JWKB Representation for Equations with Infinite Order Turning Point

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MPI / 95-66

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Abstract

Linear ordinary differential equation of order m with a parameter and with smooth coefficients is considered. It is assumed that equation has turning point of infinite order. The fundamental system of the solutions with JWKB-representations is constructed.

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*Supported by Max-Planck-Gesellschaft.

0. Introduction

We consider the linear ordinary differential equation

$$D_t^m u + \sum_{j+|\alpha|\le m, j<m} a_{j,\alpha}(t)\xi^\alpha D_t^j u = 0 \tag{0.1}$$

with the parameter $\xi \in \mathbb{R}^n$ and smooth coefficients $a_{j,\alpha} \in C^\infty(J)$. Here $J = [0,T], T > 0, D_t = -id/dt, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index, $\mid \alpha \mid = \alpha_1 + \alpha_2 + ... + \alpha_n$. We call the point $t_0 \in J$ a turning point of the equation (0.1) if there is a $\xi \neq 0$ such that the roots $\lambda_l(t,\xi)\,(l = 1, ..., m)$ of the characteristic equation

$$\lambda^m + \sum_{j+|\alpha|=m,\, j<m} a_{j,\alpha}(t)\xi^\alpha \lambda^j = 0, \tag{0.2}$$

coincide at the point t_0. In the present paper we consider the equation (0.1) with the single turning point $t_0 = 0$ and such that

$$\lambda_j(0,\xi) = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n \quad \text{and all} \quad j = 1, ..., m. \tag{0.3}$$

Further, the turning point $t_0 = 0$ is said to have the order $\mathcal{K}$ (infinite order), if

$$D_t^l \lambda_j(0,\xi) = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^n, \qquad j = 1, \dots, m, \quad l = 0, ..., \mathcal{K} - 1 \tag{0.4}$$
$$(\text{for all} \quad l = 0, 1, ...).$$

Our goal is the construction of linear independent solutions $u_j(t,\xi), j = 1, 2, ..., m$, of (0.1) which can be represented in the following way:

$$u_j(t,\xi) = e^{\Phi_j(t,\xi)} a_j(t,\xi), \quad j = 1, \dots, m, \tag{0.5}$$

where $\Phi_j(t,\xi)$, $j = 1, \dots, m$, are phase functions will be described later and where $a_j(t,\xi)$, $j = 1, \dots, m$, are the amplitude functions such that with some nonnegative numbers $m_j, j = 1, \dots, m$, for every k, α, $k \le m$, they satisfy with a constant $C_{k,\alpha}$ following inequality

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha a_j(t,\xi) \right| \le C_{k,\alpha} <\xi>^{m_j - |\alpha| + k/2} \tag{0.6}$$

for all $t \in J$ and $\xi \in \mathbb{R}^n$. Here $<\xi> = (1 + |\xi|^2)^{1/2}$.

We describe the class of equations (0.1) by means of a real-valued function $\lambda \in C^\infty(J)$ such that $\lambda(0) = \lambda'(0) = 0, \lambda'(t) > 0$ when $t > 0$. In the following λ' means $d\lambda/dt$. For $\lambda(t)$ we define $\Lambda(t) = \int_0^t \lambda(r)\,dr$ and assume that

$$\lambda^m \Lambda^{1-m} \in C^\infty(J), \tag{0.7}$$

$$c\lambda(t)/\Lambda(t) \le \lambda'(t)/\lambda(t) \le c_0\lambda(t)/\Lambda(t) \quad \text{for all} \quad t \in J \setminus 0, \tag{0.8}$$

$$c_0'^{-1} |\ln \lambda(t)| \le \lambda'(t)/\lambda(t) \le c_0' |\ln \lambda(t)|^{c_0'} \quad \text{for all} \quad t \in J \setminus 0, \tag{0.9}$$

$$|\lambda^{(k)}(t)| \le c_k |\lambda'(t)/\lambda(t)|^{k-1} \lambda'(t) \quad \text{for all} \quad k = 1, 2, ..., \ t \in J \setminus 0, \tag{0.10}$$

with non-negative constants c, c_0, c_0', c_k, where $c > (m-1)/m$.

It is easy to see that (0.9) implies

$$\lambda(t) \leq \exp\{-\varepsilon_0 t^{-\varepsilon_1}\} \quad \text{for all} \quad t \in J \setminus 0,$$

with some positive ε_0 and ε_1.

Furthermore, we assume that the coefficients $a_{j,\alpha}$ satisfy for every $k, j, \alpha, |\alpha| \neq 0$, an inequality

$$|D_t^k\, a_{j,\alpha}(t)| \leq C_k \lambda(t)^{m-j} \left(\frac{|\ln \lambda(t)|}{\Lambda(t)}\right)^{m-j-|\alpha|} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k, \tag{0.11}$$

for all $t \in J \setminus 0$ with some constant C_k. Moreover, we assume that some positive constant δ

$$|\mathrm{Im}\,(\lambda_l(t,\xi) - \lambda_k(t,\xi))\,| \geq \delta\lambda(t)|\xi|, \quad l \neq k, \quad \text{for all} \quad \xi \in \mathbb{R}^n,\, t \in J, \tag{0.12}$$

$$|\mathrm{Im}\lambda_l(t,\xi)| \geq \delta\lambda(t)|\xi|, \quad l = 1, \ldots, m, \quad \text{for all} \quad \xi \in \mathbb{R}^n,\, t \in J. \tag{0.13}$$

Equation with a turning point of finite order has been extensively studied by many authors, see, for instance, the books [3], [7], [14], [15], [18], [20], and articles [10],[11],[12],[13]. Equations with a turning point of infinite order is studied in the case of real characteristic roots $\lambda_l(t,\xi)$ only [22], [23]. Therefore in the present paper we deal with the equations with non-real $\lambda_l(t,\xi)$ which have one turning point $t_0 = 0$ of infinite order.

The methods giving the uniform asymptotic developing for the equations with turning points are based on the reduction of (0.1) in a small neighbourhood of turning point either to well-known special differential equation or to ordinary differential equation with polynomial coefficients [18], which have solutions with already determined asymptotic behaviour. These methods are quite successful for the second order equations with a turning point of finite order when one can apply to Malgrange's preparation theorem [6]. Thus, the applications of these methods are restricted to the case of finite order turning points. Nevertheless we will give an example (Example 1) of an equation with a turning point of infinite order which can be reduced by Langer transformation [14] to Kummer's equation [2] for confluent hypergeometrical function. Firstly this example was considered by Alexandrian [1] (with $\phi = \pi/2$) for the sake of an investigation a propagation of singularities of solutions of weakly hyperbolic equation and then by Hoshiro [5] (with $\phi = 0$) and independently by Reissig and author [16] (for $\phi \in [0, \pi/2)$) in the investigation of hypoellipticity property of partial differential operators of second order.

It should be noted that equations with one turning point play a special role in quantum mechanics [8], [9], geometric optics and in hydrodynamical stability [20], and in the theory of partial differential equations.

Inequality (0.11) for $a_{j,\alpha}$ with $j + |\alpha| < m, |\alpha| \neq 0$, is called Levi condition in the theory of partial differential operators. This condition has strong influence on well-posedness of the problems for partial differential operators with multiple characteristics.

The following example hints at construction we are looking for.

Example 1. [16] Let us consider a second order equation

$$(d/dt)^2 u - \xi^2 \lambda_\phi^2(t) u + \xi b \frac{\lambda_\phi^2(t)}{\Lambda_\phi(t)} u = 0\,, \tag{0.14}$$

where with $b \in \mathbb{C}, \xi \in \overline{\mathbb{R}^+}$

$$\Lambda_\phi(t) = \exp(i\phi - t^{-1}), \quad \phi \in [0, \pi/2], \quad \lambda_\phi(t) := (d/dt)\Lambda(t) = t^{-2}\exp(i\phi - t^{-1})\,.$$

At the point $t = 0$ some of the coefficients of operator have a zero of infinite order. Equation (0.14) has the following two independent solutions :

$$u_1(t,\xi) = te^{\Lambda_\phi(t)\xi}\Psi(\alpha, 1; -2\Lambda_\phi(t)\xi)\,, \quad u_2(t,\xi) = te^{-\Lambda_\phi(t)\xi}\Psi(1-\alpha, 1; 2\Lambda_\phi(t)\xi)\,, \tag{0.15}$$

where $\alpha = (1-b)/2$ and $\Psi(\alpha, \gamma; z)$ is a solution of confluent hypergeometric equation having an integral representation

$$\Psi(\alpha, \gamma; z) = \frac{1}{2i\pi} e^{-i\pi\alpha}\Gamma(1-\alpha) \int_{\infty e^{i\varphi}}^{(0+)} e^{-zt} t^{\alpha-1}(1+t)^{\gamma-\alpha-1} dt, \tag{0.16}$$

$-\pi/2 < \varphi + \arg z < \pi/2$, $\arg t = \varphi$ at the starting point, and $\Gamma(\alpha)$ is Euler's function [2]. In the case when $\alpha = -n, \gamma = 1$, n is non-negative integer,

$$\Psi(-n, 1; z) = (-1)^n n! L_n^0(z), \quad n = 0, 1, \dots\,, \tag{0.17}$$

where $L_n^0(z) = \frac{1}{n!} e^z D_z^n(e^{-z} z^n)$ are Laguerre's polynomials.

