MAPS BETWEEN p–COMPLETIONS OF THE CLARK–EWING SPACES X(W,p,n)

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Abstract. Let Z_p denote the ring of p–adic integers. Let $W \subset GL(n,Z_p)$ be a finite group such that p does not divide the order of W. The group W acts on $K((Z_p)^n,2)$. Let $X(W,p,n)_p$ be the p–completion of the space $K((Z_p)^n,2) \times_W EW$. We classified homotopy classes of maps between spaces $X(W,p,n)_p$.

0. INTRODUCTION

Let Z_p denote the ring of p–adic integers. Let Y_p denote the p–completion of a space Y.

Let T be a torus and let $W \subset GL(\pi_1(T)\otimes Z_p)$ be a fintie group. The group W acts on the space $(BT)_p$. Let

$$X(W,p,T) := ((BT)_p \times_W EW)_p$$

where EW is a contractible space equipped with a free action of W.

The aim of this paper is to apply the program from [1] to study maps between spaces X(W,p,T). The starting point was an attempt to generalize one result of Hubbuck (see [8] Theorem 1.1.). The plan of work will follow closely that of [3] and [13].

Example. *Let* G *be a connected, compact Lie group,* T *its maximal torus and* W *its Veyl group. If* p *does not divide the order of* W *then* $(BG)_p \approx (BT \times_W EW)_p$.

This example suggests the following defintion.

Definition. Let us set $X = X(W,p,T)$. *We shall call* T *a maximal torus of* X *and* W *a Weyl group of* X.

The projection $(BT)_p \times EW \longrightarrow (BT)_p \times_W EW$ *induces a map* $i : BT \longrightarrow X$. *We shall call* $i : BT \longrightarrow X$ *a structure map of* X.

We point out that in [5] A. Clark and J. Ewing studied cohomology algebras of spaces $(BT)_p \times_W EW$. We warn the reader that our notation is different from the notation used in [5]. The space $X(W,p,T)$ is the p–completion of the Clark-–Ewing space $X(W,p,\text{rank } T)$.

Through the whole paper we shall assume that p is an odd prime. We need this assumption to show Proposition 1.1. It is clear that this assumption is not essential, however we were not able to overcome technical difficulties for $p = 2$.

Now we shall state our main results.

Let us set $X = X(W,p,T)$ and $X' = X(W',p,T')$.

THEOREM 1. Assume that p *does not divide the orders of* W *and* W'. *Then for any map* $f : X \longrightarrow X'$ *there is a map* $\Upsilon : (BT)_p \longrightarrow (BT')_p$ *such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ (BT)_p & \xrightarrow{\quad \Upsilon \quad} & (BT')_p \end{array}$$

commutes up to homotopy. Moreover we have:

a) *if* $\Upsilon' : (BT)_p \longrightarrow (BT')_p$ *is such that* $f \circ i$ *is homotopic to* $i' \circ \Upsilon'$ *then there is* $w \in W'$ such that $w \circ \Upsilon'$ *is homotopic to* Υ,

b) *for any* $w \in W$ *there is* $w' \in W'$ *such that* $\Upsilon \circ w$ *is homotopic to* $w' \circ \Upsilon$.

The group W acts on $\pi_1(T) \otimes Z_p$, hence W acts on $\pi_1(T) \otimes R$ for any Z_p–module R.

DEFINITION 1. Let R *be a* Z_p*–algebra. We say that a homomorphism of* R*–modules*

$$\varphi : \pi_1(T) \otimes R \longrightarrow \pi_1(T') \otimes R$$

is admissible if for any $w \in W$ *there is* $w' \in W'$ *such that* $\varphi \circ w = w' \circ \varphi$.
We say that two admissible maps φ *and* ψ *from* $\pi_1(T) \otimes R$ *to* $\pi_1(T') \otimes R$ *are equivalent if there is* $w \in W'$ *such that* $w \circ \varphi = \psi$.

