AN INVERSE LIMIT APPROACH TO GROUP HOMOLOGY

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Abstract. In this paper, we consider for any free presentation $G = F/R$ of a group G the coinvariance $H_0(G, R_{ab}^{\otimes n})$ of the n-th tensor power of the relation module R_{ab} and show that the homology group $H_{2n}(G, \mathbb{Z})$ may be identified with the inverse limit of the groups $H_0(G, R_{ab}^{\otimes n})$, where the limit is taken over the category of these presentations of G. We also consider the free Lie ring generated by the relation module R_{ab}, in order to relate the inverse limit of the groups $\gamma_n R/[\gamma_n R, F]$ to the n-torsion subgroup of $H_{2n}(G, \mathbb{Z})$.

0. Introduction

It is well-known that one may use a presentation of a group G as the quotient F/R, where F is a free group, in order to calculate its (co-)homology. Besides Hopf's formula for the second homology group $H_2(G, \mathbb{Z})$ (cf. [2, Chapter II, Theorem 5.3]), another example supporting that claim is the existence of the Gruenberg resolution [4]. Using Quillen's description of the cyclic homology of an algebra over a field of characteristic 0 as the inverse limit of a suitable functor over the category of extensions of the algebra (cf. [7]), the homology groups $H_n(G, \mathbb{Q})$ are described in [3] as the inverse limit of certain functors over the category of group extensions $G = K/H$ (here, the group K is not necessarily free).

Working in the same direction, we obtain in this paper a description of the even homology groups of G with coefficients in an arbitrary $\mathbb{Z}G$-module M as the inverse limit of a functor over the category $\mathfrak{P}$ of all free presentations $G = F/R$. More precisely, we use the associated relation module $R_{ab} = R/[R, R]$ and prove that there is an isomorphism

$$H_{2n}(G, M) \simeq \varprojlim H_0(G, M \otimes R_{ab}^{\otimes n}),$$

where the inverse limit is taken over $\mathfrak{P}$. Together with the free associative ring TR_{ab} on R_{ab} (which is built up by the tensor powers $R_{ab}^{\otimes n}$, $n \geq 0$), we may also consider the free Lie ring $\mathfrak{L}R_{ab}$ on R_{ab}. The Lie ring $\mathfrak{L}R_{ab}$ is graded and its homogeneous component in degree $n \geq 1$ consists of the abelian group $\gamma_n R/\gamma_{n+1} R$, where $(\gamma_i R)_{i \geq 1}$ is the lower central series of R. Then, the inclusion $\mathfrak{L}R_{ab} \subseteq TR_{ab}$ induces a natural map

$$l_n : \gamma_n R/[\gamma_n R, F] \longrightarrow H_0(G, R_{ab}^{\otimes n})$$

for all $n \geq 1$. The group $\gamma_n R/[\gamma_n R, F]$ is the kernel of the free central extension

$$1 \longrightarrow \gamma_n R/[\gamma_n R, F] \longrightarrow F/[\gamma_n R, F] \longrightarrow F/\gamma_n R \longrightarrow 1$$

and can be identified, in view of Hopf's formula, with the homology group $H_2(F/\gamma_n R, \mathbb{Z})$. It has been studied by many authors; a survey of the corresponding results may be found in [10]. As an example, we note that the torsion subgroup of $\gamma_n R/[\gamma_n R, F]$, which is shown in [loc.cit.] to be an n-torsion group if $n \geq 3$, may be identified with the kernel of the so-called Gupta representation of $F/[\gamma_n R, F]$ (cf. [9,11]). Confirming the existence of a close relationship between the groups $\gamma_n R/[\gamma_n R, F]$ and the torsion in the homology of G, we show that the l_n's induce an additive map

$$\ell_n : \varprojlim \gamma_n R/[\gamma_n R, F] \longrightarrow \varprojlim H_0(G, R_{ab}^{\otimes n}) \simeq H_{2n}(G, \mathbb{Z}),$$

whose image is contained in the n-torsion subgroup $H_{2n}(G, \mathbb{Z})[n]$ of $H_{2n}(G, \mathbb{Z})$.

The contents of the paper are as follows: In Section 1, we explain how one can use dimension shifting by the powers of the relation module R_{ab}, which is associated with a presentation $G = F/R$, in order to embed the homology groups $H_{2n}(G, \_)$ into $H_0(G, \_ \otimes R_{ab}^{\otimes n})$ for all $n \geq 1$. In the following Section, we record some generalities about inverse limits and prove a simple criterion for these limits to vanish. In Section 3, we define the presentation category $\mathfrak{P}$ of G and prove the existence of an isomorphism between $H_{2n}(G, \_)$ and the inverse limit of the $H_0(G, \_ \otimes R_{ab}^{\otimes n})$'s. Finally, in the last Section, we consider the free Lie ring on the relation module R_{ab} and relate the inverse limit of the quotients $\gamma_n R/[\gamma_n R, F]$ to the n-torsion subgroup of $H_{2n}(G, \mathbb{Z})$.

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1. Relation modules and dimension shifting in homology

In this Section, we consider a group G and fix a presentation of it as the quotient of a free group $F = F(S)$ on a set S by a normal subgroup R. We shall denote by π the corresponding surjective homomorphism from F to G. We note that the conjugation action of F on R induces an action of F on the abelianization $R_{ab} = R/[R, R]$, which is obviously trivial when restricted to R. Therefore, the latter action induces an action of G on R_{ab}. The abelian group R_{ab}, endowed with the G-action defined above, is referred to as the relation module of the given presentation.