The function $\Psi(\alpha, \gamma; z)$ has for small z the following behaviour

$$\Psi(\alpha, \gamma; z) = -\frac{1}{\Gamma(\alpha)}[\ln z + \psi(\alpha) - 2\gamma] + o(|z \ln z|), \tag{0.18}$$

where γ is Euler's constant and $\psi(z)$ is the digamma function (psi function of Gauss) $\psi(z) := \Gamma'(z)/\Gamma(z)$ while for large z there is the following asymptotic expansion:

$$\Psi(\alpha, \gamma; z) \sim z^{-\alpha}\left[1 + \sum_{k=1}^{\infty}(-1)^k \frac{(\alpha)_k(\alpha - \gamma + 1)_k}{k!} z^{-k}\right], \tag{0.19}$$

when $z \to \infty$, $-\frac{3\pi}{2} \leq \arg z \leq \frac{3\pi}{2}$. Here $(\alpha)_k := \alpha(\alpha+1)\cdots(\alpha+k-1)$.

Thus formulas (0.15), (0.16), (0.17),(0.18),(0.19) give the complete asymptotic representations of the solutions (0.15).

The outline of our construction is the following. The main difficulty in carrying out classical construction is that inserting *Anzatz* (0.5) into (0.1) and using the classical approach we get for the first terms of asymptotic developments of a_j equations with unbounded coefficients which, in general, are not of Fucshian equations. On the other hand any solution $u(t,\xi)$ of (0.1) generates a solution $\mathcal{U}(t,\xi) := (\mathcal{U}_1(t,\xi),\ldots,\mathcal{U}_m(t,\xi)) := {}^t(u(t,\xi),\ldots,D^{m-1}u(t,\xi))$ of the system

$$\frac{d}{dt}\mathcal{U} = \mathcal{A}(t,\xi)\mathcal{U}$$

where $\mathcal{A}(t,\xi)$ can be written explicitly by means of $a_{j,\alpha}(t)$, and conversely. Then, there exists an "explicit" representation formula (so called *matritzant* [4])

$$\begin{aligned} \mathcal{U}(t,\xi) &= \mathcal{U}(0,\xi) + \left(\sum_{l=1}^{\infty}\int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \right. \\ &\left. \cdots \int_0^{s_{l-1}} ds_l \mathcal{A}(s_1,\xi)\cdots\mathcal{A}(s_l,\xi)\right)\mathcal{U}(0,\xi). \end{aligned} \tag{0.20}$$

Nevertheless, it is very difficult to get from the last representation uniform with respect to $t \in J$ asymptotic behaviour of $\mathcal{U}(t,\xi)$ when $\xi \to \infty$, even in the case when $\mathcal{U}(0,\xi)$ is independent of ξ. At the same time if we restrict ourselves to consideration a set

$$Z_{int}(M,N) = \{(t,\xi) \in J \times \mathbb{R}^n \mid \Lambda(t) <\xi> \leq N \ln <\xi>, <\xi> \geq M\}, \tag{0.21}$$

where M and N are positive constants, from the formula (0.20), keeping in mind

$$\int_{(t,\xi)\in Z_{int}(M,N)} |a_{j,\alpha}(t,\xi)\xi^\alpha| \left(1 + <\xi> \frac{\lambda^m(t)}{\Lambda^{m-1}(t)}(\ln <\xi>)^{m-1}\right)^{\frac{1+j-m}{m}} dt \leq const \ln <\xi>$$

for all $\xi \in \mathbb{R}^n$, $<\xi> \geq M > 0$, without difficulties we can get polynomial asymptotical behaviour of $\mathcal{U}(t,\xi)$ and corresponding JWKB representation for $u(t,\xi)$. It remains to consider the set

$$Z_{ext}(M,N) = \{(t,\xi) \in J \times \mathbb{R}^n \mid \Lambda(t) <\xi> \geq N \ln <\xi>, <\xi> \geq M\}. \tag{0.22}$$

But this set is far away from the points $(0,\xi) \in Z_{int}(M,N)$ whose projections on the base coincide with the turning point $t = 0$. That gives the chance to get a success using "almost" classical approach with modified definitions of symbol classes, asymptotic summation and integral equation. In this way we get global with respect both t and ξ asymptotic representation and behaviour.

To formulate main result of present paper we consider zeros of complete symbol of the operator (0.1) that is the continuous roots $\tau_l(t,\xi)$, $l = 1, ..., m$, of the equation

$$\tau^m + \sum_{j+|\alpha|\leq m,\, j<m} a_{j,\alpha}(t)\xi^\alpha \tau^j = 0. \tag{0.23}$$

These roots are smooth functions (Proposition 1.1) in the domain $Z_{ext}(M,N)$ for M and N large enough. Further, let $\chi(x)$ be a C^∞-function on the real line satisfying $0 \leq \chi(x) \leq 1$, $\chi(x) = 1$ for $|x| \leq 1$, and $\chi(x) = 0$ for $|x| \geq 2$.

Theorem 0.1. *Assume that (0.7)-(0.10),(0.11)-(0.13) are satisfied. Then there are linear independent solutions $u_j(t,\xi), j = 1,\dots,m$, having representations (0.5) with phase functions*

$$\Phi_j(t,\xi) = \int_0^t \left\{ \chi\left(\frac{\Lambda(s) <\xi>}{N \ln <\xi>}\right) \lambda_j(s,\xi) + \left(1 - \chi\left(\frac{\Lambda(s) <\xi>}{N \ln <\xi>}\right)\right) \tau_j(s,\xi) \right\} ds, \tag{0.24}$$

and with amplitude functions $a_j(t,\xi), j = 1,\dots,m$, satisfying (0.6).

According to the following theorem one can get a representations with a homogeneous with respect to ξ phase functions provided that coefficients $a_{m-1-|\alpha|,\alpha}$ satisfy more restrictive conditions.

Theorem 0.2. *Assume that (0.7)-(0.10),(0.11)-(0.13) are satisfied. Moreover, let the coefficients $a_{m-1-|\alpha|,\alpha}$, $|\alpha| \neq 0$, satisfy for every k,α, an inequality*

$$|D_t^k a_{m-1-|\alpha|,\alpha}(t)| \leq C_k \lambda(t)^{|\alpha|} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{k+1}, \tag{0.25}$$

for all $t \in J \setminus 0$ with some constant C_k. Then there are linear independent solutions $u_j(t,\xi), j = 1,\dots,m$, having representations (0.5) with phase functions

$$\Phi_j(t,\xi) = \int_0^t \lambda_j(s,\xi) ds, \quad j = 1,\dots,m, \tag{0.26}$$

and with amplitude functions $a_j(t,\xi), j = 1,\dots,m$, satisfying (0.6).

We notice here that the condition (0.11) is sharp. In further paper we prove that if (0.11) is violated for some $a_{j,\alpha}, j + |\alpha| < m, |\alpha| \neq 0$, then there do not exist fundamental system linear independent solutions of the equation (0.1) with the representation (0.5) and amplitude functions a_j satisfying (0.6).

Then it should be noted that our main goal is a construction presented below. Following it one can get the quantities m_j, as well as more precise estimates in (0.6) for $k = 1, 2, \dots$.

1. On the zeros of the complete symbol

Firstly we consider the exterior zone $Z_{ext}(M,N)$ defined for positive numbers M and N in (0.22). It is evident that if $M' \geq M$ and $N' \geq N$ then $Z_{ext}(M',N') \subset Z_{ext}(M,N)$.

Further, let us denote for $\xi \in \mathbb{R}^n_M := \{\, \xi \in \mathbb{R}^n \mid \, <\xi> \geq M > 0 \,\}$, $M \geq e$, by t_ξ a root of

$$\Lambda(t) <\xi> = N \ln <\xi> . \tag{1.1}$$

Lemma 1.1. *The function t_ξ is a smooth function of ξ defined on $\mathbb{R}^n_M$ and such that one has*

$$\frac{\partial t_\xi}{\partial \xi_j} = \frac{N(1-\ln <\xi>)}{<\xi>^3 \lambda(t_\xi)}\xi_j \quad j = 1,\dots,n, \tag{1.2}$$

while for every multi-index α, $|\alpha| \neq 0$, the following estimate holds:

$$\left|\left(\frac{\partial}{\partial \xi}\right)^\alpha t_\xi\right| \leq C_\alpha <\xi>^{1-|\alpha|}\left|\frac{\partial t_\xi}{\partial <\xi>}\right| \quad \text{for all } \xi \in \mathbb{R}^n_M . \tag{1.3}$$

Proof. First formula is obvious, while to prove the last estimate an induction can be applied. The lemma is proved.

