Cyclic coverings and higher order embeddings of algebraic varieties

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Abstract

Let Y be a cyclic covering of an algebraic variety X. Given a line bundle on X we give criteria for its pull-back to Y to define a higher order embedding.

Introduction

In recent years there has been considerable interest in understanding under which circumstances linear series on algebraic varieties restrict surjectively to zero-dimensional subschemes and to collections of fat points. Equivalently, one asks for the *order* of a given projective embedding of an algebraic variety. The concepts of higher order embeddings are captured by the notions of k-very ampleness, introduced in [5], and k-jet ampleness, studied in [9] and [13]. More geometrically, an algebraic variety embedded in $\mathbb{P}^N$ via a $k-$very ample line bundle has no $(k+1)-$secant $(k-1)-$plane $\mathbb{P}^{k-1} \subset \mathbb{P}^N$. The embedding given by a $k-$jet ample line bundle has stronger geometrical constraints related to the higher osculating planes.

By now, the situation on surfaces is quite well understood, mainly thanks to the availability of powerful methods such as a Reider type criterion for k-very ampleness [5] and the use of $\mathbb{Q}$-divisors in connection with the Kawamata-Viehweg vanishing theorem [17]. In higher dimensions, however, the problem seems to be much more difficult. While there are highly interesting general results on the separation of jets due to Demailly [13], there is still a lack of practical criteria that allow to determine the order of a given embedding. The purpose of this paper is to study higher order embeddings of cyclic coverings $\pi : Y \longrightarrow X$, via line bundles given by pulling back "sufficiently positive" line bundles on X. Given a line bundle L on X we relate the order of the embedding defined by π^*L to that of L and certain rank 1 summands of the vector bundle $L \otimes \pi_*\mathcal{O}_Y$. The main results are expressed in the Theorems 2.1 and 3.1.

In Section 1 we recall briefly the concepts of higher order embeddings. Section 2 is devoted to an exposition of the $k-$jet ample case. In Section 3 we prove our result

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concerning $k-$very ampleness. Although the rigorous proof is technically involved, we hope that the ideas behind are fairly transparent. Finally, the examples provided in the last section show that our results are sharp.

We work throughout over the field $\mathbb{C}$ of complex numbers.

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1 Higher order embeddings

We start by recalling the notions of k-jet ampleness and k-very ampleness that capture the concept of a *higher order embedding* in two different ways: the first notion requires the simultaneous separation of jets at finitely many points, whereas the second asks for the surjectivity of the restriction to 0-dimensional subschemes of certain length.

A line bundle L on a smooth projective variety X is called *k-jet ample*, if the evaluation map

$$H^0(X,L) \longrightarrow H^0\left(X, L\otimes \mathcal{O}_X/\left(\mathfrak{m}_{y_1}^{k_1}\otimes\ldots\otimes\mathfrak{m}_{y_r}^{k_r}\right)\right)$$

is surjective for any choice of distinct points $y_1,\ldots,y_r$ in X and positive integers $k_1,\ldots,k_r$ with $\sum k_i = k+1$. Further, L is called *k-very ample*, if for every zero-dimensional subscheme $(\mathcal{Z},\mathcal{O}_Z)\subset(X,\mathcal{O}_X)$ of length $k+1$ the natural map

$$H^0(X,L) \longrightarrow H^0\left(X, L\otimes\mathcal{O}_Z\right)$$

is surjective.

As the definitions suggest, $k-$jet ampleness implies $k-$very ampleness [9, 2.2] and both notions are equivalent to global generation in the case $k=0$ and very ampleness in the case $k=1$.

In the case when $X=\mathbb{P}^2$ the notion of k-the order embedding is another way of dealing with non "superabundant embeddings". We refer to [16] and [7] for more details.

For the purposes of the proof of Theorem 3.1 below, it will be useful to have a slightly more general definition of k-very ampleness:

A subspace $V\subset H^0(X,L)$ will be called k-very ample on an open subset $U\subset X$, if for every zero-dimensional subscheme $(\mathcal{Z},\mathcal{O}_\mathcal{Z})\subset(U,\mathcal{O}_U)$ of length $k+1$ the natural map

$$\mathrm{ev}_\mathcal{Z}: V\longrightarrow H^0\left(X, L\otimes\mathcal{O}_\mathcal{Z}\right)$$

is surjective. It is then obvious that if $V\subset H^0(X,L)$ is k-very ample on U, and if $(\mathcal{Z},\mathcal{O}_Z)\subset(U,\mathcal{O}_U)$ is a 0-dimensional subscheme of length ℓ, then the subspace

$$\mathrm{ev}_\mathcal{Z}^{-1}(g)\subset V$$

is $(k-\ell)$-very ample on $U\setminus\mathrm{supp}(\mathcal{Z})$ for every $g\in H^0(X,L\otimes\mathcal{O}_\mathcal{Z})$.

2 Coverings and k-jet ampleness

Theorem 2.1 *Let X be a smooth projective variety and $B \subset X$ a smooth divisor. Let M be a line bundle on X such that $\mathcal{O}_X(dM) \cong \mathcal{O}_X(B)$ and let $\pi : Y \longrightarrow X$ be the cyclic covering of degree d defined by M. Let L be a line bundle on X and k a non-negative integer. If $L - qM$ is $(k-q)$-jet ample for $q = 0, \ldots, \min(k, d-1)$, then $\pi^* L$ is k-jet ample.*

Proof. Let $\overline{M}$ be the total space of M and $p : \overline{M} \longrightarrow X$ the bundle projection. Let $s_B \in H^0(X, \mathcal{O}_X(B))$ be a section whose divisor of zeros is B, and let $\tau \in H^0(\overline{M}, p^* M)$ be the tautological section. As usual, Y may be viewed as the divisor of zeros of the section $p^* s_B - \tau^d$. Letting t be the restriction of τ to Y and $\pi : Y \longrightarrow X$ the restriction of p, the projection formula gives the decomposition

$$H^0(Y, \pi^* L) = \bigoplus_{q=0}^{d-1} t^q \pi^* H^0(X, L - qM) \tag{1}$$

corresponding to the eigen-values of the action of the primitive covering automorphism φ.