The augmentation ideal $\mathfrak{f} \subseteq \mathbb{Z}F$ of F is well-known to be free as a $\mathbb{Z}F$-module; in fact, it is free on the set $\{s - 1 : s \in S\}$. In particular, the $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}$ is free on the set $\{1 \otimes (s - 1) : s \in S\}$. Moreover, it follows from [2, Chapter II, Proposition 5.4] that there is an exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow R_{ab} \xrightarrow{\mu} \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f} \xrightarrow{\sigma} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{1}$$

where μ maps $r[R, R]$ onto $1 \otimes (r - 1)$ for all $r \in R$, σ maps the basis element $1 \otimes (s - 1)$ onto $\pi(s) - 1$ for all $s \in S$ and ε is the augmentation homomorphism. We note that R, being a subgroup of the free group F, is itself free; therefore, the relation module R_{ab} is $\mathbb{Z}$-free. Since this is also the case for the other three terms of the exact sequence (1), we conclude that the latter is $\mathbb{Z}$-split. We shall refer to the exact sequence (1) as the relation sequence associated with the given presentation of G. The map μ therein was defined by Magnus in [6]; it will be referred to as the Magnus embedding.

Lemma 1.1. *Let M be a $\mathbb{Z}G$-module. Then, there are natural isomorphisms $H_i(G, M) \simeq H_{i-2}(G, M \otimes R_{ab})$ for all $i \geq 2$, where G acts on $M \otimes R_{ab}$ diagonally.*

Proof. Since the relation sequence (1) is $\mathbb{Z}$-split, we may tensor it with M and obtain the exact sequence of $\mathbb{Z}G$-modules (with diagonal action)

$$0 \longrightarrow M \otimes R_{ab} \longrightarrow M \otimes (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}) \longrightarrow M \otimes \mathbb{Z}G \longrightarrow M \longrightarrow 0. \tag{2}$$

If N is a free $\mathbb{Z}G$-module, then the $\mathbb{Z}G$-module $M \otimes N$ (with diagonal action) is known to be isomorphic with an induced module (cf. [2, Chapter III, Corollary 5.7]); in particular, the homology of G with coefficients in $M \otimes N$ vanishes in positive degrees. Since the $\mathbb{Z}G$-modules $\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}$ and $\mathbb{Z}G$ are free, we may use the exact sequence (2) and dimension shifting, in order to obtain the existence of natural isomorphisms, as claimed. □

Corollary 1.2. *Let M be a $\mathbb{Z}G$-module. Then, there are natural isomorphisms $H_{2n}(G, M) \simeq H_2(G, M \otimes R_{ab}^{\otimes n-1})$ and $H_{2n+1}(G, M) \simeq H_1(G, M \otimes R_{ab}^{\otimes n})$ for all $n \geq 1$.*

Proof. The result follows by induction on n, using Lemma 1.1. □

Corollary 1.3. *There are isomorphisms $H_{2n}(G, \mathbb{Z}) \simeq H_2(G, R_{ab}^{\otimes n-1})$ and $H_{2n+1}(G, \mathbb{Z}) \simeq H_1(G, R_{ab}^{\otimes n})$ for all $n \geq 1$.* □

Remark 1.4. The dimension shifting in the homology of a group G, which is associated with the relation module R_{ab} as above, may be alternatively described by using cap products; see, for example, [12, §2.3]. More precisely, let $\chi \in H^2(G, R_{ab})$ be the cohomology class that classifies the group extension

$$1 \longrightarrow R/[R, R] \longrightarrow F/[R, R] \longrightarrow G \longrightarrow 1$$

as in [2, Chapter IV, Theorem 3.12]. Then, the dimension shifting isomorphisms above are induced by the cap product maps with χ or with suitable powers of it.

We consider a $\mathbb{Z}G$-module M and note that the Lyndon-Hochschild-Serre spectral sequence associated with the extension

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

induces in low degrees the exact sequence

$$0 \longrightarrow H_2(G, M) \longrightarrow H_0(G, H_1(R, M)) \longrightarrow H_1(F, M) \longrightarrow H_1(G, M) \longrightarrow 0.$$

Since M is trivial as a $\mathbb{Z}R$-module, we have $H_1(R, M) = M \otimes R_{ab}$ and hence the latter exact sequence reduces to

$$0 \longrightarrow H_2(G, M) \longrightarrow H_0(G, M \otimes R_{ab}) \longrightarrow H_1(F, M) \longrightarrow H_1(G, M) \longrightarrow 0.$$

We note that the above embedding of $H_2(G, M)$ into $H_0(G, M \otimes R_{ab})$, which is provided by the d^2-differential of the spectral sequence, is known to coincide (up to a sign) with the cap product map with the cohomology class $\chi \in H^2(G, R_{ab})$ defined in Remark 1.4. In particular, replacing M by $M \otimes R_{ab}^{\otimes n-1}$, we conclude that there is an exact sequence

$$0 \to H_2(G, M \otimes R_{ab}^{\otimes n-1}) \xrightarrow{\chi \cap -} H_0(G, M \otimes R_{ab}^{\otimes n}) \to H_1(F, M \otimes R_{ab}^{\otimes n-1}) \to H_1(G, M \otimes R_{ab}^{\otimes n-1}) \to 0$$

for all $n \geq 1$.

Taking into account Corollary 1.2 and Remark 1.4, we may state the following result.

