A generalization of the topological Brauer group

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Abstract

In the present paper we introduce some generalization of the topological Brauer group $H^3_{tors}(X;\,\mathbb{Z})$. We hope that this construction can be transferred to algebraic geometry and would provide a basis for introducing new birational invariants of varieties.

Contents

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Introduction

The aim of this paper is to introduce a homotopy functor **GBr** which can be viewed as a generalization of the "topological" Brauer group **Br**.

Suppose X is a finite CW-complex. The topological Brauer group $\mathbf{Br}(X)$ can be defined as the group of equivalence classes of locally-trivial bundles A_k over X with fibers $M_k(\mathbb{C})$ (for arbitrary $k \in \mathbb{N}$) with respect to the following equivalence relation:

$$A_k \sim B_l \;\Leftrightarrow\; A_k \otimes \mathrm{End}(\xi_m) \cong B_l \otimes \mathrm{End}(\eta_n)$$

for some complex vector bundles ξ_m, η_n of rank m, n respectively (in particular, $km = ln$). Roughly speaking, we can say that $\mathbf{Br}(X)$ is the group of obstructions to the lifting of locally trivial bundles A_k over X with fiber the matrix algebra $M_k(\mathbb{C})$ to bundles of the form $\mathrm{End}(\xi_k)$ for some locally trivial $\mathbb{C}^k$-bundle ξ_k over X.

In the case of our generalized Brauer group **GBr**, instead of bundles with fibers $M_k(\mathbb{C})$, $k \in \mathbb{N}$, we consider bundles $\mathfrak{A}_{k,\,l}$ whose fibers are "fans" $\mathbb{M}_{k,\,l}$, $k,\, l \in \mathbb{N}$, $(k, l) = 1$. Similar to the usual Brauer group, we define $\mathbf{GBr}(X)$ as the group of obstructions for the lifting of bundles $\mathfrak{A}_{k,\,l}$ to bundles of the form $\mathbf{F}^X_{k,\,l}(\mathrm{End}(\xi_k))$, where ξ_k is some locally trivial $\mathbb{C}^k$-bundle over X. Here $\mathbf{F}^X_{k,\,l}\colon \mathfrak{AB}_k(X) \to \mathfrak{Fan}_{k,\,l}(X)$ is a functor from the category of $M_k(\mathbb{C})$-bundles to the category of $\mathbb{M}_{k,\,l}$-bundles over X; the functor is induced by the map $M_k(\mathbb{C}) \hookrightarrow \mathbb{M}_{k,\,l}$ (see Subsection 2.1). Compared to the classical case, a whole new step is added to the procedure of the lifting. At first, we have to lift locally trivial bundles $\mathfrak{A}_{k,\,l}$ with fibers $\mathbb{M}_{k,\,l}$ to bundles of the form $\mathbf{F}^X_{k,\,l}(A_k)$ for some A_k. The second step essentially coincides with the lifting of bundles A_k to bundles of the form $\mathrm{End}(\xi_k)$ for some locally trivial bundle ξ_k, i.e. with the classical case.

According to a classical theorem of J.-P. Serre, there is an isomorphism $\mathbf{Br}(X) \cong H^3_{tors}(X;\, \mathbb{Z})$ [1]. In other words, all obstructions for the lifting of locally-trivial bundles A_k over X with fiber $M_k(\mathbb{C})$ to bundles of the form $\mathrm{End}(\xi_k)$ are independent of the higher-dimensional integer cohomologies of dimensions greater than 3. In contrast to the classical case, the generalized Brauer group $\mathbf{GBr}(X)$ actually depends on the higher-dimensional cohomologies of X.

On the other hand, just as in the classical group $\mathbf{Br}(X)$, any element of $\mathbf{GBr}(X)$ has finite order (moreover, the order of the equivalence class represented by a fan bundle $\mathfrak{A}_{k,l}$ with fiber $\mathbb{M}_{k,\,l}$ is a power of k).
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1 Fans $\mathbb{M}_{k,\,l}$

1.1 The definition of $\mathbb{M}_{k,\,l}$

Let us fix two coprime integers $k,\,l$. By $M_n(\mathbb{C})$ we denote the algebra of all $n\times n$ matrices over $\mathbb{C}$. Let $\mathbb{A}_{k,\,l}$ be the set of all central simple subalgebras in $M_{kl}(\mathbb{C})$ that are isomorphic to $M_k(\mathbb{C})$, and for any $\alpha\in\mathbb{A}_{k,\,l}$ let $M_{k,\,\alpha}\subset M_{kl}(\mathbb{C})$ be the corresponding subalgebra.

Definition 1. The set of matrices

$$\mathbb{M}_{k,\,l}:=\bigcup_{\alpha\in\mathbb{A}_{k,\,l}}M_{k,\,\alpha}\subset M_{kl}(\mathbb{C})$$

is called a *fan* of k-subalgebras in $M_{kl}(\mathbb{C})$.

A fan $\mathbb{M}_{k,\,l}$ will be equipped with some algebraic and topological structures but first, let us give a few other (equivalent) descriptions of $\mathbb{M}_{k,\,l}$.

Noether-Scolem's theorem claims that all subalgebras in $M_{kl}(\mathbb{C})$ isomorphic to $M_k(\mathbb{C})$ are conjugate to each other. In particular, any one of them is conjugate to the so-called "standard subalgebra" $M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l\subset M_{kl}(\mathbb{C})$, where E_l is the unit $l\times l$ matrix. Hence

$$\mathbb{M}_{k,\,l}=\bigcup_{g\in\mathrm{GL}_{kl}(\mathbb{C})}g(M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l)g^{-1}\subset M_{kl}(\mathbb{C}).$$

In order to obtain yet another description of $\mathbb{M}_{k,\,l}$, we consider the matrix Grassmannian $\mathrm{Gr}_{k,l}:=\mathrm{PGL}_{kl}(\mathbb{C})/\,\mathrm{PGL}_k(\mathbb{C})\otimes\mathrm{PGL}_l(\mathbb{C})$ (it is a homogeneous space and there is a canonical bijection of sets $\mathrm{Gr}_{k,l}\cong\mathbb{A}_{k,\,l}$, see Appendix 1). Let $\mathcal{A}_{k,l}\to\mathrm{Gr}_{k,l}$ be the tautological bundle over $\mathrm{Gr}_{k,l}$ with fiber $M_k(\mathbb{C})$ (see Appendix 1). There is a canonical embedding μ of $\mathcal{A}_{k,l}$ into the trivial bundle $\mathrm{Gr}_{k,l}\times M_{kl}(\mathbb{C})$:

$$\begin{tikzcd}
\mathcal{A}_{k,l} \arrow[rr, "\mu"] \arrow[dr] & & \mathrm{Gr}_{k,l}\times M_{kl}(\mathbb{C}) \arrow[dl, "\pi_1"] \\
& \mathrm{Gr}_{k,l}, &
\end{tikzcd}$$

where for any $x \in \mathrm{Gr}_{k,l}$ the fiber $(\mathcal{A}_{k,l})_x$ is embedded by μ into the fiber $(\mathrm{Gr}_{k,l} \times M_{kl}(\mathbb{C}))_x = M_{kl}(\mathbb{C})$ as the central simple subalgebra corresponding to x. Consider the composite map

$$\mathcal{A}_{k,\,l} \xrightarrow{\mu} \mathrm{Gr}_{k,\,l} \times M_{kl}(\mathbb{C}) \xrightarrow{\pi_2} M_{kl}(\mathbb{C}).$$

Clearly, $\mathbb{M}_{k,\,l} = \mathrm{im}(\pi_2 \circ \mu) \subset M_{kl}(\mathbb{C})$.

Remark 2. Notice the following property of fans. Consider the fans $\mathbb{M}_{k,\,l}$ and $\mathbb{M}_{l,\,k}$ as contained in the same algebra $M_{kl}(\mathbb{C})$. Then their intersection $\mathbb{M}_{k,\,l} \cap \mathbb{M}_{l,\,k}$ is just the center $\mathbb{C}E_{kl}$ of $M_{kl}(\mathbb{C})$ (the reason being that k, l are assumed to be coprime). Indeed, this result can be proved by using Jordan's normal form of matrices and the assumption $(k, l) = 1$.

1.2 The topology on $\mathbb{M}_{k,l}$

Any fan $\mathbb{M}_{k,\,l}$ is a subset in the Euclidean space $M_{kl}(\mathbb{C})$. By $\mathcal{O}$ denote the induced topology on $\mathbb{M}_{k,\,l}$. It follows immediately from the definition that $\mathcal{O}$ is the weakest topology on $\mathbb{M}_{k,\,l}$ such that the embedding $\mathbb{M}_{k,\,l} \hookrightarrow M_{kl}(\mathbb{C})$ is continuous. Furthermore, for any $\alpha \in \mathbb{A}_{k,\,l}$ the embedding $M_{k,\alpha} \hookrightarrow \mathbb{M}_{k,\,l}$ is a continuous map.

1.3 The algebraic structure on $\mathbb{M}_{k,l}$

The set $\mathbb{M}_{k,\,l}$ is not a $\mathbb{C}$-algebra, moreover, not even a vector space, in any natural sense. But for elements in the same k-subalgebra in $\mathbb{M}_{k,\,l}$ we have partial operations "$\cdot$" and "$+$" induced by the usual multiplication and addition.

Since all k-subalgebras in $\mathbb{M}_{k,\,l}$ contain the identity element, their intersection includes the set of scalar matrices $\mathbb{C}E_{kl}$. In particular, the vector space structures over the center $\mathbb{C}E_{kl} \cong \mathbb{C}$ on distinct k-subalgebras are concordant.

Proposition 3. *The intersection of all k-subalgebras in $\mathbb{M}_{k,\,l}$ is the center $\mathbb{C}E_{kl} \subset \mathbb{M}_{k,\,l}$, where E_{kl} is the unit $kl \times kl$-matrix. In the previous notation,*

$$\bigcap_{\alpha \in \mathbb{A}_{k,\,l}} M_{k,\alpha} = \mathbb{C}E_{kl}.$$

Proof. Clearly, $\cap_\alpha M_{k,\alpha}$ is a subalgebra in $M_{kl}(\mathbb{C})$; moreover, it is invariant with respect to all automorphisms of the algebra $M_{kl}(\mathbb{C})$. The required assertion follows from the well-known decomposition of the adjoint representation Ad of $\mathrm{GL}_n(\mathbb{C})$ into irreducible ones. $\square$

Definition 4. An *isomorphism of fans* $\vartheta \colon \mathbb{M}_{k,\,l} \to \mathbb{M}_{k,\,l}$ is a *homeomorphism* of the topological spaces $\{\mathbb{M}_{k,\,l}, \mathcal{O}\}$ such that for any k-subalgebra $M_{k,\alpha} \subset \mathbb{M}_{k,\,l}$ the restriction $\vartheta\mid_{M_{k,\alpha}}$ is a $\mathbb{C}$-algebras isomorphism with some k-subalgebra $M_{k,\beta} \subset \mathbb{M}_{k,\,l}$ (α, $\beta \in \mathbb{A}_{k,\,l}$), i.e. for any $\alpha \in \mathbb{A}_{k,\,l}$ the corresponding diagram

$$\begin{array}{ccc} \mathbb{M}_{k,\,l} & \xrightarrow{\vartheta} & \mathbb{M}_{k,\,l} \\ \cup & & \cup \\ M_{k,\alpha} & \xrightarrow{\cong} & M_{k,\beta} \end{array}$$

commutes (here $\beta = \bar{\vartheta}(\alpha)$, where $\bar{\vartheta}$ is the induced map on $\mathbb{A}_{k,\,l}$).

By $\mathfrak{G}_{k,l}$ denote the full group of automorphisms $\mathrm{Aut}(\mathbb{M}_{k,\,l})$. We shall consider $\mathfrak{G}_{k,l}$ as a topological group with respect to the *compact-open topology* related to the action $\mathfrak{G}_{k,l} \times \mathbb{M}_{k,\,l} \to \mathbb{M}_{k,\,l}$. In other words, a subbase of the open sets of $\mathfrak{G}_{k,l}$ is formed by sets $W(C,\,U) := \{g \in \mathfrak{G}_{k,l} \mid g{\cdot}C \subset U\}$ for all open $U \subset \mathbb{M}_{k,\,l}$ and compact $C \subset \mathbb{M}_{k,\,l}$. Since $\mathbb{M}_{k,\,l}$ is a Hausdorff space, so is $\mathfrak{G}_{k,l}$. Furthermore, since $\mathbb{M}_{k,\,l}$ is regular and locally compact, then the natural maps, the multiplication $\mathfrak{G}_{k,l} \times \mathfrak{G}_{k,l} \to \mathfrak{G}_{k,l}$ and the action $\mathfrak{G}_{k,l} \times \mathbb{M}_{k,\,l} \to \mathbb{M}_{k,\,l}$, $(g,x) \mapsto gx$, are continuous.

Remark 5. Note that the continuity of the reversal map $\mathfrak{G}_{k,l} \to \mathfrak{G}_{k,l}$, $g \mapsto g^{-1}$ is not obvious in our case. One way to eliminate this defect is to use a modification of the compact-open topology given by Arens (see [2], p.20).

Acting on $M_{kl}(\mathbb{C})$, the group $\mathrm{PGL}_{kl}(\mathbb{C})$ preserves the subset $\mathbb{M}_{k,\,l} \subset M_{kl}(\mathbb{C})$. Hence we have a group homomorphism

$$\kappa_{k,l} \colon \mathrm{PGL}_{kl}(\mathbb{C}) \to \mathfrak{G}_{k,l}.$$

Proposition 6. *The homomorphism $\kappa_{k,l}$ is injective. Moreover, it is a continuous map.*

Proof. Obviously, $\mathrm{Z}_{\mathrm{PGL}_{kl}(\mathbb{C})}(M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l) = E_k \otimes \mathrm{PGL}_l(\mathbb{C}) \subset \mathrm{PGL}_{kl}(\mathbb{C})$, where by Z we denote a centralizer and by $E_k \otimes \mathrm{PGL}_l(\mathbb{C})$ we denote the image of the embedding

$$\mathrm{PGL}_l(\mathbb{C}) \ni A \mapsto E_k \otimes A \in \mathrm{PGL}_{kl}(\mathbb{C}),$$

induced by the Kronecker product of matrices. It is sufficient to prove that

$$\bigcap_{g \in \mathrm{GL}_{kl}(\mathbb{C})} g(E_k \otimes \mathrm{PGL}_l(\mathbb{C}))g^{-1} = \{e\}.$$

We have already proven that

$$\bigcap_{g \in \mathrm{GL}_{kl}(\mathbb{C})} g(\mathbb{C}E_k \underset{\mathbb{C}}{\otimes} M_l(\mathbb{C}))g^{-1} = \mathbb{C}E_{kl}.$$

Hence

$$\mathrm{GL}_{kl}(\mathbb{C}) \cap \bigcap_{g \in \mathrm{GL}_{kl}(\mathbb{C})} g(\mathbb{C}E_k \underset{\mathbb{C}}{\otimes} M_l(\mathbb{C}))g^{-1} = \mathrm{GL}_{kl}(\mathbb{C}) \cap \mathbb{C}E_{kl} = \mathbb{C}^* E_{kl}.$$

Let us factorize by $\mathbb{C}^* E_{kl}$. The right-hand side is clearly equal to

$$(\mathrm{GL}_{kl}(\mathbb{C}) \cap \mathbb{C}E_{kl})/\mathbb{C}^* E_{kl} = \mathbb{C}^* E_{kl}/\mathbb{C}^* E_{kl} = \{e\}.$$

The left-hand side is actually equal to

$$\bigcap_{g \in \mathrm{GL}_{kl}(\mathbb{C})} g(E_k \otimes \mathrm{GL}_l(\mathbb{C}))g^{-1},$$

therefore when factorized by $\mathbb{C}^* E_{kl}$, it gives us

$$\{e\} = (\bigcap_{g \in \mathrm{GL}_{kl}(\mathbb{C})} g(E_k \otimes \mathrm{GL}_l(\mathbb{C}))g^{-1})/\mathbb{C}^* E_{kl} = \bigcap_{g \in \mathrm{GL}_{kl}(\mathbb{C})} g(E_k \otimes \mathrm{GL}_l(\mathbb{C}))g^{-1}/\mathbb{C}^* E_{kl}$$

$$= \bigcap_{g\in \mathrm{GL}_{kl}(\mathbb{C})} g(E_k\otimes \mathrm{GL}_l(\mathbb{C})/\mathbb{C}^* E_{kl})g^{-1} = \bigcap_{g\in \mathrm{GL}_{kl}(\mathbb{C})} g(E_k \otimes \mathrm{PGL}_l(\mathbb{C}))g^{-1}.$$

Thus, we get the desired equality.

Recall that the standard topology on $\mathrm{PGL}_{kl}(\mathbb{C})$ coincides with the compact-open topology related to the action $\mathrm{PGL}_{kl}(\mathbb{C}) \times M_{kl}(\mathbb{C}) \to M_{kl}(\mathbb{C})$. Take a compact subset $C \subset \mathbb{M}_{k,\,l}$, and an open one $U \subset \mathbb{M}_{k,\,l}$. Put $W'(C,\,U) := \{g \in \mathrm{PGL}_{kl}(\mathbb{C}) \mid g\cdot C \subset U\} \subset \mathrm{PGL}_{kl}(\mathbb{C})$. Take an open subset $V \in M_{kl}(\mathbb{C})$ such that $U = V \cap \mathbb{M}_{k,\,l}$. Consider C as a subset in $M_{kl}(\mathbb{C})$. Then the subset $W''(C,\,V) \subset \mathrm{PGL}_{kl}(\mathbb{C})$ (related to the action $\mathrm{PGL}_{kl}(\mathbb{C}) \times M_{kl}(\mathbb{C}) \to M_{kl}(\mathbb{C})$) is open with respect to the standard topology on $\mathrm{PGL}_{kl}(\mathbb{C})$. From the other hand, since $\mathbb{M}_{k,\,l}$ is a $\mathrm{PGL}_{kl}(\mathbb{C})$-invariant subset in $M_{kl}(\mathbb{C})$, we have $W'(C,\,U) = W''(C,\,V)$. Clearly, $W'(C,\,U) = W(C,\,U)\cap \mathrm{PGL}_{kl}(\mathbb{C})$, where $\mathfrak{G}_{k,\,l} \supset W(C,\,U) := \{g \in \mathfrak{G}_{k,\,l} \mid g\cdot C \subset U\}$. Finally note that subsets of the form $W(C,\,U)$ form a subbase of the compact-open topology in $\mathfrak{G}_{k,\,l}$. □

Thus, the group $\mathfrak{G}_{k,\,l}$ contains $\mathrm{PGL}_{kl}(\mathbb{C})$ as a subgroup, which consists of all automorphisms of $\mathbb{M}_{k,\,l}$ that can be extended to automorphisms of the matrix algebra $M_{kl}(\mathbb{C})$.

Consider an embedding $M_k(\mathbb{C}) \to \mathbb{M}_{k,\,l}$, which maps $M_k(\mathbb{C})$ isomorphically onto a k-subalgebra in $\mathbb{M}_{k,\,l}$.

Proposition 7. *The embedding $M_k(\mathbb{C}) \to \mathbb{M}_{k,\,l}$ is functorial.*

Proof. Any automorphism θ of a matrix k-subalgebra $A_k \subset M_{kl}(\mathbb{C})$ can be uniquely extended to the automorphism $\widetilde{\theta}$ of the whole algebra $M_{kl}(\mathbb{C})$ such that $\widetilde{\theta}\mid_{\mathrm{Z}(A_k)} = \mathrm{id}_{\mathrm{Z}(A_k)}$, where $\mathrm{Z}(A_k) \cong M_l(\mathbb{C})$ is the centralizer of the subalgebra A_k in $M_{kl}(\mathbb{C})$. Actually, the assignment $\theta \mapsto \widetilde{\theta}$ corresponds to the group monomorphism $\mathrm{PGL}_k(\mathbb{C}) \overset{\ldots\otimes E_l}{\longrightarrow} \mathrm{PGL}_{kl}(\mathbb{C})$. Thus, we obtain the automorphism $\vartheta = \kappa_{k,\,l}(\widetilde{\theta})$ of the fan $\mathbb{M}_{k,\,l}$, where $\kappa_{k,\,l}\colon \mathrm{PGL}_{kl}(\mathbb{C}) \to \mathfrak{G}_{k,\,l}$ is the group monomorphism as above. □

Remark 8. Let us remark that the group monomorphism $\mathrm{PGL}_k(\mathbb{C}) \hookrightarrow \mathfrak{G}_{k,\,l}$ corresponded to the embedding $M_k(\mathbb{C}) \to \mathbb{M}_{k,\,l}$, is continuous. Indeed, it is the composite homomorphism $\mathrm{PGL}_k(\mathbb{C}) \overset{\ldots\otimes E_l}{\hookrightarrow} \mathrm{PGL}_{kl}(\mathbb{C}) \overset{\kappa_{k,\,l}}{\hookrightarrow} \mathfrak{G}_{k,\,l}$ and therefore the required assertion follows from the previous proposition.

