Removable Singularities of Holomorphic Vector Bundles

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MPI/89-41

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In the pioneering work [U1], Uhlenbeck proved the following removable singularities theorem for Yang-Mills connections.

Theorem. *Let A be a Yang-Mills connection on a bundle P over the punctured ball $\mathbf{B}^* = \mathbf{B}\backslash\{0\}$ in $\mathbf{R}^4$. If the square integral of the curvature tensor R_A of A is finite*

$$\int_{\mathbf{B}^*} |R_A|^2 < \infty,$$

then the bundle P and the connection A extend smoothly to the whole ball $\mathbf{B}$.

Since as pointed out by Itoh [I], any Einstein Hermitian connection of a holomorphic vector bundle on a Kähler surfaces is a Yang-Mills connection, we get

Corollary. *Let $(E,h) \longrightarrow \mathbf{B}^*$ be an Einstein Hermitian holomorphic vector bundle over the punctured ball $\mathbf{B}^* \subset \mathbf{C}^2$. If its curvature is square integrable, then (E,h) extends to the whole ball $\mathbf{B}$ as an Einstein Hermitian holomorphic vector bundle.*

In a sense the assumption of the corollary is too strong. It assumes not only the Yang-Mills equation but also the equation comming from the holomorphicity. So it would be natural to try to get rid of the Einstein condition. In this direction there are works by Cornalba-Griffiths [CG], Siu [S1] and Uhlenbeck [U2]. They assumed pointwise estimates of the curvature; boundedness or positivity. We only assume the curvature belongs to L^2 and get,

Theorem 10. *Let $(E,h) \longrightarrow \mathbf{B}^*$ be a Hermitian holomorphic vector bundle over the punctured ball $\mathbf{B}^* \subset \mathbf{C}^2$. If it satisfies*

$$\int_{\mathbf{B}^*} |R_h|^2 < \infty,$$

then E extends to a holomorphic vector bundle $\bar{E}$ defined on the whole ball $\mathbf{B}$. And every holomorphic section of $\bar{E}$ is locally square integrable.

The idea of the proof is rather standard. First we show E and its dual vector bundle E^* have sufficiently many holomorphic sections on $\mathbf{B}^*$ so that we can imbedd E into a trivial vector bundle of sufficiently high rank. Then we extend E as a torsion free sheaf $\mathcal{E}$ over $\mathbf{B}$. Since the dimension of the base space is 2, the double dual $\mathcal{E}^{**}$ of $\mathcal{E}$ defines the desired vector bundle $\bar{E}$. The last statement of the theorem is an easy consequence of an anayitical lemma.

We remark that for an open set U it holds that

$$\begin{aligned}\Gamma(\bar{E};U) &\cong \{\, s \in \Gamma(E; U \cap \mathbf{B}^*) \mid s \text{ is locally square integrable} \,\} \\ &\cong \Gamma(E; U \cap \mathbf{B}^*).\end{aligned}$$

The author would like to express his gratitude for the hospitality of Max-Planck-Institut für Mathematik. This work is done during his stay there.

1. Extension of line bundles.

Theorem 1. *Let S be an analytic subset of at least codimension 2 of the ball $\mathbf{B}$ in $\mathbf{C}^n$, and (L, h) be a holomorphic Hermitian line bundle defined on $\mathbf{B}\backslash S$. Assume that the curvature ω of (L, h) is square integrable, then L extends to the whole ball $\mathbf{B}$ as a holomorphic line bundle.*

Proof. It is easy to see that the following definition gives a well-defined d-closed $(1,1)$-current $\bar{\omega}$ on $\mathbf{B}$. For a smooth $2(n-1)$-form θ with compact support in $\mathbf{B}$,

$$\bar{\omega}(\theta) = \int_{\mathbf{B}\backslash S} \omega \wedge \theta.$$

Then there exists a $(0,0)$-current u such that

$$\sqrt{-1}\partial\bar{\partial}u = \bar{\omega}.$$

The regularity theorem says that u is smooth on $\mathbf{B}\backslash S$. We replace the Hermitian metric h on L by he^u, then its curvature vanishes. It means that L comes from a representation of $\pi_1(\mathbf{B}\backslash S) = \{1\}$. Thus L is a trivial line bundle on $\mathbf{B}\backslash S$, which clearly extends to the whole ball $\mathbf{B}$ as a line bundle.

2. Solving $\bar\partial$-equations.

From now on we work under the assumption of theorem. We may assume that $\mathbf{B} = \{z \in \mathbf{C}^2 \,|\, |z|^2 < 1\}$ and (E, h) is defined on a larger punctured ball.