Further a surface $t = t_\xi$ splits set $[0,T] \times \mathbb{R}^n_M$ into two domains (zones):*exterior zone* $Z_{ext}(M,N)$ and *interior zone* $Z_{int}(M,N)$. In each domain the equation (0.1) is to be examined separately by suitable technique. We start from $Z_{ext}(M,N)$ where "almost" semiclassical approach works.

Proposition 1.1. *The assumptions (0.11)-(0.13) are equivalent to the following: there exist positive constants M,N such that the zeros $\tau_l(t,\xi), l = 1,\dots,m$, are defined in zone $Z_{ext}(M,N)$ smooth functions, $\tau_l \in C^\infty(Z_{ext}(M,N))$, $l = 1,\dots,m$, which satisfy for every k,l,α, an inequalities*

$$\left|\left(\frac{\partial}{\partial t}\right)^k\left(\frac{\partial}{\partial \xi}\right)^\alpha \tau_l(t,\xi)\right| \leq C_{k,\alpha} <\xi>^{1-|\alpha|}\lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^k . \tag{1.4}$$

Moreover with some positive constant δ_1

$$\begin{aligned} |\mathrm{Im}\,(\tau_l(t,\xi) - \tau_j(t,\xi))\,| &\geq \delta_1|\lambda(t)||\xi|, \quad l \neq j, & (1.5)\\ |\mathrm{Im}\,\tau_l(t,\xi)| &\geq \delta_1\lambda(t)|\xi| & (1.6)\end{aligned}$$

for all $(t,\xi) \in Z_{ext}(M,N)$, $l,j = 1,\dots,m,$.

Proof. First of all we note that there are the positive constants c_1, c_2 such that for all sufficiently large M and N

$$c_1 \ln <\xi> \leq |\ln \lambda(t_\xi)| \leq c_1^{-1} \ln <\xi> \quad \text{for all} \quad \xi \in \mathbb{R}^n_M, \tag{1.7}$$

$$c_2 \ln <\xi_t> \leq |\ln \lambda(t)| \leq c_2^{-1} \ln <\xi_t> \quad \text{for all} \quad t > 0. \tag{1.8}$$

Here $<\xi_t>$ denotes the root of the equation (1.1) with respect to $<\xi>$.

Let us proof an implication (1.4)&(1.5)&(1.6) $\Rightarrow$ (0.11)&(0.12)&(0.13). If $j + |\alpha| = m$, $t \geq t_\xi$ and $\xi \in \mathbb{R}^n_M$, then, for every k, the inequality

$$\begin{aligned} |D_t^k a_{j,\alpha}(t)| &= \left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \sum_{i_1 < \dots < i_{m-j}} \tau_{i_1}(t,\xi) \cdots \tau_{i_{m-j}}(t,\xi) \right| / |\alpha|! \\ &\leq c_k (\lambda(t))^{m-j} (\lambda(t)/\Lambda(t))^k \end{aligned}$$

holds with some constant c_k which does not depend on t and ξ. The inequality (0.11) for the coefficients $a_{j,\alpha}$, with $j + |\alpha| = m - l \, (l = 1, \dots, m-1)$, can be proved by induction. Indeed, for $j + |\alpha| = m - 1 - l$ it holds

$$a_{j,\alpha}(t) = \frac{1}{\alpha!} i^{|\alpha|} D_\xi^\alpha \sum_{|\gamma| \leq m-j} a_{j,\gamma}(t) \xi^\gamma - \frac{1}{\alpha!} i^{|\alpha|} D_\xi^\alpha \sum_{|\gamma| \geq m-j-l} a_{j,\gamma}(t) \xi^\gamma.$$

Hence,

$$\begin{aligned} |D_t^k a_{j,\alpha}(t)| &\leq \\ &\leq \frac{1}{\alpha!} \left| D_t^k D_\xi^\alpha \sum_{i_1 < \dots < i_{m-j}} \tau_{i_1}(t,\xi) \cdots \tau_{i_{m-j}}(t,\xi) \right| + \frac{1}{\alpha!} \left| D_t^k D_\xi^\alpha \sum_{|\gamma| \geq m-j-l} a_{j,\gamma}(t) \xi^\gamma \right| \\ &\leq c_k \lambda^{m-j}(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \Bigg\{ <\xi>^{m-j-|\alpha|} \\ &\qquad + \sum_{|\gamma| \geq m-j-l} <\xi>^{|\gamma - \alpha|} \left| \frac{\ln|\lambda(t)|}{\Lambda(t)} \right|^{m-j-|\gamma|} \Bigg\}. \end{aligned}$$

For the completion of the proof of (0.9) it is enough to put $<\xi>=<\xi_t>$ and to use the induction assumption in view of (1.7),(1.8).

In order to prove (0.12),(0.13) we make change of variables $\lambda = \lambda(t)|\xi|\mu$, $\tau = \lambda(t)|\xi|\gamma$ and the equation (0.2) and the equation (0.23) for the zeros of the principal symbol become

$$\mu^m + \sum_{0 \leq j < m} \left\{ \sum_{|\alpha| = m-j} (\lambda(t)|\xi|)^{j-m} a_{j,\alpha}(t) \xi^\alpha \right\} \mu^j = 0, \tag{1.9}$$

$$\gamma^m + \sum_{0\le j<m} \left\{ \sum_{|\alpha|=m-j} (\lambda(t)|\xi|)^{j-m} a_{j,\alpha}(t)\xi^\alpha + B_{j+1} + \Delta B_{j+1} \right\} \gamma^j = 0, \tag{1.10}$$

respectively, where $\Delta B_{j+1} = 0$ and

$$B_{j+1} = B_{j+1}(t,\xi) = \sum_{|\alpha|\le m-j-1} (\lambda(t)|\xi|)^{j-m} a_{j,\alpha}(t)\xi^\alpha, \quad j = 0,...,m-1. \tag{1.11}$$

In accordance with (0.9), for $(t,\xi) \in Z_h(M,N)$ we have

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha B_{j+1}(t,\xi)\right| \le C_{k,\alpha} <\xi>^{-|\alpha|} (\lambda(t)/\Lambda(t))^k/N \qquad (N>1), \tag{1.12}$$

and therefore we consider (1.9) as perturbed equation (1.10) with the perturbations $\Delta B_j = -B_j(t,\xi)$, $j = 1,...,m$, in coefficients. Due to (1.5) the roots of (1.10) depend analytically on the perturbation $\Delta B = (\Delta B_1, ..., \Delta B_m) \in \mathbb{C}^m$ in some neighborhood of the origin. Clearly,

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \gamma_j(t,\xi)\right| \le C_{k,\alpha} <\xi>^{-|\alpha|} (\lambda(t)/\Lambda(t))^k, \tag{1.13}$$

$$|Im\,(\gamma_l(t,\xi) - \gamma_j(t,\xi))\,| \ge \delta_1 > 0, \quad l \ne j, \tag{1.14}$$

$$|\mathrm{Im}\,\gamma_j(t,\xi)| \ge \delta_1, \quad j = 1,\dots,m, \tag{1.15}$$

for all $(t,\xi) \in Z_h(M,N)$ and all $j,l = 1,\dots,m$.

