscr{L}_n(f_i(z))$, where $\deg f_i < \deg f$ and perhaps terms of lower degree and constants.

Proposition 2.5. Let $f(z) = a \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m_j}$ be a rational function in irreducible form. The function $\mathscr{L}_2(f(z))$ can be expressed as a sum of $\mathscr{L}_2(f_i(z))$ where $\deg f_i = 1$, products of logarithms and constants.

Proof. The divisor $f^{-1}(1)$ on $\mathbb{C}$ is equal to $\sum_{k=1}^{r} r_k \cdot c_k$. The functon f defines a regular map

$$f : X = P^1(\mathbb{C}) \backslash \{a_1,\dots,a_n;b_1,\dots,b_m;c_1,\dots,c_k;\infty\} \longrightarrow Y = P^1(\mathbb{C}) \backslash \{0,1,\infty\} \ .$$

Let $f_{ij}(z) = \frac{z-a_i}{b_j-a_i}$, $g_{ik}(z) = \frac{z-a_i}{c_k-a_i}$, $h_{ij}(z) = \frac{z-b_j}{c_k-b_j}$ $i = 1,\dots,n$; $j = 1,\dots,m$; $k = 1,\dots,r$.
One checks that

$$f_* = -\sum_{i=1}^{n} \sum_{j=1}^{m} n_i \cdot m_j (f_{ij})_* + \sum_{i=1}^{n} \sum_{k=1}^{r} n_i \cdot r_k (g_{ih})_* - \sum_{j=1}^{m} \sum_{k=1}^{r} n_i \cdot r_k (h_{ij})_*$$

on $\mathscr{L}(X)_2$.
The proposition now follows directly from Theorem 2.

Proposition 2.6. Let f(z) be a rational function. Then $\mathscr{L}_3(f(z))$ can be expressed as a sum of $\mathscr{L}_3(f_i(z))$, where $\deg f_i = 1$, constants and products of dilogarithms and logarithms.

We shall omit the proof which is similar to the proof of Proposition 2.4. However in the next section we give an explicit formual for $\mathrm{Li}_3(f(z))$.

To simplify notations we set $\mathscr{L}_1(z) := \log(1-z)$ and $\mathscr{L}_0(z) := \log(z)$.
The symbols $f_i(z)$, $g_j(z)$, $h_k(z)$ will denote rational functions on $P^1(\mathbb{C})$.

Lemma 2.7. Let us assume that we have a functional equation

$$(*) \qquad \sum_{i=1}^{N} n_i \ \mathcal{L}_n(f_i(z)) + \text{l.d.t.} = 0$$

where l.d.t. is a sum of constants and products of $\mathcal{L}_k(g_j(z))$ with $k < n$.

Then we have

$$n_1(e_n^* \circ f_{1*}) + \ldots + n_N(e_n^* \circ f_{N*}) = 0$$

in $(\mathcal{L}(X)_n)^*$ where

$$X = P^1(\mathbb{C}) \setminus \{ \bigcup_{i=1}^{N} f_i^{-1}\{0,1,\infty\} \cup \bigcup_j g_j^{-1}\{0,1,\infty\} \cup \infty \} \ .$$

Proof. The formula $\sum_{i=1}^{N} n_i \ \mathcal{L}_n(f_i(z)) + \text{l.d.t.}$ is understood in the following way. For each function $\mathcal{L}_k(h_i(z))$ which appears in it we choose a path γ_h from 0 to $h(z)$ and we calculate the value $\mathcal{L}_k(h(z))$ along this path. The equality (*) means that there is a choice of paths $\Gamma = (\gamma_h)$ such that the left hand side vanishes.

Let us observe that then for any family of paths $\Delta = (\delta_h)$ there is l.d.t. which depends on Δ such that the left hand side with this new Δ vanishes.

We choose a path γ from x to z in X. Then we can rewrite the equation (*) in the form

$$\sum_{i=1}^{N} n_i \ \mathcal{L}_n(f_i(z); f_i(x), f_i(\gamma)) + \text{l.d.t.} = 0 \ .$$

It follows from Theorem 1.4 that

$$\sum_{i=1}^{N} n_i \hat{e}_n \circ f_{i*} + p(\hat{e}_k \circ g_{i_k *}; k<n) = 0 \ .$$

After restriction to $\mathscr{L}(X)_n$ we get $\sum\limits_{i=1}^{N} n_i \overset{*}{e}_n \circ f_{i*} = 0$ in $(\mathscr{L}(X)_n)^*$.