It is clear that the relation defined above is an equivalence relation on the set of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$. We shall denote by $\mathrm{Ahom}_R(T,T')$ the set of equivalence classes of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

Let us notice that the map $\pi_1(\Upsilon)$ induced by Υ from Theorem 1 on fundamental groups is admissible for $R = Z_p$. This map is unique up to the action of W', so any map $f : X \longrightarrow X'$ determines uniquely an equivalence class of $\pi_1(\Upsilon)$ in $\mathrm{Ahom}_{Z_p}(T,T')$ which we shall denote by $\chi(f)$.

THEOREM 2. Let us assume that p *does not divide the orders of* W *and* W'*. Then the natural map*

$$\chi : [X,X'] \longrightarrow \mathrm{Ahom}_{Z_p}(T,T')$$

is bijective.

For any space Y we set

$$H^*(Y,Q_p) := H^*(Y,Z_p) \otimes Q ,$$

where Q_p is a field of p–adic numbers.

THEOREM 3. Let us assume that p *does not divide the orders of* W *and* W'*. Then the natural map*

$$\phi : [X,X'] \longrightarrow \mathrm{Hom}(H^*(X',\mathbb{Q}_p), H^*(X,\mathbb{Q}_p))$$

is injective.

We denote by $K^0(\ ,R)$ the 0^{th}–term of complex K–theory with R–coefficients. Let $\mathcal{O}_R$ be the set of operations in $K^0(\ ,R)$. The functor $K^0(\ ,R)$ is equipped with the natural augmentation $K^0(\ ,R) \longrightarrow R$. Let $\mathrm{Hom}_{\mathcal{O}_R}(K^0(X',R),K^0(X,R))$ be the set of R–algebra homomorphisms which commute with the action of $\mathcal{O}_R$ and augmentations.

THEOREM 4. If p *does not divides the order of* W *and* W′, *then the natural map*

$$\psi : [X,X'] \longrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_p}}(K^0(X',Z_p),K^0(X,Z_p))$$

is bijective.

We can formulate our results in a nice categorical way.

We shall define a category $Z_p - \mathrm{Rep}$ in the following way. Objects of the category Z_p–Rep are representations $\rho : W \longrightarrow GL(M)$ where M is a free, finitely generated Z_p–module, W is a finite group and p does not divide the order of W. It remains to define morphisms in this category. If $\theta : W \longrightarrow GL(M)$ and $\theta' : W' \longrightarrow GL(M')$ are two objects of $Z_p - \mathrm{Rep}$, we say that a homomorphism of Z_p–modules $f : M \longrightarrow M'$ is admissible if for each $w \in W$ there is $w' \in W'$ such that $f \circ w = w' \circ f$. We say that two admissible homomorphisms f and g from M to M′ are equivalent if there is $w \in W'$ such that $f = w' \circ g$. We shall denote by $\mathrm{Ahom}(\theta,\theta')$ the set of equivalence classes of admissible homomorphisms from M to M′. The set $\mathrm{Ahom}(\theta,\theta')$ is the set of morphisms from θ to θ' in the category Z_p– Rep. The category $Z_p - \mathrm{Rep}$ is equipped with the product defined in the following way:

$$(\theta{:}W \longrightarrow GL(M)) \oplus (\theta'{:}W' \longrightarrow GL(M')) = \theta\oplus\theta' : W \times W' \longrightarrow GL(M \oplus M').$$

The product of morphisms is defined in the obvious way.

We denote by Ht(p) the category whose objects are spaces X(W,p,T) such that p does not divide the order of W. Morphisms in Ht(p) are homotopy classes of maps. The category Ht(p) has products defined in the obvious way.

THEOREM 5. *There is an equivalence of categories*

$$R : Z_p - \mathrm{Rep} \longrightarrow \mathrm{Ht}(p)$$

with products.

