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CONFORMAL EMBEDDING AND TWISTED THETA FUNCTIONS AT LEVEL ONE.

SWARNAVA MUKHOPADHYAY AND HACEN ZELACI

ABSTRACT. In this paper, we consider the conformal embedding of $\mathfrak{so}(r)$ into $\mathfrak{sl}(r)$ and study relations between level one $\mathrm{SO}(r)$-theta functions and twisted $\mathrm{SL}(r)$-theta functions coming from parahoric moduli spaces. In particular, we give another proof of a theorem by Pauly-Ramanan [PR01].

1. INTRODUCTION

Let V be a complex vector space of dimension $r \geqslant 5$ equipped with a non-degenerate symmetric bilinear form q and consider the group $\mathrm{SO}(V)$ of linear transformations with trivial determinant preserving the quadratic form. We consider the canonical map of $\mathrm{SO}(V) \to \mathrm{SL}(V)$ and the corresponding map $\mathfrak{so}(r) \to \mathfrak{sl}(r)$ on the level of Lie algebras. This embedding is known to be a *conformal embedding* [BB87a, SW86] i.e. the following equality holds:

$$\frac{2\dim \mathfrak{so}(r)}{r} = \frac{\dim \mathfrak{sl}(r)}{1+r}.$$

We refer the reader to Section 3 for a more precise definition and properties of conformal embeddings. Given a conformal embedding $\varphi : \mathfrak{p} \to \mathfrak{g}$, physcists suggests that the *rational conformal field theory* associated to $\mathfrak{p}$ and $\mathfrak{g}$ are closely related. This has been explored in the works of several authors [NT92, Abe08, Bel09, BP10, Muk16b, Muk16c, MW17]. The main motivation of this paper is to understand the relations among the space of conformal blocks (see Section 3) coming from the embedding $\mathfrak{so}(r) \to \mathfrak{sl}(r)$. We describe the details below.

Let Y be a smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 2$. Consider the moduli stack $\mathcal{M}_Y(\mathrm{SO}(r))$ of $\mathrm{SO}(r)-$bundles over Y and the moduli stack $\mathcal{M}_Y(\mathrm{SL}(r))$ of $\mathrm{SL}(r)-$bundles over Y. The canonical inclusion $f : \mathrm{SO}(r) \subset \mathrm{SL}(r)$ induces a natural map between the moduli stacks

$$f : \mathcal{M}_Y(\mathrm{SO}(r)) \to \mathcal{M}_Y(\mathrm{SL}(r)).$$

Let $\mathcal{D}$ be the determinant line bundle over these stacks. Then there is a natural map induced by global sections. We have the following result:

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Theorem 1.1. *The map* $f^* : H^0(\mathcal{M}_Y(\mathrm{SL}(r)), \mathcal{D}) \to H^0(\mathcal{M}_Y(\mathrm{SO}(r)), \mathcal{D})$ *is an isomorphism.*

In [Bea06], A. Beauville proves that $H^0(\mathcal{M}_Y(\mathrm{SO}(r)), \mathcal{D})$ is dual to the space of $H^0(J_Y^{g-1}, r\Theta)$, where J_Y^{g-1} is the variety parametrizing degree $g-1$ line bundles on Y and Θ is the canonical theta divisor. The result of [Bea06] combined with a result in [BNR89] can be used to prove Theorem 1.1. However our approach is different. We rephrase everything using the language of conformal blocks associated to the Lie algebras $\mathfrak{so}(r)$ and $\mathfrak{sl}(r)$ and prove a more general statement on nodal curves with marked points. We refer the reader to the statements of Theorem 2.1 and Theorem 3.4 for generalizations of Theorem 1.1. As a consequence, we can compare the Hitchin connections for the groups $\mathrm{SO}(r)$ and $\mathrm{SL}(r)$. We refer the reader to Corollary 3.6 for a precise result.

1.1. Twisted non abelian theta functions for $\mathrm{SL}(r)$.

Let as before Y be a smooth projective curve of genus g. We denote by ω_Y its canonical bundle. Let $\eta \in J_2(Y)$ be a $2-$torsion line bundle over Y, it defines an étale double cover $\pi : X_\eta \to Y$. Now, let σ be the Galois involution on X_η. A vector bundle E on X_η is called anti-invariant if there exists an isomorphism

$$\psi : \sigma^* E \xrightarrow{\sim} E^*.$$

Moreover, if $\sigma^*\psi = {}^t\psi$ (resp. $\sigma^*\psi = -{}^t\psi$) the pair (E, ψ) is called $\sigma-$symmetric (resp. $\sigma-$alternating). We refer to [Zel16, Section 3] for more detail and properties of these bundles.
Consider the moduli stack $\mathcal{SU}_{X_\eta}^{\sigma,+}(r)$ (resp. $\mathcal{SU}_{X_\eta}^{\sigma,-}(r)$) of $\sigma-$symmetric (resp. $\sigma-$alternating) anti-invariant vector bundles over X_η with trivial determinant. These stacks are connected and $\mathcal{SU}_{X_\eta}^{\sigma,-}(2m+1) = \emptyset$ (see [Zel16]). Moreover, these stacks are isomorphic to moduli stacks $\mathscr{M}_Y(\mathscr{G}_\pm)$ of parahoric $\mathscr{G}_\pm-$torsors over Y, where $\mathscr{G}_\pm$ are some parahoric non constant group schemes over Y (see [Zel17b]). Parahoric torsors are a natural generalization of parabolic bundles and theta functions for parahoric torsors are of considerable interest. In this paper we consider the moduli stack $\mathcal{SU}_{X_\eta}^{\sigma,+}(r)$.

Now consider the Norm map $\mathrm{Nm} : \mathrm{Pic}(X_\eta) \to \mathrm{Pic}(Y)$ and the subvariety $\mathrm{Nm}^{-1}(\omega_Y)$. Since the cover $X_\eta \to Y$ is étale, $\mathrm{Nm}^{-1}(\omega_Y)$ have two connected components P_η^{ev} and P_η^{od}, where the superscript corresponds to the parity of dimensions of the global sections line bundles. Similarly let $\mathrm{Nm}^{-1}(\mathcal{O}_Y) = P_\eta^0 \sqcup P'_\eta$.

It was shown in [Zel17a] that the restriction of the *determinant of cohomology* line bundle on $\mathcal{M}_{X_\eta}(\mathrm{SL}(r))$ to $\mathcal{SU}_{X_\eta}^{\sigma,\pm}(r)$ has a square root $\mathcal{P}_\lambda$ associated to each $\lambda \in P_\eta^{ev} \sqcup P_\eta^{od}$, called *Pfaffian* of cohomology line bundle. For a fixed theta characteristics κ such that $\pi_\eta^*\kappa \in P_\eta^{ev}$, we denote the corresponding line bundle by $\mathcal{P}$.