Proposition 2.8. Let $f(z)$ be a rational function of degree $k > 1$. Let us assume that $f(z)$ is not a k–th power. Let n be a natural number greater than 3. Then the function $\mathscr{L}_n(f(z))$ cannot be expressed as a sum of $\pm \mathscr{L}_n(f_i(z))$ with $\deg f_i = 1$, constants and products of $\pm \mathscr{L}_i(g(z))$ with $i < n$ and g rational.

Proof. It follows from Introduction, Example 4 that we can assume $f(z) = \alpha \prod\limits_{i=1}^{n} (z-a_i)^{n_i} / \prod\limits_{i=1}^{m} (z-b_j)^{m_j}$ and $a_1 \neq a_2$. Let $c \in \mathbb{C}$ be such that $f(c) = 1$ with the multiplicity r . We consider f as a regular map $f : X = P^1(\mathbb{C}) \setminus \{f^{-1}(0) \cup f^{-1}(\infty) \cup \infty\} \longrightarrow Y = P^1(\mathbb{C}) \setminus \{0,1,\infty\}$. We choose a base of $H_1(X)$ given by loops around missing points except ∞ . Let A_i be a loop around a_i , and let $\mathbb{C}$ be a loop around c . Let $\alpha_2 := [A_2,\mathbb{C}]$, $\alpha_n = [A_1,\alpha_{n-1}]$ and $\beta_3 = [A_2[A_2,\mathbb{C}]]$, $\beta_n = [A_1,\beta_{n-1}]$. We have $f_*(\alpha_n) = n_1^{n-2} \cdot n_2 \cdot r \cdot e_n$, $f_*(\beta_n) = n_1^{n-3} \cdot n_2^2 \cdot r \cdot e_n$. The only degree one maps which involve α_n and β_n are $g(z) = \frac{z-a_1}{z-a_2} \cdot \frac{c-a_2}{c-a_1}$ and $h(z) = \frac{z-a_2}{z-a_1} \cdot \frac{c-a_1}{c-a_2}$. For these maps we have $g_*(\alpha_n) = -e_n$, $g_*(\beta_n) = e_n + sc$, $h_*(\alpha_n) = (-1)^{n-2} e_n + sc$, $h_*(\beta_n) = (-1)^{n-3} e_n + sc$, where sc is a linear combination of basic elements all different from e_n . Now it is clear that any relation of the form $\overset{*}{e}_n \circ f_* + \sum\limits_{i}^{N} q_i \overset{*}{e}_n \circ f_{i*} = 0$ with $q_i \in \mathbb{Q}$ is impossible. Therefore Lemma

2.7 implies the proposition.

Lemma 2.9. Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_n,\infty\}$. Let $G(X)$ be $\pi(X)$ or $\mathbb{C}[[H_1(X)]]^*$ and let $\mathcal{V} \in \mathrm{Alg}(G(X)/\Gamma^3 G(X))$. Then $\mathcal{V}(\ell_X(z;x,\gamma))$ can be expressed by dilogarithms, logarithms and constants.

Proof. The integral $\int_x^z \frac{-dz}{z-a}, \frac{dz}{z-b}$ can easily be expressed by dilogarithms, logarithms and constants.

Proposition 2.10. Let a, b, c be three different points in $\mathbb{C}$. The function $N(z) = \int_x^z \frac{-dz}{z-a}, \frac{dz}{z-b}, \frac{dz}{z-c}$ can be expressed by classical polylogarithms Li_n .

Proof. We set $X = P^1(\mathbb{C})\setminus\{a,b,c,\infty\}$ and $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$. In degree 3 $\mathcal{L}(X)$ has the following base given by elementary basic elements $a_1 = ((BA)A)$, $a_2 = ((CA)A)$, $a_3 = ((BA)B)$, $a_4 = ((CA)B)$, $a_5 = ((CB)C)$, $a_6 = ((BA)C)$, $a_7 = ((CA)C)$. Let a_i^* $i = 1,\dots,7$ be dual linear forms. Let $f_1(z) = \frac{a-z}{a-b}$, $f_2(z) = \frac{c-z}{c-b}$, $f_3(z) = \frac{a-z}{c-z}$, $f_4(z) = \frac{a-z}{a-b}\cdot\frac{c-b}{c-z}$ be maps from X to Y . Then we have

