# ON FANO-ENRIQUES THREEFOLDS

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Abstract. Let $U \subset \mathbb{P}^N$ be a projective variety which is not a cone and whose hyperplane sections are smooth Enriques surfaces. We prove that the degree of such $U$ is at most 32 and the bound is sharp.

## 1. Introduction

In this paper we consider three-dimensional varieties whose hyperplane sections are Enriques surfaces. Such varieties where studied in a number of papers started with works of G. Fano [1] (see also [2]).

Let $U$ be a normal projective three-dimensional variety and let $A$ be a prime Cartier divisor on $U$ such that $\mathcal{O}_U(A)$ is ample. We say that $(U, A)$ is a *Fano-Enriques threefold* if $A$ is a smooth Enriques surface and $U$ is not a generalized cone over $A$ (i.e., $U$ is not obtained by contraction of the negative section on a $\mathbb{P}^1$-bundle over $A$). Define the *genus* of a Fano-Enriques threefold by $g := A^3/2 + 1$.

The main result of this paper is the following:

**Theorem 1.1.** *The genus $g$ of a Fano-Enriques therefold $(U, A)$ is at most 17 and the bound is sharp. Moreover, up to isomorphism there exists a unique Fano-Enriques therefold $(U_{32}, A_{32})$ of genus 17.*

Fano asserted that Fano-Enriques threefolds exists only for $g = 4, 6, 7, 9, 13$. However his arguments are unsatisfactory from the modern point of view and, in fact, contain some gaps. In particular, it was found latter that Fano-Enriques threefolds of genus $g = 10$ do exist. Fano's constructions were recovered by Conte and Murre [3], [4] but they still assumed that singularities of $U$ are sufficiently general and, in fact, did not complete the classification. New approach to the classification problem was brought by the minimal model theory. Under additional assumption that the singularities of $U$ are cyclic quotients, Fano-Enriques threefolds $(U, A)$ were classified by Bayle [5] and Sano [6]. According to [7] every Fano-Enriques threefold with only terminal singularities admits a $\mathbb{Q}$-smoothing. This means that such a threefold is a deformation of that contained in a Bayle-Sano's list. In particular, the genus of terminal Fano-Enriques threefolds takes on the following values: $2 \leq g \leq 10$ and $g = 13$. An important result was obtained

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by Cheltsov [8] who proved that any Fano-Enriques threefold $(U, A)$ has only ($\mathbb{Q}$-Gorenstein) canonical singularities. Thus to complete the classification one has to consider the case of non-terminal canonical singularities.

Note that our results are stronger than that stated in the main theorem. Actually we prove many facts about Fano-Enriques threefolds, especially, when the genus is sufficiently large. Note also that the bound $g \leq 17$ follows also from a recent result of I. Karjemanov (in preparation).

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## 2. Preliminaries

Throughout the paper the ground field is supposed to be the complex number field $\mathbb{C}$. This paper is a continuation of our previous paper [9], so we keep almost all the notation. Additionally to the techniques of [9] we use Shokurov's connectedness lemma and inversion of adjunction (see [10, Ch. 17]). These two key results will be used freely without additional reference.

The following facts can be found in [4]. We only note that there the authors assumed that $A$ is very ample and gives an embedding of $U$ as a projectively normal variety. However these hypothesis are not needed to prove (i)-(iii) below.

**Proposition 2.1.** *Let $(U, A)$ be a Fano-Enriques threefold of genus $g$. Then*

(i) *$U$ has only isolated singularities,*
(ii) *the Weil divisor $K_U$ is not Cartier,*
(iii) $\dim |A| = g$ *and* $\dim |-K_U| = g - 1$.

**Theorem 2.2** ([8])**.** *Let $(U, A)$ be a Fano-Enriques threefold. Then $U$ has only ($\mathbb{Q}$-Gorenstein) canonical singularities and $K_X + A$ is a 2-torsion element in the Weil divisor class group.*

2.3. Let $(U, A)$ be a Fano-Enriques threefold of genus $g$. Let $\pi\colon V \to U$ be the global log canonical cover. Then $\pi$ is a finite étale in codimension two morphism such that $\pi^*(K_U + A) \sim 0$. In particular, $\pi^* K_U = K_V$ is Cartier. Hence $V$ is a Fano threefold of degree $4g - 4$ with canonical Gorenstein singularities. Let $\tau$ be the Galous involution on $V$. Then $\tau$ acts freely outside of a finite number of points and $U = V/\tau$.

**Theorem 2.4** ([11], [12])**.** *In the above notation, if $g = g(U) \geq 12$, then the linear system $|-K_V|$ is very ample and defines an embedding $V \subset \mathbb{P}^{2g}$.*

2.5. From now on we assume that $-K_V$ is very ample. (However do not assume that $g \geq 12$).

2.6. If the singularities of $V$ are terminal, they are isolated cDV. Then we can apply [7, Th. 4.2]. By this result there is an one-parameter deformation $\mathfrak{U} \to \mathfrak{T} \ni 0$ with central fibre $\mathfrak{U}_0 = U$ and nearby fibres having only cyclic quotient singularities. Then the total space $\mathfrak{U}$ is $\mathbb{Q}$-Gorenstein and $-K_{\mathfrak{U}_t}^3$, $t \in \mathfrak{T}$ is a constant. By Bayle-Sano's classification [5], [6] we have $-K_{\mathfrak{U}_t}^3 \leq 24$. Hence, $g(U) \leq 13$ (and moreover $g \neq 11, 12$).

2.7. Thus we assume that $V$ has at least one non-terminal point $P$. We distinguish two main cases:

(I) $\tau P \neq P$,

(II) the point $P$ is $\tau$-invariant.

Note that (I) is equivalent to the following:

(I′) $U$ has a non-terminal Gorenstein point.

2.8. Let $\mathfrak{L} := |-K_V|$. This linear system has two $\tau$-invariant subsystems:

$$\begin{array}{lll} \mathfrak{L}^+ := & \pi^*|A|, & \dim \mathfrak{L}^+ = g, \\ \mathfrak{L}^- := & \pi^*|-K_U|, & \dim \mathfrak{L}^- = g-1. \end{array}$$

The following lemma is an immediate consequence of our construction.

**Lemma 2.9.** *A general member $L^+ \in \mathfrak{L}^+$ is a smooth $\tau$-invariant K3 surface. Furthermore, the pair $(V, \mathfrak{L}^+)$ is canonical.*

2.10. We will apply different kinds of the log minimal model program (LMMP) in the category of $G$-varieties (with $G = \boldsymbol{\mu}_2 = \langle \tau \rangle$). For a very brief introduction we refer to [13, §2.2]. In fact, there is no big difference between the $G$-LMMP and standard LMMP. We only emphasize the following:

- $G$-LMMP deals with $G\mathbb{Q}$-factorial varieties. The latter means that every $\tau$-invariant divisor is $\mathbb{Q}$-Cartier.
- If we work with log divisor $K + D$, where $D$ is a boundary (resp. linear system), this $D$ should be $G$-invariant.
- Instead of the Picard group $\operatorname{Pic} X$, Mori cone $\overline{\mathrm{NE}}(X)$, etc., we should consider their $G$-invariant analogs $\operatorname{Pic}^G X$, $\overline{\mathrm{NE}}^G(X)$, etc.
- Every divisorial contraction decreases the invariant Picard number $\rho^G(X)$ by 1 and contracts a $G$-invariant divisor.

In particular, for every threefold $V$ with canonical singularities, one can construct a $G\mathbb{Q}$-factorial terminal modification $\phi\colon W \to V$. This is by definition a birational $G$-equivariant contraction such that $W$ has only terminal $G\mathbb{Q}$-factorial singularities and $K_W = \phi^* K_V$. Such a modification is not unique but every two of them are related by a sequence of $G$-equivariant flops.