THEOREM 6. *In Theorems* 1,2,3 *and* 4 *we can drop the assumption* "p *does not divide the order of* W′ " *if* $X' = (BG)_p$, *where* G *is a connected, compact Lie group.*

COROLLARY 7. *Let* $X = X(W,p,T)$ *and let* p *be a prime not dividing the order of* W. *Let us assume that the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$ *is irreducible. Then there is a finite number of self-maps* $I_1,\dots,I_n$ *of* X *such that for any* $f : X \longrightarrow X$ *there is* k *for which* $f \circ I_k$ *is an Adams* ψ^α*–map i.e. the map induced by* $f \circ I_k$ *on* $H^{2i}(X,\mathbb{Q}_p)$ *is a multiplication by* α^i. *The number* n *is smaller or equal to the number of elements of* Aut(W)/Inn(W) *which preserve the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$.

Example. (see also [3])
Let $X = BSU(n)_p$. *The Weyl group of* SU(n) *is* Σ_n. *If* $n \neq 6$ *then* $\mathrm{Aut}\,\Sigma_n = \mathrm{Inn}\,\Sigma_n$ *and for* $n = 6$ *the outer automorphism does not preserve the natural representation of* Σ_6 *on* $\pi_1(T) \otimes \mathbb{Q}_p$. *This implies that the self-maps of* $BSU(n)_p$ *are Adams* ψ^α*–maps.*

We point out that Corollary 7 can be view as a generalization of a result of Hubbuck (see [8] Theorem 1.1.) The example is a special case of the result of Hubbuck. However, it concerns maps between p–completed spaces $BSU(n)_p$ while Hubbuck is dealing with classical spaces BG.

We would like to thank very much A. Zabrodsky who during the Barcelona conference on algebraic topology 1986 shared with us his unpublished papers and notes. We would like to express our gratitude to the referee for his patient readings of the manuscript, for his useful suggestions which allowed us to generalize substantially our results, and for pointing out several misprints in the manus-

cript.

1. THE LANNES T FUNCTOR FOR SPACES X(W,p,T)

Let $X = X(W,p,T)$. Let us assume that p does not divide the order of W. In this section we shall compute the cohomology of the mapping space map(BV,X) and its connected component $\mathrm{map}_f(BV,X)$ where V is an elementary abelian p–group and $f: BV \longrightarrow X$ is a map

It follows from [5] (see Proposition on p. 425) that

$$H^*(X,F_p) = H^*(BT,F_p)^W .$$

The map $f: BV \longrightarrow X$ induces a map $f^*: H^*(X,F_p) \longrightarrow H^*(BV,F_p)$. Let us notice that $\mathrm{Im}\, f^*$ is contained in the kernel of the Bockstein homomorphism. Hence it suffices to look at the polynomial part of $H^*(BV,F_p)$ when extending f^* to $H^*(BT,F_p)$. It follows from [2] Proposition 1.10 that there is $g^*: H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ such that $f^* = g^* \circ i^*$ where $i^*: H^*(X,F_p) \longrightarrow H^*(BT,F_p)$ is the inclusion induced by a structure map $i: BT \longrightarrow X$.

For a torus T, the solutions in T of $t^p = 1$ make up a subgroup T(1). The map g^* is induced by a homomorphism $\varphi: V \longrightarrow T(1)$. This follows from [9] Theorem 0.4. Let $\Lambda_f: V \otimes T(1)^* \longrightarrow F_p$ be an adjoint map of φ. The group W acts on $\mathrm{Hom}(V \otimes T(1)^*, F_p)$ through its action on $T(1)^*$. Let W_f be the isotropy subgroup of Λ_f.

PROPOSITION 1.1. *Let* $X = X(W,p,T)$. *Let us assume that* p *does not divide the order of* W. *Let* V *be an elementary abelian* p*–group and let* $f: BV \longrightarrow X$ *be any map. Then we have an isomorphism*

$$H^*(\mathrm{map}_f(BV,X);F_p) = H^*(BT,F_p)^{W_f} .$$

PROOF: For a vector space U over F_p let us denote by P(U) the polynomial algebra on U, by $\Lambda(U)$ the exterior algebra on U and by $A(U)$ the symmetric algebra on U divided by the ideal generated by all polynomials $x^p - x$ for $x \in U$. The polynomial $x^p - x$ splits completely over F_p. Hence we have an isomorphism of F_p–algebras $A(U) = \bigoplus_{a \in U^*} F_p$. We point out that $A(U)$ is concentrated in degree zero.