The pull-back via π_η induces a morphism (see [Zel17c, Proposition B.0.4])

$$\pi_\eta^* : \mathcal{M}_Y(\mathrm{SO}(r)) \to \mathcal{SU}_{X_\eta}^{\sigma,+}(r).$$

If $\eta = 0$, then $X_\eta = Y \sqcup Y$ and $\mathcal{SU}_{X_\eta}^{\sigma,+}(r) = \mathcal{M}_Y(\mathrm{SL}(r))$. Motivated by the identifications in Theorem 1.1, we show the following:

Theorem 1.2. *Let $\eta \neq 0$, the pull-back map π_η^* induce*

- *an isomorphism $H^0(\mathcal{SU}_{X_\eta}^{\sigma,+}(2m+1), \mathcal{P}) \simeq H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D} \otimes \mathcal{L}_\eta)$,*
- *and an injection $H^0(\mathcal{SU}_{X_\eta}^{\sigma,+}(2m), \mathcal{P}) \hookrightarrow H^0(\mathcal{M}_Y(\mathrm{SO}(2m)), \mathcal{D} \otimes \mathcal{L}_\eta))$,*

where $\mathcal{L}_\eta$ is the 2-torsion sheaf on $\mathcal{M}_Y(\mathrm{SO}(r))$ associated to η.

Observe that if $\eta = 0$, Theorem 1.2 is a generalization of Theorem 1.1. However the strategy of the proof of Theorem 1.2 is different from the proof of Theorem 1.1. Using the same strategy in [BNR89], H. Zelaci [Zel17a] showed that the space of twisted theta functions $H^0(\mathcal{SU}_{X_\eta}^{\sigma,+}(r), \mathcal{P})$ is identified with the dual space of abelian theta functions $H^0(P_\eta^{ev}, r\Xi^{ev})^\vee$, where Ξ^{ev} is the canonical Pfaffian divisor in P_η^{ev} (see Section 4). We generalize an idea of Beauville ([Bea06]) to give an alternate proof of a conjecture in [Oxb99] which was originally proved by C. Pauly and S. Ramanan [PR01]:

Theorem 1.3. *Let $\eta \neq 0$ in $J_2(Y)$, then there is a natural duality between*

$$H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D} \otimes \mathcal{L}_\eta) \simeq H^0(P_\eta^{ev}, (2m+1)\Xi^{ev})^\vee$$

$$H^0(\mathcal{M}_Y(\mathrm{SO}(2m), \mathcal{D} \otimes \mathcal{L}_\eta)) \simeq H^0(P_\eta^{ev}(2m)\Xi^{ev})^\vee \oplus H^0(P_\eta^{od}, (2m)\Xi_\alpha^{od})^\vee,$$

where $P_\eta^{ev} \sqcup P_\eta^{od} := \mathrm{Nm}^{-1}(\omega_Y)$ and Ξ_α^{od} is a translate of Ξ^{ev} by a two torsion point $\alpha \in P_\eta'$.

Once this is done, we apply the result in [Zel17a] and a commutative diagram to Theorem 1.3 to give a proof of Theorem 1.2.

2. Comparing SO(r) and SL(r) theta functions at level one

Let G be a complex Lie group and Y be a smooth curve. We denote by $\mathcal{M}_Y(G)$-the moduli stack of principal G bundles on a curve Y. Consider the map of complex Lie groups $\mathrm{SO}(r) \hookrightarrow \mathrm{SL}(r)$. This map lifts to a map $\mathrm{Spin}(r) \to \mathrm{SL}(r)$. By functoriality, we get a map between the corresponding moduli stack of principal bundles on Y.

$$\begin{tikzcd} \mathcal{M}_Y(\mathrm{Spin}(r)) \arrow[d] \arrow[dr] & \\ \mathcal{M}_Y(\mathrm{SO}(r)) \arrow[r] & \mathcal{M}_Y(\mathrm{SL}(r)) \end{tikzcd}$$

Since the moduli stack $\mathcal{M}_Y(\mathrm{SO}(r))$ has two connected components, it turns out that $\mathcal{M}_Y(\mathrm{Spin}(r))$ maps only to the component of $\mathcal{M}_Y(\mathrm{SO}(r))$ containing the trivial bundle. To take the other components of $\mathcal{M}_Y(\mathrm{SO}(r))$ into account, following [BLS98] we consider the twisted Spin bundles. More precisely, consider the *Special Clifford Group*

$$\mathrm{SC}(r) := \mathrm{Spin}(r) \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{C}^*,$$

where we have identified $\mathbb{Z}/2\mathbb{Z}$ as a subgroup of the center of $\mathrm{Spin}(r)$. Now the projection from $\mathrm{SC}(r)$ to $\mathbb{C}^*$ is the *Norm* map and it is well known that it's kernel is $\mathrm{Spin}(r)$. We consider the corresponding map between the moduli stacks:

$$\mathrm{Norm} : \mathcal{M}_Y(\mathrm{SC}(r)) \to \mathrm{Pic}(Y)$$

We define the stack of twisted $Spin(r)$ bundles $\mathcal{M}_Y^-(\mathrm{Spin}(r))$ to be the sub-stack of $\mathcal{M}_Y(\mathrm{SC}(r))$ such that

$$\mathcal{M}_Y^-(\mathrm{Spin}(r)) := \{P \in \mathcal{M}_Y(\mathrm{SC}(r)) |\, \mathrm{Norm}(P) = \mathcal{O}_Y(p)\},$$

where p is a fixed point on the curve X. Now let us define

$$\mathcal{N}_Y(r) := \mathcal{M}_Y(\mathrm{Spin}(r)) \sqcup \mathcal{M}_Y^-(\mathrm{Spin}(r)). \tag{2.1}$$

Then we have a natural map $\mathrm{SC}(r) \to \mathrm{SL}(r)$ which factors $\mathrm{SO}(r)$, we have the following diagram of moduli stacks in which the vertical arrow is surjective:

$$\begin{tikzcd} \mathcal{N}_Y(r) \arrow[d, "\pi"] \arrow[dr, "\tilde{f}"] & \\ \mathcal{M}_Y(\mathrm{SO}(r)) \arrow[r, "f"] & \mathcal{M}_Y(\mathrm{SL}(r)) \end{tikzcd}$$

As in the introduction, let $\mathcal{D}$ be the ample generator of the Picard group of $\mathcal{M}_Y(\mathrm{SL}(r))$, we prove have the following Theorem:

Theorem 2.1. *The pull back of $\tilde{f}$ gives an injection between the space of theta functions:*

$$\tilde{f}^* : H^0(\mathcal{M}_Y(\mathrm{SL}(r)), \mathcal{D}) \hookrightarrow H^0(\mathcal{N}_Y(r), \mathcal{D})$$

We postpone the proof of Theorem 2.1 to the Section 3, where we generalize the statements in the language of conformal blocks on nodal curves. We now explain, how the statement of Theorem 1.1 follows from Theorem 2.1.