1.4 The space of k-frames in $M_{kl}(\mathbb{C})$

Definition 9. A k-frame in $M_{kl}(\mathbb{C})$ is an ordered collection of k^2 linearly independent matrices $\{\alpha_{i,\,j} \mid 1 \leq i,\, j \leq k\}$ such that $\alpha_{i,\,j}\alpha_{m,n} = \delta_{jm}\alpha_{i,n}$ for all $1 \leq i,\, j,\, m,\, n \leq k$, where δ_{jm} is the Kronecker delta-symbol.

Clearly, every such frame is a basis in a certain k-subalgebra in $M_{kl}(\mathbb{C})$. We have the "standard" k-frame $\{e_{i,\,j} \mid 1 \leq i,\, j \leq k\}$, where $e_{i,\,j} := E_{ij} \otimes E_l$ is the Kronecker product of a "matrix unit" E_{ij} of order k with the unit $l \times l$ matrix E_l. It is a frame in the subalgebra $M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l \subset M_{kl}(\mathbb{C})$. Applying Noether-Scolem's theorem, we see that all k-frames in

$M_{kl}(\mathbb{C})$ are conjugate to each other. Thus, the set of k-frames in $M_{kl}(\mathbb{C})$ is the homogeneous space $\mathrm{Fr}_{k,\,l} = \mathrm{PGL}_{kl}(\mathbb{C})/E_k \otimes \mathrm{PGL}_l(\mathbb{C})$. It is the total space of the following principal bundle

$$\begin{array}{ccc} \mathrm{PGL}_k(\mathbb{C}) & \xrightarrow{\ldots\otimes E_l} & \mathrm{Fr}_{k,\,l} \\ & & \downarrow \\ & & \mathrm{Gr}_{k,\,l} \end{array}$$

over the matrix Grassmannian $\mathrm{Gr}_{k,\,l}$. The fiber of the bundle over $x \in \mathrm{Gr}_{k,\,l}$ consists of all k-frames contained in the k-subalgebra $M_{k,\,x} \subset M_{kl}(\mathbb{C})$ parameterized by x. The tautological $M_k(\mathbb{C})$-bundle $\mathcal{A}_{k,\,l}$ over $\mathrm{Gr}_{k,\,l}$ is associated with this principal bundle.

1.5 The action $\nu_{k,\,l}\colon \mathfrak{G}_{k,\,l} \times \mathrm{Fr}_{k,\,l} \to \mathrm{Fr}_{k,\,l}$

The group $\mathfrak{G}_{k,\,l}$ acts upon $\mathbb{M}_{k,\,l}$ and therefore upon the space of k-frames $\mathrm{Fr}_{k,\,l}$. We shall consider $\mathrm{Fr}_{k,\,l}$ as a topological space with respect to the standard topology of the homogeneous space $\mathrm{PGL}_{kl}(\mathbb{C})/E_k \otimes \mathrm{PGL}_l(\mathbb{C})$.

Proposition 10. *The natural action $\nu_{k,\,l}\colon \mathfrak{G}_{k,\,l} \times \mathrm{Fr}_{k,\,l} \to \mathrm{Fr}_{k,\,l}$ is continuous.*

Proof. We have k^2 continuous maps $\widetilde{\alpha}_{m,\,n}\colon \mathrm{Fr}_{k,\,l} \to \mathbb{M}_{k,\,l},\ 1 \leq m,\, n \leq k$ which to any k-frame $\{\alpha_{i,\,j} \mid 1 \leq i,\, j \leq k\}$ assign its $m,\,n$-component $\alpha_{m,\,n} \in \mathbb{M}_{k,\,l}$. Clearly, the product map $\widetilde{\alpha} := \widetilde{\alpha}_{1,\,1} \times \ldots \times \widetilde{\alpha}_{k,\,k}\colon\ \mathrm{Fr}_{k,\,l} \to \mathbb{M}_{k,\,l}^{\times k^2}$ is an embedding. Obviously, the topology on $\mathrm{Fr}_{k,\,l}$ coincides with the induced topology related to the embedding $\mathrm{Fr}_{k,\,l} \overset{\widetilde{\alpha}}{\hookrightarrow} \mathbb{M}_{k,\,l}^{\times k^2} \subset M_{kl}(\mathbb{C})^{\times k^2}$. Now the required assertion follows from the commutative diagram

$$\begin{array}{ccc} \mathfrak{G}_{k,\,l} \times \mathrm{Fr}_{k,\,l} & \overset{\nu_{k,\,l}}{\longrightarrow} & \mathrm{Fr}_{k,\,l} \\ \mathrm{id}_{\mathfrak{G}_{k,\,l}} \times \widetilde{\alpha} \downarrow & & \downarrow \widetilde{\alpha} \\ \mathfrak{G}_{k,\,l} \times \mathbb{M}_{k,\,l}^{\times k^2} & \longrightarrow & \mathbb{M}_{k,\,l}^{\times k^2}, \end{array} \tag{2}$$

where $\mathfrak{G}_{k,\,l} \times \mathbb{M}_{k,\,l}^{\times k^2} \to \mathbb{M}_{k,\,l}^{\times k^2}$ is the diagonal action. $\square$

Remark 11. Note that the action $\nu_{k,\,l}\colon \mathfrak{G}_{k,\,l} \times \mathrm{Fr}_{k,\,l} \to \mathrm{Fr}_{k,\,l}$ preserves the structure of the fiber bundle $\mathrm{Fr}_{k,\,l} \overset{\mathrm{PGL}_k(\mathbb{C})}{\longrightarrow} \mathrm{Gr}_{k,\,l}$. Since the topology on $\mathrm{Gr}_{k,\,l}$ coincides with the quotient topology, the corresponding action $\mathfrak{G}_{k,\,l} \times \mathrm{Gr}_{k,\,l} \to \mathrm{Gr}_{k,\,l}$ is continuous. Furthermore, naturally acting on the direct product $\mathrm{Gr}_{k,\,l} \times \mathbb{M}_{k,\,l}$, the group $\mathfrak{G}_{k,\,l}$ preserves the tautological $M_k(\mathbb{C})$-subbundle $\mathcal{A}_{k,\,l} \subset \mathrm{Gr}_{k,\,l} \times \mathbb{M}_{k,\,l}$. Since the standard topology on $\mathcal{A}_{k,\,l}$ coincides with the topology induced by the natural embedding $\mathcal{A}_{k,\,l} \hookrightarrow \mathrm{Gr}_{k,\,l} \times \mathbb{M}_{k,\,l}$, the induced action $\mathfrak{G}_{k,\,l} \times \mathcal{A}_{k,\,l} \to \mathcal{A}_{k,\,l}$ is also continuous.

Suppose $\mathrm{St}(x) \subset \mathfrak{G}_{k,\,l}$ is the stabilizer of $x \in \mathrm{Fr}_{k,\,l}$, then $\mathrm{St}(y) = g\,\mathrm{St}(x)g^{-1}$ for $y = gx$.

Proposition 12.
$$\bigcap_{y \in \mathrm{Fr}_{k,\,l}} \mathrm{St}(y) = \bigcap_{g \in \mathfrak{G}_{k,\,l}} g\,\mathrm{St}(x)g^{-1} = \{e\} \subset \mathfrak{G}_{k,\,l}.$$
In other words, the action $\nu_{k,\,l}\colon \mathfrak{G}_{k,\,l} \times \mathrm{Fr}_{k,\,l} \to \mathrm{Fr}_{k,\,l}$ is effective.

Proof. Suppose $h \in \cap_{g \in \mathfrak{G}_{k,l}} g \operatorname{St}(x) g^{-1}$, then h keeps all points of $\mathrm{Fr}_{k,l}$ fixed $\Rightarrow h$ keeps all k-frames in $\mathbb{M}_{k,\,l}$ fixed $\Rightarrow$ for any k-subalgebra $A_k \subset \mathbb{M}_{k,\,l}$ the restriction $h\,|_{A_k}$ is the identity homomorphism id_{A_k} $\Rightarrow$ h is the identity map of the fan $\mathbb{M}_{k,\,l}$ $\Rightarrow$ $h = e$. □

Thus, the group $\mathfrak{G}_{k,l}$ is a group of homeomorphisms of $\mathrm{Fr}_{k,l}$.

Proposition 13. *The action map $\nu_{k,l}$ equips $\mathrm{Fr}_{k,l}$ with the structure of homogeneous space over the group $\mathfrak{G}_{k,l}$.*

Proof easily follows from the observation (which is a consequence of Noether-Scolem's theorem) that even the subgroup $\mathrm{PGL}_{kl}(\mathbb{C}) \subset \mathfrak{G}_{k,l}$ acts transitively on $\mathrm{Fr}_{k,l}$. □

Remark 14. Consider the standard Hermitian metric on $M_{kl}(\mathbb{C})$ given by $\langle A,\, B\rangle = \mathrm{tr}(A\overline{B}^t)$. We say that a k-subalgebra $M_{k,\alpha} \subset M_{kl}(\mathbb{C}),\ \alpha \in \mathbb{A}_{k,\,l}$, is *unitary* if there is a unitary transformation in $\mathrm{Aut}(M_{kl}(\mathbb{C}))$ which takes the standard subalgebra $M_k(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathbb{C}E_l$ onto the subalgebra $M_{k,\alpha}$. In other words, $M_{k,\alpha} = g(M_k(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathbb{C}E_l)g^{-1}$ for some $g \in \mathrm{U}(kl) \subset \mathrm{GL}_{kl}(\mathbb{C})$. Let $\mathbb{A}^U_{k,\,l}$ be the set of unitary k-subalgebras in $M_{kl}(\mathbb{C})$. By definition, put

$$\mathbb{M}^U_{k,\,l} := \bigcup_{\alpha \in \mathbb{A}^U_{k,\,l}} M_{k,\alpha} = \bigcup_{g \in \mathrm{U}(kl)} g(M_k(\mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathbb{C}E_l)g^{-1} \subset \mathbb{M}_{k,\,l}.$$

Automorphisms of $\mathbb{M}^U_{k,\,l}$ can be defined in the same way as for $\mathbb{M}_{k,\,l}$. By $\mathfrak{G}^U_{k,\,l}$ denote the group $\mathrm{Aut}(\mathbb{M}^U_{k,\,l})$. In this "unitary" version we also replace the spaces $\mathrm{Gr}_{k,l}$ and $\mathrm{Fr}_{k,l}$ by the homotopy equivalent compact ones $\mathrm{Gr}^U_{k,\,l} := \mathrm{PU}(kl)/\,\mathrm{PU}(k) \otimes \mathrm{PU}(l)$ and $\mathrm{Fr}^U_{k,\,l} := \mathrm{PU}(kl)/E_k \otimes \mathrm{PU}(l)$ respectively. Note that $\mathrm{Gr}^U_{k,\,l}$ and $\mathrm{Fr}^U_{k,\,l}$ are just the Grassmannian of unitary k-subalgebras in $M_{kl}(\mathbb{C})$ and the space of unitary k-frames in $\mathbb{M}^U_{k,\,l}$ respectively. We endow $\mathfrak{G}^U_{k,\,l}$ with the compact-open topology, using the action $\mathfrak{G}^U_{k,\,l} \times \mathbb{M}^U_{k,\,l} \to \mathbb{M}^U_{k,\,l}$. Note that this "unitary" version of the stable theory (see below) coincides with the "noncompact" one. In particular, the topological groups $\mathfrak{G}_{k,\,l^\infty}$ and $\mathfrak{G}^U_{k,\,l^\infty}$ are homotopy equivalent to each other.

2 $\mathbb{M}_{k,\,l}$-bundles

2.1 An exact sequence of functors

Fix a pair of coprime integers k, l. Let X be a finite CW-complex. By $\mathfrak{FAB}_{k,l}(X)$, $\mathfrak{AB}_k(X)$, $\mathfrak{Fan}_{k,l}(X)$ we denote the categories of floating algebra bundles of the form $(A_k,\, \mu,\, \widetilde{M}_{kl})$ (see Appendix 1), locally trivial bundles with fiber $M_k(\mathbb{C})$, and locally trivial bundles with fiber $\mathbb{M}_{k,\,l}$ over X, respectively. We stress that any $\mathbb{M}_{k,\,l}$-bundle is assumed to be a locally trivial bundle with the structural group $\mathfrak{G}_{k,\,l}$.

We have the forgetful functor

$$\mathbf{T}^X_{k,\,l} \colon \mathfrak{FAB}_{k,l}(X) \to \mathfrak{AB}_k(X),\ (A_k,\, \mu,\, \widetilde{M}_{kl}) \mapsto A_k,$$

and the functor

$$\mathbf{F}^X_{k,\,l} \colon \mathfrak{AB}_k(X) \to \mathfrak{Fan}_{k,l}(X)$$

defined by the embedding of $M_k(\mathbb{C})$ as the "standard" subalgebra $M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l \subset \mathbb{M}_{k,\,l}$.

We also have the functor

$$\mathbf{R}^X_{k,\,l}\colon \mathfrak{AB}_{kl}(X) \to \mathfrak{Fan}_{k,\,l}(X)$$

defined by the assignment $M_{kl}(\mathbb{C}) \mapsto \mathbb{M}_{k,\,l}$. In other words, to every bundle A_{kl} over X with fiber $M_{kl}(\mathbb{C})$, the functor $\mathbf{R}^X_{k,l}$ assigns the corresponding $\mathbb{M}_{k,\,l}$-bundle.

Remark 15. Note that functors $\mathbf{F}^X_{k,l}$ and $\mathbf{R}^X_{k,l}$ correspond to the continuous monomorphisms of the structural groups $\mathrm{PGL}_k(\mathbb{C}) \to \mathfrak{G}_{k,\,l}$ (see Proposition 7) and $\mathrm{PGL}_{kl}(\mathbb{C}) \overset{\kappa_{k,\,l}}{\longrightarrow} \mathfrak{G}_{k,\,l}$, respectively.

Given an $M_k(\mathbb{C})$-bundle A_k over X, let $\Gamma(X;\, A_k)$ be the Banach algebra of its global sections. Let $\mathbf{GL}_l(A_k)$ be the group $\mathrm{GL}_l(\Gamma(X;\, A_k))$ of all invertible $l \times l$ matrices with coefficients in $\Gamma(X;\, A_k)$. In other words, $\mathbf{GL}_l(A_k) = \mathrm{GL}_1(M_l(\Gamma(X;\, A_k))) = \mathrm{GL}_1(\Gamma(X;\, A_k \otimes \widetilde{M_l}))$, where $\widetilde{M_l}$ is a trivial $M_l(\mathbb{C})$-bundle over X. We have the group monomorphism

$$\mathrm{GL}_1(\Gamma(X;\, \mathbb{C})) \hookrightarrow \mathbf{GL}_l(A_k),\ \alpha \mapsto \alpha E_{kl},$$

where $\mathrm{GL}_1(\Gamma(X;\, \mathbb{C})) = \Gamma(X;\, \mathbb{C}^*)$ and E_{kl} is the unit matrix. By $\mathbf{PGL}_l(A_k)$ we denote the corresponding quotient group.

Proposition 16. *Let $A_k,\ B_{kl},\ B'_{kl}$ be locally trivial bundles over X with fibers $M_k(\mathbb{C}),\ M_{kl}(\mathbb{C}),\ M_{kl}(\mathbb{C})$, respectively. Put $\mathfrak{A}_{k,l} := \mathbf{R}^X_{k,l}(B_{kl}),\ \mathfrak{A}'_{k,\,l} := \mathbf{R}^X_{k,l}(B'_{kl})$. Suppose we have two embeddings $\mu\colon A_k \hookrightarrow B_{kl},\ \mu'\colon A_k \hookrightarrow B'_{kl}$ such that for any point $x \in X$ their restrictions $\mu_x,\ \mu'_x$ take the fiber $(A_k)_x$ into $(B_{kl})_x,\ (B'_{kl})_x$, as a central simple subalgebra. Then the corresponding fan bundles $\mathfrak{A}_{k,l},\ \mathfrak{A}'_{k,\,l}$ are isomorphic to each other. Furthermore, they both have the form $\bigcup\limits_{g\in \mathbf{GL}_l(A_k)} g(A_k\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l)g^{-1}$ (i.e. the corresponding sections form a "fan of subbundles" $\bigcup\limits_{g\in \mathbf{GL}_l(A_k)} g(\Gamma(X;\, A_k) \underset{\Gamma(X;\mathbb{C})}{\otimes} \Gamma(X;\, \mathbb{C})E_l)g^{-1}$). Moreover, any isomorphism α of the subbundles $\mathfrak{A}_{k,\,l} \supset \mu(A_k) \overset{\alpha}{\to} \mu'(A_k) \subset \mathfrak{A}'_{k,\,l}$ can be extended to a canonical $\mathbf{PGL}_l(A_k)$-equivariant isomorphism $\widetilde{\alpha}\colon \mathfrak{A}_{k,\,l} \to \mathfrak{A}'_{k,\,l}$.*

Proof. First, let us prove that the $\mathrm{Gr}_{k,\,l}$-bundle $\mathrm{Gr}_{k,l}(B_{kl})$ associated with B_{kl} is trivial as a fiber space with structural group $\mathrm{PGL}_{kl}(\mathbb{C})$. Indeed, it follows easily from our assumptions that there is an $M_l(\mathbb{C})$-bundle C_l such that the embedding $A_k \hookrightarrow B_{kl}$ has the form $A_k \hookrightarrow A_k \otimes C_l = B_{kl}$ (here we use the fact that every locally trivial bundle with fiber matrix algebra has a canonical unit section). By $\mathrm{Prin}(B_{kl}),\ \mathrm{Prin}(A_k),\ \mathrm{Prin}(C_l)$ denote the principal bundles corresponding to the bundles $B_{kl},\ A_k,\ C_l$, respectively. We have the embedding $\mathrm{Prin}(A_k) \otimes \mathrm{Prin}(C_l) \hookrightarrow \mathrm{Prin}(B_{kl})$ corresponding to the morphism of the bundles $A_k \times C_l \overset{\otimes}{\to} B_{kl}$. We have free actions of groups $\mathrm{PGL}_{kl}(\mathbb{C}),\ \mathrm{PGL}_k(\mathbb{C}),\ \mathrm{PGL}_l(\mathbb{C})$ upon $\mathrm{Prin}(B_{kl}),\ \mathrm{Prin}(A_k),\ \mathrm{Prin}(C_l)$, respectively. Moreover, the following diagram

$$\begin{array}{ccc} \mathrm{PGL}_{kl}(\mathbb{C}) \times \mathrm{Prin}(B_{kl}) & \longrightarrow & \mathrm{Prin}(B_{kl}) \\ \uparrow & & \uparrow \\ (\mathrm{PGL}_k(\mathbb{C}) \otimes \mathrm{PGL}_l(\mathbb{C})) \times (\mathrm{Prin}(A_k) \otimes \mathrm{Prin}(C_l)) & \longrightarrow & \mathrm{Prin}(A_k) \otimes \mathrm{Prin}(C_l) \end{array}$$

commutes, i.e. the action is compatible with the embedding of the subbundles. Let $\mathrm{Prin}(B_{kl})/(\mathrm{Prin}(A_k)\otimes \mathrm{Prin}(C_l))$ be the homogeneous space of cosets $\{g\cdot(\mathrm{Prin}(A_k)\otimes \mathrm{Prin}(C_l))\mid g\in \mathrm{PGL}_{kl}(\mathbb{C})\}$. Consider the $\mathrm{PGL}_{kl}(\mathbb{C})$-equivariant map

$$\mathrm{Prin}(B_{kl})/(\mathrm{Prin}(A_k)\otimes \mathrm{Prin}(C_l))\to \mathrm{PGL}_{kl}(\mathbb{C})/(\mathrm{PGL}_k(\mathbb{C})\otimes \mathrm{PGL}_l(\mathbb{C})),$$

$$g\cdot(\mathrm{Prin}(A_k)\otimes \mathrm{Prin}(C_l))\mapsto g\cdot(\mathrm{PGL}_k(\mathbb{C})\otimes \mathrm{PGL}_l(\mathbb{C})),\quad g\in \mathrm{PGL}_{kl}(\mathbb{C}).$$

Clearly, it gives us a trivialization of the bundle $\mathrm{Gr}_{k,\,l}(B_{kl})$.