Let us notice that we have the following natural identifications

$$H^*(BT,F_p) = P(T(1)^*)$$

and

$$H^*(BV,F_p) = P(V^*) \otimes \Lambda(\beta^{-1}V^*).$$

To simplify the notation let us set $A := A(V \otimes T(1)^*)$ and $H := H^*(BT,F_p) = P(T(1)^*)$. It follows from Corollary 2 in [4] that for any unstable A_p–algebra M and any A_p–algebra homomorphism $h : P((Z/p)^*) \longrightarrow M \otimes H^*(BZ/p,F_p)$ we have

$$h(t^*) = m_{t*} \otimes 1 + m_{v*} \otimes v^*.$$

This implies that we have a natural isomorphism

$$\Phi_M : \mathrm{Hom}(H;M \otimes H^*(BV)) \approx \mathrm{Hom}\,(A \otimes H;M).$$

where Hom(;) is in the category of unstable A_p–algebras. If $h(t^*) =$

$m_{t*} \otimes 1 + \sum_{v* \in V*} m_{v*} \otimes v^*$ then $\Phi_M(h)([v \otimes t^*] \otimes 1) = \sum_{v* \in V*} m_{v*} \cdot v^*(v)$

and $\Phi_M(h)(1 \otimes t^*) = m_{t*}$.

Hence it follows that

$$(*) \qquad T_V(H) = A \otimes H.$$

If $M = F_p$ then we have an isomorphism

Φ_{F_p} : $\mathrm{Hom}(H;H^*(BV)) \approx \mathrm{Hom}(A \otimes H;F_p)$. The group W acts on H and A through its action on $T(1)^*$. The isomorphism (*) and the fact that the functor $T_V(-)$ is exact implies that

$$(**) \qquad T_V(H^W) = (A \otimes H)^W$$

(see [4] Proposition 3).

Let $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$ be the map induced by f on cohomology. Let $\lambda : T_V(H^*(X,F_p)) \longrightarrow F_p$ be the adjoint map of f^* and let $\bar{\lambda} : T_V(H) \longrightarrow F_p$ be the adjoint map of g^* We recall that $g^* : H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ is such that $f^* = g^* \circ i^*$. The restriction of $\bar{\lambda}$ to $V \otimes T(1)^*$ is equal to Λ_f, where
$\Lambda_f : V \otimes T(1)^* \longrightarrow F_p$ is an adjoint map of $\varphi : V \longrightarrow T(1)$.

It follows from [6] 2.3 Theorem and the equality (**) that

$$H^*(\mathrm{map}_f(BV,X),F_p) \approx T_V(H^*(X,F_p)) \underset{T_V^0(H^*(X,F_p))}{\otimes} F_p \approx (A \otimes H)^W \underset{A^W}{\otimes} F_p .$$

If $V^* \otimes T(1) = \coprod_{W'} W/W'$, as a W–set then $A \approx \bigoplus_{W'} F_p[W/W']$ as a W–module. This follows from the isomorphism $A(U) = \bigoplus_{a \in U^*} F_p$ mentioned at the beginning of the proof. For any $W' \subset W$, $F_p[W/W']^W \approx F_p$. The maps $\bar{\lambda}$ and λ induce $\tilde{\lambda} : A \longrightarrow F_p$ and $\tilde{\tilde{\lambda}} : A^W = \oplus F_p \longrightarrow F_p$. The algebra homomorphism $\tilde{\tilde{\lambda}}$ is the identity on one's of F_p's and it is zero on all others. We recall that the isotropy subgroup of Λ_f is W_f. The fact that $\tilde{\lambda}$ restricts to Λ_f on $V \otimes T(1)^*$ implies that $\tilde{\tilde{\lambda}}$ is the identity on $F_p[W/W_f]^W$. Hence we have the following isomorphisms

$$(A \otimes H)^W \underset{A^W}{\otimes} F_p \approx (F_p[W/W_f] \otimes H)^W \underset{F_p}{\otimes} F_p \approx H^{W_f} . \qquad \square$$