Furthermore, it is sufficient to show that, upon perturbation by some particular $\Delta B_{j+1} = -B_{j+1}(t,\xi)$ with $\Delta B_1 = ... = \Delta B_j = \Delta B_{j+2} = ... = \Delta B_m = 0$, the roots $\mu_l(l = 1,...,m)$ of the equation

$$P(t,\xi;\mu) - \mu^j B_{j+1}(t,\xi) = 0,$$

where $P(t,\xi;\mu) = (\mu - \gamma_1(t,\xi))\cdots(\mu - \gamma_m(t,\xi))$, inherit properties (1.13)-(1.15), may be with new constants $M, N, \delta_1, C_{k,\alpha}$. Indeed, by virtue of (1.14) we have

$$\mu_l(t,\xi) = \gamma_l(t,\xi) + \sum_{n=1}^{\infty} c_n^{(l)}(t,\xi)(-\Delta B_{j+1})^n, \quad l = 1,...,m, \tag{1.16}$$

where

$$c_n^{(l)}(t,\xi) = \frac{1}{2\pi i} \oint_{|w-\gamma_l|=\rho} \frac{(w-\gamma_l(t,\xi))(wP'_w(t,\xi,w) - jP(t,\xi,w))w^{jn-1}}{(P(t,\xi,w))^{n+1}} dw$$

$$= \frac{1}{(n-1)!}$$

$$\times \left[\frac{d^{n-1}}{dw^{n-1}} \left\{ \left[\frac{w-\gamma_l(t,\xi)}{P(t,\xi,w)}\right]^{n+1} (wP'_w(t,\xi,w) - jP(t,\xi,w))w^{jn-1} \right\}\right]_{w=\gamma_l(t,\xi)}.$$

Therefore for $0 < 2\rho < \delta_1$ we have the inequality

$$|c_n^{(l)}(t,\xi)| \le c\delta_1^{1-m}(c2^{m-1}/(\rho\delta_1^{m-1}))^n \quad \text{for all} \quad (t,\xi) \in Z_h(M,N), \tag{1.17}$$

with a constant c independent of t, ξ and j. Therefore, the radius of convergence r_{j+1} of the series in (1.16) ($|\Delta B_{j+1}| < r_{j+1}$) is independent of $(t,\xi) \in Z_h(M,N)$, j, l, provided that N is large enough.

Furthermore, (1.16) and (1.17) yield

$$|\mathrm{Im}(\mu_l(t,\xi) - \mu_j(t,\xi))| \ge \delta_2 > 0, \quad l \ne j, \tag{1.18}$$

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \mu_j(t,\xi)\right| \le C_{k,\alpha} <\xi>^{-|\alpha|} (\lambda(t)/\Lambda(t))^k, \tag{1.19}$$

for all $(t,\xi) \in Z_h(M,N)$ may be with new constant N. Indeed, (1.18) is evident while for the proof of (1.19) one can apply the formula of derivative of an implicit function. To this end we denote for fixed l

$$z_r(t,\xi) = \mu_l(t,\xi) - \gamma_r(t,\xi), \quad r = 1, ..., m,$$

and $y = (t,\xi) \in J \times \mathbb{R}^n \subseteq \mathbb{R}^{n+1}$. According to above mentioned formula one has

$$\partial_y^\delta \mu_l = (P'_w(y;\mu_l(y)) - j\mu_l^{j-1}(y)B_{j+1}(y))^{-1}\mathcal{E}_l^\delta(y),$$

where

$$\begin{aligned}
\mathcal{E}_l^\delta(y) &= \{\sum_{k=1}^m (\partial_y^\delta \gamma_k(y))(\prod_{r\ne k} z_r(y)) \\
&- \sum_{\delta_1+...+\delta_m=\delta,\, \delta_1\ne\delta,...,\delta_m\ne\delta} \frac{\delta!}{\delta_1!\cdots\delta_m!}\{\partial_y^{\delta_1} z_1(y)\}\cdots\{\partial_y^{\delta_m} z_m(y)\} \\
&- \sum_{\delta_1+...+\delta_j+\delta_{j+1}=\delta,\, \delta_1\ne\delta,...,\delta_j\ne\delta,\delta_{j+1}\ne\delta} \frac{\delta!}{\delta_1!\cdots\delta_j!\delta_{j+1}!}\{\partial_y^{\delta_1}\mu_l(y)\} \\
&\qquad \cdots\{\partial_y^{\delta_j}\mu_l(y)\}\{\partial_y^{\delta_{j+1}} B_{j+1}(y)\}\}.
\end{aligned}$$

Hence (1.19) follows from (1.16),(1.17),(1.18) by induction on $|\delta|$. Furthermore,

$$\begin{aligned}
|\mathrm{Im}\,\mu_l| &\ge |\mathrm{Im}\,\gamma_l| - |\mathrm{Im}\,c_1^{(l)}||B_{j+1}| \\
&- |c_1^{(l)}||\mathrm{Im}B_{j+1}| - |B_{j+1}^2|\sum_{n=2}^\infty c_n^{(l)}||B_{j+1}|^{n-2}.
\end{aligned}$$

It follows from (1.11) that if M and N are large enough then

$$|B_{j+1}(t,\xi)|^2 \le C\left(\frac{|\ln\lambda(t)|}{<\xi>\Lambda(t)}\right)^2, \quad \sum_{n=2}^\infty c_n^{(l)}||B_{j+1}|^{n-2} \le C\,, \tag{1.20}$$

for all $(t,\xi) \in Z_h(M,N)$, $l = 0, ..., m-1$.

The implication (1.4)&(1.5)&(1.6) $\Rightarrow$ (0.11)&(0.12)&(0.13) has been proved.

The proof of the implication (0.11)&(0.12)&(0.13) $\Rightarrow$ (1.4)&(1.5)&(1.6) is almost identical: the equation (1.10) is considered to be a perturbed equation (1.9). This completes the proof of the proposition.

Corollary 1.1. *If coefficients $a_{m-1-|\alpha|,\alpha}$, $|\alpha| \neq 0$, satisfy inequalities (0.25) then*

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha (\tau_j(t,\xi) - \lambda_j(t,\xi)) \right| \leq C_{k,\alpha} <\xi>^{-|\alpha|} \left\{ \frac{\lambda(t)}{\Lambda(t)} + (m-2)\frac{\lambda(t)\ln^2\lambda(t)}{\Lambda^2(t) <\xi>} \right\} \tag{1.21}$$

for all $(t,\xi) \in Z_{ext}(M,N)$.

Proof. It is consequence of the representations (1.16) and the estimates (1.20).

2. Classes of Symbols

For the constructions in the exterior zone $Z_{ext}(M,N)$ where "almost semiclassical" approach will be carried out, we need some tools. These tools are special classes of symbols as well as corresponding asymptotic calculus.

The classes of symbols $a_j(t,\xi)$ with parameter t will be presented below are basing on the estimates (1.4).

Definition 2.1. *Let m_1, m_2, m_3, ρ be real numbers while M and N are positive numbers. By $\mathcal{S}_\rho\{m_1,m_2,m_3\}_{M,N}$ we denote the set of all functions $a(t,\xi) \in C^\infty(Z_{ext}(M,N))$ such that for any k,α there exists a constant $C_{k,\alpha}$ such that*

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha a(t,\xi) \right| \leq C_{k,\alpha} <\xi>^{m_1-\rho|\alpha|} \lambda(t)^{m_2} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m_3+k}$$
$$\text{for all} \quad (t,\xi) \in Z_{ext}(M,N). \tag{2.1}$$

We also denote

$$\mathcal{H}_\rho\{m_1,m_2,m_3\}_{M,N} = \bigcap_{k=0}^{\infty} \mathcal{S}_\rho\{m_1-k, m_2-k, m_3+k\}_{M,N}\,.$$

In what follows we drop ρ and write shortly $\mathcal{S}\{m_1,m_2,m_3\}_{M,N}$, $\mathcal{H}\{m_1,m_2,m_3\}_{M,N}$ when $\rho = 1$.

Proposition 2.1. *Suppose that* $a_k(t,\xi) \in \mathcal{S}_\rho\{m_1 - k, m_2 - k, m_3 + k\}_{M,N}$, $k = 0, 1, ...,$. *Then there exists a symbol* $a(t,\xi) \in \mathcal{S}_\rho\{m_1, m_2, m_3\}_{M,N}$ *such that*

$$a \sim a_0 + a_1 + a_2 + \ldots \qquad \mathrm{mod}\, \mathcal{H}_\rho\{m_1, m_2, m_3\}_{M,N}\,, \tag{2.2}$$

in the sense that

$$(a - a_0 - a_1 - \ldots - a_{k-1}) \in \mathcal{S}_\rho\{m_1 - k, m_2 - k, m_3 + k\} \quad \text{for all} \quad k\,, \tag{2.3}$$

and any symbols with the property (2.3) differ by the elements of $\mathcal{H}_\rho\{m_1, m_2, m_3\}_{M,N}$.