Let $y_1, \ldots, y_r \in Y$ be points and $k_1, \ldots, k_r$ positive integers with $\sum k_i = k + 1$. Given a simultaneous jet

$$J \in \bigoplus_{i=1}^{r} H^0(Y, \pi^* L \otimes \mathcal{O}_Y / \mathfrak{m}_{y_i}^{k_i}) = H^0(Y, \pi^* L \otimes \mathcal{O}_Y / \mathfrak{m}_{y_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{y_r}^{k_r}) \ ,$$

we decompose J into a sum of simultaneous jets $J = J_1 + \ldots + J_r$, where $J_i = (j_{1i}, \ldots, j_{ii}, \ldots, j_{ri})$, with j_{li} the zero jet of order k_l at y_l for $l \neq i$. It is enough to find sections $s_i \in H^0(Y, \pi^* L)$ such that

$$s_i \bmod \mathfrak{m}_{y_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{y_r}^{k_r} = J_i \quad \text{for } i = 1, \ldots, r \ ,$$

since then $s = s_1 + \ldots + s_r$ has the desired jet J. We may therefore assume $J = J_1$. Now, the idea is to construct a section s explicitly out of sections in the line bundles $L - qM$, $q \geq 0$. We distinguish three cases.

Case 1. Suppose that $y_1 \notin R$ and that none of the points $y_2, \ldots, y_r$ lies in the orbit $\pi^{-1}(\pi(y_1))$. This is the easiest case. The desired sections is obtained as a pull-back of a section in L. More precisely we can view the jet j_{11} as a jet at $\pi(y-1)$, via the isomorphism of local rings $\mathcal{O}_{Y,y_1} \cong \mathcal{O}_{X,\pi(y_1)}$ induced by π. Since L is $k-$jet ample, there is a section $s \in H^0(X, L)$ such that

$$s \bmod \mathfrak{m}_{\pi(y_1)}^{k_1} = j_{11}$$

and

$$s \bmod \mathfrak{m}_{\pi(y_i)}^{k_i} = 0 \quad \text{for } i = 2, \ldots, r \ .$$

Then of course $\pi^* s \bmod \mathfrak{m}_{y_1}^{k_1} \otimes \ldots \otimes \mathfrak{m}_{y_r}^{k_r} = J_1$.

Case 2. Suppose that $y_1, \ldots, y_l$ (with $l \geq 2$) lie in the orbit $\pi^{-1}(\pi(y_1))$ and that none of the points $y_{l+1}, \ldots, y_r$ does. This case is more difficult since we have to separate points in a fiber of the covering. The construction of s builds upon the cyclic group at hand. Let $k' = \max(k_1, \ldots, k_l)$. Observe that $k' \leq k + 1/(l-1) - \sum_{i=l+1}^r k_i$. Let $\widehat{j}_{11}$ be any preimage of the jet j_{11} under the surjective map

$$\mathcal{O}/\mathfrak{m}_{y_1}^{k'} \longrightarrow \mathcal{O}/\mathfrak{m}_{y_1}^{k_1} .$$

Using now again the identification $\mathcal{O}_{Y,y_1} \cong \mathcal{O}_{X,\pi(y_1)}$ as well as the facts that $t(y_1) \neq 0$ and that the line bundle $L - qM$ is $(k-q)$-jet ample for $0 \leq q \leq l-1$, we can find sections $s_1, \ldots, s_l$ such that

$$\begin{aligned} \pi^* s_1 \bmod \mathfrak{m}_{y_1}^{k'} &= \alpha_1 \cdot \widehat{j}_{11} \\ t \cdot \pi^* s_2 \bmod \mathfrak{m}_{y_1}^{k'} &= \alpha_2 \cdot \widehat{j}_{11} \\ &\vdots \\ t^{l-1} \cdot \pi^* s_l \bmod \mathfrak{m}_{y_1}^{k'} &= \alpha_l \cdot \widehat{j}_{11} \end{aligned}$$

and

$$t^q \cdot \pi^* s_{q+1} \bmod \mathfrak{m}_{y_i}^{k_i} = 0 \quad \text{for } 0 \leq q \leq l-1 \text{ and } l+1 \leq i \leq r .$$

for any given complex numbers $\alpha_1, \ldots, \alpha_l$. Now, let $(\alpha_1, \ldots, \alpha_l)$ be a solution of the system of linear equations

$$\begin{aligned} \alpha_1 + \ldots + \alpha_l &= 1 \\ \alpha_1 + \varepsilon^{\beta_2}\alpha_2 + \ldots + \varepsilon^{(l-1)\beta_2}\alpha_l &= 0 \\ &\vdots \\ \alpha_1 + \varepsilon^{\beta_l}\alpha_2 + \ldots + \varepsilon^{(l-1)\beta_l}\alpha_l &= 0 \end{aligned} \tag{2}$$

where $\beta_2, \ldots, \beta_l$ are integers such that $y_i = \varphi^{\beta_i}(y_1)$. (Note that the determinant of the system (2) is just the Vandermonde determinant of the numbers $1, \varepsilon^{\beta_2}, \ldots, \varepsilon^{\beta_l}$.) Consider the section $s = \pi^* s_1 + \ldots + t^{l-1}\pi^* s_l$. We have $s \bmod \mathfrak{m}_{y_1}^{k'} = \widehat{j}_{11}$, which implies