2. MAPS FROM BP TO X

Let T be a torus. For a torus T the solutions in T of $t^{p^n} = 1$ make up a subgroup T(n); let $T(\infty) = \bigcup_n T(n)$. Let us set $M = \pi_1(T) \otimes Z_p$. Let $W \subset GL_{Z_p}(M)$ be a finite group. The action of W on M extends to the action of W on $M \otimes \mathbb{Q}$. The lattice M in $M \otimes \mathbb{Q}$ is invariant therefore W acts also on $M \otimes \mathbb{Q}/_M$. Observe that $M \otimes \mathbb{Q}/_M = T(\infty)$. From the action of W on $T(\infty)$ we can recover the original action of W on M if we take the induced action of W on $(H^2(BT(\infty);Z_p))^*$. Hence any finite subgroup of $GL_{Z_p}(M)$ can be realized as a subgroup of $Aut(T(\infty))$.

PROPOSITION 2.1. *Let* W *be a finite subgroup of* $Aut(T(\infty))$. *Let us assume that* p *does not divide the order of* W. *If* P *is a finite* p*-group then any map* $f: BP \longrightarrow (B(T(\infty)\tilde{\times}W))_p$ *is induced by a homomorphism* $\varphi: P \longrightarrow T(\infty)\tilde{\times}W$.

We were informed that a similar result was also known to W. Dwyer. This proposition is an analog of the theorem of Dwyer and Zabrodsky (see [7] 1.1. Theorem). The proof will follow closely the proof of the Dwyer and Zabrodsky theorem contained in [14], which depends very much on [10].

Let us set $G = T(\infty) \tilde{\times} W$.

LEMMA 2.2. *Let* $V = Z/p$, *let* $\varphi: V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\varphi$ *in* G *and let* $\varphi_0: V \longrightarrow G_0$ *be the map induced by* φ. *Then the map*

$$map_{B\varphi_0}(BV,(BG_0)_p) \longrightarrow map_{B\varphi}(BV,(BG)_p)$$

is a homotopy equivalence.

PROOF: It follows from Proposition 1.1 that $H^*(map_{B\varphi}(BV,(BG)_p),F_p) \approx P^{W_0}$ where $P \approx H^*(BT,F_p)$ and $W_0 = G_0/T(\infty)$ is the isotropy subgroup of $\varphi: V \longrightarrow T(\infty)$. In the same way we get

$H^*(\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p),F_p) = P^{W_0}$. Hence the map considered by us is a homotopy equivalence. □

LEMMA 2.3. *Let* P *be a* p*–group, let* $Z/p = V$ *be a subgroup of the center of* P. *Let* $\varphi : V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\,\varphi$ *in* G *and let* $\varphi_0 : V \longrightarrow G_0$ *be the induced homomorphism. Let*

$$[BP,(BG)_p](B\varphi) = \{f \in [BP,(BG)_p] : f_{|BV} \sim B\varphi\}$$

and let $[BP,(BG_0)_p](B\varphi_0)$ *be defined in an analogous way. Then the inclusion map* $i : G_0 \longrightarrow G$ *induces a bijection*

$$(*) \qquad [BP,(BG_0)_p](B\varphi_0) \longrightarrow [BP,(BG)_p](B\varphi) .$$

PROOF: We have a fibration $BV \longrightarrow BP \longrightarrow B(P/V)$. Let $BV \longrightarrow EP/V \longrightarrow E(P/V)$ be the fibration induced by pulling back over $pr : E(P/V) \longrightarrow B(P/V)$. The group P/V acts on EP/V through maps homotopics to the identity and the space EP/V is a model for BV. It follows from Lemma 2.2 that the map