2.1. **Proof of Theorem 1.1.** We begin the following proposition whose proof follows easily from [BLS98]:

Proposition 2.2. *The map $\pi^* : H^0(\mathcal{M}_Y(\mathrm{SO}(r)), \mathcal{D}) \to H^0(\mathcal{N}_Y(r), \mathcal{D})$ is injective.*

Proof. Let for any group G, denote by $\underline{M}_Y^{reg}(\mathrm{G})$ the coarse moduli space of stable G bundles P on a curve Y such that $\mathrm{Aut}(P)$ is equal to the center of G. By [BLS98], the map π restricted to the regularly stable locus $\underline{M}_Y^{reg}(\mathrm{SO}(r))$ is a étale Galois cover with Galois group $J_2(Y)$. Now since the line bundle $\mathcal{D}$

on $\mathcal{N}_Y(r)$ is pulled back from $\mathcal{M}_Y(\mathrm{SO}(r))$, by taking invariants with respect to $J_2(Y)$, we get an injection

$$\pi^* : H^0(\underline{M}_Y^{reg}(\mathrm{SO}(r)), \mathcal{D}) \hookrightarrow H^0(\pi^{-1}(\underline{M}_Y^{reg}(\mathrm{SO}(r))), \mathcal{D})$$

Now since the codimension of the complement of the regularly stable locus in the moduli space of semistable bundles is at least 2, the result follows by Hartogs theorem. □

To prove Theorem 1.1, we need to show that f^* is an isomorphism. The map $\tilde{f}^*$ factors through f^* and we have the following diagram

$$\begin{tikzcd}
 & H^0(\mathcal{N}_Y(r), \mathcal{D}) \\
H^0(\mathcal{M}_Y(\mathrm{SL}(r)), \mathcal{D}) \arrow[ur, "\tilde{f}^*"] \arrow[r, "f^*"] & H^0(\mathcal{M}_Y(\mathrm{SO}(r)), \mathcal{D}) \arrow[u, "\pi^*"']
\end{tikzcd}$$

Now by Theorem 2.1, we know that $\tilde{f}^*$ is injective. Now Proposition 2.2 tells us that π^* is injective. This implies f^* is injective. Now by the Verlinde formula [AMW02, AMW01], we get

$$\dim H^0(\mathcal{M}_Y(\mathrm{SL}(r)), \mathcal{D}) = \dim H^0(\mathcal{M}_Y(\mathrm{SO}(r)), \mathcal{D}) = r^{g_Y},$$

where g_Y is the genus of Y. This completes the proof of Theorem 1.1.

3. Generalization with conformal blocks

Let $\mathfrak{g}$ be a simple complex Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra. Let Δ be the set of root associated to $(\mathfrak{g}, \mathfrak{h})$. We choose a Cartan-Killing form ( ) normalized such that $(\theta, \theta) = 2$ for the highest root $\theta \in \Delta$.

Associated to $\mathfrak{g}$, we consider the corresponding Kac-Moody Lie algebra

$$\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c,$$

with Lie bracket given by the formula.

$$[X \otimes f, Y \otimes g] := [X, Y] \otimes fg + \mathrm{Res}_{t=0}\, g\frac{df}{dt}(X, Y).c$$

Let $P_+(\mathfrak{g})$ denote the set of dominant integral weights of $\mathfrak{g}$. Consider the set of *level ℓ weights*:

$$P_\ell(\mathfrak{g}) := \{\lambda \in P_+(\mathfrak{g}) | (\lambda, \theta) \leq \ell\}$$

For each $\lambda \in P_\ell(\mathfrak{g})$, there is a unique, irreducible, integrable, highest weight $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda(\mathfrak{g}, \ell)$ on which c acts as scalar multiplication by ℓ.

3.1. **Conformal Blocks.** Consider a projective algebraic curve Y (not necessarily smooth) with n-distinct marked points $\vec{p} = (p_1, \ldots, p_n)$ with chosen formal coordinates $\vec{t} = (t_1, \ldots, t_n)$. We further assume that the curve Y has at most nodal singularities and $(X, \vec{p})$ satisfy the Deligne-Mumford stability conditions.

Let $\mathfrak{g}(Y, \vec{p})$ denote the Lie algebra of $\mathfrak{g}$ valued functions on the punctured curve $Y \backslash \vec{p}$. By Laurent-expansion by local coordinates around the points $\vec{p}$, we get can realize $\mathfrak{g}(Y, \vec{p})$ as a Lie subalgebra of $\bigoplus_{i=1}^{n} \mathfrak{g} \otimes \mathbb{C}((t_i)) \oplus \mathbb{C}c$. Now for an n-tuple of level ℓ weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, consider the tensor product of highest weight representations

$$\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1}(\mathfrak{g}, \ell) \otimes \cdots \otimes \mathcal{H}_{\lambda_n}(\mathfrak{g}, \ell).$$

We recall the following definition from [TUY89].

Definition 3.1. We define the *space of covacua* $\mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{g}, \ell)$ to be the largest quotient of $\mathcal{H}_{\vec{\lambda}}$ on which $\mathfrak{g}(Y, \vec{p})$ acts trivially. We define the space of *conformal blocks* $\mathcal{V}^{\dagger}_{\vec{\lambda}}(Y, \mathfrak{g}, \ell)$ to be the vector space dual of $\mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{g}, \ell)$.

We recall some important properties of conformal blocks:

- As Y varies, the space of conformal blocks give a vector bundle $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ on the Deligne-Mumford compactification $\overline{\mathrm{M}}_{g,n}$ of the moduli stack of genus g curves with n-marked points [TUY89]
- We can compute the dimension of the space of conformal blocks by the Verlinde formula [AMW01, AMW02, TUY89, Fal94]
- The bundle $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ restricted to the interior $\mathrm{M}_{g,n}$ carries a flat projective connection [TUY89] known as the TUY/WZW/Hitchin connection.

3.2. **Conformal Embedding.** Let $\mathfrak{p}$ be a simple complex Lie algebra and $\mathfrak{g}$ be a simple algebra with an embedding $\varphi : \mathfrak{p} \to \mathfrak{g}$. The Dynkin index d_{φ} of the embedding φ is the ratio of the normalized Cartan-Killing forms. If $\mathfrak{p}$ is semisimple, then we define the Dynkin multi-index of the embedding to be the collection of the Dynkin indices for each semisimple component.

Definition 3.2. An embedding $\varphi : \mathfrak{p} \to \mathfrak{g}$ is called conformal if the following identities holds:

$$\frac{d_{\varphi} \dim \mathfrak{p}}{d_{\varphi} + h^{\vee}(\mathfrak{p})} = \frac{\dim \mathfrak{g}}{1 + h^{\vee}(\mathfrak{g})},$$

where $h^{\vee}(\mathfrak{s})$ is the dual Coxeter of a Lie algebra $\mathfrak{s}$.