2.11. If $\mathfrak{L}$ is a linear system on $V$ without fixed component and $X$ is a birational model of $V$, we denote by $\mathfrak{L}_X$ the birational transform of $\mathfrak{L}$ on $X$. By the abuse of notation we often will write simply $\mathfrak{L}$ instead of $\mathfrak{L}_W$.

## 3. An example

Let $x$ and $y_{i,j}$, $0 \le i, j \le 2$ be homogeneous coordinates in $\mathbb{P}^9$. Consider the anti-canonical embedding of $S = \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^8 = \{x = 0\} \subset \mathbb{P}^9$:

$$\begin{aligned}(u_0 : u_1) \times (v_0 : v_1) \quad &\longmapsto \quad (y_{0,0} : \cdots : y_{2,2}),\\ &\qquad y_{i,j} = u_0^i u_1^{2-i} v_0^j v_1^{2-j}.\end{aligned}$$

Let $V \subset \mathbb{P}^9$ be the projective cone over $S$ and let $P = (0 : \cdots : 0 : 1)$ be its vertex.

**Lemma 3.1.** *The variety $V$ is a Gorenstein Fano threefold with canonical singularities. Moreover, $-K_V = 2M$, where $M$ is the class of hyperplane sections.*

*Proof.* Since $S \subset \mathbb{P}^8$ is projectively normal, $V$ is normal. Let $\sigma\colon W \to V$ be the blowup of $P$. Then $W$ is a $\mathbb{P}^1$-bundle over $S$ and $\sigma$ contracts its negative section $E$ to $P$. More precisely, $W \simeq \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-K_S))$ and the map $\sigma\colon W \to V \subset \mathbb{P}^9$ is given by the tautological linear system $|\mathcal{O}(1)|$. Hence, $\mathcal{O}(1) \sim \sigma^* M$. Since $K_W \sim \mathcal{O}(-2)$ and this divisor is trivial on $E$, $K_V$ is Cartier, $K_W = \sigma^* K_V$ and $K_V = -2M$. $\square$

Define the action of cyclic group $\boldsymbol{\mu}_2 = \{1, \tau\}$ on $\mathbb{P}^8$ and $\mathbb{P}^9$ via

$$\tau\colon x \longmapsto -x, \qquad y_{i,j} \longmapsto (-1)^{i+j} y_{i,j}.$$

Then both $S$ and $V$ are $\tau$-invariant and the induced action on $S$ is

$$(u_0 : u_1) \times (v_0 : v_1) \longmapsto (-u_0 : u_1) \times (-v_0 : v_1).$$

The locus of $\tau$-fixed points in $\mathbb{P}^9$ consists of two projective subspaces

$$\begin{aligned}\mathbb{P}^4_+ \quad &:= \quad \{y_{0,0} = y_{0,2} = y_{2,0} = y_{2,2} = y_{1,1} = 0\},\\ \mathbb{P}^4_- \quad &:= \quad \{y_{0,1} = y_{2,1} = y_{1,0} = y_{1,2} = x = 0\}.\end{aligned}$$

Clearly,

$$\mathbb{P}^4_+ \cap V = \{P\}, \qquad \mathbb{P}^4_- \cap V = \{P_{0,0}, P_{0,2}, P_{2,0}, P_{2,2}\},$$

where $P_{i,j} := \{x = 0, y_{k,l} = 0 \mid (k,l) \neq (i,j)\}$. In particular, the action of $\tau$ on $V$ is free in codimension two.

**Proposition 3.2.** *The quotient of $V$ by $\boldsymbol{\mu}_2 = \{1, \tau\}$ is a Fano-Enriques threefold $U_{32}$ of genus 17.*

*Proof.* Let $\mathfrak{L}^+$ be the linear system that cuts out on $V$ by quadrics of the following form

$$q_1(y_{0,0}, y_{0,2}, y_{2,0}, y_{2,2}, y_{1,1}) + q_2(y_{0,1}, y_{2,1}, y_{1,0}, y_{1,2}, x) = 0,$$

where $q_1$ and $q_2$ are quadratic homogeneous forms. It is easy to see that $\mathfrak{L}^+$ is base point free and each member of $\mathfrak{L}^+$ is $\tau$-invariant. In particular, a general member $L \in \mathfrak{L}^+$ is smooth and does not contain any of $P$, $P_{0,0}$, $P_{0,2}$, $P_{2,0}$, $P_{2,2}$. Therefore the action of $\tau$ on $L$ is free. Since $\mathfrak{L}^+ \subset |-K_V|$, $L$ is a K3 surface. Let $\pi\colon V \to U = V/\tau$ be the quotient morphism and let $A := \pi(L) = L\tau$. Then $A$ is a smooth Enriques surface. Finally, we have $L = \pi^*A$ and $2g - 2 = A^3 = \frac{1}{2}L^3 = \frac{1}{2}(2M)^3 = 32$. Hence, $g = 17$. □

**Remark 3.3.** A similar construction can be applied to anticanonically embedded del Pezzo surface $S = S_6 \subset \mathbb{P}^6$ of degree 6. We get a new Fano-Enriques threefold of genus $g = 13$ with canonical singularities.

## 4. Case (I)

In this section we consider the case when $U$ has a non-terminal Gorenstein point. Recall that $-K_V$ is very ample by our assumption (see 2.5). We prove the following

**Proposition 4.1.** *Under the above assumption $g \le 5$.*

**Lemma 4.2.** *Let $(U, A)$ be a Fano-Enriques threefold. Assume that $U$ has a non-terminal Gorenstein point $O$ (in this lemma we do not assume that $-K_V$ is very ample). Then for a general member $A^0 \in |A|$ passing through $O$, the pair $(U, A^0)$ is lc. Moreover, $O$ is the only log canonical center of codimension $> 1$.*

*Proof.* Let $\mathcal{A}^O \subset |A|$ be the linear subsystem consisting of divisors passing through $O$ and let $\mathfrak{L}^O := \pi^*\mathcal{A}^O$. A general member $C$ of the ample linear system $|\mathcal{O}_A(A)|$ is a smooth irreducible curve (see, e.g., [14, Th. 4.10.2]). Since $H^1(\mathcal{O}_U(A)) = 0$, the restriction map $H^0(\mathcal{O}_U(A)) \to H^0(\mathcal{O}_A(A))$ is surjective. Therefore, there is an element $A^0 \in \mathcal{A}^O$ such that $A^0|_A = C$. The surface $A^0$ is reduced and has only isolated singularities. Since $A^0$ is an ample divisor, it is connected.

Let $\pi^{-1}(O) = \{P, P'\}$. Then points $P, P' \in V$ are non-terminal. Let $L^0 := \pi^*A^0 \in \mathfrak{L}^O$. This surface is also reduced, irreducible and has only isolated singularities. Since $L^0$ is a Cartier divisor, it is normal. Clearly, $L^0$ is Gorenstein and $K_{L^0} \sim 0$. Further, points $P, P' \in L^0$ are not Du Val. Indeed, otherwise points $P, P' \in V$ are isolated cDV, hence terminal. By Shokurov's connectedness result (see [10, 12.3.1]) the singularities of $L^0$ are log canonical and $L^0$ has no other non-Du Val singularities. By the inversion of adjunction and connectedness lemma the pair $(V, L^0)$ is lc. Moreover, the only log canonical centers of codimension $> 1$ are points $P$ and $P'$. Now the assertion follows by the ramification formula (see, e.g., [10, proof of 20.3] or [13, proof of 5.20]). $\square$

*Proof of Proposition* 4.1. We use notation of the proof of Lemma 4.2. Let $\Delta := \frac{1}{2}L^0 + \frac{1}{2}L^1$, where $L^0$, $L^1 \in \mathcal{L}^O$ are general members. Then the pair $(V, \Delta)$ is lc (see [10, 2.17]) and its log canonical locus coincides with $\{P, P'\}$. Since $L^0$ and $L^1$ are Cartier, each crepant exceptional divisor over $P$ or $P'$ has discrepancy $-1$ with respect to $K_V + \Delta$.

