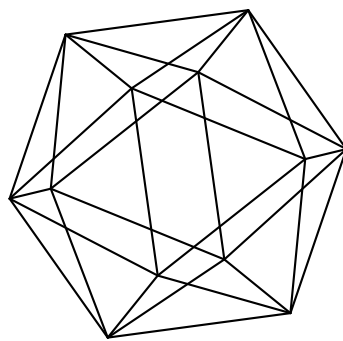# Max-Planck-Institut für Mathematik Bonn

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by

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ON THE VERLINDE FORMULA FOR TWISTED CONFORMAL BLOCKS: I

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ABSTRACT. In this paper we give a conjectural Verlinde formula for computing the ranks of the bundles of twisted conformal blocks associated with a simple Lie algebra equipped with an action of a finite group Γ and a positive integral level ℓ. As a motivation for this conjectural Verlinde formula, we prove a categorical Verlinde formula which computes the fusion coefficients for any Γ-crossed modular fusion category as defined by Turaev. To relate these two versions of the Verlinde formula, we formulate the notion of a Γ-crossed modular functor and conjecture that it is very closely related to the notion of a Γ-crossed modular fusion category. We conjecture that the bundles of Γ-twisted conformal blocks associated with a twisted affine Lie algebra define a Γ-crossed modular functor. Along the way, we give a coordinate free description of the space of twisted conformal blocks and show that it satisfies the some of the axioms of a Γ-crossed modular functor. We also explicitly describe the crossed S-matrices that appear in the conjectural Verlinde formula for twisted conformal blocks. Finally we apply these results to derive the conjectural Verlinde formula from the conjectures about Γ-crossed modular functors.

CONTENTS

1. INTRODUCTION

The Wess-Zumino-Witten (WZW) model is a two dimensional rational conformal theory and conformal blocks associated to these models were explicitly constructed in the phenomenal works of Tsuchiya, Ueno and Yamada [76]. To a simple Lie algebra $\mathfrak{g}$, a positive integer ℓ, an n-tuple $\vec{\lambda}$ of dominant integral weights of level ℓ (see Section 3.3) of the untwisted affine Kac-Moody Lie algebra, the WZW model in [76] associates a vector bundle $\mathbb{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{g}, \ell)$ of finite rank on the Deligne-Mumford-Knudsen moduli stack $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves of genus g. These vector bundles $\mathbb{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{g}, \ell)$ are known as the *bundles of conformal blocks* and their duals $\mathbb{V}_{\vec{\lambda}}(\mathfrak{g}, \ell)$ are referred to as the *sheaf of covacua*. Moreover these bundles are endowed with a flat projective connection with logarithmic singularities along the boundary divisor of $\overline{\mathcal{M}}_{g,n}$ and satisfy [76] all the key axioms of conformal field theory like *factorization* and *propagation of vacua* (see Section 4). Now we briefly point out some major interactions between algebraic geometry, representation theory and mathematical physics via conformal blocks.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$ and let $\operatorname{Bun}_G(C)$ be the moduli stack of principal G-bundles on a smooth, projective curve C of genus g. It is well known [57, 59] that the Picard group of $\operatorname{Bun}_G(C)$ is $\mathbb{Z}\mathcal{L}$, where $\mathcal{L}$ is the ample generator. The stack $\operatorname{Bun}_G(C)$ generalizes the Jacobian of a curve and hence the spaces $H^0(\operatorname{Bun}_G(C), \mathcal{L}^{\otimes \ell})$ are known as the spaces of non abelian theta functions. It is well known that the space of theta functions at level ℓ on the Jacobian of a curve is ℓ^g-dimensional. Hence it is natural to ask for a formula for the dimension of $H^0(\operatorname{Bun}_G(C), \mathcal{L}^{\otimes \ell})$. Uniformization theorems of [27, 42] and the works of [10, 32, 57] produce canonical isomorphisms between $H^0(\operatorname{Bun}_G, \mathcal{L}^{\otimes \ell})$ and the fiber of the vector bundle $\mathbb{V}_0^{\dagger}(\mathfrak{g}, \ell)$ at a smooth curve C. Several extensions of these results to the parabolic case [59] and to nodal curves [15] are also known. Hence the questions on Bun_G can be approached using conformal blocks and representation theory.

From the representation theoretic viewpoint, we get that the fibers of the conformal blocks $\mathbb{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{g}, \ell)_{|\mathbb{P}^1;(p_1,\ldots,p_n)}$ over n-pointed genus zero curves embed naturally [31, 76] in the space of invariants of $\mathfrak{g}$-representations $\operatorname{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, \mathbb{C})$. Moreover if ℓ is large enough, then the vector spaces $\mathbb{V}_{\vec{\lambda}}^{\dagger}(\mathfrak{g}, \ell)_{|\mathbb{P}^1;(p_1,\ldots,p_n)}$ and $\operatorname{Hom}_{\mathfrak{g}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}, \mathbb{C})$

are isomorphic. In particular, one can study the space of invariants of tensor product of representations using conformal blocks. Motivated by various formulas for the Littlewood-Richardson coefficients, it is again natural to ask for a formula for the rank of the conformal block bundles.

Dualities in coset conformal field theory associated to a conformal pair $(\mathfrak{g}_1, \mathfrak{g}_2)$ predict that conformal blocks for a Lie algebra $\mathfrak{g}_1$ at level ℓ_1 are often dual to conformal blocks for another Lie algebra $\mathfrak{g}_2$ at level ℓ_2. These questions/conjectures are known as rank-level duality/strange duality conjectures. As a first step to check validity of these conjectures, one needs to check that the dimensions of the two conformal blocks arising in rank level duality have same dimensions. We refer the reader [1, 14, 63, 64, 65, 82] to for more details.

In 1987, E. Verlinde conjectured [78] an explicit formula for computing the rank of these conformal blocks bundles which became well known as the Verlinde formula. More precisely, he conjectured that the *S-matrix* given by the rules of the modular transformations $\tau \to -\frac{1}{\tau}$ *diagonalizes the fusion rules*. Verlinde's conjectural rank formula was proved for $SL(2)$ independently by the works of Bertram, Bertram-Szenes, Daskalopoulos-Wentworth and M. Thaddeus [16, 17, 22, 74]. The Verlinde formula for classical groups and G_2 was proved by G. Faltings [32] and by C. Teleman [73] in full generality. A proof for the Verlinde rank formula was also given later using methods from symplectic geometry [4, 51]. In 1988, Moore-Seiberg gave an outline of the proof of Verlinde's conjecture about diagonalization of fusion rules and a conceptual connection between modular tensor categories and the Verlinde formula. Verlinde's conjecture on diagonalization of fusion rules by S-matrices has been recently proved by Y. Huang [48, 49].

Automorphisms of affine Kac-Moody Lie algebras inherited from the underlying finite dimensional Lie algebras have been closely related to the constructions of orbifold conformal blocks also known as twisted WZW models [19, 58]. Twisted WZW models associated to order two diagram automorphisms of a simple Lie algebra has been constructed and studied by Shen-Wang [79] and also by Birke-Fuchs-Schweigert [19]. Starting with a vertex algebra along with an action of a finite group Γ, Frenkel-Szczesny [39] constructed orbifold conformal blocks. More precisely, given a pointed Γ-cover $(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$ (along with a lift $\widetilde{\mathbf{p}}$) of a smooth curve C along with n-marked points $\mathbf{p} = (p_1, \ldots, p_n)$, they attached a module for the twisted vertex algebra. This construction can be carried out in families of stable curves to get a quasi-coherent sheaf $\mathbb{V}_\Gamma$ on the stack $\overline{\mathcal{M}}_{g,n}^\Gamma$ of pointed Γ-cover introduced in [51]. Later Szczesny [72] realized $\mathbb{V}_\Gamma$ as a twisted $\mathcal{D}$-module on the open part $\mathcal{M}_{g,n}^\Gamma$ of the stack $\overline{\mathcal{M}}_{g,n}^\Gamma$ corresponding to smooth curves. It is natural to consider whether the Γ-twisted WZW model associated to level ℓ-representations of a Γ-twisted affine Kac-Moody Lie algebra [53] gives a vector bundle on $\overline{\mathcal{M}}_{g,n}^\Gamma$ which is also a twisted $\mathcal{D}$-module with logarithmic singularities at the boundary $\Delta_{g,n}^\Gamma$.

In [46], Hong-Kumar studied the Γ-twisted WZW-models associated to an arbitrary Γ-twisted affine Kac-Moody Lie algebra. Adapting the constructions of Tsuchiya-Ueno-Yamada [76] in the twisted setting along with assumption that Γ preserves a Borel subalgebra of $\mathfrak{g}$, Hong-Kumar [46] construct a locally free sheaf over $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ that satisfies factorization and propagation of vacua. Similar results for diagram automorphisms of $\mathfrak{g}$ were also obtained by the second coauthor independently. In her thesis, C. Damiolini [21] considered the twisted conformal blocks where the marked points are never ramified and proved factorization, propagation of vacua and local freeness over a open substack of $\overline{\mathcal{M}}_{g,n}^{\Gamma}$.

We briefly discuss the terminology [21, 46] and refer the reader to Section 4 for a coordinate free construction. Let $\mathfrak{g}$ be a simple Lie algebra equipped with a Γ-action $\Gamma \hookrightarrow \mathrm{Aut}(\mathfrak{g})$. For any $\gamma \in \Gamma$, we consider the twisted affine Kac-Moody Lie algebra (see Section 3.1) $\widehat{L}(\mathfrak{g}, \gamma)$. The set of irreducible, integrable, highest weight representations of $\widehat{L}(\mathfrak{g}, \gamma)$ of level $\ell \in \mathbb{Z}_{\geq 0}$ are denoted by $P^{\ell}(\mathfrak{g}, \gamma)$ (see Section 3.1). If γ is trivial, then $P^{\ell}(\mathfrak{g}, \mathrm{id})$ will often be denoted by $P_{\ell}(\mathfrak{g})$. Now consider a stable nodal curve $\widetilde{C}$ with n marked point $\widetilde{\mathbf{p}}$ with a Γ action. Assume that $\widetilde{C} \setminus \Gamma \cdot \widetilde{\mathbf{p}}$ is affine on which Γ acts freely and each component has at least one marked point. Let $(\gamma_1, \ldots, \gamma_n)$ be generators of stabilizers in Γ of the points $(\widetilde{p}_1, \ldots, \widetilde{p}_n)$ determined by using the orientation on the complex curve $\widetilde{C}$. The element γ_i will be called the monodromy around the point $\widetilde{p}_i$. Suppose for each point $\widetilde{p}_i$ we have attached a highest weight integrable module $\mathcal{H}_{\lambda_i}(\mathfrak{g}, \gamma_i)$ of weight $\lambda_i \in P^{\ell}(\mathfrak{g}, \gamma_i)$. Let $\mathcal{H}_{\vec{\lambda}} := \mathcal{H}_{\lambda_1}(\mathfrak{g}, \gamma_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_n}(\mathfrak{g}, \gamma_n)$. Then the *space of twisted covacua* can be defined as

$$\mathcal{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{z}}) := \mathcal{H}_{\vec{\lambda}} / \left(\mathfrak{g} \otimes H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\Gamma.\widetilde{\mathbf{p}})) \right)^{\Gamma} \mathcal{H}_{\vec{\lambda}},$$

where $\widetilde{\mathbf{z}}$ denote a choice of a formal parameter along the points $\widetilde{\mathbf{p}}$. The corresponding vector bundles on $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ will be denoted by $\mathbb{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$. Starting with the work of Pappas-Rapoport [67, 68], the moduli stack of Bruhat-Tits torsor $\mathscr{G}$ associated to a pair $(G, \Gamma \subseteq \mathrm{Aut}(G))$ has been studied by several authors [8, 43]. Results of [46] (with some restrictions on level) and [83] (order two automorphisms of $\mathrm{SL}(r)$) connect twisted conformal blocks with global sections of line bundles on $\mathrm{Bun}\,\mathscr{G}$. Pappas-Rapoport [67, 68] ask if there is a Verlinde formula for these spaces of non-abelian theta functions or twisted conformal blocks?

Apart from the works of Birke-Fuchs-Schweigert [19] for $\Gamma = \mathbb{Z}/2$ and a double cover of $\mathbb{P}^1$ ramfied at two points, there is not even a conjectural description of the ranks of the above vector bundles for arbitrary Γ-twisted WZW conformal blocks. In this article, we give a conjectural twisted Verlinde formula that computes the rank of the Γ-twisted conformal blocks for an arbitrary finite group Γ. We now discuss our conjecture and an approach to this conjecture via Verlinde formula for Γ-crossed braided fusion categories.

Let $(C, \mathbf{p})$ be a smooth genus g curve with n-marked points $\mathbf{p}$. The fundamental group of $C \setminus \mathbf{p}$ has a presentation of the form

$$\pi_1(C \setminus \mathbf{p}, \star) = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle.$$

Let us fix a group homomorphism $\chi : \pi_1(C \setminus \mathbf{p}, \star) \twoheadrightarrow \Gamma^\circ \subseteq \Gamma$ with image Γ°. Let $m_i \in \Gamma$ be the image of γ_i. This determines (see [51, §2.3]) an n-pointed admissible Γ-cover $(\widetilde{C} \to C, \widetilde{\mathbf{p}}, \mathbf{p})$ such that all the lifts $\widetilde{\mathbf{p}}$ lie in the same connected component of $\widetilde{C}$ and the monodromy around the points $\widetilde{\mathbf{p}}$ is given by $(m_1, \cdots, m_n)$. As before, let us assume that we have an action of Γ on the simple lie algebra $\mathfrak{g}$ and we fix a level $\ell \in \mathbb{Z}_{\geq 0}$. Let $\vec{\lambda} = (\lambda_1, \cdots \lambda_n)$ with $\lambda_i \in P^\ell(\mathfrak{g}, m_i)$. We state the following conjectural Verlinde formula in the notation used above:

Conjecture 1.1. *The rank of the twisted conformal blocks bundle* $\mathbb{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$ *at level* ℓ *is given by the following formula:*

$$\operatorname{rank} \mathbb{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}) = \sum_{\mu \in P_\ell(\mathfrak{g})^{\Gamma^\circ}} \frac{S^{m_1}_{\lambda_1, \mu} \cdots S^{m_n}_{\lambda_n, \mu}}{\left(S_{0, \mu}\right)^{n+2g-2}},$$

where the summation is over the set of the untwisted dominant integral level-ℓ weights of $\mathfrak{g}$ fixed by the subgroup $\Gamma^\circ \subseteq \Gamma$, the matrix S that appears in the denominator is the untwisted S-matrix of type $\mathfrak{g}$ at level ℓ, the matrices S^{m_i} that appear in the numerator are the m_i-crossed S-matrices (see Chapter 13 in [53]) at level ℓ corresponding to the diagram automorphism class of $m_i \in \Gamma$ and which are defined explicitly in Section 8.10.1 and where 0 is the trivial element of $P_\ell(\mathfrak{g})$.

We refer the reader to Section 11 where we give a more special form of the above formula in some special cases and provide evidence by directly computing the dimensions using factorizations and dimensions of invariants of representations.

Remark 1.2. It is well known that the untwisted S-matrix S of type $\mathfrak{g}$ and level ℓ is a $P_\ell(\mathfrak{g}) \times P_\ell(\mathfrak{g})$ symmetric unitary matrix. On the other hand, for each $m \in \Gamma$, the m-crossed S-matrix S^m is a $P^\ell(\mathfrak{g}, m) \times P_\ell(\mathfrak{g})^m$ matrix. It is not immediately clear, but nevertheless true that S^m is a square matrix. Moreover, S^m is also unitary.

We now discuss the remaining results of this article which are motivated by Conjecture 1.1. There has been comprehensive study by several authors [26, 61, 70, 76, 80, 81] trying to understand the relations between modular tensor categories, 3-dimensional topological quantum field theory and 2-dimensional modular functors. A key bridge between these three topics has been conformal blocks associated to untwisted affine Lie algebras. Moore-Seiberg [61] has proved a Verlinde formula for any modular fusion category. In the twisted set up, Γ-crossed modular fusion categories have been introduced by V. Turaev [77]. Motivated by the result of [61], we prove a Verlinde formula for Γ-crossed modular fusion categories. The first coauthor introduced for each $\gamma \in \Gamma$, the notion of a γ-crossed S-matrix denoted by S^γ and twisted characters for any Γ-graded Frobenius $*$-algebra arising from a Grothendieck group of a Γ-crossed braided fusion category (see [77]). Extending earlier works of the first coauthor [25], we prove a general twisted Verlinde formula in the setting of Γ-crossed braided fusion categories which computes fusion coefficients in terms of the crossed S-matrices. We refer to Section 2.2, Theorem 2.15 for more details and to Corollary 2.17 for a higher genus analogue of the following 3-point genus zero version.