We refer the reader to [KW88] for more details. An salient feature of conformal embeddings is the equality of Virasoro operators for $\mathfrak{p}$ and $\mathfrak{g}$ and finiteness of branching of level one representations. Conformal subalgebras of simple Lie algebras has been classified [BB87b, SW86]. Given a conformal subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Dynkin index d, let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ (respectively

$\vec{\Lambda} = (\Lambda_1, \dots, \Lambda_n))$ be an n-tuple of level d(respectively 1) weights of $\mathfrak{p}$ (respectively $\mathfrak{g}$) such that $\mathcal{H}_{\lambda_i}(\mathfrak{p}, d) \hookrightarrow \mathcal{H}_{\Lambda_i}(\mathfrak{g}, 1)$. Taking tensor products, we get a map

$$\varphi : \mathcal{H}_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\Lambda}}.$$

Now let Y be a stable nodal curves with n-distinct marked points. Taking invariants with respect to $\mathfrak{p}(Y, \vec{p})$ on the left and $\mathfrak{g}(Y, \vec{p})$ on the right, we get a map of conformal blocks

$$\varphi : \mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{p}, d) \to \mathcal{V}_{\vec{\Lambda}}(Y, \mathfrak{g}, 1).$$

The map φ can be defined as a map of locally free sheaves on $\overline{\mathrm{M}}_{g,n}$. We refer the reader to [Muk16a].

The standard embedding of $\varphi : \mathfrak{so}(r) \to \mathfrak{sl}(r)$ is known to be conformal and the Dynkin index of the embedding is $d_\varphi = 2$. We now recall the branching rules for the above mentioned embedding.

3.2.1. *Branching rules.* The level one weights of the Lie algebra are the following:

$$P_1(\mathfrak{sl}(r)) = \{\omega_0, \omega_1, \dots, \omega_{r-1}\}$$

We now describe how the module $\mathcal{H}_{\omega_i}(\mathfrak{sl}(r), 1)$. The following result is due to [KP81, KW88]:

Proposition 3.3. *The module $\mathcal{H}_{\omega_i}(\mathfrak{sl}(r), 1)$ restricted to $\widehat{\mathfrak{so}}(r)$ as follows:*

1. *If $i = 0$, then $\mathcal{H}_{\omega_0}(\mathfrak{sl}(r), 1) \simeq \mathcal{H}_{2\omega_0}(\mathfrak{so}(r), 2) \oplus \mathcal{H}_{2\omega_1}(\mathfrak{so}(r), 2)$.*
2. *If $1 \le i \le \lfloor r/2 \rfloor - 2$, then $\mathcal{H}_{\omega_i}(\mathfrak{sl}(r), 1) \simeq \mathcal{H}_{\omega_i}(\mathfrak{so}(r), 2)$.*
3. *If $r = 2m + 1$, then*
   (i) $\mathcal{H}_{\omega_{m-1}}(\mathfrak{sl}(2m + 1), 1) \simeq \mathcal{H}_{\omega_{m-1}}(\mathfrak{so}(2m + 1), 2)$,
   (ii) $\mathcal{H}_{\omega_m}(\mathfrak{sl}(2m + 1), 1) \simeq \mathcal{H}_{\omega_{2\omega_m}}(\mathfrak{so}(2m + 1), 2)$.
4. *If $r = 2m$, then*
   (i) $\mathcal{H}_{\omega_{m-1}}(\mathfrak{sl}(2m), 1) \simeq \mathcal{H}_{(\omega_{m-1}+\omega_m)}(\mathfrak{so}(2m), 2)$,
   (ii) $\mathcal{H}_{\omega_m}(\mathfrak{sl}(2m), 1) \simeq \mathcal{H}_{2\omega_{m-1}}(\mathfrak{so}(2m), 2) \oplus \mathcal{H}_{\omega_{2\omega_m}}(\mathfrak{so}(2m), 2)$.

3.3. **Generalization of Theorem 2.1.** Let $P_+(\mathrm{SO}(r))$ be the set of dominant integral weights of the group $\mathrm{SO}(r)$. Then we define

$$P_\ell(\mathrm{SO}(r)) := \{\lambda \in P_+(\mathrm{SO}(r)) | (\lambda, \theta) \le \ell\}.$$

We have the following description of the $P_2(\mathrm{SO}(r))$:

- $P_2(\mathrm{SO}(2m + 1)) = \{\omega_0, \omega_1, \dots, \omega_{m-1}, 2\omega_1, 2\omega_m\}$.
- $P_2(\mathrm{SO}(2m)) = \{\omega_0, \omega_1, \dots, \omega_{m-2}, \omega_{m-1} + \omega_m, 2\omega_1, 2\omega_{m-1}, 2\omega_m\}$.

Let Y be a stable nodal curve with n distinct marked points and consider the conformal embedding $\mathfrak{so}(r) \to \mathfrak{sl}(r)$. Let $\vec{\lambda} = (\lambda_1, \dots, \lambda_n) \in P_2^n(\mathrm{SO}(r))$ and and similarly let $\vec{\Lambda} = (\Lambda_1, \dots, \Lambda_n) \in P_1^n(\mathfrak{sl}(r))$ such that for each $1 \le i \le n$, $\mathcal{H}_{\lambda_i}(\mathfrak{so}(r), 2) \hookrightarrow \mathcal{H}_{\Lambda_i}(\mathfrak{sl}(r), 1)$.

Theorem 3.4. *With the above notation, we have the following statements about surjectivity of conformal blocks arising from the branching rules of the conformal embedding $\mathfrak{so}(r) \to \mathfrak{sl}(r)$.*

(1) *Assume $r = 2m+1$ is odd and each $\lambda_i \in \{\omega_0, \dots, \omega_{m-1}, 2\omega_m\}$ and $\vec{\Lambda} \neq (\omega_0, \dots, \omega_0)$, then the map of conformal blocks is surjective.*

$$\mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\vec{\Lambda}}(Y, \mathfrak{sl}(r), 1).$$

(2) *Assume $r = 2m+1$ and $n = 1$ and $\Lambda_1 = \omega_0$, then the map of conformal blocks is surjective*

$$\mathcal{V}_{\omega_0}(Y, \mathfrak{so}(r), 2) \oplus \mathcal{V}_{2\omega_1}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\omega_0}(Y, \mathfrak{sl}(r), 1).$$

(3) *Assume $r = 2m$ is even and assume that for each $1 \leq i \leq n$, we have $\lambda_i \in \{\omega_0, \dots, \omega_{m-2}, \omega_{m-1} + \omega_{m-2}\}$. Additionally assume that $\Lambda_i \neq \omega_m$ for all $1 \leq i \leq n$ and $\vec{\Lambda} \neq (\omega_0, \dots, \omega_0)$, then the map of conformal blocks is surjective*

$$\mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\vec{\Lambda}}(Y, \mathfrak{sl}(r), 1).$$

(4) *Suppose $n = 1$, then the map of conformal blocks is surjective:*
   (i) $\mathcal{V}_{\omega_0}(Y, \mathfrak{so}(r), 2) \oplus \mathcal{V}_{2\omega_1}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\omega_0}(Y, \mathfrak{sl}(r), 1)$.
   (ii) $\mathcal{V}_{2\omega_m}(Y, \mathfrak{so}(r), 2) \oplus \mathcal{V}_{2\omega_{m-1}}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\omega_m}(Y, \mathfrak{sl}(r), 1)$.