Proposition 1.5. *Let M be a $\mathbb{Z}G$-module and consider the cohomology class $\chi \in H^2(G, R_{ab})$ defined in Remark 1.4. Then, there is an exact sequence*

$$0 \to H_{2n}(G, M) \xrightarrow{\chi^n \cap -} H_0(G, M \otimes R_{ab}^{\otimes n}) \to H_1(F, M \otimes R_{ab}^{\otimes n-1}) \to H_1(G, M \otimes R_{ab}^{\otimes n-1}) \to 0$$

for all $n \geq 1$. In particular, there is an exact sequence

$$0 \longrightarrow H_{2n}(G,\mathbb{Z}) \xrightarrow{\chi^n \cap -} H_0(G, R_{ab}^{\otimes n}) \longrightarrow H_1(F, R_{ab}^{\otimes n-1}) \longrightarrow H_1(G, R_{ab}^{\otimes n-1}) \longrightarrow 0$$

for all $n \geq 1$. □

2. Some generalities on inverse limits

Let C be a small category, Ab the category of abelian groups and $\mathfrak{F} : C \longrightarrow Ab$ a functor. Then, the inverse limit $\varprojlim \mathfrak{F}$ of $\mathfrak{F}$ is the subgroup of the direct product $\prod_{c \in C} \mathfrak{F}(c)$, consisting of those families $(x_c)_c$ which are compatible in the following sense: For any two objects $c, c' \in C$ and any morphism $a \in \mathrm{Hom}_C(c, c')$, we have $\mathfrak{F}(a)(x_c) = x_{c'} \in \mathfrak{F}(c')$. We often denote the abelian group $\varprojlim \mathfrak{F}$ by $\varprojlim \mathfrak{F}(c)$.

Let $\mathfrak{F}, \mathfrak{G}$ be two functors from C to Ab. Then, a natural transformation $\eta : \mathfrak{F} \longrightarrow \mathfrak{G}$ induces an additive map

$$\varprojlim \eta : \varprojlim \mathfrak{F} \longrightarrow \varprojlim \mathfrak{G},$$

by mapping any element $(x_c)_c \in \varprojlim \mathfrak{F}$ onto $(\eta_c(x_c))_c \in \varprojlim \mathfrak{G}$. In this way, $\varprojlim$ itself becomes a functor from the functor category Ab^C to Ab. The proof of the following well-known result is straightforward.

Lemma 2.1. *The inverse limit functor $\varprojlim : Ab^C \longrightarrow Ab$ is left exact.* □

We recall that the coproduct of two objects a and b of C is an object $a \star b$, which is endowed with two morphisms $\iota_a : a \longrightarrow a \star b$ and $\iota_b : b \longrightarrow a \star b$ having the following universal property: For any object c of C and any pair of morphisms $f : a \longrightarrow c$ and $g : b \longrightarrow c$, there is a unique morphism $h : a \star b \longrightarrow c$, such that $h \circ \iota_a = f$ and $h \circ \iota_b = g$. The morphism h is usually denoted by (f, g).

As an example, we note that the coproduct of two abelian groups M and N in the category Ab is the direct sum $M \oplus N$, endowed with the obvious inclusion maps. For any abelian group T and any pair of additive maps $f : M \longrightarrow T$ and $g : N \longrightarrow T$, the additive map $(f, g) : M \oplus N \longrightarrow T$ is given by $(m, n) \mapsto f(m) + g(n)$, $(m, n) \in M \oplus N$.

The following elementary vanishing criterion will be used twice in the sequel.

Lemma 2.2. *Let C be a small category and $\mathfrak{F} : C \longrightarrow Ab$ a functor to the category of abelian groups. We assume that:*

(i) Any two objects a, b of C have a coproduct $(a \star b, \iota_a, \iota_b)$ as above.

(ii) For any two objects a, b of C the morphisms $\iota_a : a \longrightarrow a \star b$ and $\iota_b : b \longrightarrow a \star b$ induce a monomorphism

$$(\mathfrak{F}(\iota_a), \mathfrak{F}(\iota_b)) : \mathfrak{F}(a) \oplus \mathfrak{F}(b) \longrightarrow \mathfrak{F}(a \star b)$$

of abelian groups.

Then, the inverse limit $\varprojlim \mathfrak{F}$ is the zero group.

Proof. Let $(x_c)_c \in \varprojlim \mathfrak{F}$ be a compatible family and fix an object a of C. We consider the coproduct $a \star a$ of two copies of a and the morphisms $\iota_1 : a \longrightarrow a \star a$ and $\iota_2 : a \longrightarrow a \star a$. Then, we have

$$\mathfrak{F}(\iota_1)(x_a) = x_{a \star a} = \mathfrak{F}(\iota_2)(x_a)$$

and hence the element $(x_a, -x_a)$ is contained in the kernel of the additive map

$$(\mathfrak{F}(\iota_1), \mathfrak{F}(\iota_2)) : \mathfrak{F}(a) \oplus \mathfrak{F}(a) \longrightarrow \mathfrak{F}(a \star a).$$

In view of our assumption, this latter map is injective and hence $x_a = 0$. Since this is the case for any object a of C, we conclude that the compatible family $(x_c)_c$ vanishes, as needed. $\square$

3. An inverse limit formula for $H_{2n}(G, \_)$

We fix a group G and define the category of presentations $\mathfrak{P} = \mathfrak{P}(G)$ as follows: The objects of $\mathfrak{P}$ are pairs of the form (F, π), where F is a free group and π a surjective group homomorphism from F onto G. Given two objects (F, π) and (F', π') of $\mathfrak{P}$, a morphism from (F, π) to (F', π') is a group homomorphism $\varphi : F \longrightarrow F'$ such that $\pi' \circ \varphi = \pi$. Since the groups that are involved are free, we note that for any two objects (F, π) and (F', π') of $\mathfrak{P}$ there is at least one morphism from (F, π) to (F', π').