Now consider the fibered product $\mathrm{Gr}_{k,\,l}(B_{kl})\underset{X}{\times}B_{kl}$ and the corresponding tautological $M_k(\mathbb{C})$-subbundle $\widetilde{\mathcal{A}}_{k,\,l}(B_{kl})\subset \mathrm{Gr}_{k,\,l}(B_{kl})\underset{X}{\times}B_{kl}$. Note that for any $x_0\in X$ we have the diagrams

$$\begin{array}{ccc}\mathrm{Gr}_{k,\,l}\times M_{kl}(\mathbb{C}) & \hookrightarrow & \mathrm{Gr}_{k,\,l}(B_{kl})\underset{X}{\times}B_{kl}\\ \downarrow & & \downarrow\\ x_0 & \hookrightarrow & X\end{array}$$

and

$$\begin{array}{ccc}\mathcal{A}_{k,\,l} & \hookrightarrow & \widetilde{\mathcal{A}}_{k,\,l}(B_{kl})\\ \downarrow & & \downarrow\\ \mathrm{Gr}_{k,\,l} & \hookrightarrow & \mathrm{Gr}_{k,\,l}(B_{kl}).\end{array}$$

Moreover, the subbundle $\mu(A_k)\subset B_{kl}$ determines a canonical section $\sigma\colon X\to \mathrm{Gr}_{k,\,l}(B_{kl})$ of the projection $\mathrm{Gr}_{k,\,l}(B_{kl})\overset{\mathrm{Gr}_{k,\,l}}{\longrightarrow}X$ such that $\sigma^*(\widetilde{\mathcal{A}}_{k,\,l}(B_{kl}))=A_k$.

The group $\mathbf{PGL}_l(A_k)$ obviously acts on the canonical $M_k(\mathbb{C})$-bundle $\widetilde{\mathcal{A}}_{k,\,l}(B_{kl})$ over the trivial $\mathrm{Gr}_{k,\,l}$-bundle $\mathrm{Gr}_{k,\,l}(B_{kl})$ (note that this action induces the nontrivial one on $\mathrm{Gr}_{k,\,l}(B_{kl})$). Consider the projection $\pi\colon \mathcal{A}_{k,\,l}\to \mathbb{M}_{k,\,l}$ onto $\mathbb{M}_{k,\,l}$ such that the diagram

$$\begin{array}{ccc}\mathcal{A}_{k,\,l} & \overset{\pi}{\to} & \mathbb{M}_{k,\,l}\\ \downarrow & & \downarrow\\ \mathrm{Gr}_{k,\,l} & \to & \mathrm{pt}\end{array}$$

commutes. Clearly, π is a $\mathrm{PGL}_{kl}(\mathbb{C})$-equivariant map. We have a similar $\mathbf{PGL}_l(A_k)$-equivariant projection $\widetilde{\pi}$ for $\widetilde{\mathcal{A}}_{k,\,l}(B_{kl})$. It can be shown that $\mathfrak{A}_{k,\,l}=\widetilde{\pi}(\widetilde{\mathcal{A}}_{k,\,l}(B_{kl}))$ and the diagram

$$\begin{array}{ccc}\widetilde{\mathcal{A}}_{k,\,l}(B_{kl}) & \overset{\widetilde{\pi}}{\to} & \mathfrak{A}_{k,\,l}\\ \downarrow & & \downarrow\\ \mathrm{Gr}_{k,\,l}(B_{kl}) & \to & X\end{array}$$

commutes. Now it follows easily that

$$\mathfrak{A}_{k,\,l}=\bigcup_{g\in \mathbf{GL}_l(A_k)} g(A_k\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l)g^{-1}. \tag{8}$$

In particular, it follows from (8) that the fan bundle $\mathfrak{A}_{k,\,l}:=\mathbf{R}^X_{k,\,l}(B_{kl})$ does not depend on the choice of C_l. $\square$

Corollary 17. *For any $M_{kl}(\mathbb{C})$-bundle B_{kl} containing A_k as a matrix algebra subbundle (i.e. for any point $x \in X$ the fiber $(A_k)_x$ is a central simple subalgebra in $(B_{kl})_x$) we have* $\mathbf{R}^X_{k,\,l}(B_{kl}) \cong \mathbf{F}^X_{k,\,l}(A_k)$.

Proof. Let B_{kl} be an arbitrary $M_{kl}(\mathbb{C})$-bundle containing A_k as a matrix algebra subbundle. For example, we can take $A_k \otimes \widetilde{M}_l$ for B_{kl}, where $\widetilde{M}_l$ is a trivial $M_l(\mathbb{C})$-bundle over X. Then the $\mathbb{M}_{k,\,l}$-bundle $\mathbf{R}^X_{k,\,l}(B_{kl})$ contains A_k as an $M_k(\mathbb{C})$-subbundle. It follows from Proposition 16 that $\mathbf{R}^X_{k,\,l}(B_{kl}) \cong \mathbf{F}^X_{k,\,l}(A_k)$. $\square$

Proposition 18. *A bundle $\mathbf{F}^X_{k,\,l}(A_k) \in \mathfrak{Fan}_{k,\,l}(X)$ is trivial iff A_k belongs to the image of $\mathbf{T}^X_{k,\,l}$, i.e. A_k is the core (see Appendix 1) of a floating algebra bundle $(A_k,\, \mu,\, \widetilde{M}_{kl})$.*

Remark 19. In other words, the sequence

$$\Psi_{k,\,l}(X) \stackrel{\widetilde{\mathbf{T}}^X_{k,\,l}}{\longrightarrow} \Phi_k(X) \stackrel{\widetilde{\mathbf{F}}^X_{k,\,l}}{\longrightarrow} \Upsilon_{k,\,l}(X)$$

is exact as a sequence of pointed sets. Here $\Psi_{k,\,l}(X)$, $\Phi_k(X)$, and $\Upsilon_{k,\,l}(X)$ are the sets of isomorphism classes of objects of $\mathfrak{FAB}_{k,\,l}(X), \mathfrak{AB}_k(X), \mathfrak{Fan}_{k,\,l}(X)$ respectively, and $\widetilde{\mathbf{T}}^X_{k,\,l}$, $\widetilde{\mathbf{F}}^X_{k,\,l}$ are the maps, induced by the corresponding functors.

Proof. Suppose $\mathbf{F}^X_{k,\,l}(A_k) \cong X \times \mathbb{M}_{k,\,l}$. By $\mu\colon A_k \hookrightarrow X \times M_{kl}(\mathbb{C})$ denote the composite map $A_k \hookrightarrow \mathbf{F}^X_{k,\,l}(A_k) \stackrel{\cong}{\to} X \times \mathbb{M}_{k,\,l} \subset X \times M_{kl}(\mathbb{C})$. Clearly, the triple $(A_k,\, \mu,\, \widetilde{M}_{kl})$ is a floating algebra bundle over X.

Now suppose A_k is the core of a floating algebra bundle $(A_k,\, \mu,\, \widetilde{M}_{kl})$. Using Corollary 17, we see that $\mathbf{F}^X_{k,\,l}(A_k) \cong \mathbf{R}^X_{k,\,l}(\widetilde{M}_{kl})$ is a trivial bundle. $\square$

Corollary 20. *Consider the sequence*

$$\mathrm{Gr}_{k,\,l} \stackrel{\widehat{\mathbf{T}}_{k,\,l}}{\longrightarrow} \mathrm{BPGL}_k(\mathbb{C}) \stackrel{\widehat{\mathbf{F}}_{k,\,l}}{\longrightarrow} \mathrm{B}\,\mathfrak{G}_{k,\,l}, \tag{9}$$

where $\widehat{\mathbf{T}}_{k,\,l}$ is the classifying map for the core $\mathcal{A}_{k,\,l}$ of the tautological floating algebra bundle over $\mathrm{Gr}_{k,\,l}$ as a $\mathrm{PGL}_k(\mathbb{C})$-bundle, and $\widehat{\mathbf{F}}_{k,\,l}$ is the classifying map for the $\mathbb{M}_{k,\,l}$-bundle obtained by application of functor $\mathbf{F}^{\mathrm{BPGL}_k(\mathbb{C})}_{k,\,l}$ to the universal $M_k(\mathbb{C})$-bundle A^{univ}_k over $\mathrm{BPGL}_k(\mathbb{C})$. Suppose X is a finite CW-complex such that $\dim(X) \leq 2\min\{k,\, l\}$, then the sequence of sets

$$[X;\, \mathrm{Gr}_{k,\,l}] \stackrel{\widehat{\mathbf{T}}_{k,\,l*}}{\longrightarrow} [X;\, \mathrm{BPGL}_k(\mathbb{C})] \stackrel{\widehat{\mathbf{F}}_{k,\,l*}}{\longrightarrow} [X;\, \mathrm{B}\,\mathfrak{G}_{k,\,l}]$$

is exact.

Proof. Let us recall that $\mathrm{Gr}_{k,\,l}$ is a classifying space for floating algebra bundles of the form $(A_k,\, \mu,\, \widetilde{M}_{kl})$ over finite CW-complexes of dimension $\leq 2\min\{k,\, l\}$ (see Appendix 1). Similarly, $\mathrm{BPGL}_k(\mathbb{C})$ and $\mathrm{B}\,\mathfrak{G}_{k,\,l}$ are classifying spaces for locally trivial bundles with fibers $M_k(\mathbb{C})$ and $\mathbb{M}_{k,\,l}$, respectively. Now the required assertion follows from the previous proposition. $\square$

To any $\mathbb{M}_{k,\,l}$-bundle $\mathfrak{A}_{k,\,l}$ over X we can associate a fiber bundle $\mathrm{Gr}(\mathfrak{A}_{k,\,l}) \stackrel{p}{\to} X$ over X with fiber $\mathrm{Gr}_{k,\,l}$. Let $\mathcal{A}_{k,\,l}$ be the core of the tautological floating algebra bundle over $\mathrm{Gr}_{k,\,l}$ (see Appendix 1). We have the canonical $M_k(\mathbb{C})$-bundle $\widetilde{\mathcal{A}}_{k,\,l}$ over $\mathrm{Gr}(\mathfrak{A}_{k,\,l})$ which is the tautological subbundle $\widetilde{\mathcal{A}}_{k,l} \subset \mathrm{Gr}(\mathfrak{A}_{k,l})\underset{X}{\times}\mathfrak{A}_{k,l}$ (in particular, for any fiber $\mathrm{Gr}_{k,l}$ of the fibration $\mathrm{Gr}(\mathfrak{A}_{k,l}) \xrightarrow{p} X$ the restriction $\widetilde{\mathcal{A}}_{k,l}\mid_{\mathrm{Gr}_{k,l}}$ coincides with $\mathcal{A}_{k,l} \subset \mathrm{Gr}_{k,l} \times \mathbb{M}_{k,l}$). Moreover, there is the $M_{kl}(\mathbb{C})$-bundle $\mathcal{B}_{kl}$ over $\mathrm{Gr}(\mathfrak{A}_{k,l})$ such that $\mathbf{R}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\mathcal{B}_{kl}) = p^*(\mathfrak{A}_{k,l})$. The bundle $\mathcal{B}_{kl}$ is equipped with the canonical embedding $\widetilde{\mu}\colon \widetilde{\mathcal{A}}_{k,l} \hookrightarrow \mathcal{B}_{kl}$ such that for any fiber $\mathrm{Gr}_{k,l}$ the restriction $(\widetilde{\mathcal{A}}_{k,l}\mid_{\mathrm{Gr}_{k,l}}, \widetilde{\mu}\mid_{\mathrm{Gr}_{k,l}}, \mathcal{B}_{kl}\mid_{\mathrm{Gr}_{k,l}})$ is just the canonical FAB $(\mathcal{A}_{k,l}, \mu, \widetilde{\mathcal{M}}_{kl})$ over $\mathrm{Gr}_{k,l}$. Moreover, $\mathbf{R}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\mathcal{B}_{kl}) \cong \mathbf{F}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\widetilde{\mathcal{A}}_{k,l})$ (compare with Remark 29).

Lemma 21. The pull-back $p^*(\mathfrak{A}_{k,l})$ has the form $\mathbf{F}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\widetilde{\mathcal{A}}_{k,l})$.

Proof. Clearly, $p^*(\mathfrak{A}_{k,l})$ contains $\widetilde{\mathcal{A}}_{k,l}$ as an $M_k(\mathbb{C})$-subbundle. Moreover, $p^*(\mathfrak{A}_{k,l}) = \mathbf{R}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\mathcal{B}_{kl})$ for the $M_{kl}(\mathbb{C})$-bundle $\mathcal{B}_{kl}$ defined above. Now it follows from Corollary 17 that $p^*(\mathfrak{A}_{k,l}) \cong \mathbf{F}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\widetilde{\mathcal{A}}_{k,l})$. □

Remark 22. Recall that we have the map $\mathcal{A}_{k,l} \xrightarrow{\pi} \mathbb{M}_{k,l}$ induced by the map $\mathrm{Gr}_{k,l} \to \mathrm{pt}$ (see Subsection 1.1). There is the map $\widetilde{\pi}\colon \widetilde{\mathcal{A}}_{k,l} \to \mathfrak{A}_{k,l}$ such that for any fiber $\mathrm{Gr}_{k,l}$ the restriction $\widetilde{\pi}\mid_{\mathrm{Gr}_{k,l}}$ coincides with π. Thus, the initial bundle $\mathfrak{A}_{k,l}$ can be reconstructed from $\widetilde{\mathcal{A}}_{k,l}$ by fibered application of π.

Given an $\mathbb{M}_{k,l}$-bundle $\mathfrak{A}_{k,l}$ over X, one can ask the following question: is there an $M_k(\mathbb{C})$-bundle A_k over X such that $\mathfrak{A}_{k,l} = \mathbf{F}_{k,l}^X(A_k)$?

Proposition 23. *An $\mathbb{M}_{k,l}$-bundle $\mathfrak{A}_{k,l}$ over X has the form $\mathfrak{A}_{k,l} = \mathbf{F}_{k,l}^X(A_k)$ for some $M_k(\mathbb{C})$-bundle A_k iff the fiber bundle $\mathrm{Gr}(\mathfrak{A}_{k,l}) \xrightarrow{p} X$ has a section σ.*

Proof. According to the above lemma, $p^*(\mathfrak{A}_{k,l}) = \mathbf{F}_{k,l}^{\mathrm{Gr}(\mathfrak{A}_{k,l})}(\widetilde{\mathcal{A}}_{k,l})$. Hence, given a section σ, we can take $A_k = \sigma^*(\widetilde{\mathcal{A}}_{k,l})$.

Conversely, if $\mathfrak{A}_{k,l} = \mathbf{F}_{k,l}^X(A_k)$, then A_k is an $M_k(\mathbb{C})$-subbundle in $\mathfrak{A}_{k,l}$. Clearly, any such subbundle determines a section of p. □

Let us remark that if $\mathfrak{A}_{k,l} = \mathbf{F}_{k,l}^X(A_k)$ for $A_k = \sigma^*(\widetilde{\mathcal{A}}_{k,l})$ then $\mathfrak{A}_{k,l} = \mathbf{R}_{k,l}^X(B_{kl})$, where $B_{kl} := \sigma^*(\mathcal{B}_{kl})$.

Given an $\mathbb{M}_{k,l}$-bundle $\mathfrak{A}_{k,l}$ over X, let $\mathrm{Fr}(\mathfrak{A}_{k,l}) \xrightarrow{\widehat{p}} X$ be the corresponding fiber bundle of k-frames over X.

Proposition 24. *An $\mathbb{M}_{k,l}$-bundle $\mathfrak{A}_{k,l}$ is trivial iff $\mathrm{Fr}(\mathfrak{A}_{k,l}) \xrightarrow{\widehat{p}} X$ has a section $\widehat{\sigma}$.*

Proof. Note that actually we have three fiber bundles: $\mathrm{Gr}(\mathfrak{A}_{k,l}) \xrightarrow{p} X$ with fiber $\mathrm{Gr}_{k,l}$, $\mathrm{Fr}(\mathfrak{A}_{k,l}) \xrightarrow{\widehat{p}} X$ with fiber $\mathrm{Fr}_{k,l}$, and $\mathrm{Fr}(\mathfrak{A}_{k,l}) \xrightarrow{\widetilde{p}} \mathrm{Gr}(\mathfrak{A}_{k,l})$ with fiber $\mathrm{PGL}_k(\mathbb{C})$ (see Subsection 1.4). Clearly, lifting $\widetilde{p}^*(\widetilde{\mathcal{A}}_{k,l})$ is a trivial $M_k(\mathbb{C})$-bundle $\widetilde{M}_k$ over $\mathrm{Fr}(\mathfrak{A}_{k,l})$. Furthermore, bundle $\widehat{p}^*(\mathfrak{A}_{k,l})$ has the form $\mathbf{F}_{k,l}^{\mathrm{Fr}(\mathfrak{A}_{k,l})}(\widetilde{p}^*(\widetilde{\mathcal{A}}_{k,l}))$.

Suppose $\widehat{p}$ has a section $\widehat{\sigma}$, then $\mathfrak{A}_{k,l} = \mathbf{F}_{k,l}^X(\widehat{\sigma}^*(\widetilde{M}_k))$ is a trivial bundle. The converse implication is clear. □

2.2 Relation to the frame spaces

Now we describe the functors $\mathbf{F}_{k,l}^X\colon \mathfrak{AB}_k(X) \to \mathfrak{Fan}_{k,l}(X)$, $\mathbf{R}_{k,l}^X\colon \mathfrak{AB}_{kl}(X) \to \mathfrak{Fan}_{k,l}(X)$ in terms of frames.

Definition 25. An n-frame in $M_n(\mathbb{C})$ is an ordered collection of n^2 linearly independent matrices $\{\alpha_{i,j} \mid 1 \le i,\, j \le n\}$ such that $\alpha_{i,j}\alpha_{k,l} = \delta_{jk}\alpha_{i,l}$ for all $1 \le i,\, j,\, k,\, l \le n$.

For example, the "matrix units" $\{E_{ij} \mid 1 \le i,\, j \le n\}$ form the "standard" frame in $M_n(\mathbb{C})$. Clearly, the set of all n-frames in $M_n(\mathbb{C})$ is the principal homogeneous space over $\mathrm{PGL}_n(\mathbb{C})$.

First consider the functor $\mathbf{F}^X_{k,l}$. To any $M_k(\mathbb{C})$-bundle $A_k \in \mathfrak{AB}_k(X)$ we can associate the corresponding principal bundle of k-frames. Similarly, to any $\mathbb{M}_{k,\,l}$-bundle $\mathfrak{A}_{k,l} \in \mathfrak{Fan}_{k,l}(X)$ we can associate the corresponding k-frames bundle with fiber $\mathrm{Fr}_{k,l}$. The functor $\mathbf{F}^X_{k,l}\colon \mathfrak{AB}_k(X) \to \mathfrak{Fan}_{k,l}(X)$ in these terms can be described as follows. To any k-frame $\{\alpha_{i,j} \mid 1 \le i,\, j \le k\}$ in $M_k(\mathbb{C})$ it assigns the k-frame $\{\alpha_{i,j} \otimes E_l \mid 1 \le i,\, j \le k\}$ in $\mathbb{M}_{k,\,l}$, where $\ldots \otimes E_l$ denotes the Kronecker product with the unit $l \times l$-matrix.

Now consider the functor $\mathbf{R}^X_{k,l}$. To a given a kl-frame $\{\gamma_{i,j} \mid 1 \le i,\, j \le kl\}$ in $M_{kl}(\mathbb{C})$ it assigns the k-frame $\{\alpha_{i,j} \mid 1 \le i,\, j \le k\}$ in $\mathbb{M}_{k,\,l}$, where $\alpha_{i,j} = \sum_{1\le m\le l} \gamma_{l(i-1)+m,\, l(j-1)+m},\ 1 \le i,\, j \le k$.

3 A stable theory of $\mathbb{M}_{k,\,l}$-bundles

3.1 A weak homotopy equivalence $\mathfrak{G}_{k,\,l^\infty} \simeq \mathrm{Fr}_{k,\,l^\infty}$

Recall that $\mathfrak{G}_{k,\,l}$ is a topological group, therefore there is the universal principal $\mathfrak{G}_{k,\,l}$-fibration $(\mathrm{E}\,\mathfrak{G}_{k,l},\, \mathrm{B}\,\mathfrak{G}_{k,l},\, \mathfrak{G}_{k,l},\, p_{k,l})$. Using the continuous action $\nu_{k,l}\colon \mathfrak{G}_{k,l} \times \mathrm{Fr}_{k,l} \to \mathrm{Fr}_{k,l}$, we can associate a $\mathrm{Fr}_{k,l}$-fibration with the universal $\mathfrak{G}_{k,\,l}$-fibration. We denote the obtained $\mathrm{Fr}_{k,\,l}$-fibration by $(\mathrm{E}_{k,l},\, \mathrm{B}\,\mathfrak{G}_{k,l},\, \mathrm{Fr}_{k,l},\, p'_{k,l})$.

Let n be a positive integer.

Proposition 26. *The total space* E_{k,l^n} *is* $(2l^n-1)$*-connected.*

Proof. Looking at the homotopy sequence of the fibration $(\mathrm{E}_{k,l^n},\, \mathrm{B}\,\mathfrak{G}_{k,l^n},\, \mathrm{Fr}_{k,l^n},\, p'_{k,l^n})$, we see that it is sufficient to prove that $\partial\colon \pi_r(\mathrm{B}\,\mathfrak{G}_{k,l^n}) \to \pi_{r-1}(\mathrm{Fr}_{k,l^n})$ is an isomorphism for dimensions $r \le 2l^n$.

