# KLOOSTERMAN SUMS FOR CLIFFORD ALGEBRAS, POINCARE SERIES AND A LOWER BOUND FOR THE SMALLEST POSITIVE EIGENVALUE OF THE LAPLACIAN FOR CONGRUENCE SUBGROUPS ACTING ON HYPERBOLIC SPACES

by

J. Elstrodt, F. Grunewald and J. Mennicke

Max–Planck–Institut
für Mathematik
Gottfried–Claren–Str. 26
5300 Bonn 3
Federal Republic of Germany

MPI/89 –14

# Kloosterman sums for Clifford algebras, Poincaré series and a lower bound for the smallest positive eigenvalue of the Laplacian for congruence subgroups acting on hyperbolic spaces

J. Elstrodt[1]
F. Grunewald[2]
J. Mennicke[3]

[1] Mathematisches Institut der Universität Münster, Einsteinstr. 62, D-4400 Münster, FRG

[2] Mathematisches Institut der Universität Bonn, Wegelerstr. 10, D-5300 Bonn, FRG

[3] Fakultät für Mathematik der Universität Bielefeld, Universitätsstr., D-4800 Bielefeld, FRG

## CONTENTS

**Introduction**



****************************************************

The authors thank the Max-Planck-Institute for Mathematics in Bonn for its support.

## Introduction

For an integer $k \geq 0$ let

$$\mathcal{H}^{k+2} \tag{0.1}$$

stand for $(k+2)$-dimensional hyperbolic space. Here we shall be concerned with two models of $\mathcal{H}^{k+2}$. We write:

$$\mathbb{K}^{k+2} \quad , \quad \mathbb{H}^{k+2} \tag{0.2}$$

for the hyperboloid or upper half-space model,respectively.

Assume that $\Gamma$ is a discontinuos group of hyperbolic motions so that $\Gamma\backslash\mathcal{H}^{k+2}$ has finite hyperbolic volume. A central theme in Selberg's theory of harmonic analysis on weakly symmetric spaces is the study of the spectral theory of $-\Delta$ acting on a suitable dense subspace of $L^2(\Gamma\backslash\mathcal{H}^{k+2})$. Here $\Delta$ denotes the Laplace-Beltrami operator attached to the hyperbolic metric. The operator

$$-\Delta : C_0^\infty(\Gamma\backslash\mathcal{H}^{k+2}) \to L^2(\Gamma\backslash\mathcal{H}^{k+2})$$

is known to be essentially self-adjoint and positive. Here $C_0^\infty(\Gamma\backslash\mathcal{H}^{k+2})$ stands for the space of $\Gamma$-invariant functions on $\mathcal{H}^{k+2}$ which are infinitely differentiable and have compact support modulo $\Gamma$. Let $-\tilde{\Delta}$ be the unique self-adjoint extension of $-\Delta$. The continuous spectrum of $-\tilde{\Delta}$ is easy to describe. It is empty if $\Gamma\backslash\mathcal{H}^{k+2}$ is compact and equal to the intervall $[(k+1)^2/4, \infty[$ otherwise. Even a description of the continuous part of the spectral family can be given in terms of the analytically continued Eisenstein series. The discrete spectrum of $-\tilde{\Delta}$ is a discrete subset of $[0, \infty[$ containing 0. Apart from the usual theory for spectra of Laplace-Beltrami operators on Riemannian manifolds of finite volume very little more is known about the discrete spectrum. As a general notation we write:

$$\lambda_1^\Gamma \tag{0.3}$$

for the smallest eigenvalue $\neq 0$ of $-\tilde{\Delta}$ on $L^2(\Gamma\backslash\mathcal{H}^{k+2})$.

In dimension two there is a famous conjecture of Selberg ([62]) which can be stated in the following way. The group $\mathbf{SL}_2(\mathbb{R})$ acts in a natural way on the upper half-plane $\mathbb{H}^2$ and all subgroups of finite index in $\mathbf{SL}_2(\mathbb{Z})$ act discontinuously and with a quotient of finite volume on $\mathbb{H}^2$. The conjecture of Selberg says: If $\Gamma$ is a congruence subgroup of the modular group $\mathbf{SL}_2(\mathbb{Z})$, then we have

$$\lambda_1^\Gamma \geq \frac{1}{4}, \tag{0.4}$$

that is, the non-trivial discrete spectrum is contained in the continuous one. This conjecture is known to be very deep and to have profound consequences. It is known to be a consequence of the general conjectures of Langlands. It is also known that there are subgroups $\Gamma$ of finite index in $\mathbf{SL}_2(\mathbb{Z})$ so that $\lambda_1^\Gamma$ is arbitrarily small. See for example the chapters by Randol in [9]. So far, Selberg's conjecture has been proved only for a rather small set of congruence subgroups of the modular group. The first general non-trivial lower bound for $\lambda_1^\Gamma$ was established by Selberg [62] who proved that:

$$\lambda_1^\Gamma \geq \frac{3}{16} \tag{0.5}$$

for all congruence subgroups of the modular group. The strict inequality $\lambda_1^\Gamma > 3/16$ was proved by Gelbart and Jacquet [24] by means of a lift from $\mathbf{GL}(2)$ to $\mathbf{GL}(3)$. Despite considerable efforts a strictly larger lower bound that holds for all congruence subgroups has not been established so far. We refer to the recent papers of Iwaniec [34-39] and the references in these papers for a survey on the state of the art with respect to Selberg's conjecture.

Selberg's proof of (0.5) is based on an ingenious argument that relates the existence of small eigenvalues of $-\tilde{\Delta}$ to the abscissa of convergence of the Dirichlet series:

$$Z(m,n;s) = \sum_{c\neq 0}^{\infty} \frac{S(m,n;c)}{|c|^{2s}}.$$

where $S(m,n;c)$ denotes the usual Kloosterman sum. Then Weil's estimate of Kloosterman sums [68] guaranties that $Z(m,n;s)$ is holomorphic in a suitable half-plane. This yields that the crucial poles do not exist and (0.5) follows.

In the case of dimension 3 the groups $\mathbf{SL}_2(\mathcal{O})$ serve as a kind of substitute for the rational modular group. Here, $\mathcal{O}$ denotes the ring of integers in some imaginary quadratic number field. The continuous spectrum of $-\tilde{\Delta}$ on $L^2(\Gamma\backslash\mathbb{H}^3)$ is equal to the interval $[1,\infty[$. So far, the analogue of Selberg's conjecture for dimension 3, that is

$$\lambda_1^\Gamma \geq 1, \tag{0.6}$$

has been proved only in the special examples $\Gamma = \mathbf{SL}_2(\mathbb{Z}[i])$ and $\Gamma = \mathbf{SL}_2(\mathbb{Z}[\sqrt{-2}])$ (see [18]). The analogue of (0.5) for all groups $\mathbf{SL}_2(\mathcal{O})$ is due to Sarnak [58], who proved that

$$\lambda_1^\Gamma \geq \frac{3}{4}. \tag{0.7}$$

Sarnak's proof adapts Selberg's arguments to the three-dimensional case. His method also works for all congruence subgroups of the groups $\mathbf{SL}_2(\mathcal{O})$.

In the present paper we generalize Selberg's result to congruence groups acting on the higher dimensional spaces.

Our results can conveniently be stated in terms of the hyperboloid model $\mathbb{K}^{k+2}$. The space $\mathbb{K}^{k+2}$ is naturally acted on by the component of the identity $\mathbf{SO}^\circ(1,k+2)$ of the orthogonal group $\mathbf{SO}(1,k+2)$. Identifying the group $\mathbf{SO}^\circ(1,k+2)$ with the group of orientation preserving isometries of $\mathbb{K}^{k+2}$ we get an identification:

$$\mathbb{K}^{k+2} \quad = \quad \mathbf{SO}^\circ(1,k+2)/\mathbf{SO}(k+2). \tag{0.8}$$

The orthogonal group $\mathbf{SO}(k+2)$ is the stabilizer of a suitable point in $\mathbb{K}^{k+2}$. The congruence subgroups of $\mathbf{SO}^\circ(1,k+2)$ are constructed in the following way. Let $Q$ be a quadratic form with rational coefficients which is of signature $(1,k+2)$. The group of units

$$\mathbf{SO}^\circ_{k+3}(\mathbb{Z},Q) := \mathbf{SO}_{k+3}(\mathbb{Z},Q) \cap \mathbf{SO}^\circ_{k+3}(\mathbb{R},Q)$$

acts via

$$\mathbf{SO}^\circ_{k+3}(\mathbb{Z},Q) < \mathbf{SO}^\circ_{k+3}(\mathbb{R},Q) \cong \mathbf{SO}^\circ(1,k+2)$$

on $\mathbb{K}^{k+2}$. A subgroup $\Gamma < \mathbf{SO}^\circ_{k+3}(\mathbb{Z},Q)$ is called a congruence subgroup if it contains

$$\mathbf{SO}^\circ_{k+3}(\mathbb{Z},Q) \cap (I + \ell(M_{k+3}(\mathbb{Z}))$$

for some $\ell \in \mathbb{Z}\backslash\{0\}$. The symbol $M_{k+3}(\mathbb{Z})$ stands for the ring of $(k+3)\times(k+3)$ matrices over $\mathbb{Z}$. For each group $\Gamma$ the quotient $\Gamma\backslash\mathbb{K}^{k+2}$ is always of finite volume, it is compact if and only if $Q$ is $\mathbb{Q}$-anisotropic.

**Theorem A.** Let $k \geq 0$ be an integer. Let $Q$ be a quadratic form with rational coefficients. Assume that $Q$ is of signature $(1,k+2)$ and isotropic over $\mathbb{Q}$. For any congruence subgroup $\Gamma < \mathbf{SO}^\circ_{k+3}(\mathbb{Z},Q)$ we have

$$\lambda_1^\Gamma \quad \geq \quad \begin{cases} \dfrac{3}{16} & \text{if} \quad k=0, \\ \dfrac{2k+1}{4} & \text{if} \quad k>0. \end{cases}$$

The case $k=0$ is easily derived from Selberg's result (0.5) using the exceptional isomorphism:

$$\mathbf{PSL}_2(\mathbb{R}) \cong \mathbf{SO}^\circ(1,2).$$

The result in Theorem A was announced by the authors in [16]. Li, Piatetski- Shapiro and Sarnak [45] have recently also announced the same result.The proof indicated by Li, Piatetski- Shapiro and Sarnak seems to be quite different from ours.

Notice that the Theorem of Hasse and Minkowski implies that there are no rational anisotropic quadratic forms $Q$ of signature $(1,k+2)$ if $k \geq 2$. In the remaining cases $(k=1,2)$ a result analogous to (0.5) and (0.7) can be derived from Langlands' correspondence; see for example [17].

In principle, our proof of Theorem A is based on Selberg's approach using the upper half-space model $\mathbb{H}^{k+2}$ of hyperbolic space.Instead of $\mathbf{SO}^\circ_{k+3}(\mathbb{Q},Q)$ we work with $\mathbf{Spin}_{k+3}(\mathbb{Q},Q)$ and its congruence subgroups $\Gamma$.Theorem A is then deduced using the spin-homomorphism:

$$\Theta : \mathbf{Spin}_{k+3}(\mathbb{Q},Q) \to \mathbf{O}^\circ_{k+3}(\mathbb{Q},Q). \tag{0.9}$$

We use a particular representation of $\mathbf{Spin}_{k+3}(\mathbb{Q}, Q)$ as a Vahlen group $SV_k(\mathbb{Q}, q)$ of certain $(2 \times 2)$- matrices over a rational Clifford algebra $\mathcal{C}(q)$ defined for a negative definite quadratic form $q$ suitably related with $Q$. If $q$ is any negative definite quadratic form over $\mathbb{Q}$ in $k$ variables, then the group $SV_k(\mathbb{R}, q)$ acts naturally on the upper half-space model $\mathbb{H}^{k+2}$. We develop the necessary background on Clifford algebras, the Vahlen group and arithmetic subgroups of the Vahlen group in paragraphs 1, 2. In particular, we define suitable congruence subgroups of $SV_k(\mathbb{Q}, q)$ in paragraph 3 and relate them to the congruence subgroups of $O^{\circ}_{k+3}(\mathbb{Q}, Q)$ for an appropriate quadratic form $Q$.

An important idea in Selberg's approach is to introduce suitable Poincaré series and to compute the inner product of these series. Here our work was inspired by recent developments due to Bump,Deshouillers, Friedberg,Goldfeld,Iwaniec,Kuznetsov,Piatetski-Shapiro and Stevens (see [6,7,13,23,25-28,34-39,43,52,65]).

We develop the necessary machinery for our case in paragraphs 8, 9. There a certain Dirichlet series

$$Z(\mu, \nu; s) = \sum_{\gamma} \frac{S(\mu, \nu; \gamma)}{(\bar{\gamma}\gamma)^s}, \tag{0.10}$$

the so-called Linnik-Selberg series, comes up, and the half-plane of absolute convergence of this series is intimately connected with the small eigenvalues of $-\tilde{\Delta}$ (see §10). The coefficients of (0.10) are a new type of Kloosterman sums attached to Vahlen groups, and one of the most difficult parts of this work is to establish non-trivial bounds for these generalized Kloosterman sums. We define the generalized Kloosterman sums in paragraph 4, and in paragraph 5 we decompose them into certain local factors $S_p(\mu, \nu; \gamma)$. These local factors are related with classical Kloosterman sums in paragraph 6, and then Weil's estimate [68] gives estimates for the $S_p(\mu, \nu; \gamma)$.

This approach is different from the method indicated in [16]. In [16] our computations of the coefficients was less specific than in the present paper. We then had to use other estimates for exponential sums from Deligne's theory [12]. We here only need Weil's bound. This, of course, means a noticeable simplification with respect to the tools used. Our estimates for the $S_p(\mu, \nu; \gamma)$ combined with certain results on congruential representation numbers of quadratic forms lead in paragraph 7 to the result that

$$Z(\mu, \nu; s) \quad \text{converges absolutely for} \quad Re\, s > k + 1/2 \tag{0.11}$$

whenever $k \geq 1$ and $\mu, \nu$ are both nonzero. We also have proved that $Z(\mu, \nu; s)$ has a meromorphic continuation to the entire complex plane. This will be reported on in a subsequent publication.

In paragraph 10 we then deduce from (0.11) that:

$$\lambda_1^{\Gamma} \geq \frac{2k+1}{4} \tag{0.12}$$

whenever $k > 0$ is an integer , $q$ is a negatively definite rational quadratic form in $k$ variables and $\Gamma < \mathrm{SV}_k(\mathbb{Q}, Q)$ is a congruence subgroup.

The case $k = 1$ is just Sarnak's result (0.7). Our proof works uniformly only in case $k \geq 2$. We indicate in paragraph 7 the changes which have to be made to obtain Sarnak's result.

The result (0.12) implies Theorem A and also the following:

**Theorem B.** Let $k > 0$ be an integer and $Q$ a rational quadratic form which is of signature $(1, k+2)$. Assume that $Q$ is $\mathbb{Q}$-isotropic. Let

$$\Theta : \mathbf{Spin}_{k+3}(\mathbb{Q}, Q) \to \mathbf{O}^{\circ}_{k+3}(\mathbb{Q}, Q)$$

be the spin homomorphism. Let $\Gamma < \mathbf{O}^{\circ}_{k+3}(\mathbb{Q}, Q)$ be a subgroup so that $\Theta^{-1}(\Gamma)$ is a congruence subgroup of $\mathbf{Spin}_{k+3}(\mathbb{Q}, Q)$. Then

$$\lambda_1^{\Gamma} \geq \frac{2k+1}{4}.$$

Theorem B is slightly better than Theorem A since the image under $\Theta$ of a congruence subgroup in $\mathbf{Spin}_{k+3}(\mathbb{Q}, Q)$ need not be a congruence subgroup. See paragraph 3 for more explanations.

Using our approach we have proved asymptotic relations for sums of certain exponential sums. These results are inspired by the paper of Deshouillers and Iwaniec [13]. We hope to come back to this in a future publication.

Another application of our result is that the error terms in various asymptotic laws connected with the hyperbolic lattice point theorem can be explicitly estimated. See [17] for some number theory in this connection.

In contrast to the Selberg conjecture in dimension 2 and 3 ((0.4),(0.6)) we want to point out that the naive generalization of Selberg's conjecture to the higher-dimensional cases, namely

$$\lambda_1^{\Gamma} \geq (k+1)^2/4 \tag{0.13}$$

does **not** hold for all congruence subgroups of $\mathrm{SV}_k(\mathbb{Q}, q)$ for $k \geq 2$. Eigenforms of the Laplace-Beltrami operator can be constructed by lifting certain holomorphic modular forms. See [45] for an announcement. This phenomenon was also noticed in conversations of S. Rallis and the second author. So perhaps all eigenvalues in the interval $[(2k+1)/4, (k+1)^2/4[$ are obtained by this process.

## §1. Clifford Algebras, Vahlen's Groups and Hyperbolic Spaces

This paragraph gives the construction of Vahlen's groups and their action on hyperbolic spaces. The present work relies heavily on [15]. We briefly recall the basic definitions and fix the notations. Suppose that $K$ is a field with $\operatorname{char} K \neq 2$ and $E$ is a $k$-dimensional ($k \geq 0$) $K$-vector space with non-degenerate quadratic form $q: E \to K$. Let $\mathcal{C}(q)$ be the associated Clifford algebra (see [5], [10], [14]); for $k = 0$ we have $\mathcal{C}(q) = K$. We identify $K$ and $E$ with their canonical images in $\mathcal{C}(q)$ and define

(1.1) $$V_q := K \cdot 1 \oplus E \subset \mathcal{C}(q).$$

Let $e_1, \ldots, e_k$ be a basis of $E$ orthogonal with respect to $q$. Then we have in $\mathcal{C}(q)$

(1.2) $$e_\mu^2 = q(e_\mu), \quad e_\mu e_\nu = -e_\nu e_\mu \quad (\mu, \nu = 1, \ldots, k, \quad \mu \neq \nu).$$

Denote by $\mathcal{E}_k$ the set of all subsets of $\{1, \ldots, k\}$. For $M \in \mathcal{E}_k$, $M = \{\nu_1, \ldots, \nu_r\}$ with $\nu_1 < \ldots < \nu_r$ we define

(1.3) $$e_M := e_{\nu_1} \cdot \ldots \cdot e_{\nu_r}, \quad e_\emptyset := 1 \in \mathcal{C}(q).$$

Then $\{e_M : M \in \mathcal{E}_k\}$ is a basis of $\mathcal{C}(q)$ over $K$. There are three important linear involutions defined on $\mathcal{C}(q)$ by means of

(1.4) $$\begin{cases} \overline{e}_M = (-1)^{r(r+1)/2}\, e_M, \\ e'_M = (-1)^r\, e_M, \\ e^*_M = (-1)^{r(r-1)/2}\, e_M \end{cases} \qquad (M \in \mathcal{E}_k,\ r = |M|).$$

The linear extensions of $^-$, $'$ and $^*$ are commuting linear automorphisms of $\mathcal{C}(q)$ such that

(1.5) $$\begin{cases} x^* = \overline{x'} \quad \text{for all} \quad x \in \mathcal{C}(q), \\ \overline{x} = -x,\ x' = -x,\ x^* = x \quad \text{for all} \quad x \in E, \\ \overline{xy} = \overline{y}\,\overline{x},\ (xy)' = x'y',\ (xy)^* = y^*x^* \quad \text{for all} \quad x \in \mathcal{C}(q). \end{cases}$$

The automorphism $^*$ is called the main antiinvolution, and $'$ is called the main involution of $\mathcal{C}(q)$. The **trace map** $tr: \mathcal{C}(q) \to \mathcal{C}(q)$ is defined by

(1.6) $$tr\, x = x + \overline{x}.$$

In the following we always assume $K = \mathbb{Q}$ or $K = \mathbb{R}$. We need the following concept:

**1.1 Definition.** Assume that $q$ is negative definite. An element $v \in \mathcal{C}(q)$ with $v \neq 0$ is called a **transformer** if there exists a linear automorphism $\varphi_v \colon V_q \to V_q$ such that

$$vx = \varphi_v(x)v' \quad \text{for all} \quad x \in V_q. \tag{1.7}$$

Let $T(q)$ denote the set of all transformers contained in $\mathcal{C}(q)$.

We know from [15], Proposition 3.6 that $T(q)$ is the multiplicative subgroup of $\mathcal{C}(q)$ generated by $V_q \backslash \{0\}$. It follows that every $x \in T(q)$ satisfies $\bar{x}x \in K \backslash \{0\}$.

Let $k$ be a non-negative integer. For $K = \mathbb{R}, E = \mathbb{R}^k$ and

$$q = -I_k = \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 \end{pmatrix}$$

the negative definite unit form, we write $\mathcal{C}_k$, $V_{k+1}$, $T_k$ for $\mathcal{C}(q)$, $V_q$, $T(q)$. We denote by $i_1, \dots, i_k$ the canonical basis of $\mathbb{R}^k$ and by $\{i_M : M \in \mathcal{E}_k\}$ the associated basis of $\mathcal{C}_k$. We tacitly identify $\mathcal{C}_k$ with the obvious subalgebra of $\mathcal{C}_{k+1}$ by means of the inclusion $\mathcal{E}_k \subset \mathcal{E}_{k+1}$. By definition, $\mathcal{C}_0 = \mathbb{R}$, $\mathcal{C}_1 = \mathbb{C}$.

We equip $V_{k+1}$ with the **scalar product**

$$\langle v, w \rangle := \frac{1}{2} \, tr \, v\overline{w} \quad (v, w \in V_{k+1}). \tag{1.8}$$

Then $\{1, i_1, \dots, i_k\}$ becomes an orthonormal basis of $V_{k+1}$. For

$$x = \sum_{M \in \mathcal{E}_k} \lambda_M i_M \in \mathcal{C}_k \quad (\lambda_M \in \mathbb{R})$$

let

$$|x| := \Big( \sum_{M \in \mathcal{E}_k} \lambda_M^2 \Big)^{1/2} \tag{1.9}$$

denote the Euclidean norm of $x$. Then we have

$$|v|^2 = v\overline{v} = \overline{v}v \quad \text{for all} \quad v \in T_k \tag{1.10}$$

and in particular

$$|vw| = |v||w| \quad \text{for all} \quad v, w \in T_k. \tag{1.11}$$

Moreover,

$$|x| = \langle x, x\rangle^{1/2} \quad \text{for all} \quad x \in V_{k+1}. \tag{1.12}$$

For any lattice $M \subset V_{k+1}$ we denote by

$$M^{\#} := \{y \in V_{k+1} \mid \langle x, y\rangle \in \mathbb{Z} \quad \text{for all} \quad x \in M\} \tag{1.13}$$

the dual lattice of $M$.