$$\mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi_0}(EP/V,(BG_0)_p) \to \mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi}(EP/V,(BG)_p))$$

is a homotopy equivalence. There is a bijective correspondence between P/V–maps $E(P/V) \longrightarrow \mathrm{map}_{B\varphi_0}(EP/V,(BG_0)_p)$ and maps $E(P/V) \times_{(P/V)} EP/V \longrightarrow (BG_0)_p$ which composed with $E(P/V) \times EP/V \longrightarrow E(P/V) \times_{(P/V)} EP/V$ are homotopic to $B\varphi_0$. The same bijection holds if we replace φ_0 by φ and G_0 by G. This implies that the induced map on π_0 is the map (*). This finishes the proof. □

LEMMA 2.4. (see [15] 1.5. Lemma) *Let* $\varphi : L \longrightarrow K$ *be a simplicial map. Let* $V_0^{\varphi}(L,X)$ *be the subspace of the space* map.(L,X) *of pointed maps from* L *to* X *consisting of maps* $f : L \longrightarrow X$ *such that*

$f_{|\varphi^{-1}(k)} \sim *$ *for every* $k \in K$. *Let* $map_*(\varphi^{-1}(k),X)$ *be the path component of the constant map in the space of pointed maps* $map.(\varphi^{-1}(k),X)$. *Let us assume that for every* $k \in K$, *the space* $map_*(\varphi^{-1}(k),X)$ *is weakly homotopy equivalent to* $*$. *Then* φ *induces a weak homotopy equivalence*

$$\varphi^* : map.(K,X) \xrightarrow{\approx} V_0^{\varphi}(L,X) .$$

PROOF OF PROPOSITION 2.1: Let us assume that $P = Z/p$. It follows from [2] Proposition 1.10 that $f^* : H^*(BG,F_p) \longrightarrow H^*(BP,F_p)$ factors through $H^*(BT(\infty),F_p)$. But any morphism $H^*(BT(\infty),F_p) \longrightarrow H^*(BP,F_p)$ is of the form $B\varphi$ (see [9] Theorem 0.4). Hence f is induced by a homomorphism.

Let us suppose that any map $f : BP \longrightarrow (BG)_p$ is induced by a homomorphism if the order of P is less or equal to p^{n-1}.

Let the order of P be equal to p^n and let $f : BP \longrightarrow (BG)_p$ be a map. Let $V = Z/p$ be contained in the center of P and let $i : V \longrightarrow P$ be the inclusion.

Assume that the composition

$$BV \xrightarrow{Bi} BP \xrightarrow{f} X$$

is null homotopic. We want to show that f is homotopic to $f_1 \circ Bpr$ where $pr : P \longrightarrow P/_V$ is the natural homomorphism and $f_1 : B(P/_V) \longrightarrow X$ is a map. First we show that the space of pointed maps homotopic to $*$ $map_*(BV,X)$ is weakly contractible. This space is p–local because BV and X are p–local. Let $map_{const}(BV,X)$ be the connected component containing a constant map of map (BV,X). It follows from Proposition 1.1 that

$$H^*(map_{const}(BV,X),F_p) = H^*(BT(\infty),F_p)^W .$$

The last group is of course $H^*(X,F_p)$. Hence the evaluation map $map_{const}(BV,X) \longrightarrow X$ is a weak homotopy equivalence and consequently the space $nap_*(BV,X)$ is weakly contractible. Lemma 2.4 implies that f is homotopic to $f_1 \circ Bpr$. By the inductive assumption f_1 is induced by a homomorphism.