Theorem 1.3. *Let* $\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$ *be a* Γ-*crossed modular fusion category. Let* $\gamma_1, \gamma_2 \in \Gamma$ *and let* $A \in \mathcal{C}_{\gamma_1}, B \in \mathcal{C}_{\gamma_2}$ *and* $C \in \mathcal{C}_{\gamma_1\gamma_2}$ *be simple objects. Then the multiplicity* $\nu_{A,B}^C$ *of* C *in the tensor product* $A \otimes B$ *is given by*

$$\nu_{A,B}^C = \sum_{D \in P_1^{\langle \gamma_1, \gamma_2 \rangle}} \frac{\overline{S_{A,D}^{\gamma_1}} \cdot \overline{S_{B,D}^{\gamma_2}} \cdot S_{C,D}^{\gamma_1\gamma_2}}{S_{\mathbb{1},D}} = \sum_{D \in P_1^{\langle \gamma_1, \gamma_2 \rangle}} \frac{S_{A,D}^{\gamma_1} \cdot S_{B,D}^{\gamma_2} \cdot \overline{S_{C,D}^{\gamma_1\gamma_2}}}{S_{\mathbb{1},D}},$$

where the summation is over the simple objects $D \in \mathcal{C}_1$ *fixed by both* γ_1, γ_2 *and where the crossed S-matrices are chosen in a compatible way.*

Remark 1.4. In general a γ-crossed S-matrix depends on certain choices and is only well defined up to rescaling rows by roots of unity. In the above formula, we have assumed that the crossed S-matrices are chosen in a compatible way. In general some cocycles appear in the Verlinde formula Corollary 2.17. However in the set up of Γ-twisted conformal blocks, the cocycles are trivial (see also Remark 7.5).

In the untwisted set up, Bakalov-Kirillov [12] introduce the notion of a 2-dimensional complex analytic modular functor and show that it produces a weakly ribbon braided tensor category. In Section 6 we define the notion of a Γ-crossed complex analytic modular functor generalizing the notion due to [12]. To define this notion we need to work with some operations on the stacks $\overline{\mathcal{M}}_{g,n}^\Gamma, \widetilde{\mathcal{M}}_{g,n}^\Gamma$ of stable n-marked admissible Γ-covers. We briefly recall the relevant constructions in the Appendices A and B. We refer the reader to Sections 6.5 and 6.6 for the definition of the notion of a Γ-crossed modular functor and for a more precise version of the following:

Conjecture 1.5. *Let* $\mathcal{C}$ *be a finite semisimple* Γ-*crossed abelian category (see* §6.1*). A* $\mathcal{C}$-*extended* Γ-*crossed analytic modular functor defines the structure of a braided* Γ-*crossed weakly ribbon category on the category* $\mathcal{C}$.

We now address the question of constructing a Γ-crossed modular fusion category given the action of Γ on $\mathfrak{g}$ and a level ℓ. For this firstly we need the underlying finite semisimple Γ-crossed abelian category $\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$. For $\gamma \in \Gamma$, we take $\mathcal{C}_\gamma$ to be the finite semisimple category whose simple objects are the irreducible, integrable, highest weight representations of $\widehat{L}(\mathfrak{g}, \gamma)$ of level ℓ parametrized by $P^\ell(\mathfrak{g}, \gamma)$. This is the underlying Γ-crossed abelian category (see §7.1) on which we want to define the structure of a Γ-crossed modular category.

In Section 4, we give a coordinate free construction of the twisted conformal blocks and discuss the associated descent data coming form *Propagation of Vacua*. We show that like the untwisted case [31], twisted conformal blocks with at least one trivial weights are pull backs (see Proposition 4.4) along along forgetful-stabilization morphism $\overline{\mathcal{M}}_{g,n+1}^\Gamma(\mathbf{m}, 1) \to \overline{\mathcal{M}}_{g,n}^\Gamma(\mathbf{m})$. Following the approach of the Beilinson-Bernstein localization functors in [36], we extend the construction of twisted $\mathcal{D}$-modules in [72] to the compactification of the

moduli stack $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ as a twisted $\mathcal{D}$-module with logarithmic singularities along the boundary $\Delta_{g,n}^{\Gamma}$ of $\overline{\mathcal{M}}_{g,n}^{\Gamma}$. We determine the Atiyah algebra (Theorem 5.14) of the twisted logarithmic $\mathcal{D}$-module on $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$.

Theorem 1.6. *Let Λ be the pull back of the Hodge line bundle to $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ and $\widetilde{\mathcal{L}}_i$'s denote the line bundles corresponding to Psi-classes, then the logarithmic Atiyah algebra*

$$\frac{\ell \dim \mathfrak{g}}{2(\ell + h^{\vee}(\mathfrak{g}))} \mathcal{A}_{\Lambda}(\log \Delta_{g,n}^{\Gamma}) + \sum_{i=1}^{n} N_i \Delta_{\lambda_i} \mathcal{A}_{\widetilde{\mathcal{L}}_i}(\log \Delta_{g,n}^{\Gamma})$$

acts on $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$. Here Δ_{λ_i} is the eigenvalue of the twisted Virasoro operators $L_0^{\langle \gamma_i \rangle}$ and $h^{\vee}(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$.

The Atiyah algebra in Theorem 1.6 appears in our formulation (Section 6.5) of the axioms of $\mathcal{C}$-extended Γ-crossed analytic modular functor. Motivated by results of [11, 12], we conjecture the following:

Conjecture 1.7. *Let Γ be a finite group acting on the simple Lie algebra $\mathfrak{g}$ and let $\ell \in \mathbb{Z}_{\geq 1}$. Then the corresponding Γ-twisted conformal blocks define a $\mathcal{C}$-extended Γ-crossed modular functor.*

We now discuss how Conjectures 1.5 and 1.7 imply Conjecture 1.1. Let $\Gamma, \mathfrak{g}, \ell$ be as before and let $\mathcal{C}$ be the corresponding level ℓ Γ-crossed abelian category. If we assume that Conjecture 1.5 and Conjecture 1.7 hold, then $\mathcal{C}$ can be given the structure of a braided Γ-crossed weakly ribbon category. We can now derive Conjecture 1.1 using (6.10) if we are in a position to apply Theorem 1.3. However, we still need to answer the following question:

Question 1.8. *Are the braided Γ-crossed categories arising from Γ-twisted conformal blocks rigid?*

For the untwisted case, it is well known that conformal blocks form a weakly rigid braided tensor category [12]. Rigidity for these categories has been proved by Y. Huang [48, 49] and also by Finkelberg [34, 35]. However in both the proofs of Huang and Finkelberg, some variant of the Verlinde formula was used as an input. Hence in principal, the same problem persists in the twisted setting. We circumvent this issue by proving a general fact that if the underlying untwisted braided tensor category $\mathcal{C}_1$ is rigid, then the weakly rigid braided Γ-crossed category $\mathcal{C}$ must also be rigid. This follows from the following:

Proposition 1.9. *Let $\mathcal{C}$ be a weakly fusion category. Let $M \in \mathcal{C}$ be a simple object such that $M \otimes {}^*M$ has a rigid left dual. Then M has a rigid left dual.*

We refer the reader to Section 2.1, Proposition 2.5 and Corollary 2.6 for precise statements and proofs. Thus to prove Conjecture 1.1, we are left to determine the γ-crossed S-matrices. We first compare the present situation with the untwisted situation.

The Verlinde formula for conformal blocks for untwisted Lie algebra was proved [32, 73] by studying the characters of the fusion ring $R_{\ell}(\mathfrak{g})$ and realizing it as a based commutative

Frobenius $*$-algebra. The fusion ring has a basis parametrized by $P_\ell(\mathfrak{g})$ and its structure constants are given by dimensions of the untwisted 3-pointed conformal blocks of untwisted affine Kac-Moody Lie algebra associated to $\mathfrak{g}$. For any commutative Frobenius $*$-algebra (see Section 2.2), the structure constants are determined from the set of characters. Works of [32, 73] determine the set of characters of $R_\ell(\mathfrak{g})$ as restrictions of representations of the Grothendieck ring $R(\mathfrak{g})$ of representations of the lie algebra $\mathfrak{g}$. This determines the S-matrices in the untwisted setting.

First we approach our problem as in the untwisted case. With the assumptions that Γ preserves a Borel subalgebra of $\mathfrak{g}$, it is easy to directly observe that the twisted conformal blocks give a Γ-graded Frobenius $*$-algebra $R_{\ell,\Gamma}(\mathfrak{g}) := \bigoplus_{\gamma \in \Gamma} R_\ell(\mathfrak{g}, \gamma)$, where $R_\ell(\mathfrak{g}, 1)$ is the untwisted fusion ring $R_\ell(\mathfrak{g})$ and for each $\gamma \in \Gamma$, $R_\ell(\mathfrak{g}, \gamma)$ are modules over the untwisted fusion ring $R_\ell(\mathfrak{g})$.

As a vector space, $R_\ell(\mathfrak{g}, \gamma)$ has a basis parametrized by $P^\ell(\mathfrak{g}, \gamma)$. Conjectures 1.5 and 1.7 imply that $R_{\ell,\Gamma}(\mathfrak{g})$ is also the Grothendieck ring of the Γ-crossed modular fusion category given by twisted conformal blocks. Hence in the twisted set-up, we are reduced to determine the twisted characters of the Γ-graded Frobenius algebra $R_{\ell,\Gamma}(\mathfrak{g})$, since (see §2.2) these twisted characters determine the fusion coefficients. The module $R_\ell(\mathfrak{g}, \gamma)$ does not a priori have the structure of a Frobenius $*$-algebra. This presents a significant challenge. To address this issue, we consider the twisted fusion ring (see [23]) $\mathcal{R}_\gamma$ associated to a Γ-crossed rigid braided tensor category and an element $\gamma \in \Gamma$. It was shown [23, 24], that the γ-crossed S-matrix S^γ is essentially the character table of the twisted fusion ring $\mathcal{R}_\gamma$. Hence Conjecture 1.1 reduces to the following:

Question 1.10. *Determine the characters of the twisted fusion ring $\mathcal{R}_\gamma$.*

Jiuzhu Hong [44, 45] has studied twisted fusion rings $\mathcal{R}_\sigma$ defined using the trace of the action of diagram automorphisms of the Lie algebra on untwisted conformal blocks. Using the same strategy as in [9, 32, 73], Hong gives a complete description of the character table of $\mathcal{R}_\sigma$ in terms of the characters of the representations rings of the fixed point algebra $\mathfrak{g}_\sigma$ [39]. We observe that the twisted fusion ring $\mathcal{R}_\gamma$ constructed from rigid Γ-crossed braided tensor category for conformal blocks only depends on the diagram automorphism class σ of γ. We apply these results directly to the categorical Verlinde formula in Section 2.2 to derive the statement of Conjecture 1.1.

Remark 1.11. It might be possible to directly determine the set of twisted characters for the modules $R_\ell(\mathfrak{g}, \gamma)$ using representation theoretic methods. Our work gives a conjectural description of the twisted characters in terms of the crossed S-matrices. However our approach facilitates a conceptual understanding by building a bridge between Γ-crossed modular fusion categories, Γ-crossed modular functors and twisted conformal blocks as in the untwisted case.

Remark 1.12. In [45] it is conjectured that the coefficients of the twisted fusion rings $\mathcal{R}_\sigma$ are non-negative integers and are related to the dimensions of the twisted conformal blocks described in [46]. This was disproved by Alejandro Ginnory [41] who showed that these fusion coefficients could be negative. It is important to point out that the statement

of Conjecture 1.1 involves both crossed and uncrossed S-matrices for various elements of Γ whereas the Verlinde formula (see also [23, Thm. 2.12(iii)]) for the structure constants of $\mathcal{R}_\sigma$ involves only the σ-crossed S-matrix.

We study the twisted fusion rings, their character tables and the crossed S-matrices in Sections 8, 9 and 10. We give an explicit description of all the crossed S-matrices. Also in the cases of the twisted affine Lie algebras $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$ and $D_4^{(3)}$, we express the crossed S-matrices as submatrices of certain larger uncrossed S-matrices. We also refer the reader to Section 10 for the relation between unitary character table of the fusion ring R_σ and the crossed S-matrices defined in [53, Chapter 13].

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2. A VERLINDE FORMULA FOR BRAIDED CROSSED CATEGORIES

Let Γ be a finite group. In this section we will recall the notion of a braided Γ-crossed category and prove a Verlinde formula for such categories. We will also recall the notions of rigidity and weak rigidity in monoidal categories.

2.1. **Rigidity in weakly fusion categories.** We begin by reviewing the notions of weak duality and rigidity in monoidal categories and prove a useful criterion for rigidity. We refer the reader to [30] for more details on the theory of monoidal categories and fusion categories and to [20] for more on weak duality and pivotal/ribbon structures in this setting.

2.1.1. *Monoidal r-categories and weakly fusion categories.* We begin by recalling the notion of a monoidal r-category and a weakly fusion category.

Definition 2.1. A monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is said to be an r-category if (cf. [20]):
(i) For each $X \in \mathcal{C}$, the functor $\mathcal{C} \ni Y \mapsto \mathrm{Hom}(\mathbb{1}, X \otimes Y)$ is representable by an object X^*, i.e. we have functorial identifications $\mathrm{Hom}(\mathbb{1}, X \otimes Y) = \mathrm{Hom}(X^*, Y)$.
(ii) The functor $\mathcal{C} \ni X \mapsto X^* \in \mathcal{C}^{\mathrm{op}}$ is an equivalence of categories, with the inverse functor being denoted by $X \mapsto {}^*X$.
We say that the monoidal category $\mathcal{C}$ is a weakly fusion category over an algebraically closed field k if $\mathcal{C}$ is a finite semisimple k-linear abelian r-category such that the unit $\mathbb{1}$ is a simple object.

Remark 2.2. Using both (i) and (ii), it follows that we have functorial identifications

$$\mathrm{Hom}(\mathbb{1}, X \otimes Y) = \mathrm{Hom}(X^*, Y) = \mathrm{Hom}({}^*Y, X)$$

for any pair of objects X, Y in a monoidal r-category $\mathcal{C}$.

Note that this notion of r-category is dual to the one considered in [20], namely it corresponds to the weak duality for the "second tensor product" constructed in *op. cit.* §*3.1.*

Let $\mathcal{C}$ be any monoidal r-category. By definition, for each object $X \in \mathcal{C}$, we have $\mathrm{Hom}(\mathbb{1}, X \otimes X^*) = \mathrm{Hom}(X^*, X^*)$. In particular we have a canonical coevaluation morphism $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$, which we denote pictorially by

$$\mathrm{coev}_X = \overset{\mathbb{1}}{\underset{X \quad X^*}{\overset{X}{\frown}}} = \overset{X}{\underset{X \quad\quad X^*}{\frown}} \,,$$

often dropping the unit $\mathbb{1}$ from the diagram. Using the identification $\mathrm{Hom}(\mathbb{1}, X \otimes Y) = \mathrm{Hom}(X^*, Y) = \mathrm{Hom}(^*Y, X)$, any morphism $f : \mathbb{1} \to X \otimes Y$ corresponds to a unique morphism $\widetilde{f} : X^* \to Y$ and $^*\widetilde{f} : {}^*Y \to X$ such that we have the equality of morphisms

$$\boxed{f} = \left(\overset{\mathbb{1}}{\underset{X \quad Y}{\frown}} \right) \begin{matrix} X^* \\ \boxed{\widetilde{f}} \end{matrix} = \begin{matrix} {}^*Y \\ \boxed{{}^*\widetilde{f}} \end{matrix} \left(\overset{\mathbb{1}}{\underset{X \quad Y}{\frown}} \right) .$$

Let us further assume that $\mathcal{C}$ is weakly fusion over an algebraically closed field k. Now by the semisimplicity of $\mathcal{C}$ and weak duality, it follows that for each simple object $X \in \mathcal{C}$, the tensor product $X \otimes X^*$ contains $\mathbb{1}$ as a direct summand with multiplicity one. Hence the map coev_X has a unique splitting $e_X : X \otimes X^* \to \mathbb{1}$, which we denote pictorially as

$$e_X = \overset{X \quad\quad X^*}{\underset{X}{\smile}} \quad \text{such that we have} \quad \overset{X}{\underset{X}{\circlearrowleft}} \, X \quad X^* = 1.$$

2.1.2. *Rigidity in monoidal r-categories.* Let us now consider the notion of rigid duals. Recall that an object X in a monoidal category $\mathcal{C}$ is said to have a left (rigid) dual X^* if and only if the functor $X^* \otimes (\cdot)$ is left adjoint to the functor $X \otimes (\cdot)$. This is equivalent to the existence of two morphisms $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$ and $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ denoted

pictorially as $\mathrm{coev}_X =$ and $\mathrm{ev}_X =$ such that we have

(2.1) and .