(i) *∂ is a monomorphism.* Let $\alpha\colon S^r \to \mathrm{B}\,\mathfrak{G}_{k,l^n}$ be a map and let $\mathfrak{A}_{k,l^n}$ be the $\mathbb{M}_{k,\,l^n}$-bundle over S^r classified by α. Suppose $\partial \circ \alpha \simeq *$, then the k-frame fibration $\mathrm{Fr}(\mathfrak{A}_{k,l^n})$ related to $\mathfrak{A}_{k,l^n}$ has a section $\sigma\colon S^r \to \mathrm{Fr}(\mathfrak{A}_{k,l^n})$. It follows from Proposition 24 that the bundle $\mathfrak{A}_{k,\,l^n}$ is trivial, hence $\alpha \simeq *$.

(ii) *∂ is an epimorphism.* Let $\beta \in \pi_{r-1}(\mathrm{Fr}_{k,l^n})$ be an arbitrary element. We must find a bundle with fiber $\mathbb{M}_{k,\,l^n}$ over S^r such that β is the obstruction for the corresponding k-frame fibration over S^r. It follows from the surjectivity of p_* in the homotopy sequence of the fibration

$$\begin{array}{ccc} \mathrm{PGL}_{l^n}(\mathbb{C}) & \xrightarrow{E_k\otimes\ldots} & \mathrm{PGL}_{kl^n}(\mathbb{C}) \\ & & \Big\downarrow{\scriptstyle p} \\ & & \mathrm{Fr}_{k,\,l^n} \end{array}$$

that there is $\widetilde{\beta} \in \pi_{r-1}(\mathrm{PGL}_{kl^n}(\mathbb{C}))$ such that $p_*(\widetilde{\beta}) = \beta$, if $r \leq 2l^n$. Since the boundary homomorphism

$$\pi_r(\mathrm{BPGL}_{kl^n}(\mathbb{C})) \stackrel{\cong}{\to} \pi_{r-1}(\mathrm{PGL}_{kl^n}(\mathbb{C}))$$

of the homotopy sequence of the universal $\mathrm{PGL}_{kl^n}(\mathbb{C})$-fibration is an isomorphism, we see that there is a bundle A_{kl^n} over S^r with fiber $M_{kl^n}(\mathbb{C})$ such that the obstruction to the corresponding principal bundle coincides with $\widetilde{\beta}$. Consider the $\mathbb{M}_{k,\,l^n}$-bundle $\mathbf{R}^{S^r}_{k,\,l^n}(A_{kl^n})$ corresponding to A_{kl^n} (recall that $\mathrm{PGL}_{kl^n}(\mathbb{C}) \subset \mathfrak{G}_{k,\,l^n}$). Let $\alpha\colon S^r \to \mathrm{B}\,\mathfrak{G}_{k,\,l^n}$ be its classifying map. It can be easily checked that the obtained bundle is what is required, i.e. with the obstruction β. (More precisely, consider the $\mathrm{Fr}_{k,\,l^n}$-fibration $\mathrm{Fr}_{k,\,l^n}(A^{univ}_{kl^n}) \stackrel{\mathrm{Fr}_{k,\,l^n}}{\longrightarrow} \mathrm{BPGL}_{kl^n}(\mathbb{C})$ over $\mathrm{BPGL}_{kl^n}(\mathbb{C})$ associated with the universal principal $\mathrm{PGL}_{kl^n}(\mathbb{C})$-bundle. Then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Fr}_{k,\,l^n}(A^{univ}_{kl^n}) \arrow[r] \arrow[d, "\mathrm{Fr}_{k,\,l^n}"'] & \mathrm{E}_{k,\,l^n} \arrow[d, "\mathrm{Fr}_{k,\,l^n}"] \\
\mathrm{BPGL}_{kl^n}(\mathbb{C}) \arrow[r, "\widehat{\mathbf{R}}_{k,\,l^n}"] & \mathrm{B}\,\mathfrak{G}_{k,\,l^n}.
\end{tikzcd}$$

Now the required assertion follows from the commutative diagram

$$\begin{tikzcd}
\pi_r(\mathrm{BPGL}_{kl^n}(\mathbb{C})) \arrow[r, "\partial"] \arrow[d, "\widehat{\mathbf{R}}_{k,\,l^n*}"'] & \pi_{r-1}(\mathrm{Fr}_{k,\,l^n}) \arrow[d, "\mathrm{id}"] \\
\pi_r(\mathrm{B}\,\mathfrak{G}_{k,\,l^n}) \arrow[r, "\partial"] & \pi_{r-1}(\mathrm{Fr}_{k,\,l^n})
\end{tikzcd}$$

obtained from the homotopy sequences of the above fibrations.) Hence, $\partial([\alpha]) = [\beta]$, i.e. $\partial\colon \pi_r(\mathrm{B}\,\mathfrak{G}_{k,l^n}) \to \pi_{r-1}(\mathrm{Fr}_{k,\,l^n})$ is surjective if $r \leq 2l^n$. $\square$

By definition, put $\mathfrak{G}_{k,\,l^\infty} := \varinjlim_n \mathfrak{G}_{k,\,l^n}$, $\mathrm{Fr}_{k,\,l^\infty} := \varinjlim_n \mathrm{Fr}_{k,\,l^n}$.

Theorem 27. *There is a weak homotopy equivalence* $\mathfrak{G}_{k,\,l^\infty} \simeq \mathrm{Fr}_{k,\,l^\infty}$.

Proof. It follows from the previous proposition that the total space of the direct limit fibration $(\mathrm{E}_{k,\,l^\infty}, \mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}, \mathrm{Fr}_{k,\,l^\infty}, p'_{k,\,l^\infty})$, where $\mathrm{E}_{k,\,l^\infty} := \varinjlim_n \mathrm{E}_{k,\,l^n}$, $p'_{k,\,l^\infty} := \varinjlim_n p'_{k,\,l^n}$, etc, is aspherical. Therefore there is a weak homotopy equivalence $\mathrm{Fr}_{k,\,l^\infty} \simeq \Omega(\mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}) \simeq \mathfrak{G}_{k,\,l^\infty}$. $\square$

Corollary 28. *(compare with Corollary 20) The sequence*

$$\mathrm{Gr}_{k,\,l^\infty} \stackrel{\widehat{\mathbf{T}}_{k,\,l^\infty}}{\longrightarrow} \mathrm{BPGL}_k(\mathbb{C}) \stackrel{\widehat{\mathbf{F}}_{k,\,l^\infty}}{\longrightarrow} \mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}$$

is a fibration.

Proof. Since $\mathrm{Fr}_{k,\,l^\infty} \simeq \Omega(\mathrm{B}\,\mathfrak{G}_{k,\,l^\infty})$, we see that the fibered sequence

$$\mathrm{PGL}_k(\mathbb{C}) \stackrel{\alpha}{\to} \mathrm{Fr}_{k,\,l^\infty} \to \mathrm{Gr}_{k,\,l^\infty} \to \mathrm{BPGL}_k(\mathbb{C})$$

can be extended to the right. More precisely, let A_k^{univ} be the universal $M_k(\mathbb{C})$-bundle over $\mathrm{BPGL}_k(\mathbb{C})$. Then the map $\mathrm{BPGL}_k(\mathbb{C}) \stackrel{\widehat{\mathbf{F}}_{k,\,l^\infty}}{\longrightarrow} \mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}$ is a classifying map for the $\mathbb{M}_{k,\,l^\infty}$-bundle $\mathbf{F}_{k,\,l^\infty}^{\mathrm{BPGL}_k(\mathbb{C})}(A_k^{univ})$ over $\mathrm{BPGL}_k(\mathbb{C})$ (see Remark 15), i.e. $\alpha = \Omega(\widehat{\mathbf{F}}_{k,\,l^\infty})$. More precisely, for any n we have the classifying map $\mathrm{BPGL}_k(\mathbb{C}) \to \mathrm{B}\,\mathfrak{G}_{k,\,l^n}$ of $\mathbf{F}_{k,\,l^n}^{\mathrm{BPGL}_k(\mathbb{C})}(A_k^{univ})$ as an $\mathbb{M}_{k,\,l^n}$-bundle. We have the corresponding map of k-frames $\mathrm{PGL}_k(\mathbb{C}) \stackrel{\ldots\otimes E_{l^n}}{\longrightarrow} \mathrm{Fr}_{k,\,l^n}$. Let $\alpha\colon \mathrm{PGL}_k(\mathbb{C}) \to \mathrm{Fr}_{k,\,l^\infty}$ be the direct limit of such maps as $n \to \infty$. Then the following diagram

$$\begin{tikzcd}
 & \mathrm{Fr}_{k,\,l^\infty} \arrow[rr] & & * \arrow[d] \\
\mathrm{PGL}_k(\mathbb{C}) \arrow[ur, "\alpha"] \arrow[rr] & & * \arrow[ur] \arrow[d] & \mathrm{B}\,\mathfrak{G}_{k,\,l^\infty} \\
 & & \mathrm{BPGL}_k(\mathbb{C}) \arrow[ur, "\widehat{\mathbf{F}}_{k,\,l^\infty}"] &
\end{tikzcd}$$

commutes. □

Remark 29. Note that we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{BPGL}_{kl}(\mathbb{C}) \arrow[r, "\widehat{\mathbf{R}}_{k,\,l}"] & \mathrm{B}\,\mathfrak{G}_{k,\,l} \\
\mathrm{BPGL}_k(\mathbb{C}). \arrow[u, "\ldots\otimes\widetilde{M}_l"] \arrow[ur, "\widehat{\mathbf{F}}_{k,\,l}"'] &
\end{tikzcd}$$

Remark 30. Now we want to establish some relation of the fibration of Corollary 28 with the results of Section 2. Let $\mathfrak{A}_{k,\,l^n}^{univ}$ be the universal $\mathbb{M}_{k,\,l^n}$-bundle over $\mathrm{B}\,\mathfrak{G}_{k,\,l^n}$. It was shown prior to Lemma 21 that there is $\mathrm{Gr}_{k,\,l^n}$-fibration $\mathrm{Gr}(\mathfrak{A}_{k,\,l^n}^{univ}) \stackrel{p_n}{\to} \mathrm{B}\,\mathfrak{G}_{k,\,l^n}$ associated with the $\mathbb{M}_{k,\,l^n}$-bundle $\mathfrak{A}_{k,\,l^n}^{univ}$. Moreover, we can define the canonical $M_k(\mathbb{C})$-bundle $\widetilde{\mathcal{A}}_{k,\,l^n}^{univ}$ over $\mathrm{Gr}(\mathfrak{A}_{k,\,l^n}^{univ})$ which is the tautological subbundle $\widetilde{\mathcal{A}}_{k,\,l^n}^{univ} \subset \mathrm{Gr}(\mathfrak{A}_{k,\,l^n}^{univ}) \underset{\mathrm{B}\,\mathfrak{G}_{k,\,l^n}}{\times} \mathfrak{A}_{k,\,l^n}^{univ}$ (in particular, its restriction to a fiber $\mathrm{Gr}_{k,\,l^n}$ of the fibration $\mathrm{Gr}(\mathfrak{A}_{k,\,l^n}^{univ}) \stackrel{p_n}{\to} \mathrm{B}\,\mathfrak{G}_{k,\,l^n}$ coincides with the core $\mathcal{A}_{k,\,l^n}$ of the tautological floating algebra bundle over $\mathrm{Gr}_{k,\,l^n}$).

Consider the fibration

$$\begin{tikzcd}
\mathrm{Gr}_{k,\,l^n} \arrow[r] & \mathrm{Gr}(\mathfrak{A}_{k,\,l^n}^{univ}) \arrow[d, "p_n"] \\
 & \mathrm{B}\,\mathfrak{G}_{k,\,l^n}.
\end{tikzcd}$$

It follows from the obtained results that this fibration turns into the fibration of Corollary 28 as we pass to the limit with $n \to \infty$. In particular, $\mathrm{Gr}(\mathfrak{A}_{k,\,l^\infty}^{univ}) := \varinjlim_n \mathrm{Gr}(\mathfrak{A}_{k,\,l^n}^{univ}) \simeq \mathrm{BPGL}_k(\mathbb{C})$ and, moreover, under this identification the $M_k(\mathbb{C})$-bundle $\widetilde{\mathcal{A}}_{k,\,l^\infty}^{univ}$ over $\mathrm{Gr}(\mathfrak{A}_{k,\,l^\infty}^{univ})$ coincides with the universal $M_k(\mathbb{C})$-bundle A_k^{univ} over $\mathrm{BPGL}_k(\mathbb{C})$ and the projection $\mathrm{Gr}(\mathfrak{A}_{k,\,l^\infty}^{univ}) \stackrel{p_\infty}{\longrightarrow} \mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}$ is just $\widehat{\mathbf{F}}_{k,\,l^\infty}$.

Corollary 31. *$\pi_r(\mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}) = \mathbb{Z}/k\mathbb{Z}$ for even $r \geq 2$ and $\pi_r(\mathrm{B}\,\mathfrak{G}_{k,\,l^\infty}) = 0$ for odd r.*

Proof follows from Theorem 27 and from the homotopy sequence of the fibration

$$\begin{tikzcd} \mathrm{PGL}_{l^\infty}(\mathbb{C}) \arrow[r, "E_k\otimes\ldots"] & \mathrm{PGL}_{kl^\infty}(\mathbb{C}) \arrow[d] \\ & \mathrm{Fr}_{k,\,l^\infty}. \quad \square \end{tikzcd}$$

3.2 The tensor product of fans

Let k, l, m, n be positive integers such that $(km, ln) = 1$. Denote

$$\mathbb{M}_{k,\,l}\underset{\mathbb{C}}{\otimes}\mathbb{M}_{m,\,n} := \mathbb{M}_{km,\,ln}.$$

We have the map

$$t_{k,l;\,m,n}\colon \mathbb{M}_{k,\,l}\times\mathbb{M}_{m,\,n}\to\mathbb{M}_{k,\,l}\underset{\mathbb{C}}{\otimes}\mathbb{M}_{m,\,n}$$

such that its restriction to each pair of subalgebras $M_k(\mathbb{C})\times M_m(\mathbb{C})\subset\mathbb{M}_{k,\,l}\times\mathbb{M}_{m,\,n}$ coincides with the usual bilinear map $M_k(\mathbb{C})\times M_m(\mathbb{C})\stackrel{\otimes}{\to}M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}M_m(\mathbb{C}) = M_{km}(\mathbb{C})\subset\mathbb{M}_{km,\,ln}$. (Indeed, consider the commutative diagram

$$\begin{tikzcd} \mathcal{A}_{k,l}\times\mathcal{A}_{m,n} \arrow[r, "\boxtimes"] \arrow[d] & \mathcal{A}_{km,ln} \arrow[d] \\ \mathrm{Gr}_{k,\,l}\times\mathrm{Gr}_{m,\,n} \arrow[r, "\mu_{k,l;\,m,n}"] & \mathrm{Gr}_{km,\,ln}, \end{tikzcd}$$

where $\mu_{k,l;m,n}$ is the map induced by the tensor product of algebras $M_{kl}(\mathbb{C})\times M_{mn}(\mathbb{C})\stackrel{\otimes}{\to}M_{klmn}(\mathbb{C})$. Now we define $t_{k,l;m,n}$ as the map such that the corresponding diagram

$$\begin{tikzcd} \mathcal{A}_{k,l}\times\mathcal{A}_{m,n} \arrow[r, "\boxtimes"] \arrow[d, "\pi_{k,\,l}\times\pi_{m,\,n}"'] & \mathcal{A}_{km,ln} \arrow[d, "\pi_{km,\,ln}"] \\ \mathbb{M}_{k,\,l}\times\mathbb{M}_{m,\,n} \arrow[r, "t_{k,l;\,m,n}"] & \mathbb{M}_{km,\,ln} \end{tikzcd}$$

commutes, where $\pi_{k,l}$ is the composite map $\mathcal{A}_{k,l}\hookrightarrow\mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l}\stackrel{\pi_2}{\to}\mathbb{M}_{k,\,l}$, etc.) Note that this definition makes sense due to Proposition 3. Clearly, the tensor product of fans is associative.

Now we must prove that the tensor product of fans is functorial.

Lemma 32. Suppose $\vartheta\colon\mathbb{M}_{k,\,l}\to\mathbb{M}_{k,\,l}$ is a fan automorphism. Then there is a unique automorphism $\widetilde{\vartheta}\colon\mathcal{A}_{k,l}\to\mathcal{A}_{k,l}$ of the core of the tautological bundle over $\mathrm{Gr}_{k,l}$ such that $\widetilde{\vartheta}\mid_{(\mathcal{A}_{k,\,l})_x}=\vartheta\mid_{M_{k,\,x}}$ for any $x\in\mathrm{Gr}_{k,l}$, where $M_{k,x}$ is the k-subalgebra in $\mathbb{M}_{k,\,l}$ parameterized by $x\in\mathrm{Gr}_{k,l}$. Moreover, the assignment $\vartheta\mapsto\widetilde{\vartheta}$ is a group homomorphism $\mathrm{Aut}(\mathbb{M}_{k,\,l})\to\mathrm{Aut}(\mathcal{A}_{k,l})$.

Proof. To a given $\vartheta\colon\mathbb{M}_{k,\,l}\to\mathbb{M}_{k,\,l}$ we assign the continuous map $\overline{\vartheta}\colon\mathrm{Gr}_{k,\,l}\to\mathrm{Gr}_{k,\,l}$ such that $\vartheta\mid_{M_{k,\,x}}\colon M_{k,x}\stackrel{\cong}{\to}M_{k,\overline{\vartheta}(x)}$ for any $x\in\mathrm{Gr}_{k,\,l}$. Then $\widetilde{\vartheta}\colon\mathcal{A}_{k,l}\to\mathcal{A}_{k,l}$ can be defined as a unique map such that the diagram

$$\begin{tikzcd} \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \arrow[r, "\overline{\vartheta}\times\vartheta"] & \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \\ \mathcal{A}_{k,l} \arrow[u] \arrow[r, "\widetilde{\vartheta}"] & \mathcal{A}_{k,l} \arrow[u] \end{tikzcd}$$

commutes (the vertical arrows are the canonical embeddings). Obviously, $\widetilde{\vartheta}$ is a continuous map. $\square$

Lemma 33. Conversely, suppose τ is an automorphism $\tau\colon \mathcal{A}_{k,l}\to\mathcal{A}_{k,l}$ (which is not necessarily identity on the base $\mathrm{Gr}_{k,l}$). Then there is an automorphism $\widehat{\tau}\colon \mathbb{M}_{k,\,l}\to\mathbb{M}_{k,\,l}$ such that the diagram

$$\begin{array}{ccc} \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} & \xrightarrow{\overline{\tau}\times\widehat{\tau}} & \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \\ \uparrow & & \uparrow \\ \mathcal{A}_{k,\,l} & \xrightarrow{\tau} & \mathcal{A}_{k,\,l} \end{array}$$

commutes, where $\overline{\tau}$ is the map covered by τ:

$$\begin{array}{ccc} \mathcal{A}_{k,\,l} & \xrightarrow{\tau} & \mathcal{A}_{k,\,l} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{k,\,l} & \xrightarrow{\overline{\tau}} & \mathrm{Gr}_{k,\,l}\,. \end{array}$$

(In other words, by definition $\widehat{\tau}\mid_{M_{k,\,x}}=\tau\mid_x\colon (\mathcal{A}_{k,l})_x\to(\mathcal{A}_{k,l})_{\overline{\tau}(x)}$ for any $x\in\mathrm{Gr}_{k,l}$, if we identify $M_{k,x}$ with $(\mathcal{A}_{k,l})_x$.) Moreover, the assignment $\tau\mapsto\widehat{\tau}$ is a group homomorphism $\mathrm{Aut}(\mathcal{A}_{k,l})\to\mathrm{Aut}(\mathbb{M}_{k,\,l})$.