$$s \bmod \mathfrak{m}_{y_1}^{k_1} = j_{11} ,$$

as required. Moreover we have

$$\begin{aligned} s \bmod \mathfrak{m}_{y_i}^{k'} &= (\varphi^{\beta_i})^* s \bmod \mathfrak{m}_{y_1}^{k'} \\ &= \pi^* s_1 + \varepsilon^{\beta_i} t \pi^* s_2 + \ldots + \varepsilon^{(l-1)\beta_i} t^{l-1} \pi^* s_l \bmod \mathfrak{m}_{y_1}^{k'} \\ &= (\alpha_1 + \varepsilon^{\beta_i}\alpha_2 + \ldots + (\varepsilon^{\beta_i})^{l-1}\alpha_l)\widehat{j}_{11} = 0 , \end{aligned}$$

so that $s \bmod \mathfrak{m}_{y_i}^{k_i} = 0$ for $i = 2, \ldots, l$, and of course

$$s \bmod \mathfrak{m}_{y_i}^{k_i} = 0 \text{ for } i = l+1, \ldots, r .$$

Case 3. Suppose $y_1 \in R$. Since B (and hence R) is smooth, there are local coordinates $u_1, \ldots, u_n$ at y_1 and $v_1, \ldots, v_n$ at $\pi(y_1)$ such that π is locally given as

$$\pi(u_1, \ldots, u_n) = (u_1^d, u_2, \ldots, u_n) = (v_1, \ldots, v_n) \ .$$

In these coordinates t is given by u_1 and j_{11} may be written as

$$j_{11} = \sum_{i_1+\ldots+i_n \le k_1} a_{i_1,\ldots,i_n} u_1^{i_1} \cdot \ldots \cdot u_n^{i_n} = \sum_{l=0}^{d-1} u_1^l \sum_{\substack{i_1 \equiv l \bmod d \\ i_1+\ldots+i_n \le k_1}} a_{i_1\ldots i_n} (u_1^d)^{\frac{i_1-l}{d}} u_2^{i_2} \cdot \ldots \cdot u_n^{i_n}.$$

This splitting reflects the fact that only jets containing powers of u_1 divisible by d arise as pull-backs of jets on X. Since $L - qM$ is $(k-q)$-jet ample for $q = 0, \ldots, \min(d-1, k)$, there are sections $s_q \in H^0(X, L - qM)$ such that

$$s_q \bmod \mathfrak{m}_{\pi(y_1)}^{k_1-q} = \sum_{\substack{i_1 \equiv l \bmod d \\ \frac{i_1-q}{d}+i_2+\ldots+i_n \le k_1-q}} a_{i_1\ldots i_n} v_1^{\frac{i_1-l}{d}} v_2^{i_2} \cdot \ldots \cdot v_n^{i_n} \ .$$

and

$$s_q \bmod \mathfrak{m}_{\pi(y_i)}^{k_i} = 0 \quad \text{for } i = 2, \ldots, r \ .$$

It is now easy to check that the section

$$s = \sum_{q \ge 0} t^q \pi^* s_q$$

has the prescribed jets. $\square$

Working as in cases 1 and 2 of the above proof, one gets immediately the same result for unbranched coverings:

Theorem 2.2 *(Unbranched case) Let $\pi : Y \longrightarrow X$ be an unbranched cyclic covering of degree d, defined by a line bundle M with $\mathcal{O}_X(dM) = \mathcal{O}_X$. Let L be a line bundle on X and k a non-negative integer. If $L - qM$ is $(k-q)$-jet ample for $q = 0, \ldots, \min(k, d-1)$, then $\pi^* L$ is k-jet ample.*

3 Coverings and k-very ampleness

In Theorem 3.1 below we will assert the k-very ampleness of a pullback of an ample line bundle L under suitable hypotheses on the line bundles $L - qM$, $q \ge 0$. In order to get a useful criterion, one needs to formulate a delicate numerical hypothesis. To this end, given positive integers k and ℓ, we introduce the abbreviations $\gamma(k,\ell) = 1$ if k/ℓ is an integer, and $\gamma(k,\ell) = 0$ otherwise, and we set

$$\tau(k,\ell) = k - \left\lfloor \frac{k}{\ell} \right\rfloor - \ell + \gamma(k,\ell) + 1 \ .$$

Further, we use the notation

$$\sigma(k,d,q) = \max\left\{ \tau(k+1,\ell) \;\middle|\; q+1 \le \ell \le \min(d,k+1) \right\} - 1$$

if $q > 0$ and $\sigma(k,d,0) = k$.

Our result can then be stated as follows:

Theorem 3.1 *Let X be a smooth projective variety and $B \subset X$ a smooth divisor. Let M be a line bundle on X such that $\mathcal{O}_X(dM) \cong \mathcal{O}_X(B)$ and let $\pi : Y \longrightarrow X$ be the cyclic covering of degree d defined by M. Let L be a line bundle on X and k a non-negative integer. If $L - qM$ is $\sigma(k,d,q)$-very ample for $q = 0,\dots,\min(k,d-1)$ then π^*L is k-very ample.*

One can prove the k-very ampleness of π^*L under the assumption that $L - qM$ is $(k-q)$-very ample for $0 \le q \le \min(k,d+1)$ by arguments very similar to the ones used in the proof of Theorem 2.1. The point however here is that k-very ampleness holds already under the much weaker (but also more intricate) hypotheses involving the numbers $\sigma(k,d,q)$. For instance, if L is 2-very ample, then π^*L is also 2-very ample as soon as both $L - M$ and $L - 2M$ are globally generated. If π is of degree 2 it is enough to check that $L - M$ is globally generated. We provide a list of explicit values of $\sigma(k,d,q)$ in Remark 3.5.