Let ρ be a smooth $\bar\partial$-closed E-valued $(0,1)$-form which has compact support in $\mathbf{B}^*$. We want to solve the following equation;

$$\bar\partial u = \rho \qquad \text{on } \mathbf{B}^*,$$

with $u \in H^1$, namely u and its covariant derivative ∇u are square integrable. First we solve the $\bar\partial$-Neumann problem; with the formal adjoint ϑ of $\bar\partial$

$$\Box\phi = (\bar\partial\vartheta + \vartheta\bar\partial)\phi = \rho \qquad \text{on } \mathbf{B}^*,$$

with $\phi \in H^1$ which satisfies the $\bar\partial$-Neumann condition at the boundary $\partial\mathbf{B}$. We need to specify a base metric and a fibre metric. We fix the base metric to be the standard Euclidian one and the fibre metric to be $h_K = he^{-K|z|^2}$ with a sufficiently large constant K to be chosen later.

For a small number $\epsilon > 0$, we solve the the Dirichlet-$\bar\partial$-Neumann problem on $\mathbf{B}^*_\epsilon = \{\, z \in \mathbf{C}^2 \mid \epsilon < |z|^2 < 1\,\}$, i.e. we put the $\bar\partial$-Neumann condition on $\{\, |z|^2 = 1\,\}$ and the Dirichlet condition on $\{\, |z|^2 = \epsilon\,\}$.

Lemma 2. *If we take K large enough, then for a section ϕ which satisfies the Dirichlet-$\bar\partial$-Neumann condition, we get that*

$$(\Box\phi, \phi) = \|\bar\partial\phi\|^2 + \|\vartheta\phi\|^2 \geq \|\phi\|^2.$$

In particular we can solve the equation $\Box\phi = \rho$, with $\|\phi\|, \|\vartheta\phi\|, \|\bar\partial\phi\| \leq \|\rho\|$.

Proof. Let $R_K = R_h + K$ be the curvature tensor of the metric h_K, and η be a cut-off function which is equal to 1 near the origin, then,

$$\begin{aligned}
(\Box\phi, \phi) &= \|\bar\partial\phi\|^2 + \|\vartheta\phi\|^2 \\
&\geq \int |\nabla^{0,1}\phi|^2 + (R_K\phi, \phi) \\
&\geq \int \frac{1}{2}|\nabla^{0,1}(\eta\phi)|^2 - |\nabla^{0,1}\eta|^2|\phi|^2 + (R_K\phi, \phi) \\
&= \int \frac{1}{4}(|\nabla^{0,1}(\eta\phi)|^2 + |\nabla^{1,0}(\eta\phi)|^2) \\
&\qquad\qquad - |\nabla^{0,1}\eta|^2|\phi|^2 - \frac{1}{4}(\mathrm{tr}R_K\phi, \phi) + (R_K\phi, \phi)
\end{aligned}$$

$$= \int \frac{1}{4}|\nabla(\eta\phi)|^2 + \frac{K}{2}|\phi|^2 - |\nabla^{0,1}\eta|^2|\phi|^2 - \frac{1}{4}(\mathrm{tr}R_h\phi,\phi) + (R_h\phi,\phi),$$

where tr is taken in the form part. Since $\eta\phi$ has compact support, we can apply the Sobolev inequality and get that with a positive constant S,

$$\left(\int |\eta\phi|^4\right)^{1/2} \leq S \int |\nabla(\eta\phi)|^2.$$

Choose η such that whose support is so small that

$$\int_{\mathrm{supp}\,\eta} |R_h|^2 \leq \frac{1}{64S^2},$$

and take K large enough, then we get that

$$\begin{aligned}(\Box\phi,\phi) &\geq \int (\frac{K}{2} - |\nabla^{0,1}\eta|^2)|\phi|^2 + (1-\eta^2)(-\frac{1}{4}(\mathrm{tr}R_h\phi,\phi) + (R_h\phi,\phi)) \\ &\geq \|\phi\|^2.\end{aligned}$$

Letting $\epsilon \longrightarrow 0$ we get

Lemma 3. *We have a solution $\phi \in H^1$ of the $\bar{\partial}$-Neumann problem on $\mathbf{B}^*$.*

By Moser's iteration arguement one can get the following lemma. (c.f. [BKN] Lemma(5.8), Lemma(5.9).)

Lemma 4. *Let f be a square integrable non-negative function on $\mathbf{B}^*$, and u be a locally H^1 non-negative function on $\mathbf{B}^*$ such that with a positive constant c*

$$\Delta u \geq -fu - c, \qquad \text{on } \mathbf{B}^*,$$

where $\Delta = \sum \frac{\partial^2}{(\partial x^i)^2}$. If $u \in H^1$ or $\int_{B(r)} u^p = o(r^2)$ as $r \longrightarrow 0$ with $p > 1$,where $B(r) = \{\,|z| < r\,\}$, then we get that $u \in L^q$ for all $q > 1$ on $B(1/2)$.