The set underlying our model of $(k+2)$-dimensional hyperbolic space is the upper half-space

$$\mathbb{H}^{k+2} := \{x_0 + x_1 i_1 + \ldots + x_{k+1} i_{k+1} \mid x_0, \ldots, x_{k+1} \in \mathbb{R}, x_{k+1} > 0\}.$$

Define maps $z\colon \mathbb{H}^{k+2} \to V_{k+1}$, $r\colon \mathbb{H}^{k+2} \to ]0, \infty[$ by

$$\begin{cases} z(P) := x_0 + x_1 i_1 + \ldots + x_k i_k, \\ r(P) := x_{k+1} \end{cases} \tag{1.14}$$

for $P = x_0 + x_1 i_1 + \ldots + x_{k+1} i_{k+1} \in \mathbb{H}^{k+2}$. Then (1.10) yields

$$|P|^2 = |z(P)|^2 + r(P)^2 \quad \text{for all} \quad P \in \mathbb{H}^{k+2}. \tag{1.15}$$

We endow $\mathbb{H}^{k+2}$ with the Riemannian metric

$$ds^2 = \frac{{dx_0}^2 + \ldots + dx_{k+1}^2}{x_{k+1}^2}$$

and obtain the model $(\mathbb{H}^{k+2}, d)$ of $(k+2)$-dimensional hyperbolic space. The hyperbolic distance $d(P, Q)$ of two points $P, Q \in \mathbb{H}^{k+2}$ is given by

$$2 \cosh d(P, Q) = \delta(P, Q), \tag{1.16}$$

where $\delta(P, Q)$ is defined by

$$\delta(P, Q) = \frac{|z(P) - z(Q)|^2 + r(P)^2 + r(Q)^2}{r(P)\, r(Q)} \tag{1.17}$$

(see [17], Prop. 1.4). The volume measure associated with the hyperbolic metric $ds^2$ is

$$dv = \frac{dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{k+1}}{x_{k+1}^{k+2}} \tag{1.18}$$

and the corresponding Laplace-Beltrami operator is

$$\Delta = x_{k+1}^2 \left(\frac{\partial^2}{\partial x_0^2} + \ldots + \frac{\partial^2}{\partial x_{k+1}^2}\right) - k x_{k+1} \frac{\partial}{\partial x_{k+1}}. \tag{1.19}$$

The group of orientation preserving isometries of $\mathbb{H}^{k+2}$ can be described by means of a certain group of $(2 \times 2)$ - matrices over $\mathcal{C}_k$, the so-called Vahlen group. This description is due to Vahlen [66]; see also [1], [2], [3], [15], [17], [47].

**1.2 Definition.** Let $M_2(\mathcal{C}_k)$ be the set of $(2 \times 2)$- matrices over $\mathcal{C}_k$ and define the **Vahlen group** $SV_k$ by

$$SV_k := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathcal{C}_k) \;\middle|\; \begin{array}{ll} \text{(i)} & \alpha,\beta,\gamma,\delta \in T_k \cup \{0\}, \\ \text{(ii)} & \overline{\alpha}\beta, \overline{\gamma}\delta \in V_{k+1}, \\ \text{(iii)} & \alpha\delta^* - \beta\gamma^* = 1 \end{array} \right\}.$$

Clearly, $SV_0 = SL_2(\mathbb{R}), SV_1 = SL_2(\mathbb{C})$. We infer from [1-3] or [15] or [47] that $SV_k$ is a group under matrix multiplication with inverse

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}. \tag{1.20}$$

The group $SV_k$ was discussed in a more general context in [15], and Theorem 3.7 of the latter paper gives two more descriptions of $SV_k$. We remark that $SV_k$ is generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \qquad (x \in V_{k+1})$$

(see [15, Proposition 3.8]). The importance of $SV_k$ for hyperbolic geometry is manifest from the next theorem which collects some results from [15], §5; these results may also be drawn from [1-3, 47, 66].

**1.3 Theorem.** Suppose that $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k$ and $P \in \mathbb{H}^{k+2}$. Then $\gamma P + \delta \in T_{k+1}$,

$$\sigma P := (\alpha P + \beta)(\gamma P + \delta)^{-1} \in \mathbb{H}^{k+2}, \tag{1.21}$$

and the map $P \to \sigma P$ is an orientation preserving isometry of $\mathbb{H}^{k+2}$. The corresponding group homomorphism induces an isomorphism of $SV_k/\{I, -I\}$ onto the group of orientation-preserving motions of $\mathbb{H}^{k+2}$. The action of $SV_k$ on $\mathbb{H}^{k+2}$ is both transitive and transitive on pairs of points with fixed hyperbolic distance.

Let $z = z(P), r = r(P)$. Then (1.21) may be rewritten in the form

$$z(\sigma P) = \frac{(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha\overline{\gamma} r^2}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}, \tag{1.22}$$

$$r(\sigma P) = \frac{r}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}. \tag{1.23}$$

The following technical lemma is contained in [15], Theorem 3.7.

**1.4 Lemma.** If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k$, then

(i) $\alpha\overline{\gamma}, \beta\overline{\delta} \in V_{k+1}$,

(ii) $\alpha x\overline{\delta} + \beta\overline{x}\,\overline{\gamma} \in V_{k+1}$ for all $x \in V_{k+1}$,

(iii) $\alpha x\overline{\beta} + \beta\overline{x}\,\overline{\alpha}, \gamma x\overline{\delta} + \delta\overline{x}\,\overline{\gamma} \in \mathbb{R}$ for all $x \in V_{k+1}$.

Recall that

$$Re\frac{aw+b}{cw+d} = \frac{a}{c} - \frac{Re(cw+d)}{c|cw+d|^2}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ with $c \neq 0$ and $w \in \mathbb{H}^2$. The following lemma is a kind of substitute for this rule which holds for the Vahlen group.

**1.5 Lemma.** If $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k$ with $\gamma \neq 0$ and $P \in \mathbb{H}^{k+2}, z(P) = z, r(P) = r$, then

$$z(\sigma P) = \alpha\gamma^{-1} - (\gamma^*)^{-1} \frac{\overline{(\gamma z + \delta)}}{|\gamma z + \delta|^2 + |\gamma|^2 r^2}. \tag{1.24}$$

**Proof.** Using (1.10) and the "determinant condition" $\beta\gamma^* = \alpha\delta^* - 1$, we rewrite the numerator in (1.22) as follows:

$$\begin{aligned}(\alpha z + \beta)\overline{(\gamma z + \delta)} + \alpha\overline{\gamma}r^2 &= (\alpha z + |\gamma'|^{-2}\beta\gamma^*\gamma')\overline{(\gamma z + \delta)} + \alpha\overline{\gamma}r^2 \\ &= (\alpha z + |\gamma'|^{-2}\alpha\delta^*\gamma')\overline{(\gamma z + \delta)} + \alpha\overline{\gamma}r^2 - (\gamma^*)^{-1}\overline{(\gamma z + \delta)}.\end{aligned}$$

Here, the last term yields the second term on the right-hand side of (1.24). Now the assertion follows from

$$\begin{aligned}&(\alpha z + |\gamma'|^{-2}\alpha\delta^*\gamma')\overline{(\gamma z + \delta)} + \alpha\overline{\gamma}r^2 \\ &= \alpha\gamma^{-1}((\gamma z + |\gamma'|^{-2}\gamma\delta^*\gamma')\overline{(\gamma z + \delta)} + |\gamma|^2 r^2) \\ &= \alpha\gamma^{-1}(|\gamma z + \delta|^2 + |\gamma|^2 r^2),\end{aligned}$$

since $|\gamma'|^{-2}\gamma\delta^*\gamma' = \delta$, because of Lemma 1.4 (i) applied to $\sigma^{-1}$ and since the elements of $V_{k+1}$ are invariant under the anti-involution $^*$. □

## §2. Some Arithmetic Subgroups of the Vahlen Group

Let $K$ be a field with $\operatorname{char} K \neq 2$ and $k \geq 0$ an integer. $E$ is a k dimensional vector space over $K$ with non-degenerate quadratic form $q: E \to K$, and let $C(q)$, $V_q$ be defined as in paragraph 1. The **Vahlen group** for $q$ is defined by

$$(2.1) \qquad SV_k(K,q) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(C(q)) \;\middle|\; \begin{array}{l} (i)\ \alpha\delta^* - \beta\gamma^* = 1, \\ (ii)\ \alpha\beta^* = \beta\alpha^*,\ \gamma\delta^* = \delta\gamma^*, \\ (iii)\ \alpha\overline{\alpha},\ \beta\overline{\beta},\ \gamma\overline{\gamma},\ \delta\overline{\delta} \in K, \\ (iv)\ \alpha\overline{\gamma},\ \beta\overline{\delta} \in V_q, \\ (v)\ \alpha x\overline{\beta} + \beta\overline{x}\,\overline{\alpha},\ \gamma x\overline{\delta} \\ \quad + \delta\overline{x}\,\overline{\gamma} \in K\ (\forall x \in V_q), \\ (vi)\ \alpha x\overline{\delta} + \beta\overline{x}\,\overline{\gamma} \in V_q\ (\forall x \in V_q) \end{array} \right\},$$

see [15]. Then $SV_k(K,q)$ is a group under matrix multiplication with inverse (1.20), and we proved in [15], Theorem 3.7 that

$$(2.2) \qquad SV_k(\mathbb{R}, -I_k) = SV_k$$

with $SV_k$ from Definition 1.2. Note that $SV_k(K,q)$ is the set of $K$-rational points of an affine algebraic group defined over $K$.

Suppose now that $q_1$ is another non-degenerate quadratic form on the $K$-vector space $E_1$ such that $q_1 \circ f = q$ for some linear isomorphism $f: E \to E_1$. Then there exists a unique isomorphism

$$(2.3) \qquad \hat{f}: C(q) \to C(q_1)$$

such that $\hat{f} \mid E = f$ (see [5], §9, no. 2, p. 140). Moreover, $\hat{f}$ commutes with $^*$, $'$ and $^-$. Hence $\hat{f}$ induces an isomorphism

$$(2.4) \qquad \hat{f}: SV_k(K,q) \to SV_k(K,q_1).$$

For the rest of paragraph 2 let $K = \mathbb{Q}$, assume that $q$ is negative definite, and let $E, q, C(q), SV_k(\mathbb{Q}, q)$ be as above. Recall that a **$\mathbb{Z}$-order** in a $\mathbb{Q}$-algebra $A$ is a subring R such that the additive group of R is finitely generated and contains a $\mathbb{Q}$-basis of $A$. Rings and algebras are always (tacitly) assumed to have a unit element and substructures are supposed to contain the unit element.

## §2. *Some Arithmetic Subgroups of the Vahlen group*

**2.1 Definition.** A subring $R \subset \mathcal{C}(q)$ is called **compatible** if it is stable under the involutions $^*$ and $'$ of $\mathcal{C}(q)$. For a compatible subring $R \subset \mathcal{C}(q)$ let

$$V(R) := R \cap V_q, \quad T(R) := R \cap T(q), \tag{2.5}$$

$$SV_k(R) := SV_k(\mathbb{Q}, q) \cap M_2(R). \tag{2.6}$$

For $n \in \mathbb{N}$ and $J \subset \mathcal{C}(q)$ a compatible $\mathbb{Z}$- order let

$$SV_k(J; n) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(J) \;\middle|\; \alpha - 1, \beta, \gamma, \delta - 1 \in nJ \right\} \tag{2.7}$$

denote the **principal congruence subgroup** of $SV_k(J)$ of **level** $n$. A subgroup $\Gamma < SV_k(J)$ is called a **congruence subgroup** if $SV_k(J; n) < \Gamma$ for some $n \in \mathbb{N}$.

The set $SV_k(\mathbb{Q}, q)$ is the group of $\mathbb{Q}$-rational points of a connected and simply connected affine algebraic $\mathbb{Q}$-group of both rational and real rank one. (In fact, we show in §3 that $SV_k(\mathbb{Q}, q)$ is isomorphic to a certain spin-group which is known to be simply connected.) Let $f$ be an isomorphism of $E \otimes_{\mathbb{Q}} \mathbb{R}$ onto $\mathbb{R}^k$ such that $q = -I_k \circ f$. (Such an $f$ exists by Sylvester's inertia law.) The groups $\hat{f}(SV_k(J; n)) \subset SV_k \quad (n \in \mathbb{N})$ (see (2.4)) are discrete and hence act discontinuously on $\mathbb{H}^{k+2}$. If $f_1$ and $f_2$ are two choices for $f$, then the groups $\hat{f}_1(SV_k(J; n))$, and $\hat{f}_2(SV_k(J; n))$ are conjugate in $SV_k$. Our next aim is to show that the groups $\hat{f}(SV_k(J; n))$ are cofinite. For this we need a little lemma.

**2.2 Lemma.** Let $q$ be negative definite, $n \in \mathbb{N}$, and assume that $J_1, J_2$ are compatible $\mathbb{Z}$-orders in $\mathcal{C}(q)$. Then there exists an $m \in \mathbb{N}$ such that

$$SV_k(J_2; m) < SV_k(J_1; n).$$

**Proof.** There exists an $\ell \in \mathbb{N}$ such that

$$\ell J_1 + \ell J_2 \subset J_1 \cap J_2. \qquad \square$$

Sometimes it is convenient to restrict to a special set of forms $q$ which contains a representative of each equivalence class of quadratic forms. For $d = (d_1, \ldots, d_k)^t \in \mathbb{N}^k \quad (k \geq 0)$ let $q_d : \mathbb{Q}^k \to \mathbb{Q}$,

$$q_d(x) := \sum_{j=1}^{k} d_j x_j^2 \qquad \text{for} \qquad (x_1, \ldots, x_k)^t \in \mathbb{Q}^k \tag{2.8}$$

and

$$\mathcal{C}_d := \mathcal{C}(-q_d). \tag{2.9}$$

The generators $e_1, \dots e_k$ of $\mathcal{C}_d$ satisfy

$$e_\nu^2 = -d_\nu, \quad e_\mu e_\nu = -e_\nu e_\mu \quad (\mu, \nu = 1, \dots, k,\ \mu \neq \nu). \tag{2.10}$$

Hence we may embed $\mathcal{C}_d$ as a subalgebra of $\mathcal{C}_k$ via the map

$$e_\nu \mapsto \sqrt{d_\nu}\, i_\nu, \qquad \nu = 1, \dots, k. \tag{2.11}$$

With respect to this embedding we have the identifications

$$V_{-q_d} = \mathcal{C}_d \cap V_{k+1}, \tag{2.12}$$

$$SV_k(\mathbb{Q}, -q_d) = SV_k \cap M_2(\mathcal{C}_d). \tag{2.13}$$

In the sequel we always embed $\mathcal{C}_d$ tacitly into $SV_k$ as above. Note that

$$J_d := \bigoplus_{M \in \mathcal{E}_k} \mathbb{Z}\, e_M \tag{2.14}$$

is a compatible $\mathbb{Z}$-order of $\mathcal{C}_d$.

**2.3 Proposition.** Let $q$ be negative definite, and let $f : E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^k$ be an isomorphism such that $q = -I_k \circ f$. Assume that $J$ is a compatible $\mathbb{Z}$-order in $\mathcal{C}(q)$ and $n \in \mathbb{N}$. Then $\hat{f}(SV_k(J; n)) \subset SV_k$ is a cofinite discrete non-cocompact group.

**Proof.** By the above arguments, $\hat{f}(SV_k(J, n))$ is conjugate to a group which is commensurable with $SV_k(J_d)$ (see (2.14)). The assertion for $SV_k(J_d; n)$ was proved in [17]. □

## §3. The Isogenies of Vahlen's Groups and Orthogonal Groups

Although our later considerations will be in terms of the Vahlen group, we also want to formulate our results for orthogonal groups. For this we recall the hyperboloid model of hyperbolic space (see [15], Section 5): For $k \geq 0$ let $\tilde{q}_k$ be the following quadratic form in $k+3$ variables:

$$\tilde{q}_k(y_0, y_1, y_2, x_1, \ldots, x_k) = y_0^2 - y_1^2 - y_2^2 - x_1^2 - \ldots - x_k^2.$$

The set underlying the hyperboloid model is

$$\mathbb{K}^{k+2} := \{y = (y_0, y_1, y_2, x_1, \ldots, x_k)^t \mid y_0 > 0, \tilde{q}_k(y) = 1\}. \tag{3.1}$$

The line element

$$ds^2 = -dy_0^2 + dy_1^2 + dy_2^2 + dx_1^2 + \ldots + dx_k^2$$

defines the hyperbolic metric on $\mathbb{K}^{k+2}$. We denote by $\mathrm{Iso}(\mathbb{K}^{k+2})$ and $\mathrm{Iso}^+(\mathbb{K}^{k+2})$ the sets of isometries and orientation preserving isometries of $\mathbb{K}^{k+2}$, respectively.

If $q$ is a non-degenerate quadratic form on the vector space $K^n$ over a field $K$ and $R \subset K$ a subring of $K$, we denote by $\mathbf{O}_n(R, q)$ the set of elements of the orthogonal group $\mathbf{O}_n(K, q)$ of $q$ over $K$ with entries in $R$. This notation follows the book of Dieudonné [14]. Let

$$\mathbf{O}^\circ(1, k+2) := \mathbf{O}^\circ_{k+3}(\mathbb{R}, \tilde{q}_k),$$
$$\mathbf{SO}^\circ(1, k+2) := \mathbf{SO}^\circ_{k+3}(\mathbb{R}, \tilde{q}_k)$$

be the components of the identity element in $\mathbf{O}_{k+3}(\mathbb{R}, \tilde{q}_k)$ and $\mathbf{SO}_{k+3}(\mathbb{R}, \tilde{q}_k)$, respectively. These groups act on $\mathbb{R}^{k+3}$ by left multiplication and stabilize $\mathbb{K}^{k+2}$. The resulting bijections of $\mathbb{K}^{k+2}$ are isometries, and we have the identifications

$$\begin{cases} \mathbf{O}^\circ(1, k+2) = \mathrm{Iso}(\mathbb{K}^{k+2}), \\ \mathbf{SO}^\circ(1, k+2) = \mathrm{Iso}^+(\mathbb{K}^{k+2}), \\ \mathbb{K}^{k+2} = \mathbf{O}^\circ(1, k+2)/\mathbf{O}(k+2), \end{cases} \tag{3.2}$$

where $\mathbf{O}(k+2)$ denotes the stabilizer of $(1, 0, \ldots, 0)^t \in \mathbb{K}^{k+2}$ in $\mathbf{O}^\circ(1, k+2)$.

In the following we introduce certain arithmetic subgroups of $\mathrm{Iso}(\mathbb{K}^{k+2})$. Let $q$ be a quadratic form over $\mathbb{Q}$ in $k+3$ variables which is equivalent to $\tilde{q}_k$ over $\mathbb{R}$. If $f \in \mathbf{GL}_{k+3}(\mathbb{R})$ satisfies $\tilde{q}_k = q \circ f$, then conjugation by $f$ induces an isomorphism

$$\varphi_f \colon \mathbf{O}^\circ_{k+3}(\mathbb{R}, q) \to \mathbf{O}^\circ_{k+3}(\mathbb{R}, \tilde{q}_k).$$

We define

$$\begin{cases} \mathbf{O}^\circ_{k+3}(\mathbb{Q}, q) := \mathbf{O}_{k+3}(\mathbb{Q}, q) \cap \mathbf{O}^\circ_{k+3}(\mathbb{R}, q), \\ \mathbf{O}^\circ_{k+3}(\mathbb{Z}, q) := \mathbf{O}_{k+3}(\mathbb{Z}, q) \cap \mathbf{O}^\circ_{k+3}(\mathbb{R}, q), \end{cases} \tag{3.3}$$

where $\mathbf{O}^\circ_{k+3}(\mathbb{R},q)$ denotes the component of the identity in $\mathbf{O}_{k+3}(\mathbb{R},q)$.

The group $\varphi_f(\mathbf{O}^\circ_{k+3}(\mathbb{Z},q))$ acts discontinuously on $\mathbb{K}^{k+2}$. The various choices for $f$ lead to a $\mathbf{O}^\circ_{k+3}(\mathbb{R},q)$-conjugacy class of discontinuous subgroups of $\mathbf{O}^\circ_{k+3}(\mathbb{R},q)$. It is well known that $\varphi_f(\mathbf{O}^\circ_{k+3}(\mathbb{Z},q))$ acts with compact quotient if and only if $q$ is anisotropic over $\mathbb{Q}$ (see [4]). Note that this is only possible for $k \leq 1$ by the Hasse-Minkowski Theorem.

For $\ell \in \mathbb{N}$ the subgroups

$$(3.4)\qquad \begin{cases} \mathbf{O}_{k+3}(\mathbb{Z},q,\ell) := \{g \in \mathbf{O}_{k+3}(\mathbb{Z},q) \quad | \quad g \equiv I \bmod \ell\} \\ \mathbf{O}^\circ_{k+3}(\mathbb{Z},q,\ell) := \mathbf{O}_{k+3}(\mathbb{Z},q,\ell) \cap \mathbf{O}^\circ_{k+3}(\mathbb{Z},q) \end{cases}$$

are called **principal congruence subgroups of level** $\ell$. These groups are of finite index in $\mathbf{O}_{k+3}(\mathbb{Z},q)$ and $\mathbf{O}^\circ_{k+3}(\mathbb{Z},q)$, respectively. A subgroup $\Gamma < \mathbf{O}_{k+3}(\mathbb{Z},q)$ is called a **congruence subgroup** if it contains a principal congruence subgroup $\mathbf{O}^\circ_{k+3}(\mathbb{Z},q,\ell)$ for some $\ell \in \mathbb{N}$.

