Yang-Mills connections on quaternionic Kähler quotients

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The purpose of this note is to announce our recent results on quaternionic Kähler manifolds (see Salamon [8] for definition of quaternionic Kähler manifolds). Let (M,g) be a $4n$-dimensional connected quaternionic Kähler manifold with scalar curvature s and let $\mathbb{H}$ be the skew field of quaternions ($\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$). Furthermore, let ρ be an $Sp(n)\cdot Sp(1)$-module induced by adjoint representation of $Sp(1)$. Then the vector bundle V corresponding to ρ is a subbundle in $End(TM)$, whose rank is three. The Levi-Civita connection induces a metric connection on $End(TM)$ naturally. The subbundle V is preserved by the connection, which is restricted to the connection on V, denoted by ∇. For each point in M, there are local frames I, J, K of V associated to $i, j, k \in sp(1) \subset \mathbb{H}$ on a neighbourhood of the point. We denote by ω_α ($\alpha = I, J, K$), 2-forms $g(\alpha, \ )$ ($\alpha = I, J, K$). Then $\Sigma_{\alpha=I,J,K}\,\omega_\alpha \otimes \alpha$ defined locally can be globalized as a section on M to $\Lambda^2 T^* M \otimes V$, which is denoted by $\Omega \in \Gamma(M, \Lambda^2 T^*M \otimes V)$ (cf.[2]).

Let G be a compact Lie group which acts on M preserving the quaternionic Kähler structure g , V . Let $\mathfrak{g}$ be the Lie algebra of G .

Definition 1 (cf. [2], [5]) A section μ to $\mathfrak{g}^* \otimes V$ is a moment mapping for the action of G on M if

(i) $\nabla(\mu(X)) = \iota_{X^*}\Omega$, where X is an element of $\mathfrak{g}$ and X^* is the Killing vector field associated to X ,

(ii) μ is a G-equivariant mapping.

When the scalar curvature s of M is not zero and G is connected, the moment mapping exists uniquely (see [2] for the proof). By the condition (ii), the set $\mu^{-1}(0)$ is G-invariant. Suppose that $\mu^{-1}(0)$ is a non-empty submanifold in M and that G acts on it freely. Then the quotient $N = \mu^{-1}(0)/G$ is a manifold and g , V are naturally pushed down to the metric $\bar{g}$, the structure bundle $\bar{V}$ on N . The reduction $(N,\bar{g},\bar{V})$ is a quaternionic Kähler manifold of dimension $4m=4n-4\dim(G)$ and it is called a quaternionic Kähler reduction (or hyperkähler reduction when $s = 0$). Now we denote by

$$p : \mu^{-1}(0) \longrightarrow N$$

the principal bundle, which has a natural G-connection η as follows :

the horizontal space is the orthogonal complement to the fibre with respect to g .

On the other hand. the $Sp(m)\cdot Sp(1)$-module $\Lambda^2 \mathbb{H}^m$ is a direct sum $N_2' \oplus N_2'' \oplus L_2$ of its irreducible submodules N_2' , N_2'' , L_2 where N_2' (resp. L_2) is the submodule fixed by $Sp(m)$ (resp. $Sp(1)$) and for $m = 1$, we have $N_2'' = \{0\}$. Hence the vector bundle $\Lambda^2 T^*N$ is written as a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its holonomy invariant subbundles in such a way that A_2' , A_2'' , B_2 correspond to N_2' , N_2'' , L_2 , respectively.

Let $q : Q \longrightarrow N$ be a principal bundle whose fibre is a Lie group K ($\mathfrak{k}$:= the Lie algebra).

Definition 2 (cf. [6]). A connection on $q : Q \longrightarrow N$ is called a B_2-connection if the corresponding curvature is a $\mathfrak{k}$ -valued q^*B_2-form.

Now we obtain :

Theorem. The connection η is a B_2-connection.

Proof. The space $\mu^{-1}(0)$ is a submanifold in M. We denote the second fundamental form by π. By definition the Levi-Civita connection ∇_1 on $\mu^{-1}(0)$ is written as : for vector fields $x, y \in \mathfrak{X}(\mu^{-1}(0))$

$$(1) \qquad \nabla^M_x y = \nabla_{1x} y + \pi(x,y),$$

where ∇^M is the Levi-Civita connection on (M,g). We denote by $\tilde{s}$ and x^V, the horizontal lift of $s \in \mathfrak{X}(N)$ and the vertical component of $x \in \mathfrak{X}(\mu^{-1}(0))$, i.e.

$$\mu(\tilde{s}) = 0 \quad , \qquad p_*(\tilde{s}) = s \quad ,$$
$$\mu(x - x^V) = 0 \quad .$$