The equation $\Box\phi = \rho$ implies $\Delta\phi = 2R_h\phi - \mathrm{tr}\,R_h\phi - 2\rho$, hence $\Delta|\phi| \geq -4|R_h||\phi| - 2|\rho|$. The lemma yields that $\phi \in L^q$ for all $q > 1$. By integration by parts with a cut-off function η we get that

$$\int |\nabla(\eta\phi)|^2 \leq \left\{4\left(\int |R_h|^2\right)^{1/2} + 2\left(\int |\eta\rho|^{4/3}\right)^{3/4}\right\}\left(\int |\eta\phi|^4\right)^{1/2} + \left(\int |\nabla\eta|^4\right)^{1/2}\left(\int_{\mathrm{supp}\nabla\eta} |\phi|^4\right)^{1/2}.$$

As $\phi \in L^q$ for any $q > 1$, it holds that $\int_{B(r)} |\phi|^4 = O(r^{4-2\delta})$ for any positive δ, and $\int_{B(r)} |\nabla\phi|^2 = O(r^{2-\delta})$. Taking $\bar\partial$ of the equation $\Box\phi = \rho$, we get that $0 = \bar\partial\Box\phi = \bar\partial\vartheta\bar\partial\phi = \Box\bar\partial\phi$, and $\triangle|\bar\partial\phi| \geq -2|R_K||\bar\partial\phi|$. Appying Lemma 4 with $u = |\bar\partial\phi|$ and $1 < p < 2$, we get that $\bar\partial\phi \in L^q$ for any $q > 1$. Taking a cut-off function $\eta = \eta_r$ such that $\eta(z) = 1$ for $|z| > 2r$, $= 0$ for $|z| < r$ and $|\nabla\eta| < 2/r$, we get that

$$0 = (\bar\partial\vartheta\bar\partial\phi, \eta^2\phi) = (\vartheta\bar\partial\phi, \vartheta(\eta^2\bar\partial\phi)) = \|\eta\vartheta\bar\partial\phi\|^2 + 2(\eta\vartheta\bar\partial\phi, \nabla^{1,0}\eta * \bar\partial\phi),$$

$$\|\eta\vartheta\bar\partial\phi\|^2 \leq 4\left(\int |\nabla\eta|^3\right)^{2/3}\left(\int |\bar\partial\phi|^6\right)^{1/3} \longrightarrow 0 \qquad \text{as } r \longrightarrow 0.$$

Thus putting $u = \vartheta\phi$ we get

Lemma 5. *For a given smooth $\bar\partial$-closed $(0,1)$-form ρ with compact support in $\mathbf{B}^*$, we can solve the $\bar\partial$-equation*

$$\bar\partial u = \rho \qquad \text{on } \mathbf{B}^*,$$

with $\|u\| \leq \|\rho\|$.

Let s be a holomorphic section of E defined in a neighborhood of $z_0 \in \mathbf{B}^*$, and η be a cut-off function with compact support in $\mathbf{B}^*$ which is equal to 1 in a neighborhood of z_0 such that ηs makes a smooth section on $\mathbf{B}^*$ by putting $\eta s = 0$ where s is not defined. Then $\rho = \bar\partial(\eta s)$ is a $\bar\partial$-closed $(0,1)$-form which vanishes in a neighborhood of z_0. We take a pluri-subharmonic function $w(z) = \log|z - z_0|^2$ as a weight function, and use a fibre metric $he^{-k|z|^2-2w}$ instead of $he^{-k|z|^2}$ in the proof of Lemma 5. Since $\rho \in L^2$ in the new metric the solution u also belongs to the L^2-space that means u vanishes at z_0. Thus $\eta s - u$ is a holomorphic L^2-section of E on $\mathbf{B}^*$ which is equal to s at z_0.

Lemma 6. *For any point* $z_0 \in \mathbf{B}^*$ *we can fined a family* $\{s_1, s_2, \ldots, s_r\}$, $r = \operatorname{rank} E$ *of holomorphic* L^2*-sections on* $\mathbf{B}^*$ *(*L^2 *with respect to the given metric* h*) which gives a base of* E_{z_0} *at* z_0.

Lemma 7. *There exists a family* $\{s_1, s_2, \ldots, s_m\}$ *of finite numbers of hlomorphic* L^2*-sections on* $\mathbf{B}^*$ *such that* $\{s_1(z), s_2(z), \ldots, s_m(z)\}$ *spans* E_z *for each point* $z \in B(1/2)^*$.