Given an object (F, π) of $\mathfrak{P}$, we may consider the group ring $\mathbb{Z}F$, the augmentation ideal $\mathfrak{f}$, the kernel $R = \ker \pi$, the relation module R_{ab} and the associated Magnus embedding

$$\mu : R_{ab} \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}.$$

It is clear that all these depend naturally on the object (F, π) of $\mathfrak{P}$. Moreover, this is also true for the cohomology class $\chi \in H^2(G, R_{ab})$ defined in Remark 1.4. Therefore, invoking the naturality of the low degrees exact sequence which is induced by the Lyndon-Hochschild-Serre spectral sequence with respect to the group extension and the coefficient module, we conclude that the dimension shifting isomorphisms as well as the exact sequence of Proposition 1.5 is natural with respect to the morphisms of $\mathfrak{P}$. In view of the left exactness of the inverse limit functor (cf. Lemma 2.1), we thus obtain an exact sequence

$$0 \longrightarrow H_{2n}(G, M) \longrightarrow \varprojlim H_0(G, M \otimes R_{ab}^{\otimes n}) \longrightarrow \varprojlim H_1(F, M \otimes R_{ab}^{\otimes n-1}) \tag{3}$$

for all $n \geq 1$, where the inverse limits are taken over the category $\mathfrak{P}$.

Lemma 3.1. *Let (F, π) and (F', π') be two objects of the presentation category $\mathfrak{P}$ of G.*

(i) The coproduct $(F, \pi) \star (F', \pi')$ is provided by the object (F'', π'') of $\mathfrak{P}$, where F'' is the free product of F and F' and $\pi'' : F'' \longrightarrow G$ the homomorphism which extends both π and π'.

(ii) Let $\iota : (F, \pi) \longrightarrow (F'', \pi'')$ and $\iota' : (F', \pi') \longrightarrow (F'', \pi'')$ be the structural morphisms of the coproduct (F'', π''). Then, the induced maps $\iota_* : R_{ab} \longrightarrow R''_{ab}$ and $\iota'_* : R'_{ab} \longrightarrow R''_{ab}$ between the corresponding relation modules are both split monomorphisms of $\mathbb{Z}G$-modules.

Proof. Assertion (i) is clear and, because of symmetry, we only have to prove assertion (ii) for the structural morphism ι. We note that the additive map $\iota_* : R_{ab} \longrightarrow R''_{ab}$ i s obtained by restricting ι and then passing to the quotients. We choose a morphism $\varphi : (F', \pi') \longrightarrow (F, \pi)$ in $\mathfrak{P}$ and consider the morphism $\lambda = (id_F, \varphi) : (F'', \pi'') \longrightarrow (F, \pi)$, which extends both the identity of (F, π) and φ. Then, λ restricts to a group homomorphism $\lambda_0 : R'' \longrightarrow R$, which is a left inverse of the restriction $\iota_0 : R \longrightarrow R''$ of ι and satisfies the equality

$$\lambda_0(\iota(x)\, r'' \iota(x)^{-1}) = x \lambda_0(r'')\, x^{-1}$$

for all $x \in F$ and $r'' \in R''$. It follows that the additive map $\lambda_* : R''_{ab} \longrightarrow R_{ab}$ which is induced by λ_0 to the quotients is a $\mathbb{Z}G$-linear left inverse of ι_*. $\square$

We can now state and prove our first main result.

Theorem 3.2. *Let M be a $\mathbb{Z}G$-module. Then, there is an isomorphism of abelian groups*

$$H_{2n}(G,M) \xrightarrow{\sim} \lim_{\longleftarrow} H_0(G, M \otimes R_{ab}^{\otimes n}),$$

where the inverse limit is taken over the category $\mathfrak{P}$ of presentations of G for all $n \geq 1$. In particular, there is an isomorphism

$$H_{2n}(G,\mathbb{Z}) \xrightarrow{\sim} \lim_{\longleftarrow} H_0(G, R_{ab}^{\otimes n})$$

for all $n \geq 1$.

Proof. In view of the exact sequence (3), it suffices to prove the vanishing of the inverse limit of the functor from $\mathfrak{P}$ to the category of abelian groups, which maps any object (F,π) onto $H_1(F, M \otimes R_{ab}^{\otimes n-1})$. In order to prove the vanishing of that inverse limit, we shall apply the criterion established in Lemma 2.2. We have to verify that conditions (i) and (ii) therein are satisfied. To that end, we fix two objects (F,π) and (F',π') of $\mathfrak{P}$ and denote by R_{ab} and R'_{ab} the corresponding relation modules.