Now we need our assumption that $-K_V$ is very ample. Let $\mathcal{H} \subset |-K_V|$ be the linear system of hyperplane sections passing through $P$. Since $\mathcal{L}^O \subset \mathcal{H}$, the pair $(V, \mathcal{H})$ is lc. ¿From now on we ignore the $\tau$-action, so our constructions will not be $\tau$-equivariant. Let $\phi\colon W \to V$ be a terminal $\mathbb{Q}$-factorial modification such as in [9, Lemma 6.6]. We can write

$$\phi^*(K_V + \Delta) = K_W + \Delta_W + E + E',$$

where $E$ and $E'$ are (reduced) exceptional divisors over $P$ and $P$, respectively. In notation of [9, §6] we also have

$$K_W + \mathcal{H}_W + B \sim \phi^*(K_V + \mathcal{H}),$$

where $\mathrm{Supp}(B) = \mathrm{Supp}(E)$. Since $(V, \mathcal{H})$ is lc, $B$ is reduced. Moreover, since $L^0$, $L^1 \in \mathcal{H}$, $B = E$. Now as in [9, §6] we run $(K_W + \mathcal{H}_W)$-MMP. On each step, any fibre meets $E$. So by the connectedness lemma it does not meet $E'$. At the end we get an extremal contraction $f\colon X \to Z$ to a lower-dimensional variety. The log divisor $K_X + \Delta_X + E_X + E'_X$ is lc and numerically trivial. Since $E_X$ and $E'_X$ do not meet each other and $\rho(X/Z) = 1$, the only possibility is when $f$ has one-dimensional fibres. Then according to [9, Lemma 10.1] $Z$ is smooth rational surface and $f$ is a $\mathbb{P}^1$-bundle. In this case, divisors $E_X$ and $E'_X$ must be disjointed sections of $f$. Then the components of $\Delta_X$ are $f$-vertical. Let $\mathcal{L}^{P,P'}$ be the linear system of hyperplane sections of $V$ passing through $P$ and $P'$. Let $\mathcal{L}_W^{P,P'}$ and $\mathcal{L}_X^{P,P'}$ be its birational transforms on $W$ and $X$, respectively. Then $\dim \mathcal{L}_X^{P,P'} = \dim \mathcal{L} - 2 = 2g - 2$. Since $\mathcal{L}^O \subset \mathcal{L}_X^{P,P'}$, we can write $\phi^*(K_V + \mathcal{L}_W^{P,P'}) = K_W + \mathcal{L}_W^{P,P'} + E + E'$. By the above all the members of $\mathcal{L}_X^{P,P'}$ are $f$-vertical. Thus $\mathcal{L}_X^{P,P'} = f^*\mathcal{F}$, where $\mathcal{F}$ is a linear system on $Z$ whose general member is reduced and irreducible. By the adjunction formula we have $K_{E_X} = (K_X + E_X)|_{E_X} \sim -L_X^0|_{E_X}$. Hence $\mathcal{L}_X^{P,P'}|_{E_X} \subset |-K_{E_X}|$. Since $f|_{E_X}\colon E_X \to Z$ is an isomorphism, $\mathcal{F} \subset |-K_Z|$. In particular, $-K_Z$ is nef.

If $-K_Z$ is big, then by Riemann-Roch and Kawamata-Viehweg vanishing

$$2g - 2 = \dim \mathcal{L}^{P,P'} = \dim \mathcal{F} \le \dim |-K_Z| = K_Z^2 \le 9.$$

Hence $g \le 5$. Otherwise $\dim|-K_Z| \le 1$ and as above $2g-2 = \dim \mathcal{F} \le 1$, a contradiction. $\square$

## 5. Case (II)

5.1. **Construction.** Throughout this section we assume that there is a $\tau$-invariant non-terminal point $P \in V$. Recall that $-K_V$ is very ample by our assumption 2.5. Let $\phi\colon W \to V$ be a terminal $\boldsymbol{\mu}_2\mathbb{Q}$-factorial modification. By definition, $\phi$ is $\tau$-equivariant, $K_W = \phi^*K_V$, and $W$ has only terminal (Gorenstein) $\boldsymbol{\mu}_2\mathbb{Q}$-factorial singularities. Let $\mathcal{H}$ be the linear system of hyperplane sections of $V \subset \mathbb{P}^{2g}$ passing through $P$. Clearly, $\dim \mathcal{H} = 2g-1$. Following 2.11 we denote $\mathcal{H}_W$, $\mathcal{L}_W$, and $\mathcal{L}_W^+$ birational transforms on $W$ of $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{L}^+$, respectively.

**Lemma 5.2** (cf. [9, Lemma 6.6])**.** *The linear system $\mathcal{H}_W$ consists of Cartier divisors and the pair $(W, \mathcal{H}_W)$ is canonical. Moreover, one can choose $\phi$ so that $\mathcal{H}_W$ is nef.*

*Proof.* Since the class $\mathcal{H}_W$ is $\tau$-invariant, $\mathcal{H}_W$ is $\mathbb{Q}$-Cartier. Hence $\mathcal{H}_W$ consists of Cartier divisors (recall that $W$ has only terminal Gorenstein singularities). According to [9, Lemma 6.6] there is (a non-$\tau$-equivariant) terminal modification $\phi'\colon W' \to V$ such that $(W', \mathcal{H}_{W'})$ is canonical. Varieties $W$ and $W'$ are isomorphic in codimension one over $V$. Hence the corresponding map $(W, \mathcal{H}_W) \dashrightarrow (W', \mathcal{H}_{W'})$ is crepant and $(W, \mathcal{H}_W)$ is also canonical. The last statement is proved similar to the corresponding statement in [9, Lemma 6.6]. $\square$

Now starting with $(W, \mathcal{H}_W)$ we run $\boldsymbol{\mu}_2$-LMMP with respect to $K+\mathcal{H}$. Similar to [9, Lemma 3.4] we see that the properties in Lemma 5.2 are preserved. At the end we get a $\boldsymbol{\mu}_2$-extremal $K+\mathcal{H}$-negative contraction $f\colon X \to Z$ to a lower-dimensional variety. By the above and inversion of adjunction we have

**Corollary 5.3.** *The pair $(X, \mathcal{H})$ is canonical. Hence a general member $H \in \mathcal{H}$ is a normal surface with at worst Du Val singularities. Moreover, $H$ does not pass through non-Gorenstein points of $X$.*

**Lemma 5.4.** *For a general member $L \in \mathcal{L}_X^+$, the quotient $L/\tau$ is an Enriques surface with at worst Du Val singularities. In particular, the action of $\tau$ on $L$ is free.*

*Proof.* Since all birational transformations $V \dashrightarrow X$ are $K + \mathcal{L}^+$-crepant, the pair $(X, L)$ is canonical and $K_X + L \sim 0$. Hence $L$ is a K3 surface with at worst Du Val singularities. The surface $L/\tau$ has only quotient singularities and is birationally equivalent to an Enriques surface. The action of $\tau$ extends to the minimal resolution $\tilde{L}$ of $L$. We have the following diagram

$$
\begin{array}{ccc}
\tilde{L} & \xrightarrow{\tilde{\varsigma}} & \tilde{L}/\tau \\
\downarrow & & \downarrow{\scriptstyle \upsilon} \\
L & \xrightarrow{\varsigma} & L/\tau
\end{array}
$$

By the ramification formula $K_{\tilde{L}} = \tilde{\varsigma}^* K_{\tilde{L}/\tau} + \tilde{\Xi}$, where $\tilde{\Xi}$ is the branch divisor. Since $K_{\tilde{L}} = 0$ and $\tilde{L}/\tau$ is birationally equivalent to an Enriques surface, we have $\tilde{\Xi} = 0$ and $K_{\tilde{L}/\tau} \equiv 0$. It follows that the action of $\tau$ on $\tilde{L}$ and $L$ is free in codimension one and $\upsilon$ is crepant: $K_{\tilde{L}/\tau} = \upsilon^* K_{L/\tau}$. Hence the singularities of $L/\tau$ are Du Val and $K_{L/\tau} \equiv 0$. This proves the first statement. Further, for the topological Euler number we have $24 = \chi_{\mathrm{top}}(\tilde{L}) = 2\chi_{\mathrm{top}}(\tilde{L}/\tau) - s$, where $s$ is the number of $\tau$-fixed points. This gives us $\chi_{\mathrm{top}}(\tilde{L}/\tau) = 12 + s/2$. Since $\tilde{L}/\tau$ is an Enriques surface with singularities of type $A_1$, we have $s = 0$. □

Now we consider cases according to the dimension of $Z$. The main theorem is a consequence of Propositions 5.6, 5.8, and 5.15 below.