By O'Neill's formula (cf.[7]) for Riemannian submersion, if $s, w \in \mathfrak{X}(N)$,

$$(2) \qquad \widetilde{\nabla^N_s w} = \nabla_{1\tilde{s}}\tilde{w} - 1/2[\tilde{s},\tilde{w}]^V ,$$

where ∇^N is the Levi-Civita connection on N. Equations (1), (2) lead to

$$(3) \qquad \widetilde{\nabla^N_s w} = \nabla^M_{\tilde{s}}\tilde{w} - \pi(\tilde{s},\tilde{w}) - 1/2[\tilde{s},\tilde{w}]^V .$$

For any point in N , there exists a local neighbourhood U of it such that the quaternionic structure bundle on N is spanned by I , J , K on U . When we exchange w to Iw ,

$$(4) \qquad \widetilde{\nabla^N{}_s Iw} = \nabla^M{}_{\tilde{s}} \widetilde{Iw} - \pi(\tilde{s}, \widetilde{Iw}) - 1/2[\tilde{s}, \widetilde{Iw}]^V , \quad \text{on } U .$$

If we denote by $\bar{I}$, $\bar{J}$, $\bar{K}$ the pullback of I , J , K to TM on $\mu^{-1}(0)$, then

$$(5) \qquad \widetilde{Iw} = \bar{I}w \quad .$$

Since M is a quaternionic Kähler manifold,

$$(6) \qquad \nabla^M \bar{I} = a_{12} \bar{J} + a_{13} \bar{K} ,$$

where a_{12} , a_{13} are connection forms with respect to the local frame $\bar{I}$, $\bar{J}$, $\bar{K}$. We obtain by (4) , (5) , (6) ,

$$\begin{aligned}(7) \qquad & \bar{I}\nabla^N{}_{\tilde{s}}\tilde{w} + \bar{I}\pi(\tilde{s},\tilde{w}) + 1/2 I[\tilde{s},\tilde{w}]^V + a_{12}(\tilde{s})\bar{J}\tilde{w} + a_{13}(\tilde{s})\bar{K}\tilde{w} \\ & = \widetilde{\nabla^N{}_s Iw} + \pi(\tilde{s},\bar{I}\tilde{w}) + 1/2[\tilde{s},\bar{I}\tilde{w}]^V \quad .\end{aligned}$$

The vertical component of (7) is

$$(\bar{I}\pi(\tilde{s},\tilde{w}))^V = 1/2[\tilde{s},\bar{I}\tilde{w}]^V \quad .$$

Since π is summetric, we obtain :

$$
\begin{aligned}
(8) \qquad [\tilde{s},\bar{I}\tilde{w}]^V &= 2(\bar{I}\pi(\tilde{s},\tilde{w}))^V \\
&= 2(\bar{I}\pi(\tilde{w},\tilde{s}))^V \\
&= [\tilde{w},\bar{I}\tilde{s}]^V \\
&= -[\bar{I}\tilde{s},\tilde{w}]^V \quad .
\end{aligned}
$$

The curvature of η is written as $R(\tilde{s},\tilde{w}) = -\eta([\tilde{s},\tilde{w}]^V)$.
By (8),

$$
\begin{aligned}
R(\widetilde{Is},\widetilde{Iw}) &= -\eta([\widetilde{Is},\widetilde{Iw}]^V) \\
&= -\eta(-[\tilde{s},\widetilde{IIw}]^V) \\
&= -\eta([\tilde{s},\tilde{w}]^V) \\
&= R(\tilde{s},\tilde{w}) \quad .
\end{aligned}
$$

By same argument, $R(\widetilde{Is},\widetilde{Iw}) = R(\widetilde{Js},\widetilde{Jw}) = R(\widetilde{Ks},\widetilde{Kw}) = R(\tilde{s},\tilde{w})$.
Hence the connection η is a B_2-connection.

Examples. (i) Galicki and Lawson proved the reduction space $P^n\mathbb{H}//U(1)$ is complex Grassmann manifold $G_{2,n-1}(\mathbb{C})$ (cf. [2]). The natural connection on $P \longrightarrow G_{2,n-1}(\mathbb{C})$ is a B_2-connection. Furthermore Galicki showed that the reduction space $P^n\mathbb{H}//SU(2)$ is real Grassmann manifold $G_{4,n-3}(\mathbb{R})$ (cf. [1]). It has also a B_2-connection.

(ii) The argument is local. When reduction space $\mu^{-1}(0)/G$ is not a smooth manifold but an orbifold, the connection is a B_2-connection over the orbifold. Galicki and Nitta constructed many quaternionic Kähler orbifolds as quaternionic Kähler reduction spaces (cf. [3]). In these cases the connections are B_2-connections over the quaternionic Kähler orbifolds.

Remark. A corresponding result for the case of hyperkähler reductions was previously obtained by Gocho and Nakajima [4]. Our result is inspired by their result.

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