We briefly recall the relation between the Vahlen and orthogonal groups (see [15]). Equip $E_0 := \mathbb{Q}^3$ with the quadratic form

$$q_0(y_0,y_1,y_2) := y_0^2 - y_1^2 - y_2^2$$

and let $E$ be a $k$-dimensional vector space over $\mathbb{Q}$ with non-degenerate quadratic form $q$. The non-degenerate quadratic form

$$(3.5)\qquad \tilde{q} := q_0 \perp q$$

on $\tilde{E} := E_0 \oplus E$ defines an associated Clifford algebra $\mathcal{C}(\tilde{q})$. Let $\mathcal{C}^+(\tilde{q})$ be the subalgebra of $\mathcal{C}(\tilde{q})$ spanned by the basis elements $e_M (M \in \mathcal{E}_{k+3})$ with $|M| \equiv 0 \bmod 2$. Writing

$$f_0 := (1,0,0)^t, \quad f_1 := (0,1,0)^t, \quad f_2 := (0,0,1)^t$$

for the standard basis of $E_0$ we define the following elements of $\mathcal{C}(q_0)$:

$$(3.6)\qquad \begin{cases} u := \frac{1}{2}(1+f_0f_1), \quad w_1 := \frac{1}{2}(f_0-f_1)f_2, \\ w_0 := \frac{1}{2}(f_0+f_1)f_2, \quad v := \frac{1}{2}(1-f_0f_1). \end{cases}$$

The $\mathbb{Q}$-linear map

$$\begin{aligned} \dot{}: E &\to \mathcal{C}^+(\tilde{q}), \\ x &\to \dot{x} := f_0f_1f_2x \end{aligned}$$

extends by [15], Proposition 2.4 to an injective $\mathbb{Q}$-algebra homomorphism

$$\dot{} : \mathcal{C}(q) \to \mathcal{C}^+(\tilde{q})$$

which commutes with the antiautomorphism *. The map

$$\Psi: M_2(\mathcal{C}(q)) \to \mathcal{C}^+(\tilde{q}),$$
$$\Psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) := \dot{\alpha}u + \dot{\beta}w_1 + \dot{\gamma}w_0 + \dot{\delta}v$$

is a $\mathbb{Q}$-algebra isomorphism which restricts to an isomorphism

$$\Psi: SV_k(\mathbb{Q}, q) \to \mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q})$$

of the Vahlen group $SV_k(\mathbb{Q}, q)$ onto the spin-group

$$\mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q}) = \{s \in \mathcal{C}^+(\tilde{q}) \;\mid\; s\tilde{E}s^* \subset \tilde{E}, ss^* = 1\}$$

(see [15], Theorem 4.1). There is a canonical homomorphism

$$\Theta: \mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q}) \to \mathbf{SO}^\circ_{k+3}(\mathbb{Q}, \tilde{q})$$

mapping an element $s \in \mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q})$ to the endomorphism

$$\Theta(s): \tilde{E} \to \tilde{E}, \quad \Theta(s)(x) := sxs^* \quad (x \in \tilde{E}).$$

Defining $\tau := \Theta \circ \Psi$ we obtain an exact sequence

$$(3.7) \qquad 1 \to \{1, -1\} \to SV_k(\mathbb{Q}, q) \xrightarrow{\tau} \mathbf{SO}_{k+3}(\mathbb{Q}, \tilde{q}) \xrightarrow{\sigma} \mathbb{Q}^*/(\mathbb{Q}^*)^2 \to 1$$

(see [15], §5). The map $\sigma$ is the spinoral norm ([14], §7 and §10). An analogous construction holds over $\mathbb{R}$ instead of $\mathbb{Q}$.

Assume now that $q$ is negative definite. Every linear isomorphism $f: E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^k$ satisfying $q = -I_k \circ f$ extends to a linear isomorphism $\tilde{f}: \tilde{E} \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^{k+3}$ such that $\tilde{q} = \tilde{q}_k \circ \tilde{f}$ and we have the following commutative diagram (remember (2.4)):

$$\begin{array}{ccccccc} SV_k(\mathbb{Q}, q) & \subset & SV_k(\mathbb{R}, q) & \xrightarrow{\hat{f}} & SV_k & = & \mathrm{Iso}^+(\mathbb{H}^{k+2}) \\ \downarrow \tau_q & & \downarrow \tau_q & & \downarrow \tau_{-I_k} & & \\ \mathbf{SO}^\circ_{k+3}(\mathbb{Q}, \tilde{q}) & \subset & \mathbf{SO}^\circ_{k+3}(\mathbb{R}, \tilde{q}) & \xrightarrow{\varphi_f} & \mathbf{SO}^\circ_{k+3}(\mathbb{R}, \tilde{q}_k) & = & \mathrm{Iso}^+(\mathbb{K}^{k+2}), \end{array}$$

where the indices at the various maps $\tau$ indicate the underlying quadratic forms. In [15], Proposition 5.3 we explicitly wrote out a $\tau_q$-equivariant rational bijection $\Psi_0: \mathbb{H}^{k+2} \to \mathbb{K}^{k+2}$. The $\tau_q$-equivariance of $\Psi_0$ means that

$$(3.8) \qquad \Psi_0(\hat{f}(g)P) = \varphi_{\tilde{f}}(\tau_q(g))(\Psi_0(P))$$

for all $g \in SV_k(\mathbb{R}, q)$ and $P \in \mathbb{H}^{k+2}$.

**3.1 Proposition.** Let $q$ be a negative definite form in $k$ variables with rational coefficients and let $\Gamma < \mathbf{O}_{k+3}(\mathbb{Q}, \tilde{q})$ be a congruence subgroup. Then there exists a congruence subgroup $\Delta < SV_k(\mathbb{Q}, q)$ such that $\tau_q(\Delta) < \Gamma$.

**Proof**: Let $\ell \in \mathbb{Z}$ be an integer with

$$\mathbf{O}_{k+3}(\mathbb{Z}, \tilde{q}, \ell) < \Gamma.$$

The quadratic form $q$ is defined on $\mathbb{Q}^k$. We choose orthogonal elements $g_1, \ldots, g_k \in \mathbb{Z}^k$ so that $q(g_1), \ldots, q(g_k) \in \mathbb{Z}$. Then

$$\Lambda := \mathbb{Z}g_1 \oplus \ldots \oplus \mathbb{Z}g_k$$

has finite index in $\mathbb{Z}^k$. We further choose an integer $m_1$ with

$$m_1 \cdot \mathbb{Z}^k < \Lambda.$$

We define $J$ to be the subring of $\mathcal{C}(q)$ generated by $\Lambda$. $J$ is an order in $\mathcal{C}(q)$ with

$$J \cap V_q = \Lambda.$$

We define

$$\Delta := SV_k(J, m_1 \cdot \ell).$$

and prove now that $\tau_q(\Delta) \leq \Gamma$: We define $\tilde{\Lambda}$ to be the lattice in $\mathcal{C}(\tilde{q})$ generated by $\Lambda$ and $f_0, f_1, f_2$. The following inclusions are then trivial:

$$m_1 \cdot \mathbb{Z}^{k+3} < \tilde{\Lambda}, m_1 \tilde{\Lambda} < m_1 \mathbb{Z}^{k+3}. \tag{3.9}$$

For an element $g \in \Delta$ it is clear that $\tau_q(g)(\tilde{\Lambda}) < \tilde{\Lambda}$. We conclude from (3.9) that $\tau_q(g)(\mathbb{Z}^{k+3}) < \mathbb{Z}^{k+3}$ and $\tau_q(g) \equiv I \bmod \ell$. □

**3.2 Remark.** It is in general not true that the image $\tau_q(\Delta) < \mathbf{O}^{\circ}_{k+3}(\mathbb{Q}, \tilde{q})$ of a congruence subgroup $\Delta < \mathrm{SV}_k(\mathbb{Q}, q)$ is a congruence subgroup. Consider the following example. Let $q$ be negative definite definite form in $k \geq 0$ variables with rational coefficients. Let $J \subset \mathcal{C}(q)$ be any $\mathbb{Z}$-order and let $p_1, p_2$ be two distinct odd primes. Then clearly $\tau_q(\mathrm{SV}_k(J, p_1 \cdot p_2))$ is not a congruence subgroup. See [53] for some general theorems in connection with this phenomenon. It is interesting to note that the subgroup

$$\{x \in \mathbf{O}_{k+3}(\mathbb{Z}, \tilde{q}) \quad | \quad \sigma(x) = 1\}$$

of elements of spiniorial norm 1, is always a congruence subgroup of $\mathbf{O}_{k+3}(\mathbb{Z}, \tilde{q})$. This is proved in [51] and [55].

**3.3 Remark.** Let again $q$ be a negative definite form in $k \geq 0$ variables with rational coefficients. The congruence subgroups $\Delta < \mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q})$ are defined to be those subgroups which contain a group of the form

$$\{g \in \mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q}) \quad | \quad g - 1 \in nJ\}$$

for some $n \in \mathbb{N}$ and some compatible $\mathbb{Z}$-order $J < \mathcal{C}(q)$. It is then clear that both

$$\Psi : \mathrm{SV}_k(\mathbb{Q}, q) \to \mathbf{Spin}_{k+3}(\mathbb{Q}, \tilde{q})$$

and its inverse map congruence subgroups to congruence subgroups.

## §4. Generalized Kloosterman Sums and the Linnik-Selberg Series

This paragraph introduces the generalized Kloosterman sums and the Linnk-Selberg series.We fix the following hypotheses and notations for paragraph 4:

$k$ – a natural number or 0 ,

$q$ – a negative definite rational quadratic form on a $k$-dimensional rational vector space $E$,

$J < \mathcal{C}(q)$ – a compatible order in the rational Clifford algebra $\mathcal{C}(q)$,

$n$ – a natural number.

In addition we introduce

$$V_q = \mathbb{Q} \cdot 1 \oplus E \subset \mathcal{C}(q), \quad T(q)$$

as in paragraph 1. We furthermore put:

$$\begin{aligned} V(J) &= V_q \cap J, \\ T(J) &= T(q) \cap J, \\ SV_k(J) &= SV_k(\mathbb{Q}, q) \cap M_2(J), \\ \Gamma := SV_k(J, n) &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(J) \;\middle|\; \alpha - 1, \beta, \gamma, \delta - 1 \in nJ \right\}. \end{aligned}$$

The bilinear form

$$\langle x, y \rangle := \frac{1}{2} tr\, x\overline{y} \quad (x, y \in V_q)$$

is an inner product on $V_q$, and we denote by $\Lambda^\#$ the dual of the lattice $\Lambda \subset V_q$. Obviously, a matrix $\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}$ belongs to $\Gamma$ if and only if $w \in nV(J)$. Put

$$\Gamma'_\infty := \left\{ \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \;\middle|\; w \in nV(J) \right\}.$$

## §4. *Generalized Kloosterman Sums and the Linnik-Selberg Series*

One main aim of paragraph 4 is to describe explicitly a representative system of $\Gamma'_\infty\backslash\Gamma/\Gamma'_\infty$. The result will be given in Theorem 4.7. Another aim is to introduce a certain type of generalized Kloosterman sums for Clifford algebras. These sums will come up in our discussion of the scalar product of two Poincaré series in paragraph 9. The following definition will play a major role in what follows.

**4.1 Definition.** For $\gamma \in nT(J)$ let

$$T_r(\gamma) := \{\alpha \in J \mid \alpha - 1 \in nJ \quad , \quad \alpha\overline{\gamma} \in nV(J)\}, \tag{4.1}$$

$$T_\ell(\gamma) := \{\delta \in J \mid \delta - 1 \in nJ \quad , \quad \overline{\gamma}\delta \in nV(J)\}. \tag{4.2}$$

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma$ and $\gamma \neq 0$, then $\gamma \in nT(J)$ and $\alpha \in T_r(\gamma)$, $\delta \in T_\ell(\gamma)$. If $\gamma \in nT(J)$ and $k = 0$ or $k = 1$ then it is easy to see that $T_r(\gamma) \neq \emptyset$, $T_\ell(\gamma) \neq \emptyset$. However, the latter conclusion does not hold unconditionally if $k \geq 2$: By way of example, let $k = 2$ and $E = \mathbb{Q}^2$, $q(x,y) = -x^2 - y^2$ for $(x,y)^t \in E$. Put $e_3 = e_1e_2$ and define $J := \mathbb{Z} + \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$. Then for $n = 2$, we have

$$\gamma := 2(2 + 2e_1 + 2e_2 + e_3) \in 2T(J)$$

and $T_r(\gamma) = \emptyset$, $T_\ell(\gamma) = \emptyset$.

**4.2 Lemma.** Let $\gamma \in nT(J)$. Then the additive group $nV(J)\gamma$ acts on $T_r(\gamma)$ via $\alpha \mapsto \alpha + x \quad (\alpha \in T_r(\gamma), x \in nV(J)\gamma)$, and we have

$$|T_r(\gamma)/nV(J)\gamma| \leq (\overline{\gamma}\gamma)^{k+1}.$$

An analogous statement holds for $T_\ell(\gamma)$ with $n\gamma V(J)$ instead of $nV(J)\gamma$.

**Proof.** The assertion is trivially true if $T_r(\gamma) = \emptyset$. Suppose now that $T_r(\gamma) \neq \emptyset$. Obviously, $nV(J)\gamma$ acts on $T_r(\gamma)$. Fix $\alpha_0 \in T_r(\gamma)$ and let $M := \{\alpha - \alpha_0 : \alpha \in T_r(\gamma)\}$. Then $M$ is an additive abelian group, and $nV(J)\gamma$ is a subgroup of $M$. For any $x \in M$ we have $x\overline{\gamma} \in nV(J)$ and hence $x\overline{\gamma}\gamma \in nV(J)\gamma$ where $\overline{\gamma}\gamma \in \mathbb{N}$. This implies that the rank of $M$ equals $k+1$ and

$$[M : nV(J)\gamma] \leq (\overline{\gamma}\gamma)^{k+1}.$$

Since the map

$$T_r(\gamma)/nV(J)\gamma \to M/nV(J)\gamma,$$
$$\alpha + nV(J)\gamma \mapsto \alpha - \alpha_0 + nV(J)\gamma$$

is a bijection, the assertion follows. □

**4.3 Lemma.** Let $\gamma \in nT(J)$ and $y \in T_\ell(\gamma)$. Then

$$\gamma y^* \in V_q \cap J\gamma^*.$$

**Proof.** Since $\overline{\gamma} y \in nV(J) \subset V_q$, we have from (1.7):

$$\gamma\overline{\gamma}y\gamma^* = \varphi_\gamma(\overline{\gamma}y)\gamma'\gamma^* \in V_q$$

and hence

$$y\gamma^* \in V_q \cap J\gamma^*.$$

But the antiinvolution $*$ leaves every element of $V_q$ fixed and hence

$$\gamma y^* = (y\gamma^*)^* = y\gamma^* \in J\gamma^*. \qquad \square$$

**4.4 Lemma.** Assume that $\gamma \in nT(J)$ and $x \in T_r(\gamma)$, $y \in T_\ell(\gamma)$, $xy^* - 1 \in nJ\gamma^*$. Then there exists some $\beta \in nJ$ such that

$$\begin{pmatrix} x & \beta \\ \gamma & y \end{pmatrix} \in SV_k(J, n).$$

**Proof.** There exists some $\beta \in nJ$ such that $xy^* - \beta\gamma^* = 1$. In order to show that $\beta$ satisfies our requirements we use [15], p. 376, Theorem 3.7, condition 3: Obviously, $x-1, \beta, \gamma, y-1 \in nJ$ and $\gamma \in T(q)$, $x, y \in T(q) \cup \{0\}$. Now we have

$$\begin{aligned} \overline{x}\beta &= (\overline{x}xy^*\gamma' - \overline{x}\gamma')(\overline{\gamma}\gamma)^{-1}, \\ y^*\gamma' &= (\overline{\gamma}y)^* \in V_q, \\ \overline{(\overline{x}\gamma')} &= \gamma^*(x\overline{\gamma})\gamma(\overline{\gamma}\gamma)^{-1} \\ &= \varphi_{\gamma^*}(x\overline{\gamma}) \in V_q \end{aligned}$$

and hence $\overline{x}\beta \in V_q$. The condition $\overline{\gamma}y \in V_q$ is clearly satisfied. Now if $x \neq 0$, we have $\overline{x} \in T(q)$ and hence $\beta \in T(q) \cup \{0\}$. For $x = 0$ we have $-\beta\gamma^* = 1$ and hence $\beta \in T(q)$. $\square$

**4.5 Lemma.** Assume that $\gamma \in nT(J)$ and

$$(x,y),\ (u,y) \in T_r(\gamma) \times T_\ell(\gamma), \quad xy^* - 1 \in nJ\gamma^*, \quad uy^* - 1 \in nJ\gamma^*.$$

Then we have $x - u \in nV(J)\gamma$.

**Proof.** Using Lemma 4.4 we find $\beta, b \in nJ$ such that

$$\sigma := \begin{pmatrix} x & \beta \\ \gamma & y \end{pmatrix} \in SV_k(J,n), \qquad \tau := \begin{pmatrix} u & b \\ \gamma & y \end{pmatrix} \in SV_k(J,n).$$

Hence using Lemma 4.3 we obtain

$$\tau\sigma^{-1} = \begin{pmatrix} 1 & -u\beta^* + bx^* \\ 0 & 1 \end{pmatrix} \in \Gamma'_\infty$$

and hence $-u\beta^* + bx^* \in nV(J)$. □

**4.6 Theorem.** Let $\gamma \in nT(J)$. Then $\gamma$ is the lower left entry of a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(J,n)$$

if and only if $(\alpha, \delta) \in T_r(\gamma) \times T_\ell(\gamma)$ and $\alpha\delta^* - 1 \in nJ\gamma^*$.

**Proof.** If $(\alpha, \delta) \in T_r(\gamma) \times T_\ell(\gamma)$ satisfies $\alpha\delta^* - 1 \in nJ\gamma^*$, the existence of $\beta$ follows from Lemma 4.4. The converse is obvious from the remark after Definition 4.1. □

**4.7 Theorem.** For $\gamma \in nT(J)$ let

$$D(\gamma) := \{(x,y) \in (T_r(\gamma)/nV(J)\gamma) \times (T_\ell(\gamma)/n\gamma V(J)) \ \mid \ xy^* - 1 \in nJ\gamma^*\}. \tag{4.3}$$

Then $D(\gamma)$ is well-defined and

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(J,n) \ \middle| \ \gamma \in nT(J), (\alpha,\delta) \in D(\gamma) \right\} \tag{4.4}$$

is a representative system of the set of elements $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma'_\infty \backslash \Gamma / \Gamma'_\infty$ with $\gamma \neq 0$.

**Proof.** Lemma 4.3 yields that $D(\gamma)$ is well-defined. The rest follows from Lemma 4.5 and Theorem 4.6. □

We now introduce the generalized Kloosterman sums which naturally come up in the present framework.

**4.8 Definition.** For $\mu, \nu \in (nV(J))^{\#}$ and $\gamma \in nT(J)$ let the **generalized Kloosterman sum** $S(\mu,\nu;\gamma)$ be defined by

$$(4.5)\qquad \begin{aligned} S(\mu,\nu;\gamma) &:= \sum_{(x,y)\in D(\gamma)} e(\frac{1}{2}tr(x\gamma^{-1}\overline{\mu} + \gamma^{-1}y\overline{\nu})) \\ &= \sum_{(x,y)\in D(\gamma)} e(\langle x\gamma^{-1},\mu\rangle + \langle \gamma^{-1}y,\nu\rangle) \end{aligned}$$

with $D(\gamma)$ from (4.3), where $e(z) = \exp(2\pi i z)\,(z \in \mathbb{C})$.

The sum (4.5) contains only finitely many terms since

$$(4.6)\qquad |D(\gamma)| \leq |T_r(\gamma)/nV(J)\gamma| \leq (\overline{\gamma}\gamma)^{k+1}$$

by Lemmas 4.5 and 4.2. It is readily checked that $S(\mu,\nu;\gamma)$ is well-defined.

**4.9 Examples.**

a) For $k = 0$ we have $J = V(J) = \mathbb{Z}$, and for $n = 1$ one has the classical Kloosterman sum for the modular group. For $n > 1$ one obtains the general Kloosterman sums for the principal congruence subgroups of $SL_2(\mathbb{Z})$ as considered by Selberg [62].

b) Let $k = 1$, $E = \mathbb{Q}$, $D \in \mathbb{N}$ a square-free natural number, $q(x) = -Dx^2\,(x \in \mathbb{Q})$, and let $J$ be the ring of integers in $\mathbb{Q}(\sqrt{-D})$. Then the corresponding generalized Kloosterman sums coincide with the sums of Kloosterman type considered by Sarnak [58].

c) Let $q\colon \mathbb{Q}^k \to \mathbb{Q}$ be a negative definite quadratic form and define the positive definite quadratic form $Q\colon \mathbb{Q} \oplus \mathbb{Q}^k \to \mathbb{Q}$ by

$$Q(x) := x_0^2 - q(x_1,\ldots,x_k)$$

for $x = (x_0,\ldots,x_k) \in \mathbb{Q} \oplus \mathbb{Q}^k$. Suppose that $J \subset C(q)$ is a $\mathbb{Z}$-order, let $\mu,\nu \in V(J)^{\#}$, and define the linear forms $\lambda_1, \lambda_2\colon V_q \to \mathbb{Q}$,

$$\lambda_1(x) := \langle \mu, x\rangle, \quad \lambda_2(x) := \langle \nu, x\rangle$$

($x \in V_q$). Consider the special case $\gamma \in \mathbb{Z}$. Then we have $T_r(\gamma) = T_\ell(\gamma) = V(J)$, and we find

$$S(\mu,\nu;\gamma) = \sum_{x\in V(J)/\gamma V(J)} \exp\left(\frac{2\pi i}{\gamma}(\lambda_1(x) + \frac{\lambda_2(x)}{Q(x)})\right).$$

Exponential sums of these types have also been considered by Deligne [12].

Following Linnik [46] and Selberg [62] we consider the Dirichlet series associated with the generalized Kloosterman sums (4.5).

**4.10 Definition.** The Dirichlet series

$$Z(\mu,\nu;s) := \sum_{\gamma\in nT(J)} \frac{S(\mu,\nu;\gamma)}{(\overline{\gamma}\gamma)^s} \tag{4.7}$$

($\mu,\nu \in (nV(J))^\#$) is called the **Linnik-Selberg series.**

By (4.6), $Z(\mu,\nu;s)$ is termwise dominated by

$$\sum_{\gamma\in nT(J)} (\overline{\gamma}\gamma)^{k+1-Re\,s}.$$

Since $\gamma$ is contained in a lattice in $\mathcal{C}(q)$ which is a vector space of dimension $2^k$, one finds that $Z(\mu,\nu;s)$ converges absolutely for

$$Re\,s > 2^{k-1}+k+1. \tag{4.8}$$

Of course, one expects a much larger half-plane of absolute convergence. We shall spend much effort in paragraph 7 and actually prove that the Linnik-Selberg series converges absolutely for

$$Re\,s > k+\frac{1}{2}, \tag{4.9}$$

(see Theorem 7.17). This will be crucial for the lower bound for $\lambda_1$.

## §5. Factorization of Generalized Kloosterman Sums

In this paragraph we study the exponential sums $S(\mu,\nu;\gamma)$ introduced in paragraph 4. We consider the following objects to be fixed:

- $k$ – a natural number or 0 ,
- $q$ – a negative definite rational quadratic form on a $k$-dimensional rational vector space $E$,
- $J < \mathcal{C}(q)$ – a compatible order in the rational Clifford algebra $\mathcal{C}(q)$,
- $n$ – a natural number.