In view of Lemma 3.1(i), the objects (F,π) and (F',π') have a coproduct in $\mathfrak{P}$, which is provided by (F'',π''), where F'' is the free product of F and F. Let R''_{ab} denote the relation module that corresponds to the coproduct (F'',π''). We have to prove that the map

$$H_1(F, M \otimes R_{ab}^{\otimes n-1}) \oplus H_1(F', M \otimes R_{ab}^{'\otimes n-1}) \longrightarrow H_1(F'', M \otimes R_{ab}^{''\otimes n-1}),$$

which is induced by the inclusions of F and F' into F'', is injective. To that end, we note that the corresponding Mayer-Vietoris exact sequence shows that the natural map

$$H_1(F, M \otimes R_{ab}^{''\otimes n-1}) \oplus H_1(F', M \otimes R_{ab}^{''\otimes n-1}) \longrightarrow H_1(F'', M \otimes R_{ab}^{''\otimes n-1})$$

is injective. Therefore, it only remains to prove that the natural maps

$$H_1(F, M \otimes R_{ab}^{\otimes n-1}) \longrightarrow H_1(F, M \otimes R_{ab}^{''\otimes n-1})$$

and

$$H_1(F', M \otimes R_{ab}^{'\otimes n-1}) \longrightarrow H_1(F', M \otimes R_{ab}^{''\otimes n-1})$$

are injective. We may now complete the proof invoking Lemma 3.1(ii), which itself implies that the natural map $R_{ab}^{\otimes n-1} \longrightarrow R_{ab}^{''\otimes n-1}$ (resp. $R_{ab}^{'\otimes n-1} \longrightarrow R_{ab}^{''\otimes n-1}$) is a split monomorphism of $\mathbb{Z}G$-modules and hence of $\mathbb{Z}F$-modules (resp. of $\mathbb{Z}F'$-modules). □

Remark 3.3 Using a different technique than that employed above, an analogous inverse limit description of the even homology groups of G with coefficients in $\mathbb{Q}$ is established in [3]. More precisely, for any extension K of G with kernel H, one may consider the augmentation ideal $\mathfrak{h} \subseteq \mathbb{Z}H$ of H and the subgroup $B = \mathbb{Z}H \cap [\mathbb{Z}K, \mathbb{Z}K] \subseteq \mathbb{Z}H$, where $[\mathbb{Z}K, \mathbb{Z}K]$ is the subgroup of $\mathbb{Z}K$ generated by the set $\{xy - yx : x, y \in K\}$. Then, as shown in [loc.cit.], there is an isomorphism

$$H_{2n}(G,\mathbb{Q}) \xrightarrow{\sim} \lim_{\longleftarrow} \frac{\mathfrak{h}_{\mathbb{Q}}^n + B_{\mathbb{Q}}}{\mathfrak{h}_{\mathbb{Q}}^{n+1} + B_{\mathbb{Q}}} \qquad (4)$$

for all $n \geq 1$. Here, $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h} \otimes \mathbb{Q}$ and $B_{\mathbb{Q}} = B \otimes \mathbb{Q}$, whereas the inverse limit is taken over the category of all extensions $G = K/H$ of G. We note that the latter category contains the presentation category $\mathfrak{P}$ of G as a full subcategory. Even though it is not clear whether the existence of the isomorphism (4) is implied by Theorem 3.2, it seems that the two results are not unrelated. In order to describe such a relationship, we consider an object (F, π) of the presentation category $\mathfrak{P}$ of G and the kernel $R = \ker \pi$. Let $\mathfrak{r}_{\mathbb{Q}}$ and $B_{\mathbb{Q}}$ be the subspaces of the rational group ring $\mathbb{Q}R$ that are constructed as above for the extension $G = F/R$ of G. We now consider, for all $n \geq 1$, the map

$$\delta_n : R_{ab}^{\otimes n} \longrightarrow \frac{\mathfrak{r}_{\mathbb{Q}}^n + B_{\mathbb{Q}}}{\mathfrak{r}_{\mathbb{Q}}^{n+1} + B_{\mathbb{Q}}},$$

which is given by mapping an elementary tensor $r_1[R,R] \otimes \cdots \otimes r_n[R,R]$ onto the residue class of the product $(r_1 - 1) \cdot \ldots \cdot (r_n - 1) \in \mathfrak{r}_{\mathbb{Q}}^n \subseteq \mathfrak{r}_{\mathbb{Q}}^n + B_{\mathbb{Q}}$ in the quotient for all $r_1, \ldots, r_n \in R$. It is easily seen that δ_n induces, by passage to the quotient, an additive map

$$\overline{\delta_n} : H_0(G, R_{ab}^{\otimes n}) \longrightarrow \frac{\mathfrak{r}_{\mathbb{Q}}^n + B_{\mathbb{Q}}}{\mathfrak{r}_{\mathbb{Q}}^{n+1} + B_{\mathbb{Q}}},$$

which depends naturally upon the object (F, π) of $\mathfrak{P}$. Hence, the groups that appear in the inverse limit $\varprojlim H_0(G, R_{ab}^{\otimes n})$ of Theorem 3.2 map canonically into some of the groups that appear in the inverse limit of the right hand side of the isomorphism (4) above. It would be of some interest to know whether one can recover in this way the map

$$H_{2n}(G, \mathbb{Z}) \longrightarrow H_{2n}(G, \mathbb{Q}),$$

which is induced by the inclusion of $\mathbb{Z}$ into $\mathbb{Q}$.