We will now see that in a monoidal r-category, the two equations in (2.1) above are in fact equivalent. In other words, to check rigidity of an object in an r-category, it is sufficient to only check one of the above conditions.

Lemma 2.3. *Let $\mathcal{C}$ be a monoidal r-category. Let $X \in \mathcal{C}$ and let $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$ be the canonical coevaluation map. Let $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ be any morphism. Then the two equations in (2.1) are equivalent to each other.*

Proof. Let us assume that the first equality holds and deduce the second equality. We need to verify the equality of two morphisms in $\mathrm{Hom}(X^*, X^*)$. Since $\mathcal{C}$ is an r-category, this is equivalent to verifying the equality of the corresponding morphisms in $\mathrm{Hom}(\mathbb{1}, X \otimes X^*)$. The desired equality follows, since using the first equality from (2.1) we obtain:

 .

The equivalence in the other direction follows from a similar argument. $\square$

Now let us get back to our setting of a weakly fusion category $\mathcal{C}$ over an algebraically closed field k. For each simple object X we have the coevaluation map $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$, but we do not know whether the desired evaluation map necessarily exists. But instead we have the previously defined map $e_{*X} : {}^*X \otimes X \to \mathbb{1}$ which splits off the corresponding coevaluation map $\mathrm{coev}_{*X} : \mathbb{1} \to {}^*X \otimes X$. Hence for each simple object $X \in \mathcal{C}$ we can construct the following two morphisms in $\mathcal{C}$:

(2.2) and .

Lemma 2.4. *Let X be a simple object in a weakly fusion category $\mathcal{C}$. Then the following statements are equivalent:*
(i) The object X has a rigid left dual.
(ii) The first morphism in (2.2) is invertible (or equivalently, nonzero).
(iii) The second morphism in (2.2) is invertible (or equivalently, nonzero).

Proof. Let us first prove (ii)$\Rightarrow$(i). Suppose that the first morphism in (2.2) is an isomorphism with inverse $\delta : X^* \to {}^*X$. Define ev_X to be the composition $\mathrm{ev}_X : X^* \otimes X \xrightarrow{\delta \otimes \mathrm{id}_X} {}^*X \otimes X \xrightarrow{e_{*X}} \mathbb{1}$. Hence by construction, the second equation in (2.1) is satisfied. By Lemma 2.3, the first condition must also hold and we conclude that X^* must in fact be the rigid left dual of X.

To prove (i)$\Rightarrow$(ii), suppose that X^* is a rigid left dual of X. Hence we have an evaluation map $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ satisfying (2.1). In particular by semisimplicity, $\mathbb{1}$ must be a direct summand of $X^* \otimes X$. Hence we must have ${}^*X \cong X^*$. Choosing such an isomorphism we obtain a non-zero morphism ${}^*X \otimes X \to X^* \otimes X \xrightarrow{\mathrm{ev}_X} \mathbb{1}$ which differs from $e_{*X} : {}^*X \otimes X \to \mathbb{1}$ by an element of $k^\times$. Statement (ii) now follows.

The proof of (i)$\Leftrightarrow$(iii) is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.1.3. *A criterion for rigidity.* We prove the following useful criterion that guarantees the existence of rigid duals in a weakly fusion category.

Proposition 2.5. *Let $\mathcal{C}$ be a weakly fusion category. Let $M \in \mathcal{C}$ be a simple object such that $M \otimes {}^*M$ has a rigid left dual. Then M has a rigid left dual.*

Proof. Consider the morphism $e_{*M} \otimes \mathrm{id}_{*M} : {}^*M \otimes M \otimes {}^*M \to {}^*M$. It is non-zero, since the morphism $\mathrm{coev}_{*M} \otimes \mathrm{id}_{*M} : {}^*M \to {}^*M \otimes M \otimes {}^*M$ is its right inverse. Let us rewrite this non-zero morphism diagrammatically as below:

In the second equality above, we have used the existence of the rigid left dual of the object $M \otimes {}^*M \in \mathcal{C}$. Let $f : \mathbb{1} \to M \otimes ({}^*M \otimes (M \otimes {}^*M)^*)$ be the morphism represented by the red dotted box above. Let $\widetilde{f} : M^* \to {}^*M \otimes (M \otimes {}^*M)^*$ be the morphism corresponding to

f using weak duality. Hence the non-zero morphism $e_{*M} \otimes \mathrm{id}_{*M}$ can be expressed as:

By looking at the blue dotted box above, we conclude that the morphism

must be non-zero and hence an isomorphism. From Lemma 2.4 we conclude that M has a rigid left dual.   □

Finally we deduce the following corollary of the above result:

Corollary 2.6. *Let Γ be a finite group and let $\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$ be a Γ-graded weakly fusion category such that the identity component $\mathcal{C}_1$ is rigid. Then $\mathcal{C}$ is rigid and hence a fusion category.*

Proof. This follows directly from the proposition above. Let M be a simple object in $\mathcal{C}$, lying in say $\mathcal{C}_\gamma$. Then $^*M \in \mathcal{C}_{\gamma^{-1}}$ and $M \otimes {}^*M$ lies in $\mathcal{C}_1$ which we have assumed is a fusion category. Hence $M \otimes {}^*M$ has a rigid left dual. From the proposition it follows that M has a rigid left dual. Hence $\mathcal{C}$ is a fusion category.   □

2.2. **Braided crossed categories.** Let Γ be a finite group and let $\mathcal{C}$ be a braided Γ-crossed fusion category with unit object $\mathbb{1}$ as defined in [28, 77]. In particular, we have a Γ-grading $\mathcal{C} = \bigoplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$ and a monoidal Γ-action on $\mathcal{C}$ such that the action of an element $g \in \Gamma$ maps the component $\mathcal{C}_\gamma$ to $\mathcal{C}_{g\gamma g^{-1}}$. Moreover, we have functorial crossed braiding isomorphisms

$$\beta_{M,N} : M \otimes N \xrightarrow{\cong} \gamma(N) \otimes M \text{ for } \gamma \in \Gamma, M \in \mathcal{C}_\gamma, N \in \mathcal{C}$$

satisfying certain compatibility relations.

Remark 2.7. By Corollary 2.6, rigidity of such a $\mathcal{C}$ will follow if $\mathcal{C}_1$ is rigid and $\mathcal{C}$ is a monoidal r-category.

Let us denote by $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$ the $\mathbb{Q}^{\mathrm{ab}}$-algebra obtained from the Grothendieck ring of $\mathcal{C}$ by extension of scalars to $\mathbb{Q}^{\mathrm{ab}}$, the maximal abelian extension of $\mathbb{Q}$. Then $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma} K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_\gamma)$ is a Γ-crossed Frobenius $\mathbb{Q}^{\mathrm{ab}}$-$*$-algebra as described in [25]. We briefly recall what this means. Firstly we have the non-degeneratete symmetric linear functional $\nu : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}) \to \mathbb{Q}^{\mathrm{ab}}$ which records the coefficient of the class of the unit $[\mathbb{1}]$ in any element of $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$. This allows us to identify $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_\gamma)^* \cong K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$ and hence to identify $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})^* \cong K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$ as $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$-bimodules. Now we have the canonically defined 'complex conjugation' involution of $\mathbb{Q}^{\mathrm{ab}}$. The rigid duality on $\mathcal{C}$ can be used to define a $\mathbb{Q}^{\mathrm{ab}}$-semilinear anti-involution $* : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}) \to K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$. This allows us to define a positive definite Hermitian form $\langle a, b \rangle := \nu(ab^*)$ on $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$. By the results of [30], it follows that $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$ is a split semisimple $\mathbb{Q}^{\mathrm{ab}}$-algebra, in particular all its irreducible complex representations are already defined over $\mathbb{Q}^{\mathrm{ab}}$.

It follows from the braiding isomorphisms that the identity component $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ is central in $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$. For $\gamma \in \Gamma$, let P_γ denote the finite set of (representatives of isomorphism classes of) simple objects of the semisimple category $\mathcal{C}_\gamma$. We will also think of P_γ as a $\mathbb{Q}^{\mathrm{ab}}$-basis of $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_\gamma)$. It is an orthonormal basis with respect to the Hermitian form defined above. Our goal in this section is to describe the fusion coefficients of $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C})$. For $\gamma_1, \gamma_2 \in \Gamma, A \in \mathcal{C}_{\gamma_1}, B \in \mathcal{C}_{\gamma_2}$ and $C \in \mathcal{C}_{\gamma_1 \gamma_2}$, the fusion coefficient $\nu_{A,B}^C \in \mathbb{Z}_{\geq 0}$ is defined to be the multiplicity of C in the product $A \otimes B$, so that we have the equality $[A] \cdot [B] = \sum_{C \in P_{\gamma_1 \gamma_2}} \nu_{A,B}^C \cdot [C]$ in the Grothendieck ring. Equivalently, the fusion coefficient can be thought of as the multiplicity of the unit $\mathbb{1}$ in $A \otimes B \otimes C^*$, where $C^* \in \mathcal{C}_{\gamma_2^{-1} \gamma_1^{-1}}$ is the rigid dual of C, i.e. $\nu_{A,B}^C = \nu([A] \cdot [B] \cdot [C^*]) = \dim \mathrm{Hom}(\mathbb{1}, A \otimes B \otimes C^*)$.

2.2.1. *Twisted characters.* We will always assume that the grading on $\mathcal{C}$ is faithful, i.e. $\mathcal{C}_\gamma$ is non-zero for all $\gamma \in \Gamma$. We will also assume that the identity component $\mathcal{C}_1$ is non-degeneratete as a braided fusion category. Consider an element $\gamma \in \Gamma$. Under our assumptions, the component $\mathcal{C}_\gamma$ becomes an invertible $\mathcal{C}_1$-bimodule category corresponding (under the equivalence [29, Thm. 5.2]) to the braided autoequivalence $\gamma : \mathcal{C}_1 \xrightarrow{\cong} \mathcal{C}_1$ induced by the Γ-action on $\mathcal{C}$. Hence we are in the setting studied in [23, 24, 25]. Note that we also have the braided $\langle \gamma \rangle$-crossed category $\mathcal{C}_{\langle \gamma \rangle} := \bigoplus_{g \in \langle \gamma \rangle} \mathcal{C}_g \subseteq \mathcal{C}$, where $\langle \gamma \rangle \leq \Gamma$ denotes the cyclic subgroup generated by Γ. Let us recall the notion of twisted characters that can be defined in this setting.

We know from [30] that the commutative $\mathbb{Q}^{\mathrm{ab}}$-algebra $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ is semisimple and split. Let $\mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$ be the set of all irreducible representations $\rho : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}) \to \mathbb{Q}^{\mathrm{ab}}$. The Γ-action on $\mathcal{C}$ induces a Γ-action on the algebra $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ and hence on the set $\mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$. If $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma$, then we can extend ρ to a 1-dimensional character $\widetilde{\rho} : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\langle \gamma \rangle}) \to \mathbb{Q}^{\mathrm{ab}}$ (see [24]). The γ-twisted character $\rho^\gamma : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_\gamma) \to \mathbb{Q}^{\mathrm{ab}}$ is then defined to be the restriction of $\widetilde{\rho}$ to $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_\gamma)$. Using the identification $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_\gamma)^* \cong K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$, let $\alpha_\rho^\gamma \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$ be the element corresponding to ρ^γ, namely $\alpha_\rho^\gamma := \sum_{A \in P_\gamma} \widetilde{\rho}([A]) \cdot [A^*] \in$

$K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$. Note that ρ^γ, and hence α_ρ^γ are well-defined only up to scaling by m-th roots of unity, where m is the order of γ in Γ. We will call either of ρ^γ or α_ρ^γ as the γ-twisted character associated with $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma$. For $\gamma = 1$, we have the well-defined element $\alpha_\rho := \alpha_\rho^1 \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ for each $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$. The following results about these characters and γ-twisted characters are proved in [66, 24]:

Theorem 2.8. *(i) Let $\rho, \rho' \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$. Then $\rho'(\alpha_\rho) = 0$ if $\rho' \neq \rho$ and $f_\rho := \rho(\alpha_\rho) \in \mathbb{Q}^{\mathrm{ab}}$ is a totally positive cyclotomic integer known as the formal codegree of ρ. In other words, $e_\rho := \frac{\alpha_\rho}{f_\rho} \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ is the minimal idempotent corresponding to $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$.*
(ii) For $\rho, \rho' \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$, we have orthogonality of characters, $\langle \alpha_\rho, \alpha_{\rho'} \rangle = \delta_{\rho\rho'} \cdot f_\rho$.
(iii) Let $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma$ and $\alpha_\rho^\gamma \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$ the corresponding γ-twisted character. Then for any $\rho' \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$, we have

$$\alpha_{\rho'} \cdot \alpha_\rho^\gamma = \alpha_\rho^\gamma \cdot \alpha_{\rho'} = \begin{cases} 0 & \text{if } \rho' \neq \rho, \\ f_\rho \cdot \alpha_\rho^\gamma & \text{if } \rho' = \rho. \end{cases}$$

We have a direct sum decomposition

$$K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}}) = \bigoplus_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma} \mathbb{Q}^{\mathrm{ab}} \cdot \alpha_\rho^\gamma \text{ as a } K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)\text{-module.}$$

(iv) For $\rho, \rho' \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma$, we have orthogonality of twisted characters, $\langle \alpha_\rho^\gamma, \alpha_{\rho'}^\gamma \rangle = \delta_{\rho\rho'} \cdot f_\rho$. In other words the γ-twisted characters $\{\alpha_\rho^\gamma | \rho \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)^\gamma\}$ form an orthogonal basis of $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$ made up of eigenvectors for the action of $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$.
(v) For $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma$, let $\widetilde{\rho} : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\langle\gamma\rangle}) \to \mathbb{Q}^{\mathrm{ab}}$ be an extension. Let $n = |\langle\gamma\rangle|$. Then the formal codegree $f_{\widetilde{\rho}}$ equals $n \cdot f_\rho$. Define $e_\rho^\gamma := \frac{\alpha_\rho^\gamma}{f_\rho}$. Then $(e_\rho^\gamma)^n = e_\rho$.
(vi) For each $\gamma \in \Gamma$, we have $|P_1^\gamma| = |P_\gamma| = |\mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^\gamma|$.

This result describes the structure of $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}})$ as a $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$-module. Now suppose we have $\gamma_1, \gamma_2 \in \Gamma$. Then for $\rho_i \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\gamma_i}$, we now describe the product $\alpha_{\rho_2}^{\gamma_2} \cdot \alpha_{\rho_1}^{\gamma_1} \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma_2^{-1}\gamma_1^{-1}})$ of the twisted characters. We first consider the case where the ρ_1, ρ_2 above are distinct. In this case the product of the primitive idempotents $e_{\rho_2} \cdot e_{\rho_1}$ equals zero. Hence the product $\alpha_{\rho_2}^{\gamma_2} \cdot \alpha_{\rho_1}^{\gamma_1} = \alpha_{\rho_2}^{\gamma_2} \cdot e_{\rho_2} \cdot e_{\rho_1} \cdot \alpha_{\rho_1}^{\gamma_1}$ also equals zero in the case $\rho_1 \neq \rho_2$.