In order to simplify the exposition of the proof of the theorem, we start by stating two lemmas. We begin with an elementary observation on local rings:

Lemma 3.2 *Let $(\mathcal{O},\mathfrak{m})$ be a local ring, and let $I_1,\dots,I_\ell$, $\ell \ge 2$, be ideals contained in $\mathfrak{m}$ such that* $\operatorname{length} \mathcal{O}/I_1 \ge \operatorname{length} \mathcal{O}/I_i$ *for all i. Let $k = \sum_{i=1}^{\ell} \operatorname{length} \mathcal{O}/I_i$. If $k < \infty$, then*

$$\operatorname{length} \mathcal{O}/I_2 \cap \ldots \cap I_\ell \le \tau(k,\ell) \ .$$

Proof. We have the inequality

$$\begin{aligned} \operatorname{length} \mathcal{O}/I_2 \cap \ldots \cap I_\ell \ &\le\ \sum_{i=1}^{\ell} \operatorname{length} \mathcal{O}/I_i - \operatorname{length} \mathcal{O}/I_2 + (I_3 \cap \ldots \cap I_\ell) \\ &\quad - \ldots - \operatorname{length} \mathcal{O}/I_2 + \ldots + I_\ell \le k - \operatorname{length} \mathcal{O}/I_1 - (\ell - 2) \ . \end{aligned}$$

The assertion follows now from the fact that $\operatorname{length} \mathcal{O}/I_1 \ge \lfloor k/\ell \rfloor + 1 - \gamma(k,\ell)$. $\square$

We will also need the following

Lemma 3.3 *Let $\ell_1,\ldots,\ell_m$, $K_1,\ldots,K_r$ and q be positive integers, and let $K=\sum_{i=1}^r K_i$. If $\ell_i\geq 2$ for all i, then*

$$\sum_{i=1}^m \tau(K_i,\ell_i)+\sum_{i=m+1}^r \left\lfloor \frac{K_i}{q+1}\right\rfloor \leq \max\left\{\tau(K,\ell)\;\Big|\;\min(\ell_1,\ldots,\ell_q,q+1)\leq \ell\right\}\ .$$

Proof. First we note that it is enough to prove

$$\sum_{i=1}^m \tau(K_i,\ell_i)+\sum_{i=m+1}^r \left\lfloor \frac{K_i}{q+1}\right\rfloor \leq \tau(K,\ell) \tag{3}$$

with $\ell=\max(\ell_1,\ldots,\ell_m)$. Due to the fact that

$$\left\lfloor \frac{K_i}{q+1}\right\rfloor \leq K_i-\left\lfloor \frac{K_i}{2}\right\rfloor -1+\gamma(K_i,2)=\tau(K_i,2)\ .$$

we may assume $r=m$ (by setting $l_i=2$ for $m<i\leq r$). Then (3) reads $\sum_{i=1}^r\tau(K_i,\ell_i)\leq\tau(K,\ell)$ and it suffices to prove it for $r=2$. We may also assume $\ell_1\leq\ell_2=\ell$. From

$$\left\lfloor \frac{K_1}{\ell_1}\right\rfloor \geq \left\lfloor \frac{K_1}{\ell}\right\rfloor +\gamma(K_1,\ell_1)\cdot\min(1,\ell-\ell_1)$$

and

$$\left\lfloor \frac{K_1}{\ell}\right\rfloor +\left\lfloor \frac{K_2}{\ell}\right\rfloor \geq \left\lfloor \frac{K}{\ell}\right\rfloor -1+\gamma(K_1,\ell)+\gamma(K_2,\ell)-\gamma(K_1,\ell)\cdot\gamma(K_2,\ell)$$

we get

$$\begin{aligned}\tau(K_1,\ell_1)+\tau(K_2,\ell_2) \quad\leq\quad & K-\left\lfloor \tfrac{K}{\ell}\right\rfloor -\ell+\gamma(K_1,\ell)\cdot\gamma(K_2,\ell)+1+\\ & +[2-\ell_1]+[\gamma(K_1,\ell_1)-\gamma(K,\ell)-\gamma(K_1,\ell_1)\cdot\min(1,\ell-\ell_1)].\end{aligned}$$

Now, the assertion follows by observing that $\gamma(K_1,\ell)\cdot\gamma(K_2,\ell)\leq\gamma(K,\ell)$ and the terms in the square brackets are non-positive. □

Proof of Theorem 3.1. Let $(\mathcal{Z},\mathcal{O}_{\mathcal{Z}})\subset Y$ be a zero-dimensional subscheme of length $k+1$, defined by the ideal $\mathcal{I}_Z$, and supported on the points $y_1,\ldots,y_r$. Let $k_i=\mathrm{length}(\mathcal{O}_{\mathcal{Z},y_i})$. Given an element

$$g=(g_1,\ldots,g_r)\in\bigoplus_{i=1}^r H^0(\pi^*L\otimes\mathcal{O}_{\mathcal{Z},y_i})=H^0(\pi^*L\otimes\mathcal{O}_{\mathcal{Z}})\ ,$$

we will construct a section $s\in H^0(Y,\pi^*L)$ whose image in $H^0(\pi^*L\otimes\mathcal{O}_{\mathcal{Z}})$ is g.