$$a_6^* = f_1^*(e_3^*) + f_2^*(e_3^*) + f_3^*(e_3^*) - f_4^*(e_4^*)$$

on $\Gamma^3\pi(X)/\Gamma^4\pi(X)$. This implies that in $\mathrm{Alg}(\pi(X))$ we have an equality

$$a_6^* = f_1^*(\hat{e}_3) + f_2^*(\hat{e}_3) + f_3^*(\hat{e}_3) - f_4^*(\hat{e}_3) + P$$

where P is a polynomial in functions on $\pi(X)/\Gamma^3\pi(X)$. Corollary 1.5, Proposition 2.4

and Lemma 2.3 imply that $a_6^*(\ell_X(z;x,\gamma))$ can be expressed by classical polylogarithms. One shows using the method of the proof of Proposition 2.4 that the function $N(z)$ can be expressed by $a_6^*(\ell_X(z;x,\gamma))$ and classical polylogarithms.

Corollary 2.11. Let $G(X)$ be $\pi(X)$ or $\mathbb{C}[[H_1(X)]]^*$. Let $\mathscr{C} \in \mathrm{Alg}(G(X)/\Gamma^4 G(X))$. Then $\mathscr{C}(\ell_X(z;x,\gamma))$ can be expressed by classical polylogarithms.

Proof. This follows from the formulas 1.5.2 and 1.6.2 in [2] and Proposition 2.9.

Proposition 2.12. Let a_i, $i = 1,2,3,4$ be four different points in $\mathbb{C}$. Let $\text{Ł}(z) = \int_x^z \frac{dz}{z-a_1}, \frac{dz}{z-a_2}, \frac{dz}{z-a_3}, \frac{dz}{z-a_4}$. There is no polynomial $P(s,t_1,\ldots,t_r)$ such that $p(\text{Ł}(z), \mathrm{Li}_{n_1}(f_1(z)),\ldots,\mathrm{Li}_{n_r}(f_r(z))) = 0$ where $\mathrm{Li}_{n_k}(z)$ are classical polylogarithms and $f_i(z)$ are rational functions on $P^1(\mathbb{C})$.

Proof. The function $p(z) = p(\text{Ł}(z), \mathrm{Li}_{n_1}(f_1(z)),\ldots)$ is a multivalued function on $X = P^1(\mathbb{C})\setminus\{a_1,a_2,a_3,a_n \cup \text{finite set of points}\}$. The monodromy of $\text{Ł}(z)$ on one of the commutators $\alpha = [[a_{i_1},a_{i_2}],[a_{i_3},a_{i_4}]]$ with all i_k different is equal to $\pm(-2\pi i)^4$. The monodromy of $\mathrm{Li}_{n_k}(f_k(z))$ on α is trivial. This follows from the equality

$$\hat{\mathscr{C}}_n(\ell_Y(f_k(z);f_k(x),f_k(\gamma)) = \hat{\mathscr{C}}_n \circ f_{k*}(\ell_X(z;x,\gamma))$$

and from Lemma 1.6. Hence the monodromy of $p(z)$ on α is non–trivial. Therefore $p(z) \neq 0$.

3. Functional equations of low degree polylogarithm

In this section we write down functional equations for dilogarithm, trilogarithm and fourth–order polylogarithm. We shall get them in the similar way we showed Theorem 2.1. However we shall work with the group $G(\ ) = \mathbb{C}[[H_1(\ )]]^*$ instead of $\pi(\ )$. This has an advantage to deal with more familiar functions. On the other side we do not have the analog of Theorem 2.1 for $\mathbb{C}[[H_1(\ )]]^*$. This is due to the fact that coefficients of $\ell_Y(z)$ at different monomials can be related (for example coefficients at UV and VU are obviously related). Therefore we shall prove (calculate) any analog of Theorem 2.1 in every special case we consider.

3.1. Functional equations of dilogarithm.

First we show how forms of functional equations known before can be deduced from our general form from Theorem A.

Example 1. Let $f(z) = z^n$ and $x = 0$. Then we have

$$Li_2(z^n) = n \sum_{k=1} Li_2(\zeta^k z)$$

where $\zeta = e^{2\pi i/n}$.