For $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$, let $\Gamma_\rho \leq \Gamma$ be the stabilizer of ρ under the action of Γ and $e_\rho \in K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ the associated primitive idempotent. For each $\gamma \in \Gamma_\rho$, we have the γ-twisted character α_ρ^γ and the element e_ρ^γ (determined up to scaling by roots of unity). It follows from Theorem 2.8 that $e_\rho K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}) = \bigoplus_{\gamma \in \Gamma_\rho} e_\rho K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\gamma^{-1}}) = \bigoplus_{\gamma \in \Gamma_\rho} \mathbb{Q}^{\mathrm{ab}} \cdot e_\rho^\gamma$. From this Γ_ρ-graded algebra, we obtain a central extension $0 \to \mathbb{Q}^{\mathrm{ab}\times} \to \widetilde{\Gamma}_\rho \to \Gamma_\rho \to 0$. Our choice of the elements e_ρ^γ determines a 2-cocycle $\varphi_\rho : \Gamma_\rho \times \Gamma_\rho \to \mathbb{Q}^{\mathrm{ab}\times}$. Hence we obtain

Corollary 2.9. *(i) If $\gamma_1, \gamma_2 \in \Gamma$, $\rho_1 \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\gamma_1}$, $\rho_2 \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\gamma_2}$ and $\rho_1 \neq \rho_2$, then the product of the corresponding twisted characters $\alpha_{\rho_2}^{\gamma_2} \cdot \alpha_{\rho_1}^{\gamma_1}$ equals 0.*
(ii) Let $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$ and let $\gamma_1, \gamma_2 \in \Gamma_\rho$. Then $\alpha_\rho^{\gamma_2} \cdot \alpha_\rho^{\gamma_1} = \varphi_\rho(\gamma_1, \gamma_2) \cdot f_\rho \cdot \alpha_\rho^{\gamma_1\gamma_2}$ and

hence $e_\rho^{\gamma_2} \cdot e_\rho^{\gamma_1} = \varphi_\rho(\gamma_1, \gamma_2) \cdot e_\rho^{\gamma_1 \gamma_2}$, where φ_ρ is a 2-cocycle corresponding to the central extension $0 \to \mathbb{Q}^{ab \times} \to \widetilde{\Gamma}_\rho \to \Gamma_\rho \to 0$. Equivalently, for $r_i \in K_{\mathbb{Q}^{ab}}(\mathcal{C}_{\gamma_i})$ we have the relation $\rho^{\gamma_1}(r_1)\rho^{\gamma_2}(r_2) = \varphi_\rho(\gamma_1, \gamma_2)\rho^{\gamma_1 \gamma_2}(r_1 r_2)$.

Remark 2.10. Let $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{ab}}(\mathcal{C}_1))$. For any $\gamma_1, \cdots, \gamma_n$ in the stabilizer Γ_ρ of ρ, let $\varphi_\rho(\gamma_1, \cdots, \gamma_n)$ be such that $\rho^{\gamma_1}(r_1) \cdots \rho^{\gamma_n}(r_n) = \varphi_\rho(\gamma_1, \cdots, \gamma_n)\rho^{\gamma_1 \cdots \gamma_n}(r_1 \cdots r_n)$. These scalars will appear in our twisted categorical Verlinde formula. Furthermore, for each pair (ρ, γ) such that $\gamma \in \Gamma_\rho$ we have chosen the elements $e_\rho^\gamma \in K_{\mathbb{Q}^{ab}}(\mathcal{C}_{\gamma^{-1}})$ to be such that $(e_\rho^\gamma)^m = 1$ where m is the order of γ. It will be notationally convenient to further assume that we always have $e_\rho^\gamma \cdot e_\rho^{\gamma^{-1}} = 1$. It is clear that such a choice can always be made. This ensures the equality $\varphi_\rho(\gamma, \gamma^{-1}) = 1$.

2.2.2. *Crossed S-matrices.* Let us now suppose that the braided Γ-crossed fusion category $\mathcal{C}$ is also equipped with a ribbon structure. In particular, $\mathcal{C}_1$ is a modular fusion category and each $\mathcal{C}_\gamma$ is equipped with a $\mathcal{C}_1$-module trace. Hence the trace of endomorphisms in $\mathcal{C}$, and hence categorical dimensions of objects in $\mathcal{C}$ are now well-defined. In this setting we can define a $P_\gamma \times P_1^\gamma$-matrix $\widetilde{S}^\gamma$ known as the unnormalized γ-crossed S-matrix as follows. For each $C \in P_1^\gamma$, let us choose an isomorphism $\psi_C : \gamma(C) \xrightarrow{\cong} C$ as in [23, §2.1]. For $M \in P_\gamma, C \in P_1^\gamma$, we set

$$\widetilde{S}^\gamma_{M,C} = \mathrm{tr}(C \otimes M \xrightarrow{\beta_{C,M}} M \otimes C \xrightarrow{\beta_{M,C}} \gamma(C) \otimes M \xrightarrow{\psi_C \otimes \mathrm{id}_M} C \otimes M).$$

Remark 2.11. The γ-crossed S-matrix as defined above is the transpose of the crossed S-matrix defined in [23]. Also in *op. cit.* the first author only deals with unnormalized crossed S-matrices as defined above, whereas in the current paper we will mostly work with the normalized unitary crossed S-matrix as defined in Definition 2.13 below. This explains the differences of notation in the categorical Verlinde formula of the two papers.

In the presence of the ribbon structure, we have the unnormalized S-matrix $\widetilde{S}$ of the modular fusion category $\mathcal{C}_1$. Using this S-matrix, we can identify $P_1 \cong \mathrm{Irr}(K_{\mathbb{Q}^{ab}}(\mathcal{C}_1))$ as follows: $P_1 \ni C \mapsto (\rho_C : [D] \mapsto \frac{\widetilde{S}_{D,C}}{\dim C})$, i.e. the S-matrix is essentially the character table of $K_{\mathbb{Q}^{ab}}(\mathcal{C}_1)$. Similarly, the γ-crossed S-matrix is essentially the table of γ-twisted characters (see [23, 24]):

Theorem 2.12. *(i) For $M \in P_\gamma, C \in P_1^\gamma$ the numbers $\frac{\widetilde{S}^\gamma_{M,C}}{\dim M} = \frac{\widetilde{S}^\gamma_{M,C}}{\widetilde{S}^\gamma_{M,\mathbb{1}}}$ and $\frac{\widetilde{S}^\gamma_{M,C}}{\dim C} = \frac{\widetilde{S}^\gamma_{M,C}}{\widetilde{S}^\gamma_{\mathbb{1},C}}$ are cyclotomic integers. For $C \in P_1^\gamma$ the linear functional $\rho_C^\gamma : K_{\mathbb{Q}^{ab}}(\mathcal{C}_\gamma) \to \mathbb{Q}^{ab}, [M] \mapsto \frac{\widetilde{S}^\gamma_{M,C}}{\dim C}$ is the γ-twisted character associated with $\rho_C \in \mathrm{Irr}(K_{\mathbb{Q}^{ab}}(\mathcal{C}_1))^\gamma$.*
(ii) The categorical dimension $\dim \mathcal{C}_1$ is a totally positive cyclotomic integer. We have

$$\widetilde{S}^\gamma \cdot \overline{\widetilde{S}^\gamma}^T = \overline{\widetilde{S}^\gamma}^T \cdot \widetilde{S}^\gamma = \dim \mathcal{C}_1 \cdot I.$$

For $C \in P_1$, the formal codegree $f_C = f_{\rho_C}$ of the corresponding character ρ_C equals the totally positive cyclotomic integer $\frac{\dim \mathcal{C}_1}{\dim^2 C}$.

Definition 2.13. We choose a square root $\sqrt{\dim \mathcal{C}_1}$ and define the normalized γ-crossed S-matrix to be $S^\gamma := \frac{1}{\sqrt{\dim \mathcal{C}_1}} \cdot \widetilde{S}^\gamma$. The matrix S^γ is a $P_\gamma \times P_1^\gamma$ unitary matrix with entries in $\mathbb{Q}^{\mathrm{ab}}$.

2.2.3. *A twisted Verlinde formula.* We will now state and prove a Verlinde formula for braided Γ-crossed categories. For $\gamma_1, \gamma_2 \in \Gamma$, $A \in P_{\gamma_1}, B \in P_{\gamma_2}, C \in P_{\gamma_1 \gamma_2}$ we will compute the fusion coefficient $\nu_{A,B}^C$ in terms of the crossed S-matrices, or equivalently, in terms of the twisted character tables. By the rigid duality, it is clear that $\nu_{A,B}^C = \nu_{B^*,A^*}^{C^*} = \dim \mathrm{Hom}(\mathbb{1}, A \otimes B \otimes C^*) = \nu([A] \cdot [B] \cdot [C^*])$.

More generally, let $\gamma_1, \cdots, \gamma_n \in \Gamma$ with $\gamma_1 \cdots \gamma_n = 1$. We will in fact prove an n-pointed twisted Verlinde formula to compute $\dim \mathrm{Hom}(\mathbb{1}, A_1 \otimes \cdots \otimes A_n) = \nu([A_1] \cdots [A_n])$ for $A_i \in P_{\gamma_i}$. We will use the following:

Lemma 2.14. *Let $\gamma_1, \cdots, \gamma_n \in \Gamma$ and let $A_i \in \mathcal{C}_{\gamma_i}$, so that $A_1 \otimes \cdots \otimes A_n \in \mathcal{C}_{\gamma_1 \cdots \gamma_n}$. Let $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\gamma_1 \cdots \gamma_n}$ be such that (the $\gamma_1 \cdots \gamma_n$-twisted character) $\rho^{\gamma_1 \cdots \gamma_n}([A_1] \cdots [A_n]) \neq 0$. Then $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle \gamma_1, \cdots, \gamma_n \rangle}$.*

Proof. Consider any $A \in \mathcal{C}_1$.. Then by the crossed braiding isomorphisms, for each i we have $A \otimes (A_1 \otimes \cdots \otimes A_i) \cong (A_1 \otimes \cdots \otimes A_i) \otimes A \cong (\gamma_1 \cdots \gamma_i)(A) \otimes (A_1 \otimes \cdots \otimes A_i)$. Hence for each i

$$A \otimes A_1 \otimes \cdots \otimes A_n \cong (\gamma_1 \cdots \gamma_i)(A) \otimes A_1 \otimes \cdots \otimes A_n \text{ as objects of } \mathcal{C}_{\gamma_1 \cdots \gamma_n}.$$

Hence $\rho^{\gamma_1 \cdots \gamma_n}([A] \cdot [A_1] \cdots [A_n]) = \rho^{\gamma_1 \cdots \gamma_n}((\gamma_1 \cdots \gamma_n)[A] \cdot [A_1] \cdots [A_n])$. Hence

$$\rho([A]) \cdot \rho^{\gamma_1 \cdots \gamma_n}([A_1] \cdots [A_n]) = \rho(\gamma_1 \cdots \gamma_i[A]) \cdot \rho^{\gamma_1 \cdots \gamma_n}([A_1] \cdots [A_n]).$$

Since we have assumed that $\rho^{\gamma_1 \cdots \gamma_n}([A_1] \cdots [A_n]) \neq 0$, we conclude that

$$\rho([A]) = \rho(\gamma_1 \cdots \gamma_i[A])$$

for any $A \in \mathcal{C}_1$ and $1 \leq i \leq n$. Hence $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\gamma_1 \cdots \gamma_i}$ for each i. Hence we conclude that ρ is fixed by each γ_i as desired. $\qquad\square$

Theorem 2.15. *(Categorical twisted Verlinde formula.) Let $\gamma_1, \cdots, \gamma_n \in \Gamma$ with the condition $\gamma_1 \cdots \gamma_n = 1$. Let $A_i \in \mathcal{C}_{\gamma_i}$ for $1 \leq i \leq n$. Then*
(i)

$$\dim \mathrm{Hom}(\mathbb{1}, A_1 \otimes \cdots \otimes A_n) = \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle \gamma_1, \cdots, \gamma_n \rangle}} \frac{\rho^{\gamma_1}([A_1]) \cdots \rho^{\gamma_n}([A_n])}{f_\rho \cdot \varphi_\rho(\gamma_1, \cdots, \gamma_n)},$$

where f_ρ is the formal codegree and $\varphi_\rho(\gamma_1, \cdots, \gamma_n)$ are the scalars defined in Remark 2.10.
(ii) Now let $A_i \in P_{\gamma_i}$ be simple objects. If $\mathcal{C}$ is equipped with a ribbon structure, then in terms of the crossed S-matrices we have

$$\dim \mathrm{Hom}(\mathbb{1}, A_1 \otimes \cdots \otimes A_n) = \sum_{D \in P_1^{\langle \gamma_1, \cdots, \gamma_n \rangle}} \frac{S_{A_1,D}^{\gamma_1} \cdots S_{A_n,D}^{\gamma_n}}{S_{\mathbb{1},D}^{(n-2)} \cdot \varphi_D(\gamma_1, \cdots, \gamma_n)}.$$

Proof. To prove (i), observe that in the Frobenius $*$-algebra $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)$ the unit

$$1 = \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))} \frac{\alpha_\rho}{f_\rho}$$

is expressed as a sum of the minimal idempotents. Using the fact $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1) \cong K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)^*$, this translates to the equality $\nu = \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))} \frac{\rho}{f_\rho}$ in $K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1)^*$. Hence

$$
\begin{aligned}
\dim \mathrm{Hom}(\mathbb{1}, A_1 \otimes \cdots \otimes A_n) &= \nu([A_1] \cdots [A_n]) \\
&= \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))} \frac{\rho([A_1] \cdots [A_n])}{f_\rho} \\
&= \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle \gamma_1, \cdots, \gamma_n \rangle}} \frac{\rho([A_1] \cdots [A_n])}{f_\rho} \quad \text{(by Lemma 2.14)} \\
&= \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle \gamma_1, \cdots, \gamma_n \rangle}} \frac{\rho^{\gamma_1}([A_1]) \cdots \rho^{\gamma_n}([A_n])}{f_\rho \cdot \varphi_\rho(\gamma_1, \cdots, \gamma_n)}
\end{aligned}
$$

The last line follows from Remark 2.10. Now to prove (ii), observe that in the ribbon setting we have a Γ-equivariant bijection $P_1 \cong \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))$ denoted by $D \leftrightarrow \rho_D$. By Theorem 2.12 and Definition 2.13 for each $D \in P_1^{\langle \gamma_1, \cdots, \gamma_n \rangle}$ we have $\rho_D^{\gamma_i}([A_i]) = \frac{S_{A_i, D}^{\gamma_i}}{S_{\mathbb{1}, D}}$ and the formal codegree $f_D = f_{\rho_D} = \frac{\dim \mathcal{C}_1}{\dim^2 D} = \frac{1}{(S_{\mathbb{1}, D})^2}$. Statement (ii) now follows from (i). $\square$

2.2.4. *A higher genus twisted Verlinde formula.* In order to motivate the higher genus Verlinde formula for twisted conformal blocks, let us derive a categorical version of such a formula. Let $a, b \in \Gamma$. For a simple object $A \in P_a$, $b(A^*)$ is a simple object of $\mathcal{C}_{ba^{-1}b^{-1}}$. We define the object

$$(2.3) \qquad \Omega_{a,b} := \bigoplus_{A \in P_a} A \otimes b(A^*) \in \mathcal{C}_{[a,b]}, \text{ where } [a, b] = aba^{-1}b^{-1}.$$

Lemma 2.16. *Let $a, b \in \Gamma$ and let $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{[a,b]}$ be a character fixed by the commutator $[a, b]$. Then*

$$\rho^{[a,b]}([\Omega_{a,b}]) = \begin{cases} \dfrac{f_\rho}{\varphi_\rho(a, b, a^{-1}, b^{-1})} & \text{if } \rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle a,b \rangle} \text{ (see Remark 2.10)} \\ 0 & \text{else.} \end{cases}$$

Proof. We have $\rho^{[a,b]}([\Omega_{a,b}]) = \rho^{[a,b]} \left(\sum_{A \in P_a} [A] \cdot b[A^*] \right) = \sum_{A \in P_a} \rho^{[a,b]} ([A] \cdot b[A^*])$. By Lemma 2.14, if $\rho^{[a,b]} ([A] \cdot b[A^*]) \neq 0$ for some $A \in P_a$, then we must have $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle a, ba^{-1}b^{-1} \rangle}$.