We shall decompose the sums $S(\mu,\nu;\gamma)$ as a product of exponential sums which are defined over the localizations of the order $J$. These "local" exponential sums are easier to compute, as the next paragraph will show.

As for the general notation (as $V(J), T(J),\ldots$) we refer to paragraphs 1 through 4. We further introduce:

§5. *Factorization of Generalized Kloosterman Sums*

**5.1 Notation.** Let $N$ be a natural number. We put

$$\mathbb{Z}_{(N)} = \{\frac{a}{b} \quad | \quad a \in \mathbb{Z}\,,\, b \in \mathbb{N}\,,\, (b, N) = 1\}.$$

If $R$ is a subring of a $\mathbb{Q}$-algebra $A$ we define

$$R_{(N)} = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(N)} < A.$$

$\mathbb{Z}_{(N)}$ always is a semilocal ring. If $N$ is a prime number then $\mathbb{Z}_{(N)}$ is the localization of $\mathbb{Z}$ at $N \cdot \mathbb{Z}$.

**5.2 Notation.** For $M, N \in \mathbb{N}$ we define $N_M$ to be the smallest divisor of $N$ which satisfies $(N/N_M, M) = 1$.

Note that $N_{M_1} \cdot N_{M_2} = N_{M_1 \cdot M_2}$ whenever $(M_1, M_2) = 1$. The following definition is the first step for the introduction of the "local" exponential sums.

**5.3 Definition.** Let $R < \mathcal{C}(q)$ be a compatible subring and $\gamma \in nT(R)$ an element with $\overline{\gamma}\gamma = N$. We put:

$$\begin{aligned} \widehat{T}_r(R, \gamma) &= \{x \in nV(R) \mid x\gamma - N \in nNR\}, \\ \widehat{T}_\ell(R, \gamma) &= \{y \in nV(R) \mid \gamma y - N \in nNR\}, \\ \widehat{T}^\circ_r(R, \gamma) &= \{x \in nV(R) \mid x\gamma \in nNR\}, \\ \widehat{T}^\circ_\ell(R, \gamma) &= \{y \in nV(R) \mid \gamma y \in nNR\}. \end{aligned}$$

The definition makes it clear that $\widehat{T}_r(R, \gamma) \subset V(R)$ is the set of solutions of an (inhomogeneous) system of $R$-linear equations. The set of solutions of the corresponding homogeneous system is $\widehat{T}^\circ_r(R, \gamma)$.

The sets $\hat{T}_r(R, \gamma)$ and $\hat{T}_\ell(R, \gamma)$ are additively acted on by the group $nNV(R)$. Note that in case $n = 1$ the defining conditions for $\hat{T}_r(R, \gamma), \hat{T}_\ell(R, \gamma)$ can be expressed as $x\gamma \in NR$ and $\gamma y \in NR$.

**5.4 Lemma.** Let $\gamma \in n \cdot T(J)$ be an element with $\overline{\gamma}\gamma = N$. Then the following maps are (well-defined) bijections:

$$\varphi_r: T_r(\gamma)/nV(J)\gamma \to \widehat{T}_r(J,\gamma)/nNV(J),$$
$$\varphi_r: x \mapsto x\overline{\gamma},$$
$$\varphi_\ell: T_\ell(\gamma)/n\gamma V(J) \to \widehat{T}_\ell(J,\gamma)/nNV(J),$$
$$\varphi_\ell: y \mapsto \overline{\gamma}y.$$

**Proof.** It is clear that the maps $\varphi_r, \varphi_\ell$ respect the defining conditions and are well-defined. Their inverses are

$$\varphi_r^{-1}(z) = \frac{z\gamma}{N}, \qquad \varphi_\ell^{-1}(z) = \frac{\gamma z}{N}. \qquad \square$$

**5.5 Lemma.** Let $R < \mathcal{C}(q)$ be a compatible subring and $\gamma \in n \cdot T(R)$ an element with $\overline{\gamma}\gamma = N$. Then the following map is a bijection

$$\psi: \quad \widehat{T}_r(R,\gamma) \quad \to \widehat{T}_\ell(R,\gamma),$$
$$\psi: \qquad x \qquad \mapsto \frac{\overline{\gamma}\,\overline{x}\gamma'}{N}.$$

**Proof.** Any $x \in \widehat{T}_r(R,\gamma)$ is in $n \cdot V(R)$ hence the condition on $\gamma$ implies that

$$y = \frac{\overline{\gamma}\,\overline{x}\gamma'}{N} \in V_q \cap nR = nV(R).$$

By direct inspection it can also be seen that $y \in \widehat{T}_\ell(R,\gamma)$. The inverse of $\psi$ can be computed as

$$\psi^{-1}(z) = \frac{\gamma'\overline{z}\,\overline{\gamma}}{N}. \qquad \square$$

Note that the bijections $\psi : \widehat{T}_r(R,\gamma) \to \widehat{T}_\ell(R,\gamma)$ do in general not induce maps between

$$\widehat{T}_r(R,\gamma)/nNV(R) \quad \text{and} \quad \widehat{T}_\ell(R,\gamma)/nNV(R).$$

**5.6 Definition.** Let $R < \mathcal{C}(q)$ be a compatible subring and $\gamma \in nT(R)$ an element with $\overline{\gamma}\gamma = N$. We define

$$\widehat{D}(R,\gamma) = \{(x,y) \in (\widehat{T}_r(R,\gamma)/nNV(R)) \times (\widehat{T}_\ell(R,\gamma)/nNV(R)) \mid \frac{x\gamma}{N} \cdot \frac{y\gamma^*}{N} - 1 \in nR\gamma^*\}.$$

To see that Definition 5.6 makes sense we use considerations similar to those in paragraph 4 after the definition of $D(\gamma)$.

**5.7 Lemma.** Let $\gamma \in nT(J)$ be an element with $\overline{\gamma}\gamma = N$. Then the maps $\varphi_r, \varphi_\ell$ induce a bijection

$$\begin{array}{llcl} \varphi_r \times \varphi_\ell \colon & D(\gamma) & \to & \widehat{D}(J,\gamma), \\ \varphi_r \times \varphi_\ell \colon & (\alpha,\delta) & \mapsto & (\varphi_r(\alpha), \varphi_\ell(\delta)). \end{array}$$

For $\mu, \nu \in (nV(J))^\#$ the exponential sums of paragraph 4 can be computed as:

$$S(\mu,\nu;\gamma) = \sum_{(u,v)\in\widehat{D}(J,\gamma)} e\left(\frac{1}{2}tr(\frac{u\overline{\mu}+v\overline{\nu}}{N})\right).$$

**Proof.** Using the fact that every element from $V_q$ is left fixed by the involution * the first part is obvious. For the second we find

$$\begin{aligned} S(\mu,\nu;\gamma) &= \sum_{(x,y)\in D(\gamma)} e\left(\frac{1}{2}tr(x\gamma^{-1}\overline{\mu}+\gamma^{-1}y\overline{\nu})\right) \\ &= \sum_{(x,y)\in D(\gamma)} e\left(\frac{1}{2}tr(\frac{x\overline{\gamma}}{N}\overline{\mu}+\frac{\overline{\gamma}y}{N}\overline{\nu})\right) \\ &= \sum_{(u,v)\in\widehat{D}(J,\gamma)} e\left(\frac{1}{2}tr(\frac{u\overline{\mu}+v\overline{\nu}}{N})\right). \end{aligned}$$ □

The following contains our definition of the local Kloosterman sums.

**5.8 Definition.** Let $M$ be a natural number and $\gamma \in nT(J_{(M)})$ with $\overline{\gamma}\gamma = N$. The number $nN \in \mathbb{Z}_{(M)}$ can uniquely be written as

$$nN = L \cdot \eta$$

where $\eta \in \mathbb{Z}^*_{(M)}$ and $L \in \mathbb{N}$ with $L_M = L$. The inclusion $V(J) < V(J_{(M)})$ induces an isomorphism

$$i : V(J)/L \cdot V(J) \to V(J_{(M)})/nN \cdot V(J_{(M)}).$$

For $\mu, \nu \in (nV(J))^{\#}$ we define

$$S_M(\mu, \nu; \gamma) = \sum_{\substack{(x,y) \in (V(J)/LV(J))^2 \\ (i(x), i(y)) \in \widehat{D}(J_{(M)}, \gamma)}} e\left(\frac{1}{2} tr(\frac{x\overline{\mu} + y\overline{\nu}}{L})\right).$$

Note that $S_p(\mu, \nu; \gamma) = S_{p^e}(\mu, \nu; \gamma)$ for a prime $p$ and every $e \in \mathbb{N}$. The connection with our previous exponential sums is given by the following:

**5.9 Lemma.** Let $\gamma \in n \cdot T(J)$ be an element with $\overline{\gamma}\gamma = N$. For any $\mu, \gamma \in (nV(J))^{\#}$ we have:

$$S_N(\mu, \nu; \gamma) = S(\mu, \nu; \gamma).$$

**Proof.** Since $n|N$ the number $L$ from Definition 5.8 is equal to $n \cdot N$. By elementary considerations we find that for

$$(x, y) \in V(J)/nNV(J) \times V(J)/nNV(J)$$

the condition $(i(x), i(y)) \in \widehat{D}(J_{(N)}, \gamma)$ is equivalent to $(x, y) \in \widehat{D}(J, \gamma)$. The result follows from Lemma 5.7. □

**5.10 Lemma.** Let $M, \lambda$ be natural numbers with $(M, \lambda) = 1$. Let $\gamma \in nT(J_{(M)})$ be an element with $\overline{\gamma}\gamma = N$. Then $\lambda\gamma \in n \cdot T(J_{(M)})$ and for any $\mu, \nu \in (nV(J))^{\#}$ we have

$$S_M(\mu, \nu; \lambda\gamma) = S_M(\lambda\mu, \lambda\nu; \gamma).$$

**Proof.** The proof results from a straightforward check of the summation conditions. □

The following describes the multiplicative behaviour of the exponential sums $S_M(\mu, \nu; \gamma)$.

**5.11 Proposition.** Let $M_1, M_2, \in \mathbb{N}$ be coprime natural numbers. Let $\gamma \in nT(J_{(M_1 M_2)})$ be an element with $\overline{\gamma}\gamma = N$. Choose $L_1, L_2, l_1, l_2 \in \mathbb{N}$ with

$$\frac{nN}{L_1} \in \mathbb{Z}^*_{(M_1)}, \qquad \frac{nN}{L_2} \in \mathbb{Z}^*_{(M_2)}$$

and $L_i l_i - 1 \in M_i\mathbb{Z}$ for $i = 1, 2$. For $\mu, \nu \in (nV(J))^\#$ we have:

$$S_{M_1 M_2}(\mu, \nu; \gamma) = S_{M_1}(l_2\mu, l_2\nu; \gamma) \cdot S_{M_2}(l_1\mu, l_1\nu; \gamma).$$

**Proof.** Note first of all that the $L$ found in the Definition 5.8 for $S_{M_1 M_2}(\mu, \nu; \gamma)$ is equal to $L_1 \cdot L_2$. The summation in the definition of $S_{M_1 M_2}(\mu, \nu; \gamma)$ extends over certain elements

$$(x, y) \in V(J)/L_1 \cdot L_2 V(J) \times V(J)/L_1 \cdot L_2 V(J).$$

The elements $x, y$ can be written as

$$x = L_1 x_1 + L_2 x_2, \quad y = L_1 y_1 + L_2 y_2$$

where $x_1, y_1$ are uniquely determined modulo $L_2 V(J)$ and $x_2, y_2$ are uniquely determined modulo $L_1 V(J)$. It is straightforward to check that the condition

$$(i(x), i(y)) \in \widehat{D}(J_{(M_1 M_2)}, \gamma)$$

is equivalent to the following two conditions:

$$(i(L_1 x_1), i(L_1 y_1)) \in \widehat{D}(J_{(M_2)}, \gamma),$$
$$(i(L_2 x_2), i(L_2 y_2)) \in \widehat{D}(J_{(M_1)}, \gamma).$$

The proposition follows. □

**5.12 Corollary.** Let $\gamma \in nT(J)$ be an element with $\overline{\gamma}\gamma = N$. Assume that

$$N = p_1^{e_1} \dots p_r^{e_r}$$

is the prime factorization of $N$. For $i = 1, \dots, r$ there are $\lambda_i \in \mathbb{N}$ with $(\lambda_i, p_i) = 1$ such that

$$S(\mu, \nu; \gamma) = \prod_{i=1}^{r} S_{p_i}(\lambda_i \mu, \lambda_i \nu; \gamma)$$

for all $\mu, \nu \in (nV(J))^\#$.

**Proof.** The corollary follows from Lemma 5.9 and Proposition 5.11. □

Corollary 5.12 contains a formula given by Estermann [19] for the classical Kloosterman sums; see also Sarnak [58].

## §6. Relations with Classical Kloosterman Sums and Estimates for Generalized Kloosterman Sums

This paragraph contains certain estimates of the exponential sums $S_p(\mu, \nu; \gamma)$ defined in paragraph 5. These will be important for our estimates of the coefficients of the Linnik-Selberg series. Under a certain hypothesis on $\gamma$ we shall be able to compute $S_p(\mu, \nu; \gamma)$ in terms of classical Kloosterman sums. The required estimate will follow from the bounds of Weil [68] and Salié [56], [57]. In the remaining cases we study the summation condition and estimate the exponential sum $S_p(\mu, \nu; \gamma)$ by the number of summands. These last results are far from being optimal. But for our purposes here they turn out to be sufficient.

As for this paragraph we consider the following objects to be fixed:

$k$ – a natural number or 0,

$q$ – a negative definite rational quadratic form on a $k$-dimensional rational vector space $E$,

$J < \mathcal{C}(q)$ – a compatible order in the rational Clifford algebra $\mathcal{C}(q)$,

$n$ – a natural number,

$p$ – a prime number,

$n_p$ – the maximal integer such that $p^{n_p} | n$.

For our later considerations it will be useful to introduce the following classification of elements of $\mathcal{C}(q)$.

**6.1 Definition.** Let $R < \mathcal{C}(q)$ be a compatible subring. Put $R_0 = R \cap \mathbb{Q} \cdot 1$. An element $\gamma \in T(R)$ is called **$(R_0, n)$-primitive** if the equation

$$r \cdot \eta = \gamma$$

for $\eta \in R$ and $r \in R_0$ implies that $r$ divides $n$ in $R_0$. A $(R_0, n)$-primitive element $\gamma \in T(R)$ is called **$(R_0, n)$-degenerate** if there is a non-unit $\lambda \in R_0$ with:

(i) $\lambda | \gamma\overline{\gamma}$,

(ii) $\widehat{T}_r(R, \gamma) \subset \lambda \cdot nV(R)$ or $\widehat{T}_\ell(R, \gamma) \subset \lambda \cdot nV(R)$;

$\gamma$ is called **$(R_0, n)$-nondegenerate** otherwise. $R_0$-primitive or $R_0$- degenerate is defined as $(R_0, 1)$-primitive or degenerate.

For an element $\gamma \in T(J)$ the following statements are equivalent:

(i) $\gamma$ is $(\mathbb{Z}, n)$-primitive .

(ii) $\gamma$ is $(\mathbb{Z}_{(\ell)}, n)$-primitive for all primes $\ell$.

(iii) $\gamma$ is $(\mathbb{Z}_{(M)}, n)$-primitive for all $M \in \mathbb{N}$.

For a $(\mathbb{Z}, n)$-primitive element $\gamma \in T(J)$ the following statements are equivalent:

(i) $\gamma$ is $(\mathbb{Z}, n)$-nondegenerate .
(ii) $\gamma$ is $(\mathbb{Z}_{(\ell)}, n)$-nondegenerate for all primes $\ell$.
(iii ) $\gamma$ is $(\mathbb{Z}_{(M)}, n)$-nondegenerate for all $M \in \mathbb{N}$.

Every $(\mathbb{Z}_{(p)}, n)$-primitive element $v \in V(J_{(p)})$ is also $(\mathbb{Z}_{(p)}, n)$-nondegenerate. Note further that every element $\delta \in n \cdot T(J_{(p)})$ can be written as

$$\delta = p^f \gamma$$

with $f \geq 0$, where $\gamma \in n \cdot T(J_{(p)})$ is $(\mathbb{Z}_{(p)}, n)$-primitive.

The following is more or less obvious.

**6.2 Lemma.** Let $\delta, \gamma \in nT(J_{(p)})$ with $\delta = p^f \cdot \gamma$ for some $f \in \mathbb{N} \cup \{0\}$. Then:

$$\widehat{T}_r(J_{(p)}, \delta) = p^f\, \widehat{T}_r(J_{(p)}, \gamma), \quad \widehat{T}_\ell(J_{(p)}, \delta) = p^f\, \widehat{T}_\ell(J_{(p)}, \gamma),$$
$$\widehat{T}^\circ_r(J_{(p)}, \delta) = p^f\, \widehat{T}^\circ_r(J_{(p)}, \gamma), \quad \widehat{T}^\circ_\ell(J_{(p)}, \delta) = p^f\, \widehat{T}^\circ_\ell(J_{(p)}, \gamma).$$

The following lemma is useful for the final computation of our exponential sums.

**6.3 Lemma.** Let $p$ be a prime and $\gamma \in nT(\mathbb{Z}_{(p)})$ a $(\mathbb{Z}_{(p)}, n)$-primitive element which is $(\mathbb{Z}_{(p)}, n)$-degenerate. Then:

$$\widehat{D}(J_{(p)}, p^f \gamma) = \emptyset$$

for all $f \in \mathbb{N} \cup \{0\}$.

**Proof.** We first consider the case $f = 0$. Take an element

$$(x, y) \in \widehat{T}_r(J_{(p)}, \gamma) \times \widehat{T}_\ell(J_{(p)}, \gamma)$$

so that its image lies in $\widehat{D}(J_{(p)}, \gamma)$.
If $y \in pnV(J_{(p)})$ we consider the equation

$$\frac{x\gamma}{p^r\eta} \cdot \frac{y\gamma^*}{p^r\eta} - 1 \in nJ_{(p)}\gamma^*$$

where $\overline{\gamma}\gamma = p^r\eta$ with a unit $\eta$ and $r \in \mathbb{N}$. Notice that $r = 0$ is not possible because $\gamma$ is $(\mathbb{Z}_{(p)}, n)$-degenerate. Multiplication by $\gamma'$ yields

$$\frac{x\gamma}{p^r\eta} \cdot y - \gamma' \in nJ_{(p)}p^r.$$

This contradicts the fact that $\gamma$ is $(\mathbb{Z}_{(p)}, n)$ primitive.
If $x \in pnV(J_{(p)})$ then using Lemma 5.5 we write:

$$y = \frac{\overline{\gamma}\,\overline{z}\gamma'}{p^r\eta}$$

with a suitable $z \in \widehat{T}_r(J_{(p)}, \gamma)$. The defining equation for $(x, y) \in \widehat{D}(J_{(p)}, \gamma)$ yields

$$\frac{x\gamma}{p^r\eta} \cdot \frac{\overline{\gamma}\,\overline{z}\gamma'\overline{\gamma}'}{p^r\eta \cdot p^r\eta} - 1 = \frac{x\overline{z}}{p^r\eta} - 1 \in nJ_{(p)}\gamma^*.$$

We multiply by $\gamma'$ and obtain

$$x \cdot \frac{\overline{z} \cdot \gamma'}{p^r\eta} - \gamma' \in nJ_{(p)}p^r.$$

This again is a contradiction against $\gamma$ being $(\mathbb{Z}_{(p)}, n)$-primitive. This finishes the case $f = 0$.
The case $f \geq 1$ follows with the help of Lemma 6.2 and similar arguments. □

**6.4 Corollary.** Let $\gamma \in nT(J)$ be an element which is $(\mathbb{Z}, n)$-primitive and $(\mathbb{Z}, n)$-degenerate. Then:

$$S(\mu, \nu; t\gamma) = 0$$

for all $t \in \mathbb{Z}\backslash\{0\}$ and $\mu, \nu \in (nV(J))^{\#}$.

**Proof.** This is deduced from the above and Corollary 5.12 which describes the multiplicative decomposition of $S(\mu, \nu; t\gamma)$. □

To proceed with our computations we need the following definitions.

**6.5 Definition.** Let $R < \mathcal{C}(q)$ be a compatible subring. Put $R_0 = R \cap \mathbb{Q} \cdot 1$ and

$$V_1(R) = R \cap E.$$

Define $\tilde{R}$ to be the $R_0$-algebra generated by $V_1(R)$ in $\mathcal{C}(q)$.

**6.6 Definition.** We define $p^{f(p,J)}$ to be the exponent of $\tilde{J}_{(p)}$ in $J_{(p)}$, that is

$$f(p,J) = \min\{f \in \mathbb{N} \cup \{0\} \quad | \quad p^f J_{(p)} < \tilde{J}_{(p)}\}.$$

We call $J$ to $\mathbb{Z}_{(p)}$**-diagonal** if $f(p,J) = 0$.

Notice that $f(p,J)$ is well-defined since $\tilde{J}_{(p)}$ has finite index in $J_{(p)}$. Since $\tilde{J}$ also has finite index in $J$, the $\mathbb{Z}$-order $J$ is $\mathbb{Z}_{(p)}$-diagonal for almost all primes $p$.

The two following lemmas are used to compute $\widehat{T}_r(J_{(p)}, \gamma)$ for certain elements $\gamma$.

**6.7 Lemma.** Let $p$ be an odd prime and $f \in \mathbb{N} \cup \{0\}$. Let $u \in V(J_{(p)})$ be a $\mathbb{Z}_{(p)}$-primitive element with

$$\overline{u}u = p^t \cdot \kappa$$

where $t \in \mathbb{N} \cup \{0\}$ and $\kappa \in \mathbb{Z}_{(p)}$ is a unit. Then we have:

$$\{x \in V(J_{(p)}) \quad | \quad xu \in p^f J_{(p)}\} \subset \{\lambda\overline{u} + p^{f-f(p,J)}z \quad | \quad \lambda \in \mathbb{Z}_{(p)} \cap p^{f-f(p,J)-t}\mathbb{Z}_{(p)}, \quad z \in V(J_{(p)})\}.$$

Equality holds if $J$ is $\mathbb{Z}_{(p)}$-diagonal.