Example 2. Let $f(z) = \frac{yz}{(y-1)(z-1)}$ and $x = 0$. Then we have

$$\mathrm{Li}_2\left[\frac{yz}{(y-1)(z-1)}\right] = \mathrm{Li}_2\left[\frac{z}{1-y}\right] - \mathrm{Li}_2\left[\frac{1-z}{y}\right] + \mathrm{Li}_2\left[\frac{1}{y}\right] - \mathrm{Li}_2(z) - \log\left[\frac{y-1}{y}\right]\log(1-2)$$
$$-\tfrac{1}{2}\log^2(1-z) \ .$$

Example 3. Let $f(z) = \frac{(1-y)z}{z-1}$ and $x = 0$. Then we get

$$\mathrm{Li}_2\left[\frac{(1-y)z}{z-1}\right] = \mathrm{Li}_2(yz) - \mathrm{Li}_2\left[\frac{y}{y-1}(1-z)\right] + \mathrm{Li}_2\left[\frac{y}{y-1}\right] - \mathrm{Li}_2(z) +$$
$$-\log(1-y)\log(z-1) - \tfrac{1}{2}\log^2(1-z) \ .$$

Proof of Theorem A. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m_j}$ and let $f^{-1}(1) = \sum_{k=1} r_k c_k$. Let $X = P^1(\mathbb{C})\setminus\{a_1,\dots,a_n,b_1,\dots,b_m,c_1,\dots,c_r,\infty\}$ and $Y = P^1(\mathbb{C})\setminus\{0,1,\infty\}$. We set $G(X) = \mathbb{C}[[A_1,\dots,A_n,B_1,\dots,B_m,C_1,\dots,C_r]]^*$ and $G(Y) = \mathbb{C}[[U,V]]$. Let $\mathscr{C}$ be a regular function on $G(Y)$ equal to a coefficient at UV. For any regular $g : X \longrightarrow Y$ we have

$$(*) \qquad \mathscr{C}\,(g_*\ell_X(z;x,\gamma)) = \mathscr{C}\,(\ell_Y(g(z);g(x)),g(\gamma))) \ .$$

We are looking for degree one maps $f_i : X \longrightarrow Y$ such that

$$\mathscr{C}\ \mathrm{o} f_* \text{–linear combination of } \mathscr{C}\ \mathrm{o} f_{i*}$$

will vanish or at least will be possible to calculate easily.

We have

$$f_*(A_i \cdot C_k) = n_i \cdot r_k UV \ , \ f_*(B_j \cdot C_k) = -m_j \cdot r_k UV \ ,$$
$$f_*(A_i \cdot B_j) = -n_i \cdot m_j UV \ , \ f_*(B_j \cdot B_{j'}) = m_j \cdot m_{j'} UV \ .$$

We need maps of degree one $g : X \longrightarrow Y$ which induce the same maps on these products. Here there are three families of such maps

$$f_{ik}(z) = \frac{z-a_i}{c_k-a_i}, \quad (f_{ik})_*(A_iC_k) = UV \ ;$$

$$g_{ij}(z) = \frac{z-a_i}{b_j-a_i}, \quad (g_{ij})_*(A_iB_j) = UV \ ;$$

$$h_{jk}(z) = \frac{z-b_j}{c_k-b_j}, \quad (h_{jk})_*(B_jC_k) = -UV \ .$$

We have

$$\varphi \circ f_* = \sum_{i,k} n_i \cdot r_k \ \mathscr{C} \circ (f_{ik})_* - \sum_{j,k} m_i \cdot r_k \ \mathscr{C} \circ (h_{ij})_* - \sum_{i,j} n_i \cdot m_j \ \mathscr{C} \circ (g_{ij})_* + \sum_{jj'} \psi_{jj'}$$

where $\psi_{jj'}$ is a coefficient at $B_j \cdot B_{j'}$. From this formula applied to $\ell_X(z;x,\gamma)$, the equality $(*)$ and the formula $\int \frac{dz}{z-b}, \frac{dz}{z-a} + \int \frac{dz}{z-a}, \frac{dz}{z-b} = \int \frac{dz}{z-a} \cdot \int \frac{dz}{z-b}$ we get

$$\int_{f(x)}^{f(z)} \omega = \sum_{i,k} n_i \cdot r_k \int_{f_{ik}(x)}^{f_{ik}(z)} \omega - \sum_{i,j} n_i \cdot m_j \int_{g_{ij}(x)}^{g_{ij}(z)} \omega - \sum_{j,k} m_j \cdot r_k \int_{h_{ij}(x)}^{h_{ij}(z)} \omega +$$

$$-\frac{1}{2} \sum_{j,j'} m_j \cdot m_{j'} \log\left(\frac{z-b_j}{x-b_j}\right) \log\left(\frac{z-b_{j'}}{x-b_{j'}}\right)$$

where $\omega = \frac{-dz}{z-1}, \ \frac{dz}{z}$. Theorem A follows immediately from this equation.