Proof. Let $\chi(x)$ be a C^∞-function is described before Theorem 0.1. We also define the function

$$\gamma_\varepsilon(t,\xi) = 1 - \chi(\varepsilon\Lambda(t) <\xi>)$$

and note that

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \gamma_\varepsilon(t,\xi)\right| \le C_{k,\alpha} <\xi>^{-|\alpha|} \Lambda(t)^{-k}\lambda(t)^k$$

for all $(t,\xi) \in Z_{ext}(M,N)$, $0 < \varepsilon \le 1$. A sequence $\{\varepsilon_k\}_0^\infty$, $1 \ge \varepsilon_0 > \varepsilon_1 > \ldots > \varepsilon_k > \ldots$, $\varepsilon_k \to 0$, can be chosen in such a way that for for all $(t,\xi) \in Z_{ext}(M,N)$, we have

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha (\gamma_{\varepsilon_j}(t,\xi)a_j(t,\xi))\right| \le 2^{-j} <\xi>^{m_1 - j + 1 - \rho|\alpha|} \lambda(t)^{m_2 - j + 1} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m_3 + k + j - 1}$$

for all k,α, $k + |\alpha| \le j, j = 0, 1, \ldots$. Therefore, for the remainder of the series

$$\sum_{j=0}^{\infty} \gamma_{\varepsilon_j}(t,\xi)a_j(t,\xi) \tag{2.4}$$

we have

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \sum_{j=r+1}^{\infty} \gamma_{\varepsilon_j}(t,\xi)a_j(t,\xi)\right| \le C_{k,\alpha} <\xi>^{m_1 - \rho|\alpha| - r} \lambda(t)^{m_2 - r} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m_3 + k + r}$$

$$\times \sum_{j=r+1}^{\infty} (2\Lambda(t) <\xi>)^{-j} \le C'_{k,\alpha} <\xi>^{m_1 - \rho|\alpha| - r} \lambda(t)^{m_2 - r} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^{m_3 + k + r}.$$

Thus, $\sum_{j=r+1}^{\infty} \gamma_{\varepsilon_j}(t,\xi)a_j(t,\xi) \in \mathcal{S}_\rho\{m_1 - r, m_2 - r, m_3 + r\}_{M,N}$. It follows that the series (2.8) defines the function $a(t,\xi) \in \mathcal{S}_\rho\{m_1, m_2, m_3\}_{M,N}$ and that (2.2) holds. The proposition is proved.

If $A(t,\xi)$ is a matrix-function, then $A(t,\xi) \in \mathcal{S}_\rho\{m_1, m_2, m_3\}_{M,N}$ means that the elements $a_{jk}(t,\xi)$ of $A(t,\xi)$ belong to this class for all i and j.

Lemma 2.1. *Assume that a sequence of matrix-functions* $N^{(j)}(t,\xi) \in \mathcal{S}\{-j,-j,j\}_{M,N}$*,* $j = 1,2,\dots$ *, is given. Then there are the matrix-functions* $N(t,\xi), N^{\#}(t,\xi) \in \mathcal{S}\{0,0,0\}_{M,N}$ *such that*

$$N \sim I + N^{(1)} + N^{(2)} + \dots \quad \mathrm{mod}\, \mathcal{H}_\rho\{m_1,m_2,m_3\}_{M,N}\,,$$

and

$$N^{\#}(t,\xi)N(t,\xi) = I$$

may be with a new M,N.

Proof. The existence of the $N(t,\xi)$ is a consequence of the Proposition 2.1. In order to prove the existence of $N^{\#}(t,\xi)$ we note that $N(t,\xi) - I \in \mathcal{S}\{-1,-1,1\}_{M,N}$ implies

$$\|N(t,\xi) - I\| \leq const < 1 \quad \text{for all } (t,\xi) \in Z_{ext}(M_1,N)$$

when M_1 is large enough. Therefore, if we choose for $(t,\xi) \in Z_{ext}(M,N)$ the reciprocal matrix $N^{-1}(t,\xi)$ as $N^{\#}(t,\xi)$ then the last assertion of the lemma will be satisfied. The lemma is proved.

3. Reduction to a "First Order Diagonal" System in Exterior Zone

In exterior zone $Z_{ext}(M,N)$ the equation (0.1) can be reduced to a "first order" system as follows.

Let $H_{ext}(t,\xi)$ be a diagonal matrix-function

$$\begin{pmatrix} h^{m-1}(t,\xi) & 0 & \cdots & 0 \\ 0 & h^{m-2}(t,\xi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $h(t,\xi) = \lambda(t) <\xi>$. Introducing the vector $U = {}^t(U_1,U_2,\dots,U_m) = H_{ext}(t,\xi)^t(u, D_tu, \dots, D_t^{m-1}u)$, the equation (0.1) can be transformed to the system

$$L_0U = 0 \quad , \tag{3.1}$$

where

$$L_0 = D_t - A(t,\xi) + i(\partial_t H_{ext}(t,\xi))H_{ext}^{-1}(t,\xi).$$

We are restricted ourselves to exterior zone, hence with some positive M and N we have

$$A(t,\xi) \in \mathcal{S}\{1,1,0\}_{M,N}. \tag{3.2}$$

Further, for the function

$$\Delta(t,\xi) = \prod_{j<i\leq m} (\tau_i(t,\xi) - \tau_j(t,\xi))/h(t,\xi))$$

we have

$$\Delta \in \mathcal{S}\{0,0,0\}_{M,N}\,, \tag{3.3}$$

$$0 < const \leq |\Delta(t,\xi)| \leq const \quad \text{for all} \quad (t,\xi) \in Z_{ext}(M,N)\,. \tag{3.4}$$

For the system $\{\tau_j(t,\xi)/h(t,\xi)\}_{j=1}^{m}$ we form the Vandermonde matrix $M^{\#}(t,\xi) = V(\tau_1/h, \tau_2/h, \ldots, \tau_m/h)$. Let $M(t,\xi)$ be its inverse matrix. According to (3.3),(3.4) we have

$$M^{\#}(t,\xi),\ M(t,\xi) \in \mathcal{S}\{0,0,0\}_{M,N}. \tag{3.5}$$

Then the vector $V = M(t,\xi)U$ is a solution of the system

$$\begin{aligned} D_tV &- M(t,\xi)A(t,\xi)M^{\#}(t,\xi)V + iM(t,\xi)(\partial_t H_{ext}(t,\xi))H_{ext}^{-1}(t,\xi)M^{\#}(t,\xi)V \\ &- iM(t,\xi)(\partial_t M^{\#}(t,\xi))V = 0\,. \end{aligned} \tag{3.6}$$

Lemma 3.1. *The system (3.6) can be rewritten in the following form*

$$D_tV + i\mathcal{D}(t,\xi)V + \mathcal{B}(t,\xi)V = 0\,, \tag{3.7}$$

where $\mathcal{D}(t,\xi)$ is a diagonal matrix-function with elements $-i\tau_1(t,\xi), -i\tau_2(t,\xi), \ldots,$ $-i\tau_m(t,\xi)$, and where $\mathcal{B}(t,\xi) \in \mathcal{S}\{0,0,1\}_{M,N}$.

Proof. From the definition $\mathcal{D}(t,\xi)$ we get

$$\begin{aligned} \mathcal{B}(t,\xi) &= -i\mathcal{D}(t,\xi) - M(t,\xi)(A(t,\xi) \\ &- (\partial_t H_{ext}(t,\xi))H_{ext}^{-1}(t,\xi))M^{\#}(t,\xi) - iM(t,\xi)\partial_t M^{\#}(t,\xi)\,. \end{aligned}$$

The last assertion of the lemma follows immediately from definition of $H_{ext}(t,\xi)$ and from (3.5). The lemma is proved.

Proposition 3.1. *There exist matrix-functions $\mathcal{N}(t,\xi)$,$\mathcal{F}(t,\xi)$,$\mathcal{R}(t,\xi)$ such that the following operator-valued identity*

$$\begin{aligned} (D_t &+ i\mathcal{D}(t,\xi) + \mathcal{B}(t,\xi))\mathcal{N}(t,\xi) = \mathcal{N}(t,\xi)(D_t + i\mathcal{D}(t,\xi) \\ &+ \mathcal{F}(t,\xi) - \mathcal{R}(t,\xi)), \end{aligned} \tag{3.8}$$

holds and

(i) $\mathcal{N}(t,\xi) \in \mathcal{S}\{0,0,0\}_{M,N}$, $|\det \mathcal{N}(t,\xi)| \geq const > 0$ *for all* $(t,\xi) \in Z_{ext}(M,N)$;

(ii) $\mathcal{F}(t,\xi)$ *is a diagonal matrix,* $\mathcal{F}(t,\xi) \in \mathcal{S}\{0,0,1\}_{M,N}$, $\mathcal{R}(t,\xi) \in \mathcal{H}\{0,0,1\}_{M,N}$.