The strategy is different to that in the proof of Theorem 2.1. Roughly speaking, for $q\geq 0$, we will approximate s by building a sequence of affine subspaces

$$\ldots\subset V_2^q\subset V_1^q\subset V_0^q\subset H^0(X,L-qM)$$

consisting of sections $s_{q,i}$ such that the length of the subscheme of $\mathcal{Z}$ where $\sum t^q \pi^* s_{q,i} \bmod \mathcal{I}_{\mathcal{Z}}$ agrees with g increases with i reaching length $\mathcal{Z}$ in the last step. First of all we take care of subschemes of $\mathcal{Z}$ supported in the ramification locus of π. Starting with a fiber containing most of the points y_i we deal then with subschemes of $\mathcal{Z}$ supported in regular fibers of π. Each step consists now of fixing sections in V_i^q for q appropriately big and imposing new conditions on sections in V_i^q for lower q thus defining subspaces V_{i+1}^q. Positivity assumptions of the theorem assure that we never run out of sections. Let us now turn to the details.

After reordering $y_1, \ldots, y_r$ we may assume that

$$\begin{aligned} y_1, \ldots, y_{n_1 - 1} &\in R \\ y_{n_1}, \ldots, y_{n_2 - 1} &\in \pi^{-1}(x_1) \\ &\vdots \\ y_{n_m}, \ldots, y_{n_{m+1}-1} = y_r &\in \pi^{-1}(x_m) \ . \end{aligned}$$

for distinct points $x_1, \ldots, x_n \in X \setminus B$. Letting $l_i = n_{i+1} - n_i$, we may also assume that $l_1 \leq \ldots \leq l_m$ and $k_{n_i} = \max(k_{n_i}, \ldots, k_{n_{i+1}-1})$. We will use the abbreviations $\mathcal{I}_i = \mathcal{I}_{\mathcal{Z}, y_i}$ for $i = 1, \ldots, r$ and $K_i = k_{n_1} + \ldots + k_{n_{i+1}-1}$ for $i = 1, \ldots, m$.

Identifying the local rings of the points in the same fiber by means of π, we consider the ideals

$$\mathcal{J}_i =_{\text{def}} \mathcal{I}_{n_i + 1} \cap \ldots \cap \mathcal{I}_{n_{i+1}-1} \subset \mathcal{O}_{Y, y_{n_i}} .$$

and for $i = 1, \ldots, m$ we denote by

$$\rho_i : \mathcal{O}_{Y, y_{n_i}} / (\mathcal{I}_{n_i} \cap \mathcal{J}_i) \longrightarrow \mathcal{O}_{Y, y_{n_i}} / \mathcal{J}_i$$

and

$$\eta_i : \mathcal{O}_{Y, y_{n_i}} / (\mathcal{I}_{n_i} \cap \mathcal{J}_i) \longrightarrow \mathcal{O}_{Y, y_{n_i}} / \mathcal{I}_{n_i}$$

the quotient maps. Applying Lemma 3.2 we get the inequalities

$$\text{length}(\mathcal{O}_{X, x_i} / \mathcal{J}_i) \leq \tau(K_i, l_i) \tag{4}$$

for all $i \in \{1, \ldots, m\}$ such that $l_i \geq 2$.

Let y_i be one of the points $y_1, \ldots, y_{n_1 - 1}$ on the ramification divisor R and let $\mathcal{W}$ be the restriction of $\mathcal{Z}$ to y_i. Choosing coordinates $u_1, \ldots, u_n$ at the point y_i as in *Case 3* of the proof of Theorem 2.1, we have a decomposition

$$\mathcal{I}_{\mathcal{W}_i} = \bigoplus_{q=0}^{d-1} u^q \pi^{-1} \mathfrak{a}_{i,q} \ ,$$

where $\mathfrak{a}_{i,q}$ are ideals in $\mathcal{O}_{X, \pi(y_i)}$, hence we get an isomorphism

$$\mathcal{O}_{\mathcal{W}_i} \cong \bigoplus_{q=0}^{d-1} \mathcal{O}_X / \mathfrak{a}_{i,q} \tag{5}$$

Note that $\mathfrak{a}_{i,q} \subset \mathfrak{a}_{i,q+1}$ for $q = 0, \ldots, d-2$, giving a cofiltration

$$\mathcal{O}_X/\mathfrak{a}_{i,0} \twoheadrightarrow \mathcal{O}_X/\mathfrak{a}_{i,1} \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{O}_X/\mathfrak{a}_{i,d-1} \to 0 \ .$$

This implies $\operatorname{length}(\mathcal{O}_X/\mathfrak{a}_{i,q}) \leq \left\lfloor \frac{\operatorname{length}(\mathcal{O}_{\mathcal{W}_i})}{q+1} \right\rfloor$. Under the isomorphism (5) the element g_i corresponds to a d-tuple $(g_{i,0}, \ldots, g_{i,d-1})$ for $i = 1, \ldots, n_1 - 1$. Since $L - qM$ is $\sigma(k,d,q)$-very ample, the subspace

$$V_0^q = \{s \in H^0(X, L - qM) \mid s \operatorname{mod} \mathfrak{a}_{i,q} = g_{i,q} \text{ for } i = 1, \ldots, n_1 - 1\}$$

is at least $\sigma_0(k,d,q)$-very ample on $X \setminus R$, where

$$\sigma_0(k,d,q) = \sigma(k,d,q) - \left\lfloor \frac{\operatorname{length}(\mathcal{O}_{\mathcal{W}_i})}{q+1} \right\rfloor$$

for $q = 0, \ldots, \min(k, d-1)$. For any section s of the form

$$s = \pi^* s_0 + t\pi^* s_1 + \ldots + t^{\min(k,d-1)} \pi^* s_{\min(k,d-1)}$$

with $s_q \in V_0^q$ we then have

$$s \operatorname{mod} \mathcal{I}_i = g_i \quad \text{ in } \mathcal{O}_{Y,y_i}$$

for $i = 1, \ldots, n_1 - 1$.