Example 4.

We shall indicate how the Newman functional equation

$$2\mathrm{Li}_2(x) + 2\mathrm{Li}_2(y) + 2\mathrm{Li}_2(z) = \mathrm{Li}_2(xy) + \mathrm{Li}_2(yz) + \mathrm{Li}_2(zx)$$

if $x+y+z = xyz + 2$ can be get by our method.

We consider five maps $f_1(z) = \frac{z(z+x-2)}{xz-1}$, $f_2(z) = \frac{z+x-2}{xz-1}$, $f_3(z) = xz$, $f_4(z) = z$, $f_5(z) = \frac{x(z+x-2)}{xz-1}$ from $P^1(\mathbb{C})\setminus\{0,1,1/x,2-x,\infty\}$ to $P^1(\mathbb{C})\setminus\{0,1,\infty\}$. Aplying the method from the proof of the theorem we get an equation

$$\begin{gathered}\mathrm{Li}_2(xy) + \mathrm{Li}_2(yz) + \mathrm{Li}_2(zx) - 2\mathrm{Li}_2(z) - 2\mathrm{Li}_2(y) + \\ + 2\mathrm{Li}_2(x-2) - \mathrm{Li}_2(x(2-x)) + \text{logarithmic terms} = 0 \ .\end{gathered}$$

After applying Theorem A to $\mathrm{Li}_2(x(2-x))$ we get the Newman functional equation.

3.2. Functional equation of trilogarithm.

We shall give only formulas. The proof is similar to the proof of Theorem A and we shall omit it.

Theorem 3.2.1. Let $f(z) = \alpha \prod_{i=1}^{n} (z-a_i)^{n_i} / \prod_{j=1}^{m} (z-b_j)^{m_j}$ and let $f^{-1}(1) = \sum_{k=1} r_k c_k$. We have the following formula:

$$\mathrm{Li}_3(f(z)) - \mathrm{Li}_3(f(x)) =$$

$$\sum_{i<i',k} n_i \cdot n_{i'} \cdot r_k \biggl[-\mathrm{Li}_3\biggl[\frac{z-a_i}{z-a_{i'}} \cdot \frac{c_k-a_{i'}}{c_k-a_i}\biggr] + \mathrm{Li}_3\biggl[\frac{x-a_i}{x-a_{i'}} \cdot \frac{c_k-a_{i'}}{c_k-a_i}\biggr] +$$

$$-\,\mathrm{Cor}\biggl[\frac{z-a_i}{z-a_{i'}} \cdot \frac{c_k-a_{i'}}{c_k-a_i}\biggr] +$$

$$\mathrm{Li}_3\biggl[\frac{z-a_i}{z-a_{i'}}\biggr] - \mathrm{Li}_3\biggl[\frac{x-a_i}{x-a_{i'}}\biggr] + \mathrm{Cor}\biggl[\frac{z-a_i}{z-a_{i'}}, \frac{x-a_i}{x-a_{i'}}\biggr]\biggr] +$$

$$\sum_{i,i',k} n_i \cdot n_{i'} \cdot r_k \biggl[\mathrm{Li}_3\biggl[\frac{z-a_i}{c_k-a_i}\biggr] - \mathrm{Li}_3\biggl[\frac{x-a_i}{c_k-a_i}\biggr] + \mathrm{Cor}\biggl[\frac{z-a_i}{c_k-a_i}, \frac{x-a_i}{c_k-a_i}\biggr]\biggr] +$$

$$\sum_{i,j,k} n_i \cdot m_j \cdot r_k \biggl[-\mathrm{Li}_3\biggl[\frac{z-a_i}{z-b_j} \cdot \frac{c_k-b_j}{c_k-a_i}\biggr] + \mathrm{Li}_3\biggl[\frac{x-a_i}{x-b_j} \cdot \frac{c_k-b_j}{c_k-a_i}\biggr] +$$

$$-\,\mathrm{Cor}\biggl[\frac{z-a_i}{z-b_j} \cdot \frac{c_k-b_j}{c_k-a_i}, \frac{x-a_i}{x-b_j} \cdot \frac{c_k-b_j}{c_k-a_i}\biggr] +$$

$$\mathrm{Li}_3\biggl[\frac{z-a_i}{z-b_j}\biggr] - \mathrm{Li}_3\biggl[\frac{x-a_i}{x-b_j}\biggr] + \mathrm{Cor}\biggl[\frac{z-a_i}{z-b_j}, \frac{x-a_i}{x-b_j}\biggr] +$$