Proof. We look for $\mathcal{N}(t,\xi)$, $\mathcal{F}(t,\xi)$ having the following representations:

$$\begin{cases} \mathcal{N}(t,\xi) \sim I + \mathcal{N}^{(1)}(t,\xi) + \mathcal{N}^{(2)}(t,\xi) + \dots \quad \mathrm{mod}\mathcal{H}\{0,0,0\}_{M,N} \\ \mathcal{N}^{(\nu)}(t,\xi) \in \mathcal{S}\{-\nu,-\nu,\nu\}_{M,N}\,, \quad \nu = 1,2,\dots\,, \end{cases}$$

$$\begin{cases} \mathcal{F}(t,\xi) \sim \mathcal{F}^{(0)}(t,\xi) + \mathcal{F}^{(1)}(t,\xi) + \dots \quad \mathrm{mod}\,\mathcal{H}\{0,0,1\}, \\ \mathcal{F}^{(\nu)}(t,\xi) \in \mathcal{S}\{-\nu,-\nu,\nu+1\}_{M,N}\,, \quad \nu = 0,1,2,\dots\,. \end{cases}$$

Let us choose $\mathcal{F}^{(0)}(t,\xi) = \mathrm{diag}[\mathcal{B}(t,\xi)]$. Here $\mathrm{diag}[\mathcal{B}]$ is a diagonal part of the matrix $\mathcal{B}$. If we set $\mathcal{B}^{(0)}(t,\xi) = \mathcal{B}(t,\xi)$,

$$\mathcal{B}^{(\nu+1)} = (D_t + i\mathcal{D} + \mathcal{B})(1 + \sum_{\mu=1}^{\nu+1} \mathcal{N}^{(\mu)}) - (I + \sum_{\mu=1}^{\nu+1} \mathcal{N}^{(\mu)})(D_t + i\mathcal{D} + \sum_{\mu=0}^{\nu} \mathcal{F}^{(\mu)})\,, \quad \nu = 0,1,\dots\,,$$

$$\mathcal{F}^{(\nu)}(t,\xi) = \mathrm{diag}[\mathcal{B}^{(\nu)}(t,\xi)], \quad \nu = 0,1,\dots\,,$$

$$\mathcal{N}^{(\nu+1)}_{j,k}(t,\xi) = \begin{cases} \mathcal{B}^{(\nu+1)}_{j,k}(t,\xi)/(\tau_j(t,\xi) - \tau_k(t,\xi)), & \text{when} \quad j \neq k, \\ 0, \quad \text{when} \quad j = k, & \text{for all} \quad j,k = 1,\dots,m, \end{cases}$$

then (i),(ii) follows from Proposition 2.1. For

$$\mathcal{R} = -\mathcal{N}^{\#}\{(D_t + i\mathcal{D} + \mathcal{B})\mathcal{N} - \mathcal{N}(D_t + i\mathcal{D} + \mathcal{F})\}$$

the property (3.8) holds. The proposition is proved.

4. Construction of exact solutions in exterior zone

Now we are going to construct linear independent solutions $Y_j(t,\xi)$, $j = 1,\dots,m$, of the system

$$(D_t + i\mathcal{D}(t,\xi) + \mathcal{F}(t,\xi) - \mathcal{R}(t,\xi))Y = 0. \tag{4.1}$$

We are looking for a solutions having representations

$$Y_j(t,\xi) = e^{\tilde{\Phi}_j(t,\xi)} A_j(t,\xi), \qquad j = 1,\dots,m, \tag{4.2}$$

where

$$\tilde{\Phi}_j(t,\xi) = \int_{t_j}^{t} (\tau_j(s,\xi) - i f_j(s,\xi))\, ds, \qquad j = 1,\dots,m\,, \tag{4.3}$$

and t_j, $j = 1,\dots,m$, are the points of $[t_\xi, T]$.

If we denote diagonal matrix $\mathcal{D}(t,\xi) - i\mathcal{F}(t,\xi)$ by $\mathcal{Q}(t,\xi)$ with the elements $q_j(t,\xi)$ then we can rewrite (4.1) in the following form

$$\partial_t Y = \mathcal{Q}(t,\xi)Y + i\mathcal{R}(\sqcup,\xi)Y \tag{4.4}$$

for the vector $Y(t,\xi) = {}^{tr}(y_1(t,\xi), y_2(t,\xi), \ldots, y_m(t,\xi))$. Let t_j, $j = 1,\ldots,m$, be the points of $[t_\xi, T]$, then from (4.4) it follows

$$\left(e^{-\int_{t_j}^t q_j(\tau,\xi)d\tau} y_j(t,\xi)\right)_t' = e^{-\int_{t_j}^t q_j(\tau,\xi)d\tau} \left(i\mathcal{R}(t,\xi)Y(t,\xi)\right)_j, \quad j = 1,\ldots,m.$$

Hence

$$y_j(t,\xi) = y_j(t_j,\xi)e^{\int_{t_j}^t q_j(\tau,\xi)d\tau} + \int_{t_j}^t e^{\int_\tau^t q_j(z,\xi)dz} \left(i\mathcal{R}(\tau,\xi)Y(\tau,\xi)\right)_j d\tau, \quad j = 1,\ldots,m. \tag{4.5}$$

For the simplicity of notations we will construct solution Y_1 of (4.2), which we denote by $\overline{Y} = {}^{tr}(\overline{y_1}, \overline{y_2}, \ldots, \overline{y_m})$ and which has representation

$$\overline{Y}(t,\xi) = e^{\int_{t_1}^t q_1(\tau,\xi)d\tau}\overline{e_1} + e^{\int_{t_1}^t q_1(\tau,\xi)d\tau} W, \tag{4.6}$$

where $\overline{e_1} = {}^{tr}(1,0,\ldots,0)$. Thus we have to find vector-function W only. If we choose

$$\overline{y_1}(t_1,\xi) = 1, \quad \overline{y_2}(t_2,\xi) = \overline{y_3}(t_3,\xi) = \ldots = \overline{y_m}(t_m,\xi) = 0, \tag{4.7}$$

then for $W = {}^{tr}(w_1,\ldots,w_m)$ we get

$$\begin{aligned} w_j(t,\xi) &= \int_{t_j}^t e^{\int_\tau^t (q_j(z,\xi)-q_1(z,\xi))dz} \left(i\mathcal{R}(\tau,\xi)\overline{e_1}\right)_j d\tau \\ &+ \int_{t_j}^t e^{\int_\tau^t (q_j(z,\xi)-q_1(z,\xi))dz} \left(i\mathcal{R}(\tau,\xi)Y(\tau,\xi)\right)_j d\tau, \quad j = 1,\ldots,m. \end{aligned} \tag{4.8}$$

The functions $\varphi_j(t,\xi) = \mathrm{Re}(q_j(t,\xi) - q_1(t,\xi))$ keep a constant sign for all $(t,\xi) \in Z_{ext}(M,N)$ if M large enough. Indeed,

$$\begin{aligned} |Re\,(q_j(t,\xi) - q_1(t,\xi))| &= |Im\,(\tau_j(t,\xi) - \tau_1(t,\xi)) + Im\,(f_j(t,\xi) - f_1(t,\xi))| \\ &\geq \delta_1\lambda(t) <\xi> - const\lambda(t)/\Lambda(t) \geq \delta_2\lambda(t) <\xi> \end{aligned}$$

for all $(t,\xi) \in Z_{ext}(M,N)$ provided that M is large enough and $j \neq 1$.

Thus one can rewrite (4.8) in the following way:

$$W = F + \mathcal{K}W, \tag{4.9}$$

where $F = {}^{tr}(f_1, f_2, \dots, f_m)$,

$$
\begin{aligned}
f_j(t,\xi) &= \int_{t_j}^{t} e^{\int_\tau^t (q_j(z,\xi)-q_1(z,\xi))dz} \left(i\mathcal{R}(\tau,\xi)\overline{e_1}\right)_j d\tau, \quad j = 1,\dots,m, && (4.10)\\
(\mathcal{K}W)_j(t,\xi) &= \int_{t_j}^{t} e^{\int_\tau^t (q_j(z,\xi)-q_1(z,\xi))dz} \left(i\mathcal{R}(\tau,\xi)W(t,\xi)\right)_j d\tau, \quad j = 1,\dots,m, && (4.11)
\end{aligned}
$$

In (4.9)-(4.11) we set $t_j = t_\xi$ when $\varphi_j(t,\xi) \le 0$, while set $t_j = T$ when $\varphi_j(t,\xi) > 0$.

Lemma 4.1. *There exist positive constant M and N such that the equation (4.9) has a solution $W(t,\xi) \in C^\infty(Z_{ext}(M,N))$ which for every R, h and α satisfies inequality*

$$
\left\| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha W(t,\xi) \right\| \le C_{R,k,\alpha} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k <\xi>^{-|\alpha|} (\ln <\xi>)^{-R} \qquad (4.12)
$$

for all $(t,\xi) \in Z_{ext}(M,N)$

Proof. From (4.11) we have

$$
\begin{aligned}
|(\mathcal{K}W)_j(t,\xi)| &\le \left| \int_{t_j}^{t} \|\mathcal{R}(\tau,\xi)\| \|W(\tau,\xi)\| e^{\int_\tau^t \varphi_j(z,\xi)dz} d\tau \right| \le \left| \int_{t_j}^{t} \|\mathcal{R}(\tau,\xi)\| \|W(\tau,\xi)\| d\tau \right| \\
&\le C_R \left| \int_{t_j}^{t} \frac{\lambda(\tau)}{\Lambda(\tau)} (\Lambda(\tau) <\xi>)^{-R} \|W(\tau,\xi)\| d\tau \right|, \quad j = 1,\dots,m,
\end{aligned}
$$

where $\|W(\tau,\xi)\|$ denotes the norm of the vector-function W at the point (τ,ξ).