Otherwise each individual term in the summation and hence $\rho^{[a,b]}([\Omega_{a,b}])$ must be zero. Hence let us assume that $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle a, ba^{-1}b^{-1}\rangle}$. In this case we get

$$(2.4) \quad \rho^{[a,b]}([\Omega_{a,b}]) = \sum_{A \in P_a} \frac{\rho^a([A]) \cdot \rho^{ba^{-1}b^{-1}}(b[A^*])}{\varphi_\rho(a, ba^{-1}b^{-1})} = \sum_{A \in P_a} \frac{\rho^a([A]) \cdot (\rho^{ba^{-1}b^{-1}} \circ b)([A^*])}{\varphi_\rho(a, ba^{-1}b^{-1})}.$$

Recall that here $\rho^{ba^{-1}b^{-1}} : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{ba^{-1}b^{-1}}) \to \mathbb{Q}^{\mathrm{ab}}$ is the restriction of some choice of character

$$\widetilde{\rho^{ba^{-1}b^{-1}}} : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\langle ba^{-1}b^{-1}\rangle}) \to \mathbb{Q}^{\mathrm{ab}} \text{ extending } \rho : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1) \to \mathbb{Q}^{\mathrm{ab}}.$$

Hence $\widetilde{\rho^{ba^{-1}b^{-1}}} \circ b : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{\langle a^{-1}\rangle}) \to \mathbb{Q}^{\mathrm{ab}}$ is a character extending $\rho \circ b : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1) \to \mathbb{Q}^{\mathrm{ab}}$. Hence the composition $\rho^{ba^{-1}b^{-1}} \circ b : K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_{a^{-1}}) \to \mathbb{Q}^{\mathrm{ab}}$ differs from the chosen twisted character $(\rho \circ b)^{a^{-1}}$ by some $|\langle a \rangle|$-th root of unity. Hence by the orthogonality of twisted characters (Theorem 2.8(iv)) and (2.4), we get that $\rho^{[a,b]}([\Omega_{a,b}]) = 0$ if $\rho \neq \rho \circ b$. In other words, we have proved that if $\rho^{[a,b]}([\Omega_{a,b}]) \neq 0$, then we must have $\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\langle a,b\rangle}$.

Hence now suppose that $a, b \in \Gamma_\rho$. Again by the twisted orthogonality relations we have $\sum\limits_{A \in P_a} \rho^a([A]) \cdot \overline{\rho^a([A])} = \sum\limits_{A \in P_a} \rho^a([A]) \cdot \rho^{a^{-1}}([A^*]) = f_\rho$. Also using Remark 2.10, we have $\rho^{ba^{-1}b^{-1}} \circ b = \frac{\rho^{a^{-1}}}{\varphi_\rho(b, a^{-1}, b)}$. Combining with (2.4) we complete the proof of the lemma. $\square$

Finally using Theorem 2.15 and Lemma 2.16 we obtain the following higher genus categorical Verlinde formula:

Corollary 2.17. *Let g, n be non-negative integers and let $a_1, \cdots, a_g, b_1, \cdots, b_g, m_1, \cdots, m_n$ be elements in Γ that satisfy the relation $[a_1, b_1] \cdots [a_g, b_g] \cdot m_1 \cdots m_n = 1$. Let $\Gamma^\circ \leq \Gamma$ be the subgroup generated by a_j, b_j, m_i. Let $M_i \in \mathcal{C}_{m_i}$. Then*

$$\dim \mathrm{Hom}(\mathbb{1}, \Omega_{a_1, b_1} \otimes \cdots \otimes \Omega_{a_g, b_g} \otimes M_1 \otimes \cdots \otimes M_n)$$

$$= \sum_{\rho \in \mathrm{Irr}(K_{\mathbb{Q}^{\mathrm{ab}}}(\mathcal{C}_1))^{\Gamma^\circ}} \frac{(f_\rho)^{g-1} \cdot \rho^{m_1}([M_1]) \cdots \rho^{m_n}([M_n])}{\varphi_\rho(a_1, b_1, a_1^{-1}, b_1^{-1}, \cdots, m_1, \cdots, m_n)}.$$

If $\mathcal{C}$ is equipped with a ribbon structure and if $M_i \in P_{m_i}$ are simple then

$$\dim \mathrm{Hom}(\mathbb{1}, \Omega_{a_1, b_1} \otimes \cdots \otimes \Omega_{a_g, b_g} \otimes M_1 \otimes \cdots \otimes M_n)$$

$$= \sum_{D \in P_1^{\Gamma^\circ}} \frac{\left(\frac{1}{S_{\mathbb{1}, D}}\right)^{n+2g-2} \cdot S_{M_1, D}^{m_1} \cdots S_{M_n, D}^{m_n}}{\varphi_D(a_1, b_1, a_1^{-1}, b_1^{-1}, \cdots, m_1, \cdots, m_n)}.$$

Remark 2.18. The fundamental group of a smooth complex genus g curve with n punctures has a presentation $\langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g, \gamma_1, \cdots, \gamma_n | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] \gamma_1 \cdots \gamma_n = 1 \rangle$. Hence the choice of the elements $a_j, b_j, m_i \in \Gamma$ in the above corollary is equivalent to the choice of a group homomorphism from the fundamental group to Γ.

3. Twisted Kac-Moody Lie algebras and their representations

Our goal is to apply the Verlinde formula for braided crossed categories to conformal blocks for twisted affine Lie algebras. We first recall following Kac [53] some notations and facts about twisted Kac-Moody Lie algebras and their representations. Our notations and conventions will be similar to [53, 46], with key differences at some places.

3.1. **Twisted affine Lie algebras.** Let γ be an automorphism of a finite dimensional simple Lie algebra $\mathfrak{g}$ of order $|\gamma|$. We fix a $|\gamma|$-th root of unity $\epsilon := e^{\frac{2\pi\sqrt{-1}}{|\gamma|}}$ of unity and then we have an eigen-decomposition $\mathfrak{g} = \oplus_{i=0}^{|\gamma|-1}\mathfrak{g}_i$, where

$$\mathfrak{g}_i := \{X \in \mathfrak{g}|\gamma(X) = \epsilon^i X\}.$$

In particular $\mathfrak{g}_0$ (also denoted by $\mathfrak{g}^\gamma$) is the Lie subalgebra of $\mathfrak{g}$ invariant under γ. The automorphism γ induces an natural automorphism γ of the affine Lie algebra $\widehat{L}(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$. The Lie bracket given by the formula

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y)_{\mathfrak{g}} \operatorname{Res}_{t=0} gdf \cdot c, \ [c, \widehat{L}(\mathfrak{g})] = 0,$$

where $(,)_{\mathfrak{g}}$ is the normalized Killing form on $\mathfrak{g}$ such that $(\theta, \theta) = 2$ for any long root θ of $\mathfrak{g}$.

We define the *twisted affine Lie algebra* $\widehat{L}(\mathfrak{g}, \gamma) := (\widehat{L}(\mathfrak{g}))^\gamma$, where γ acts on $\mathbb{C}((t))$ by the formula $\gamma(t) = \epsilon^{-1}t$. Using the eigen-decomposition of $\mathfrak{g}$, we get

$$\widehat{L}(\mathfrak{g}, \gamma) = \oplus_{i=0}^{|\gamma|-1}\mathfrak{g}_i \otimes \mathcal{A}_i \oplus \mathbb{C}c,$$

where $\mathcal{A}_i = \{f(t) \in \mathbb{C}((t))|\gamma(f(t)) = \epsilon^{-i}f(t)\}$. The classification [53, Proposition 8.1] of finite order automorphisms of $\mathfrak{g}$ tells us:

Proposition 3.1. *Let γ be an automorphism of $\mathfrak{g}$ of order $|\gamma|$. Then there exists a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{h}$ such that*

$$\gamma = \sigma \exp(\operatorname{ad} \frac{2\pi\sqrt{-1}}{m}h),$$

where σ is a diagram automorphism of $\mathfrak{g}$, such that we have the following:

- *Both γ and σ preserve $\mathfrak{h}$.*
- *h is a element of the subalgebra of $\mathfrak{h}$ fixed by σ.*
- *σ preserves a set of simple roots $\Pi' = \{\alpha'_1, \ldots, \alpha'_{\operatorname{rank}\mathfrak{g}}\}$ and $\alpha'_i(h) \in \mathbb{Z}$ for each $1 \le i \le \operatorname{rank}\mathfrak{g}$.*
- *σ and $\exp(\operatorname{ad} \frac{2\pi\sqrt{-1}}{m}h)$ commute.*

Now we know that diagram automorphisms of finite dimensional simple Lie algebras have been classified and the order m of σ is in the set $\{1, 2, 3\}$. Now suppose γ and σ are related by Proposition 3.1, then m divides $|\gamma|$. By assumption, we get that $\epsilon' := \epsilon^{|\gamma|/m}$ is an m-th root of unity. Consider the following natural map

$$(3.1) \qquad \phi_{\sigma,\gamma} : \widehat{L}(\mathfrak{g}, \sigma) \to \widehat{L}(\mathfrak{g}, \gamma), \ X[t^j] \to X[t^{\frac{|\gamma|}{m}j+k}], \ c \to c,$$

where X is an ϵ'-eigenvector of $\mathfrak{g}$ and a k-eigenvector for $\exp(\mathrm{ad}\,\frac{2\pi\sqrt{-1}}{m}h)$. By Proposition 8.6 in [53], we get that the map in Equation (3.1) is an isomorphism of Lie algebras. Thus we are reduced to the case of studying twisted affine Lie algebras when γ is a diagram automorphism of $\mathfrak{g}$.

3.2. **Twisted affine Kac-Moody Lie algebras.** Let X_N denote a type of finite dimensional complex Lie algebra of rank N (as in the classification in [53]) and the associated Lie algebra is $\mathfrak{g}(X_N)$. Similarly, let $X_N^{(m)}$ be the affine Kac-Moody Lie algebra associated to a diagram automorphism γ of $\mathfrak{g}(X_N)$ of order m. The affine Kac-Moody Lie algebra of type $X_N^{(m)}$ will be denoted by $\mathfrak{g}(X_N^{(m)})$. Our *numbering* of the vertices of the Dynkin diagram of $\mathfrak{g}(X_N)$ and $\mathfrak{g}(X_N^{(m)})$ is the same as in [53].

Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{g}(X_N^{(m)})$ and let $\mathfrak{H}^*$ denote its dual. Denote the set of simple roots $\prod = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$ and the simple coroots $\prod^\vee = \{\alpha_0^\vee, \ldots, \alpha_N^\vee\}$. The triple $(\mathfrak{H}, \Pi, \Pi^\vee)$ forms a *realization* of the affine Kac-Moody Lie algebra $\mathfrak{g}(X_N^{(m)})$. Since we have fixed our numbering on the set of the vertices of the Dynkin diagram of $X_N^{(m)}$, we let $a_0, \ldots, a_N$ be the Coxeter labels and $a_0^\vee, \ldots, a_N^\vee$ be the dual Coxeter labels of the affine Lie algebra $\mathfrak{g}(X_N^{(m)})$. We refer the reader to Page 80 in [53] for a list of the Coxeter and the dual Coxeter labels. The root lattice and coroot lattice of the Kac-Moody Lie algebra will be denoted by Q and $Q^\vee$ respectively. The center of the affine Kac-Moody Lie algebra $\mathfrak{g}(X_N^{(m)})$ is one dimensional and is generated by the element $K := \sum_{i=0}^N a_i^\vee \alpha_i^\vee$. Similarly consider the following:

(1) $\delta := \sum_{i=0}^N a_i \alpha_i \in \mathfrak{H}^*$.
(2) $\theta := \sum_{i=1}^N a_i \alpha_i$.

The following result can be found in [53, Proposition 6.4]

Proposition 3.2. *The element θ defined above can be explicitly described as*

(1) *If $m = 1$, then the element θ is the unique longest root of $\mathring{\mathfrak{g}}$ of height $\sum_{i=1}^N a_i$.*
(2) *If $m = 2$ and $X_N = A_{2n}$, the element θ is the unique longest root of $\mathring{\mathfrak{g}}$ of height $\sum_{i=1}^N a_i$.*
(3) *If $m \neq 1$ and $X_N \neq A_{2n}$, then the element θ is the unique shortest root of maximal height $\sum_{i=0}^N a_i - 1$.*

3.2.1. *Normalized Invariant Bilinear form.* We now describe the induced normalized invariant bilinear form on $\mathfrak{H}$ and $\mathfrak{H}^*$. First, we fix an element $d \in \mathfrak{H}$ which satisfies the conditions $\alpha_i(d) = 0$ for $1 \leq i \leq N$ and $\alpha_0(d) = 1$. Such an element is defined up to a summand by K. The element $\alpha_0^\vee, \ldots, \alpha_N^\vee, d$ form a basis of the Lie algebra $\mathfrak{H}$. We describe the normalized form $\kappa_{\mathfrak{g}(X_N^{(m)})}$ on $\mathfrak{H}$. It is given by the following formula: (see Page 81 in [53]):

(1) $\kappa_{\mathfrak{g}(X_N^{(m)})}(\alpha_i^\vee, \alpha_j^\vee) = a_{ij}\frac{a_j}{a_j^\vee}$, where a_{ij} are the entries of the Cartan matrix of $\mathfrak{g}(X_N^{(m)})$.
(2) $\kappa_{\mathfrak{g}(X_N^{(m)})}(\alpha_i^\vee, d) = 0$, for $1 \leq i \leq N$.

(3) $\kappa_{\mathfrak{g}(X_N^{(m)})}(\alpha_0^\vee, d) = a_0$ and $\kappa_{\mathfrak{g}(X_N^{(m)})}(d, d) = 0$.

The form $\kappa_{\mathfrak{g}(X_N^{(m)})}$ is non-degeneratete and it induces an isomorphism $\nu_{\mathfrak{g}(X_N^{(m)})} : \mathfrak{H}^* \to \mathfrak{H}$. We now describe the induced form on $\mathfrak{H}^*$. Let Λ_0 be an element in $\mathfrak{H}^*$ defined by the equations:

$$\Lambda_0(\alpha_i^\vee) = \delta_{i,0} \text{ for } 0 \leq i \leq N, \text{and } \Lambda_0(d) = 0.$$

Thus once d is fixed, the element Λ_0 is uniquely determined. Now we get a basis of $\mathfrak{h}^*$ given by $\alpha_0, \ldots, \alpha_N, \Lambda_0$. The invariant normalized bilinear form $\kappa_{\mathfrak{g}(X_N^{(m)})}$ is given by the following:

(1) $\kappa_{\mathfrak{g}(X_N^{(m)})}(\alpha_i, \alpha_j) = a_{ij}\frac{a_i^\vee}{a_i}$.
(2) $\kappa_{\mathfrak{g}(X_N^{(m)})}(\alpha_i, \Lambda_0) = 0$ for $\leq i \leq N$.
(3) $\kappa_{\mathfrak{g}(X_N^{(m)})}(\alpha_0, \Lambda_0) = a_0^{-1}$ and $\kappa_{\mathfrak{g}(X_N^{(m)})}(\Lambda_0, \Lambda_0) = 0$.

3.2.2. *Horizontal subalgebra and weight lattice.* We denote the finite dimensional Lie algebra obtained by deleting the 0-th vertex of the Dynkin diagram of $\mathfrak{g}(X_N^{(m)})$ by $\mathring{\mathfrak{g}}$. Let $\mathring{\mathfrak{h}}$ denote the Cartan subalgebra of $\mathring{\mathfrak{g}}$. Clearly $\mathring{\mathfrak{h}}$ is generated by the coroots $\alpha_1^\vee, \ldots, \alpha_N^\vee$. Let $Q(\mathring{\mathfrak{g}})$ (resp. $Q^\vee(\mathring{\mathfrak{g}})$) denote the root lattice (resp. coroot lattice) of $\mathring{\mathfrak{g}}$ and $\mathring{\Delta}$ denote the positive simple roots of $\mathring{\mathfrak{g}}$.

For $1 \leq i \leq N$, we define the affine fundamental weights Λ_i that satisfies the equation $\Lambda_i(\alpha_j^\vee) = \delta_{ij}$. Then we can write Λ_i in terms of its horizontal projection as follows:

(3.2) $$\Lambda_i := \overline{\Lambda}_i + a_i^\vee \Lambda_0.$$

Then $\overline{\Lambda}_1, \ldots, \overline{\Lambda}_N$ are the fundamental weights of the horizontal subalgebra $\mathring{\mathfrak{g}}$.

Remark 3.3. The numbering of the vertices of the Dynkin diagram of $\mathring{\mathfrak{g}}$ as in [53] may not extend to a number scheme of the Dynkin diagram of $\mathfrak{g}(X_N^{(m)})$. Hence we might need to reorder the set $\overline{\Lambda}_1, \ldots, \overline{\Lambda}_N$ to match up with the usual conventions for $\mathring{\mathfrak{g}}$.