$$\mathrm{Li}_3\biggl[\frac{z-a_i}{c_k-a_i}\biggr] - \mathrm{Li}_3\biggl[\frac{x-a_i}{c_k-a_i}\biggr] + \mathrm{Cor}\biggl[\frac{z-a_i}{c_k-a_i}, \frac{x-a_i}{c_k-a_i}\biggr] +$$

$$\mathrm{Li}_3\biggl[\frac{z-b_j}{c_k-b_j}\biggr] - \mathrm{Li}_3\biggl[\frac{x-b_j}{c_k-b_j}\biggr] + \mathrm{Cor}\biggl[\frac{z-b_j}{c_k-b_j}, \frac{x-b_j}{c_k-b_j}\biggr]\biggr] +$$

$$\sum_{j<j',k} m_j \cdot m_{j'} \cdot r_k \biggl[-\mathrm{Li}_3\biggl[\frac{z-b_j}{z-b_{j'}} \cdot \frac{c_k-b_{j'}}{c_k-b_j}\biggr] + \mathrm{Li}_3\biggl[\frac{x-b_j}{x-b_{j'}} \cdot \frac{c_k-b_{j'}}{c_k-b_j}\biggr] +$$

$$-\operatorname{Cor}\left[\frac{z-b_j}{z-b_{j'}}\cdot\frac{c_k-b_{j'}}{c_k-b_j},\frac{x-b_j}{x-b_{j'}}\cdot\frac{c_k-b_{j'}}{c_k-b_j}\right]+$$

$$\mathrm{Li}_3\left[\frac{z-b_j}{z-b_{j'}}\right]-\mathrm{Li}_3\left[\frac{x-b_j}{x-b_{j'}}\right]+\operatorname{Cor}\left[\frac{z-b_j}{z-b_{j'}},\frac{x-b_j}{x-b_{j'}}\right]\Bigg]+$$

$$\sum_{j,j',k}m_j\cdot m_{j'}\cdot r_k\left[\mathrm{Li}_3\left[\frac{z-b_j}{c_k-b_j}\right]-\mathrm{Li}_3\left[\frac{x-b_j}{c_k-b_j}\right]+\operatorname{Cor}\left[\frac{z-b_j}{c_k-b_j},\frac{x-b_j}{c_k-b_j}\right]\right]+$$

$$\sum_{i<i',k}n_i\cdot n_{i'}\cdot m_j\Bigg[-\mathrm{Li}_3\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{b_j-a_{i'}}{b_j-a_i}\right]+\mathrm{Li}_3\left[\frac{x-a_i}{x-a_{i'}}\cdot\frac{b_j-a_{i'}}{b_j-a_i}\right]+$$

$$-\operatorname{Cor}\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{b_j-a_{i'}}{b_j-a_i},\frac{x-a_i}{x-a_{i'}}\cdot\frac{b_j-a_{i'}}{b_j-a_i}\right]+$$

$$+\mathrm{Li}_3\left[\frac{z-a_i}{z-a_{i'}}\right]-\mathrm{Li}_3\left[\frac{x-a_i}{x-a_{i'}}\right]+\operatorname{Cor}\left[\frac{z-a_i}{z-a_{i'}},\frac{x-a_i}{x-a_{i'}}\right]\Bigg]+$$

$$\sum_{i,i',j}n_i\cdot n_{i'}\cdot m_j\left[\mathrm{Li}_3\left[\frac{z-a_i}{b_j-a_i}\right]-\mathrm{Li}_3\left[\frac{x-a_i}{b_j-a_i}\right]+\operatorname{Cor}\left[\frac{z-a_i}{b_j-a_i},\frac{x-a_i}{b_j-a_i}\right]\right]+$$

$$\sum_{j<j',i}m_j\cdot m_{j'}\cdot n_i\Bigg[\mathrm{Li}_3\left[\frac{z-b_j}{z-b_{j'}}\cdot\frac{a_i-b_{j'}}{a_i-b_j}\right]-\mathrm{Li}_3\left[\frac{x-b_j}{x-b_{j'}}\cdot\frac{a_i-b_{j'}}{a_i-b_j}\right]+$$

$$\operatorname{Cor}\left[\frac{z-b_j}{z-b_{j'}}\cdot\frac{a_i-b_{j'}}{a_i-b_j},\frac{x-b_j}{x-b_{j'}}\cdot\frac{a_i-b_{j'}}{a_i-b_j}\right]+$$