From now on we proceed in m steps. The recursive procedure works so that the i-th step takes care of what we generate in the $(m-i+1)$-st fiber.

Step 1. For all $q \geq l_m$ such that $V_0^q \neq H^0(X, L - qM)$ we fix a section $s_q \in H^0(X, L - qM)$. For $i = 1, \ldots, l_m$ we denote by h_i the image

$$(\varphi^{\beta(n_m, n_m+i-1)})^* (\sum_{q \leq l_m} t^q \pi^*(s_q)) \operatorname{mod} \mathcal{I}_{n_m} \cap \ldots \cap \mathcal{I}_{n_{m+1}-1}$$

in the local ring $\mathcal{O}_{Y,y_{n_m}}$. We choose preimages $\widehat{g}_i$ of g_{n_m+i-1} in $\mathcal{O}_{Y,y_{n_m}}/\mathcal{I}_{n_m} \cap \ldots \cap \mathcal{I}_{n_{m+1}-1}$. For $q \leq l_m - 1$ we let V_1^q be the subspace of V_0^q of sections s_q satisfying the following conditions in $\mathcal{O}_{Y,y_{n_m}}$:

$$\begin{aligned}
t^{l_m-1}\pi^* s_{l_m-1} \operatorname{mod} \mathcal{J}_m &= \alpha_{l_m-1,1}\rho_m(\widehat{g}_1 - h_1) + \ldots + \alpha_{l_m-1,l_m}\rho_m(\widehat{g}_{l_m} - h_{l_m}) \\
&\ \vdots \\
t\pi^* s_1 \operatorname{mod} \mathcal{J}_m &= \alpha_{1,1}\rho_m(\widehat{g}_1 - h_1) + \ldots + \alpha_{1,l_m}\rho_m(\widehat{g}_{l_m} - h_{l_m}) \\
\pi^* s_0 \operatorname{mod} \mathcal{I}_{n_m} \cap \mathcal{J}_m &= \alpha_{0,1}(\widehat{g}_1 - h_1) + \ldots + \alpha_{0,l_m}(\widehat{g}_{l_m} - h_{l_m}) \ ,
\end{aligned}$$

where $\alpha_{i,j}$ are solutions of the Vandermonde type systems of linear equations

$$\begin{aligned}
\alpha_{0,i} + \ldots + \alpha_{l_m-1,i} &= \delta_{1i} \\
\alpha_{0,i} + \varepsilon^{\beta(n_m,n_m+1)}\alpha_{1,i} + \ldots + \varepsilon^{\beta(n_m,n_m+1)}\alpha_{l_m-1,i} &= \delta_{2i} \\
&\ \vdots \\
\alpha_{0,i} + \varepsilon^{(l_m-1)\beta(n_m,n_{m+1}-1)}\alpha_{1,i} + \ldots + \varepsilon^{(l_m-1)\beta(n_m,n_{m+1}-1)}\alpha_{l_m-1,i} &= \delta_{l_m i} \ .
\end{aligned}$$

We observe that (4) implies that V_1^q remains at least $\sigma_1(k,d,q)$-very ample, where for $q \geq 1$

$$\sigma_1(k,d,q) = \sigma_0(k,d,q) - \tau(K_m, l_m)$$

and for $q = 0$

$$\sigma_1(k,d,0) = \sigma_0(k,d,0) - \sum_{i=n_m}^{n_{m+1}-1} K_i = \sum_{i=1}^{n_m-1} k_i.$$

Let s be of the form $\pi^* s_0 + \ldots + t^l \pi^* s_l$, where $s_q \in V_1^q$, for $q \leq l_m - 1$. Then in the local ring $\mathcal{O}_{Y, y_{n_m+i-1}}$ we have

$$\begin{aligned}
s \bmod \mathcal{I}_{n_m+i-1} &= (\varphi^{\beta(n_m, n_m+i-1)})^* s \bmod \mathcal{I}_{n_m} \\
&= \pi^* s_0 + \varepsilon^{\beta(n_m, n_m+i-1)} t \pi^* s_1 + \ldots + \varepsilon^{(l_m-1)\beta(n_m, n_m+i-1)} t^{l_m-1} \pi^* s_{l_m-1} + \\
&\quad + (\varphi^{\beta(n_m, n_m+i-1)})^* (\sum_{q \geq l_m} s_q) \bmod \mathcal{I}_1 = \\
&= \alpha_{0,1} \eta_m(\widehat{g}_1 - h_1) + \ldots + \alpha_{0,l_m} \eta_m(\widehat{g}_{l_m} - h_{l_m}) + \ldots + \\
&\quad + \varepsilon^{\beta(n_m, n_m+i-1)} (\alpha_{1,1} \eta_m(\widehat{g}_1 - h_1) + \ldots + \alpha_{1,l_m} \eta_m(\widehat{g}_{l_m} - h_{l_m})) + \\
&\vdots \\
&\quad + \varepsilon^{(l_m-1)\beta(n_m, n_m+i-1)} (\alpha_{l_m-1,1} \eta_m(\widehat{g}_1 - h_1) + \ldots + \alpha_{l_m-1,l_m} \eta_m(\widehat{g}_{l_m} - h_{l_m}) + \\
&\quad + \eta_m(h_i) = \\
&= (\alpha_{0,1} + \varepsilon^{\beta(n_m, n_m+i-1)} \alpha_{1,1} + \ldots + \varepsilon^{(l_m-1)\beta(n_m, n_m+i-1)} \alpha_{l_m-1,1}) \eta_m(\widehat{g}_1 - h_1) + \\
&\vdots \\
&\quad + (\alpha_{0,l_m} + \varepsilon^{\beta(n_m, n_m+i-1)} \alpha_{1,l_m} + \ldots + \varepsilon^{(l_m-1)\beta(n_m, n_m+i-1)} \alpha_{l_m-1,l_m}) \eta_m(\widehat{g}_{l_m} - h_{l_m}) + \\
&\quad + \eta_m(h_i) \\
&= g_{n_m+i-1}
\end{aligned}$$