Proof. First we discuss the case $r = 2m+1$ and we assume that none of the Λ_i's are ω_0. With the assumption, the branching rules in Proposition 3.3 tell us that $\mathcal{H}_{\Lambda_i}(\mathfrak{sl}(r), 1)$ considered as a $\widehat{\mathfrak{so}}(r)$ module is irreducible. Hence there is an isomorphism of $\oplus_{i=1}^n \mathfrak{so}(r) \otimes \mathbb{C}((t_i)) \oplus \mathbb{C}c$-modules

$$\bigotimes_{i}^{n} \mathcal{H}_{\lambda_i}(\mathfrak{so}(2m+1), 2) \simeq \bigotimes_{i=1}^{n} \mathcal{H}_{\Lambda_i}(\mathfrak{sl}(2m+1), 1),$$

where $\Lambda_i \in \{\omega_1, \dots, \omega_m\}$. Since $\mathfrak{so}(2m+1)(Y, \vec{p}) \hookrightarrow \mathfrak{sl}(2m+1)(Y, \vec{p})$, taking quotients, we get a surjective map

$$\mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\vec{\Lambda}}(Y, \mathfrak{sl}(r), 1).$$

Thus we are done in this case.

Now we discuss how to consider the general case. First by *propagation of vacua* [TUY89], we get isomorphisms

$$\Psi_1 : \mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{so}(2m+1), 2) \simeq \mathcal{V}_{\omega_0, \vec{\lambda}}(Y, \mathfrak{so}(2m+1), 2),$$

$$\Psi_2 : \mathcal{V}_{\vec{\Lambda}}(Y, \mathfrak{sl}(2m+1), 1) \simeq \mathcal{V}_{\omega_0, \vec{\Lambda}}(Y, \mathfrak{sl}(2m+1), 1).$$

Moreover the isomorphism Ψ_1 and Ψ_2 are functorial with respect to the branching rule of representations for the embedding $\mathfrak{so}(r) \to \mathfrak{sl}(r)$. In particular there is a commutative diagram

$$\begin{tikzcd}
\mathcal{V}_{\vec{\lambda}}(Y, \mathfrak{so}(2m+1), 2) \arrow[r] \arrow[d, "\Psi_1"] & \mathcal{V}_{\vec{\Lambda}}(Y, \mathfrak{sl}(2m+1), 1) \arrow[d, "\Psi_2"] \\
\mathcal{V}_{\omega_0, \vec{\lambda}}(Y, \mathfrak{so}(2m+1), 2) \arrow[r] & \mathcal{V}_{\omega_0, \vec{\Lambda}}(Y, \mathfrak{sl}(2m+1), 1).
\end{tikzcd}$$

where the vertical maps are isomorphism and the horizontal maps are obtained by restriction as in the statement of the Theorem 3.4. Now if the top horizontal map is surjective it implies that the lower horizontal arrows are also surjective. Thus we are done in this case.

Now we assume that $r = 2m+1$, $n = 1$ and $\Lambda_1 = \omega_0$. By the branching rule in Proposition 3.3, we get an isomorphism of $\widehat{\mathfrak{so}}(2m+1)$-modules

$$\mathcal{H}_{\omega_0}(\mathfrak{sl}(2m+1), 1) \simeq \mathcal{H}_{\omega_0}(\mathfrak{so}(2m+1), 2) \oplus \mathcal{H}_{2\omega_1}(\mathfrak{so}(2m+1), 2).$$

As before taking quotient, we get an surjective map

$$\mathcal{V}_{\omega_0}(Y, \mathfrak{so}(r), 2) \oplus \mathcal{V}_{2\omega_1}(Y, \mathfrak{so}(r), 2) \twoheadrightarrow \mathcal{V}_{\omega_0}(Y, \mathfrak{sl}(r), 1).$$

The proof in the case $r = 2m$ is similar with the observation that when $n = 1$ and $\vec{\Lambda}$ is (ω_0) or (ω_m), then by Proposition 3.3, there are isomorphisms of $\widehat{\mathfrak{so}}(2m)$ module of the form:

$$\mathcal{H}_{\omega_0}(\mathfrak{sl}(2m), 1) \simeq \mathcal{H}_{\omega_0}(\mathfrak{so}(2m), 2) \oplus \mathcal{H}_{2\omega_1}(\mathfrak{so}(2m), 2). \tag{3.1}$$

$$\mathcal{H}_{\omega_m}(\mathfrak{sl}(2m), 1) \simeq \mathcal{H}_{2\omega_m}(\mathfrak{so}(2m), 2) \oplus \mathcal{H}_{2\omega_{m-1}}(\mathfrak{so}(2m), 2). \tag{3.2}$$

□

Remark 3.5. In all the above cases in Theorem 3.4 as the underlying curve Y varies in $\mathrm{M}_{g,n}$, the map of conformal blocks is projectively flat with respect to the Hitchin/WZW connection for the Lie algebra $\mathfrak{so}(r)$ on the source and that of $\mathfrak{sl}(r)$ on the target. This is a formal consequence of the fact that the embedding $\mathfrak{so}(r) \to \mathfrak{sl}(r)$ is conformal [Bel09, NT92].

Thus an immediate corollary of Theorem 3.4 similar in spirit to [AF10] is the following:

Corollary 3.6. *The projective monodromy representations of $\pi_1(\mathrm{M}_g)$ under the WZW/Hitchin connection associated the groups $\mathrm{SO}(r)$ and $\mathrm{SL}(r)$ on the space $\mathcal{V}^{\dagger}_{\omega_0}(\mathfrak{sl}(r), 1)$ are isomorphic. Moreover the vector bundle $\mathbb{V}^{\dagger}_{\omega_0}(\mathfrak{sl}(r), 1)$ restricted to M_g, splits as a non-trivial direct sum of vector bundles $\mathbb{W}_1 \oplus \mathbb{W}_2$ such that each $\mathbb{W}_i$ is preserved by the WZW connection for $\mathrm{SO}(r)$.*

Remark 3.7. We do not know if the space of $\mathrm{SO}(r)$-theta functions for a semisimple, not simply connected group gives a vector bundle on $\overline{\mathrm{M}}_g$. Recent work of Belkale-Fakhruddin [BF15] show that one can get a coherent sheaf on $\overline{\mathrm{M}}_g$. A positive answer to the above question and Corollary 3.6 will imply that the Verlinde bundle $\mathbb{V}_{\omega_0}(\mathfrak{sl}(r), 1)$ is not stable on $\overline{\mathrm{M}}_g$.

3.4. **Theorem 3.4 implies Theorem 2.1.** Let first recall the following result from [MW17].