In view of (0.8) and (0.9) we obtain from the last inequality

$$
|(\mathcal{K}W)_j(t,\xi)| \le C_R (\ln <\xi>)^{-R} \left| \int_{t_j}^{t} \|W(\tau,\xi)\| d\tau \right|, \quad j = 1,\dots,m,
$$

may be with new R and C_R. For every fixed ξ let consider the Banach space $C([t_\xi, T])$ of all continuous vector-functions $W(t,\xi)$ with the norm

$$
\|W\|_{M,N,\xi} = \max_{t\in[t_\xi,T]} \|W(t,\xi)\|.
$$

Then

$$
\|\mathcal{K}W\|_{M,N,\xi} \le d \|W\|_{M,N,\xi}
$$

where constant is independent of ξ, N and can be chosen arbitrary small by increasing M_0 uniformly with respect to $M \in [M_0, \infty]$. By the principal of contraction mappings, equation (4.9) has a solution $W \in C([t_\xi, T])$ which satisfies an estimate

$$
\sup_{(t,\xi)\in Z_{ext}(M,N)} \|W(t,\xi)\| \le C \sup_{(t,\xi)\in Z_{ext}(M,N)} \|F(t,\xi)\| \qquad (4.13)
$$

where constant C is independent of M, N. Moreover, it is evident that $W \in C^\infty(Z_{ext}(M,N))$ and that in view of Lemma 1.1 induction in (4.9) can be applied to get an estimate (4.12). Lemma is proved.

Thus, going back to equation (0.1) we get the following

Theorem 4.1. *Assume that (0.7)-(0.10),(0.11)-(0.13) are satisfied. Then there exist constants M and N such that in $Z_{ext}(M,N)$ an equation (0.1) has linear independent solutions $u_j(t,\xi), j = 1,\dots,m$, having representations*

$$u_j(t,\xi) = e^{\int_{t_\xi}^t \tau_j(s,\xi)ds} a_j(t,\xi), \quad j = 1,\dots,m, \tag{4.14}$$

with amplitude functions $a_j(t,\xi), j = 1,\dots,m$, such that (0.6) holds for all $(t,\xi) \in Z_{ext}(M,N)$.

If in addition to (0.11) the condition (0.25) is satisfied then there are linear independent solutions with representations

$$u_j(t,\xi) = e^{\int_{t_\xi}^t \lambda_j(s,\xi)ds} a_j(t,\xi), \quad j = 1,\dots,m. \tag{4.15}$$

Proof. We have proved that $Y_j(t,\xi), j = 1,\dots,m$, of (4.2) exists with corresponding $W_j = {}^{tr}(w_{j1}, w_{j2},\dots,w_{jm})$ satisfying (4.12). It follows that for $A_j(t,\xi)$ of (4.2) for every $k, \alpha, \varepsilon > 0$, an inequality

$$\left\| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha A_j(t,\xi) \right\| \le C_{k,\alpha} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k <\xi>^{\varepsilon-|\alpha|} \tag{4.16}$$

holds for all $(t,\xi) \in Z_{ext}(M,N)$. Further, there exists a constant K such that for every k and α the inequality

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \exp\left(-i\int_{t_j}^t f_j(s,\xi)ds\right) \right| \le C_{k,\alpha} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k <\xi>^{K-|\alpha|} \tag{4.17}$$

holds for all $(t,\xi) \in Z_{ext}(M,N)$. This proves a representation (4.17).

If (0.25) are satisfied then according to Corollary 1.1 one has

$$\left| \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \exp\int_{t_j}^t (\tau_j(s,\xi) - \lambda_j(s,\xi))\,ds \right| \le C_{k,\alpha} \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k <\xi>^{K-|\alpha|}. \tag{4.18}$$

Hence this term can be included in the amplitude function, and we can replace τ_j in (4.17) on λ_j and get (4.18). Theorem is proved.

5. Construction in "inner" zone

Let $\rho(t,\xi)$ be a positive root of the following equation

$$\rho^m - 1 - <\xi> \lambda^m(t)\Lambda(t)^{1-m}|\ln <\xi> |^{m-1} = 0. \tag{5.1}$$

Lemma 5.1. *For every positive ε and every α, k the following inequalities:*

$$\left|\left(\frac{\partial}{\partial\xi}\right)^\alpha \rho(t,\xi)\right| \leq C_\alpha <\xi>^{\varepsilon+1/m-|\alpha|}, \tag{5.2}$$

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial\xi}\right)^\alpha \rho(t,\xi)\right| \leq C_{k,\alpha} <\xi>^{\varepsilon+k-|\alpha|}, \quad k \geq 1, \tag{5.3}$$

hold for all $t \in J$ and all $\xi \in \mathbb{R}^n$. Moreover,

$$\rho(t,\xi) \in \mathcal{S}\{1,1,0\}_{M,N}, \quad \int_0^{t_\xi} \left(\rho(t,\xi) + \frac{\rho_t(t,\xi)}{\rho(t,\xi)}\right) dt \leq K \ln <\xi> . \tag{5.4}$$

Proof. First, we prove the inequality

$$|\partial_\xi^\alpha \rho(t,\xi)| \leq C_\alpha <\xi>^{-|\alpha|} \rho(t,\xi) \quad \text{for all} \quad (t,\xi) \in J \times \mathbb{R}^n_M. \tag{5.5}$$

Indeed, if we denote $A(t,\xi) = 1+ <\xi> \lambda^m(t)\Lambda(t)^{1-m}|\ln <\xi> |^{m-1}, y = (t,\xi), \delta = (j,\alpha)$, then we have

$$\partial_y^\delta \rho = \left\{ - \sum_{\substack{\delta_1+\ldots+\delta_m=\delta,\\ \delta_1\neq\delta,\ldots\delta_m\neq\delta}} (\partial_y^{\delta_1}\rho(y))\ldots(\partial_y^{\delta_m}\rho(y))\delta!/(\delta_1!\cdots\delta_m!) + \partial_y^\delta A \right\} \rho^{1-m}/m. \tag{5.6}$$

For $\delta = (0,\alpha)$ (5.6) gives (5) by induction on $|\alpha|$. It follows from (5.4) that for every positive ε

$$|\partial_t\partial_\xi^\alpha \rho(t,\xi)| \leq C_{\alpha,\varepsilon} <\xi>^{1+\varepsilon-|\alpha|} \quad \text{for all} \quad (t,\xi) \in J \times \mathbb{R}^n_M. \tag{5.7}$$

The inequality (5.7) proves (5.2) and (5.3). The last assertion of the lemma follows from the following inequality:

$$|D_t^j D_\xi^\alpha \rho(t,\xi)| \leq C_{j,\alpha} <\xi>^{1-|\alpha|} \lambda(t)^{j+1}\Lambda(t)^{-j} \quad \text{for all} \quad (t,\xi) \in Z_{ext}(M_1,N). \tag{5.8}$$

The proof is completed.

Furthermore, by means of matrix-valued function

$$H_{int}(t,\xi) = \begin{pmatrix} \rho^{m-1}(t,\xi) & 0 & \cdots & 0 \\ 0 & \rho^{m-2}(t,\xi) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

for the vector $\mathcal{U} := {}^t(\mathcal{U}_1, \mathcal{U}_2, ..., \mathcal{U}_m) = H_{int}(t,\xi)^t(u, D_t u, ..., D_t^{m-1}u)$, the equation (0.1) leads to the system

$$\frac{d}{dt}\mathcal{U} = \mathcal{A}(t,\xi)\mathcal{U}\,. \tag{5.9}$$

Every solution $\mathcal{U}(t,\xi)$ of (5.9) can be represented by the following explicit formula

$$\begin{aligned}\mathcal{U}(t,\xi) &= \mathcal{U}(t_\xi,\xi) + \left(\sum_{l=1}^{\infty}\int_{t_\xi}^{t} ds_1 \int_{t_\xi}^{s_1} ds_2 \cdots \right.\\ &\qquad \left. \cdots \int_{t_\xi}^{s_{l-1}} ds_l \mathcal{A}(s_1)\cdots\mathcal{A}(s_l)\right)\mathcal{U}(t_\xi,\xi)\,. \end{aligned} \tag{5.10}$$

Using an operator $(Ir)(t) = \int_{t_\xi}^t r(s)ds$ one can rewrite (5.10) in the form

$$\mathcal{U}(t,\xi) = \mathcal{U}(t_\xi,\xi) + \sum_{l=1}^{\infty} \underbrace{I\mathcal{A}I\mathcal{A}\cdots I\mathcal{A}I\mathcal{A}}_{l}\mathcal{U}(t_\xi,\xi)\,.$$

For a scalar function g one has $\underbrace{IgIg\cdots Ig}_{l} = (Ig)^l/l! = \left(\int_{t_\xi}^t g(s)ds\right)^l/l!$.