Let us suppose that $f \circ Bi$ is induced by a homomorphism $\varphi : V \longrightarrow G$ and $\varphi(V) \neq 0$. Let G_0 be the centralizer of $\varphi(V)$ in G. It follows from Lemma 2.3 that up to homotopy there is a unique map $f_0 : BP \longrightarrow (BG_0)_p$ such that $BP \xrightarrow{f_0} (BG_0)_p \longrightarrow (BG)_p$ is homotopic to f and f_0 restricted to BV is induced by φ. Let $\rho : G_0 \longrightarrow G_0/\varphi(V)$ be the natural projection. The composition

$$BV \longrightarrow BP \xrightarrow{f_0} (BG_0)_p \xrightarrow{(B\rho)_p} (BG_0/\varphi(V))_p$$

is null–homotopic hence $(B\rho)_p \circ f_0$ factors uniquely as

$$BP \xrightarrow{Bpr} B(P/V) \xrightarrow{f_1} B(G_0/\varphi(V))_p \ .$$

This follows from the previous discussion.

One has the homotopy pullback

$$\begin{array}{ccc} BP & \xrightarrow{f_0} & (B(G_0)_p \\ \Big\downarrow{\scriptstyle Bpr} & & \Big\downarrow{\scriptstyle (B\rho)_p} \\ B(P/V) & \xrightarrow{f_1} & (B(G_0/\varphi(V)))_p \end{array}$$

because $\varphi(V)$ is contained in the center of G_0. By the inductive assumption f_1 is induced by a homomorphism $\varphi_1 : P/V \longrightarrow G_0/\varphi(V)$. We have a pullback of groups

$$\begin{array}{ccc} P & \xrightarrow{\psi} & G_0 \\ \Big\downarrow{\scriptstyle pr} & & \Big\downarrow{\scriptstyle \rho} \\ P/V & \xrightarrow{\varphi_1} & G_0/\varphi(V) \ . \end{array}$$

After applying the functor $(B\ )_p$ we get a homotopy pullback

$$\begin{array}{ccc} BP & \xrightarrow{(B\psi)_p} & (BG_0)_p \\ \Big\downarrow \scriptstyle Bpr & & \Big\downarrow \scriptstyle (B\rho)_p \\ B(P/V) & \xrightarrow{(B\varphi_1)_p} & B(G_0/\varphi(V))_p \end{array} .$$

The map f_0 is homotopic to $(B\psi)_p$ hence f is homotopic to $(B\rho)_p \circ (B\psi)_p$. □

COROLLARY 2.5. *Let* T' *by any torus. Then any map* $g : BT'(\infty) \longrightarrow (BG)_p$ *is induced by a homomorphism* $\alpha : T'(\infty) \longrightarrow T(\infty)$.

PROOF. It follows from Proposition 2.1 that for any n the restriction of g to $BT'(n)$, $g_n : BT'(n) \longrightarrow (BG)_p$ is induced by a homomorphism. Let $S_n = \{\beta : T'(n) \longrightarrow G \mid (B\beta)_p \sim g_n\}$. The restriction of $\beta : T'(n) \longrightarrow G$ to $T'(n-1)$ maps S_n into S_{n-1}. Each set S_n is non–empty and finite. This implies that $\varprojlim_n S_n$ is non–empty. Hence there is a homomorphism $\alpha : T'(\infty) \longrightarrow G$ such that α induces g and factorizes through $T(\infty)$. □

3. PROOFS.

We start with the following lemma.

Lemma 3.1 Let $X = X(W,p,T)$, let $i : BT(\infty) \longrightarrow X$ be a structure map of X and let $w : BT(\infty) \longrightarrow BT(\infty)$ be a map induced by $w \in W$. Then the maps i and $i \circ w$ are homotopic.

Proof. Let $\tilde{w} : BT(\infty) \times EW \longrightarrow BT(\infty) \times EW$ be w on $BT(\infty)$ and a translation by w^{-1} on EW. Observe that $\tilde{w}$ is a covering transformation of the projection $pr : BT(\infty) \times EW \longrightarrow BT(\infty) \times_W EW$. The composition

$BT(\infty) \times EW \xrightarrow{pr} BT(\infty) \times_W EW \longrightarrow (BT(\infty) \times_W EW)_p$ is homotopic to i. Hence i and $i \circ w$ are homotopic. □

PROOF OF THEOREM 1:

It follows from Corollary 2.5 that $f \circ i$ is induced by a homomorphism $\varphi : T(\infty) \longrightarrow T'(\infty)$. We set $\Upsilon = (B\varphi)_p$. □