5.5. **Case:** $\dim Z = 0$**.** Then $\rho^{\langle\tau\rangle}(X) = 1$, so $-K_X \equiv rH$, where $H \in \mathcal{H}_X$ and $r \in \mathbb{Q}$. Since $-(K_X + H)$ is ample, $r > 1$. By [9, §7] we have $\dim |-K_X| \le 33$. Therefore, $g \le 16$ in this case. Analysing the properties of action of $\tau$ we can get more precise result:

**Proposition 5.6.** *If $Z$ is a point, then $g \le 5$.*

*Proof.* By the adjunction formula $-K_H \equiv (r-1)H|_H$ is ample. Hence $H$ is a del Pezzo surface with at worst Du Val singularities. So, $K_H^2 \le 9$. By Kawamata-Viehweg vanishing $h^1(\mathcal{O}_X) = 0$. Therefore, by Riemann-Roch on $H$ we have

$$2g \le h^0(\mathcal{O}_X(H)) = h^0(\mathcal{O}_H(H)) + h^0(\mathcal{O}_X) = \frac{r}{2(r-1)^2} K_H^2 + 2.$$

If $X$ is Gorenstein, then $r \ge 2$ and $g \le 5$.

Assume that $X$ is not Gorenstein. Then according to [15] $X$ is either a weighted projective space $\mathbb{P}(1,1,1,2)$ or isomorphic to one of the following weighted hypersurfaces:

- $X_6 \subset \mathbb{P}(1,1,2,3,I)$, $I = 2,3,4,5,6$;
- $X_4 \subset \mathbb{P}(1,1,1,2,I)$, $I = 2,3$;
- $X_3 \subset \mathbb{P}(1,1,1,1,2)$.

If $X \simeq \mathbb{P}(1,1,1,2)$, then $X$ has a unique point $P := (0:0:0:1)$ of index 2. This point is contained in the base locus of $\mathcal{L}^+ \subset |-K_X|$. On the other hand, $P$ must be $\tau$-invariant. This contradicts Lemma 5.4. Similar arguments show that cases $X_6 \subset \mathbb{P}(1,1,2,3,4)$, $X_6 \subset \mathbb{P}(1,1,2,3,5)$, $X_4 \subset \mathbb{P}(1,1,1,2,3)$, and $X_3 \subset \mathbb{P}(1,1,1,1,2)$ are also impossible. In case $X_6 \subset \mathbb{P}(1,1,2,3,2)$ the curve $\Upsilon$ given by $x_1 = x_2 = x_4 = 0$ is $\tau$-invariant. The intersection $X_6 \cap \Upsilon$ is given on $\Upsilon = \mathbb{P}(2,2) \simeq \mathbb{P}^1$ by a cubic polynomial. Hence there is a $\tau$-invariant point $P \in X_6 \cap \Upsilon$. This point is of index 2, so $P \in \mathrm{Bs}\,|-K_X|$. As above we get a contradiction. Similar arguments work in the case $X_6 \subset \mathbb{P}(1,1,2,3,6)$. It remains to consider two cases:

- $X \simeq X_6 \subset \mathbb{P}(1,1,2,3,3)$, and
- $X \simeq X_4 \subset \mathbb{P}(1,1,1,2,2)$.

In the first case, we have $\mathcal{O}_X(H) \simeq \mathcal{O}_X(3)$. Then $2g \le h^0(\mathcal{O}_X(H)) = 4+4+2 = 10$. The second case is treated similarly. $\square$

5.7. **Case:** $\dim Z = 1$**.** Then $Z$ is a smooth rational curve.

**Proposition 5.8.** *If $Z$ is a curve, then $g \le 16$.*

Since $\rho^{\langle\tau\rangle}(X/Z) = 1$, we have $-K_X \equiv rH + f^*\Xi$, where $r \in \mathbb{Q}$, $r > 1$, and $\Xi$ is a $\mathbb{Q}$-divisor on $Z$. By adjunction formula the generic fibre $X_\eta$ is isomorphic either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. We need some information about the structure of singular fibres.

**Lemma 5.9.** *Let $f\colon X \to Z$ be a contraction from a terminal threefold to a curve. Assume there exists an $f$-ample Cartier divisor $H$ such that $-(K_X + H)$ is $f$-ample and $(X, H)$ is canonical. Then every fibre $f^*o$, $o \in Z$ is reduced. Moreover, $f^*o$ has at most two components and the pair $(X, f^*o)$ is dlt. If $f^*o$ is irreducible, then it is isomorphic to one of the following surfaces:*

$$\mathbb{P}^2, \quad \mathbb{P}(1,1,4), \quad \mathbb{P}^1 \times \mathbb{P}^1, \quad \mathbb{P}(1,1,2). \tag{5.9.1}$$

Note that proofs of corresponding statements in [9] (Corollary 8.6 and Lemma 9.2) have some small gaps. The above lemma is a generalization and correction of these statements.

*Proof.* As above the generic fibre $X_\eta$ is isomorphic either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Fix a point $o \in Z$ and regard $Z$ and $X$ as small neighborhoods of $o$ and the fibre $f^{-1}(o)$, respectively. By the inversion of adjunction the surface $H$ has only Du Val singularities and does not contain any non-Gorenstein points of $X$. The restriction $\varphi\colon H \to Z$ is a rational curve fibration such that $-K_H$ is $\varphi$-ample. Since $K_H$ is Cartier, every fibre of $\varphi$ is isomorphic to a plane conic. In particular, so is $\varphi^*o = f^*o \cap H$. This immediately implies that every ample divisor on $H$ is very ample (over $Z$). Further, by Kawamata-Viehweg vanishing $H^1(\mathcal{O}_X) = 0$. Hence the restriction map $H^0(\mathcal{O}_X(H)) \to H^0(\mathcal{O}_H(H))$ is surjective and $|H|$ is $f$-base point free.

If the fibre $f^*o$ is not reduced, $\varphi^*o$ is a double line. So $f^*o = 2S$, where $S$ is a prime Weil $\mathbb{Q}$-Cartier divisor and $S \cap H$ is a reduced irreducible rational curve. Since $X$ is Gorenstein (and terminal) near $H$, $S \cap H$ is Cartier on $H$. Thus $H$ is smooth near $S \cap H \simeq \mathbb{P}^1$. But then the rational curve fibration $\varphi\colon H \to Z$ cannot have multiple fibres, a contradiction. Therefore, $f^*o$ is reduced.

If the fibre $S := f^*o$ is irreducible, all the arguments [9, Lemmas 8.1, 8.5] work. Indeed, $\chi(\mathcal{O}_S(H)) = \chi(\mathcal{O}_{X_\eta}(H))$ and $h^2(\mathcal{O}_S(H)) = 0$. Hence, $h^0(\mathcal{O}_S(H)) \ge h^0(\mathcal{O}_{X_\eta}(H))$. For the $\Delta$-genus, we have

$$\Delta(S, \mathcal{O}_S(H)) := \dim S + \deg \mathcal{O}_S(H) - h^0(\mathcal{O}_S(H)) \le \Delta(X_\eta, \mathcal{O}_{X_\eta}(H)) = 0.$$

By Fujita's classification of polarized varieties of $\Delta$-genus zero we obtain the cases in (5.9.1). By the inversion of adjunction $(X, S)$ is plt in these cases.