Proposition 3.8. *Let Y be a smooth curve of genus g, then the space of section $H^0(\mathcal{N}_Y(r), \mathcal{D})$ is canonically isomorphic to the direct sum of the space of conformal blocks $\mathcal{V}^{\dagger}_{\omega_0}(Y, \mathfrak{so}(r), 2) \oplus \mathcal{V}^{\dagger}_{2\omega_1}(Y, \mathfrak{so}(r), 2)$*

The proof of the following proposition is standard and follows directly from uniformization theorems for moduli of G-bundles [BL94, KNR94, LS97, Fal94] and Proposition 3.8

Proposition 3.9. *Let Y be a smooth projective curve of genus g_Y. Then the following diagram is commutative.*

$$\begin{tikzcd}
\varphi^* : \mathcal{V}^{\dagger}_{\omega_0}(Y, \mathfrak{sl}(r), 1) \arrow[r] \arrow[d] & \mathcal{V}^{\dagger}_{2\omega_1}(Y, \mathfrak{so}(r), 2) \oplus \mathcal{V}^{\dagger}_{\omega_0}(Y, \mathfrak{so}(r), 2) \arrow[d] \\
\tilde{f}^* : H^0(\mathcal{M}_Y(\mathrm{SL}(r), \mathcal{D})) \arrow[r] & H^0(\mathcal{N}_Y, \mathcal{D}),
\end{tikzcd}$$

where the vertical arrows are isomorphisms induced from the identification of conformal blocks with non-abelian theta functions.

Since by Theorem 3.4, the horizontal map φ^* is injective, Proposition 3.9 implies that the map $\tilde{f}^*$ is also injective. Thus Theorem 3.4 implies Theorem 2.1.

4. Abelianization for $\mathrm{SO}(r)$ theta functions

The spaces of $\mathrm{SL}(r)$-theta functions at level one is connected to the space classical theta functions associated to Jacobians [BNR89]. Similarly one can try to extend the results of [BNR89] to other *abelianize* theta functions for other classical groups. In [Oxb99], W. Oxbury formulated a conjecture relating $\mathrm{SO}(r)$ theta functions with theta functions associated to Prym varieties which was proved by Pauly-Ramanan [PR01]. In this section, we give an alternate proof of a Oxbury's conjecture by applying an idea from [Bea06].

Let Y be a smooth curve and η be a non zero element in $J_2(Y)$. Let $X_\eta \to Y$ be the étale double cover associated to η and consider the norm map $\mathrm{Nm} : \mathrm{Pic}(X_\eta) \to \mathrm{Pic}(Y)$. Recall that the cover $X_\eta \to Y$ is étale of degree 2, the variety $\mathrm{Nm}(\mathcal{O}_Y)$ and $\mathrm{Nm}(\omega_Y)$ has two connected components.

$$\mathrm{Nm}^{-1}(\mathcal{O}_Y) := P^0_\eta \sqcup P^{'}_\eta \text{ and } \mathrm{Nm}^{-1}(\omega_Y) := P^{ev}_\eta \sqcup P^{od}_\eta.$$

Now like the $J^{g_{X_\eta}-1}_{X_\eta}$, the variety P^{ev}_η carries a reduced Riemann theta divisor Ξ^{ev}_η whose set theoretic support consists of line bundles in P^{ev}_η with a non global section.

Let as before denote by $\mathcal{SU}^{\sigma,+}_{X_\eta}(r)$, the moduli stack of $\sigma-$symmetric anti-invariant vector bundles on X_η, where $\sigma : X_\eta \to X_\eta$ is the involution associated to the double cover $\pi_\eta : X_\eta \to Y$.

Let $(\mathcal{E}, q, \omega)$ be an oriented orthogonal pair in $\mathcal{M}_Y(\mathrm{SO}(r))$. Then by [Zel17c, Proposition B.0.4], q induces a $\sigma-$symmetric isomorphism on the pullback bundle $\pi^*_\eta E$. Thus π^*_η gives a well defined map

$$\pi^*_\eta : \mathcal{M}_Y(\mathrm{SO}(r)) \to \mathcal{SU}^{\sigma,+}_{X_\eta}(r). \tag{4.1}$$

Note that the choice of an element of P^{ev}_η gives an isomorphism of $\mathrm{Nm}^{-1}(\mathcal{O}_Y) \cong \mathrm{Nm}^{-1}(\omega_Y)$ that identifies the Prym variety P^0_η with P^{ev}_η.

Denote by Ξ^{ev} the canonical Pfaffian divisor in P^{ev}_η. Let $\mathcal{D}$ be the determinant line bundle on $\mathcal{M}_Y(\mathrm{SO}(r))$, and consider $\mathcal{L}_\eta$ the torsion sheaf on $\mathcal{M}_Y(\mathrm{SO}(r))$ associated to η (see e.g.[BLS98]). Moreover by [Zel17a], for any line bundle $\lambda \in P_\eta^{ev} \sqcup P_\eta^{od}$, there is an associated Pfaffian line bundle $\mathcal{P}_\lambda$ over $\mathcal{SU}_X^{\sigma,+}(r)$, whose square is the determinant of cohomology line bundle.

Proposition 4.1. *Let κ be a theta characteristic over Y such that $\kappa_Y = \pi_\eta^*\kappa$ is in P_η^{ev} and $\mathcal{P}$ be the associated Pfaffian line bundle. Then we have $q_\eta^*\mathcal{P} \simeq \mathcal{D} \otimes \mathcal{L}_\eta$, where $q_\eta = \pi_\eta^*$ as in Equation (4.1).*

Proof. Let $\mathcal{U}$ (respectively $\bar{\mathcal{U}}$) be the universal family over $\mathcal{SU}_{X_\eta}^{\sigma,+}(r) \times X_\eta$ (resp. $\mathcal{M}_Y(\mathrm{SO}(r)) \times Y$). We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_Y(\mathrm{SO}(r)) \times X_\eta \arrow[r, "f"] \arrow[d, "\pi_\eta"] & \mathcal{SU}_{X_\eta}^{\sigma,+}(r) \times X_\eta \arrow[d, "\pi_\eta"] \\
\mathcal{M}_Y(\mathrm{SO}(r)) \times Y \arrow[r, "g"] & \mathcal{SU}_{X_\eta}^{\sigma,+}(r) \times Y,
\end{tikzcd}$$

where f and g are the maps given by $\pi_\eta^* \otimes id_{X_\eta}$ and $\pi_\eta^* \otimes id_Y$ respectively. For a family $\mathcal{F}$ of vector bundles parametrized by a variety S with a non-degenerate quadratic form with values in the canonical bundle, we denote by $\mathrm{Pf}(\mathcal{F})$ the Pfaffian line bundle over S which is a square root of the determinant bundle ([LS97]).
Then we have the following isomorphisms:

$$\begin{aligned}
q_\eta^*\mathcal{P} &= q_\eta^*\mathrm{Pf}(\pi_{\eta_*}(\mathcal{U} \otimes pr_2^*\pi_\eta^*\kappa)) \text{ (By definition)} \\
&= \mathrm{Pf}(g^*\pi_{\eta_*}(\mathcal{U} \otimes pr_2^*\pi_\eta^*\kappa)) \text{ (By functoriality)} \\
&= \mathrm{Pf}(\pi_{\eta_*}f^*(\mathcal{U} \otimes pr_2^*\pi_\eta^*\kappa)) \\
&= \mathrm{Pf}(\bar{\mathcal{U}} \otimes pr_2^*\kappa \oplus \bar{\mathcal{U}} \otimes pr_2^*(\kappa \otimes \eta)) \\
&= \mathcal{P}_\kappa \otimes \mathcal{P}_{\kappa\otimes\eta},
\end{aligned}$$

where pr_2 is the projection on the second factor and $\mathcal{P}_\kappa$ is the Pfaffian line bundle over $\mathcal{M}_Y(\mathrm{SO}(r))$. Now by [MW17, Proposition 3.9], we have

$$\mathcal{P}_\kappa \otimes \mathcal{P}_{\kappa\otimes\eta} \simeq \mathcal{P}_\kappa^2 \otimes \mathcal{L}_\eta \simeq \mathcal{D} \otimes \mathcal{L}_\eta.$$