$$\mathrm{Li}_3\left[\frac{z-b_j}{z-b_{j'}}\right]-\mathrm{Li}_3\left[\frac{x-b_j}{x-b_{j'}}\right]+\operatorname{Cor}\left[\frac{z-b_j}{z-b_{j'}},\frac{x-b_j}{x-b_{j'}}\right]\Bigg]+$$

$$\sum_{j,j',i} m_j \cdot m_{j'} \cdot n_i \left[\mathrm{Li}_3\left[\frac{z-b_j}{a_i-b_j}\right] - \mathrm{Li}_3\left[\frac{x-b_j}{a_i-b_j}\right] + \mathrm{Cor}\left[\frac{z-b_j}{a_i-b_j}, \frac{x-b_j}{a_i-b_j}\right]\right] +$$

$$\sum_{i,j,j'} -m_j \cdot m_{j'} \cdot n_i \frac{1}{2} \log\frac{z-a_i}{x-a_i} \log\frac{z-b_j}{x-b_j} \log\frac{z-b_{j'}}{x-b_{j'}} +$$

$$\sum_{j,j',j''} -m_j \cdot m_{j'} \cdot m_{j''} \cdot \frac{1}{6} \log\frac{z-b_j}{x-b_j} \log\frac{z-b_{j'}}{x-b_{j'}} \cdot \log\frac{z-b_{j''}}{x-b_{j''}} \ ,$$

where $\mathrm{Cor}(a,b) = -\mathrm{Li}_2(b)\log\left[\frac{a}{b}\right] - \frac{1}{2}\log(1-b)\log^2\left[\frac{a}{b}\right]$. (The summation convention: $\sum\limits_{i,j,k} = \sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{r}$, $\sum\limits_{i<i',k} = \sum\limits_{i=1}^{n-1} \sum\limits_{i'=i+1}^{n} \sum\limits_{k=1}^{r}$.)

We omit the proof of this theorem because it is the same as the proof of Theorem A. I shall indicate two more formulas.

Theorem 3.2.2.

$$\mathrm{Li}_3(\alpha \prod_{i=1}^{n} (z-a_i)^{n_i}) - \mathrm{Li}_3(\alpha \prod_{i=1}^{n} (x-a_i)^{n_i}) =$$

$$= \sum_{i<i',k} n_i \cdot n_{i'} \cdot r_k \left[\mathrm{Li}_3\left[\frac{z-a_i}{c_k-a_i} \cdot \frac{z-a_{i'}}{c_k-a_{i'}}\right] - \mathrm{Li}_3\left[\frac{x-a_i}{c_k-a_i} \cdot \frac{x-a_{i'}}{c_k-a_{i'}}\right] +\right.$$

$$\left.\mathrm{Cor}\left[\frac{z-a_i}{c_k-a_i} \cdot \frac{z-a_{i'}}{c_k-a_{i'}}, \frac{x-a_i}{c_k-a_i} \cdot \frac{x-a_{i'}}{c_k-a_{i'}}\right]\right] +$$

$$\sum_{i,k} (2n_i-N) \cdot n_i \cdot r_k \left[\mathrm{Li}_3\left[\frac{z-a_i}{c_k-a_i}\right] - \mathrm{Li}_3\left[\frac{x-a_i}{c_k-a_i}\right] + \mathrm{Cor}\left[\frac{z-a_i}{c_k-a_i}, \frac{x-a_i}{c_k-a_i}\right]\right] +$$

$$\sum_{i<i,k} n_i\cdot n_{i'}\cdot r_k\left[\mathrm{Li}_3\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{c_k-a_i}{c_k-a_{i'}}\right]-\mathrm{Li}_3\left[\frac{x-a_i}{x-a_{i'}}\cdot\frac{c_k-a_i}{c_k-a_{i'}}\right]+\right.$$

$$\left.\mathrm{Cor}\left[\frac{z-a_i}{z-a_{i'}}\cdot\frac{c_k-a_i}{c_k-a_{i'}},\frac{z-a_i}{z-a_{i'}}\cdot\frac{c_k-a_i}{c_k-a_{i'}}\right]\right]+$$

$$\sum_{i\neq i',k} -2n_i\cdot n_{i'}\cdot r_k\left[\mathrm{Li}_3\left[\frac{z-a_i}{a_{i'}-c_k}\right]-\mathrm{Li}_3\left[\frac{x-a_i}{a_{i'}-c_k}\right]+\mathrm{Cor}\left[\frac{z-a_i}{a_{i'}-c_k},\frac{x-a_i}{a_{i'}-c_k}\right]\right]$$