Step i. Proceeding as in *Step 1*, we fix sections $s_q \in V_{i-1}^q$ for $l_{m-i+1} \leq q < l_{m-i+2}$ and we construct subspaces

$$V_i^q \subset V_{i-1}^q$$

for $q < l_{m-i+1}$, such that for all $s_q \in V_i^q$ one has

$$\sum_{q=0}^{l} t^q \pi^* s_q \bmod \mathcal{I}_j = g_j$$

for all $n_{m-i+1} \leq j < n_{m-i+2}$. Then V_i^q is still $\sigma_i(k,d,q)$-very ample, where by (4)

$$\sigma_i(k,d,q) = \sigma_{i-1}(k,d,q) - \tau(K_i, l_i) = \sigma_0(k,d,q) - \sum_{\nu=i+1}^{m} \tau(K_\nu, l_\nu)$$

for $q \geq 1$ and

$$\sigma_i(k,d,0) = \sigma_0(k,d,0) - K_i = \sum_{\nu=1}^{n_{m-i+1}-1} k_\nu.$$

After $m-1$ steps, by Lemma 3.3, the space V^q_{m-1} will still be $\sigma_{m-1}(k,d,q)$-very ample with

$$\sigma_{m-1}(k,d,q) \geq \tau(K_1, l_1)$$

for $1 \leq q < l_1$ and

$$\sigma_{m-1}(k,d,0) \geq K_1.$$

This guarantees that we can find sections $s_1, \ldots, s_{l_1}$ such that

$$s =_{\text{def}} \sum_{q=0}^{l} t^q \pi^* s_q$$

satisfies

$$s \operatorname{mod} \mathcal{I}_{\mathcal{Z}} = g.$$

This completes the proof of the theorem. □

As in section 2 we get immediately the same result for unbranched coverings.

Theorem 3.4 *Let X be a smooth projective variety. Let M be a line bundle on X such that $\mathcal{O}_X(dM) \cong \mathcal{O}_X$ and let $\pi : Y \longrightarrow X$ be the cyclic covering of degree d defined by M. Let L be a line bundle on X and k a non-negative integer. If $L - qM$ is $\sigma(k,d,q)$-very ample for $q = 0, \ldots, \min(k, d-1)$ then $\pi^* L$ is k-very ample.*

Remark 3.5 In order to convey some feeling for the numbers $\sigma(k,d,q)$ appearing in the hypotheses of Theorem 3.1, we include a table for $d = 15$ and $0 \leq k \leq 15$. (The k-th column lists the values $\sigma(k, 15, q)$ for $1 \leq q \leq k$.) For instance $\pi^* L$ is 2-very ample if L is so and both $L - M$ and $L - 2M$ are globally generated, and it is 4-very ample if L is so and $L - M$ and $L - 2M$ are very ample and $L - 3M$ and $L - 4M$ are globally generated.

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$L-1M$		0	0	1	1	2	2	3	4	4	5	6	6	7	8	9
$L-2M$			0	0	1	2	2	3	4	4	5	6	6	7	8	9
$L-3M$				0	0	1	2	3	3	4	5	6	6	7	8	9
$L-4M$					0	0	1	2	3	4	4	5	6	7	8	8
$L-5M$						0	0	1	2	3	4	5	5	6	7	8
$L-6M$							0	0	1	2	3	4	5	6	6	7
$L-7M$								0	0	1	2	3	4	5	6	7
$L-8M$									0	0	1	2	3	4	5	6
$L-9M$										0	0	1	2	3	4	5
$L-10M$											0	0	1	2	3	4
$L-11M$												0	0	1	2	3
$L-12M$													0	0	1	2
$L-13M$														0	0	1
$L-14M$															0	0

4 Examples and applications

Theorems 2.1 and 3.1 state, roughly speaking, that under suitable assumptions on the bundles $L - qM$ the positivity of a pullback π^*L is at least as high as the positivity of L. It is then natural to ask whether the converse statement is also true. The following example shows that this is not the case:

Example 4.1 Let (X, L) be a principally polarized abelian variety. A d-torsion point $M \in \mathrm{Pic}^0(X)$ give rise to a cyclic covering $\pi : Y \longrightarrow X$ such that (Y, π^*L) is an abelian variety of type $(1, \ldots, 1, d)$. If d is sufficiently large, then π^*L will be very ample. (One knows by work of Debarre-Hulek-Spandaw [11] that it is enough to take $d > 2^g$.) But L is not even globally generated. In the surface case, where the generation of jets and the $k-$very ampleness is well understood (see [2]), one can even give explicit values d_k (resp. d'_k) for every $k > 0$ such that π^*L is k-jet ample (resp. $k-$very ample) whenever $d \geq d_k$ (resp. $d \geq d'_k$).