3.2.3. *Level ℓ-weights.* The weight lattice P is the $\mathbb{Z}$-lattice generated by $\Lambda_0, \ldots, \Lambda_N$. Similarly let us define the set $P^\ell(\mathfrak{g}(X_N^{(m)}))$ of dominant integral weights of level ℓ as follows:

$$P^\ell(\mathfrak{g}(X_N^{(m)})) = \{\lambda \in P | \lambda(\alpha_i^\vee) \geq 0 \text{ for all } 0 \leq i \leq N \text{ and } \lambda(K) = \ell\}.$$

The set of irreducible, integrable, highest weight representations at level ℓ of the affine Kac-Moody Lie algebra $\mathfrak{g}(X_N^{(m)})$ is in bijection with the set $P^\ell(\mathfrak{g}(X_N^{(m)}))$. If $m = 1$, then the set $P^\ell(\mathfrak{g}(X_N^{(1)})$ will often be denoted by $P_\ell(\mathfrak{g})$. The following Lemma is easy to prove:

Lemma 3.4. *Let $P_+(\mathring{\mathfrak{g}})$ denote the set of dominant integral weights of $\mathring{\mathfrak{g}}$, then*

$$P^\ell(\mathfrak{g}(X_N^{(m)})) := \{\lambda \in P_+(\mathring{\mathfrak{g}}) | \kappa_{\mathfrak{g}(X_N^{(m)})}(\lambda, \theta) \leq \ell\}.$$

Proof. The numbering of the vertices of the Dynkin-diagram of $\mathfrak{g}(X_N^{(m)})$ guarantees that $a_0^\vee = 1$. If $\lambda = \sum_{i=0}^N b_i \Lambda_i$, then using the explicit description of the form $\kappa_{\mathfrak{g}(X_N^{(m)})}$, we get

$\lambda(K) = a_0^\vee b_0 + \cdots + a_N^\vee b_N = \ell$. This implies that $P^\ell(\mathfrak{g}(X_N^{(m)}))$ is in a natural bijection with the set

$$\{\lambda = \sum_{i=1} b_i \overline{\Lambda}_i \in P_+(\mathring{\mathfrak{g}}) | a_1^\vee b_1 + \cdots + a_N^\vee b_N \leq \ell\}.$$

Now again by the definition of θ and $\kappa_{\mathfrak{g}(X_N^{(m)})}$, we get $\kappa_{\mathfrak{g}(X_N^{(m)})}(\lambda, \theta) = a_1^\vee b_1 + \cdots + a_N^\vee b_N$. Hence the result follows. $\qquad \square$

We refer the reader to Section 8 for an explicit description of the set $P^\ell(\mathfrak{g}(X_N^{(m)}))$.

3.3. **Twisted affine Lie algebras as Kac-Moody Lie algebras.** Let γ be a finite order automorphism of $\mathfrak{g}$ and σ a diagram automorphism related to γ as in Proposition 3.1. The map $\phi_{\sigma,\gamma}$ gives an isomorphism of Lie algebras $\widehat{L}(\mathfrak{g}, \sigma) \simeq \widehat{L}(\mathfrak{g}, \gamma)$. Now following [53], we relate $\widehat{L}(\mathfrak{g}, \sigma)$ with the affine Kac-Moody Lie algebra $\mathfrak{g}(X_N^{(m)})$. Let d' be a derivation on $\widehat{L}(\mathfrak{g})$ which acts on $\mathfrak{g} \otimes \mathbb{C}((t))$ by $t\frac{d}{dt}$ and commutes with c. We extend $\widehat{L}(\mathfrak{g}, \sigma)$ by d' and denote it by $\widehat{L}(\mathfrak{g}, \sigma) \ltimes \mathbb{C}d'$.

We only restrict to the following cases, when σ is non-trivial. Let $\varepsilon = 0$ when in cases

TABLE 1.

Case	X_N	σ	m	$\mathring{\mathfrak{g}}$	$\mathfrak{g}^\sigma$
1	$A_{2n},\ n \geq 2$	$i \leftrightarrow 2n - i$	2	C_n	B_n
2	$A_{2n-1},\ n \geq 3$	$i \leftrightarrow 2n - 1 - i$	2	C_n	C_n
3	$D_{n+1},\ n \geq 4$	$n \leftrightarrow n - 1$	2	B_n	B_n
4	D_4	$1 \to 4 \to 3 \to 1$	3	G_2	G_2
5	E_6	$1 \leftrightarrow 5,\ 2 \leftrightarrow 4$	2	F_4	F_4

2-5 and $\varepsilon = N$ in the case 1 from the above table. We define the set $I := \{0, \ldots, N\} \backslash \{\varepsilon\}$. In each of the above cases, [53, Section 8.3] constructs elements H_i, E_i, F_i (indexed by $\{0, \ldots, N\}$) in the Lie algebra $\mathfrak{g}(X_N)$ and an element θ_0 in $\mathfrak{h}$ such that

(1) E_i, F_i for $i \in I$ are Chevalley generators of $\mathfrak{g}^\sigma$.
(2) Roots α_i for $i \in I$ of the Lie algebra $\mathfrak{g}^\sigma$.
(3) An abelian subalgebra $\mathfrak{H} = \mathfrak{h}^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d'$ of $\widehat{L}(\mathfrak{g}, \sigma) \ltimes \mathbb{C}d'$.

The following result can be found in [53, Proposition 8.3, Theorem 8.3]

Proposition 3.5. *Let* $\Pi = \{\alpha_\varepsilon := \delta - \theta_0, \alpha_i, where\ i \in I\}$, *where* $\delta \in \mathfrak{h}^*$ *be such that* $\delta(d') = 1$ *and* $\delta(\mathfrak{h}^\sigma \oplus \mathbb{C}c) = 0$. *Moreover let* $\Pi' = \{\alpha_\epsilon^\vee := \frac{m}{a_0}c + H_\epsilon, \alpha_i^\vee = H_i, where\ i \in I\}$.
Then the triple $(\mathfrak{H}, \Pi, \Pi')$ *gives a Kac-Moody realization of type* $\mathfrak{g}(X_N^{(m)})$.

3.4. **Modules for twisted affine Lie algebras.** For any finite order automorphism γ, let $P^\ell(\mathfrak{g}, \gamma)$ be a subset of $P_+(\mathfrak{g}^\gamma)$ parametrizing the set of integrable, irreducible highest weight representations of $\widehat{L}(\mathfrak{g}, \gamma)$. By Proposition 3.5, we get a natural bijection

(3.3) $$P^\ell(\mathfrak{g}, \gamma) \leftrightarrow P^\ell(\mathfrak{g}(X_N^{(m)})).$$

We also refer the reader to [46, Section 2] for an alternate description and construction. For $\lambda \in P^\ell(\mathfrak{g}, \gamma)$, let V_λ denote the highest weight irreducible module of $\mathfrak{g}^\gamma$ of highest weight λ. Similarly, we denote the highest weight irreducible integrable modules by $\mathcal{H}_\lambda(\mathfrak{g}, \gamma)$ (also by $\mathcal{H}_\lambda$ if there is no confusion). They are characterized by the property:

(1) $V_\lambda \subseteq \mathcal{H}_\lambda(\mathfrak{g}, \gamma)$.
(2) The central element c acts on $\mathcal{H}_\lambda(\mathfrak{g}, \gamma)$ by multiplication by ℓ.

In an upcoming section, we give a coordinate free construction of the modules $\mathcal{H}_\lambda(\mathfrak{g}, \gamma)$.

4. Sheaf of Twisted conformal blocks

In this section, we give a brief *coordinate free construction* of the sheaf of twisted conformal blocks ([31], [39],[46],[76],[75]).

4.1. **Family of pointed Γ-curves.** Let T be a smooth variety and consider a proper, flat family $\pi : \widetilde{C} \to T$ of curves with at most nodal singularities. Let Γ be a finite group that acts on $\widetilde{C}$ such that the map π is Γ equivariant. Let the quotient $\overline{\pi} : C = \widetilde{C}/\Gamma \to T$ be the induced family of curves over T. The genus of the fibers of π and $\overline{\pi}$ are related by the Hurwitz formula. We further choose mutually disjoint sections $\mathbf{p} = (p_1, \ldots, p_n)$ (respectively $\widetilde{\mathbf{p}} = (\widetilde{p}_1, \ldots, \widetilde{p}_n)$) of $\overline{\pi}$ (respectively π) such that

(1) We have $pr(\widetilde{\mathbf{p}}(t)) = \mathbf{p}(t)$ for all points $t \in T$, where $pr : \widetilde{C} \to C$ is the canonical quotient map.
(2) The points $\mathbf{p}(t)$ are all smooth.
(3) For any $t \in T$, the smooth branching points of the Γ-cover $\widetilde{C}_t \to C_t$ are contained in $\cup_{i=1}^n p_i(t)$.
(4) For any point $t \in T$, the data $(\widetilde{C}_t, C_t, \widetilde{\mathbf{p}}(t), \mathbf{p}(t))$ is a n-pointed admissible Γ-cover in the sense of [51].
(5) We assume that $\widetilde{C} \setminus \Gamma \cdot \widetilde{\mathbf{p}}(T)$ is affine over T.

Let $\mathbf{m} = (m_1, \ldots, m_n) \in \Gamma^n$ be the monodromies around the sections $\widetilde{\mathbf{p}}$, in other words for all $1 \leq i \leq n$, m_i is the generator of the stabilizer $\Gamma_i \leq \Gamma$ of $\widetilde{p}_i$ determined by the orientation of the curve.

4.2. **Coordinate free highest weight integrable modules.** We start with a family of pointed Γ-curves satisfying the conditions in Section 4.1. Let $\mathcal{I}_{\widetilde{\mathbf{p}}}$ (respectively $\mathcal{I}_{\widetilde{p}_i}$) be the ideal of the image of $\widetilde{\mathbf{p}}$ (respectively $\widetilde{p}_i$) in $\widetilde{C}$. Let $\widehat{\mathcal{O}}_{\widetilde{C}, \widetilde{p}_i}$ be the formal completion of $\mathcal{O}_{\widetilde{C}}$ along the image of $\widetilde{p}_i$ and $\mathcal{K}_{\widetilde{C}, \widetilde{p}_i}$ be the sheaf of formal meromorphic functions along $\widetilde{p}_i$. Observe that if $\Gamma_i = \langle m_i \rangle$ denotes the stabilizer in Γ of $\widetilde{p}_i$, then Γ_i acts on $\mathcal{K}_{\widetilde{C}, \widetilde{p}_i}$. We define the *coordinate free twisted affine Lie algebra* as

$$(4.1) \qquad \widehat{\mathfrak{g}}_{\widetilde{p}_i} := \left(\mathfrak{g} \otimes \mathcal{K}_{\widetilde{C}, \widetilde{p}_i} \right)^{\Gamma_i} \oplus \mathcal{O}_T c,$$

Suppose $\widehat{\mathfrak{p}}_{\widetilde{p}_i} := \left(\mathfrak{g} \otimes \widehat{\mathcal{O}}_{\widetilde{C}, \widetilde{p}_i} \right)^{\Gamma_i} \oplus \mathcal{O}_T c$. For $\lambda_i \in P^\ell(\mathfrak{g}, \Gamma_i) := P^\ell(\mathfrak{g}, m_i)$, let V_λ denote the $\mathfrak{g}^{\Gamma_i}$ module of highest weight λ. We let $\left(\mathfrak{g} \otimes \widehat{\mathcal{O}}_{\widetilde{C}, \widetilde{p}_i} \right)^{\Gamma_i}$ act on V_λ via evaluation at $\widetilde{p}_i$ and c acts

on V_λ by multiplication by ℓ. Thus $\widehat{\mathfrak{p}}_{\widetilde{p}_i}$ act on V_λ. We denote by $M_{\lambda_i,\widetilde{p}_i} := \mathrm{Ind}_{\widehat{\mathfrak{p}}_{\widetilde{p}_i}}^{\widehat{\mathfrak{g}}_{\widetilde{p}_i}} V_\lambda$. It follows that $M_{\lambda_i,\widetilde{p}_i}$ admits a unique irreducible quotient which we denote by $\mathbb{H}_{\lambda_i,\widetilde{p}_i}$.

Remark 4.1. If T is a point and we choose formal coordinates around $\widetilde{p}_i$, we get an isomorphism of $\widehat{\mathfrak{g}}_{\widetilde{p}_i}$ with $\widehat{L}(\mathfrak{g},\Gamma_i) = \widehat{L}(\mathfrak{g},m_i)$. Under this isomorphism $\mathbb{H}_{\lambda_i,\widetilde{p}_i}$ gets identified with $\mathcal{H}_\lambda(\mathfrak{g},\Gamma_i)$.

Consider the Lie algebra $\widehat{\mathfrak{g}}_{p_i} := \left(\mathfrak{g} \otimes \big(\bigoplus_{\widetilde{p}_i' \in pr^{-1}(p_i)} \mathcal{K}_{\widetilde{C},\widetilde{p}_i'} \big) \right)^\Gamma \oplus \mathbb{C}c$. This is canonically identified with $\widehat{\mathfrak{g}}_{\widetilde{p}_i}$. Similarly consider the sheaf of Lie algebras $\widehat{\mathfrak{g}}_{\widetilde{C},\mathbf{p}} := (\oplus_{i=1}^n \widehat{\mathfrak{g}}_{p_i})/\mathcal{Z}$, where $\mathcal{Z}$ is a subsheaf of $\oplus_{i=1}^n \mathcal{O}_T c$ consisting of tuples $(f_1,\ldots,f_n)$ such that $f_1 + \cdots + f_n = 0$.

4.2.1. *Sheaf of twisted covacua.* Let $\vec{\lambda} = (\lambda_1,\ldots,\lambda_n)$ be an n-tuple of weights such that each $\lambda_i \in P^\ell(\mathfrak{g},\Gamma_i)$. The sheaf $\mathbb{H}_{\vec{\lambda}} := \mathbb{H}_{\lambda_1,\widetilde{p}_1} \otimes \cdots \otimes \mathbb{H}_{\lambda_n,\widetilde{p}_n}$ of $\mathcal{O}_T$-modules is also a representation of the sheaf of Lie algebras $\widehat{\mathfrak{g}}_{\widetilde{C},\mathbf{p}}$. We define the *sheaf of twisted covacua* to be

$$\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p}) := \mathbb{H}_{\vec{\lambda}}/(\mathfrak{g} \otimes \mathcal{O}_{\widetilde{C}}(*\Gamma.\widetilde{\mathbf{p}}))^\Gamma \mathbb{H}_{\vec{\lambda}},$$

where $\Gamma.\widetilde{\mathbf{p}}$ denotes the union of the Γ orbits of $\widetilde{p}_i$ for $1 \le i \le N$. Similarly we define the the sheaf of vacua as

$$\mathbb{V}_{\vec{\lambda},\Gamma}^\dagger(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p}) := \{\langle\Psi| \in \mathbb{H}_{\vec{\lambda}}^* | \langle\Psi|X[f] = 0, \text{ for all } X \otimes f \in (\mathfrak{g} \otimes \mathcal{O}_{\widetilde{C}}(*\Gamma.\widetilde{\mathbf{p}}))^\Gamma\}.$$

We now recall some basic properties of the sheaf $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p})$.

4.3. **Properties of twisted Vacua.** It can be shown that both $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p})$ and $\mathbb{V}_{\vec{\lambda},\Gamma}^\dagger(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p})$ are coherent $\mathcal{O}_T$-modules [71, Lemma 2.5.2] which are compatible with base change ([46]). Moreover like in the untwisted case, Hong-Kumar [46] (under the assumption that Γ preserves a Borel subalgebra of $\mathfrak{g}$) show that the sheaf $\mathbb{V}_{\vec{\lambda},\Gamma}^\dagger(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p})$ is locally free and $\mathbb{V}_{\vec{\lambda},\Gamma}^\dagger(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p})$ and $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p})$ are dual to each other. We also refer the reader to [72], for a more general statement on the interior of $\overline{\mathcal{M}}_{g,n}^\Gamma$.