Proof. By $\widetilde{\mathbb{M}}_{k,\,l}$ denote the trivial $\mathbb{M}_{k,\,l}$-bundle over $\mathrm{Gr}_{k,\,l}$. Using the equality $\widetilde{\mathbb{M}}_{k,\,l}=\bigcup_{g\in\mathbf{GL}_l(\mathcal{A}_{k,\,l})} g(\mathcal{A}_{k,l}\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l)g^{-1}$ of Proposition 16 (see (8)), we can extend the isomorphism τ to the $\mathbf{PGL}_l(\mathcal{A}_{k,l})$-equivariant isomorphism of trivial bundles $\mathrm{T}\colon \widetilde{\mathbb{M}}_{k,\,l}\to\widetilde{\mathbb{M}}_{k,\,l}$ which covers $\overline{\tau}\colon\ \mathrm{Gr}_{k,l}\to\mathrm{Gr}_{k,l}$ and such that the diagram

$$\begin{array}{ccc} \widetilde{\mathbb{M}}_{k,\,l} & \xrightarrow{\mathrm{T}} & \widetilde{\mathbb{M}}_{k,\,l} \\ \uparrow & & \uparrow \\ \mathcal{A}_{k,l} & \xrightarrow{\tau} & \mathcal{A}_{k,l} \end{array}$$

commutes. It is not difficult to verify that T preserves the standard trivialization $\mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l}$ of $\widetilde{\mathbb{M}}_{k,\,l}$, i.e. the diagram

$$\begin{array}{ccc} \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} & \xrightarrow{\mathrm{T}} & \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \\ {\scriptstyle\pi_2}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ \mathbb{M}_{k,\,l} & \xrightarrow{\widehat{\tau}} & \mathbb{M}_{k,\,l} \end{array}$$

commutes. Then $\widehat{\tau}\colon \mathbb{M}_{k,\,l}\to\mathbb{M}_{k,\,l}$ is the required automorphism, and $\mathrm{T}=(\overline{\tau},\,\widehat{\tau})$. $\square$

Comparing the two previous lemmas, we get the following one.

Lemma 34. The composition $\vartheta\mapsto\widehat{\tau}$, where $\tau=\widetilde{\vartheta}$, is the identity map $\mathrm{id}_{\mathrm{Aut}(\mathbb{M}_{k,\,l})}$. Furthermore, the composition $\tau\mapsto\widetilde{\vartheta}$, where $\widehat{\tau}=\vartheta$, is the identity map $\mathrm{id}_{\mathrm{Aut}(\mathcal{A}_{k,\,l})}$. In particular, $\mathrm{Aut}(\mathbb{M}_{k,\,l})=\mathfrak{G}_{k,\,l}$ is isomorphic to $\mathrm{Aut}(\mathcal{A}_{k,l})$ (recall that we consider automorphisms of the bundle $\mathcal{A}_{k,l}$ which are not necessarily identity on the base $\mathrm{Gr}_{k,l}$).

Proof. By $\pi\colon \mathcal{A}_{k,l}\to \mathbb{M}_{k,\,l}$ denote the composite map $\mathcal{A}_{k,l}\stackrel{\mu}{\to}\mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l}\stackrel{\pi_2}{\to}\mathbb{M}_{k,\,l}$. Note that it is surjective. Then it is sufficient to observe that the maps ϑ and $\widehat{\tau}$ both make the diagrams

$$\begin{tikzcd} \mathcal{A}_{k,l} \arrow[r, "\widetilde{\vartheta}=\tau"] \arrow[d, "\pi"'] & \mathcal{A}_{k,l} \arrow[d, "\pi"] \\ \mathbb{M}_{k,\,l} \arrow[r, "\vartheta"] & \mathbb{M}_{k,\,l} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A}_{k,l} \arrow[r, "\widetilde{\vartheta}=\tau"] \arrow[d, "\pi"'] & \mathcal{A}_{k,l} \arrow[d, "\pi"] \\ \mathbb{M}_{k,\,l} \arrow[r, "\widehat{\tau}"] & \mathbb{M}_{k,\,l} \end{tikzcd}$$

commutative. Furthermore, the maps τ and $\widetilde{\vartheta}$ both make the diagrams

$$\begin{tikzcd} \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \arrow[r, "\overline{\vartheta}\times\vartheta=\overline{\widehat{\tau}}\times\widehat{\tau}"] & \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \\ \mathcal{A}_{k,l} \arrow[u] \arrow[r, "\tau"] & \mathcal{A}_{k,l} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \arrow[r, "\overline{\vartheta}\times\vartheta=\overline{\widehat{\tau}}\times\widehat{\tau}"] & \mathrm{Gr}_{k,\,l}\times\mathbb{M}_{k,\,l} \\ \mathcal{A}_{k,l} \arrow[u] \arrow[r, "\widetilde{\vartheta}"] & \mathcal{A}_{k,l} \arrow[u] \end{tikzcd}$$

commutative. $\square$

Remark 35. It is interesting to compare the result of the previous lemma $\mathfrak{G}_{k,\,l}\cong\mathrm{Aut}(\mathcal{A}_{k,l})$ with Remark 30. In particular, consider the total space of the tautological bundle $\widetilde{\mathcal{A}}^{univ}_{k,l}$ as a fibration over $\mathrm{B}\,\mathfrak{G}_{k,\,l}$ with fiber the total space of the tautological bundle $\mathcal{A}_{k,l}$ over $\mathrm{Gr}_{k,\,l}$. Then the isomorphism $\mathfrak{G}_{k,\,l}\cong\mathrm{Aut}(\mathcal{A}_{k,l})$ implies that the fibration $\widetilde{\mathcal{A}}^{univ}_{k,l}$ can be glued by the same $\mathfrak{G}_{k,\,l}$-cocycle over $\mathrm{B}\,\mathfrak{G}_{k,\,l}$ as the universal $\mathbb{M}_{k,\,l}$-bundle $\mathfrak{A}^{univ}_{k,\,l}$, i.e. it is the universal $\mathcal{A}_{k,l}$-fibration.

For an upcoming important proposition we need one trivial lemma.

Lemma 36. Let $\tau_1\colon M_{k,\alpha}\to M_{k,\alpha}$ be an automorphism of the standard subalgebra $M_{k,\alpha}=M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l\subset\mathbb{M}_{k,\,l}$ in the fan $\mathbb{M}_{k,\,l}$ and let $\tau_2\colon M_{m,\beta}\to M_{m,\beta}$ be an automorphism of the standard subalgebra $M_{m,\beta}=M_m(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_n\subset\mathbb{M}_{m,\,n}$ in the fan $\mathbb{M}_{m,\,n}$, respectively. Then τ_1 can be canonically extended to the $\mathrm{PGL}_{kl}(\mathbb{C})$-equivariant automorphism $\widehat{\tau}_1\colon\mathbb{M}_{k,\,l}\to\mathbb{M}_{k,\,l}$ and τ_2 can be canonically extended to the $\mathrm{PGL}_{mn}(\mathbb{C})$-equivariant automorphism $\widehat{\tau}_2\colon\mathbb{M}_{m,\,n}\to\mathbb{M}_{m,\,n}$, respectively. Furthermore, the automorphism $\tau_1\otimes\tau_2\colon M_{k,\alpha}\otimes M_{m,\beta}\to M_{k,\alpha}\otimes M_{m,\beta}$ can be canonically extended to the $\mathrm{PGL}_{klmn}(\mathbb{C})$-equivariant automorphism $\widehat{\tau_1\otimes\tau_2}\colon\mathbb{M}_{km,\,ln}\to\mathbb{M}_{km,\,ln}$ of the fan $\mathbb{M}_{km,\,ln}=\mathbb{M}_{k,\,l}\otimes\mathbb{M}_{m,\,n}$ such that $\widehat{\tau_1\otimes\tau_2}\mid_{\mathbb{M}^{(1)}_{k,\,l}}=\widehat{\tau}_1$, $\widehat{\tau_1\otimes\tau_2}\mid_{\mathbb{M}^{(2)}_{m,\,n}}=\widehat{\tau}_2$, where $\mathbb{M}^{(1)}_{k,\,l}$, $\mathbb{M}^{(2)}_{m,\,n}$ are the images of the embeddings (see Proposition 3) $\mathbb{M}_{k,\,l}\overset{\ldots\underset{\mathbb{C}}{\otimes}\mathbb{C}E_{mn}}{\hookrightarrow}\mathbb{M}_{km,\,ln}$ and $\mathbb{M}_{km,\,ln}\overset{\mathbb{C}E_{kl}\underset{\mathbb{C}}{\otimes}\ldots}{\hookleftarrow}\mathbb{M}_{m,\,n}$, respectively.

Proof is trivial and is based on the equality $\mathbb{M}_{k,\,l}=\bigcup\limits_{g\in\mathrm{GL}_{kl}(\mathbb{C})}g(M_k(\mathbb{C})\underset{\mathbb{C}}{\otimes}\mathbb{C}E_l)g^{-1}$. Let us only remark that the proof of the next proposition can be considered as a generalization of the proof of this trivial lemma to the replacements

$$M_{k,\alpha}\leftrightarrow\mathcal{A}_{k,l},\ M_{m,\beta}\leftrightarrow\mathcal{A}_{m,n},\ \mathrm{GL}_{kl}(\mathbb{C})\leftrightarrow\mathbf{GL}_l(\mathcal{A}_{k,l}),$$

$$\mathrm{GL}_{mn}(\mathbb{C})\leftrightarrow\mathbf{GL}_n(\mathcal{A}_{m,n}),\ \mathrm{GL}_{klmn}(\mathbb{C})\leftrightarrow\mathbf{GL}_{ln}(\mathcal{A}_{k,l}\boxtimes\mathcal{A}_{m,n}),$$

etc. $\square$

Proposition 37. *For any pair $(\vartheta_1, \vartheta_2)$, $\vartheta_1 \in \mathrm{Aut}(\mathbb{M}_{k,l})$, $\vartheta_2 \in \mathrm{Aut}(\mathbb{M}_{m,n})$ there exists an element $\vartheta_1 \otimes \vartheta_2 \in \mathrm{Aut}(\mathbb{M}_{km,\,ln})$ canonically determined by the pair and such that $\vartheta_1 \otimes \vartheta_2 \mid_{\mathbb{M}^{(1)}_{k,\,l}} = \vartheta_1$, $\vartheta_1 \otimes \vartheta_2 \mid_{\mathbb{M}^{(2)}_{m,\,n}} = \vartheta_2$, where $\mathbb{M}^{(1)}_{k,\,l}$, $\mathbb{M}^{(2)}_{m,\,n}$ are the images of the embeddings $\mathbb{M}_{k,\,l} \overset{\dots\underset{\mathbb{C}}{\otimes}\mathbb{C}E_{mn}}{\hookrightarrow} \mathbb{M}_{km,\,ln}$ and $\mathbb{M}_{km,\,ln} \overset{\mathbb{C}E_{kl}\underset{\mathbb{C}}{\otimes}\dots}{\hookleftarrow} \mathbb{M}_{m,\,n}$, respectively.*

Proof. Applying Lemma 32, we get automorphisms $\tau_1 \colon \mathcal{A}_{k,l} \to \mathcal{A}_{k,l}$, $\tau_2 \colon \mathcal{A}_{m,n} \to \mathcal{A}_{m,n}$. Thus, we have automorphism

$$\tau_1 \boxtimes \tau_2 \colon \mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n} \to \mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n}$$

of the bundle $\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n}$ over $\mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n}$ with fiber $M_{km}(\mathbb{C})$. Note that we have the canonical embedding

$$\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n} \to \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln}$$

such that the diagram

$$\begin{tikzcd}
\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n} \arrow[r] \arrow[d] & \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln} \arrow[d, "\mu_{k,l;\,m,n} \times \mathrm{id}_{\mathbb{M}_{km,ln}}"] \\
\mathcal{A}_{km,\,ln} \arrow[r] & \mathrm{Gr}_{km,\,ln} \times \mathbb{M}_{km,\,ln}
\end{tikzcd}$$

commutes, where $\mu_{k,l;m,n} \colon \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \to \mathrm{Gr}_{km,\,ln}$ is the map induced by the tensor product of algebras.

Applying Proposition 16 and using the same arguments as in the proof of Lemma 33, we get the automorphism T of the trivial bundle $\mathrm{T} \colon \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln} \to \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln}$ such that the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln} \arrow[r, "\mathrm{T}"] & \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln} \\
\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n} \arrow[u] \arrow[r, "\tau_1 \boxtimes \tau_2"] & \mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n} \arrow[u]
\end{tikzcd}$$

commutes. Now we can define $\vartheta_1 \otimes \vartheta_2$ as the map that makes the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln} \arrow[r, "\mathrm{T}"] \arrow[d, "\pi_2"'] & \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \times \mathbb{M}_{km,\,ln} \arrow[d, "\pi_2"] \\
\mathbb{M}_{km,\,ln} \arrow[r, "\vartheta_1 \otimes \vartheta_2"] & \mathbb{M}_{km,\,ln}
\end{tikzcd}$$

commutative. Note that having $\vartheta_1 \otimes \vartheta_2$ and using Lemma 32, we can construct the automorphism $\widetilde{\vartheta_1 \otimes \vartheta_2} \colon \mathcal{A}_{km,ln} \to \mathcal{A}_{km,ln}$ of the tautological $M_{km}(\mathbb{C})$-bundle over $\mathrm{Gr}_{km,ln}$. Clearly, its restriction $\widetilde{\vartheta_1 \otimes \vartheta_2} \mid_{\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n}}$ coincides with $\tau_1 \boxtimes \tau_2$ (here we consider $\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n}$ as a subset in $\mathcal{A}_{km,ln}$ due to the canonical embedding $\mathcal{A}_{k,l} \boxtimes \mathcal{A}_{m,n} \hookrightarrow \mathcal{A}_{km,ln}$ which covers the map $\mu_{k,l;m,n} \colon \mathrm{Gr}_{k,l} \times \mathrm{Gr}_{m,n} \hookrightarrow \mathrm{Gr}_{km,\,ln}$).

Using the above lemmas, one can easily verify that $\vartheta_1 \otimes \vartheta_2$ has the required property, i.e. $\vartheta_1 \otimes \vartheta_2 \mid_{\mathbb{M}^{(1)}_{k,\,l}} = \vartheta_1$, $\vartheta_1 \otimes \vartheta_2 \mid_{\mathbb{M}^{(2)}_{m,\,n}} = \vartheta_2$. $\square$

It easily follows from the corresponding properties of the usual tensor product of algebras that the tensor product of fans is associative and commutative up to canonical equivalence.

Corollary 38. *For any positive integers k, l, m, n such that $(km, ln) = 1$ we have a group monomorphism $\mathfrak{G}_{k,l} \times \mathfrak{G}_{m,n} \hookrightarrow \mathfrak{G}_{km,ln}$ and therefore a mapping of classifying spaces $\mathrm{B}\,\mathfrak{G}_{k,l} \times \mathrm{B}\,\mathfrak{G}_{m,n} \hookrightarrow \mathrm{B}\,\mathfrak{G}_{km,ln}$.*

3.3 A stable equivalence relation

Let X be a finite CW-complex. It follows from Proposition 37 that the tensor product of fan bundles with fibers $\mathbb{M}_{k,\,l}$, $\mathbb{M}_{m,\,n}$, $(km, ln) = 1$ is a well-defined fan bundle with fiber $\mathbb{M}_{km,\,ln}$.

By $\widetilde{\mathbb{M}}_{k,\,l}$ we denote a trivial $\mathbb{M}_{k,\,l}$-bundle over X.

Definition 39. Two fan bundles $\mathfrak{A}_{k^m,\,l^n}$ and $\mathfrak{B}_{k^r,\,l^s}$ over X are said to be *stable equivalent* if there are integers t, u, v, w such that

$$\mathfrak{A}_{k^m,\,l^n} \otimes \widetilde{\mathbb{M}}_{k^t,\,l^u} \cong \mathfrak{B}_{k^r,\,l^s} \otimes \widetilde{\mathbb{M}}_{k^v,\,l^w}$$

(of course, $m + t = r + v$, $n + u = s + w$).

For a fan bundle $\mathfrak{A}_{k^m,\,l^n}$, by $[\mathfrak{A}_{k^m,\,l^n}]$ we denote its stable equivalence class.

Let $\mathrm{Fan}^{k,\,l}(X)$ be the set of stable equivalence classes of fan bundles with fibers of the form $\mathbb{M}_{k^m,\,l^n}$ (for arbitrary m, n) over X.

By $\mathfrak{G}_{k^\infty,\,l^\infty}$, $\mathrm{Fr}_{k^\infty,\,l^\infty}$ we denote the direct limits $\varinjlim_n \mathfrak{G}_{k^n,\,l^\infty}$, $\varinjlim_n \mathrm{Fr}_{k^n,\,l^\infty}$, respectively.

Remark 40. We can also define the space $\mathrm{Fr}_{k^\infty,\,l^\infty}$ by taking the direct limit $\varinjlim_n \mathrm{Fr}_{k^n,\,l^n}$ with respect to the maps α_n:

$$\begin{tikzcd}
\mathrm{PGL}_{l^{n+1}}(\mathbb{C}) \arrow[r, "E_{k^{n+1}}\otimes\ldots"] & \mathrm{PGL}_{(kl)^{n+1}}(\mathbb{C}) \arrow[r] & \mathrm{Fr}_{k^{n+1},\,l^{n+1}} \\
\mathrm{PGL}_{l^n}(\mathbb{C}) \arrow[r, "E_{k^n}\otimes\ldots"] \arrow[u, "\ldots\otimes E_l"] & \mathrm{PGL}_{(kl)^n}(\mathbb{C}) \arrow[r] \arrow[u, "E_k\otimes\ldots\otimes E_l"'] & \mathrm{Fr}_{k^n,\,l^n}\,. \arrow[u, "\alpha_{n+1}"']
\end{tikzcd}$$

One can easily verify that in the stable dimensions r $\pi_r(\mathrm{Fr}_{k^n,\,l^n}) = \mathbb{Z}/k^n\mathbb{Z}$ for odd $r \geq 1$ and 0 otherwise. Furthermore, it follows from the homotopy sequence of the above diagram that α_{n+1} induces the usual embedding

$$\pi_{2s-1}(\mathrm{Fr}_{k^n,\,l^n}) \hookrightarrow \pi_{2s-1}(\mathrm{Fr}_{k^{n+1},\,l^{n+1}}), \quad \mathbb{Z}/k^n\mathbb{Z} \hookrightarrow \mathbb{Z}/k^{n+1}\mathbb{Z},\ s \geq 1.$$

Hence $\pi_r(\mathrm{Fr}_{k^\infty,\,l^\infty}) = \varinjlim_n \mathbb{Z}/k^n\mathbb{Z}$ for odd r and 0 otherwise.

We can simplify our notation with the help of the following lemma.

Lemma 41. The homotopy type of the space $\mathrm{Fr}_{k^\infty,\,l^\infty}$ does not depend on the choice of l, $(k, l) = 1$.

Proof. Clearly, $\mathrm{Fr}_{k^\infty,\,l^\infty}$ is a fiber of the fibration

$$\mathrm{Gr}_{k^\infty,\,l^\infty} \overset{\widehat{\mathbf{T}}_{k^\infty,\,l^\infty}}{\longrightarrow} \mathrm{BPGL}_{k^\infty}(\mathbb{C}), \tag{10}$$

where $\widehat{\mathbf{T}}_{k^\infty,l^\infty}$ is the classifying map for the core $\mathcal{A}_{k^\infty,l^\infty}$ of the universal FAB over $\mathrm{Gr}_{k^\infty,l^\infty}$. But the homotopy type of $\mathrm{Gr}_{k^\infty,l^\infty}$ does not depend on the choice of $k,l,\ (k,l)=1$, moreover, $\mathrm{Gr}_{k^\infty,l^\infty}\simeq \mathrm{BSU}$. The map $\widehat{\mathbf{T}}_{k^\infty,l^\infty}$ actually does not depend on l, because for $k,\ l,\ m,\ (k,lm)=1$ the diagram

$$
\begin{tikzcd}
& \mathrm{Gr}_{k^\infty,(lm)^\infty} \arrow[dd, "\widehat{\mathbf{T}}_{k^\infty,(lm)^\infty}"] & \\
\mathrm{Gr}_{k^\infty,l^\infty} \arrow[ur, "\simeq"] \arrow[dr, "\widehat{\mathbf{T}}_{k^\infty,l^\infty}"'] & & \mathrm{Gr}_{k^\infty,m^\infty} \arrow[ul, "\simeq"'] \arrow[dl, "\widehat{\mathbf{T}}_{k^\infty,m^\infty}"] \\
& \mathrm{BPGL}_{k^\infty}(\mathbb{C}) &
\end{tikzcd}
$$

commutes. Thus, we see that $\mathrm{Gr}_{k^\infty,l^\infty}$ and $\widehat{\mathbf{T}}_{k^\infty,l^\infty}$ in (10) are independent of l with $(l,k)=1$, and therefore this is also true for the fiber $\mathrm{Fr}_{k^\infty,l^\infty}$. $\square$

Remark 42. Another proof of the previous lemma can be obtained in the following way. Suppose $(k,lm)=1$, then the maps in the diagram

$$
\begin{tikzcd}
& \mathrm{Fr}_{k,lm} & \\
\mathrm{Fr}_{k,l} \arrow[ur] & & \mathrm{Fr}_{k,m} \arrow[ul]
\end{tikzcd}
\tag{11}
$$

are homotopy equivalences in small dimensions. Thus, we have $\mathrm{Fr}_{k^\infty,l^\infty}\simeq \mathrm{Fr}_{k^\infty,(lm)^\infty}\simeq \mathrm{Fr}_{k^\infty,m^\infty}$.