The previous example shows that the positivity can very well increase after taking a pullback. In general, however, it does not need to increase at all, as the following example shows:

Example 4.2 Consider a product of elliptic curves $X = E_1 \times \ldots \times E_n$ and the product polarization

$$L = \mathcal{O}_X\left(\sum_{i=1}^{n} pr_i^*(0)\right) ,$$

where pr_i is the i-th projection. The choice of a d-torsion point on E_n determines a covering $\widetilde{E}_n \longrightarrow E_n$ which in turn induces a cyclic covering $Y = E_1 \times \ldots \times E_{n-1} \times \widetilde{E}_n \longrightarrow X$ such that

$$\pi^*L = \mathcal{O}_Y\left(\sum_{i=1}^{n-1} pr_i^*(0) + d \cdot pr_n^*(0)\right) .$$

The line bundles $\mathcal{O}_X((k+2)L)$ and $\mathcal{O}_Y((k+2)\pi^*L)$ are then k-jet ample (by [1]), but neither of them is ℓ-jet ample (or even ℓ-very ample) for any $\ell > k$, since the restricted bundles $\mathcal{O}_X((k+2)L)|E_n$ and $\mathcal{O}_Y((k+2)\pi^*L)|\widetilde{E}_n$ are only of degree $k+2$ (cf. [8, Proposition 2.1]).

Even if L is positive, the positivity of π^*L may drop if the assumptions on the line bundles $L - qM$ are not satisfied. The following example illustrates their importance.

Example 4.3 Let $X \cong \mathbb{P}^n$ and let r and d be integers greater or equal 2. Let $M = \mathcal{O}_{\mathbb{P}^n}(r)$ and let B be a smooth member of $|dM|$. Let $\pi : Y \longrightarrow X$ be the cyclic covering defined by B and let $L = \mathcal{O}_{\mathbb{P}^n}((d-1)r)$. Then L is $d-$jet ample but π^*L fails to be that positive. The reason is that $L - qM$ fails to be $(d-q)-$jet ample for $q = d-1$. Indeed, as in the proof of the case 3 in Theorem 2.1 let $y \in R$ be a point on the ramification divisor

$R \subset Y$ and $u_1, \ldots, u_n$ be local coordinates such that R is locally defined by $u_1 = 0$. From the decomposition

$$H^0(\pi^* L) \cong H^0(\mathcal{O}_{\mathbb{P}^n}((d-1)r)) \oplus u_1 \cdot H^0(\mathcal{O}_{\mathbb{P}^n}((d-2)r)) \oplus \ldots \oplus u_1^{d-1} \cdot H^0(\mathcal{O}_{\mathbb{P}^n})$$

we infer that no section of $\pi^* L$ generates a jet of the form $u_1^{d-1} u_i$ for $i = 2, \ldots, n$. In fact Theorem 2.1 implies that $\pi^* L$ is $(d-1)-$ jet ample.

Remark 4.4 More generally, in the case of branched coverings, using the decomposition (1) one can show as in the previous example that if $L - qM$ fails to generate $(k-q)-$jets at the points of the branched locus, $\pi^* L$ fails to be $k-$jet ample.

Now, we want to show that the assumptions of Theorem 3.1 cannot be weakened.

Example 4.5 *(Geiser involution)* Let $B \subset X \cong \mathbb{P}^2$ be a smooth plane quartic and let $\pi : Y \longrightarrow X$ be the double covering branched over B. Then Y is the Del Pezzo surface of degree 2 and $-K_Y \cong \pi^* \mathcal{O}_{\mathbb{P}^2}(1)$, (see [6, 10.2.4]). Theorem 3.1 states that $-kK_Y$ is $k-$very ample for $k \geq 2$. This reproves a result of [14]. Moreover, in [14] it was shown that $-kK_Y$ is not $(k+1)-$very ample. This implies that our theorem is sharp.

Finally, we show how our results can be applied in certain situations to study higher order embeddings of blown up varieties.

Example 4.6 *(Bertini involution)* Let S be the Del Pezzo surface of degree 1. Then $|-K_S|$ has a base point x and $|-2K_S|$ is globally generated and defines a $2:1$ mapping $p : S \longrightarrow Q$ onto a quadric cone $Q \subset \mathbb{P}^3$. It is known (cf. [6, 10.4.3]) that p is branched over a smooth curve and the vertex $v = p(x)$ of Q. Let $\tau : \mathbb{F}_2 \longrightarrow Q$ be the blowing up of Q at v with the exceptional divisor D and let $\sigma : X \longrightarrow S$ be the blowing up of S at x with the exceptional divisor E. Then the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi} & \mathbb{F}_2 \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \tau} \\ S & \xrightarrow{p} & Q \end{array}$$

is commutative and π is a double covering branched over a smooth divisor homologous to $2(2D + 3f)$, where f denotes the pull-back of the ruling on Q by τ. It is interesting to investigate the positivity of the line bundles of the form $-sK_X - tE$, for $s, t > 0$. Notice that for $s, t \leq 0$ the line bundle is not even ample. We observe that $-sK_X - tE = s\sigma^*(-K_S) - (s+t)E$ but since the jet ampleness of $-sK_S$ is not known for we cannot apply the results from [10] directly. On the other hand $-sK_X - tE = s\pi^* f - t\pi^{-1}(D)$. This suggests to look for the positivity of the line bundles coming from $\mathbb{F}_2$ via π. We recall that a line bundle $L = aD + bf$ is $k-$jet ample on $\mathbb{F}_2$ (equivalently $k-$very ample, cf. e.g. [15], [8]) if and only if $L.D = -2a + b \geq k$ and $L.f = a \geq k$. Then Theorem 2.1 implies that $\pi^* L = -bK_X - 2aE$ is $k-$jet ample if $a \geq k+1$ and $b \geq 3k$ and Theorem 3.1 asserts that $\pi^* L$ is $k-$very ample for $a \geq \left\lfloor \frac{k-1}{2} \right\rfloor + 2$ and $b \geq 3 \left\lfloor \frac{k-1}{2} \right\rfloor + 3$.

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