4. The inverse limit of the $\gamma_n R/[\gamma_n R, F]$'s

Let H be a group. We recall that the lower central series $(\gamma_n H)_{n \geq 1}$ of H is given by $\gamma_1 H = H$ and $\gamma_{n+1} H = [\gamma_n H, H]$ for all $n \geq 1$. Then, the graded Lie ring $Gr\, H = \bigoplus_{n=1}^{\infty} Gr^n H$ of H is defined in degree n to be the (additively written) abelian group $Gr^n H = \gamma_n H / \gamma_{n+1} H$. The Lie bracket on $Gr\, H$ is defined by letting

$$(x\gamma_{n+1}H, y\gamma_{m+1}H) = [x, y]\gamma_{n+m+1}H,$$

where $[x, y] = x^{-1}y^{-1}xy$ for all $x \in \gamma_n H$ and $y \in \gamma_m H$ (cf. [8, Chapter 2]).

On the other hand, if A is an abelian group then we may consider the free associative ring on A, i.e. the tensor ring $TA = \bigoplus_{n=0}^{\infty} A^{\otimes n}$. We recall that the multiplication in TA is defined by concatenation of tensors. The associated Lie ring LTA is equal to TA as an abelian group, whereas its Lie bracket is defined by letting $(x, y) = xy - yx$ for all $x, y \in TA$. The free Lie ring on A is the Lie subring $\mathfrak{L}A$ of LTA generated by A. In fact, $\mathfrak{L}A$ is a graded subring of LTA, whose homogeneous component $\mathfrak{L}_n A \subseteq A^{\otimes n}$ of degree n is generated as an abelian group by the left normed n-fold commutators $(x_1, \ldots, x_n)$, $x_1, \ldots, x_n \in A$.

We now consider a group H and its abelianization $H_{ab} = H/[H, H]$. Then, in view of the universal property of the free Lie ring $\mathfrak{L}H_{ab}$, the identity map of $H_{ab} = \mathfrak{L}_1 H_{ab}$ into $H_{ab} = Gr^1 H$ extends to a graded Lie ring homomorphism

$$\kappa : \mathfrak{L}H_{ab} \longrightarrow Gr\, H.$$

It is clear that κ depends naturally on H. In particular, for all $n \geq 1$ there is an additive map

$$\kappa_n : \mathfrak{L}_n H_{ab} \longrightarrow \gamma_n H / \gamma_{n+1} H, \tag{5}$$

which is natural in H. We note that if the group H is free then the map κ (and hence all of the κ_n's) is bijective; cf. [8, Chapter 4, Theorem 6.1].

We shall now specialize the discussion above by letting (F, π) be an object of the presentation category $\mathfrak{P}$ of the group G and considering the kernel $R = \ker \pi$. Then, the terms of the lower central series of R are normal subgroups of F; in particular, F acts on each quotient $Gr^n R = \gamma_n R/\gamma_{n+1} R$ by letting $x \cdot y\gamma_{n+1}R = xyx^{-1}\gamma_{n+1}R$ for all $x \in F$ and $y \in \gamma_n R$. The latter action being trivial on R, it induces an action of G on the $Gr^n R$'s. Endowed with that action, the abelian group $Gr^n R$ is referred to as the n-th higher relation module associated with the given presentation. (For $n = 1$, we recover the relation module $Gr^1 R = R_{ab}$.) It is clear that the induced action of G on $Gr\, R$ is compatible with the Lie bracket. On the other hand, the diagonal action of G on the tensor powers $R_{ab}^{\otimes n}$ induces a G-action on TR_{ab}, which is compatible with multiplication. In particular, G acts on the associated Lie ring LTR_{ab} by Lie ring automorphisms. It is easily seen that the action of any group element on LTR_{ab} restricts to a Lie ring automorphism of the free Lie ring $\mathfrak{L}R_{ab}$. In particular, $\mathfrak{L}R_{ab}$ is a $\mathbb{Z}G$-submodule of LTR_{ab} and the homogeneous component $\mathfrak{L}_n R_{ab}$ is a $\mathbb{Z}G$-submodule of $R_{ab}^{\otimes n}$ for all $n \geq 1$.

In view of the naturality of the additive map (5) with respect to group homomorphisms, we conclude that the additive map

$$\kappa_n : \mathfrak{L}_n R_{ab} \longrightarrow \gamma_n R/\gamma_{n+1} R$$

is $\mathbb{Z}G$-linear for all $n \geq 1$. Moreover, since the group R is free (being a subgroup of the free group F), the latter map is an isomorphism. For all $n \geq 1$ we consider the $\mathbb{Z}G$-linear map

$$\lambda_n : \gamma_n R/\gamma_{n+1} R \longrightarrow R_{ab}^{\otimes n},$$

which is defined as the composition

$$\gamma_n R/\gamma_{n+1} R \xrightarrow{\kappa_n^{-1}} \mathfrak{L}_n R_{ab} \hookrightarrow R_{ab}^{\otimes n}.$$

Since the coinvariance $H_0(G, \gamma_n R/\gamma_{n+1} R)$ is identified with $\gamma_n R/[\gamma_n R, F]$, the $\mathbb{Z}G$-linear maps λ_n defined above induce additive maps

$$l_n : \gamma_n R/[\gamma_n R, F] \longrightarrow H_0(G, R_{ab}^{\otimes n})$$

for all $n \geq 1$. As shown in [11, Proposition 2], the abelian groups $J_n^G(R_{ab}, \mathbb{Z}) = \ker l_n$ are n-torsion groups. It is clear that l_n depends naturally upon the object (F, π) of the presentation category $\mathfrak{P}$ of G. Therefore, taking inverse limits over $\mathfrak{P}$, we may consider the additive map

$$\ell_n = \varprojlim l_n : \varprojlim \gamma_n R/[\gamma_n R, F] \longrightarrow \varprojlim H_0(G, R_{ab}^{\otimes n}).$$

We can now state our second main result.