¿From now on we assume that $f^*o$ has at least two components. Since $|H|$ is base point free, replacing $H$ with a general member of $|H|$ we also may assume that $\varphi^*o$ is reduced (and reducible). Thus $\varphi^*o$ is isomorphic to a pair of lines meeting in a point $P$: $\varphi^*o = l + l'$, $l_i \simeq \mathbb{P}^1$, $l \cap l' = \{P\}$. In particular, this implies that $f^*o$ has exactly two components: $f^*o = S + S'$. Note that $S \cap S'$ is of pure dimension one (see for example [9, Lemmas 8.7]). We claim that $(H, \varphi^*o)$ is lc. Indeed, by adjunction

$$(K_H + l + l')|_l = K_l + \mathrm{Diff}_l(l').$$

Hence $\deg \mathrm{Diff}_l(l') = 2 + K_H \cdot l < 2$. Clearly, $\mathrm{Supp}\, \mathrm{Diff}_l(l') = P$. Since $K_H + l + l'$ is Cartier, $\mathrm{Diff}_l(l')$ is an integral divisor. Thus, $\mathrm{Diff}_l(l') = P$ and the pair $(l, \mathrm{Diff}_l(l'))$ is lc. By the inversion of adjunction so is $(H, \varphi^*o)$.

Again by the inversion of adjunction the pair $(X, H + f^*o)$ is lc near $H$. By Shokurov's connectedness lemma $(X, H + f^*o)$ is lc everywhere. Let $\Gamma$ be the locus of log canonical singularities of the pair $(X, f^*o)$. Then $\Gamma \cap H = \{P\}$. Since $|H|$ is base point free, the one-dimensional component $\Gamma_0 \subset \Gamma$ is an irreducible curve. Assume that $(X, f^*o)$ has a zero-dimensional center $P$ of log canonical singularities. By the above $P \notin H$. Take a general hyperplane section $M$ passing through $P$. Then, for any $\epsilon > 0$, the pair $(X, f^*o + H + \epsilon M)$ is not lc at $P$ and lc near $H$. Again applying connectedness lemma to $(X, (1-\delta)f^*o + H + \epsilon M)$ for $0 < \delta \ll \epsilon \ll 1$ we derive a contradiction. Thus $(X, f^*o)$ has no any zero-dimensional centers of log canonical singularities and its log canonical locus is an irreducible curve $\Gamma = S \cap S'$. Hence $(X, f^*o)$ is dlt (see [13, Prop. 2.40]). □

Now by [10, 16.15] we have the following two corollaries.

**Corollary 5.10.** *In notation of Lemma* 5.9 *each component of $f^*o$ is normal.*

**Corollary 5.11.** *Notation as in Lemma* 5.9*. Assume that the fibre $f^*o$ is reducible and let $S$, $S'$ be its irreducible components. Let $P \in S \cap S' \subset X$ be a point of index $m > 1$. Then, near $P$, each component $S$, $S' \subset f^*o$ is analytically isomorphic to $\mathbb{C}^2/\boldsymbol{\mu}_m(1, q_i)$, $\gcd(q_i, m) = 1$ and the intersection curve $S \cap S'$ is smooth.*

**Lemma 5.12.** *Let $S$ be a normal surface. Assume that $-K_S \equiv \alpha h + C$, where $\alpha > 1$, $h$ is an ample Cartier divisor, and $C$ is an effective Weil divisor. Then $S$ has at most one singular point. If $\alpha \geq 2$, then $S \simeq \mathbb{P}^2$.*

*Proof.* Let $\mu\colon \tilde{S} \to S$ be the minimal resolution, let $\tilde{C}$ be the proper transform of $C$, and let $h^* = \mu^* h$. We can write

$$K_{\tilde{S}} \equiv \mu^* K_S + \Delta, \quad \mu^* C = \tilde{C} + \Theta,$$

where $\Delta$ and $\Theta$ are $\mu$-exceptional effective $\mathbb{Q}$-divisors. Therefore,

$$K_{\tilde{S}} + \alpha h^* \equiv -(\tilde{C} + \Delta + \Theta).$$

Let $R$ be a $K_{\tilde{S}} + \alpha h^*$-negative extremal ray on $\tilde{S}$, let $\varphi\colon \tilde{S} \to Q$ be the corresponding contraction, and let $\ell$ be a curve contracted by $\varphi$. Then $h^* \cdot \ell \ge 1$, $\Delta \cdot \ell \ge 0$, and $\Theta \cdot \ell \ge 0$. Hence $K_{\tilde{S}} \cdot \ell \le -\alpha < -1$. There are two possibilities:

(i) $Q$ is a point. Then $S = \tilde{S} \simeq \mathbb{P}^2$.
(ii) $Q$ is a curve and $\varphi$ is a $\mathbb{P}^1$-bundle. Then $\rho(\tilde{S}) = 2$.

In the first case we are done. Consider the second case. Since $\rho(\tilde{S}) = 2$, $\mu$ either is an identity map or contracts an irreducible curve. Hence $S$ has at most one singular point. Assume $\alpha \ge 2$. Since $K_{\tilde{S}} \cdot \ell = -2$, $h^* \cdot \ell = 1$. Then $\alpha = 2$ and $(\tilde{C} + \Delta + \Theta) \cdot \ell = 0$. This is possible only if $\Delta = \Theta = 0$ and $\tilde{C}$ is contained in fibres of $\varphi$. If $\mu$ is not an identity map, then $\tilde{C}$ meets the $\mu$-exceptional curve. Then $\Theta \ne 0$, a contradiction. Hence $S$ is smooth and $-K_S \equiv 2h + C$ is ample. Thus there is a $K$-negative extremal ray $R' \ne R$. For corresponding extremal curve $\ell'$ we have $C \cdot \ell' > 0$, so $-K_S \cdot \ell' \ge 3$, a contradiction. □

**Corollary 5.13.** *Assume that $f^*o$ is reducible. Then*

(i) *If $X_\eta \simeq \mathbb{P}^2$, then $X$ has a unique non-Gorenstein point on $f^*o$.*
(ii) *If $X_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then $X$ is Gorenstein near $f^*o$.*

*Proof.* Let $f^*o = S + S'$. Near $f^*o$ we have $-K_X \equiv \alpha H$, where $\alpha = 3/2$ (resp., $\alpha = 2$) in the case $X_\eta \simeq \mathbb{P}^2$ (resp., in the case $X_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$). By the adjunction formula we have

$$-K_S \equiv -\left(K_X + S\right)|_S \equiv \left(\alpha H + S'\right)|_S \equiv \alpha h + C,$$

where $h := H|_S$. If $X_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1$, then by Lemma 5.12 we have $S \simeq \mathbb{P}^2$ and by symmetry $S' \simeq \mathbb{P}^2$. By Corollary 5.11 $X$ is Gorenstein near $f^*o$ in this case. Consider the case $X_\eta \simeq \mathbb{P}^2$. If $X$ is Gorenstein near $f^*o$, then $K_X^2 \cdot S = 1/2 K_X^2 \cdot f^*o = 9/2$ must be an integer, a contradiction. Then the assertion follows by Lemma 5.12 and Corollary 5.11. □

*Proof of Proposition* 5.8. Assume $g \ge 17$. Consider the case when $X_\eta$ is isomorphic to $\mathbb{P}^2$. Let $o \in Z$ be a $\tau$-fixed point and let $S := f^*o$ be the fibre. If $S \simeq \mathbb{P}^2$, then the locus of $\tau$-fixed points on $S$ consists of a line $\Gamma$ and an isolated point. But then each member of $\mathcal{L}^+$ meets $\Gamma$. This contradicts Lemma 5.4. If $S \not\simeq \mathbb{P}^2$, then by Lemma 5.9 and Corollary 5.13 $X$ has only one non-Gorenstein point $P \in f^*o$. This point must be $\tau$-invariant. On the other hand, $P \in \operatorname{Bs}|-K_X| \subset \operatorname{Bs}\mathcal{L}^+$. Again we have a contradiction by Lemma 5.4.