Corollary 43. *The homotopy type of the space $\mathfrak{G}_{k^\infty,l^\infty}$ does not depend on the choice of $l,\ (k,l)=1$.*

Proof follows from the previous lemma and Theorem 27. $\square$

Thus we can omit l in the notation $\mathrm{Fr}_{k^\infty,l^\infty}$, $\mathfrak{G}_{k^\infty,l^\infty}$, etc.

The proof of the following theorem is standard.

Theorem 44. *The functor $X\mapsto \mathrm{Fan}^{k,l}(X)$ is represented by space $\mathrm{B}\,\mathfrak{G}_{k^\infty}$.*

Corollary 45. *The functor $X\mapsto \mathrm{Fan}^{k,l}(X)$ is independent of l.*

Thus we can write just $\mathrm{Fan}^k(X)$ instead of $\mathrm{Fan}^{k,l}(X)$.

We define the *product* of two stable equivalence classes $[\mathfrak{A}_{k^m,l^n}]$, $[\mathfrak{B}_{k^r,l^s}]$ over X as

$$[\mathfrak{A}_{k^m,l^n}]\cdot[\mathfrak{B}_{k^r,l^s}]:=[\mathfrak{A}_{k^m,l^n}\otimes\mathfrak{B}_{k^r,l^s}].$$

Clearly, the product is well defined.

Lemma 46. The order of any stable equivalence class of fan bundles over a finite CW-complex X with respect to the above-defined operation is finite, i.e. there is $w\in\mathbb{N}$ such that $[\mathfrak{A}_{k^m,l^n}]^{k^w}$ coincides with the stable equivalence class of a trivial bundle.

Proof follows from Theorem 44 and from the fact that $\pi_r(\mathrm{B}\,\mathfrak{G}_{k^\infty}) = \varinjlim_n \mathbb{Z}/k^n\mathbb{Z}$ for even r and 0 otherwise. $\square$

Thus, every stable equivalence class is invertible.

Corollary 47. *The product of stable equivalence classes induces a structure of Abelian group on the set* $\mathrm{Fan}^k(X)$*. In other words,* $\mathrm{B}\,\mathfrak{G}_{k^\infty}$ *is an* H*-space (with inversion).*

Remark 48. Note that the multiplication on the H-space $\mathrm{B}\,\mathfrak{G}_{k^\infty}$ is determined by the tensor product map

$$\mathrm{B}\,\mathfrak{G}_{k^m,\,l^n} \times \mathrm{B}\,\mathfrak{G}_{k^r,\,l^s} \to \mathrm{B}\,\mathfrak{G}_{k^{m+r},\,l^{n+s}}$$

(which is a classifying map for the tensor product of the universal fan bundles over $\mathrm{B}\,\mathfrak{G}_{k^m,\,l^n}$ and $\mathrm{B}\,\mathfrak{G}_{k^r,\,l^s}$) on finite subspaces.

Recall that in Corollary 28 we have considered the fibration

$$\mathrm{Gr}_{k^n,\,l^\infty} \overset{\widehat{\mathbf{T}}_{k^n,\,l^\infty}}{\longrightarrow} \mathrm{BPGL}_{k^n}(\mathbb{C}) \overset{\widehat{\mathbf{F}}_{k^n,\,l^\infty}}{\longrightarrow} \mathrm{B}\,\mathfrak{G}_{k^n,\,l^\infty}.$$

Taking the limit as $n \to \infty$, we obtain the following fibration

$$\mathrm{Gr} \overset{\widehat{\mathbf{T}}_{k^\infty}}{\longrightarrow} \mathrm{BPGL}_{k^\infty}(\mathbb{C}) \overset{\widehat{\mathbf{F}}_{k^\infty}}{\longrightarrow} \mathrm{B}\,\mathfrak{G}_{k^\infty}, \tag{12}$$

where $\mathrm{Gr} := \mathrm{Gr}_{k^\infty,\,l^\infty}$, $\mathrm{BPGL}_{k^\infty}(\mathbb{C}) := \varinjlim_n \mathrm{BPGL}_{k^n}(\mathbb{C})$ with respect to the maps induced by the canonical group homomorphisms $\mathrm{PGL}_{k^n}(\mathbb{C}) \to \mathrm{PGL}_{k^{n+1}}(\mathbb{C})$, $A \mapsto A \otimes E_k$, and $\widehat{\mathbf{T}}_{k^\infty} := \varinjlim_n \widehat{\mathbf{T}}_{k^n,\,l^\infty}$, $\widehat{\mathbf{F}}_{k^\infty} := \varinjlim_n \widehat{\mathbf{F}}_{k^n,\,l^\infty}$.

Recall that Gr and $\mathrm{BPGL}_{k^\infty}(\mathbb{C})$ are H-spaces. More precisely, Gr is a classifying space for stable equivalence classes of floating algebra bundles with the multiplication induced by the tensor product of FABs (see Appendix 1). The H-space $\mathrm{BPGL}_{k^\infty}(\mathbb{C})$ is a classifying space for locally trivial bundles with fibers $M_{k^m}(\mathbb{C})$, $m \in \mathbb{N}$, with respect to the following equivalence relation,

$$A_{k^m} \sim B_{k^n} \;\Leftrightarrow\; \exists r,\, s \text{ such that } A_{k^m} \otimes \widetilde{M}_{k^r} \cong B_{k^n} \otimes \widetilde{M}_{k^s},$$

where $\widetilde{M}_{k^m}$ is a trivial bundle. The multiplication on it is induced by the tensor product of bundles. This gives us an interpretation of the groups $\widetilde{\mathrm{AB}}^1(X) := [X;\, \mathrm{Gr}]$ and $\widetilde{\mathrm{AB}}^k(X) := [X;\, \mathrm{BPGL}_{k^\infty}(\mathbb{C})]$.

Theorem 49. *For any finite CW-complex X the sequence of Abelian groups*

$$\widetilde{\mathrm{AB}}^1(X) \overset{\widehat{\mathbf{T}}_{k^\infty *}}{\longrightarrow} \widetilde{\mathrm{AB}}^k(X) \overset{\widehat{\mathbf{F}}_{k^\infty *}}{\longrightarrow} \mathrm{Fan}^k(X)$$

is exact.

Proof follows from the fact that the composition (12) is a fibration. $\square$

3.4 A GL-version

Consider spaces $\widehat{\mathrm{Gr}}_{k,l}$ and $\widehat{\mathrm{Gr}}$ which are defined in Appendix 1. It easily follows from the definition that there is a fibered sequence

$$\mathrm{GL}_k(\mathbb{C}) \hookrightarrow \mathrm{Fr}_{k,l} \to \widehat{\mathrm{Gr}}_{k,l} \xrightarrow{\widehat{\mathbf{T}}'_{k,l}} \mathrm{BGL}_k(\mathbb{C}),$$

where $\widehat{\mathbf{T}}'_{k,l}$ is the classifying map for the tautological $\mathrm{GL}_k(\mathbb{C})$-bundle over $\widehat{\mathrm{Gr}}_{k,l}$. It is mentioned in Appendix 1 that the map $\widehat{\mathbf{T}}'_{k^\infty}(:= \varinjlim_n \widehat{\mathbf{T}}'_{k^n,l^n})\colon \widehat{\mathrm{Gr}}(= \varinjlim_n \widehat{\mathrm{Gr}}_{k^n,l^n}) \to \mathrm{BGL}_{k^\infty}(\mathbb{C})$ is just the localization at k (in particular, $\mathrm{BGL}_{k^\infty}(\mathbb{C})$ is a $\mathbb{Z}[\frac{1}{k}]$-local space).

Now consider the sequence

$$\widehat{\mathrm{Gr}} \xrightarrow{\widehat{\mathbf{T}}'_{k^\infty}} \mathrm{BGL}_{k^\infty}(\mathbb{C}) \xrightarrow{\widehat{\mathbf{F}}'_{k^\infty}} \mathrm{B}\,\mathfrak{G}_{k^\infty}, \tag{13}$$

where the map $\widehat{\mathbf{F}}'_{k^\infty}$ is defined as follows. For any n we have the group homomorphism $\mathrm{GL}_{k^n}(\mathbb{C}) \to \mathfrak{G}_{k^n,l^n}$ determined by the embedding $M_{k^n}(\mathbb{C}) \hookrightarrow \mathbb{M}_{k^n,l^n}$ (more precisely, it is determined by the composition

$$\mathrm{GL}_{k^n}(\mathbb{C}) \to \mathrm{PGL}_{k^n}(\mathbb{C}) = \mathrm{Aut}(M_{k^n}(\mathbb{C})) \to \mathrm{Aut}(\mathbb{M}_{k^n,l^n}) = \mathfrak{G}_{k^n,l^n})$$

and the direct limit of such maps determines the required map $\widehat{\mathbf{F}}'_{k^\infty}\colon \mathrm{BGL}_{k^\infty}(\mathbb{C}) \to \mathrm{B}\,\mathfrak{G}_{k^\infty}$ of classifying spaces. In particular, the diagram

$$\begin{tikzcd}
\mathrm{BGL}_{k^\infty}(\mathbb{C}) \arrow[d] \arrow[dr, "\widehat{\mathbf{F}}'_{k^\infty}"] & \\
\mathrm{BPGL}_{k^\infty}(\mathbb{C}) \arrow[r, "\widehat{\mathbf{F}}_{k^\infty}"] & \mathrm{B}\,\mathfrak{G}_{k^\infty}
\end{tikzcd}$$

commutes.

The proof of the following proposition easily follows from the fact that the space $\mathrm{Fr}_{k^\infty} \simeq \Omega(\mathrm{B}\,\mathfrak{G}_{k^\infty})$ is a fiber of the localization map $\widehat{\mathbf{T}}'_{k^\infty}\colon \widehat{\mathrm{Gr}} \to \mathrm{BGL}_{k^\infty}(\mathbb{C})$.

Proposition 50. *The sequence (13) is a fibration.*

Let us remark that the sequence (13) is a GL-counterpart of the sequence (12):

$$\mathrm{Gr} \xrightarrow{\widehat{\mathbf{T}}_{k^\infty}} \mathrm{BPGL}_{k^\infty}(\mathbb{C}) \xrightarrow{\widehat{\mathbf{F}}_{k^\infty}} \mathrm{B}\,\mathfrak{G}_{k^\infty}.$$

3.5 $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

Define the direct limits

$$\mathrm{B}\,\mathcal{GL}(\mathbb{C}) := \varinjlim_k \mathrm{BGL}_k(\mathbb{C}), \quad \mathrm{B}\,\mathcal{PGL}(\mathbb{C}) := \varinjlim_k \mathrm{BPGL}_k(\mathbb{C}), \quad \text{and} \quad \mathrm{B}\,\mathfrak{G} := \varinjlim_k \mathrm{B}\,\mathfrak{G}_k,$$

where k runs over all positive integers, with respect to the maps corresponding to the tensor product of bundles (in particular, $\mathrm{B}\,\mathcal{GL}(\mathbb{C})$ is a $\mathbb{Q}$-space).

Remark 51. In this remark we give a more detailed definition of the direct limit $\mathrm{B}\,\mathfrak{G} = \varinjlim_{(k,l)=1} \mathrm{B}\,\mathfrak{G}_{k,l}$.

Let $p_1, p_2, \dots, p_r, \dots$ be the sequence of primes $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots$. For any $r \in \mathbb{N}$ we can choose q_r such that $(p_1 \dots p_r, q_r) = 1$. Suppose $(p_1 \dots p_r p_{r+1}, q_r) > 1$. It follows from Proposition 26 and (11) that there are homotopy equivalences

$$\mathrm{B}\,\mathfrak{G}_{(p_1\dots p_r)^\infty,\, q_r^\infty} \overset{\simeq}{\to} \mathrm{B}\,\mathfrak{G}_{(p_1\dots p_r)^\infty,\, (q_r q_{r+1})^\infty} \overset{\simeq}{\leftarrow} \mathrm{B}\,\mathfrak{G}_{(p_1\dots p_r)^\infty,\, q_{r+1}^\infty}.$$

Therefore we have the composite map

$$\mathrm{B}\,\mathfrak{G}_{(p_1\dots p_r)^\infty,\, q_r^\infty} \overset{\simeq}{\to} \mathrm{B}\,\mathfrak{G}_{(p_1\dots p_r)^\infty,\, q_{r+1}^\infty} \to \mathrm{B}\,\mathfrak{G}_{(p_1\dots p_r p_{r+1})^\infty,\, q_{r+1}^\infty},$$

and we can take the direct limit with respect to such maps.

The sequences (12) and (13) give us the following fibrations:

$$\mathrm{Gr} \overset{\widehat{\mathbf{T}}}{\to} \mathrm{B}\,\mathcal{PGL}(\mathbb{C}) \overset{\widehat{\mathbf{F}}}{\to} \mathrm{B}\,\mathfrak{G} \tag{14}$$

and

$$\widehat{\mathrm{Gr}} \overset{\widehat{\mathbf{T}}'}{\to} \mathrm{B}\,\mathcal{GL}(\mathbb{C}) \overset{\widehat{\mathbf{F}}'}{\to} \mathrm{B}\,\mathfrak{G}, \tag{15}$$

respectively.

The relation between (14) and (15) is very clear. More precisely, recall that

$$\widehat{\mathrm{Gr}} = \mathrm{Gr} \times \mathbb{C}P^\infty, \quad \mathrm{B}\,\mathcal{GL}(\mathbb{C}) = \prod_{q\geq 1} \mathrm{K}(2q,\, \mathbb{Q}), \quad \mathrm{B}\,\mathcal{PGL}(\mathbb{C}) = \mathrm{K}(2,\, \mathbb{Q}/\mathbb{Z}) \times \prod_{q\geq 2} \mathrm{K}(2q,\, \mathbb{Q}), \tag{16}$$

$\mathbb{C}P^\infty = \mathrm{K}(2,\, \mathbb{Z})$, and we have the fibration $\mathbb{C}P^\infty \hookrightarrow \mathrm{K}(2,\, \mathbb{Q}) \to \mathrm{K}(2,\, \mathbb{Q}/\mathbb{Z})$ corresponding to the exact sequence of groups in the title of this subsection. Therefore (14) can be obtained from (15) with the help of the "factorization" by $\mathbb{C}P^\infty$.

Note that the sequence (15) relates something "integer" ($\widehat{\mathrm{Gr}}$), something "rational" ($\mathrm{B}\,\mathcal{GL}(\mathbb{C})$) and something "finite" ($\mathrm{B}\,\mathfrak{G}$). Indeed, the sequence of homotopy groups $\pi_r(\widehat{\mathrm{Gr}}) \overset{\widehat{\mathbf{T}}'_*}{\to} \pi_r(\mathrm{B}\,\mathcal{GL}(\mathbb{C})) \overset{\widehat{\mathbf{F}}'_*}{\to} \pi_r(\mathrm{B}\,\mathfrak{G})$ coincides with $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ for even r and with $0 \to 0 \to 0$ for odd r.

4 A generalization of the topological Brauer group

4.1 More about H-spaces $\mathrm{B}\,\mathcal{GL}(\mathbb{C})$ and $\mathrm{B}\,\mathcal{PGL}(\mathbb{C})$

Recall that we consider the spaces $\mathrm{B}\,\mathcal{GL}(\mathbb{C})$ and $\mathrm{B}\,\mathcal{PGL}(\mathbb{C})$ as H-spaces with respect to the multiplication induced by the tensor product of vector GL-bundles and PGL-bundles with fiber a matrix algebra, respectively.

Consider $\prod_{q\geq 1} \mathrm{K}(2q,\, \mathbb{Q})$ as an H-space with respect to the multiplication induced by the product of formal series of the form $1 + x_1 + x_2 + \dots,\ x_i \in H^{2i}(X;\, \mathbb{Q})$. We claim that H-spaces $\mathrm{B}\,\mathcal{GL}(\mathbb{C})$ and $\prod_{q\geq 1} \mathrm{K}(2q,\, \mathbb{Q})$ are isomorphic to each other. Indeed, it follows from the identity $ch(\xi \otimes \eta) = ch(\xi)ch(\eta)$ for any vector bundles $\xi,\, \eta$, where ch is the Chern character.

Note that for the special linear group SL we have similar isomorphism

$$\mathrm{B}\,\mathcal{SL}(\mathbb{C}) \stackrel{ch}{\to} \prod_{q\geq 2} \mathrm{K}(2q,\,\mathbb{Q}),$$

where $\prod_{q\geq 2} \mathrm{K}(2q,\,\mathbb{Q})$ is considered as an H-space with respect to the multiplication induced by the product of formal series of the form $1 + x_2 + x_3 + \ldots,\ x_i \in H^{2i}(X;\,\mathbb{Q})$.

Lemma 52. The H-space $\mathrm{B}\,\mathcal{PGL}(\mathbb{C})$ is isomorphic to the product of H-spaces $\mathrm{K}(2,\,\mathbb{Q}/\mathbb{Z}) \times \prod_{q\geq 2} \mathrm{K}(2q,\,\mathbb{Q})$.

Proof. Consider the commutative diagram of the H-spaces:

$$\begin{tikzcd}
\mathrm{B}\,\mathcal{PGL}(\mathbb{C}) \arrow[r, "ch"] & \prod_{q\geq 2} \mathrm{K}(2q,\,\mathbb{Q}) \\
\mathrm{B}\,\mathcal{SL}(\mathbb{C}). \arrow[u] \arrow[ur, "\cong"] &
\end{tikzcd}$$

It follows from the last isomorphism in (16) that $\ker(ch)$ coincides with the H-space $\mathrm{K}(2,\,\mathbb{Q}/\mathbb{Z})$. $\square$

4.2 Reminder: the classical Brauer group

Let X be a finite CW-complex. Consider the set of isomorphism classes of locally trivial bundles over X with fiber $M_k(\mathbb{C})$ with an arbitrary integer $k > 1$. On the set of such bundles consider the following equivalence relation:

$$A_k \sim B_l \;\Leftrightarrow A_k \otimes \widetilde{M}_m \cong B_l \otimes \widetilde{M}_n \quad \text{for some } m,\, n > 1, \tag{17}$$

where by $\widetilde{M}_r$ we denote the trivial bundle over X with fiber $M_r(\mathbb{C})$. By $\{C_m\}$ denote the stable equivalence class of C_m. It can easily be checked that the product $\{A_k\} \circ \{B_l\} := \{A_k \otimes B_l\}$ is well defined and it equips the set of stable equivalence classes with the structure of Abelian group. We denote this group by $\widetilde{\mathrm{AB}}(X)$.

The H-space $\mathrm{B}\,\mathcal{PGL}(\mathbb{C})$ is a represented space for the homotopy functor $X \mapsto \widetilde{\mathrm{AB}}(X)$ on the category of finite CW-complexes. Since we take the direct limit $\mathrm{B}\,\mathcal{PGL}(\mathbb{C}) = \varinjlim_k \mathrm{BPGL}_k(\mathbb{C})$ by the maps of classifying spaces $\mathrm{BPGL}_k(\mathbb{C}) \stackrel{i_{k,\,m}}{\longrightarrow} \mathrm{BPGL}_{km}(\mathbb{C})$ such that $i^*_{k,\,m}(A^{univ}_{km}) = A^{univ}_k \otimes \widetilde{M}_m$, where by A^{univ}_r we denote the universal $M_r(\mathbb{C})$-bundle over $\mathrm{BPGL}_r(\mathbb{C})$, the stable equivalence relation (17) appears.

Now consider more coarse stable equivalence relation, where

$$A_k \sim B_l \;\Leftrightarrow A_k \otimes \mathrm{End}(\xi_m) \cong B_l \otimes \mathrm{End}(\eta_n) \tag{18}$$

for some vector bundles ξ_m and η_n over X of rank $m,\, n$, respectively. By $[C_m]$ we denote the stable equivalence class of $M_m(\mathbb{C})$-bundle C_m. The tensor product of bundles induces a group structure on the set of such stable equivalence classes. It is just the classical topological Brauer group $\mathbf{Br}(X)$.