This implies that $q_\eta^*\mathcal{P} \simeq \mathcal{D} \otimes \mathcal{L}_\eta$. This completes the proof. $\square$

Let $\mathcal{U}_{X_\eta}^{\sigma,+}(r, \omega_{X_\eta})$ be the moduli stack of pairs (E, ψ) such that

$$\psi : \sigma^*E \xrightarrow{\sim} E^* \otimes \omega_{X_\eta} \text{ and } \psi \text{ symmetric .}$$

This stack has two connected components ([Zel16]). The identity component (i.e. the component that contains the bundle $\kappa^{\oplus r}$ with its trivial isomorphism ψ, where κ is a $\sigma-$invariant even theta characteristic) of this space is denoted $\mathcal{U}_{X_\eta,0}^{\sigma,+}(r, \omega_{X_\eta})$. Following [Zel17a], we get a natural reduced divisor $\widetilde{\Xi}$ on $\mathcal{U}_{X_\eta,0}^{\sigma,+}(r, \omega_{X_\eta})$ supported on the set

$$\{E \in \mathcal{U}_{X_\eta,0}^{\sigma,+}(s, \omega_{X_\eta}) \mid \dim H^0(X, E) > 0\}.$$

Now taking the multiplication map $\mathfrak{m}$ with respect to elements in $\mathrm{Nm}^{-1}(\omega_Y)$, we consider the following:

4.1. **The case $r = 2m+1$.**

$$\begin{tikzcd} P_\eta^{ev} \times \mathcal{SU}_{X_\eta}^{\sigma,+}(2m+1) \arrow[r, "\mathfrak{m}"] \arrow[d] & \mathcal{U}_{X_\eta,0}^{\sigma,+}(2m+1, \omega_{X_\eta}) \arrow[d, equal] \\ P_\eta^{ev} \times \mathcal{M}_Y(\mathrm{SO}(2m+1)) \arrow[r] & \mathcal{U}_{X_\eta,0}^{\sigma,+}(2m+1, \omega_{X_\eta}) \end{tikzcd}$$

When r is odd, the pull back $\mathfrak{m}^*\widetilde{\Xi}$ gives a divisor in $P_\eta^{ev} \times \mathcal{SU}_{X_\eta}^{\sigma,+}(2m+1)$ which by [Oxb99] restricts to a divisor in $P_\eta^{ev} \times \mathcal{M}_Y(\mathrm{SO}(2m+1))$. Now since the stack $\mathcal{M}_Y(\mathrm{SO}(2m+1))$ is smooth and algebraic, the divisor associated to the restriction of $\mathfrak{m}^*\widetilde{\Xi}$ is a Cartier divisor [GS15] (this fails to hold if we work with the moduli space $\underline{M}_Y(\mathrm{SO}(2m+1))$).

Thus we have a section of $H^0(P_\eta^{ev} \times \mathcal{M}_Y(\mathrm{SO}(2m+1)), (2m+1)\Xi^{ev} \boxtimes \mathcal{D} \otimes \mathcal{L}_\eta)$. Now by the Künneth formula, we get the following map:

$$\iota_\eta : H^0(P_\eta^{ev}, (2m+1)\Xi^{ev})^\vee \longrightarrow H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D} \otimes \mathcal{L}_\eta). \tag{4.2}$$

4.2. **The case $r = 2m$.**

$$\begin{tikzcd} P_\eta^{ev} \sqcup P_\eta^{od} \times \mathcal{SU}_{X_\eta}^{\sigma,+}(2m) \arrow[r, "\mathfrak{m}"] & \mathcal{U}_{X_\eta,0}^{\sigma,+}(2m, \omega_{X_\eta}) \ . \arrow[d, equal] \\ P_\eta^{ev} \sqcup P_\eta^{od} \times \mathcal{M}_Y(\mathrm{SO}(2m)) \arrow[r] \arrow[u, "(\mathrm{id}, q_\eta)"] & \mathcal{U}_{X_\eta,0}^{\sigma,+}(2m, \omega_{X_\eta}) \end{tikzcd}$$

In this case, the pull back $\mathfrak{m}^*\widetilde{\Xi}$ gives a divisor of $P_\eta^{ev} \sqcup P_\eta^{od} \times \mathcal{SU}_{X_\eta}^{\sigma,+}(r)$ which restricts to a section [Oxb99] in $P_\eta^{ev} \sqcup P_\eta^{od} \times \mathcal{M}_Y(\mathrm{SO}(2m))$. Thus we get the following map $\iota_\eta \oplus \iota'_\eta$:

(4.3)

$$(H^0(P_\eta^{ev}, (2m)\Xi^{ev}) \oplus H^0(P_\eta^{ev}, (2m)\Xi_\alpha^{od}))^\vee \to H^0(\mathcal{M}_Y(\mathrm{SO}(2m)), \mathcal{D} \otimes \mathcal{L}_\eta).$$

With the above notation, we have the following proposition:

Proposition 4.2. *The map ι_η is injective. If r is even, then the map $\iota_\eta \oplus \iota'_\eta$ is also injective.*

Proof. We will show that the dual of the above maps are surjective. We first observe the following: Let $s \in H^0(\mathcal{M}_1 \times \mathcal{M}_2, \mathcal{L}_1 \boxtimes \mathcal{L}_2)$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are appropriate spaces and $\mathcal{L}_1$ and $\mathcal{L}_2$ are line bundles on them. Then to check that the map $H^0(\mathcal{M}_1, \mathcal{L}_1)^\vee \to H^0(\mathcal{M}_2, \mathcal{L}_2)$ induced by s is surjective, it is enough to produce points $x_1, \dots, x_m$ of $\mathcal{M}_1$ such that the sections $s(x_1, \ ), \dots, s(x_m, \ )$ spans the space $H^0(\mathcal{M}_2, \mathcal{L}_2)$. Here the role $\mathcal{M}_1$ will be played by $\mathcal{M}_Y(\mathrm{SO}(r))$ and $\mathcal{M}_2$ will be played by Prym varieties.