Now we consider the case when $X_\eta$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. By Lemma 5.9 and Corollary 5.13 $X$ is Gorenstein and each fibre of $f$ is a reduced quadric in $\mathbb{P}^3$. In this case the linear system $|H|$ determines a $\tau$-equivariant embedding into $\mathbb{P} := \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 4 vector bundle over $Z \simeq \mathbb{P}^1$. We may assume that $\mathcal{E} = \oplus \mathcal{O}(d_i)$, where $d_1 \geq d_2 \geq d_3 \geq d_4 = 0$. As in [9, §9] we introduce the following notation. Put $d = \sum d_i$. Let $M$ and $F$ be classes of the tautological divisor and a fibre of the projection $\mathbb{P} \to Z$. Then $X \sim 2M + rF$ for some $r \in \mathbb{Z}$. Let $G$ and $Q$ be restrictions on $X$ of $M$ and $F$, respectively. Clearly, $\operatorname{Pic}^{\langle\tau\rangle} X = \mathbb{Z} \cdot G \oplus \mathbb{Z} \cdot Q \simeq \operatorname{Pic} \mathbb{P}$. Hence,

$$H \sim G + \alpha Q, \quad B \sim G - (d + r + \alpha - 2)Q, \quad \alpha \in \mathbb{Z}.$$

Since $R^i f_* \mathcal{O}_X(H) = R^i f_* \mathcal{O}_X(B) = 0$ for $i > 0$,

$$H^0(\mathcal{O}_X(H)) \simeq H^0(\mathcal{E}(\alpha)), \quad H^0(\mathcal{O}_X(B)) \simeq H^0(\mathcal{E}(2 - d - r - \alpha)).$$

Since $B$ is effective, $d + r + \alpha - 2 \leq d_1$. Further,

$$L \cdot B \cdot H = (-K_X) \cdot B \cdot H = 2(6 - d - 2r) \geq 0$$

(because $H$ is nef, see [9, Lemma 9.8]). Hence, $d + 2r \leq 6$. Further,

$$\begin{aligned} 2g - 1 \leq h^0(\mathcal{O}_X(H)) &= d + 4 + 4\alpha \leq d + 4 + 4(2 + d_1 - d - r) \\ &= 12 + 4d_1 - 3d - 4r \leq 12 + d_1 - 4r. \end{aligned} \tag{5.13.2}$$

We claim that $r < 0$. Indeed, assume $r \geq 0$. Then $d_1 \leq d \leq 6$, $2g - 1 \leq 18$, and $g \leq 9$, a contradiction.

Let $C \subset \mathbb{P}(\mathcal{E})$ be the subscrol $\mathbb{P}(\oplus_{d_i=0} \mathcal{O}(d_i))$. The linear system $|M|$ is base point free and contracts $C$. Since $X \sim 2M + rF$, where $r < 0$, $C$ is contained in $X$. Clearly, $\dim C = 1$. In particular, $d_2 \geq 1$ and $d_1 \leq d - 2$. Thus (5.13.2) can be rewritten as follows

$$2g - 1 \leq d + 4 + 4\alpha \leq 12 + 4d_1 - 3d - 4r \leq 4 + d - 4r. \tag{5.13.3}$$

If $K_X \cdot C > 0$, then $C$ is contained in the base locus of $\mathcal{L}^+$. On the other hand, $\tau$ has two fixed points on $C \simeq \mathbb{P}^1$. This contradicts Lemma 5.4. Hence $K_X \cdot C < 0$ and $-K_X$ is nef and big. From the main result of [9] we get

$$\begin{aligned} 72 \geq -K_X^3 = (2G + (2 - d - r)Q)^3 &= (2M + rF) \cdot (2M + (2 - d - r)F)^3 = \\ &= 16d + 24(2 - d - r) + 8r = 8(6 - d - 2r). \end{aligned}$$

Hence, $-3 \leq d + 2r \leq 6$. Since $G \cdot C = 0$, we have $0 \leq -K_X \cdot C = 2 - d - r$. This gives us $d + r \leq 2$, $d \leq 7$, and $r \geq -5$. By (5.13.3) we get $g \leq 16$. $\square$

5.14. Consider the case when $\dim Z = 2$. Then $\mathcal{H}_X$ is a linear system of sections of $f$. Since the invariant Picard number of $X$ over $Z$ is equal to 1, $f$ is equi-dimensional. In this situation, the proof of [9, Lemma 10.1] works and we get that both $X$ and $Z$ are smooth and $f$ is a $\mathbb{P}^1$-bundle. Thus $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 vector bundle over $Z$. By construction the pair $(X, \mathcal{L} \subset |-K_X|)$ is canonical. There is a decomposition $-K_X \sim H + B$, where $H$ and $B$ are effective $f$-ample divisors (sections of $f$), the divisor $H$ is nef, and the image of the map $\Phi_{|H|}$ given by the linear system $|H|$ is three-dimensional. We have

$$2g-1=\dim\mathcal{H}\le\dim|H|.$$

**Proposition 5.15.** *In the above notation we have* $2g-1\le\dim|H|\le 33$. *Moreover, if* $\dim|H|=33$, *then* $g(U)=17$ *and* $U\simeq U_{32}$ *(see §3).*

5.16. The involution $\tau$ acts on $Z$ effectively. Let $L\in\mathcal{L}_X^+$ be a general member and let $f_L\colon L\to\bar{L}\to Z$ be the Stein factorization. Here $\bar{L}\to L$ is a finite of degree two morphism with branch divisor $\Theta\in|-2K_Z|$. The involution $\tau$ naturally acts on $L$, $\bar{L}$ and $\Theta$. Moreover, by Lemma 5.4 $\tau$ has no fixed points on $\Theta$.

5.17. Now we run $\boldsymbol{\mu}_2$-MMP on $Z$: $Z=Z_1\to Z_2\to\cdots\to Z_N=Z'$. Each step is a contraction of an $\tau$-invariant set of disjointed $(-1)$-curves. At the end we get one of the following cases [16]:

(i) $\operatorname{Pic}^{\langle\tau\rangle}Z'\simeq\mathbb{Z}$ and $Z'$ is a del Pezzo surface,

(ii) $\operatorname{Pic}^{\langle\tau\rangle}Z'\simeq\mathbb{Z}\oplus\mathbb{Z}$ and there exists a contraction $Z'\to\mathbb{P}^1$. Each fibre is isomorphic to a reduced plane conic.

According to [9, Lemmas 5.4, 10.4] be can construct a sequence of $\tau$-equivariant birational $K+\mathcal{L}$-crepant transformations

$$\begin{tikzcd}
X=X_1 \arrow[r, dashed] \arrow[d, "f"] & X_2 \arrow[r, dashed] \arrow[d, "f_2"] & \cdots \arrow[r, dashed] & X_N \arrow[d, "f_N"] \\
Z=Z_1 \arrow[r] & Z_2 \arrow[r] & \cdots \arrow[r] & Z_N
\end{tikzcd}$$

where each square is one of transformations (iii)-(v) of [9, Lemma 5.4] over either a $\tau$-invariant $(-1)$-curve or $\tau$-invariant pair of disjointed $(-1)$-curves. These transformations preserve all the properties of $X$, $\mathcal{L}$, $\mathcal{L}^+$, and $\mathcal{H}$ (except for the canonical property of $(X,\mathcal{H})$). Moreover, by construction all the transformations are $\tau$-equivariant and the action of $\tau$ on a general member of $\mathcal{L}^+$ is free (see Lemma 5.4). Replacing $X/Z$ with $X_N/Z_N$ we may assume that $Z$ satisfies one of the conditions (i) or (ii). Let $\Omega\subset Z$ be the locus of $\tau$-fixed points.