Theorem 4.1. *Under the isomorphism between the homology group $H_{2n}(G, \mathbb{Z})$ and the inverse limit $\varprojlim H_0(G, R_{ab}^{\otimes n})$, which is established in Theorem 3.2, the image of the additive map ℓ_n defined above is contained in the n-torsion subgroup $H_{2n}(G, \mathbb{Z})[n]$ of $H_{2n}(G, \mathbb{Z})$.*

The proof of the Theorem will occupy the remaining of the Section. Let (F, π) be an object of the presentation category $\mathfrak{P}$ and consider the associated Magnus embedding

$$\mu : R_{ab} \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}$$

and the n-th tensor power map

$$\mu^{\otimes n} : R_{ab}^{\otimes n} \longrightarrow (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n},$$

which is also $\mathbb{Z}G$-linear. As shown in [11, Lemma 8], the kernel of the induced additive map

$$\overline{\mu^{\otimes n}} : H_0(G, R_{ab}^{\otimes n}) \longrightarrow H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n}\big)$$

is identified with the homology group $H_{2n}(G, \mathbb{Z})$. The composition

$$\gamma_n R/\gamma_{n+1} R \xrightarrow{\lambda_n} R_{ab}^{\otimes n} \xrightarrow{\mu^{\otimes n}} (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n}$$

is a $\mathbb{Z}G$-module map, which induces, by applying the functor $H_0(G, \_)$, an additive map

$$\gamma_n R/[\gamma_n R, F] \xrightarrow{l_n} H_0(G, R_{ab}^{\otimes n}) \xrightarrow{\overline{\mu^{\otimes n}}} H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n}\big)\,.$$

We shall denote the latter map by φ_n. As shown in [11, Proposition 1], the kernel of φ_n can be identified with the kernel of a certain matrix representation of the group $F/[\gamma_n R, F]$, which was defined by C.K. Gupta and N.D. Gupta in [5]. Since the kernel of $\overline{\mu^{\otimes n}}$ is identified with the homology group $H_{2n}(G, \mathbb{Z})$, as we have already noted above, there is an exact sequence

$$0 \longrightarrow J_n^G(R_{ab}, \mathbb{Z}) \longrightarrow \ker \varphi_n \longrightarrow H_{2n}(G, \mathbb{Z})$$

for all $n \geq 1$. In fact, as shown in [9], the homology group $H_{2n}(G, \mathbb{Z})$ in the exact sequence above may be replaced by its n-torsion subgroup $H_{2n}(G, \mathbb{Z})[n]$. Since the n-torsion subgroup $H_{2n}(G, \mathbb{Z})[n]$ of $H_{2n}(G, \mathbb{Z}) = \ker \overline{\mu^{\otimes n}}$ is contained in the n-torsion subgroup $H_0(G, R_{ab}^{\otimes n})[n]$ of $H_0(G, R_{ab}^{\otimes n})$, we conclude that there is an exact sequence

$$0 \longrightarrow J_n^G(R_{ab}, \mathbb{Z}) \longrightarrow \ker \varphi_n \xrightarrow{l_n|} H_0(G, R_{ab}^{\otimes n})[n],$$

where $l_n|$ denotes the restriction of l_n to the subgroup $\ker \varphi_n \subseteq \gamma_n R/[\gamma_n R, F]$. We shall now consider the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & J_n^G(R_{ab}, \mathbb{Z}) & \longrightarrow & \ker \varphi_n & \xrightarrow{l_n|} & H_0(G, R_{ab}^{\otimes n})[n] \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & J_n^G(R_{ab}, \mathbb{Z}) & \longrightarrow & \gamma_n R/[\gamma_n R, F] & \xrightarrow{l_n} & H_0(G, R_{ab}^{\otimes n})
\end{array}$$

where both unlabelled vertical arrows are the corresponding inclusion maps. Since all maps involved are natural with respect to the given object (F, π) of the presentation category $\mathfrak{P}$ of G, we may invoke Lemma 2.1 in order to obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim J_n^G(R_{ab}, \mathbb{Z}) & \longrightarrow & \varprojlim \ker \varphi_n & \xrightarrow{\ell_n|} & \varprojlim H_0(G, R_{ab}^{\otimes n})[n] \\
 & & \| & & \downarrow & & \downarrow \\
0 & \longrightarrow & \varprojlim J_n^G(R_{ab}, \mathbb{Z}) & \longrightarrow & \varprojlim \gamma_n R/[\gamma_n R, F] & \xrightarrow{\ell_n} & \varprojlim H_0(G, R_{ab}^{\otimes n})
\end{array}$$

Since the inverse limit $\varprojlim H_0(G, R_{ab}^{\otimes n})[n]$ of the n-torsion subgroups is identified with the n-torsion subgroup of the inverse limit $\varprojlim H_0(G, R_{ab}^{\otimes n})$, the assertion in the statement of Theorem 4.1 follows from the next result.