First consider the case for ι_η and r is arbitrary. Choose elements $\alpha_1,\dots,\alpha_r$ in $P^0_\eta[2]$ such that $\bigotimes_{i=1}^r \alpha_i = \mathcal{O}_{X_\eta}$. Let $\alpha_i = \pi_\eta^*\lambda_i$ and consider the orthogonal bundle $\mathcal{E} = \lambda_1 \oplus \cdots \oplus \lambda_r$ on Y. The Pfaffian $\Xi_{\mathcal{E}}$ divisor on P^{ev}_η associated to $\pi_\eta^*\mathcal{E}$ is equal to $\Xi^{ev}_{\alpha_1} + \cdots + \Xi^{ev}_{\alpha_r}$, where $\Xi^{ev}_{\alpha_i} := T^*_{\alpha_i}(\Xi^{ev})$. Now Lemma 4.3 below (due to Beauville [Bea06]) implies that sections of this form spans $H^0(P^{ev}_\eta, r\Xi)$.

If r is even, the argument for ι'_η is similar by considering $\beta_1,\dots,\beta_r$ in $P'_\eta[2]$ such that $\beta_1 \otimes \cdots \otimes \beta_r = \mathcal{O}_{X_\eta}$ and applying the same argument as above. Moreover the Pfaffian sections constructed via β's in $P'_\eta[2]$ (resp. α's in $P^0_\eta[2]$) are only supported on P^{od}_η (resp. P^{ev}_η). Hence $\iota_\eta \oplus \iota'_\eta$ are surjective when r is even. $\square$

For completeness, we recall the following lemma from [Bea06]:

Lemma 4.3. *Let A be an abelian variety with an ample line bundle L. Then the natural multiplication map is surjective*

$$\bigoplus_{\substack{\alpha_1,\dots,\alpha_r \in \widehat{A}[2] \\ \alpha_1+\cdots+\alpha_r=0}} H^0(A, L\otimes\alpha_1)\otimes\cdots\otimes H^0(A, L\otimes\alpha_r) \to H^0(A, L^r),$$

where $\widehat{A}[2]$ denote the 2-torsion points in the Picard group of A.

As an application of Proposition 4.2 and Theorem 1.1, we get a direct proof of a theorem of Pauly-Ramanan [PR01].

Theorem 4.4. *Let $\mathcal{N}_Y(r)$ be as in Equation (2.1). Then there are natural isomorphisms*

$$\iota : \bigoplus_{\eta\in J_2(Y)} H^0(P^{ev}_\eta, (2m+1)\Xi^{ev})^\vee \simeq H^0(\mathcal{N}_Y((2m+1)), \mathcal{D})$$

$$\iota' : \bigoplus_{\eta\in J_2(Y)} H^0(P^{ev}_\eta, (2m)\Xi^{ev})^\vee \oplus H^0(P^{od}_\eta, (2m)\Xi^{od}_\alpha)^\vee \simeq H^0(\mathcal{N}_Y((2m)), \mathcal{D}),$$

where $\mathcal{D}$ is the determinant of cohomology line bundle and ι (resp. $\tilde{\iota}$) is the direct sum of maps ι_η (resp. $\iota_\eta \oplus \iota'_\eta$) as given in Equations (4.2) and (4.3). In particular, one deduces that each ι_η is an isomorphism.

Proof. We treat just the odd case. Then the argument in the even case is similar. Now Proposition 4.2 tells us that for $\eta \neq 0$, the map

$$\iota_\eta : H^0(P^{ev}_\eta, (2m+1)\Xi^{ev})^\vee \to H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D}\otimes\mathcal{L}_\eta)$$

is injective. Now suppose $\eta = 0$, then P^{ev}_0 is just J^{g-1}_Y and Ξ^{ev} is just the classical Θ divisor. Then by [Bea06], we get $\iota_0 : H^0(J^{g-1}, r\Theta)^\vee \simeq H^0(\mathcal{M}_X(\mathrm{SO}(r)), \mathcal{D})$. Combining these, we deduce that the map $\iota = \oplus_{\eta\in J_2(Y)}\iota_\eta$ is injective. Now we will be done if we can show that the target and source have the same dimension. This follows by applying the Verlinde formula for $\mathrm{Spin}(2m+1)$ and the fact that

$$H^0(\mathcal{N}_Y((r), \mathcal{D})) \simeq \bigoplus_{\eta\in J_2(Y)} H^0(\mathcal{M}_Y(\mathrm{SO}(r)), \mathcal{D}\otimes\mathcal{L}_\eta).$$

□

Corollary 4.5. *Let $\eta \neq 0$ in $J_2(Y)$ and $\pi_\eta : X_\eta \to Y$ be the corresponding étale double cover. Then the pullback $q_\eta = \pi_\eta^*$ induces an isomorphism between the spaces of global sections:*

$$H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D} \otimes \mathcal{L}_\eta) \simeq H^0(\mathcal{SU}^{\sigma,+}_{X_\eta}(2m+1), \mathcal{P}),$$
$$H^0(\mathcal{M}_Y(\mathrm{SO}(2m)), \mathcal{D} \otimes \mathcal{L}_\eta) \hookleftarrow H^0(\mathcal{SU}^{\sigma,+}_{X_\eta}(2m), \mathcal{P}).$$

Proof. First we consider the case $\mathrm{SO}(2m+1)$. Recall that we have a commutative diagram.

$$\begin{tikzcd} & H^0(\mathcal{SU}^{\sigma,+}_{X_\eta}(2m+1), \mathcal{P}) \arrow[d, "q_\eta^*"] \\ H^0(P_\eta^{ev}, (2m+1)\Xi^{ev})^\vee \arrow[ur, "\nu^*"] \arrow[r, "\iota_\eta = \rho^*"] & H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D} \otimes \mathcal{L}_\eta). \end{tikzcd}$$

By Theorem 4.4, we know that the map ι_η is an isomorphism. Similarly by [Zel17a], the map ν^* is an isomorphism. Now the commutativity of the above diagram implies that q_η^* is an isomorphism between the untwisted theta functions $H^0(\mathcal{M}_Y(\mathrm{SO}(2m+1)), \mathcal{D} \otimes \mathcal{L}_\eta)$ and the space of twisted theta functions $H^0(\mathcal{SU}^{\sigma,+}_{X_\eta}(2m+1), \mathcal{P})$.

For even rank, we have a similar commutative diagram and hence the argument follows as in the odd rank case. The vertical map ν^* is an isomorphism and the map ι_η is injective.

$$\begin{tikzcd} H^0(\mathcal{SU}^{\sigma,+}_{X_\eta}(2m), \mathcal{P}) \arrow[r, "q_\eta^*"] & H^0(\mathcal{M}_Y(\mathrm{SO}(2m)), \mathcal{D} \otimes \mathcal{L}_\eta) \\ H^0(P_\eta^{ev}, (2m)\Xi^{ev})^\vee \arrow[u, "\nu^*"] \arrow[ur, hook, "\iota_\eta = \rho^*"] & \end{tikzcd}$$

□

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