Consider case $\operatorname{Pic}^{\langle\tau\rangle}Z\simeq\mathbb{Z}$. Then the curve $\Theta$ is ample. Since $\Theta\cap\Omega=\emptyset$, $\Omega$ is finite. If $Z$ contains no $(-1)$-curves, then $Z\simeq\mathbb{P}^2$ or $\mathbb{P}^1\times\mathbb{P}^1$. It is easy to see that in both cases either the set of $\tau$-fixed points contains a curve or $\operatorname{Pic}^{\langle\tau\rangle}Z\not\simeq\mathbb{Z}$. Therefore, $Z$ contains a $(-1)$-curve $E$ and $K_Z$ is not divisible, i.e., $K_Z$ generates $\operatorname{Pic}^{\langle\tau\rangle}Z$. Since $\operatorname{Pic}^{\langle\tau\rangle}Z\simeq\mathbb{Z}$, $E':=\tau(E)\ne E$ and $E'\cap E\ne\emptyset$. Moreover, $E+E'\sim -aK_Z$ for some positive integer $a$. Then

$$aK_Z^2=-K_Z\cdot(E+E')=2.$$

The quotient $Y:=Z/\tau$ is a del Pezzo surface with Du Val singularities of type $A_1$. In particular, $K_Y$ is Cartier and $K_Y^2$ is an integer. Hence $K_Z^2=2K_Y^2$ is even. Thus we have the only possibility $K_Z^2=2$, $a=1$,

$E + E' \sim -K_Z$, and $E \cdot E' = 2$. Thus for every $(-1)$-curve $E$ its image $\tau(E)$ is uniquely defined by $\tau(E) \sim -K_Z - E$. Recall that the Geiser involution $\tau_g$ on the del Pezzo surface $Z$ degree 2 is the Galois involution of the anticanonical double cover $Z \to \mathbb{P}^2$. For this involution we also have $\tau_g(E) \sim -K_Z - E$. Hence the actions of $\tau$ and $\tau_g$ on $\operatorname{Pic} Z$ coincide. This obviously implies $\tau = \tau_g$. On the other hand, the Geiser involution has a curve of fixed point, a contradiction.

Now we consider the case when $\operatorname{Pic}^{\langle\tau\rangle} Z \simeq \mathbb{Z} \oplus \mathbb{Z}$. Let $\psi \colon Z \to \mathbb{P}^1$ be the $\tau$-equivariant conic bundle. Assume that the morphism $\psi$ is not smooth. Let $r$ be the number of degenerate fibres. By [16] we have $K_Z^2 \leq 5$. Therefore, $r = 8 - K_Z^2 \geq 3$. The involution $\tau$ interchange components of singular fibres. Hence every singular fibre is $\tau$-invariant and there are at least three $\tau$-fixed points on $\mathbb{P}^1$. This is possible only if the action of $\tau$ on $\mathbb{P}^1$ is trivial. In this case, the set $\Omega$ contains no components of fibres and meets each smooth fibre at two points and each singular fibre at one point. Hence $\Omega$ is a connected curve such that the restriction $\psi|_\Omega$ is of degree 2 ramified exactly over the intersection of $\Omega$ with singular fibres. Note also that $\Omega$ is the branch divisor of the quotient map $Z \to Z/\tau$. So, $\Omega$ is smooth and irreducible. By the Hurwitz formula, $2p_a(\Omega) - 2 = r - 4$. By [16] the group $\operatorname{Pic}^{\langle\tau\rangle} Z/\mathbb{P}^1 = 1$ is generated by $K_Z$ and the class of the fibres. Hence we can write $\Omega \sim -K_Z + m\psi^*P$, where $P \in \mathbb{P}^1$ is a point and $m \in \mathbb{Z}$. Since $\Omega \cap \Theta = \emptyset$, we have $K_Z \cdot \Omega = 0$. Then

$$0 = -K_Z \cdot \Omega = -K_Z \cdot (-K_Z + m\psi^*P) = K_Z^2 + 2m,$$

$$K_Z^2 + 4m = (-K_Z + m\psi^*P)^2 = \Omega^2 = (K_Z + \Omega) \cdot \Omega = 2p_a(\Omega) - 2.$$

Using this by Noether formula we obtain

$$8 - r = K_Z^2 = 2 - 2p_a(\Omega) = 4 - r,$$

a contradiction.

Finally, we assume that $\psi \colon Z \to \mathbb{P}^1$ is a smooth morphism. Then $Z \simeq \mathbb{F}_e$, $e \geq 0$, $e \neq 1$. If the action of $\tau$ on $\mathbb{P}^1$ is trivial, then as above $\Omega$ is a curve meeting each fibre at two points. Therefore, $\Omega$ is a disjointed union of two sections. But in this case at least one of these sections meets $\Theta \sim -2K_Z$. Thus the action of $\tau$ on $\mathbb{P}^1$ is nontrivial. Since $\Omega \cap \Theta = \emptyset$, $\Omega$ has no $\psi$-vertical components. Therfore, $\Omega$ is a finite set. We claim that $e = 0$. Indeed, assume that $e \geq 2$. Let $\Sigma$ be the negative section of the ruling. Then $\Theta \cdot \Sigma = -2K_Z \cdot \Sigma = 2(2 - e)$. If $e > 2$, then $\Sigma$ is a component of $\Theta$. On the other hand, $\Sigma$ is $\tau$-invariant. But then $\tau$ must have two fixed points on $\Sigma \subset \Theta$. This contradicts 5.16. Therefore $e = 2$. As above, $\Sigma$ is not a component of $\Theta$. Hence, $\Theta \cap \Sigma = \emptyset$. Let $F := f^{-1}(\Sigma)$. Clearly, $F \simeq \mathbb{F}_m$ for some $m \geq 0$. Let $\Sigma' \subset F$ be a minimal section of the ruling. Then

$$m - 2 = K_F \cdot \Sigma' = K_X \cdot \Sigma' + F \cdot \Sigma' = K_X \cdot \Sigma' - 2.$$

Thus $K_X \cdot \Sigma' = m$. If $m > 0$, then $K_X \cdot \Sigma' < 0$ and $\Sigma'$ is contained in the base locus of $\mathcal{L}_X \subset |-K_X|$. On the other hand, $\Sigma'$ is $\tau$-invariant and $\tau$ has two fixed points on $\Sigma'$. This contradicts Lemma 5.4. Therefore, $m = 0$, $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and $\Sigma' \cdot K_X = 0$. This implies that $F$ does not intersect a general member of $\mathcal{L}_X$. In particular, $F \cap \operatorname{Bs} \mathcal{L}_X = \emptyset$. This means that the map $\Psi\colon X \dashrightarrow V$ given by $\mathcal{L}_X$ is a morphism near $F$. By the above $\Psi$ contracts $F$ to a curve. Since $F \cdot \Sigma' = \Sigma^2 = -2$, $V$ is singular along $\Psi(F)$. This contradicts Proposition 2.1, (i). Thus $e = 0$ and $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

For simplicity we put $c_i := c_i(\mathcal{E})$, $i = 1, 2$. Then

$$-K_X = 2H + f^*(-K_Z - c_1),$$

$$H^2 = H \cdot f^*c_1 - f^*c_2, \qquad H^3 = c_1^2 - c_2.$$

Let $\Sigma$ and $l$ be generators of $\mathbb{P}^1 \times \mathbb{P}^1$. We may assume that $c_1 = a\Sigma + bl$ and $c_2 = c$, where $a, b, c$ are integers. By Riemann-Roch we have:

$$\chi(\mathcal{E}) = \frac{1}{2}(c_1^2 - 2c_2 - K_Z \cdot c_1) + 2 = (a+1)(b+1) - c + 1. \qquad (*)$$

Since $\mathcal{E}$ is nef, $c_1 = a\Sigma + bl$ is a nef class, so $a$, $b \geq 0$. Up to permutation we may assume that $a \leq b$.