Let us give a homotopic description of $\mathbf{Br}(X)$. Consider the fibration

$$\begin{array}{ccc} \mathbb{C}P^\infty & \hookrightarrow & \mathrm{BGL}_k(\mathbb{C}) \\ & & \downarrow \\ & & \mathrm{BPGL}_k(\mathbb{C}) \end{array} \tag{19}$$

corresponding to the exact sequence of groups

$$1 \to \mathbb{C}^* \to \mathrm{GL}_k(\mathbb{C}) \to \mathrm{PGL}_k(\mathbb{C}) \to 1.$$

After taking the limit with $k \to \infty$, the above fibration gives us the fibration

$$\begin{array}{ccc} \mathbb{C}P^\infty & \overset{i}{\hookrightarrow} & \mathrm{B}\,\mathcal{GL}(\mathbb{C}) \\ & & \downarrow p \\ & & \mathrm{B}\,\mathcal{PGL}(\mathbb{C}), \end{array} \quad \text{i.e.} \quad \begin{array}{ccc} \mathbb{C}P^\infty & \hookrightarrow & \prod_{q\geq 1}\mathrm{K}(\mathbb{Q},\, 2q) \\ & & \downarrow \\ & & \mathrm{K}(\mathbb{Q}/\mathbb{Z},\, 2)\times \prod_{q\geq 2}\mathrm{K}(\mathbb{Q},\, 2q). \end{array} \tag{20}$$

It is easy to see that $\mathbf{Br}(X) = \operatorname{coker}\{p_*\colon [X;\, \mathrm{B}\,\mathcal{GL}(\mathbb{C})] \to [X;\, \mathrm{B}\,\mathcal{PGL}(\mathbb{C})]\}$, i.e. $\mathbf{Br}(X) = \operatorname{coker}\{[X;\, \mathrm{K}(\mathbb{Q},\, 2)] \to [X;\, \mathrm{K}(\mathbb{Q}/\mathbb{Z},\, 2)]\} = \operatorname{coker}\{H^2(X;\, \mathbb{Q}) \to H^2(X;\, \mathbb{Q}/\mathbb{Z})\}$. Using the exact sequence of cohomology groups induced by the sequence of coefficients

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0,$$

we see that $\operatorname{coker}\{H^2(X;\, \mathbb{Q}) \to H^2(X;\, \mathbb{Q}/\mathbb{Z})\} = \operatorname{im}\{H^2(X;\, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\delta} H^3(X;\, \mathbb{Z})\}$ (here δ is the coboundary homomorphism), i.e. $H^3_{tors}(X;\, \mathbb{Z})$.

The explicit form of the isomorphism $\mathbf{Br}(X) \cong H^3_{tors}(X;\, \mathbb{Z})$ can be described as follows. Recall that the structural group of a bundle A_k with fiber $M_k(\mathbb{C})$ is $\mathrm{Aut}(M_k(\mathbb{C})) = \mathrm{PGL}_k(\mathbb{C})$. The obstruction theory asserts that the first (and unique!) obstruction for the lifting of $\mathrm{PGL}_k(\mathbb{C})$-bundle to a $\mathrm{GL}_k(\mathbb{C})$-bundle belongs to the group $H^3(X;\, \pi_2(\mathbb{C}P^\infty))$ (see fibration (19)). Therefore the assignment

$$A_k \mapsto \{\text{the first obstruction for the lifting}\}$$

gives us the required description of the isomorphism $\mathbf{Br}(X) \cong H^3_{tors}(X;\, \mathbb{Z})$.

Thus, any element of the group $H^3_{tors}(X;\, \mathbb{Z})$ can be realized as an obstruction for the lifting of some PGL-bundle to a GL-bundle.

Remark 53. Now suppose A_k has a lifting ξ_k (i.e. $\mathrm{End}(\xi_k) = A_k$). Then the obstruction theory says that ξ_k is determined by A_k uniquely up to taking the tensor product with a line bundle ζ over X. Indeed, $H^2(X;\, \pi_2(\mathbb{C}P^\infty)) = H^2(X;\, \mathbb{Z})$ is isomorphic to the group of line bundles with respect to the tensor product; from the other hand, for any line bundle ζ we have $\mathrm{End}(\xi_k \otimes \zeta) = \mathrm{End}(\xi_k)$.

Remark 54. Note that for any fixed integer $k > 1$ one can develop the corresponding theory of bundles with fibers of the form $M_{k^n}(\mathbb{C})$ for arbitrary $n \in \mathbb{N}$. In this way one can define the k-component of the Brauer group $\mathbf{Br}_k(X) = \operatorname{coker}\{p_{k*}\colon [X;\, \mathrm{BGL}_{k^\infty}(\mathbb{C})] \to [X;\, \mathrm{BPGL}_{k^\infty}(\mathbb{C})]\}$ (which is the k-primary subgroup in the "whole" group $\mathbf{Br}(X)$).

4.3 The generalized Brauer group

Let us return to fibration (15):

$$\widehat{\mathrm{Gr}} \xrightarrow{\widehat{\mathbf{T}}'} \mathrm{B}\,\mathcal{GL}(\mathbb{C}) \xrightarrow{\widehat{\mathbf{F}}'} \mathrm{B}\,\mathfrak{G}.$$

By Fan we denote the homotopy functor to the category of Abelian groups represented by the H-space $\mathrm{B}\,\mathfrak{G}$, i.e. $X \overset{\mathrm{Fan}}{\mapsto} [X;\, \mathrm{B}\,\mathfrak{G}]$.

Thus, we have exact sequence of groups

$$[X;\, \widehat{\mathrm{Gr}}] \xrightarrow{\widehat{\mathbf{T}}'_*} [X;\, \mathrm{B}\,\mathcal{GL}(\mathbb{C})] \xrightarrow{\widehat{\mathbf{F}}'_*} \mathrm{Fan}(X).$$

Definition 55. The *generalized topological Brauer group* $\mathbf{GBr}(X)$ of X is $\operatorname{coker} \widehat{\mathbf{F}}'_* = \mathrm{Fan}(X)/\operatorname{im}(\widehat{\mathbf{F}}'_*)$.

Remark 56. Since $\mathrm{Fan}(X)$ is a finite (Abelian) group for a finite CW-complex X, so is $\mathbf{GBr}(X)$. Moreover, the k-primary component $\mathbf{GBr}_k(X)$ is just $\operatorname{coker} \widehat{\mathbf{F}}'_{k^\infty *} = \mathrm{Fan}^k(X)/\operatorname{im}(\widehat{\mathbf{F}}'_{k^\infty *})$.

Let us also remark that fibration (15) is closely connected with fibration (20):

$$\mathbb{C}P^\infty \overset{i}{\hookrightarrow} \mathrm{B}\,\mathcal{GL}(\mathbb{C}) \xrightarrow{p} \mathrm{B}\,\mathcal{PGL}(\mathbb{C}).$$

In particular, the space $\widehat{\mathrm{Gr}} \cong \mathrm{BU}_\otimes$ represents the homotopy functor

$$X \mapsto \{\text{ the group of virtual U-bundles of virtual dimension 1 over } X\},$$

where the group operation is the tensor product of virtual bundles of virtual dimension 1, therefore it is a counterpart of the H-space $\mathbb{C}P^\infty$, which represents the group (of isomorphism classes) of *geometric* line bundles with respect to the operation induced by the tensor product.

Now consider fibration (14):

$$\mathrm{Gr} \xrightarrow{\widehat{\mathbf{T}}} \mathrm{B}\,\mathcal{PGL}(\mathbb{C}) \xrightarrow{\widehat{\mathbf{F}}} \mathrm{B}\,\mathfrak{G}.$$

By definition, put $\overline{\mathbf{GBr}}(X) := \operatorname{coker}\{\widehat{\mathbf{F}}_*\colon [X;\, \mathrm{B}\,\mathcal{PGL}(\mathbb{C})] \to [X;\, \mathrm{B}\,\mathfrak{G}]\}$. Using the fibrations (15) and (14), one can easily prove the following proposition.

Proposition 57. *There is an exact sequence of functors*

$$0 \to \mathbf{Br} \to \mathbf{GBr} \to \overline{\mathbf{GBr}} \to 0.$$

In other words, $\mathbf{GBr}$ *is an extension of the classical Brauer group* $\mathbf{Br}$ *by* $\overline{\mathbf{GBr}}$.

Proof follows from the commutative diagram

$$\begin{tikzcd}
\mathrm{B}\,\mathcal{GL}(\mathbb{C}) \arrow[d, "\mathbb{C}P^\infty"'] \arrow[dr, "\widehat{\mathbf{F}}'"] & \\
\mathrm{B}\,\mathcal{PGL}(\mathbb{C}) \arrow[r, "\widehat{\mathbf{F}}"] & \mathrm{B}\,\mathfrak{G}.
\end{tikzcd}$$ □

Now we want to obtain a description of the generalized Brauer group in terms of equivalence classes of fan bundles with respect to some equivalence relation which is a counterpart of relation (18).

Recall that the H-space $\mathrm{B}\,\mathfrak{G}_{k^\infty}$ is a classifying space for equivalence classes of fan bundles with fibers of the form $\mathbb{M}_{k^m,\, l^n}$ (for some l coprime with k) with respect to the following equivalence relation:

$$\mathfrak{A}_{k^m,\, l^n} \sim \mathfrak{B}_{k^r,\, l^s} \Leftrightarrow \mathfrak{A}_{k^m,\, l^n} \otimes \widetilde{\mathbb{M}}_{k^t,\, l^u} \cong \mathfrak{B}_{k^r,\, l^s} \otimes \widetilde{\mathbb{M}}_{k^v,\, l^w}, \tag{23}$$

where by $\widetilde{\mathbb{M}}_{k^m,\, l^n}$ we denote the trivial $\mathbb{M}_{k^m,\, l^n}$-bundle over X. This equivalence relation is a counterpart of relation (17).

The counterpart of relation (18) is the following one:

$$\mathfrak{A}_{k^m,\, l^n} \sim \mathfrak{B}_{k^r,\, l^s} \Leftrightarrow \text{ there are vector bundles } \xi_{k^t},\ \eta_{k^v}$$

$$\text{with fibers } \mathbb{C}^{k^t},\ \mathbb{C}^{k^v},\ \text{respectively, and integers } u,\, w \geq 1 \text{ such that}$$

$$\mathfrak{A}_{k^m,\, l^n} \otimes \mathbf{F}^X_{k^t,\, l^u}(\mathrm{End}(\xi_{k^t})) \cong \mathfrak{B}_{k^r,\, l^s} \otimes \mathbf{F}^X_{k^v,\, l^w}(\mathrm{End}(\eta_{k^v})) \tag{24}$$

(recall that $\mathbf{F}^X_{k,\, l}$ is the functor $\mathfrak{AB}_k(X) \to \mathfrak{Fan}_{k,\, l}(X)$, i.e. $\mathbf{F}^X_{k,\, l}(\mathrm{End}(\xi_k))$ is a fan bundle with fiber $\mathbb{M}_{k,\, l}$). It is natural to denote the composite functor $\xi_k \mapsto \mathrm{End}(\xi_k) \mapsto \mathbf{F}^X_{k,\, l}(\mathrm{End}(\xi_k))$ by ${\mathbf{F}'}^X_{k,l}$.

Note that the tensor product of fan bundles induces a well-defined operation on equivalence classes of equivalence relation (24). Note also that the equivalence classes of bundles of the form $\mathbf{F}^X_{k^t,\, l^u}(\mathrm{End}(\xi_{k^t}))$ form the image of the homomorphism $\widehat{\mathbf{F}}'_{k^\infty *}\colon [X;\, \mathrm{BGL}_{k^\infty}(\mathbb{C})] \to [X;\, \mathrm{B}\,\mathfrak{G}_{k^\infty}]$. Now the proof of the following proposition is clear.

Proposition 58. *The k-primary component $\mathbf{GBr}_k(X)$ of the generalized Brauer group $\mathbf{GBr}(X)$ coincides with the group of equivalence classes of equivalence relation (24).*

The similar description can be given for the group $\overline{\mathbf{GBr}}(X)$. More precisely, consider the following equivalence relation:

$$\mathfrak{A}_{k^m,\, l^n} \sim \mathfrak{B}_{k^r,\, l^s} \Leftrightarrow \text{ there are bundles } A_{k^t},\ B_{k^v}$$

$$\text{with fibers } M_{k^t}(\mathbb{C}),\ M_{k^v}(\mathbb{C}),\ \text{respectively, and integers } u,\, w \geq 1 \text{ such that}$$

$$\mathfrak{A}_{k^m,\, l^n} \otimes \mathbf{F}^X_{k^t,\, l^u}(A_{k^t}) \cong \mathfrak{B}_{k^r,\, l^s} \otimes \mathbf{F}^X_{k^v,\, l^w}(B_{k^v}). \tag{25}$$

Note that the equivalence classes of bundles of the form $\mathbf{F}^X_{k^t,\, l^u}(A_{k^t})$ form the image of the homomorphism $\widehat{\mathbf{F}}_{k^\infty *}\colon [X;\, \mathrm{BPGL}_{k^\infty}(\mathbb{C})] \to [X;\, \mathrm{B}\,\mathfrak{G}_{k^\infty}]$.

Proposition 59. *The k-primary component $\overline{\mathbf{GBr}}_k(X)$ of the group $\overline{\mathbf{GBr}}(X)$ coincides with the group of equivalence classes of relation (25).*

Remark 60. (compare with Remark 53). Suppose a fan bundle $\mathfrak{A}_{k,l}$ over X has a lifting A_k (i.e. $\mathfrak{A}_{k,l} = \mathbf{F}^X_{k,l}(A_k)$ for some $M_k(\mathbb{C})$-bundle A_k). Then the bundle A_k is determined by the equivalence class (23) of $\mathfrak{A}_{k,l}$ up to taking the tensor product with the core of some FAB. In particular, the liftings of the stable equivalence class (23) of a trivial fan bundle are precisely the cores of FABs over X; thus, we also see that FABs play the same role as (geometric) line bundles. This result follows from fibration (14). The same result is true for fibration (15).

Suppose a finite CW-complex X is obtained by applying the forgetful functor to some algebraic variety $\widetilde{X}$ over $\mathbb{C}$. Then the group $\mathbf{Br}(X)$ is not only a homotopy invariant of X but also a birational invariant of $\widetilde{X}$.

Conjecture 61. The group $\mathbf{GBr}(X)$ is a birational invariant of $\widetilde{X}$.

5 Appendix 1: The functor represented by the H-space Gr

In this appendix we give a brief survey of the stable theory of floating algebra bundles, for more details see [3].

5.1 Category $\mathfrak{FAB}(X)$

Let X be a finite CW-complex. By $\widetilde{M_n}$ denote the trivial bundle (over X) with a fiber $M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ is the algebra of all $n \times n$ matrices over $\mathbb{C}$.

Definition 62. Let A_k $(k > 1)$ be a locally trivial bundle over X with a fiber $M_k(\mathbb{C})$. Assume that there is a bundle map μ

$$\begin{tikzcd} A_k \arrow[rr, "\mu"] \arrow[dr] & & \widetilde{M_{kl}} \arrow[dl] \\ & X & \end{tikzcd}$$

such that for any point $x \in X$ it embeds the fiber $(A_k)_x \cong M_k(\mathbb{C})$ into the fiber $(\widetilde{M_{kl}})_x \cong M_{kl}(\mathbb{C})$ as a central simple subalgebra, and the positive integers k, l are relatively prime (i.e. their greatest common divisor $(k,l) = 1$) . Then the triple $(A_k, \mu, \widetilde{M_{kl}})$ is called a *floating algebra bundle* (abbrev. FAB) over X.

Remark 63. Let A be a central simple algebra over a field $\mathbb{K}$, $B \subset A$ a central simple subalgebra in A. It is well known that the centralizer $\mathrm{Z}_A(B)$ of B in A is a central simple subalgebra in A again, moreover, the equality $A = B \underset{\mathbb{K}}{\otimes} \mathrm{Z}_A(B)$ holds. Taking centralizers for all fibers of the subbundle $A_k \subset \widetilde{M_{kl}}$ in the corresponding fibers of the trivial bundle $\widetilde{M_{kl}}$, we get the complementary subbundle B_l with fiber $M_l(\mathbb{C})$ together with its embedding $\nu \colon B_l \hookrightarrow M_{kl}(\mathbb{C})$. Thus, we have the complementary subbundle $\nu(B_l) \subset \widetilde{M_{kl}}$, where B_l is a locally trivial bundle with fiber $M_l(\mathbb{C})$, and ν is its evident embedding into $\widetilde{M}_{kl}$. Moreover, $A_k \otimes B_l = \widetilde{M}_{kl}$.

Conversely, to a given pair $(A_k,\, B_l)$ consisting of $M_k(\mathbb{C})$-bundle A_k and $M_l(\mathbb{C})$-bundle B_l over X such that $A_k \otimes B_l = \widetilde{M}_{kl}$, we can construct a unique triple $(A_k,\, \mu,\, \widetilde{M}_{kl})$, where μ is the embedding $A_k \hookrightarrow A_k \otimes B_l,\ a \mapsto a \otimes 1_{B_l}$.

Definition 64. A *morphism from a FAB $(A_k,\, \mu,\, \widetilde{M}_{kl})$ to a FAB $(C_m,\, \nu,\, \widetilde{M}_{mn})$* over X is a pair $(f,\, g)$ of bundle maps $f\colon A_k \hookrightarrow C_m,\ g\colon \widetilde{M}_{kl} \hookrightarrow \widetilde{M}_{mn}$ such that

- $f,\, g$ are fiberwise homomorphisms of algebras (i.e. they are embeddings in fact);
- the square diagram
$$\begin{array}{ccc} \widetilde{M}_{kl} & \xrightarrow{g} & \widetilde{M}_{mn} \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ A_k & \xrightarrow{f} & C_m \end{array}$$
commutes;
- let $B_l \subset \widetilde{M}_{kl},\ D_n \subset \widetilde{M}_{mn}$ be the complementary subbundles for $A_k,\ C_m$, respectively (see the remark above), then g maps B_l into D_n.

Note that a morphism $(f,\, g)\colon (A_k,\, \mu,\, \widetilde{M}_{kl}) \to (C_m,\, \nu,\, \widetilde{M}_{mn})$ exists only if $k|m,\ l|n$.

In particular, an *isomorphism between FABs $(A_k,\, \mu,\, \widetilde{M}_{kl})$ and $(C_k,\, \nu,\, \widetilde{M}_{kl})$* is a pair of bundle maps $f : A_k \to C_k$, $g : \widetilde{M}_{kl} \to \widetilde{M}_{kl}$ which are fiberwise isomorphisms of algebras such that the diagram
$$\begin{array}{ccc} \widetilde{M}_{kl} & \xrightarrow{g} & \widetilde{M}_{kl} \\ {\scriptstyle\mu}\uparrow & & \uparrow{\scriptstyle\nu} \\ A_k & \xrightarrow{f} & C_k \end{array}$$
commutes.

Clearly, FABs over X with morphisms just defined form a category, which we shall denote by $\mathfrak{FAB}(X)$.

For a continuous map $\varphi\colon X \to Y$ we have the natural transformation $\varphi^*\colon \mathfrak{FAB}(Y) \to \mathfrak{FAB}(X)$.

5.2 Classifying spaces

For any pair $k,\, l > 1$ there is the "matrix Grassmannian" $\mathrm{Gr}_{k,l}$, parameterizing k-subalgebras (i.e. those isomorphic to $M_k(\mathbb{C})$) in the fixed matrix algebra $M_{kl}(\mathbb{C})$. As a homogeneous space it can be represented as follows:

$$\mathrm{Gr}_{k,l} = \mathrm{PGL}_{kl}(\mathbb{C})\big/\mathrm{PGL}_k(\mathbb{C}) \otimes \mathrm{PGL}_l(\mathbb{C}),$$

where by $\mathrm{PGL}_k(\mathbb{C}) \otimes \mathrm{PGL}_l(\mathbb{C})$ we denote the image of the embedding $\mathrm{PGL}_k(\mathbb{C}) \times \mathrm{PGL}_l(\mathbb{C}) \hookrightarrow \mathrm{PGL}_{kl}(\mathbb{C})$ induced by the Kronecker product of matrices.

By $\widetilde{\mathcal{M}}_{kl}$ denote the trivial bundle $\mathrm{Gr}_{k,l} \times M_{kl}(\mathbb{C})$. There is a *tautological FAB* $(\mathcal{A}_{k,l},\, \mu\,, \widetilde{\mathcal{M}}_{kl})$ over $\mathrm{Gr}_{k,l}$ which can be defined as follows: the fiber $(\mathcal{A}_{k,l})_x$ over $x \in \mathrm{Gr}_{k,l}$ is the k-subalgebra in $M_{kl}(\mathbb{C}) = (\widetilde{\mathcal{M}}_{kl})_x$ corresponding to this point.

Remark 65. The noncompact space $\mathrm{Gr}_{k,l}$ can be replaced by a homotopy equivalent compact one $\mathrm{Gr}^U_{k,l}$. More precisely, let $\mathrm{PU}(n)$ be the projective unitary group, i.e. the quotient group $\mathrm{U}(n)/\{\exp(i\varphi)E_n \mid \varphi \in \mathbb{R}\}$. Then

$$\mathrm{Gr}^U_{k,l} = \mathrm{PU}(kl)/\mathrm{PU}(k) \otimes \mathrm{PU}(l)$$

(which is the subspace in $\mathrm{Gr}_{k,l}$ corresponding to the "unitary" subalgebras) with a tautological FAB $(\mathcal{A}^U_{k,l},\, \mu^U,\, \widetilde{\mathcal{M}}^U_{kl})$ over it.

Let $\Psi_{k,l}(X)$ be the set of isomorphism classes of FABs of the form $(A_k,\, \mu,\, \widetilde{M}_{kl})$ over X.