Lemma 5.2. *Let $\mathcal{U}_j(t,\xi)$, $j = 1,\ldots,m$ be solutions of (5.9) which are smooth continuations into zone $Z_{int}(M,N)$ of the vector-functions $H_{ext}(t)^t\left(u_j, D_t u_j, \ldots, D_t^{m-1}u_j\right)$, $j = 1,\ldots,m$. Then there exist positive constants $C_{1,j}$, such that for every k, l the following estimates*

$$\left|\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \xi}\right)^\alpha \mathcal{U}_j(t,\xi)\right| \le C_{k,\alpha} <\xi>^{C_{1,j}-|\alpha|+k/2} \quad \text{for all} \quad (t,\xi)\in Z_{int}(M,N) \tag{5.11}$$

hold.

First we prove the following

Lemma 5.3. *There exist constants m_j, $j = 1,\ldots,m$ such that for every k the following estimates*

$$\left|\left(\frac{\partial}{\partial \xi}\right)^k \mathcal{U}_j(t_\xi,\xi)\right| \le C_k <\xi>^{m_j-k} \tag{5.12}$$

hold for all $\xi \in \mathbb{R}^+_M$.

Proof. In view of (3.5) and (i) of Proposition 3.1 it is enough to consider functions $Y_j(t,\xi)$ constructed in Section 4. To estimate the derivatives $\left(\frac{\partial}{\partial\xi}\right)^k Y_j(t_\xi,\xi)$ one has to take into consideration that $Y_j(t_\xi,\xi) = \overline{e_j} + W_j(t_\xi,\xi)$ where $\overline{e_j} = {}^{tr}(0,\dots,0,1,0,\dots,0)$ has 1 on j-th place, and each of W_j satisfies the estimate (4.12). Lemma is proved.

Proof of Lemma 5.2. Let us consider a matrix-valued function $\mathcal{E}(t,\xi)$ defined as follows

$$\mathcal{E}(t,\xi) := I + \sum_{l=1}^{\infty} \underbrace{I\mathcal{A}I\mathcal{A}\cdots I\mathcal{A}I\mathcal{A}}_{l}\,.$$

Thanks to Lemma 5.3 it is enough to estimate $\mathcal{E}(t,\xi)$ ant its derivatives with respects to ξ only. As a consequence of (5.4) we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^k \mathcal{A}(t,\xi)\right| \le C_k <\xi>^{-k} g(t,\xi) \quad \text{for all} \quad (t,\xi)\in Z_{int}(M,N)\,, \tag{5.13}$$

where the notation $g(t,\xi) = \rho(t,\xi) + \frac{\rho_t(t,\xi)}{\rho(t,\xi)}$ is used. Then it is clear that

$$\|\mathcal{E}(t,\xi)\| \le \sum_{l=0}^{\infty} \frac{1}{l!}\left(C_0\int_t^{t_\xi} g(s,\xi)ds\right)^l \quad \text{for all} \quad (t,\xi)\in Z_{int}(M,N)\,. \tag{5.14}$$

Further, for the derivative $\frac{\partial}{\partial\xi}\mathcal{E}(t,\xi)$ we have

$$\begin{aligned}
\left\|\frac{\partial}{\partial\xi}\mathcal{E}(t,\xi)\right\| &\le \sum_{l=1}^{\infty}\left\|\underbrace{I\frac{\partial\mathcal{A}}{\partial\xi}I\mathcal{A}\cdots I\mathcal{A}I\mathcal{A}}_{l-1}I\mathcal{A}\right\| + \ldots + \sum_{l=1}^{\infty}\left\|\underbrace{I\mathcal{A}I\mathcal{A}\cdots I\mathcal{A}I\frac{\partial\mathcal{A}}{\partial\xi}}_{l-1}I\mathcal{A}\right\| \\
&+ \sum_{l=1}^{\infty}\left\|\underbrace{I\mathcal{A}I\mathcal{A}\cdots I\mathcal{A}I\mathcal{A}}_{l-1}I\frac{\partial\mathcal{A}}{\partial\xi}\right\| + \sum_{l=1}^{\infty}\left\|\underbrace{I\mathcal{A}I\mathcal{A}\cdots I\mathcal{A}I\mathcal{A}}_{l-1}I\frac{\partial t_\xi}{\partial\xi}\mathcal{A}(t_\xi,\xi)\right\| \\
&\le C <\xi>^{-1}\sum_{l=1}^{\infty}\frac{l}{l!}\left(C_1\int_t^{t_\xi} g(s,\xi)ds\right)^{l+1} \\
&+ g(t_\xi,\xi)\left|\frac{\partial t_\xi}{\partial\xi}\right| C\sum_{l=0}^{\infty}\frac{1}{l!}\left(C_1\int_t^{t_\xi} g(s,\xi)ds\right)^l.
\end{aligned}$$

Then, due to (1.1),(1.2),(1.3) and to definition of $g(t,s)$, one has

$$g(t_\xi,\xi)\left|\frac{\partial t_\xi}{\partial\xi}\right| \le C <\xi>^{-1} \quad \text{for all} \quad \xi\in\mathbb{R}^n_M\,.$$

Thus, according to (5.4) we get

$$\begin{aligned}
\left\|\frac{\partial}{\partial\xi}\mathcal{E}(t,\xi)\right\| &\le C <\xi>^{-1}\sum_{l=0}^{\infty}\frac{1}{l!}\left(C_2\int_t^{t_\xi} g(s,\xi)ds\right)^l \\
&\le C <\xi>^{m-1} \quad \text{for all} \quad (t,\xi)\in Z_{int}(M,N)
\end{aligned}$$

with some positive constant m. All other derivatives can be considered in a similar way. Lemma is proved.

Theorem 5.1. *Assume that (0.7)-(0.10),(0.11)-(0.13) are satisfied. Then there exist constants M and N such that smooth continuations in $Z_{int}(M,N)$ of linear independent solutions $u_j(t,\xi), j = 1,\ldots,m$, constructed in Theorem 4.1 admit representations*

$$u_j(t,\xi) = e^{\int_0^t \lambda_j(s,\xi)ds} a_j(t,\xi), \quad j = 1,\ldots,m, \tag{5.15}$$

with amplitude functions $a_j(t,\xi), j = 1,\ldots,m$, such that (0.6) holds for all $(t,\xi) \in Z_{int}(M,N)$.

Proof. Indeed, any solution $\mathcal{U}_j(t,\xi)$ of (5.9) can be written in the form $\mathcal{U}(t,\xi) = \mathcal{V}(t,\xi)\exp\Phi(t,\xi)$, where $\mathcal{V}(t,\xi) = \mathcal{U}(t,\xi)\exp(-\Phi(t,\xi))$. If we set for $\mathcal{U}_j$ the phase function $\Phi(t,\xi) = \Phi_j(t,\xi) = \int_0^t \lambda_j(s,\xi)ds$ then, according to (0.21) and due to Lemma 5.2 we obtain for $\mathcal{V}_j(t,\xi) = \mathcal{U}_j(t,\xi)\exp(-\Phi_j(t,\xi))$ in $Z_{int}(M,N)$ an estimate

$$|\mathcal{V}_j(t,\xi)| \leq |\mathcal{U}_j(t,\xi)|\,|\exp(-\Phi_j(t,\xi))| \leq C <\xi>^{m_{j,int}} .$$

with positive constant $m_{j,int}$. Derivatives of $\mathcal{V}_j(t,\xi)$ can be estimated in a similar way. Theorem is proved.

The construction for $\mathcal{U}_j(t,\xi)$ is completely finished.

To finish the proof of Theorem 0.1 we note only that if $\Lambda(t_2) <\xi>= 2N\ln <\xi>$ then

$$<\xi> \int_{t_\xi}^{t_2} \lambda(s)ds \leq N\ln <\xi> .$$

Therefore the cutt-off functions in (0.26) do not bring any difficulties. Thus, Theorem 0.1 is proved.

Acknowledgements. This work was completed during my research stay in the Max Plank Research Group "Partial Differential Equations and Complex Analysis" at the University of Potsdam. I am grateful to all the members of Max Plank Research Group and especially to Prob.B.-W.Schulze for the helpful and warm hospitality.

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