The proof of point a) is the same as the proof of Theorem 1.7 in [1]. Point b) follows from a) and Lemma 3.1. □

PROOF OF THEOREM 3:

Let $f,g : X \longrightarrow X'$ be two maps such that $H^*(f,\mathbb{Q}_p) = H^*(g,\mathbb{Q}_p)$. Let $i : BT_p \longrightarrow X$ be the map induced by a structure map $i : BT \longrightarrow X$. Corollary 2.5 implies that $f \circ i$ and $g \circ i$ are induced by two homomorphisms $\varphi, \Psi : T(\infty) \longrightarrow T'(\infty) \tilde{\times} W'$. We must show that φ and Ψ are conjugate.

For a finite group π let $R(\pi)$ be its complex representation ring. Let

$$R(T(\infty)) := \varprojlim_{n} R(T(n)) \text{ and } R(T'(\infty) \tilde{\times} W) := \varprojlim_{n} R(T'(n) \tilde{\times} W').$$

The Chern character $ch : K^0(\ ;Z_p) \longrightarrow \prod_i H^{2i}(\ ;\mathbb{Q}_p)$ is injective for spaces $BT(\infty)$ and $B(T'(\infty) \tilde{\times} W') = BT'(\infty) \times_W EW$. The group $R(T(\infty))$ is mapped injectively into $K^0(BT(\infty);Z_p)$. Hence we have

$$R(\varphi) = R(\Psi) : R(T'(\infty) \tilde{\times} W') \longrightarrow R(T(\infty)).$$

For each subgroup $S = Z/p^n$ of $T(\infty)$ the restrictions of φ and to S are conjugate by an element of W' because S is cyclic. The fact that W' is finite implies that the restrictions of φ and to any subgroup Z/p^∞ of $T(\infty)$ are conjugate by some element of W'. Once more the fact that W' is finite and the set of subgroups of the form Z/p^∞ in $T(\infty)$ is uncountable if rank $T > 1$ implies that φ and are conjugate by an element of W'. Hence $f \circ i$ and $g \circ i$ are homotopic. It follows from [12] Theorem 1 that f and g are homotopic. □

PROOF OF THEOREM 2:

We set $\chi(f) = \pi_1(\Upsilon)$ where Υ is the map from Theorem 1. The injectivity of χ follows from Theorem 3. Next one observe that $K^0(X';Z_p) = K^0((BT')_p;Z_p)^W$.

Then the proof of surjectivity is the same as in Theorem 1.5 in [13]. It is a standard application of Theorem 1 from [12]. □

PROOF OF THEOREM 4:
The fact that ψ is injective follows from Theorem 3 and the injectivity of Chern character. The proof of surjectivity is the same as in Theorem 1.5 in [13]. □

PROOF OF THEOREM 5:
Theorem 5 is a direct consequence of Theorem 2. □

PROOF OF THEOREM 6:
Let G be a connected, compact Lie group. Observe that any map $BT(\infty) \longrightarrow (BG)_p$ is induced by a homomorphism $T(\infty) \longrightarrow G$ what is an immediate consequence of [7] 1.1. Theorem. This was the crucial point to prove Theorems 1,2,3 and 4 for $X' = X(W',p,T')$. The proofs of Theorems 1,2 and 3 for $X' = (BG)_p$ are the same. Observe that $K^0((BG)_p;Z_p) = K^0((BT)_p;Z_p)^W$. Hence the proof of Theorem 4 carry over to the case $X' = (BG)_p$. □

PROOF OF COROLLARY 7: If the natural representation of W on $\pi_1(T) \otimes \mathbb{Q}_p$ is irreducible then $\pi_1(\Upsilon) : \pi_2((BT)_p) \longrightarrow \pi_2((BT)_p)$ is an isomorphism or a trivial map. The correspondence $w \longrightarrow w'$ from Theorem 7 point b) is then an isomorphism. The rest is obvious. □

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