Let $\mathbf{q} = (q_1,\ldots,q_m)$ be m-sections of $\overline{\pi} : C \to T$ marking étale points of the Γ-cover $\widetilde{C} \to C$ and let $\widetilde{\mathbf{q}}$ be a choice of lifts to $\widetilde{C}$. This data endows the family $\widetilde{C} \to T$ as a $n+m$-pointed Γ-cover with monodromy data $\mathbf{m}' = (\mathbf{m},1,\ldots,1)$. Then we have following [39, 46]:

Proposition 4.2. *Let $\vec{0} = (0,\ldots,0) \in P_\ell(\mathfrak{g})^n$ and assume that Γ preserves a Borel subalgebra of $\mathfrak{g}$. Then there is a natural isomorphism between $\mathcal{O}_T$-modules $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C},C,\widetilde{\mathbf{p}},\mathbf{p}) \simeq \mathbb{V}_{\vec{\lambda}\sqcup\vec{0},\Gamma}(\widetilde{C},C,\widetilde{\mathbf{p}}\sqcup\widetilde{\mathbf{q}},\mathbf{p}\sqcup\mathbf{q})$. Moreover these isomorphisms are compatible with each other for different choices of the étale points chosen.*

4.3.1. *Descent data.* Following the discussion in [31], we use Proposition 4.2 to drop the condition that $\widetilde{C} \setminus \Gamma \cdot \widetilde{\mathbf{p}}(T)$ is affine. Let $\widetilde{C} \to T$ be a family satisfying conditions 1-4 in Section 4.1. We can find an étale cover T' of T and m-sections marking étale points $\widetilde{\mathbf{q}}$ (respectively $\mathbf{q}$) of the induced family $\widetilde{C}'$ such that it satisfies all the conditions in Section 4.1. Thus we can associate a sheaf $\mathbb{V}_{\vec{\lambda} \sqcup \vec{0}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}} \sqcup \widetilde{\mathbf{q}}, \mathbf{p} \sqcup \mathbf{q})$ on T'. We can define $\mathbb{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$ to be the natural descent of $\mathbb{V}_{\vec{\lambda} \sqcup \vec{0}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}} \sqcup \widetilde{\mathbf{q}}, \mathbf{p} \sqcup \mathbf{q})$ given by Proposition 4.2. The same discussion (see Proposition 2.1) in [31], tells us that the $\mathbb{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$ is independent of the choice of the étale cover of T' of T. Thus, we get a well defined sheaf of covacua $\mathbb{V}_{\vec{\lambda}, \Gamma}$ on the moduli stack $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ of n-pointed admissible covers defined in [51] (see also Appendix A).

4.3.2. *Factorization.* Let $\widetilde{C} \to C \to T$ be a family of stable $n+2$ pointed stable covers with group Γ. Let $\widetilde{\mathbf{p}}' = (\widetilde{\mathbf{p}}, \widetilde{q}_1, \widetilde{q}_2)$ be the $n+2$ sections such that the monodromies around the sections $\widetilde{q}_1$ and $\widetilde{q}_2$ are inverse to each other, say γ and γ^{-1} respectively. Then identifying the family $\widetilde{C}$ along the sections $\widetilde{q}_1$ and $\widetilde{q}_2$, we get a new family of stable n-pointed cover of curves $\widetilde{D} \to T$ with group Γ along with n-sections $\widetilde{\mathbf{p}}$. Assume that Γ preserves a Borel subalgebra of $\mathfrak{g}$. Then by the factorization theorem in [46], we have the following isomorphisms of locally free sheaves on T.

Proposition 4.3.

$$\mathbb{V}_{\vec{\lambda}, \Gamma}(\widetilde{D}, D, \widetilde{\mathbf{p}}, \mathbf{p}) \simeq \bigoplus_{\mu \in P^{\ell}(\mathfrak{g}, \gamma)} \mathbb{V}_{\vec{\lambda} \sqcup \{\mu, \mu^*\}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}', \mathbf{p}')$$

4.3.3. *Global Properties.* We continue to assume that the group Γ preserves a Borel subalgebra of $\mathfrak{g}$. The bundles of twisted covacua are compatible with natural morphisms on $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ which we state below.

Proposition 4.4. *Let $\vec{\lambda}$ be an n-tuple of weights corresponding to $\mathbf{m}$ and let $\mathbb{V}_{\vec{\lambda}, \Gamma}$ denote the sheaf of twisted covacua on $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$, where $\mathbf{m} = (m_1, \ldots, m_n) \in \Gamma^n$. Then the following holds:*

(1) *Assume that for some i, we have $m_i = 1$, $\lambda_i = 0$ and that $(g, n-1)$ is a stable pair. Consider the natural forgetful stabilization map $f_i : \overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m}) \to \overline{\mathcal{M}}_{g,n-1}^{\Gamma}(\mathbf{m}')$ obtained by forgetting the point $\widetilde{p}_i$ (and contracting any unstable component), then there is a natural isomorphism*

$$\mathbb{V}_{\vec{\lambda}, \Gamma} \simeq f_i^* \mathbb{V}_{\vec{\lambda}', \Gamma},$$

where $\vec{\lambda}'$ (respectively $\mathbf{m}'$) is obtained by deleting $\lambda_i = 0$ (respectively $m_i = 1$) from $\vec{\lambda}$ (respectively $\mathbf{m}$).

(2) *Let $\xi_{1,2,\gamma} : \overline{\mathcal{M}}_{g_1,n_1+1}^{\Gamma}(\mathbf{m}_1, \gamma) \times \overline{\mathcal{M}}_{g_2,n_2+1}^{\Gamma}(\mathbf{m}_2, \gamma^{-1}) \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}^{\Gamma}(\mathbf{m_1}, \mathbf{m_2})$ be the morphism obtained by gluing two curves of genus g_1 and g_2 along the last marked*

point with monodromy γ and γ^{-1}, then there is a natural isomorphism

$$\xi_{1,2,\gamma}^* \mathbb{V}_{\vec{\lambda},\Gamma} \simeq \bigoplus_{\mu \in P^\ell(\mathfrak{g},\gamma)} \mathbb{V}_{\vec{\lambda}',\Gamma} \boxtimes \mathbb{V}_{\vec{\lambda}'',\Gamma},$$

where $\vec{\lambda}' = (\lambda_1, \ldots, \lambda_{n_1}, \mu)$ and $\vec{\lambda}'' = (\lambda_{n_1+1}, \ldots, \lambda_{n_2}, \mu^*)$.

(3) Let $\xi_\gamma : \overline{\mathcal{M}}_{g-1,n+2}^\Gamma(\mathbf{m}, \gamma, \gamma^{-1}) \to \overline{\mathcal{M}}_{g,n}^\Gamma(\mathbf{m})$ be the morphism obtained by gluing a curve along two points with opposite monodromies. Then there is a canonical isomorphism

$$\xi_\gamma^* \mathbb{V}_{\vec{\lambda},\Gamma} \simeq \bigoplus_{\mu \in P^\ell(\mathfrak{g},\gamma)} \mathbb{V}_{\vec{\lambda} \sqcup \{\mu,\mu^*\},\Gamma}.$$

Moreover these isomorphisms induced by $\xi_{1,2,\gamma}$, f_i and ξ_γ are compatible with each other.

Proof. The second and the third part of the proposition follows directly from Proposition 4.3. We now discuss the proof of the first part which is similar to the discussion in Section 2.2 of [31].

Let $\widetilde{C} \to T$ be a family of stable $n+1$-pointed covers with group Γ and $\widetilde{\mathbf{p}}$ are the sections. Assume that the points in the fibers of $\widetilde{C} \to T$ marked by the $n + 1$-th section in $\widetilde{\mathbf{p}}$ are always étale. Let $\vec{\lambda} = (\lambda_1, \ldots, \lambda_{n+1})$ be such that $\lambda_{n+1} = 0$ in $P_\ell(\mathfrak{g})$. Then the forgetting the $n + 1$-th section, gives a family of n-pointed Γ covers which may not be stable. To make the new family stable, we have to contract unstable components to a point. Hence two situations can occur.

First we can contract a rational curve that has two marked points and p_n and p_{n+1} and meets the other component E at a nodal point q. In the case after stabilization we obtain the curve E is smooth at q and we declare q to be the n-th marked point p_n. Secondly we can contract a rational component which has one marked point p_{n+1} and meets the other components E_1 and E_2 at two distinct nodal points q_1 and q_2. In this case, after stabilization we get a curve obtained by joining E_1 and E_2 by identifying q_1 with q_2. We refer the reader to Section 2 in [51] for more details on these stabilization morphisms. Now in both these cases, part (1) of Proposition 4.4 follows from Proposition 4.3 and Lemma 4.5. $\qquad\square$

Lemma 4.5. Let $\mathbf{m} = (\gamma, \gamma^{-1}, 1) \in \Gamma^3$ and $\vec{\lambda} = (\lambda, \lambda^*, 0)$, where $\lambda \in P^\gamma(\mathfrak{g}, \gamma)$. Assume that the marked points $\widetilde{\mathbf{p}}$ are in the same connected component of $\widetilde{C}$. Then the fibers of the vector bundle $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p})$ restricted to such points are one dimensional.

4.3.4. *Equivariance with respect to permutations and conjugation.* As before let $\mathbf{m} = (m_1, \cdots, m_n) \in \Gamma^n$ and consider the moduli stack $\overline{\mathcal{M}}_{g,n}^\Gamma(\mathbf{m})$. Let $\sigma \in S_n$ be a permutation. Then permutation of the marked points induces an isomorphism $\xi_\sigma : \overline{\mathcal{M}}_{g,n}^\Gamma(\mathbf{m}) \xrightarrow{\cong} \overline{\mathcal{M}}_{g,n}^\Gamma(\mathbf{m}_\sigma)$, where $\mathbf{m}_\sigma := (m_{\sigma(1)}, \cdots, m_{\sigma(n)})$. If $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ is an n-tuple of weights with $\lambda_i \in P^\ell(\mathfrak{g}, m_i)$, we set $\vec{\lambda}_\sigma$ to be the permuted weights. Then we have a natural isomorphism between the sheaves of covacua

$$(4.2) \qquad\qquad\qquad \mathbb{V}_{\vec{\lambda},\Gamma} \simeq \xi_\sigma^* \mathbb{V}_{\vec{\lambda}_\sigma,\Gamma}.$$

Now let $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n) \in \Gamma^n$. Let $^{\boldsymbol{\gamma}}\mathbf{m} = (^{\gamma_1}m_1, \cdots, ^{\gamma_n}m_n)$ be the conjugated n-tuple, where $^{\gamma_i}m_i := \gamma_i m_i \gamma_i^{-1}$. Then acting on the marked points $\widetilde{\mathbf{p}}$ in $\widetilde{C}$ by $\boldsymbol{\gamma}$ induces an isomorphism $\xi_{\boldsymbol{\gamma}} : \overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m}) \xrightarrow{\cong} \overline{\mathcal{M}}_{g,n}^{\Gamma}(^{\boldsymbol{\gamma}}\mathbf{m})$. If $\vec{\lambda} = (\lambda_1, \cdots, \lambda_n)$ is an n-tuple of weights as before, then we obtain the weights $\boldsymbol{\gamma} \cdot \vec{\lambda}$ with $\gamma_i \cdot \lambda_i \in P^\ell(\mathfrak{g}, ^{\gamma_i}m_i)$ (see also Section 7.1.1). Then we have a natural isomorphism between the sheaves of covacua

$$(4.3) \qquad \mathbb{V}_{\vec{\lambda}, \Gamma} \simeq \xi_{\boldsymbol{\gamma}}^* \mathbb{V}_{\boldsymbol{\gamma} \cdot \vec{\lambda}, \Gamma}.$$

Combining the two above equations, we see that the sheaves of covacua are equivariant for the action of the wreath product $S_n \ltimes \Gamma^n$ on the moduli stacks of pointed admissible Γ-covers.

5. ATIYAH ALGEBRA OF THE TWISTED WZW CONNECTION ON $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}$

As described in Appendix A, let $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}$ be the moduli stack of n-pointed admissible Γ-covers of genus g stable pointed curves along with a choice of local coordinates around the marked points. Let $\pi : \widetilde{C} \to C$ be a Γ-cover of nodal curves and let $\mathbf{p} = (p_1, \ldots, p_n)$ be a sequence of n-distinct points of C and $\widetilde{\mathbf{p}}$ be a lift of $\mathbf{p}$ to $\widetilde{C}$.

Let $\widetilde{\mathbf{z}} = (\widetilde{z}_1, \ldots, \widetilde{z}_n)$ be an n-tuple of formal special coordinates of $\widetilde{\mathbf{p}}$ and $\mathbf{z} = (z_1, \ldots, z_n)$ denote an n-tuple of formal coordinates of $\mathbf{p}$. The local coordinates $\widetilde{\mathbf{z}}$ and $\mathbf{z}$ are related by the formula $z_i = \widetilde{z}_i^{N_i}$, where N_i is the order of the stabilizer of Γ_i at the points $\widetilde{p}_i$. More precisely, the functor $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ along with a choice of formal special coordinates will be denoted by $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}$. Similarly, we denote the functor $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ along with a choice of one-jets along the marked points by $\widetilde{\overline{\mathcal{M}}}_{g,n}^{\Gamma}$. Let $\mathbf{m} = (m_1, \ldots, m_n)$ be an n-tuple of elements of Γ^n. The coherent sheaf of twisted conformal blocks on $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ along with choice of formal coordinates will be denoted by $\mathcal{V}_{\vec{\lambda}, \Gamma}^{\dagger}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{z}})$. We refer the reader to [46, 72, 79] for more details. By taking invariant pushforward, the coherent sheaf $\mathcal{V}_{\vec{\lambda}, \Gamma}^{\dagger}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{z}})$ descends to a coherent sheaf on $\widetilde{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ which we denote by $\mathcal{V}_{\vec{\lambda}, \Gamma}^{\dagger}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}})$. Moreover the coherent sheaf $\mathcal{V}_{\vec{\lambda}, \Gamma}^{\dagger}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}})$ on $\widetilde{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ carries a flat projective connection with logarithmic singularities along the boundary $\widetilde{\Delta}_{g,n,\Gamma}$ of $\widetilde{\overline{\mathcal{M}}}_{g,n}^{\Gamma}$. Following the construction of the twisted WZW connections and the Virasoro uniformization theorem in [72], we describe the Atiyah algebra of this projective connection. The corresponding result in the untwisted case is due to Tsuchimoto [75]. In this section, we do not assume that Γ preserves a Borel subalgebra of $\mathfrak{g}$.

5.1. Atiyah Algebras.
In this section, we recall the basic notion of Atiyah algebras from [11, 18, 75]. Let E be a vector bundle of rank r on a smooth projective variety X. We denote by $\mathcal{D}^{\leq 1}(E)$ to be the sheaf of differential operators of E of order at most 1

Definition 5.1. The Atiyah algebra $\mathcal{A}_E$ of E is the subsheaf of Lie algebras of $\mathcal{D}^{\leq 1}(E)$ such that the symbol map restricted to $\mathcal{A}_E$ surjects on to $\Theta_X \otimes \mathrm{id}$, where Θ_X is the tangent sheaf of X. The Atiyah algebra $\mathcal{A}_E$ can be put in an short exact sequence of sheaves of Lie algebras of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{E}nd_{\mathcal{O}_X}(E) \to \mathcal{A}_E \to \Theta_X \otimes \mathrm{id}_E \to 0.$$

Definition 5.2. Let R be a sheaf of associative $\mathcal{O}_X$-algebras endowed with the obvious non-trivial Lie bracket. An R-Atiyah algebra consists of the following data:

- A Lie algebra extension $\mathcal{A}$ of the tangent sheaf Θ_X by $\mathcal{O}_X$-module R, also known as the fundamental exact sequence: $0 \to R \to \mathcal{A} \xrightarrow{\epsilon} \Theta_X \to 0$.
- A left $\mathcal{O}_X$-module structure on $\mathcal{A}$ compatible with the $\mathcal{O}_X$ module structures on R and Θ_X. For these structures the following identities must hold:
  - $[\alpha, ab] = [\alpha, a]b + a[\alpha, b]$,
  - $[\alpha, f] = \epsilon(\alpha)(f)$, where $\alpha \in \mathcal{A}$, $f \in \mathcal{O}_X$ and $a, b \in R$.

It was observed in [18] that R-Atiyah algebras form a category where the objects are Atiyah algebras and morphisms are maps of Lie algebra extensions.

5.1.1. *Operations on Atiyah algebras.* We will be focusing on Atiyah algebras where $R = \mathcal{O}_X$ but we will need some fundamental operations on Atiyah algebras which we recall below.

(1) (Functoriality) Let $\mathcal{A}$ be an R-Atiyah algebra on Y. If $X \to Y$ be a map of smooth varieties, then $f^*(R) = f^{-1}R \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_Y$. We define the pull back of $\mathcal{A}$ by $f^\bullet(\mathcal{A}) := \Theta_X \times_{f^*\Theta_Y} f^*(\mathcal{A})$.