**Proof.** Since $J_{(p)}$ is a compatible subring of $C(q)$ and $p$ is odd we have

$$V(J_{(p)}) = \mathbb{Z}_{(p)} \oplus V_1(J_{(p)}).$$

Take a $\mathbb{Z}_{(p)}$-basis $f_1, \ldots, f_k$ of $V_1(J_{(p)}) = J_{(p)} \cap E$ and write

$$x = x_0 + x_1 f_1 + \ldots + x_k f_k,$$
$$u = u_0 + u_1 f_1 + \ldots + u_k f_k.$$

We compute

$$xu = a + \sum_{i=1}^{k}(x_0u_i + u_0x_i)f_i + \sum_{i<j}(x_iu_j - u_ix_j)f_if_j \tag{6.1}$$

with some $a \in \mathbb{Z}_{(p)}$. Clearly for $xu \in p^f J_{(p)}$ it is necessary that $p^{f-f(p,J)}$ divides the coefficients of the expression (6.1). Hence the $(2 \times 2)$-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 \ldots x_k \\ -u_0 & u_1 \ldots u_k \end{pmatrix}$$

are all divisible by $p^{f-f(p,J)}$. By linear algebra ($u$ is $\mathbb{Z}_{(p)}$-primitive!) this implies that

$$x = \lambda\overline{u} + p^{f-f(p,J)}z \tag{6.2}$$

for some $\lambda \in \mathbb{Z}_{(p)}$ and $z \in V(J_{(p)})$. From (6.2) we compute $x \cdot u$ and find the condition for $\lambda$.
If $f(p,J) = 0$ then the other inclusion is obvious. □

The prime 2 plays a somewhat particular role:

**6.8 Lemma.** Let $f \in \mathbb{N} \cup \{0\}$ and $u \in V(J_{(2)})$ a $\mathbb{Z}_{(2)}$-primitive element with

$$\overline{u}u = 2^t \cdot \kappa$$

where $t \in \mathbb{N} \cup \{0\}$ and $\kappa \in \mathbb{Z}_{(2)}$ is a unit. Then

$$\{x \in V(J_{(2)}) \quad | \quad xu \in 2^f J_{(2)}\} \subset \\ \{\lambda\overline{u} + p^{f-f(2,J)-2}z \quad | \quad \lambda \in \mathbb{Z}_{(2)} \cap 2^{f-f(2,J)-t-2}\mathbb{Z}_{(2)}, \quad z \in V(J_{(2)})\}.$$

**Proof.** The compatibility of $J_{(2)}$ implies that

$$2 \cdot V(J(2)) < \mathbb{Z}_{(2)} \oplus V_1(J_{(2)}).$$

As in the proof of Lemma 6.7 the elements $2x$ und $2u$ can be suitably expressed and the proof is finished by the same argument. □

**6.9 Corollary.** Let $\gamma \in nT(J_{(p)})$ be a $(\mathbb{Z}_{(p)}, n)$-primitive and $(\mathbb{Z}_{(p)}, n)$-nondegenerate element with $\overline{\gamma}\gamma = N = p^r\eta$ where $r \in \mathbb{N} \cup \{0\}$ and $\eta$ is a unit in $\mathbb{Z}_{(p)}$. Then

$$|\widehat{T}_r^0(J_{(p)}, \gamma)/p^r nV(J_{(p)})| \leq p^{r+(k+1)(f(p,J)+n_p+a_p)},$$
$$|\widehat{T}_\ell^0(J_{(p)}, \gamma)/p^r nV(J_{(p)})| \leq p^{r+(k+1)(f(p,J)+n_p+a_p)}.$$

Here, $a_p = 0$ if $p$ is odd and $a_2 = 2$.

**Proof.** Take an element $x \in \widehat{T}_r(J_{(p)}, \gamma)$ which is not contained in $p^{n_p+1}V(J_{(p)})$. Let $y \in \widehat{T}_r^0(J_{(p)}, \gamma)$. The two equations

$$x\gamma - N \in nNJ_{(p)} \quad , \quad y\gamma \in nNJ_{(p)} \tag{6.3}$$

imply

$$y\frac{\overline{x}}{p^{n_p}} \in p^r J_{(p)},$$

and since $\overline{x}/p^{n_p}$ is $\mathbb{Z}_{(p)}$ primitive Lemmas 6.7 and 6.8 apply.

The statement about $\widehat{T}_\ell^0(J_{(p)}, \gamma)$ follows by using the antiinvolution $^-$. □

For the following note that if $p$ does not divide $n$ (i.e. $n_p = 0$) we have

$$\widehat{T}_r(J_{(p)}, \gamma) = \widehat{T}_r^0(J_{(p)}, \gamma)$$

for all $\gamma \in nT(J_{(p)}) = T(J_{(p)})$.

**6.10 Corollary.** Assume that $p$ is an odd prime and that $J$ is $\mathbb{Z}_{(p)}$-diagonal. Assume further that $n_p = 0$. Let $\gamma \in T(J_{(p)})$ be $\mathbb{Z}_{(p)}$-primitive and $\mathbb{Z}_{(p)}$-nondegenerate with $\overline{\gamma}\gamma = p^r\eta$ where $r \in \mathbb{N} \cup \{0\}$ and $\eta \in \mathbb{Z}_{(p)}$ is a unit. Then the groups

$$\widehat{T}_r(J_{(p)}, \gamma)/p^r V(J_{(p)}) \quad \text{and} \quad \widehat{T}_\ell(J_{(p)}, \gamma)/p^r V(J_{(p)})$$

are cyclic of order $p^r$.

**Proof.** Follows as in Corollary 6.9 from Lemma 6.7. □

**6.11 Remark.** The structure of the groups $\widehat{T}_r^\circ(J, \gamma)/nNV(J)$ can also be described in the case where $\gamma$ is $\mathbb{Z}$-degenerate. We wish to mention the following result which we shall not prove here. Let $n = 1$ and assume that $J$ is $\mathbb{Z}$-diagonal. Let $\gamma \in T(J)$ be a $\mathbb{Z}$-primitive element with $\overline{\gamma}\gamma = N$. Then

$$|\widehat{T}_r^\circ(J, \gamma)/NV(J)| = N. \tag{6.4}$$

We also have obtained results on the exact cycle structure of the groups $\widehat{T}_r^\circ(J, \gamma)/NV(J)$.

Under the assumptions of Corollary 6.10 we shall now compute the exponential sums $S_p(\mu, \nu; \gamma)$. We remind the reader now of the definition of the classical Kloosterman sums [13], [56], [68].

**6.12 Notation.** Let $N$, $a$, $b$, $c$ be integers with $N \neq 0$. Then the (classical) **Kloosterman sum** is defined as:

$$KS(a,b;c,N) = \sum_{\substack{x,y\in\mathbb{Z}/N\mathbb{Z} \\ xyc-1\in N\mathbb{Z}}} e(\frac{xa+yb}{N}). \tag{6.5}$$

The following definition is made possible by Corollary 6.10.

**6.13 Definition.** Assume that $p$ is an odd prime and that $J$ is $\mathbb{Z}_{(p)}$-diagonal. Assume further that $n_p = 0$. Let $\gamma \in T(J_{(p)})$ be a $\mathbb{Z}_{(p)}$-primitive and nondegenerate element with $\overline{\gamma}\gamma = p^r\eta$ where $r \in \mathbb{N} \cup \{0\}$ and $\eta \in \mathbb{Z}_{(p)}$ is a unit. Define

$$v_\gamma \in \widehat{T}_r(J_{(p)},\gamma), \quad w_\gamma \in \widehat{T}_\ell(J_{(p)},\gamma)$$

so that their residue classes $[v_\gamma]$ and $[w_\gamma]$ generate $\widehat{T}_r(J_{(p)},\gamma)/p^rV(J_{(p)})$ or $\widehat{T}_\ell(J_{(p)},\gamma)/p^rV(J_{(p)})$ , respectively.

**6.14 Proposition.** Assume that $p$ is an odd prime number and that $J$ is $\mathbb{Z}_{(p)}$-diagonal. Assume further that $p \not| n$. Let $\gamma \in T(J_{(p)})$ be a $\mathbb{Z}_{(p)}$-primitive and $\mathbb{Z}_{(p)}$-nondegenerate element with $\overline{\gamma}\gamma = p^r\eta$ where $r \in \mathbb{N} \cup \{0\}$ and $\eta \in \mathbb{Z}_{(p)}$ is a unit. Choose $v_\gamma$, $w_\gamma$ as in Definition 6.13. Let $t_1, t_2, t_3 \in \mathbb{Z}_{(p)}$ and $u \in V(J_{(p)})$ be chosen so that

$$w_\gamma = \frac{\overline{\overline{\gamma}(t_1v_\gamma + p^r\eta u)\gamma'}}{p^r\eta}, \tag{6.6}$$

$$t_2 = \frac{\overline{v}_\gamma v_\gamma}{p^r\eta}, \qquad t_3 = v_\gamma\overline{u} + \overline{v}_\gamma. \tag{6.7}$$

Define $d = d(\gamma, v_\gamma, w_\gamma)$ to be $d = t_1t_2 + t_3$. Then the following statements hold:

(i) For $x, y \in \mathbb{Z}_{(p)}$ the conditions

$$(x[v_\gamma], y[w_\gamma]) \in \widehat{D}(J_{(p)},\gamma)$$

and

$$dxy - 1 \in p^r\mathbb{Z}_{(p)}$$

are equivalent.

(ii) For $\mu, \nu \in (nV(J))^{\#}$ put

$$a = \langle v_\gamma, \mu \rangle \quad , \quad b = \langle v_\gamma, \nu \rangle.$$

Then we have

$$S_p(\mu, \nu; \gamma) = KS(a, b; d, p^r).$$

**Proof.** Notice first of all that the choice of $t_1, t_2, t_3$ is possible because of Lemmas 5.5 and 6.7. The condition $(x[v_\gamma], y[w_\gamma]) \in \widehat{D}(J_{(p)}, \gamma)$ is equivalent to

$$\text{(6.8)} \qquad \frac{xv_\gamma\gamma}{p^r\eta} \cdot \frac{yw_\gamma\gamma^*}{p^r\eta} - 1 \in J_{(p)}\gamma^*.$$

We substitute (6.6) and (6.7) into (6.8) and multiply by $\gamma'$ from the right and find that (6.8) is equivalent to

$$(xyd - 1)\,\gamma' \in p^r J_{(p)}.$$

Since $\gamma$ is $\mathbb{Z}_{(p)}$-primitive the claim (i) follows. (ii) is an obvious consequence of (i). □

The formula for the $S_p(\mu, \nu; \gamma)$ can now be used to evaluate our generalized Kloosterman sums in terms of classical Kloosterman sums.

**6.15 Corollary.** Assume that $n = 1$ and that $J$ is $\mathbb{Z}$-diagonal. Let $\gamma \in T(J)$ be a $\mathbb{Z}$-primitive and $\mathbb{Z}$-nondegenerate element with $\overline{\gamma}\gamma = N$. For every $\mu, \nu \in V(J)^{\#}$ there are $a, b, d \in \mathbb{Z}$ so that

$$S(\mu, \nu; \gamma) = KS(a, b; d, N).$$

**Proof.** This fact follows from the multiplicative decomposition in Lemma 5.5. Notice that for the classical Kloosterman sums the same formulas hold, [19]. □

The assumption $n = 1$ in Corollary 6.15 can easily be removed. One gets a similar formula where the Kloosterman sum on the right has to be replaced by a somewhat generalized exponential sum.

**6.16 Remark.** Proposition 6.14 is a special case of a great variety of special formulas for the $S_p(\mu,\nu;\gamma)$. As these play no role in the present proofs we do not treat them in detail. We want to mention here the case that $\gamma = p^f\gamma_0$ where $\gamma_0$ is $\mathbb{Z}$-primitive and $f \geq 1$. In this case we report on the following (generic) special case. Assume that $p$ is a prime and $r, f \in \mathbb{N} \cup \{0\}$. Assume further that $J_{(p)}$ is $\mathbb{Z}_{(p)}$-diagonal and that $V(J_{(p)})/p^{r+2f}V(J_{(p)})$ has a basis

$$1, \overline{v} = f_1, f_2, \dots, f_k \tag{6.9}$$

with

$$f_i f_j = -f_j f_i \quad \text{for} \quad i,j = 1,\dots,k \text{with } i \neq j$$

and

$$f_i \overline{f}_i = \lambda_i \quad \text{for} \quad i = 2,\dots,k$$

and

$$\overline{f}_1 f_1 = \overline{v}v = p^r.$$

It follows from the above that

$$\widehat{T}_r(J_{(p)}, p^f v) = \widehat{T}_\ell(J_{(p)}, p^f v) =$$
$$= \left\{ xp^f\overline{v} + p^{r+f}(x_0 + \sum_{i=2}^{k} x_i f_i) \;\middle|\; x, x_0, x_2, \dots, x_k \in \mathbb{Z}_{(p)} \right\}.$$

We write $[u]$ for the residue class of $u \in V(J_{(p)})$ in $V(J_{(p)})/p^{r+2f}V(J_{(p)})$. Then the condition

$$\left( [xp^f\overline{v} + p^{r+f}(x_0 + \sum_{i=2}^{k} x_i f_i)] \quad , \quad [yp^f\overline{v} + p^{r+f}(y_0 + \sum_{i=2}^{k} y_i f_i)] \right) \in \widehat{D}(J_{(p)}, p^f v)$$

is equivalent to the following set of equations:

$$\begin{cases} xy - 1 + p^r(x_0 y_0 + \sum_{i=2}^{k} \lambda_i x_i y_i) \in p^{r+f}\mathbb{Z}_{(p)}, \\ \qquad xy_i + yx_i \in p^f\mathbb{Z}_{(p)} \quad \text{for} \quad i = 0,2,\dots,k, \\ \qquad x_0 y_i + y_0 x_i \in p^f\mathbb{Z}_{(p)} \quad \text{for} \quad i = 2,\dots,k, \\ \qquad x_i y_j - y_j x_i \in p^f\mathbb{Z}_{(p)} \quad \text{for} \quad i,j = 2,\dots,k with i \neq j. \end{cases} \tag{6.10}$$

From (6.10) a description of the exponential sums $S_p(\mu,\nu;p^f v)$ can be derived.

For the formulation of our final results we give the following definition.

**6.17 Definition.** For $\mu \in (nV(J))^{\#}$ and $N \in \mathbb{N}$ we write $\mu_N$ for the following homomorphism

$$\mu_N: nV(J)/nNV(J) \longrightarrow \mathbb{Z}/N\mathbb{Z}, \quad x + nNV(J) \longmapsto \langle x, \mu \rangle + N\mathbb{Z}.$$

**6.18 Theorem.** Let $p$ be a prime and let $\gamma_0 \in nT(J_{(p)})$ be a $(\mathbb{Z}_{(p)}, n)$-primitive element with $\overline{\gamma_0}\gamma_0 = p^r\eta$ with $r \in \mathbb{N} \cup \{0\}$ and $(p, \eta) = 1$. For $f \in \mathbb{N} \cup \{0\}$ consider $\gamma = p^f\gamma_0$. Then the following statements hold:

(i) There is a constant $c_p$ (depending only on $p$ and not on $\gamma$) such that

$$|S_p(\mu, \nu; \gamma)| \leq c_p p^{r+f(k+1)}$$

for all $\mu, \nu \in (nV(J))^{\#}$ and $\gamma$ as above.

(ii) If $J_{(p)}$ is $\mathbb{Z}$-diagonal and if $n_p = 0$ then

$$|S_p(\mu, \nu; \gamma)| \leq p^{r+f(k+1)}$$

for all $\mu, \nu \in V(J)^{\#}$ and $\gamma$ as above.

(iii) Let $J_{(p)}$ be $\mathbb{Z}_{(p)}$-diagonal and assume that $r = 1$ and $f = 0$. Let $v_\gamma, w_\gamma$ be the elements defined in Definition 6.13. Then

$$|S_p(\mu, \nu; \gamma)| \leq \begin{cases} p-1 & \text{if } [v_\gamma] \in \ker \mu_p \text{ and } [w_\gamma] \in \ker \nu_p, \\ 1 & \text{if } [v_\gamma] \notin \ker \mu_p \text{ and } [w_\gamma] \in \ker \nu_p, \\ 1 & \text{if } [v_\gamma] \in \ker \mu_p \text{ and } [w_\gamma] \notin \ker \nu_p, \\ 2\sqrt{p} & \text{if } [v_\gamma] \notin \ker \mu_p \text{ and } [w_\gamma] \notin \ker \nu_p. \end{cases}$$

**Proof.** In case $f = 0$ the statements (i) and (ii) follow from Corollary 6.9 and 6.10. The case $f > 0$ is easily derived from this and Lemma 6.2. Statement (iii) follows from Proposition 6.14 together with either trivial considerations or as in the last case from Weil's estimate [68] for Kloosterman sums. □

## §7. An upper bound for the abscissa of absolute convergence of the Linnik-Selberg series

Here we shall use our estimates of the generalized Kloosterman sums from paragraph 6 to obtain results about the region of absolute convergence of the Linnik-Selberg series defined in paragraph 4.

We consider the following objects to be fixed:

$k$ – a natural number or 0,

$q$ – a negative definite rational quadratic form on a $k$-dimensional vector space $E$,

$J < \mathcal{C}(q)$ – a compatible order in the rational Clifford algebra $\mathcal{C}(q)$,

$J^1$ – is the set of transformers of norm 1 in $T(J)$,

$w_J$ – is the cardinality of $J^1$,

$n$ – a natural number.

Notice that $J^1 = \{\varepsilon \in T(J) \quad | \quad \varepsilon\bar{\varepsilon} = 1\}$ is finite because of (1.10).

We start by introducing the coefficients of the Dirichlet series represented by the Linnk-Selberg series.

**7.1 Definition.** For every $\mu, \nu \in (nV(J))^{\#}$ and every $N \in \mathbb{N}$ we put

$$A(\mu,\nu;N) = \sum_{\substack{\gamma \in nT(J) \\ \bar{\gamma}\gamma = N}} S(\mu,\nu;\gamma).$$

For a $d \in \mathbb{N}$ with $d^2|N$ we put

$$A_d(\mu,\nu;N) = \sum_{\substack{\delta \in nT(J) \\ \delta \text{ is } (\mathbb{Z},n)-\text{primitive} \\ d^2\bar{\delta}\delta = N}} S(\mu,\nu;d\delta).$$

With our definitions from paragraph 4 we find

$$Z(\mu,\nu;s) = \sum_{N=1}^{\infty} \frac{A(\mu,\nu;N)}{N^s}. \tag{7.1}$$

Obviously we have

$$A(\mu,\nu;N) = \sum_{d^2|N} A_d(\mu,\nu;N). \tag{7.2}$$

Next we shall replace the summation over $\gamma$ in the definition of $A(\mu,\nu;N)$ by the summation over certain subgroups of $V(J)/nNV(J)$. In the following we shall think of $N, d, M$ as natural numbers satisfiying $N = Md^2$.

**7.2. Definition.** Let $M \geq 1$ be an integer. A subgroup $U \leq V(J)/nMV(J)$ is called a **liftable 1-dimensional isotropic subspace** if there is an $a \in nV(J)$ with the following properties:

(i) the residue class $a + nMV(J)$ generates $U$,

(ii) $a\bar{a} = Mt$ and $(M, t) = 1$,

(iii) $a/n$ is $\mathbb{Z}_{(M)}$-primitive.

The element $a$ is called a **distinguished generator** for $U$. We write ***I(M)*** for the set of liftable 1-dimensional isotropic subspaces in $V(J)/nMV(J)$.

**7.3 Definition.** Let $N, d, M \geq 1$ be integers with $N = Md^2$. Let $\gamma \in nT(J)$ be an element with $\overline{\gamma}\gamma = N$ so that $\gamma/d$ is $(\mathbb{Z}, n)$-primitive. A liftable 1-dimensional isotropic subspace $U \in I(M)$ is called **attached** to $\gamma$ if there is a distinguished generator $a$ of $U$ with

$$da\gamma - N \in nNJ.$$

We write $I(\gamma)$ for the set of $U \in I(M)$ attached to $\gamma$.

**7.4 Lemma.** Let $N \geq 1$ be an integer and $\gamma \in nT(J)$ be an element with $\overline{\gamma}\gamma = N$. Assume that

$$\widehat{D}(J, \gamma) \neq \emptyset.$$

Then $I(\gamma) \neq \emptyset$.

**Proof.** The hypothesis being satisfied choose $(x, y) \in \widehat{T}_r(J, \gamma) \times \widehat{T}_\ell(J, \gamma)$ with

$$\frac{x\gamma}{N} \cdot \frac{y\gamma^*}{N} - 1 \in nJ\gamma^*. \tag{7.3}$$

By Lemma 5.5 we find a $z \in \widehat{T}_r(J, \gamma)$ with

$$y = \frac{\overline{\gamma}\,\overline{z}\gamma'}{N}.$$

Equation (7.3) implies:

$$\frac{x\overline{z}}{N} - 1 \in nJ\gamma^*. \tag{7.4}$$

It follows from the definitions that $N | u\overline{u}$ for every $u \in \widehat{T}_r(J, \gamma)$. We define $t_u$ to be the integer satisfying $Nt_u = u\overline{u}$. From (7.4) we get by taking norms:

$$t_x t_z - \frac{x\overline{z} + z\overline{x}}{N} + 1 \in nN \cdot \mathbb{Z}. \tag{7.5}$$

Using the Chinese remainder theorem we see that there are $r, s \in \mathbb{Z}$ with

$$(7.6)\qquad \begin{cases} r - s - 1 \in n\mathbb{Z}, \\ (r^2 t_x - rst_x t_z - rs + s^2 t_z, N) = 1. \end{cases}$$

We define

$$b = rx - sz.$$

Clearly we have:

$$(7.7)\qquad b\gamma - N \in nNJ.$$

Let $d$ be the natural number so that $\gamma/nd$ is in $J$ and is $\mathbb{Z}$-primitive. We infer from (7.7): $b - \overline{\gamma} \in nJ\overline{\gamma}$. This implies that $b/nd \in V(J)$. Multiplying (7.7) from the left by $\overline{b}$ we find $t\gamma - \overline{b} \in \overline{b}nJ$ where t satisfies $\overline{b}b = Nt$. This implies that $a = b/d$ is $\mathbb{Z}_{(N)}$-primitive. It is easy to check now using (7.5), (7.6) that $a$ is a distinguished generator for a liftable 1-dimensional isotropic subspace in $I(M)$ which is attached to $\gamma$. □

**7.5 Lemma.** Let $N, d \geq 1$ be integers with $d^2 | N$. Suppose that $\gamma_1, \gamma_2 \in nT(J)$ are elements with $\overline{\gamma}_1\gamma_1 = \overline{\gamma}_2\gamma_2 = N$ and such that $\gamma_2/d$ and $\gamma_2/d$ are $(\mathbb{Z}, n)$-primitive. Assume that

$$I(\gamma_1) \cap I(\gamma_2) \neq \emptyset$$

then there is a unit $\varepsilon$ in $J^1$ with

$$\gamma_2 = \gamma_1 \cdot \varepsilon.$$

**Proof.** Take $U \in I(\gamma_1) \cap I(\gamma_1)$ and distinguished generators $a, b$ for $U$ with

$$da\gamma_1 = Nj_1 \quad \text{and} \quad db\gamma_2 = Nb_1$$

with suitable $j_1, b_1 \in J$. Since we have

$$a = \lambda b + nMc$$

with some $c \in V(J)$ and $\lambda \in \mathbb{Z}$ we have

$$(7.8)\qquad a\gamma_1 = Nj_1 \quad \text{and} \quad a\gamma_2 = Nj_2$$

with suitable $j_2 \in J$. Let $t$ be the integer satisfying $a\overline{a} = Nt$, $(N, t) = 1$. Put

$$\varepsilon = \frac{\overline{\gamma}_1\gamma_2}{N}.$$

From (7.8) we get

$$\varepsilon = \frac{\overline{j}_1 j_2}{t}.$$

Since the denominators $N$ and $t$ are coprime we have $\varepsilon \in J$. Clearly $\varepsilon$ satisfies the conclusion of the lemma. □

The following sets of primes play a somewhat exceptional role in our arguments.