$$+\sum_{i<i',k} -n_i\cdot n_{i'}\cdot r_k\left[\mathrm{Li}_3\left[\frac{z-a_i}{z-a_{i'}}\right]-\mathrm{Li}_3\left[\frac{x-a_i}{x-a_{i'}}\right]+\mathrm{Cor}\left[\frac{z-a_i}{z-a_{i'}},\frac{x-a_i}{x-a_{i'}}\right]\right]$$

where $c_1,\dots,c_k,\dots,c_r$ are roots of $a\prod_{i=1}^{n}(z-a_i)^{n_i}-1=0$ with the multiplicities $r_1,\dots,r_k,\dots$, and $N=\sum_{i=1}^{n} n_i$.

The Corollary below is the special case of Theorem B. We write it down to have the implicit formula in the simplest case.

Corollary 3.2.3. Let $a,b\in\mathbb{C}$ and $a\neq b$. We have a formula:

$$\mathrm{Li}_3((a-z)(b-z))=2\mathrm{Li}_3\left[\frac{z-a}{c_1-a}\right]-\mathrm{Li}_3\left[\frac{z-a}{z-b}\cdot\frac{c_1-b}{c_1-a}\right]+$$

$$+2\mathrm{Li}_3\left[\frac{z-b}{c_1-b}\right]-2\mathrm{Li}_3\left[\frac{z-a}{c_2-a}\right]-\mathrm{Li}_3\left[\frac{z-a}{z-b}\cdot\frac{c_2-b}{c_2-a}\right]+2\mathrm{Li}_3\left[\frac{z-b}{c_2-b}\right]+$$

$$2\mathrm{Li}_3\left[\frac{z-a}{z-b}\right] - 2\mathrm{Li}_3\left[\frac{a-b}{c_1-b}\right] - 2\mathrm{Li}_3\left[\frac{a-b}{c_2-b}\right] +$$

$$-\log\left[\frac{a-c_1}{b-c_1}\right]\log^2\left[\frac{z-b}{a-b}\right] - \log\left[\frac{a-c_2}{b-c_2}\right]\log^2\left[\frac{z-b}{a-b}\right] +$$

$$-2\mathrm{Li}_2\left[\frac{a-b}{c_1-b}\right]\log\frac{z-b}{a-b} - 2\mathrm{Li}_2\left[\frac{a-b}{c_2-b}\right]\log\left[\frac{z-b}{a-b}\right] ,$$

where $(a-z)(b-z)-1 = (z-c_1)(z-c_2)$.

3.3. The fourth–order polylogarithm.

Theorem 3.3.1. We have the formula

$$\mathrm{Li}_4\left[-\frac{1}{(b-a)^2}(z-a)(z-b)\right] + \mathrm{Li}_4\left[\frac{1}{b-a}\,\frac{(z-a)^2}{z-b}\right] =$$

$$3\mathrm{Li}_4\left[\frac{z-a}{z-b}\cdot\frac{c_1-b}{c_1-a}\right] + 3\mathrm{Li}_4\left[\frac{z-a}{z-b}\cdot\frac{c_2-b}{c_2-a}\right] + 6\mathrm{Li}_4\left[\frac{z-a}{c_1-a}\right] +$$

$$6\mathrm{Li}_4\left[\frac{z-a}{c_2-a}\right] + 3\mathrm{Li}_4\left[\frac{z-b}{c_1-b}\right] + 3\mathrm{Li}_4\left[\frac{z-b}{c_2-b}\right] - 3\mathrm{Li}_4\left[\frac{a-b}{c_1-b}\right] +$$

$$-3\mathrm{Li}_4\left[\frac{a-b}{c_2-b}\right] + 2\mathrm{Li}_4\left[\frac{z-a}{z-b}\right] - 8\mathrm{Li}_4\left[\frac{z-a}{b-a}\right] + 4\int_{\frac{z-b}{a-b}}^{1}\frac{-dz}{z-1},\frac{dz}{z},\frac{dz}{z},\frac{dz}{z}$$

$$-\frac{2}{4!}(\log(z-b)-\log(a-b))^4$$

where c_1, c_2 are roots of the equation

$$-\frac{1}{(b-a)^2}(z-a)(z-b)-1=0\ .$$

Let us notice that this functional equation has less quadratic terms then the Kummer's functional equation of the fourth–order polylogarithm (see [5]).

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