**Lemma 5.18** ([9, Lemma 10.6])**.** *Let $\Gamma \subset Z$ be a smooth rational curve such that $\dim |\Gamma| > 0$ and let $\mathcal{E}|_\Gamma \simeq \mathcal{O}_{\mathbb{P}^1}(d_1) + \mathcal{O}_{\mathbb{P}^1}(d_2)$. Then $|d_1 - d_2| \leq 2 + \Gamma^2$.*

**Lemma 5.19.** *If $\mathcal{E}$ is decomposable, then $g \leq 17$. Moreover, equality holds only if $\mathcal{E} \simeq \mathcal{O}(4,2) \oplus \mathcal{O}(2,4)$ and in this case $U$ is isomorphic to the variety $U_{32}$ from §3.*

*Proof.* Let $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, where $\mathcal{E}_0 = \mathcal{O}(\alpha, \beta)$, $\mathcal{E}_1 = \mathcal{O}(\gamma, \delta)$. We have

$$a = \alpha + \gamma \geq 0, \qquad b = \beta + \delta \geq 0,$$

$$c_2 = \alpha\delta + \beta\gamma, \quad |\alpha - \gamma| \leq 2, \quad |\beta - \delta| \leq 2,$$

(by Lemma 5.18). Put $r := \alpha - \gamma$ and $s := \beta - \delta$. Then $c_2 = \frac{1}{2}(ab - rs)$ and

$$2g \leq \chi(\mathcal{E}) = \frac{1}{2}ab + a + b + \frac{1}{2}rs + 2 = \frac{1}{2}(a+2)(b+2) + \frac{1}{2}rs.$$

Since $-2 \leq r, s \leq 2$, we have $g \leq 17$. The equality holds only if $a = b = 6$ and $r = s = \pm 2$. Thus we may assume that $\mathcal{E} \simeq \mathcal{O}(4,2) \oplus \mathcal{O}(2,4)$. Then $-K_X = 2(H - 2f^*(\Sigma + l))$, $\dim \mathcal{H} = 33$, and $\dim \mathcal{L} = 34 = \dim |-K_X|$. In this case, $V$ is the anti-canonical image of $X = \mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(2,2))$. Clearly, this image coincides with the variety $V$ from §3. There is only one choice for the action of $\tau$. Hence we get the Fano-Enriques threefold $U_{32}$ from Proposition 3.2. $\square$

¿From now on we assume that $\mathcal{E}$ is indecomposable. Our arguments below are very similar to that in [9, §10]. Let $B_0$ be the $f$-horizontal component of $B$. Since $L \cap B$ is an effective 1-cycle,

$$\begin{aligned} 0 \le -K_X \cdot B \cdot f^*\Sigma = \\ (2H + f^*(-K_Z - c_1)) \cdot (H + f^*(-K_Z - c_1)) \cdot f^*\Sigma = \\ 2H^2 \cdot f^*\Sigma + 3(-K_Z - c_1) \cdot \Sigma = \\ 2c_1 \cdot \Sigma + 3(-K_Z - c_1) \cdot \Sigma = -3K_Z \cdot \Sigma - c_1 \cdot \Sigma. \end{aligned}$$

This gives us $0 \le a \le b \le 6$.

Moreover, if $a = b = 6$, then

$$\begin{aligned} -K_X \cdot B \cdot f^*(\Sigma + l) = \\ (2H + f^*(-K_Z - c_1)) \cdot (H + f^*(-K_Z - c_1)) \cdot f^*(\Sigma + l) = \\ 2H^2 \cdot f^*\Sigma + 3(-K_Z - c_1) \cdot (\Sigma + l) = \\ = 2c_1 \cdot (\Sigma + l) - 24 = 0. \end{aligned}$$

Hence, $L \cap B_0 = \emptyset$. It follows that $-K_X$ is nef. Indeed, otherwise there is a curve $R$ such that $K_X \cdot R > 0$. Then $L \cdot R < 0$ and $B \cdot R < 0$, so $R \subset L \cap B$ and $R \cdot f^*(\Sigma + l) > 0$, a contradiction. Since $L \cap B_0 = \emptyset$, the map $\Psi \colon X \dashrightarrow V$ given by $\mathcal{L} \subset |-K_X|$ is a morphism near $B_0$ and contracts $B_0$. Moreover, since $V$ has only isolated singularities the image of $B_0$ is a point. In this situation the vector bundle $\mathcal{E}$ must be decomposable. This contradicts our assumption.

Thus we may assume that $0 \le a \le 5$. Put

$$p := \lfloor a/2 \rfloor + 1, \quad q := \lfloor b/2 \rfloor + 1, \quad a' := a - 2p, \quad b' := b - 2q.$$

Then

$$-2 \le a', b' \le -1.$$

Consider the twisted bundle $\mathcal{E}' := \mathcal{E} \otimes \mathcal{O}(-p\Sigma - ql)$. We have

$$c_1(\mathcal{E}') = a'\Sigma + b'l, \quad c_2(\mathcal{E}') = c - aq - bp + 2pq. \tag{5.19.4}$$

**Claim 5.20.** *We have* $c_2(\mathcal{E}') < -3$.

*Proof.* Assume that $c_2(\mathcal{E}') \ge -3$. Then

$$c = c_2(\mathcal{E}') + aq + bp - 2pq \ge aq + bp - 2pq - 3.$$

By Riemann-Roch (*) we have

$$\begin{aligned} 34 \le 2g \le \chi(\mathcal{E}) \le (a+1)(b+1) - aq - bp + 2pq + 4 \\ = \frac{1}{2}ab + a + b + 5 + \frac{1}{2}a'b' = \frac{1}{2}(a+2)(b+2) + 3 + \frac{1}{2}a'b' \le 33, \end{aligned}$$

a contradiction. □

**Claim 5.21.** $\chi(\mathcal{E}') > 0$.

*Proof.* Put $c' := c(\mathcal{E}')$. By Riemann-Roch we have

$$\chi(\mathcal{E}') = b'(a'+1) + a' - c' + 2.$$

Assume that $\chi(\mathcal{E}') \le 0$. Taking into account Claim 5.20 we obtain

$$b'(a'+1) + a' \le c' - 2 < -5.$$

Hence $a' \ne -1$. Therefore, $a' = -2$ and

$$\chi(\mathcal{E}') = -b' - c' > 3 - b' > 0.$$

□

**Claim 5.22.** $H^0(\mathcal{E}') \ne 0$.

*Proof.* Assume that $H^0(\mathcal{E}') = 0$. By Claim 5.21 we have $H^2(\mathcal{E}') \ne 0$. By Serre Duality

$$H^2(\mathcal{E}')^* \simeq H^0(\mathcal{E}'^* \otimes \omega_Z) \simeq H^0(\mathcal{E}' \otimes \det \mathcal{E}'^* \otimes \omega_Z).$$

On the other hand, $(\det \mathcal{E}'^* \otimes \omega_Z)^* = \mathcal{O}_Z((a'+2)\Sigma + (b' + (e+2))l)$ and $H^0((\det \mathcal{E}'^* \otimes \omega_Z)^*) \ne 0$, a contradiction. □

Now we finish the proof of Proposition 5.15. Consider a nonzero section $s \in H^0(\mathcal{E}')$. If $s$ does not vanish anywhere, then $\mathcal{E}'$ is an extension of some line bundle $\mathcal{E}_1$ by $\mathcal{O}$. But then $c_2(\mathcal{E}') = c_2(\mathcal{E}_1) = 0$. This contradicts Claim 5.20. Therefore, the zero locus of $s$ contains a curve $Y$. Let $Y \sim q_1\Sigma + q_2 l$. Then restrictions $\mathcal{E}'$ to general curves $l \in |l|$ and $l' \in |\Sigma|$ are $\mathcal{E}'|_l = \mathcal{O}(q_1) \oplus \mathcal{O}(a' - q_1)$ and $\mathcal{E}'|_{l'} = \mathcal{O}(q_2) \oplus \mathcal{O}(b' - q_2)$. Since $-a', -b' \ge 1$, at least one of the following holds: $2q_1 - b' \ge 3$, or $2q_2 - a' \ge 3$. This contradicts Lemma 5.18. Proposition 5.15 is proved.

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