**7.6 Definition.** For the order $J < \mathcal{C}(q)$ we define the following set of primes.

$$\begin{aligned} B(J,n) = \{2\} &\cup \{\, p \mid J_{(p)} \text{ is not } \mathbb{Z}_{(p)}\text{-diagonal}\} \\ &\cup \{\, p \mid \text{the quadratic form } x \mapsto x\overline{x} \text{ on } V(J)/pV(J) \text{ is degenerate}\} \\ &\cup \{\, p \mid p|n \,\} \end{aligned}$$

For $\mu \in (nV(J))^{\#}\backslash\{0\}$ we define

$$B(\mu) = \{p \mid \mu_p = 0\}.$$

For the definition of $\mu_p$ see Definition 6.17. Notice that both sets of primes $B(J,n)$ and $B(\mu)$ are finite.
Let $N$ be an integer and $p$ a prime then we use

$$p \parallel N \tag{7.9}$$

to indicate that $p$ divides $N$ exactly once.

**7.7 Definition.** For an integer $M \geq 1$ and an element $\mu \in (nV(J))^{\#}$ with $\mu \neq 0$ we define the following set of primes:

$$P(\mu, M) = \{\, p \mid p \parallel M \ , \ p \notin B(J,n) \cup B(\mu)\}.$$

Let $N, d, M$ be a natural numbers with $N = Md^2, P \subset P(\mu, M)$ a subset and $\mu \in (nV(J))^{\#}$. We define

$$T(N,d;P,\mu) = \left\{ \gamma \in nT(J) \;\middle|\; \begin{array}{llll} (i) & \gamma\overline{\gamma} = N, & & \\ (ii) & \gamma = d \cdot \gamma_0 & \text{with} & \gamma_0 \ \ (\mathbb{Z}, n)\text{-primitive}, \\ (iii) & v(p,\gamma_0) \notin \ker\mu_p & \text{if} & p \in P, \\ (iv) & v(p,\gamma_0) \in \ker\mu_p & \text{if} & p \in P(\mu, M)\backslash P. \end{array} \right\}$$

The elements $v(p,\gamma_0) \in V(J)/pV(J)$ are the residue classes of the elements $v_{\gamma_0}$ defined in Definition 6.13.

Obviously we have

$$A_d(\mu, \nu; N) = \sum_{P \subset P(\mu,M)} \ \sum_{\gamma \in T(N,d;P,\mu)} S(\mu, \nu; \gamma). \tag{7.10}$$

In Theorem 6.18 we have given an estimate of $S_p(\mu, \nu; \gamma)$ for $\gamma \in T(N,d;P,\mu)$ which does not depend on $\gamma$. Considering an estimate for $A_d(\mu, \nu; N)$ and the equation (7.10) we are left with finding the sizes of the sets $P(\mu, M)$ and of $T(N,d;P,\mu)$ for a subset $P \subset P(\mu, N)$. To do this we define:

**7.8 Definition.** For an integer $M \geq 1$ , a subset $P \subset P(\mu, M)$ and an element $\mu \in (nV(J))^{\#}$ with $\mu \neq 0$ we define

$$I(M;P,\mu) = \left\{ U \in I(M) \;\middle|\; \begin{array}{ll} U \text{ has a distinguished generator } a \text{ with} & \\ (i)\; a + pV(J) \notin \ker \mu_p & \text{if } p \in P, \\ (ii)\; a + pV(J) \in \ker \mu_p & \text{if } p \in P(\mu, M) \backslash P \end{array} \right\}.$$

**7.9 Lemma.** Let $N, d, M \geq 1$ be integers with $N = Md^2$ and $P \subset I(M)$ a subset and $\mu \in (nV(J))^{\#} \backslash \{0\}$ an element. We have

$$|T(N, d; P, \mu)| \leq w_J \cdot |I(M; P, \mu)|.$$

**Proof.** This is an obvious consequence of Lemmas 7.4, 7.5. □

For our estimates we need now some considerations about representation numbers of quadratic forms.

Let $p$ be an odd prime number and let $V$ be a vector space of dimension $m \geq 0$ over the field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let $q$ be a quadratic form on $V$. A subspace $U < V$ is called isotropic if $q(u) = 0$ for all $u \in U$. If $q$ is nondegenerate it follows from [32], Theorem 5.2.6 that

$$(7.11) \qquad |\{ W < V \mid W \text{ is a 1-dimensional isotropic subspace}\}| \leq m \cdot p^{m-2}.$$

See also [60]. We notice that (7.11) implies

$$(7.12) \qquad |\{x \in V \mid q(x) = 0\}| \leq c_1 p^{m-1}$$

with a constant $c_1$ independent of $q$ and $p$.

Every quadratic form $q$ over $\mathbb{F}_p$ can be decomposed as orthogonal sum $q = q_1 \perp q_0$, where $q_0$ is 0 and $q_1$ is nondegenerate, [8]. Hence (7.11) holds for every nonzero quadratic form $q$. This implies:

**7.10 Lemma.** Let $p$ be an odd prime number and $k \geq 2$ an integer. Let $V$ be a $(k+1)$-dimensional $\mathbb{F}_p$-vectorspace with nondegenerate quadratic form $q$. Let $U$ be a $k$-dimensional subspace. Then

$$|\{W < U \mid W \text{ is 1-dimensional isotropic subspace}\}| \leq (k+1) \cdot p^{k-2}.$$

**Proof.** By the above remarks it is enough to notice that $q$ restricted to $U$ cannot be the zero form. □

We also need the following result on congruential representation numbers which is intimately related to (7.12).

**7.11 Lemma.** Let $p$ be a prime and let $\mathbb{Z}_p$ be the ring of $p$-adic integers. Let $Q$ be a nondegenerate quadratic form in $m \geq 0$ variables with coefficients in $\mathbb{Z}_p$. For an $e \in \mathbb{N}$ define

$$N(Q,e) = |\,\{x \in \mathbb{Z}_p^m/p^e\mathbb{Z}_p^m \quad | \quad x \notin p\cdot\mathbb{Z}_p^m + p^e\mathbb{Z}_m^m \quad , \quad Q(x) \equiv 0 \bmod p^e\}\,|.$$

Then there is a $\kappa > 0$ so that

$$p^{-e(m-1)}\cdot N(Q,e)$$

is independent of $e$ for all $e \geq \kappa$.

**Proof.** For an element $v \in \mathbb{Z}_p^m$ write $\Lambda(v)$ for the $\mathbb{Z}_p$-ideal generated by

$$\left\{\frac{\partial Q}{\partial x_i}(v) \quad | \quad i = 1,\dots,m\right\}.$$

There is a $t \in \mathbb{N}$ depending only on $Q$ so that

$$p^t\mathbb{Z}_p < \Lambda(v)$$

for every vector $v \in \mathbb{Z}_p^m$ with $v \notin p\mathbb{Z}_p^m$. For $p$ odd this follows since $Q$ is equivalent to a diagonal form [40]. For $p = 2$ the result can easily be checked for the normal forms over $\mathbb{Z}_2$ in [40], Theorems 33, 33a.

We finish the argument by the usual Hensel's lemma type considerations. See for example [33], Appendix to Chapter 3. □

**7.12 Lemma.** There is a constant $c_2$ (independent of $p$) such that for all primes $p$ and all integers $r \geq 1$

$$\left|\left\{x \in V(J)/p^rV(J) \quad \middle| \quad \begin{matrix}(i)\; x \notin pV(J) + p^rV(J),\\ (ii)\; x\overline{x} \equiv 0 \bmod p^r\end{matrix}\right\}\right| \leq c_2 p^{rk}.$$

**Proof.** If $p$ is odd and if the quadratic form induced by $x \mapsto x\overline{x}$ on $V(J)/pV(J)$ is nondegenerate we prove our result from (7.12) by means of Hensel's lemma and obtain a constant $c_2$ independent of $p$. In the finitely many remaining cases we use Lemma 7.11. □

**7.13 Lemma.** Assume that $k \geq 2$. There is a constant $c_3 > 0$ so that

$$|I(M;P,\mu)| \leq \prod_{p^2|M} c_3 p^{r_p(k-1)} \cdot \prod_{\substack{p\|M \\ p\notin P(\mu,M)}} c_3 p^{k-1} \cdot \prod_{p\in P} c_3 p^{k-1} \cdot \prod_{p\in P(\mu,M)\setminus P} c_3 p^{k-2}$$

for every integer $M \geq 1$ with prime number decomposition

$$M = \prod_p p^{r_p} \quad (r_p \in \mathbb{N} \cup \{0\})$$

and all subsets $P \subset P(\mu, M)$ and all $\mu \in (nV(J))^\# \setminus \{0\}$.

**Proof.** Let $U \in I(M;P,\mu)$ and let $a \in U$ be a distinguished generator. For every prime $p$ dividing $M$ we associate to $U$ the subgroup of $V(J)/p^{r_p}V(J)$ generated by the $\mathbb{Z}_{(p)}$-primitive element $a/n$. The residue class of $a$ in $V(J)/nMV(J)$ is determined by the collection of these elements in the various $V(J)/p^{r_p}V(J)$. Using Lemmas 7.10 and 7.12 we find our constant $c_3$. □

**7.14 Proposition.** Assume that $k \geq 2$. For $\mu, \nu \in (nV(J))^\# \setminus \{0\}$ put

$$B = B(J,n) \cup B(\mu).$$

(i) There is a constant $c_4$ so that

$$|A_d(\mu,\nu;N)| \leq \prod_{\substack{p\|N \\ p\notin B}} c_4 p^{k-\frac{1}{2}} \cdot \prod_{\substack{p\|N \\ p\in B}} c_4 p^k \cdot \prod_{p^2|N} c_4 p^{g_p k}$$

for all natural numbers $N$ with prime number decomposition

$$N = \prod_p p^{g_p}.$$

(ii) For every $\epsilon > 0$ there is a constant $c_\epsilon > 0$ so that

$$|A(\mu,\nu;N)| \leq \prod_{\substack{p\|N \\ p\notin B}} c_\epsilon p^{k-\frac{1}{2}+\epsilon} \prod_{\substack{p\|N \\ p\in B}} c_\epsilon p^{k+\epsilon} \prod_{p^2|N} c_\epsilon p^{g_p(k+\epsilon)}$$

for all natural numbers $N$ with prime number decomposition

$$N = \prod_p p^{g_p}.$$

**Proof.** (i) With our usual conventions (in particular $N = Md^2$) we find from (7.10) and Corollary 5.12:

$$|A_d(\mu,\nu;N)| \le \sum_{P\subset P(\mu,M)} \sum_{\gamma\in T(N,d;P,\mu)} |S(\mu,\nu;\gamma)|$$
$$\le \sum_{P\subset P(\mu,M)} \sum_{\gamma\in T(N,d;P,\mu)} \prod_{p|N} |S_p(\lambda_p\mu,\lambda_p\nu;\gamma)|$$

with suitable elements $\lambda_p \in \mathbb{Z}^*_{(p)}$. We use Theorem 6.18 to conclude:

$$|A_d(\mu,\nu;N)| \le \sum_{P\subset P(\mu,M)} |T(N,d;P,\mu)| \cdot \prod_{\substack{p|N\\ p\in B}} c_p p^{r_p+f_p(k+1)}$$
$$\cdot \prod_{\substack{p^2|N\\ p\notin B}} p^{r_p+f_p(k+1)} \cdot \prod_{\substack{p\|N\\ p\in P}} 2p^{1/2} \cdot \prod_{\substack{p\|N\\ p\in P(\mu,M)\setminus P}} p\,.$$

We use Lemmas 7.9 and 7.13 and the fact $2k \ge k+1$ to find

$$|A_d(\mu,\nu;N)| \le \sum_{P\subset P(\mu,M)} \prod_{\substack{p|N\\ p\in B}} c_5 p^{g_p k} \cdot \prod_{\substack{p^2|N\\ p\notin B}} c_5 p^{g_p k}\cdot$$
$$\cdot \prod_{\substack{p\|N\\ p\in P}} c_5 p^{k-1/2} \cdot \prod_{\substack{p\|N\\ p\in P(\mu,M)\setminus P}} p^{k-1}$$

with some constant $c_5$. We then find a constant $c_6 \ge 1$ so that:

$$|A_d(\mu,\nu;N)| \le \prod_{\substack{p\|N\\ p\in B}} c_6 p^k \cdot \prod_{p^2|N} c_6 p^{g_p k}\cdot$$
$$\cdot\Big( \sum_{P\subset P(\mu,M)} \prod_{\substack{p\|N\\ p\in P}} c_6 p^{k-1/2} \cdot \prod_{\substack{p\|N\\ p\in P(\mu,M)\setminus P}} c_6 p^{k+1}\Big)$$
$$\le \prod_{\substack{p\|N\\ p\in B}} c_6 p^k \cdot \prod_{p^2|N} c_6 p^{g_p k}$$
$$\prod_{p\in P(\mu,M)} (c_6 p^{k-1/2} + c_6 p^{k-1}).$$

Part (i) is now proved since

$$P(\mu,M) = \{p \mid p \| N\,,\ p \notin B\}.$$

Part (ii) follows from the fact that the number of divisors of a natural number $N$ is $O(N^\epsilon)$ for every $\epsilon > 0$, (see [29],Theorem 315). □

To obtain the desired result on the abscissa of convergence for $Z(\mu,\nu;s)$ we need the following elementary lemma on Dirichlet series.

**7.15 Lemma.** Let

$$D(s) = \sum_{N=1}^{\infty} \frac{a_N}{N^s}$$

be a Dirichlet series so that there are constants $c > 0$ and $r \in \mathbb{R}$ and a finite set of primes $B$ so that

$$|a_N| \leq \prod_{\substack{p \| N \\ p \notin B}} cp^{r-\frac{1}{2}} \prod_{\substack{p \| N \\ p \in B}} cp^r \prod_{p^2 | N} cp^{g_p r}$$

for all natural numbers $N$ with prime number decomposition

$$N = \prod_p p^{g_p}.$$

Then $D(s)$ converges for

$$\operatorname{Re} s > r + \frac{1}{2}.$$

**Proof.** For any $\epsilon > 0$, the various factors $c$ are less than $p^\epsilon$ for almost all primes $p$ and hence yield a factor in the upper bound for $|a_n|$ which is $O(n^\epsilon)$. This factor does not affect the final bound for the abscissa of absolute convergence. Hence we may assume without loss of generality that $c = 1$.

Now we have for $s \in \mathbb{R}$

$$\sum_{n=1}^{\infty} \left( \prod_{\substack{p \| N \\ p \in B}} p^r \prod_{\substack{p \| N \\ p \notin B}} p^{r-\frac{1}{2}} \prod_{p^2 | N} p^{r g_p} \right) N^{-s}$$
$$\leq \prod_{p \in B} (1 - p^{r-s})^{-1} \prod_{p \notin B} (1 + p^{r-\frac{1}{2}-s}) \cdot$$
$$\cdot \prod_{p \notin B} (1 + \sum_{m=2}^{\infty} p^{m(r-s)}),$$

and the product on the right-hand side converges if the series

$$\sum_{p \notin B} p^{r-\frac{1}{2}-s}, \sum_{p \notin B} \sum_{m=2}^{\infty} p^{m(r-s)}$$

converge. Obviously, both series converge for $s > k + \frac{1}{2}$. □

**7.16 Remark.** The present proof for Theorem 7.17 excludes the cases $k = 0, 1$. The case $k = 1$ can be handled by the same argument together with the remark that in this case for a fixed $\mu \in (nV(J))^{\#} \backslash \{0\}$ there are only finitely many primes $p$ so that the kernel of $\mu_p$ contains a nonzero element $x$ with $x\bar{x} \equiv 0 \bmod p$. Enlarging our set $B(\mu)$ we can prove Lemma 7.13 also for $k = 1$, and the rest of the argument survives. For $k = 0$ the arguments are correct until Proposition 7.14 (i). Here it is necessary to give better estimates of some of the Kloosterman sums. But this would be the old proof of Selberg.

An obvious consequence is:

**7.17 Theorem.** Assume that $k \geq 1$ and let $\mu, \nu \in (nV(J))^{\#}$ be both nonzero, then $Z(\mu, \nu; s)$ converges absolutely for $\operatorname{Re} s > k + \frac{1}{2}$.

To obtain a proof of Theorem 7.17 in case $k = 1$ we could either carry out the content of Remark 7.16 or quote the paper of Sarnak [58] which contains this result.

## §8. Poincaré Series and Eigenfunctions of the Laplacian

The following assumptions and notations will be kept fixed throughout paragraphs 8–10:

- $k$ – a natural number or 0 ,
- $q$ – a negative definite rational quadratic form on a $k$-dimensional rational vectorspace $E$,
- $J < \mathcal{C}(q)$ – a compatible order in the rational Clifford algebra $\mathcal{C}(q)$,
- $n$ – a natural number.

We furthermore define:

$$V(J) := V_q \cap J \quad , \quad T(J) := T(q) \cap J,$$
$$SV_k(J) := SV_k(\mathbb{Q}, q) \cap M_2(J),$$
$$\Gamma := SV_k(J; n) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SV_k(J) \;\middle|\; \alpha - 1, \beta, \gamma, \delta - 1 \in nJ \right\}.$$

Once and for all we fix an isomorphism $f: E \otimes_{\mathbb{Q}} \mathbb{R} \to \mathbb{R}^k$ such that $q = -I_k \circ f$, and we fix the associated monomorphism (see (2.3), (2.4)) $\hat{f}: SV_k(\mathbb{Q}, q) \to SV_k$. The elements

$\sigma \in SV_k(\mathbb{Q}, q)$, $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ act on $\mathbb{H}^{k+2}$ via this monomorphism $\hat{f}$, and we simply write e.g.

$$\sigma P = (\alpha P + \beta)(\gamma P + \delta)^{-1} \qquad (P \in \mathbb{H}^{k+2})$$

instead of

$$(\hat{f}(\sigma))(P) = (\hat{f}(\alpha)P + \hat{f}(\beta))(\hat{f}(\gamma)P + \hat{f}(\delta))^{-1}.$$

In this way $\Gamma$ is a cofinite discrete non-cocompact subgroup of $SV_k$ (Proposition 2.3). In the special case $q = -q_d$ (see 2.8)) an embedding $\hat{f}$ is explicitly given by (2.11). Finally, we define $\Gamma'_\infty$ to be the set of all elements $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in \Gamma$. Recall that $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in \Gamma$ iff $\omega \in nV(J)$. The set $\Lambda := \hat{f}(nV(J))$ is a lattice in $V_{k+1}$, and we denote by $\Lambda^\#$ the dual lattice. Sometimes we tacitly identify $\Lambda$ with $nV(J)$.

**8.1 Definition.** For $\mu \in \Lambda^\#, P \in \mathbb{H}^{k+2}$ and $s \in \mathbb{C}$ with $\operatorname{Re} s > k+1$ let

$$U_\mu(P, s) := \sum_{\sigma \in \Gamma'_\infty \backslash \Gamma} r(\sigma P)^s e(\, i|\mu| r(\sigma P) + \langle z(\sigma P), \mu\rangle \,), \tag{8.1}$$

where $e(w) := \exp(2\pi i w)$ for $w \in \mathbb{C}$.

The function (8.1) generalizes the series introduced by Selberg [62] and Sarnak [58], [59]; compare also [28], [43]. The function $U_0$ is an Eisenstein series, and $U_\mu$ converges absolutely and uniformly on compact sets in $\mathbb{H}^{k+2} \times \{s \in \mathbb{C} \mid \operatorname{Re} s > k+1\}$. Moreover,

$$U_\mu(\cdot, s) \in L^2(\Gamma \backslash \mathbb{H}^{k+2}) \quad \text{if} \quad \mu \in \Lambda^\#,\ \mu \neq 0,\ \operatorname{Re} s > k+1; \tag{8.2}$$

in fact, we shall compute the inner product of two Poincaré series in paragraph 9.
A quick check proves that the function $\varphi_\mu(\cdot, s)\colon \mathbb{H}^{k+2} \to \mathbb{C}$ which is defined by:

$$\varphi_\mu(P, s) := r(P)^s e(i|\mu| r(P) + \langle z(P), \mu\rangle)$$

($P \in \mathbb{H}^{k+2}$) satisfies the differential equation

$$(-\Delta - s(k+1-s))\varphi_\mu(P, s) = 2\pi|\mu|(2s-k)\varphi_\mu(P, s+1),$$

and hence we have

$$(-\Delta - s(k+1-s))U_\mu(P, s) = 2\pi|\mu|(2s-k)U_\mu(P, s+1) \tag{8.3}$$

for all $\mu \in \Lambda^\#$ and $\operatorname{Re} s > k+1$. For $\mu = 0$ this is the usual differential equation of the Eisenstein series.

**8.2 Proposition.** Let $f \in C^2(\Gamma\backslash\mathbb{H}^{k+2})$ satisfy the differential equation

$$-\Delta f = \lambda f \tag{8.4}$$

for some $\lambda \in \mathbb{C}$, and assume that $f$ is of polynomial growth at the cusp infinity, that is:

$$f(z + ri_{k+1}) = O(r^\alpha) \quad \text{as} \quad r \to \infty$$

uniformly in $z \in V_{k+1}$ for some constant $\alpha > 0$. Put

$$t = \frac{k+1}{2} + \sqrt{\left(\frac{k+1}{2}\right)^2 - \lambda} = \frac{k+1}{2} + \rho, \tag{8.5}$$

that is:

$$\lambda = t(k+1-t) = \left(\frac{k+1}{2}\right)^2 - \rho^2. \tag{8.6}$$

Then $f$ has a Fourier expansion of the form

$$f(z + ri_{k+1}) = a_0 r^t + b_0 r^{k+1-t} + \sum_{\omega \in \Lambda^\#, \omega \neq 0} a(\omega) r^{\frac{k+1}{2}} K_\rho(2\pi|\omega|r) e(\langle \omega, z\rangle) \tag{8.7}$$

provided that $\lambda \neq (k+1)^2/4$. For $\lambda = (k+1)^2/4$ the zeroth term must be replaced by $a_o r^t + b_0 r^t \log r$.