Lemma 4.2. *The additive map $\varprojlim \ker \varphi_n \longrightarrow \varprojlim \gamma_n R/[\gamma_n R, F]$, which is induced by the inclusions $\ker \varphi_n \hookrightarrow \gamma_n R/[\gamma_n R, F]$, is an isomorphism for all $n \geq 1$.*

Proof. In view of Lemma 2.1, the exact sequence

$$0 \longrightarrow \ker \varphi_n \longrightarrow \gamma_n R/[\gamma_n R, F] \xrightarrow{\varphi_n} H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n}\big) ,$$

which is associated with an object (F, π) of $\mathfrak{P}$ as above, induces an exact sequence

$$0 \longrightarrow \varprojlim \ker \varphi_n \longrightarrow \varprojlim \gamma_n R/[\gamma_n R, F] \xrightarrow{\phi_n} \varprojlim H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n}\big) ,$$

where $\phi_n = \varprojlim \varphi_n$. Therefore, the result will follow if we show the vanishing of the inverse limit $\varprojlim H_0(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n})$. To that end, we shall apply the criterion established in Lemma 2.2. We have to verify that conditions (i) and (ii) therein are satisfied. In view of Lemma 3.1, any two objects (F, π) and (F', π') of $\mathfrak{P}$ have a coproduct, which is provided by (F'', π''), where F'' is the free product of F and F'. Therefore, if F (resp. F') is free on the set S (resp. S'), then F'' is free on the disjoint union S'' of S and S'. It follows that the $\mathbb{Z}G$-modules $\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}$, $\mathbb{Z}G \otimes_{\mathbb{Z}F'} \mathfrak{f}'$ and $\mathbb{Z}G \otimes_{\mathbb{Z}F''} \mathfrak{f}''$ are free on the sets $\{1 \otimes (s-1) : s \in S\}$, $\{1 \otimes (s'-1) : s' \in S'\}$ and $\{1 \otimes (s''-1) : s'' \in S''\}$ respectively. Hence, the inclusions of F and F' into F'' induce an isomorphism of $\mathbb{Z}G$-modules

$$(\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f}) \oplus (\mathbb{Z}G \otimes_{\mathbb{Z}F'} \mathfrak{f}') \xrightarrow{\sim} \mathbb{Z}G \otimes_{\mathbb{Z}F''} \mathfrak{f}''.$$

Therefore, considering n-th tensor powers, we conclude that the natural map

$$(\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n} \oplus (\mathbb{Z}G \otimes_{\mathbb{Z}F'} \mathfrak{f}')^{\otimes n} \longrightarrow (\mathbb{Z}G \otimes_{\mathbb{Z}F''} \mathfrak{f}'')^{\otimes n}$$

is a split monomorphism of $\mathbb{Z}G$-modules. Therefore, applying the functor $H_0(G, \_)$, we conclude that the natural map

$$H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathfrak{f})^{\otimes n}\big) \oplus H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F'} \mathfrak{f}')^{\otimes n}\big) \longrightarrow H_0\big(G, (\mathbb{Z}G \otimes_{\mathbb{Z}F''} \mathfrak{f}'')^{\otimes n}\big)$$

is a (split) monomorphism of abelian groups, as needed. □

Remarks 4.3 (i) Let (F, π) be an object of the presentation category $\mathfrak{P}$ of G. Then, as shown in [9, Theorem 2], the kernel $\ker \varphi_n$ of the additive map φ_n constructed above coincides with the torsion subgroup of $\gamma_n R/[\gamma_n R, F]$. Therefore, it follows from [10] that $\ker \varphi_n$ is an n-torsion group if $n \geq 3$ and a 4-torsion group if $n = 2$. Since this is also the case for the inverse limit of these groups, we may invoke Lemma 4.2 in order to conclude that the inverse limit $\varprojlim \gamma_n R/[\gamma_n R, F]$ is an n-torsion group if $n \geq 3$ and a 4-torsion group if $n = 2$. The latter assertion provides another proof of Theorem 4.1, in the case where $n \geq 3$.

(ii) It follows from the proof of Theorem 4.1 given above that for all $n \geq 1$ there is a short exact sequence of abelian groups

$$0 \longrightarrow \varprojlim J_n^G(R_{ab}, \mathbb{Z}) \longrightarrow \varprojlim \gamma_n R/[\gamma_n R, F] \longrightarrow H_{2n}(G, \mathbb{Z})[n],$$

where the inverse limits are taken over the presentation category $\mathfrak{P}$ of G. In order to obtain an embedding of the inverse limit $\varprojlim \gamma_n R/[\gamma_n R, F]$ into the n-torsion subgroup $H_{2n}(G, \mathbb{Z})[n]$ of the homology group $H_{2n}(G, \mathbb{Z})$, at least in the case where $n \geq 3$, one may ask whether the abelian group $\varprojlim J_n^G(R_{ab}, \mathbb{Z})$ is zero. Following Thomson, who studied the vanishing of the group $J_n^G(R_{ab}, \mathbb{Z})$ in [11], we consider the following special cases:

(ii1) Assume that G is a finite group of order relatively prime to n. Then, the homology group $H_{2n}(G, \mathbb{Z})$ has no non-trivial n-torsion elements and the group $J_n^G(R_{ab}, \mathbb{Z})$ vanishes for any presentation $G = F/R$ (cf. [11, Theorem 2(ii)]). Therefore, taking into account the exact sequence above, it follows that $\varprojlim \gamma_n R/[\gamma_n R, F] = 0$.

(ii2) Assume that the cohomological dimension of G is ≤ 2. Then, the group $J_n^G(R_{ab}, \mathbb{Z})$ vanishes for any presentation $G = F/R$ (cf. [11, Theorem 2(iii)]), whereas the homology group $H_{2n}(G, \mathbb{Z})$ vanishes for all $n \geq 2$. Therefore, taking into account the exact sequence above, it follows that $\varprojlim \gamma_n R/[\gamma_n R, F] = 0$ for all $n \geq 2$.

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