Proof. Theorem 1 says that the determinant line bundle $\det E$ extends to the whole ball as a holomorphic line bundle, which we still call $\det E$. Fix a point $z_0 \in \mathbf{B}^*$ and construct holomorphic L^2-sections $s_1, s_2, \ldots, s_r$ on $\mathbf{B}^*$ which gives a base at z_0. Then $\sigma = s_1 \wedge s_2 \wedge \ldots \wedge s_r$ gives a section of $\det E$ on $\mathbf{B}^*$. Since by Hartogs' theorem σ extends to $\mathbf{B}$ as a holomorphic section, we still call it σ, the divisor $(\sigma) = \{ z \in \mathbf{B} \,|\, \sigma(z) = 0 \}$ has a finitely many irreduceble components D_i $(i = 1, \ldots, l)$ in $B(4/5)$. Take a point $z_i \in B(4/5)^*$ in each component D_i, and construct a family $\{s_{i,1}, s_{i,2}, \ldots, s_{i,r}\}$ of hplomorphic L^2-sections on $\mathbf{B}^*$, which spans E_{z_i} at z_i. Then $\{ s_1, s_2, \ldots, s_r, s_{i,1}, s_{i,2}, \ldots, s_{i,r} \, (i = 1, \ldots, l)\}$ spans E in $B(3/5)^*$ except finite numbers of points $\{z'_j\}$. Again we construct finite numbers of holomorphic L^2-sections $\{ s'_1, s'_2, \ldots, s'_{l'} \}$ on $\mathbf{B}^*$ to make them span E at $\{z'_j\}$. Then $\{ s_1, s_2, \ldots, s_r, s_{i,1}, s_{i,2}, \ldots, s_{i,r}, s'_1, s'_2, \ldots, s'_{l'} \}$ is the desired family.

Since E^* also have the square integrable curvature, we have

Lemma 8. *There exists a family* $\{t_1, t_2, \ldots, t_n\}$ *of finite numbers of hlomorphic* L^2*-sections on* $\mathbf{B}^*$ *such that* $\{t_1(z), t_2(z), \ldots, t_n(z)\}$ *spans* E^*_z *for each point* $z \in B(1/2)^*$.

3. Extension of Holomorphic vector bundles.

We embedd the vector bundle $E\big|_{B^*}$, $B = B(1/2)$ in the trivial vector bundle $\mathbf{C}^n$ by

$$E \ni s \longmapsto (< s, t_1 >, < s, t_2 >, \ldots, < s, t_n >) \in \mathbf{C}^n.$$

Then it is generated by the images $\{\tilde{s}_i\}$ of $\{s_i\}_{i=1}^m$. By Hartogs' theorem $\tilde{s}_i$ extends to the whole ball B as a holomorphic section. We define a coherent subsheaf $\mathcal{E}$ of $\mathbf{C}^n$ on B as the one generated by $\{\tilde{s}_i\}$. Since $\dim B = 2$, the double dual $\mathcal{E}^{**}$ of $\mathcal{E}$, which coincides $\mathcal{E}$ ecxept at the origin, comes from a holomorphic vector bundle $\bar{E}$. Then there exists a non-zero polynomial P such that for any holomorphic section $\tilde{s}$ of $\bar{E}$, $P\tilde{s}$ belongs to $\mathcal{E}$. It implies that the restriction of $P\tilde{s}$ to B^* is square integrable with respect to the metric h. We fix an arbitrary smooth fibre metric $\bar{h}$ on $\bar{E}$. Then $\log^+ \mathrm{tr}_{\bar{h}} h$ belongs to the L^q-space for any $q > 1$. A calculation shows

$$\triangle \log^+ \mathrm{tr}_{\bar{h}} h \geq -2(|\mathrm{tr} R_h| + |\mathrm{tr} R_{\bar{h}}|).$$

We solve the equation

$$\Delta v = -2(|\mathrm{tr} R_h| + |\mathrm{tr} R_{\bar{h}}|) \in L^2,$$
$$v\big|_{\partial B} = \log^+ \mathrm{tr}_{\bar{h}} h\big|_{\partial B}$$

Then we get that $v \in H^2$ and $\log^+ \mathrm{tr}_{\bar{h}} h \leq v$. We apply the following lemma to see $\mathrm{tr}_{\bar{h}} h$ belongs to the L^q-space for any $q > 1$.

Lemma 9. *Let v be a function in the H^2-space on a real 4-dimensional ball, then $\exp v$ belongs to the L^q-space for any $q > 1$.*

Now we get our

Theorem 10. *Let $(E, h) \longrightarrow \mathbf{B}^*$ be a Hermitian holomorphic vector bundle over the punctured ball $\mathbf{B}^* \subset \mathbf{C}^2$. If it satisfies*

$$\int_{\mathbf{B}^*} |R_h|^2 < \infty,$$

then E extends to a holomorphic vector bundle $\bar{E}$ defined on the whole ball $\mathbf{B}$. And every holomorphic section of $\bar{E}$ is locally square integrable.

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