The proof requires a routine computation based on [48], p. 77, or the result may be drawn from [47]. The next proposition is now obvious.

**8.3 Proposition.** Suppose that $f \in C^2(\Gamma\backslash\mathbb{H}^{k+2})$ satisfies the differential equation (8.4) and is square integrable over a cusp sector at infinity:

$$\int_{\mathcal{P}\times[1,\infty[} |f(P)|^2 dv(P) < \infty,$$

where $\mathcal{P}$ is a fundamental parallelotope for $\Lambda$. Then $f$ has a Fourier expansion of the form (8.7), where

(i) $a_0 = b_0 = 0$ if $\lambda \geq \left(\frac{k+1}{2}\right)^2$,
(ii) $b_0 = 0$ if $\operatorname{Re}\lambda < 0$,
(iii) $a_0 = 0$ if $\operatorname{Re}\lambda > 0$.

**8.4 Proposition.** Assume that $f \in L^2(\Gamma\backslash\mathbb{H}^{k+2}) \cap C^2(\Gamma\backslash\mathbb{H}^{k+2})$ satisfies (8.4) and (8.7), and let $\mu \in \Lambda^\#, \mu \neq 0$, and $\operatorname{Re} s > k+1$. Then we have

$$\langle f, U_\mu(\cdot, \bar{s}) \rangle_\Gamma = \eta\sqrt{\pi}(4\pi|\mu|)^{\frac{k+1}{2}-s} \cdot \frac{\Gamma(s-t)\Gamma(s-(k+1-t))}{\Gamma(s-\frac{k}{2})} V\, a(\mu), \tag{8.8}$$

where $V$ denotes the covolume of the lattice $\Lambda$ in $V_{k+1}$ and $\eta = |\Gamma \cap \{\pm I\}|$.

**Proof.** Let $\mathcal{P}$ be as in Proposition 8.3. Unfolding the Poincaré series we obtain

$$\begin{aligned}&\langle f, U_\mu(\cdot, \bar{s})\rangle_\Gamma \\ &= \eta \int_{\mathcal{P}\times]0,\infty[} f(P) r^s e^{-2\pi i(z,\mu) - 2\pi|\mu| r} dv(P) \\ &= \eta a(\mu)\, V \int_0^\infty r^{s-\frac{k+1}{2}} K_\rho(2\pi|\mu|r) e^{-2\pi|\mu|r} \frac{dr}{r},\end{aligned}$$

and [31], p. 50, (26) yields the result. □

## §9. The Inner Product of Two Poincaré Series

The aim of this paragraph is to compute the inner product of two Poincaré series. We maintain the notations established at the beginning of paragraph 8. In addition, let $\mathcal{P}$ denote a fundamental parallelotope for $\Lambda$ and $V$ its volume. For $P \in \mathbb{H}^{k+2}$ we always write $P = z + r i_{k+1}$ with $z \in V_{k+1}$ and $r > 0$, and we denote by $dz = dx_0 \wedge \ldots \wedge dx_k$ the Lebesgue measure on $V_{k+1}$. Further, we tacitly write $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Let $\mu, \nu \in \Lambda^\#$ be elements with $\mu \neq 0, \nu \neq 0$ and let $s, t \in \mathbb{C}, \operatorname{Re} s > k+1$, $\operatorname{Re} t > k+1$. The goal of the present paragraph is to compute the inner product

$$\langle U_\mu(\cdot, s), U_\nu(\cdot, \bar{t}) \rangle_\Gamma = \int_{\mathcal{F}} U_\mu(P, s) \overline{U_\nu(P, \bar{t})}\, dv(P), \tag{9.1}$$

where $\mathcal{F}$ denotes a fundamental domain for $\Gamma$ in $\mathbb{H}^{k+2}$. The final result will be collected in Theorem 9.1. First we do the necessary computations. Additional hypotheses will be introduced when required.

Unfolding the Poincaré series in (9.1) we obtain:

$$(9.2)\qquad \langle U_\mu\,(\cdot,s)\,,\,U_\nu(\cdot,\bar{t})\rangle_\Gamma = \\ = \int_0^\infty r^t e^{-2\pi|\nu|r} \int_{\mathcal{P}} U_\mu(P,s)\, e^{-2\pi i\langle z,\nu\rangle}\, dz \frac{dr}{r^{k+2}},$$

$$(9.3)\qquad \int_{\mathcal{P}} U_\mu(P,s) e^{-2\pi i\langle z,\nu\rangle} dz \\ = \sum_{\sigma\in\Gamma'_\infty\backslash\Gamma} \int_{\mathcal{P}} r(\sigma P)^s e^{-2\pi|\mu| r(|\gamma z+\delta|^2+|\gamma|^2 r^2)^{-1}+2\pi i(\langle z(\sigma P),\mu\rangle-\langle z,\nu\rangle)}\, dz.$$

First we determine the contribution of the elements $\sigma\in\Gamma'_\infty\backslash\Gamma$ with $\gamma=0$: For $\gamma=0$ we have $\alpha\delta^*=1$ and hence by (1.11) $|\alpha||\delta|=1$, that is, $|\alpha|=|\delta|=1$ (since $|\alpha|^2\in\mathbb{Z}, |\delta|^2\in\mathbb{Z}$). By (1.22) and (1.7) this implies:

$$\begin{aligned} z(\sigma P) &= (\alpha z+\beta)\,\overline{\delta} \\ &= \varphi_\alpha(z)\,\alpha'\overline{\delta}+\beta\overline{\delta} \\ &= \varphi_\alpha(z)+\beta\overline{\delta} \end{aligned}$$

where $\varphi_\alpha\colon V_{k+1}\to V_{k+1}$ is an orthogonal linear map (by [15], Proposition 3.6) and $\beta\overline{\delta}\in V_q\cap nJ = nV(J)$ (by Lemma 1.5). Denoting by $\varphi^*_\alpha$ the dual map of $\varphi_\alpha$ with respect to $\langle\cdot,\cdot\rangle$ we obtain for the contribution of $\sigma$ to the right-hand side of (9.3)

$$(9.4)\qquad \int_{\mathcal{P}} r^s e^{-2\pi|\mu|r+2\pi i\langle z,\varphi^*_\alpha(\mu)-\nu\rangle} dz = r^s e^{-2\pi|\mu|r} V\delta_{\varphi^*_\alpha(\mu),\nu}$$

(with Kronecker's symbol $\delta$). Incidentally, we have used that $\varphi^*_\alpha(\mu)\in\Lambda^\#$, and this follows from

$$\langle v,\varphi^*_\alpha(\mu)\rangle = \langle\varphi_\alpha(v)\,\alpha'\overline{\delta},\mu\rangle = \langle\alpha v\overline{\delta},\mu\rangle\in\mathbb{Z}$$

for all $v\in\Lambda$. Moreover, the consideration above yields that every $\sigma\in\Gamma'_\infty\backslash\Gamma$ with $\gamma=0$ has a representative with $\beta=0$. Hence the number of elements $\sigma\in\Gamma'_\infty\backslash\Gamma$ with $\gamma=0$ is finite, and we let $C_{\mu,\nu}$ denote the number of elements $\sigma\in\Gamma'_\infty\backslash\Gamma$ with $\gamma=0$ such that $\varphi^*_\alpha(\mu)=\nu$. (Note that $C_{\mu,\nu}=0$ if $|\mu|\neq|\nu|$.) We have now finished to compute the contribution of the elements $\sigma\in\Gamma'_\infty\backslash\Gamma$ with $\gamma=0$ to the right-hand side of (9.2)

$$\sum_{\substack{\sigma\in\Gamma'_\infty\backslash\Gamma\\ \gamma=0}} \int_0^\infty r^t e^{-2\pi|\nu|r} \int_{\mathcal{P}} r^s e^{-2\pi|\mu|r+2\pi i\langle z,\varphi^*_\alpha(\mu)-\nu\rangle}\, dz \frac{dr}{r^{k+2}}$$

$$(9.5)\qquad \begin{aligned} &= C_{\mu,\nu} V \int_0^\infty r^{s+t-k-1} e^{-2\pi(|\mu|+|\nu|)r} \frac{dr}{r} \\ &= (2\pi(|\mu|+|\nu|))^{k+1-s-t}\,\Gamma(s+t-k-1) C_{\mu,\nu} V \\ &= (4\pi|\mu|)^{k+1-s-t}\,\Gamma(s+t-k-1) C_{\mu,\nu} V. \end{aligned}$$

Second we deal with the contribution of the elements $\sigma \in \Gamma'_\infty\backslash\Gamma$ with $\gamma \neq 0$ to the right-hand side of (9.3). To that end, let

$$(9.6) \qquad H(r) := \sum_{\substack{\sigma\in\Gamma'_\infty\backslash\Gamma \\ \gamma\neq 0}} \int_{\mathcal{P}} r(\sigma P)^s e^{-2\pi|\mu|r(|\gamma z+\delta|^2+|\gamma|^2r^2)^{-1}+2\pi i(\langle z(\sigma P),\mu\rangle - \langle z,\nu\rangle)}\,dz.$$

Using (1.24) we obtain

$$H(r) = \sum_{\substack{\sigma\in\Gamma'_\infty\backslash\Gamma \\ \gamma\neq 0}} \frac{r^s}{|\gamma|^{2s}}\, e(\langle \alpha\gamma^{-1},\mu\rangle) \int_{\mathcal{P}} \frac{\exp(-\frac{2\pi|\mu|}{|\gamma|^2}\frac{r}{|z+\gamma^{-1}\delta|^2+r^2})}{(|z+\gamma^{-1}\delta|^2+r^2)^s}\cdot$$
$$\cdot\, e\left(-\left\langle \frac{(\gamma^*)^{-1}}{|\gamma|^2}\frac{\overline{(z+\gamma^{-1}\delta)}\,\overline{\gamma}}{|z+\gamma^{-1}\delta|^2+r^2},\mu\right\rangle - \langle z,\nu\rangle\right) dz.$$

Using Theorem 4.7 we rewrite the summation condition as follows:

$$H(r) = \sum_{\gamma\in nT(J)} \frac{r^s}{|\gamma|^{2s}} \sum_{\substack{(\alpha,\delta)\in D(\gamma)\\ \omega\in\Lambda}} e(\langle \alpha\gamma^{-1},\mu\rangle) \int_{\mathcal{P}} \frac{\exp(-\frac{2\pi|\mu|}{|\gamma|^2}\frac{r}{|z+\gamma^{-1}\delta+\omega|^2+r^2})}{(|z+\gamma^{-1}\delta+\omega|^2+r^2)^s}\cdot$$
$$\cdot\, e\left(-\left\langle \frac{(\gamma^*)^{-1}}{|\gamma|^2}\frac{\overline{(z+\gamma^{-1}\delta+\omega)}\,\overline{\gamma}}{|z+\gamma^{-1}\delta+\omega|^2+r^2},\mu\right\rangle - \langle z,\nu\rangle\right) dz$$

$$= \sum_{\gamma\in nT(J)} \frac{r^{k+1-s}}{|\gamma|^{2s}} \sum_{(\alpha,\delta)\in D(\gamma)} e(\langle \alpha\gamma^{-1},\mu\rangle + \langle \gamma^{-1}\delta,\nu\rangle)\cdot$$
$$\cdot \int_{V_{k+1}} \frac{\exp\left(-\frac{2\pi|\mu|}{r|\gamma|^2}\frac{1}{|z|^2+1}\right)}{(|z|^2+1)^s}\, e\left(-\left\langle \frac{(\gamma^*)^{-1}\overline{z}\,\overline{\gamma}}{r|\gamma|^2(|z|^2+1)},\mu\right\rangle - r\langle z,\nu\rangle\right) dz$$

$$= r^{k+1-s} \sum_{\gamma\in nT(J)} \frac{S(\mu,\nu;\gamma)}{|\gamma|^{2s}}\cdot$$
$$\cdot \int_{V_{k+1}} \frac{\exp\left(-\frac{2\pi|\mu|}{r|\gamma|^2}\frac{1}{|z|^2+1}\right)}{(|z|^2+1)^s}\, e\left(-\left\langle \frac{(\gamma^*)^{-1}\overline{z}\gamma^{-1}}{r(|z|^2+1)},\mu\right\rangle - r\langle z,\nu\rangle\right) dz,$$

where the generalized Kloosterman sum $S(\mu,\nu;\gamma)$ is defined by (4.5). Consider the last integral: For $|\gamma|$ large the exponentials are nearly equal to one. Hence we decompose the contribution of $H(r)$ to the right-hand side of (9.2) as follows:

$$\int_0^\infty r^t e^{-2\pi|\nu|r} H(r)\frac{dr}{r^{k+2}}$$

(9.7)
$$= \sum_{\gamma\in nT(J)} \frac{S(\mu,\nu;\gamma)}{|\gamma|^{2s}} \int_0^\infty r^{t-s-1}e^{-2\pi|\nu|r} \int_{V_{k+1}} \frac{e^{-2\pi i r\langle z,\nu\rangle}}{(|z|^2+1)^s} dz\, dr + \sum_{\gamma\in nT(J)} \frac{S(\mu,\nu;\gamma)}{|\gamma|^{2s}} \cdot R_{\mu,\nu}(s,t;\gamma)$$

with

(9.8)
$$R_{\mu,\nu}(s,t;\gamma) \doteq \int_0^\infty r^{t-s-1}e^{-2\pi|\nu|r} \int_{V_{k+1}} \frac{e^{-2\pi i r\langle z,\nu\rangle}}{(|z|^2+1)^s} \cdot \left(\exp\left(-\frac{2\pi|\mu|}{r|\gamma|^2}\frac{1}{|z|^2+1} - 2\pi i\left\langle \frac{(\gamma^*)^{-1}\bar{z}\gamma^{-1}}{r(|z|^2+1)}, \mu\right\rangle\right) - 1\right) dz\, dr.$$

We evaluate the integral over $V_{k+1}$ on the right-hand side of (9.7): Obviously,

(9.9)
$$\int_{V_{k+1}} \frac{e^{-2\pi i r\langle z,\nu\rangle}}{(|z|^2+1)^s}dz = \int_{\mathbb{R}} e^{-2\pi i r|\nu|x_0} \int_{\mathbb{R}^k} \frac{1}{(x_0^2+x_1^2+\ldots+x_k^2+1)^s}\, dx_1\cdot\ldots\cdot dx_k\, dx_0 = \int_{\mathbb{R}} \frac{e^{-2\pi i r|\nu|u}}{(u^2+1)^{s-k/2}}\, du \int_{\mathbb{R}^k} \frac{1}{(u_1^2+\ldots+u_k^2+1)^s}\, du_1\cdot\ldots\cdot du_k.$$

Here we have by [48], p. 85:

(9.10)
$$\int_{-\infty}^{+\infty} \frac{e^{-2\pi i r|\nu|u}}{(u^2+1)^{s-k/2}}du = \frac{2}{\Gamma(s-k/2)}\pi^{s-k/2}(|\nu|r)^{s-(k+1)/2}\cdot K_{s-(k+1)/2}(2\pi|\nu|r).$$

Further, putting $I_0 := 1$ we have for $k \geq 1$:

(9.11)
$$I_k(s) := \int_{\mathbb{R}^k} \frac{1}{(u_1^2+\ldots+u_k^2+1)^s}\, du_1\cdot\ldots\cdot du_k = I_1(s-\frac{k-1}{2})I_{k-1}(s),$$

that is,

(9.12)
$$I_k(s) = I_1(s-\frac{k-1}{2})\cdot I_1(s-\frac{k-2}{2})\cdot\ldots\cdot I_1(s).$$

Since

$$I_1(s) = \frac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \tag{9.13}$$

(see [48], p.6), we get from (9.9) - (9.13)

$$\begin{aligned} &\int_0^\infty r^{t-s-1} e^{-2\pi|\nu|r} \int_{V_{k+1}} \frac{e^{-2\pi i r(z,\nu)}}{(|z|^2+1)^s} dz\, dr \\ &= 2\pi^s |\nu|^{s-(k+1)/2} \Gamma^{-1}(s) \int_0^\infty r^{t-(k+1)/2-1} e^{-2\pi|\nu|r} K_{s-(k+1)/2}(2\pi|\nu|r)\, dr. \end{aligned} \tag{9.14}$$

Now we introduce the additional hypothesis

$$\operatorname{Re} t > \operatorname{Re} s. \tag{9.15}$$

Then the last integral may be evaluated by means of [31], p. 50, (26), and we obtain

$$\begin{aligned} &\int_0^\infty r^{t-s-1} e^{-2\pi|\nu|r} \int_{V_{k+1}} \frac{e^{2\pi i r(z,\nu)}}{(|z|^2+1)^s}\, dz\, dr \\ &= 2^{k+2-2t} \pi^{s-t+1+k/2} |\nu|^{s-t} \frac{\Gamma(t+s-k-1)\Gamma(t-s)}{\Gamma(s)\Gamma(t-k/2)}. \end{aligned} \tag{9.16}$$

Collecting terms from (9.2), (9.3) (9.5), (9.6), (9.7), (9.16) we obtain the following result.

**9.1 Theorem.** Suppose that $s, t$ are complex numbers with $\operatorname{Re} t > \operatorname{Re} s > k+1$, and let $\mu, \nu \in \Lambda^\# = (nV(J))^\#$ satisfy $\mu \neq 0, \nu \neq 0$. Then we have

$$\begin{aligned} &\langle\, U_\mu(\cdot, s)\,,\, U_\nu(\cdot, \bar{t})\,\rangle_\Gamma \\ &= (4\pi|\mu|)^{k+1-s-t}\,\Gamma(s+t-k-1) C_{\mu,\nu} V \\ &+ 2^{k+2-2t} \pi^{s-t+1+k/2} |\nu|^{s-t} \frac{\Gamma(t+s-k-1)\Gamma(t-s)}{\Gamma(s)\Gamma(t-k/2)} Z(\mu,\nu;s) \\ &+ \sum_{\gamma \in nT(J)} \frac{S(\mu,\nu;\gamma)}{|\gamma|^{2s}} R_{\mu,\nu}(s,t;\gamma), \end{aligned} \tag{9.17}$$

where $R_{\mu,\nu}(s,t;\gamma)$ is given by (9.8) and where $Z(\mu,\nu;s)$ denotes the Linnik-Selberg series (4.7). The function $R_{\mu,\nu}(s,t;\gamma)$ satisfies the inequality

$$|R_{\mu,\nu}(s,t;\gamma)| \leq M|\gamma|^{-2} \tag{9.18}$$

for some constant $M > 0$ (depending on $s, t, \mu, \nu$), provided that $\operatorname{Re} t > \operatorname{Re} s + 1$.

**Proof:** Having established (9.17) above, we still have to prove (9.18). To that end we split in (9.8) the interval $]0, \infty[$ into the subintervals $]0, |\gamma|^{-2}]$ and $[|\gamma|^{-2}, \infty[$. For brevity, put

$$w = \frac{-2\pi|\mu|}{r|\gamma|^2(|z|^2+1)} - \frac{2\pi i}{r|\gamma|^2(|z|^2+1)}\langle(\gamma^*)^{-1}\overline{z}\,\overline{\gamma}, \mu\rangle.$$

Then the integral over $]0, |\gamma|^{-2}]$ gives a contribution to $R_{\mu,\nu}(s,t;\gamma)$ which is dominated by

(9.19)
$$\begin{aligned}
&\int_0^{|\gamma|^{-2}} r^{\operatorname{Re}(t-s)-1} e^{-2\pi|\nu|r} \int_{V_{k+1}} (|z|^2+1)^{-\operatorname{Re} s} |e^w - 1| \, dz \, dr \\
&\qquad \leq 2 I_{k+1}(\operatorname{Re} s) \int_0^{|\gamma|^{-2}} r^{\operatorname{Re}(t-s)-1} \, dr \\
&\qquad = \frac{2\pi^{(k+1)/2}\Gamma(\operatorname{Re} s - (k+1)/2)}{\operatorname{Re}(t-s)\Gamma(\operatorname{Re} s)} |\gamma|^{-2\operatorname{Re}(t-s)}.
\end{aligned}$$

Since $|(\gamma^*)^{-1}\overline{z}\,\overline{\gamma}| = |z|$, we have for $r \geq |\gamma|^{-2}$

$$|w| \leq \frac{2\pi|\mu|}{r|\gamma|^2} \frac{1+|z|}{1+|z|^2} \leq \frac{3\pi|\mu|}{r|\gamma|^2} \leq 3\pi|\mu|$$

and hence

$$\begin{aligned}
|e^w - 1| &\leq \sum_{\nu=1}^{\infty} \frac{|w|(3\pi|\mu|)^{\nu-1}}{\nu!} \\
&\leq \frac{|w|}{3\pi|\mu|} e^{3\pi|\mu|} \\
&\leq \frac{1}{r|\gamma|^2} e^{3\pi|\mu|}.
\end{aligned}$$

This implies, that the integral over $[|\gamma|^{-2}, \infty[$ gives a contribution to $R_{\mu,\nu}(s,t;\gamma)$ which is dominated by

$$\begin{aligned}
&|\gamma|^{-2} e^{3\pi|\mu|} \int_{|\gamma|^{-2}}^{\infty} r^{Re(t-s)-2} e^{-2\pi|\nu|r} \, dr \cdot \int_{V_{k+1}} (|z|^2+1)^{-\operatorname{Re} s} \, dz \\
&\qquad = I_{k+1}(\operatorname{Re} s) e^{3\pi|\mu|} |\gamma|^{-2} \int_{|\gamma|^{-2}}^{\infty} r^{Re(t-s)-2} e^{-2\pi|\nu|r} dr.
\end{aligned}$$

If moreover $\mathrm{Re}\,t > \mathrm{Re}\,s + 1$, then the last expression is dominated by

$$(9.20)\qquad \begin{aligned} &I_{k+1}(\mathrm{Re}\,s)e^{3\pi|\mu|}|\gamma|^{-2} \int_{|\gamma|^{-2}}^{\infty} r^{Re(t-s)-2}e^{-2\pi|\nu|r}\,dr \\ &\le I_{k+1}(\mathrm{Re}\,s)e^{3\pi|\mu|}\Gamma(Re(t-s)-1)(2\pi|\nu|)^{-Re(t-s-1)}|\gamma|^{-2}. \end{aligned}$$

Now (9.19) and (9.20) yield (9.18). □

Combining (9.17) with (9.18) we get an interesting consequence: Suppose that $\mathrm{Re}\,t > \mathrm{Re}\,s + 1$ and fix $t$ such that $\mathrm{Re}\,t > k+2$. Then (9.18) and Theorem 7.17 imply that

$$\sum_{\gamma\in nT(J)} \frac{S(\mu,\nu;\gamma)}{|\gamma|^{2s}} R_{\mu,\nu}(s,t;\gamma)$$

is a holomorphic function of $s$ for $\mathrm{Re}\,t - 1 > \mathrm{Re}\,s > (k+\frac{1}{2})-2$, whereas the term involving the Linnik-Selberg series on the right-hand side of (9.17) is holomorphic for

$$\mathrm{Re}\,t - 1 > \mathrm{Re}\,s > k + \frac{1}{2}.$$

## §10. A Lower Bound for the Smallest Positive Eigenvalue of the Laplacian for Congruence Subgroups Acting on Hyperbolic Spaces

The aim of paragraph 10 is to prove the main results described in the introduction. We maintain the notations introduced in paragraph 8.