Proposition 66. *If* $\dim X \leq 2\min\{k,\, l\}$*, then the assignment*

$$[X,\, \mathrm{Gr}_{k,l}] \to \Psi_{k,l}(X), \quad \varphi \mapsto \varphi^*(\mathcal{A}_{k,l},\, \mu,\, \widetilde{\mathcal{M}}_{kl})$$

is a bijection.

5.3 Stabilization

Define the product $\circ$ of two FABs over X $(A_k,\, \mu,\, \widetilde{M}_{kl}),\ (B_m,\, \nu,\, \widetilde{M}_{mn})$ such that $(km,\, ln) = 1$ as

$$(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (B_m,\, \nu,\, \widetilde{M}_{mn}) = (A_k \otimes B_m,\, \mu \otimes \nu,\, \widetilde{M}_{kl} \otimes \widetilde{M}_{mn})$$

(notice that $\widetilde{M}_{kl} \otimes \widetilde{M}_{mn} = \widetilde{M}_{klmn}$).

Definition 67. A FAB of the form $(\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$ is called *trivial* if the map $\tau\colon \widetilde{M}_k \to \widetilde{M}_{kl}$ acts as follows,

$$X \times M_k(\mathbb{C}) \to X \times M_{kl}(\mathbb{C}), \qquad (x,\, T) \mapsto (x,\, T \otimes E_l)$$

(under some choice of trivializations on $\widetilde{M}_k$ and $\widetilde{M}_{kl}$) for any point $x \in X$, where E_l is the unit $l \times l$-matrix and $T \otimes E_l$ denotes the Kronecker product of matrices. In other words, the bundle $\widetilde{M}_k$ is embedded into $\widetilde{M}_{kl}$ as a fixed subalgebra.

Definition 68. Two FABs $(A_k,\, \mu,\, \widetilde{M}_{kl})$ and $(B_m,\, \nu,\, \widetilde{M}_{mn})$ over X are said to be *stable equivalent*, if there is a sequence of pairs $\{t_i,\, u_i\} \in \mathbb{N}^2$, $1 \leq i \leq s$ such that

- $\{t_1,\, u_1\} = \{k,\, l\},\ \{t_s,\, u_s\} = \{m,\, n\}$;
- $(t_i t_{i+1},\, u_i u_{i+1}) = 1$ if $s > 1$, $1 \leq i \leq s-1$,

and a corresponding sequence of FABs $(A_{t_i},\, \mu_i,\, \widetilde{M}_{t_i u_i})$ over X such that

- $(A_{t_1},\, \mu_1,\, \widetilde{M}_{t_1 u_1}) = (A_k,\, \mu,\, \widetilde{M}_{kl}),\ (A_{t_s},\, \mu_s,\, \widetilde{M}_{t_s u_s}) = (B_m,\, \nu,\, \widetilde{M}_{mn})$;

- $(A_{t_i},\, \mu_i,\, \widetilde{M}_{t_iu_i}) \circ (\widetilde{M}_{t_{i+1}},\, \tau,\, \widetilde{M}_{t_{i+1}u_{i+1}}) \cong (A_{t_{i+1}},\, \mu_{i+1},\, \widetilde{M}_{t_{i+1}u_{i+1}}) \circ (\widetilde{M}_{t_i},\, \tau,\, \widetilde{M}_{t_iu_i})$,

where $1 \leq i \leq s-1$ and $(\widetilde{M}_{t_i},\, \tau,\, \widetilde{M}_{t_iu_i})$ are trivial FABs.

By $\widetilde{\mathrm{AB}}^1(X)$ denote the set of stable equivalence classes of FABs over X.

The following theorem justifies the previous definition.

Theorem 69. *1) For all sequences of pairs of positive integers $\{k_j,\, l_j\}_{j\in\mathbb{N}}$ such that*

$$\text{(i)}\quad k_j,\, l_j \to \infty; \quad \text{(ii)}\ k_j|k_{j+1},\ l_j|l_{j+1}; \quad \text{(iii)}\ (k_j,\, l_j) = 1$$

for every j, the corresponding direct limits $\varinjlim_j \mathrm{Gr}_{k_j,l_j}$ are homotopy-equivalent. This unique homotopy type we denote by Gr.
2) Gr *is a classifying space for stable equivalence classes of FABs over a finite CW-complex X. In other words, the functor $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ from the homotopy category of finite CW-complexes to the category $\mathfrak{Set}$ is represented by the space* Gr.

The proof is based on Proposition 66 and on the following lemma (recall that we consider embeddings $\mathrm{Gr}_{k,l} \to \mathrm{Gr}_{m,n}$ induced by homomorphisms of algebras $M_{kl}(\mathbb{C}) \hookrightarrow M_{mn}(\mathbb{C})$; in particular, $k|m,\ l|n$).

Lemma 70. If $(m,n) = 1$ ($\Rightarrow (k,l) = 1$), then the embedding

$$\mathrm{Gr}_{k,l} \to \mathrm{Gr}_{m,n} \tag{31}$$

is a homotopy equivalence in dimensions $\leq 2\min\{k,\, l\}$.

5.4 The group structure

Let $(A_k,\, \mu,\, \widetilde{M}_{kl})$ be a FAB over X. By $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$ we denote its stable equivalence class. Define the product $\diamond$ of two classes $[(A_k,\, \mu,\, \widetilde{M}_{kl})],\ [(B_m,\, \nu,\, \widetilde{M}_{mn})]$ as

$$[(A_k,\, \mu,\, \widetilde{M}_{kl})] \diamond [(B_m,\, \nu,\, \widetilde{M}_{mn})] = [(A_k,\, \mu,\, \widetilde{M}_{kl}) \circ (B_m,\, \nu,\, \widetilde{M}_{mn})]$$

$$= [(A_k \otimes B_m,\, \mu \otimes \nu,\, \widetilde{M}_{klmn})].$$

Clearly, this product is well defined if $(km, ln) = 1$. The following lemma allows us to reject this restriction.

Lemma 71. For any pair $\{k,\, l\}$ such that (i) $(k,l)\ = 1$, (ii) $2\min\{k,\, l\} \geq \dim X$, any stable equivalence class of FABs over X has a representative of the form $(A_k,\, \mu,\, \widetilde{M}_{kl})$.

Clearly, the product $\diamond$ is associative, commutative, and has identity element $[(\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})]$, where $(\widetilde{M}_k,\, \tau,\, \widetilde{M}_{kl})$ is the trivial FAB. Moreover, for any class $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$ there exists the inverse element. In order to find it, let us recall the following fact. The centralizer $\mathrm{Z}_P(Q)$ of a central simple subalgebra Q in a central simple algebra P (over some field $\mathbb{K}$) is a central simple subalgebra again, moreover, the equality $P = Q \underset{\mathbb{K}}{\otimes} \mathrm{Z}_P(Q)$ holds.

Therefore by taking centralizers for every fiber of the subbundle A_k in $\widetilde{M_{kl}}$, we obtain the complementary subbundle B_l with fiber $M_l(\mathbb{C})$ together with its embedding $\nu\colon B_l \hookrightarrow M_{kl}(\mathbb{C})$ into the trivial bundle. Moreover, $A_k \otimes B_l = \widetilde{M_{kl}}$. It is not hard to prove that $[(B_l,\, \nu,\, \widetilde{M_{kl}})]$ is the inverse element for $[(A_k,\, \mu,\, \widetilde{M_{kl}})]$. Thus, the functor $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ takes values in the category of Abelian groups $\mathfrak{Ab}$.

Proposition 72. *The space* Gr *can be equipped with an H-space structure such that there is a natural equivalence of functors $X \mapsto [X,\, \mathrm{Gr}]$ and $X \mapsto \widetilde{\mathrm{AB}}^1(X)$ taking values in the category* $\mathfrak{Ab}$.

5.5 One interesting property of a bundle A_k which is a part of FAB $(A_k,\, \mu,\, \widetilde{M_{kl}})$

Definition 73. For a FAB $(A_k,\, \mu,\, \widetilde{M_{kl}})$, the locally trivial $\mathrm{Aut}(M_k(\mathbb{C})) \cong \mathrm{PGL}_k(\mathbb{C})$-bundle A_k is said to be a *core* of $(A_k,\, \mu,\, \widetilde{M_{kl}})$.

Lemma 74. If A_k is the core of some FAB $(A_k,\, \mu,\, \widetilde{M_{kl}})$, then its structural group can be reduced from $\mathrm{Aut}\, M_k(\mathbb{C}) \cong \mathrm{PGL}_k(\mathbb{C})$ to $\mathrm{SL}_k(\mathbb{C})$.

Lemma 75. Let X be a finite CW-complex. Suppose $\dim X \leq 2\min\{k,\, m\}$; then the following conditions are equivalent:

- A_k is the core of some FAB over X;
- for arbitrary m such that $2m \geq \dim X$ there is a bundle B_m with fiber $M_m(\mathbb{C})$ such that $A_k \otimes \widetilde{M_m} \cong B_m \otimes \widetilde{M_k}$;
- $A_k \otimes \widetilde{M_m} \cong B_m \otimes \widetilde{M_k}$ for some locally trivial bundle B_m with fiber $M_m(\mathbb{C})$, where $(k, m) = 1$.

Moreover, for any pair of bundles $A_k,\, B_m$ such that $(k, m) = 1$ and $A_k \otimes \widetilde{M_m} \cong B_m \otimes \widetilde{M_k}$, there exists a unique stable equivalence class of FABs over X which has (for sufficient large n, $(km, n) = 1$) FABs of the forms $(A_k,\, \mu,\, \widetilde{M_{kn}})$, $(B_m,\, \nu,\, \widetilde{M_{mn}})$ as representatives (for some embeddings $\mu,\, \nu$).

5.6 Localization

Let X be a finite CW-complex, let $k \geq 2$ be a fixed integer. The set of isomorphism classes of bundles of the form A_{k^m} (for arbitrary $m \in \mathbb{N}$) over X with fiber $M_{k^m}(\mathbb{C})$ is a monoid with respect to the operation $\otimes$ (with the identity element $M_{k^0}(\mathbb{C}) \cong \mathbb{C}$).

Let us consider the following equivalence relation

$$A_{k^m} \sim B_{k^n} \Longleftrightarrow \exists r, s \in \mathbb{N} \text{ such that } A_{k^m} \otimes \widetilde{M_{k^r}} \cong B_{k^n} \otimes \widetilde{M_{k^s}} \tag{32}$$

($\Rightarrow m + r = n + s$). The set of equivalence classes $[A_{k^m}]$ of such bundles is a group with respect to the operation induced by $\otimes$. This group we denote by $\widetilde{\mathrm{AB}}^k(X)$.

Let us consider the direct limit $\varinjlim_n \mathrm{BPGL}_{k^n}(\mathbb{C})$ with respect to the maps induced by

$$\begin{array}{ccc} \mathrm{PGL}_{k^n}(\mathbb{C}) & \hookrightarrow & \mathrm{PGL}_{k^{n+1}}(\mathbb{C}), \\ A & \mapsto & A \otimes E_k. \end{array}$$

Clearly, the functor $X \mapsto \widetilde{\mathrm{AB}}^k(X)$ is represented by $\varinjlim_n \mathrm{BPGL}_{k^n}(\mathbb{C})$.

According to Lemma 71, for any stable equivalence class of FABs over X there is a representative of the form $(A_{k^m},\, \mu,\, \widetilde{M}_{(kl)^m})$. Therefore for any k we have the forgetful functor

$$\widehat{\mathbf{T}}^X_{k^\infty,\, l^\infty *}\colon \widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{AB}}^k(X), \quad [(A_{k^m},\, \mu,\, \widetilde{M}_{(kl)^m})] \mapsto [A_{k^m}].$$

This functor corresponds to the following diagram

$$\begin{tikzcd} {} & {} \\ \mathrm{Gr}_{k^2,\, l^2} \arrow[r] \arrow[u] & \mathrm{BPGL}_{k^2}(\mathbb{C}) \arrow[u] \\ \mathrm{Gr}_{k,\, l} \arrow[r] \arrow[u] & \mathrm{BPGL}_{k}(\mathbb{C}), \arrow[u] \end{tikzcd}$$

where the horizontal arrows are classifying maps for the cores $\mathcal{A}_{k,l}$ and $\mathcal{A}_{k^2,l^2}$ as $\mathrm{PGL}_k(\mathbb{C})$ and $\mathrm{PGL}_{k^2}(\mathbb{C})$-bundles, respectively.

The kernel of the homomorphism $\widetilde{\mathrm{AB}}^1(X) \to \widetilde{\mathrm{AB}}^k(X)$ is just the k-torsion subgroup in $\widetilde{\mathrm{AB}}^1(X)$.

5.7 Relation between $\widetilde{\mathrm{AB}}^1$ and $\widetilde{\mathrm{KSU}}$-theory

Recall that $\mathrm{BSU}_\otimes$ is the space BSU with the structure of H-space related to the tensor product of virtual SU-bundles of virtual dimension 1.

Theorem 76. *There is an H-space isomorphism* $\mathrm{Gr} \cong \mathrm{BSU}_\otimes$.

By $\widetilde{\mathrm{KSU}}(X)$ denote the reduced K-functor constructed by means of SU-bundles over X. Recall that $\widetilde{\mathrm{KSU}}(X)$ is a ring with the multiplication induced by the tensor product of bundles.

The previous theorem shows that the group $\widetilde{\mathrm{AB}}^1(X)$ is isomorphic to the multiplicative group of the ring $\widetilde{\mathrm{KSU}}(X)$, i.e. to the group ($X$ is a finite CW-complex) of elements of $\widetilde{\mathrm{KSU}}(X)$ with respect to the operation $\xi * \eta = \xi + \eta + \xi\eta$ ($\xi, \eta \in \widetilde{\mathrm{KSU}}(X)$).

This gives us a geometric description of the H-structure on $\mathrm{BSU}_\otimes$. For example, the construction of the inverse stable equivalence class $[(B_m,\, \nu,\, \widetilde{M}_{mn})]$ for a given one $[(A_k,\, \mu,\, \widetilde{M}_{kl})]$ is closely connected with taking centralizer for a subalgebra in a fixed matrix algebra.

5.8 A GL-version

Consider the canonical map $\mathrm{BGL}_k(\mathbb{C}) \to \mathrm{BPGL}_k(\mathbb{C})$ induced by the group homomorphism $\mathrm{GL}_k(\mathbb{C}) \to \mathrm{PGL}_k(\mathbb{C})$. By $\widehat{\mathrm{Gr}}_{k,\,l}$ denote the total space of the $\mathrm{Fr}_{k,\,l}$-fibration induced by the fibration $\mathrm{Gr}_{k,\,l} \xrightarrow{\mathrm{Fr}_{k,\,l}} \mathrm{BPGL}_k(\mathbb{C})$ and the map $\mathrm{BGL}_k(\mathbb{C}) \to \mathrm{BPGL}_k(\mathbb{C})$ (as ever, the integers $k,\,l$ are assumed to be coprime).

It follows easily that there is a $\mathbb{C}P^\infty$-fibration $\widehat{\mathrm{Gr}}_{k,\,l} \to \mathrm{Gr}_{k,\,l}$.

Consider the following morphism of $\mathrm{GL}_k(\mathbb{C})$-fibrations:

$$\begin{tikzcd}
 & \mathrm{GL}_k(\mathbb{C}) \arrow[r] & * \arrow[d] \\
\mathrm{GL}_k(\mathbb{C}) \arrow[ur, "="] \arrow[r] & \mathrm{Fr}_{k,\,l} \arrow[ur] \arrow[d] & \mathrm{BGL}_k(\mathbb{C}) \\
 & \widehat{\mathrm{Gr}}_{k,\,l}, \arrow[ur, "\widehat{\mathbf{T}}'_{k,\,l}"] &
\end{tikzcd}$$

where $\widehat{\mathbf{T}}'_{k,\,l}$ is a classifying map for the canonical $\mathrm{GL}_k(\mathbb{C})$-bundle over $\widehat{\mathrm{Gr}}_{k,\,l}$ and by $*$ we denote a contractible space. A simple computation with homotopy sequences of the fibrations shows that $\widehat{\mathbf{T}}'_{k,\,l*}\colon \pi_{2r}(\widehat{\mathrm{Gr}}_{k,\,l}) \to \pi_{2r}(\mathrm{BGL}_k(\mathbb{C}))$, $r \leq \min\{k,\,l\}$ is just the monomorphism $\mathbb{Z} \to \mathbb{Z},\ 1 \mapsto k\cdot 1$ (note that the odd-dimensional stable homotopy groups of both spaces are equal to 0). This implies that the limit map $\widehat{\mathbf{T}}'_{k^\infty}\colon \widehat{\mathrm{Gr}} \to \mathrm{BGL}_{k^\infty}(\mathbb{C})$ is just the localization at k (in the sense that k is invertible; in particular, $\mathrm{BGL}_{k^\infty}(\mathbb{C})$ is a $\mathbb{Z}[\frac{1}{k}]$-local space), where $\widehat{\mathrm{Gr}} := \varinjlim_{(k,l)=1} \widehat{\mathrm{Gr}}_{k,\,l}$.

The space $\widehat{\mathrm{Gr}}$ is an H-space with respect to the multiplication induced by the tensor product of bundles. It can be proved that $\widehat{\mathrm{Gr}} \cong \mathrm{BU}_\otimes$ as H-spaces. Let us also recall that $\mathrm{BU}_\otimes \cong \mathrm{BSU}_\otimes \times \mathbb{C}P^\infty$ and $\mathrm{Gr} \cong \mathrm{BSU}_\otimes$ as H-spaces, hence $\widehat{\mathrm{Gr}} \cong \mathrm{Gr} \times \mathbb{C}P^\infty$. In particular, the H-space $\widehat{\mathrm{Gr}}$ represents the functor of "multiplicative group" of the ring $\widetilde{\mathrm{K}}_{\mathbb{C}}$, i.e. the functor $X \mapsto \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$, where $\widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ is considered as a group with respect to the operation $\xi * \eta = \xi + \eta + \xi\eta,\ \xi,\,\eta \in \widetilde{\mathrm{K}}_{\mathbb{C}}(X)$ (here $\widetilde{\mathrm{K}}_{\mathbb{C}}$ is the reduced complex K-functor).

6 Appendix 2: The space Fr_{k^∞} is a loop space

Here we consider another way to prove that Fr_{k^∞} is a loop space.

First we recall some properties of the localization. Let X be a CW-complex such that $X = \Omega^N Z$ for some space Z and positive integer N. For a CW-complex Y by $\lambda[\frac{1}{k}]_Y\colon Y \to Y[\frac{1}{k}]$ we denote the localization at k (i.e. homotopy and homology groups of $Y[\frac{1}{k}]$ are $\mathbb{Z}[\frac{1}{k}]$-modules).

Lemma 77. There is a weak homotopy equivalence $\alpha\colon X[\frac{1}{k}] \to \Omega^N(Z[\frac{1}{k}])$ such that the diagram

$$\begin{tikzcd} X = \Omega^N Z \arrow[r, "\Omega^N(\lambda[\frac{1}{k}]_Z)"] \arrow[dr, "\lambda[\frac{1}{k}]_X"'] & \Omega^N(Z[\frac{1}{k}]) \\ & X[\frac{1}{k}] \arrow[u, "\alpha", "\cong"'] \end{tikzcd}$$

commutes. In other words, "delooping and localization commute".

Proof. Since the space $\Omega^N(Z[\frac{1}{k}])$ is $\mathbb{Z}[\frac{1}{k}]$-local, the existence of α follows from the universal property of the localization. It is easy to see that α induces an isomorphism of homotopy groups. $\square$

The following obvious fact is needed for the sequel.

Lemma 78. In the above notation, let Y_k be a fiber of the fibration $Y_k \to Z \xrightarrow{\lambda[\frac{1}{k}]_Z} Z[\frac{1}{k}]$. Then a fiber of the fibration $X \xrightarrow{\lambda[\frac{1}{k}]_X} X[\frac{1}{k}]$ is just $\Omega^N Y_k$.

Now, applying the above lemmas to the localization map

$$\widehat{\mathrm{Gr}} \xrightarrow{\widehat{\mathbf{T}}'_{k^\infty}} \mathrm{BGL}_{k^\infty}(\mathbb{C}), \tag{33}$$

we obtain the following result.

Proposition 79. *For any positive integer N the space Fr_{k^∞} is an N-fold loop space.*

Proof. Recall that

(i) map (33) is the localization at k;

(ii) fibration (33) has fiber Fr_{k^∞}.

Furthermore, it is well-known that $\widehat{\mathrm{Gr}} \cong \mathrm{BU}_\otimes$ is an infinite loop space. Now, applying the above lemmas, we complete the proof. $\square$

Remark 80. In particular, $\mathrm{Fr} = \varinjlim_{(k,l)=1} \mathrm{Fr}_{k,l} = \Omega(\mathrm{B}\,\mathfrak{G})$, and therefore $\mathrm{B}\,\mathfrak{G}$ is an infinite loop space too. Thus, the group $[X;\, \mathrm{B}\,\mathfrak{G}]$ can be considered as 0^{th} term of some generalized cohomology theory.

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