**10.1 Theorem.** Suppose that $\Gamma < SV_k(\mathbb{Q}, q)$ is a congruence subgroup. Then the least positive eigenvalue $\lambda_1^\Gamma$ of the operator $-\Delta$ acting on its domain in $L^2(\Gamma\backslash\mathbb{H}^{k+2})$ satisfies

$$\lambda_1^\Gamma \ge \frac{1}{4}(2k+1), \quad \text{if} \quad k \ge 1,$$
$$\lambda_1^\Gamma \ge \frac{3}{16}, \quad \text{if} \quad k = 0.$$

**Proof.** The proof is based on ideas of Selberg [62] which were also used in [28], [43],[58], [59]. We restrict to the case $k \ge 1$; the case $k = 0$ is dealt with in the same way.

It suffices to prove the assertion for $\Gamma = SV_k(J,n)$. Assume to the contrary that there is an eigenvalue $\lambda_1$ of $-\tilde{\Delta}$ with $0 < \lambda_1 < (2k+1)/4$. Then we have

$$k+1 > t_1 := \frac{k+1}{2} + \sqrt{\left(\frac{k+1}{2}\right)^2 - \lambda_1} \quad > \quad k + \frac{1}{2},$$

that is, $t_1$ belongs to the half-plane of absolute convergence of the Linnik-Selberg series. Denote by $\rho(-\tilde{\Delta})$ the resolvent set of the unique self-adjoint extension $-\tilde{\Delta}: \mathcal{D} \to L^2(\Gamma\backslash\mathbb{H}^{k+2})$ of the operator $-\Delta: C_0^\infty(\Gamma\backslash\mathbb{H}^{k+2}) \to L^2(\Gamma\backslash\mathbb{H}^{k+2})$, and let

$$R_\lambda := (-\tilde{\Delta} - \lambda)^{-1} \qquad (\lambda \in \rho(-\tilde{\Delta}))$$

be the resolvent operator. As is well known from Selberg's theory of the eigenvalue problem of the automorphic Laplacian, the operator $R_{s(k+1-s)}$ is holomorphic for $\operatorname{Re} s > k+1$ and meromorphic for $\operatorname{Re} s > (k+1)/2$ with at most finitely many poles of order one in $](k+1)/2, k+1]$ (see [21], [22],[30],[42], [44],[54],[61],[63], [67]).
Invoking (8.3), we have for $\mu \in \Lambda^\#, P \in \mathbb{H}^{k+2}$

$$U_\mu(P,s) = 2\pi|\mu|(2k-k)R_{s(k+1-s)}U_\mu(P,s+1).$$

Here, $U_\mu(P,s+1)$ is even holomorphic in $s$ for $\operatorname{Re} s > k$, whereas the resolvent operator is meromorphic in this half-plane. (Note that $k \geq (k+1)/2$ since $k \geq 1$.) This implies that $U_\mu(P,s)$ is meromorphic in $s$ for $\operatorname{Re} s > k$. In particular, $U_\mu(P,s)$ has at most a simple pole at $s = t_1$. For brevity, put

$$\lambda = s(k+1-s),$$

and we have

$$\lambda - \lambda_1 = (s-t_1)(k+1-(s+t_1)).$$

Recall that for any $f \in L^2(\Gamma\backslash\mathbb{H}^{k+2})$,

$$\operatorname{res}(R_\lambda; \lambda = \lambda_1)f = \operatorname{pr}_{\lambda_1}(f),$$

where $\operatorname{pr}_{\lambda_1}$ denotes the orthogonal projection of $L^2(\Gamma\backslash\mathbb{H}^{k+2})$ onto the eigenspace of $-\Delta$ for the eigenvalue $\lambda_1$ (see [41, V, §3, 5]). Computing the residue of $U_\mu(P,s)$ at $s = t_1$, we find

$$\begin{aligned}
&\operatorname{res}(U_\mu(P,s); s = t_1)\\
&= \frac{2\pi|\mu|(2t_1-k)}{k+1-2t_1} \lim_{\lambda\to\lambda_1} ((\lambda-\lambda_1)R_\lambda)\, U_\mu(\cdot, t_1+1)|_P\\
&= \frac{2\pi|\mu|(2t_1-k)}{k+1-2t_1} \operatorname{pr}_{\lambda_1}(U_\mu(\cdot, t_1+1))|_P\\
&= \frac{2\pi|\mu|(2t_1-k)}{k+1-2t_1} \sum_{j=1}^{p} \langle U_\mu(\cdot, t_1+1), v_j\rangle_\Gamma v_j(P),
\end{aligned}$$

where $v_1, \dots, v_p$ is an orthonormal basis of $\mathrm{pr}_{\lambda_1}(L^2(\Gamma\backslash\mathbb{H}^{k+2})$. The function $v_j$ has a Fourier expansion of the form (8.7) with $a_0 = 0$ (Proposition 8.3). For $w \in \Lambda^{\#}, w \neq 0$ let $a_j(w)$ be the $w$-th Fourier coefficient of $v_j$ $(1 \leq j \leq p)$. Then we have by (8.8)

$$\begin{aligned} &\mathrm{res}(U_\mu(P,s); s = t_1) \\ &= -\eta\sqrt{\pi}V(4\pi|\mu|)^{(k+1)/2-t_1}\frac{\Gamma(2t_1-k-1)}{\Gamma(t_1-k/2)}\sum_{j=1}^{p}\overline{a_j(\mu)}v_j(P). \end{aligned} \tag{10.1}$$

Now we fix $t \in \mathbb{C}$ with $\mathrm{Re}\, t > k+2$, fix $\nu \in \Lambda^{\#}$ and obtain from (10.1) and (8.8)

$$\begin{aligned} &\langle \mathrm{res}(U_\mu(\cdot,s); s = t_1), U_\nu(\cdot,\bar{t})\rangle_\Gamma \\ &= -\eta^2\pi V^2(4\pi|\mu|)^{(k+1)/2-t_1}(4\pi|\nu|)^{(k+1)/2-t}\cdot \\ &\cdot\frac{\Gamma(2t_1-k-1)}{\Gamma(t_1-k/2)}\cdot\frac{\Gamma(t-t_1)\Gamma(t+t_1-k-1)}{\Gamma(t-k/2)}\cdot \\ &\cdot\sum_{j=1}^{p}\overline{a_j(\mu)}a_j(\nu). \end{aligned} \tag{10.2}$$

In particular, (10.2) reads for $\mu = \nu$ as follows:

$$\begin{aligned} &\big\langle \mathrm{res}(U_\mu(\cdot,s); s = t_1), U_\mu(\cdot,\bar{t})\big\rangle_\Gamma \\ &= -\eta^2\pi V^2(4\pi|\mu|)^{k+1-t-t_1}\frac{\Gamma(2t_1-k-1)\Gamma(t-t_1)\Gamma(t+t_1-k-1)}{\Gamma(t_1-k/2)\Gamma(t-k/2)} \\ &\sum_{j=1}^{p}|a_j(\mu)|^2. \end{aligned} \tag{10.3}$$

Since $\lambda_1$ is an eigenvalue of $-\Delta$, we may choose $\mu$ in such a way that the right-hand side of (10.3) is different from zero. Then $\langle U_\mu(\cdot,s), U_\mu(\cdot,\bar{t})\rangle_\Gamma$ has a pole at $s = t_1$. On the other hand, Theorem 9.1 combined with Theorem 7.17 implies that the latter scalar product is holomorphic for $\mathrm{Re}\, s > k + 1/2$. This contradiction proves Theorem 10.1. □

We proceed to reformulate Theorem 10.1 in terms of orthogonal groups. For that end we need the results of paragraph 3. This reformulation of Theorem 10.1 is contained in Theorems A,B of the introduction. We shall indicate a proof now:

**Proof of Theorems A,B:** Let $Q$ be rational quadratic form of signature $(1, k+2)$ which is $\mathbb{Q}$-isotropic. It is elementary to see that $Q$ is $\mathbb{Q}$-equivalent to a quadratic form

$$\lambda\cdot(q_0 \perp q) = \lambda\cdot\tilde{q}$$

where $q$ is negative definite with rational coefficients and

$$q_0(y_0, y_1, y_2) = y_0^2 - y_1^2 - y_2^2$$

and $\lambda \in \mathbb{Q}, \lambda > 0$. The statements then follow from Theorem 10.1 together with Lemma 3.1 and Remark 3.3. □

## REFERENCES

[1] Ahlfors, L.V.: Old and new in Möbius groups. Ann. Acad. Sci. Fenn. **9**, 93-105 (1984).

[2] Ahlfors, L.V.: Möbius transformations and Clifford numbers. In: Differential geometry and complex analysis, a volume dedicated to the memory of H.E. Rauch, ed. by I. Chavel, H.M. Farkas, pp. 65-73. Berlin-Heidelberg-New York: Springer 1985.

[3] Ahlfors, L.V.: Clifford numbers and Möbius transformations in $\mathbb{R}^n$. In: J.S.R. Chisholm, A.K. Common (eds.): Clifford algebras and their applications in mathematical physics, pp. 167-175. Amsterdam: D. Reidel Publ. Comp. 1986.

[4] Borel, A.: Introduction aux groupes arithmétiques. Paris: Hermann 1969.

[5] Bourbaki, N.: Algèbre, chapitre 9: Formes sesquilinéaires et formes quadratiques. Paris: Hermann 1959.

[6] Bump, D., Friedberg, S., Goldfeld, D.: Poincaré series and Kloosterman sums. Contemporary Math. **53**, 39-49. Providence, R.I.: Amer. Math. Soc. 1986.

[7] Bump, D., Friedberg, S., Goldfeld, D.: Poincaré series and Kloosterman sums for $\mathbf{GL}(3,\mathbb{Z})$. Acta Arith.**50**, No. 1, 31-89 (1988).

[8] Cassels, J.S.W.: Rational quadratic forms. London-New York: Academic Press 1978;Russian edition: Moscow: Mir 1982.

[9] Chavel, I.: Eigenvalues in Riemannian Geometry. New York- London: Academic Press 1984.

[10] Chevalley, C.C.: The algebraic theory of spinors. New York, N.Y.: Columbia University Press 1954.

[11] Christian, U.: Über die analytische Fortsetzung gewisser Poincaréscher Reihen zu elliptischen Modulgruppen. Mathematica Gottingensis, Schriftenreihe des Sonderforschungsbereichs Geometrie und Analysis, Heft **13**, 52 pp. Göttingen: Mathematisches Institut der Universität Göttingen 1987.

[12] Deligne, P.: Application de la formule de traces aux sommes trigonométriques. In: Cohomologie étale. Séminaire de géométrie algébrique du Bois-Marie SGA $4\frac{1}{2}$. Lect. Notes Math. **569**, pp. 168-232. Berlin-Heidelberg- New York: Springer 1977.

[13] Deshouillers, J.M., Iwaniec, H.: Kloosterman sums and Fourier coefficients of cusp forms. Invent. Math. **70**, 219-288 (1982).

[14] Dieudonné, J.: La géométrie des groupes classiques. Troisième édition. Berlin-Heidelberg-New York: Springer 1971.

[15] Elstrodt, J., Grunewald, F., Mennicke, J.: Vahlen's group of Clifford matrices and spin-groups. Math. Z. **196**, 369-390 (1987).

[16] Elstrodt, J., Grunewald, F., Mennicke, J.: Séries de Poincaré, sommes de Kloosterman et valeurs propres du laplacien pour les groupes de congruence agissant sur un espace hyperbolique. C.R. Acad. Sci. Paris, Sér. I, **305**, 577-581 (1987).

[17] Elstrodt, J., Grunewald, F., Mennicke, J.: Arithmetic applications of the hyperbolic lattice point theorem. Proc. London Math. Soc. (3) **57**, 239-283 (1988).

[18] Elstrodt, J., Grunewald, F., Mennicke, J.: Some remarks on discrete subgroups of $\mathbf{SL}_2(\mathbb{C})$ (Russian). Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova 162, 77-106 (1987); English transl.: J. Soviet Math: (to appear).

[19] Estermann, T.: Vereinfachter Beweis eines Satzes von Kloosterman. Abh. Math. Sem. Univ. Hamburg **7**, 82-98 (1930).

[20] Estermann, T.: On Kloosterman's sum. Mathematika **8**, 83-86 (1961).

[21] Faddeev, L.D.: Expansion in eigenfunctions of the Laplace operator on the fundamental domain of a discrete group on the Lobachevskii plane. Trudy Moskov. Mat. Obshch. **17**, 323-350 (1967) (Russian); English transl.: Trans. Moscow Math. Soc. **17**, 357-386 (1967).

[22] Fay, J.D.: Fourier coefficients of the resolvent for a Fuchsian group. J. Reine Angew. Math. **293/294**, 143-203 (1977).

[23] Friedberg, S.: Poincaré series for $\mathbf{GL}(n)$: Fourier expansion, Kloosterman sums, and algebro-geometric estimates. Math. Z. **196**, 165-188 (1987).

[24] Gelbart, S.: Jacquet, H.: A relation between automorphic representations of $\mathbf{GL}(2)$ and $\mathbf{GL}(3)$. Ann. Sci. Éc. Norm. Sup. $4^e$ série, **11**, 471-452 (1978).

[25] Goldfeld, D.: Analytic and arithmetic theory of Poincaré series. Proc. Conf. Journées Arithmétiques, Astérisque **61**, 95-107 (1979).

[26] Goldfeld, D.: Kloosterman zeta functions for $\mathbf{GL}(n, \mathbb{Z})$. In: Proc. Internat. Congr. Math., Berkeley 1986, ed. by A.M. Gleason, pp. 417-424. Providence, R.I.: Amer. Math. Soc. 1987.

[27] Goldfeld, D.: Analytic number theory on $\mathbf{GL}(r, \mathbb{R})$, with an appendix by S. Friedberg. In: Analytic number theory and diophantine problems, Proc. Conf. Oklahoma State Univ. 1984, ed. by A.C. Adolphson, J.B. Conrey, A. Ghosh, R.I. Yager, pp. 159-182. Boston-Basel-Stuttgart: Birkhäuser 1987.

[28] Goldfeld, D., Sarnak, P.: Sums of Kloosterman sums. Invent. Math. **71**, 243-250 (1983).

[29] Hardy, G.H., Wright, E.M.: An introduction to the theory of numbers. Fifth edition. Oxford: Oxford University Press 1979.

[30] Hejhal, D.A.: The Selberg trace formula for $\mathbf{PSL}(2, \mathbb{R})$. Lect. Notes Math. **1001.** Berlin-Heidelberg-New York- Tokyo: Springer 1983.

[31] Higher Transcendental Functions, Vol. II. California Institute of Technology, Bateman Manuscript Project, ed. A. Erdélyi, research associates W. Magnus, F. Oberhettinger, F.G. Tricomi. New York-Toronto-London: McGraw-Hill Book Comp. 1953.

[32] Hirschfeld, J.W.P.: Projective geometries over finite fields. Oxford Mathematical Monographs. Oxford: Clarendon Press 1979.

[33] Igusa, J.-I.: Lectures on forms of higher degree. Berlin-Heidelberg-New York: Springer 1978.

[34] Iwaniec, H.: Non-holomorphic modular forms and their applications. In: Modular forms, ed. by R.A. Rankin, pp. 157-196. Chichester: Ellis Horwood 1984.

[35] Iwaniec, H.: Promenade along modular forms and analytic number theory. In: Topics in analytic number theory, ed. by S.W. Graham, J.D. Vaaler, pp. 221-303. Austin, Texas: University of Texas Press 1985.

[36] Iwaniec, H.: Selberg's lower bound of the first eigenvalue for congruence groups. Preprint 1987.

[37] Iwaniec, H.: Spectral theory of automorphic functions and recent developments in analytic number theory. In: Proc. Internat. Congr. Math., Berkeley 1986, ed. by A.M. Gleason, pp. 444-456. Providence, R.I.: Amer. Math. Soc. 1987.

[38] Iwaniec, H.: Small eigenvalues for congruence groups. Preprint 1988.

[39] Iwaniec, H.: Small eigenvalues of Laplacian for $\Gamma_0(N)$. Preprint 1989.

[40] Jones, B.W.: The arithmetic theory of quadratic forms. New York-London: J. Wiley & Sons 1967.

[41] Kato, T.: Perturbation theory for linear operators. Berlin-Heidelberg-New York: Springer 1966.

[42] Kubota, T.: Elementary theory of Eisenstein series. Tokyo: Kodansha Ltd.; New York-London-Sydney-Toronto: J. Wiley & Sons 1973. Russian ed.: Moscow: Nauka 1986.

[43] Kuznetsov, N.V.: Petersson's conjecture for cusp forms of weight zero and Linnik's conjecture. Sums of Kloosterman sums. Mat. Sb., Nov. Ser. **111**, (153), no. 3, 334-383 (1980) (Russian); English transl.: Math. USSR, Sb. **39**, no. 3, 299-342 (1981).

[44] Langlands, R.P.: On the functional equations satisfied by Eisenstein series. Lect. Notes Math. **544**. Berlin-Heidelberg- New York: Springer 1976.

[45] Li, J.-S., Piatetski-Shapiro, I., Sarnak, P.: Poincaré series for $\mathbf{SO}(n,1)$. Proc. Indian Acad. Sci. (Math. Sci.) **97**, 231-237 (1987).

[46] Linnik, Ju. V.: Additive problems and eigenvalues of the modular operators. In: Proc. Int. Congr. Math., Stockholm 1962, pp. 270-284. Djursholm, Sweden: Institut Mittag-Leffler.

[47] Maaß, H.: Automorphe Funktionen in mehreren Veränderlichen und Dirichletsche Reihen. Abh. Math. Sem. Univ. Hamburg **16**, 72-100 (1949).

[48] Magnus, W., Oberhettinger, F., Soni, R.P.: Formulas and theorems for the special functions of mathematical physics. Third ed. Berlin-Heidelberg-New York: Springer 1966.

[49] Malyshev, A.V.: Generalized Kloosterman sums and their estimates (Russian). Vestnik Leningradskogo Univ., Ser. I, **13**, 59-75 (1960).

[50] Malyshev, A.V.: On the representation of integers by definite quadratic forms (Russian). Trudy Mat. Inst. Steklova **65**, 1-212 (1962).

[51] Millson, J., Raghunathan, M.S.: Geometric construction of cohomology for arithmetic groups I. In: Geometry and Analysis, papers dedicated to the memory of V.K. Patodi. Berlin-Heidelberg-New York: Springer 1981.

[52] Piatetski-Shapiro, I.: Invariant theory and Kloosterman sums. In: Algebraic groups, Utrecht 1986, Proc. Symp. in honour of T.A. Springer, ed. by A.M. Cohen, W.H. Hesselink, W.L.J. van der Kallen, J.R. Strooker. Lect. Notes Math. **1271**, pp. 229-236. Berlin-Heidelberg-New York: Springer 1987.

[53] Raghunathan, S.: Isogenies and Congruence subgroups. In: Manifolds and Lie Groups, Papers in Honour of Y.Matsushima, ed. by J. Hano et al. Basel-Boston-Stuttgart: Birkhäuser Verlag 1981.

[54] Roelcke, W.: Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene, I. Math. Ann. **167**, 292-337 (1966). II. Ibid. **168**, 261-324 (1967).

[55] Rohlfs, J., Speh, B.: Representations with cohomology in the discrete spectrum of subgroups of $\mathbf{SO}(n,1)(\mathbb{Z})$ and Lefschetz numbers. Ann. Sci. Éc. Norm. Sup., 4[e] série, **20**, 89-136 (1987).

[56] Salié, H.: Über die Kloostermanschen Summen $S(u,v;q)$. Math. Z. **34**, 91-109 (1932).

[57] Salié, H.: Zur Abschätzung der Fourierkoeffizienten ganzer Modulformen. Math. Z. **36**, 263-278 (1933).

[58] Sarnak, P.: The arithmetic and geometry of some hyperbolic three manifolds. Acta Math. **151**, 253-295 (1983).

[59] Sarnak, P.: Additive number theory and Maaß forms. In: Number theory, New York 1982, ed. by D.V. Chudnovsky, G.V. Chudnovsky, H. Cohn, M.B. Nathanson. Lect. Notes Math. **1052**, pp. 286-309. Berlin-Heidelberg-New York: Springer 1984.

[60] Scharlau, R.: Buildings. To appear.

[61] Selberg, A.: Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. J. Indian Math. Soc. **20**, 47-87 (1956).

[62] Selberg, A.: On the estimation of Fourier coefficients of modular forms. In: Proc. Sympos. Pure Math., Vol. **8**: Theory of numbers, pp. 1-15. Providence, R.I.: Amer. Math. Soc. 1965.

[63] Selberg, A.: The Selberg trace formula. Lectures, Massachusetts Institute of Technology, Cambridge, Mass., 1980. (Unpublished lecture notes.)

[64] Siegel, C.L.: Über die analytische Theorie der quadratischen Formen I. Ann. of Math. 36, 527-606 (1935) (= C.L. Siegel: Gesammelte Abhandlungen, Bd. I, pp. 326-405. Berlin-Heidelberg- New York: Springer 1966).

[65] Stevens, G.: Poincaré series on $\mathbf{GL}(r)$ and Kloosterman sums. Math. Ann. **277**, 25-51 (1987).

[66] Vahlen, R.: Über Bewegungen und complexe Zahlen. Math. Ann. **55**, 585-593 (1902).

[67] Venkov, A.B.: Spectral theory of automorphic functions. Trudy Mat. Inst. Steklova **153** (1981) (Russian); English transl.: Proc. Steklov Inst. Math. **1982**, Issue 4.

[68] Weil, A.: On some exponential sums. Proc. Nat. Acad. Sci. USA **34**, 204-207 (1948) (= A. Weil: Oeuvres scientifiques, Vol. I, pp. 386-389. Berlin-Heidelberg-New York: Springer, Second Printing 1980).