(2) ($\mathcal{O}_X$-Atiyah algebras) We set $R = \mathcal{O}_X$ and we consider the following:
   (i) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two Atiyah algebras, we define the sum

   $$\mathcal{A}_1 + \mathcal{A}_2 = (\mathcal{A}_1 \times_{\Theta_X} \mathcal{A}_2)/\{(f, -f)|f \in \mathcal{O}_X\}.$$

   (ii) For $\lambda \in \mathbb{C}$, we define $\lambda\mathcal{A} = (\mathcal{O}_X \oplus \mathcal{A})/(\lambda, 1)\mathcal{O}_X$. The fundamental exact sequence of the Atiyah algebra $\lambda\mathcal{A}$ is the following: $0 \to \mathcal{O}_X \xrightarrow{\lambda^{-1}} \mathcal{A} \xrightarrow{\epsilon} \Theta_X \to 0$.

(3) (Action on coherent sheaf) Let E be a coherent sheaf on X and $\mathcal{A}$ be an $\mathcal{O}_X$-Atiyah algebra. We say that we have an action of $\mathcal{A}$ on E if the following are satisfied:
   (i) Each section of $\mathcal{A}$ acts on E as a first order differential operator.
   (ii) The principal symbol of α consider as a first order differential operator coincided with $\epsilon(\alpha) \otimes \mathrm{id}_E$.
   (iii) Let $f \in \mathcal{O}_X$, then the action of $\mathcal{A}$ restricts to multiplication by f.

(4) Let $\mathcal{A}$ be an Atiyah algebra acting on a coherent sheaf E on X, then the following holds:
   (i) E is locally free coherent sheaf.
   (ii) The Atiyah algebra $\mathcal{A}$ is canonically isomorphic to $\frac{1}{rk(E)}\mathcal{A}_{\det E}$, where $\mathrm{rk}(E)$ is the rank of the locally free sheaf E.

5.2. **Atiyah Algebras with Logarithmic singularities.** Let X be a smooth variety and D be a normal crossing divisor. Let $\Theta_{X,D}$ be the subsheaf of vector fields that preserve the divisor D, i.e

$$\Theta_{X,D}(U) := \{s \in \Theta_X(U) | s(\mathfrak{I}_D(u)) \subseteq \mathfrak{I}_D(u)\},$$

where $\mathfrak{I}_D$ is the ideal sheaf of the divisor D, U is an open set of X. If $\mathcal{A}$ is a Atiyah algebra over X, we define a log D-Atiyah algebra $\mathcal{A}(\mathrm{logD}) := \epsilon^{-1}(\Theta_{X,D}) \subseteq \mathcal{A}$.

If $(\mathcal{A}, R)$ and $(\mathcal{B}, S)$ are two Atiyah algebras, then we can call a morphism with logarithmic singularities at D, or simple a log D-morphism between $\mathcal{A}$ and $\mathcal{B}$ to be an $\mathcal{O}_X$-linear Lie algebra map $f : \mathcal{A}(\mathrm{logD}) \to \mathcal{B}(\mathrm{logD})$ such that $f(R) \subseteq S$ is an algebra map and the induced map $\mathcal{A}/R \to \mathcal{B}/S$ is an identity on $\Theta_{X,D}$.

Let $\mathcal{A} = \mathcal{A}_{\mathcal{O}_X}$ and $\mathcal{B} = \mathcal{A}_E$, where E is a vector bundles on X. A log D-morphism is the same as an integrable connection on E with logarithmic singularities at D. Let $\mathcal{A}$ be any $\mathcal{O}_X$-Atiyah algebra and E be a vector bundle on X, then the following are equivalent:

- A log D morphism between $\mathcal{A}(\log D) \to \mathcal{A}_E(\log D)$
- A flat log-projective connection on the projective bundle E.

Remark 5.3. The operations on Atiyah algebras described in Section 5.1.1 can be easily generalized to the case of log-D Atiyah algebras.

5.3. **Localization of twisted $\mathcal{D}$-modules.** In this section, we briefly recall the definition of Harish-Chandra pairs and the yoga of Beilinson-Bernstein localization functors [13] to produce twisted $\mathcal{D}$-modules. We mostly follow the book [36], however we discuss the adaptation to the logarithmic settings. First we recall the definition of a Harish-Chandra pair [36, Sections 17.1-17.2]:

Definition 5.4. A Harish-Chandra pair is a pair $(\mathfrak{s}, K)$, where $\mathfrak{s}$ is a Lie algebra and K is a Lie group, along with the following data:

(1) An embedding of the Lie algebra $\mathfrak{k}$ of K into $\mathfrak{s}$.
(2) An action Ad of K on $\mathfrak{s}$ which is compatible with the adjoint action of K on $\mathfrak{k}$.

Let Z be a smooth variety and let Θ_Z be the tangent sheaf of Z. Let Δ_Z be a normal crossing divisor in Z and let Θ_{Z,Δ_Z} be the sheaf of tangent vector fields preserving the divisor Δ_Z. By a *log $(\mathfrak{s}, K)$-action* on Z we mean an action of K on Z that preserves Δ_Z along with a Lie algebra map $\alpha : \mathfrak{s} \otimes \mathcal{O}_Z \to \Theta_{Z,\Delta_Z}$, that satisfies the following:

(1) The differential of the map $K \times Z \to Z$ is the restriction of α to $\mathfrak{k}$.
(2) The action intertwines the representation of K on $\mathfrak{s}$ with the action of K on Θ_{Z,Δ_Z}.

Motivated by the setting in [36], we say that the $(\mathfrak{s}, K)$-action on (Z, Δ_Z) is *log-transitive* if α is surjective. Let $\widehat{\mathfrak{s}}$ be a central extension of $\mathfrak{s}$ and consider the short exact sequence over Z

(5.1) $$0 \to \mathcal{O}_Z \to \widehat{\mathfrak{s}} \otimes \mathcal{O}_Z \to \mathfrak{s} \otimes \mathcal{O}_Z \to 0.$$

We assume the following:

(1) The exact sequence (5.1) splits over $\mathrm{Ker}\,\alpha$.

(2) The central extension defining $\widehat{\mathfrak{s}}$ splits over $\mathfrak{k}$ making the pair $(\widehat{\mathfrak{s}}, K)$ a new Harish-Chandra pair.

The above assumptions on the central extension $\widehat{\mathfrak{s}}$ imply that the quotient of $\widehat{\mathfrak{s}} \otimes \mathcal{O}_Z$ by $\operatorname{Ker}\alpha$ is a logarithmic Atiyah algebra $\mathcal{A}(\log \Delta_Z)$ on Z. Now if V is a module for the Harish-Chandra pair $(\widehat{\mathfrak{s}}, K)$, then the formalism of Beilinson-Bernstein first considers the sheaf $V \otimes \mathcal{O}_Z / \operatorname{Ker}\alpha(\mathcal{O}_Z \otimes V)$ on which $\mathcal{A}(\log \Delta_Z)$ acts. More generally, let $\widehat{\mathfrak{l}}$ be any Lie algebra containing $\widehat{\mathfrak{s}}$ and carrying a compatible adjoint K action and suppose $\widetilde{\mathfrak{l}}$ is a subsheaf of $\widehat{\mathfrak{l}} \otimes \mathcal{O}_Z$ that satisfies the following:

(1) it is preserved by $\widehat{\mathfrak{s}} \otimes \mathcal{O}_Z$;
(2) it is preserved by the action of K;
(3) it contains $\operatorname{Ker}\alpha$, where $\alpha : \mathfrak{s} \otimes \mathcal{O}_Z \to \Theta_{Z,\Delta_Z}$.

Let V be a $\widehat{\mathfrak{l}}$-module which is also a $(\widehat{\mathfrak{s}}, K)$-module, then the assumptions on $\widetilde{\mathfrak{l}}$ guarantee that the sheaf $V \otimes \mathcal{O}_Z / \widetilde{\mathfrak{l}} \cdot (V \otimes \mathcal{O}_Z)$ is a $\mathcal{A}(\log \Delta_Z)$-module.

5.3.1. *Twisted log-localization functor.* We recall following Section 17.2.5 in [36] how to descend the above twisted $\mathcal{A}(\log \Delta_Z)$ module to a free group quotient T of Z by K.

Let Z be a principal K-bundle over a variety T and let Δ_T be a normal crossing divisor whose inverse image in Z is Δ_Z. We say a pair (T, Δ_T) has a log-$(\mathfrak{s}, K)$ structure if the Harish-Chandra pair $(\mathfrak{s}, K)$ acts log-transitively on Z extending the action of K on the fibers of the map $\pi : Z \to T$. In this set up, the anchor map α gives a surjective map $\overline{\alpha} : \overline{\mathfrak{s}} \to \Theta_{T,\Delta_T}$, where $\overline{\mathfrak{s}} = (\mathfrak{s}/\mathfrak{k} \otimes \mathcal{O}_Z)^K$. We denote the corresponding log-Atiyah algebra by $\mathcal{A}(\log \Delta_T)$.

Remark 5.5. Unlike in [36], we do not require the action of $\mathfrak{s}$ to have trivial stabilizers.

Further let $\widehat{\mathfrak{s}}$, $\widehat{\mathfrak{l}}$, $\widetilde{\mathfrak{l}}$ and V be as above. The $\mathcal{A}(\log \Delta_Z)$-module $V \otimes \mathcal{O}_Z / \widetilde{\mathfrak{l}} \cdot (V \otimes \mathcal{O}_Z)$ is K-equivariant. Now since Z is a K-torsor over S, it follows that $(V \otimes \mathcal{O}_Z) / \widetilde{\mathfrak{l}} \cdot (V \otimes \mathcal{O}_Z)$ descends as a $\mathcal{O}_T$-module on T which we denote by $\widehat{BB}(V)$. Moreover $\widehat{BB}(V)$ is also a $\mathcal{A}(\log \Delta_T)$-module. We now describe the fibers of the module $\widehat{BB}(V)$ at a point $t \in T$.

Consider the set $Z_t = \pi^{-1}(t)$. The group K acts freely transitively on Z_t and we consider the vector space $\mathcal{V}_t := Z_t \times_K V$. Similarly for every $t \in T$, set $\widehat{\mathfrak{l}}_t := Z_t \times_K \widehat{\mathfrak{l}}$. Since $\widetilde{\mathfrak{l}}$ is preserved by the K-action, we also a get a Lie subalgebra $\widetilde{\mathfrak{l}}_t$ of $\widehat{\mathfrak{l}}_t$. The fibers of $\widehat{BB}(V)$ at a point $t \in T$ are the coinvariants $\mathcal{V}_t / \widetilde{\mathfrak{l}}_t \cdot \mathcal{V}_t$.

5.4. **Coordinate Torsors.** In this section, we recall (following [36], [37]) natural coordinate torsors arising out of ramified cover of disks. First we discuss the absolute situation without covers.

5.4.1. *Formal coordinate on disks.* Let R be a ring, then following Section 4 in [72], we let $\mathcal{O}_R$ (respectively $\mathcal{K}_R$) denote the rings $R[[z]]$ (respectively $R((z))$). Then $\operatorname{Aut}\mathcal{O}_R$ (respectively $\operatorname{Aut}\mathcal{K}_R$) can be identified with space of formal R-power series (respectively formal

R-Laurent series) of the form

$$f(z) = \sum_{n \geq 1} c_n z^n \text{ (resp. } f(z) = \sum_{n \geq k} c_n z^n),$$

where $c_n \in R$ (respectively c_n is nilpotent if $n \leq 0$) and c_1 is an unit. The functor $R \to \mathrm{Aut}\, \mathcal{O}_R$ (respectively $R \to \mathrm{Aut}\, \mathcal{K}_R$) is represented by a group scheme (resp. ind-group scheme) which will be denoted by $\mathrm{Aut}\, \mathcal{O}$ (respectively $\mathrm{Aut}\, \mathcal{K}$). The Lie algebra $\mathrm{Der}^{(0)} \mathcal{O}$ of $\mathrm{Aut}\, \mathcal{O}$ (respectively $\mathrm{Der}\, \mathcal{K}$ of $\mathrm{Aut}\, \mathcal{K}$) is topologically generated by elements of the form $z^k \partial_z$ for $k \geq 1$(respectively $k \in \mathbb{Z}$).

Similarly, we denote by $\mathrm{Aut}_+ \mathcal{O}$ the subgroup scheme of $\mathrm{Aut}\, \mathcal{O}$ such that for any ring R, we have $\mathrm{Aut}_+ \mathcal{O}_R$ consists of R-power series of the form

$$f(z) = \sum_{n \geq 1} c_n z^n, \ c_1 = 1 \text{ and } c_n \in R.$$

We denote the Lie algebra of $\mathrm{Aut}_+ \mathcal{O}$ by $\mathrm{Der}_+ \mathcal{O}$. The following elementary lemma can be found in [36, Lemma 5.1.1].

Lemma 5.6. *With the above notation*

(1) $\mathrm{Aut}\, \mathcal{O}$ *is a semidirect product of* $\mathbb{G}_m$ *and* $\mathrm{Aut}_+ \mathcal{O}$.
(2) *The group* $\mathrm{Aut}_+ \mathcal{O}$ *is a prounipotent proalgebraic group. Moreover the exponential map from* $\mathrm{Der}_+ \mathcal{O}$ *to* $\mathrm{Aut}_+ \mathcal{O}$ *is an isomorphism.*

5.4.2. *Formal coordinates on disks with ramification.* Consider the automorphism of the ramified covering $R[[z^{\frac{1}{N}}]] \to R[[z^{\frac{1}{N}}]]$ given by sending $z^{\frac{1}{N}} \to \varepsilon z^{\frac{1}{N}}$, where ε is an N-th root of unity. Let as before $\mathrm{Aut}\, R[[z^{\frac{1}{N}}]]$ denote the automorphisms of $R[[z^{\frac{1}{N}}]]$ preserving the ideal $(z^{\frac{1}{N}})$. Let $\mathrm{Aut}_N \mathcal{O}_R$ denote the subgroup of $\mathrm{Aut}(R[[z^{\frac{1}{N}}]])$ preserving the subalgebra $R[[z]]$. Then there is an exact sequence

$$(5.2) \qquad\qquad 0 \to \mathbb{Z}/N\mathbb{Z} \to \mathrm{Aut}_N \mathcal{O}_R \to \mathrm{Aut}\, \mathcal{O}_R \to 0.$$

Following the absolute case, we consider the group scheme (ind group-scheme) $\mathrm{Aut}_N \mathcal{O}$ (respectively $\mathrm{Aut}_N \mathcal{K}$) and its Lie algebra $\mathrm{Der}_N^{(0)} \mathcal{O}$ (respectively $\mathrm{Der}_N \mathcal{K}$). We have the following explicit descriptions:

$$(5.3) \qquad \mathrm{Der}_N^{(0)} \mathbb{C}[[z^{\frac{1}{N}}]] = z^{\frac{1}{N}} \mathbb{C}[[z]] \partial_{z^{\frac{1}{N}}}, \ \mathrm{Der}_N \mathbb{C}((z^{\frac{1}{N}})) = z^{\frac{1}{N}} \mathbb{C}((z)) \partial_{z^{\frac{1}{N}}}.$$

There are natural homomorphisms between the corresponding groups (respectively ind-groups) $\mu : \mathrm{Aut}_N \mathcal{O} \to \mathrm{Aut}\, \mathcal{O}$ (respectively $\mathrm{Aut}_N \mathcal{K} \to \mathrm{Aut}\, \mathcal{K}$). By the exact sequence (5.2), μ is an isogeny whose derivative $z^{\frac{1}{N}+k} \partial_{z^{\frac{1}{N}}} \to N z^{k+1} \partial_z$ gives an isomorphism of the following Lie algebras:

$$(5.4) \qquad\qquad d\mu : \mathrm{Der}_N^{(0)} \mathcal{O} \to \mathrm{Der}^{(0)} \mathcal{O}, \ d\mu : \mathrm{Der}_N \mathcal{K} \to \mathrm{Der}\, \mathcal{K}.$$

Let $\overline{D}_R = \mathrm{Spec}\, R[[z]]$ be a formal R-disk and let x be a R-point of $\overline{D}_R$. A formal coordinate is an automorphism of $\overline{D}_R$ that identifies x with the origin given by the maximal

ideal $(z) \subseteq R[[z]]$. Let $\mathcal{A}ut(\overline{D}_{R,x})$ denote the set of formal coordinates on $\overline{D}_R$. Clearly it is an $\operatorname{Aut} \mathcal{O}_R$-torsor.

Similarly let $D_R = \operatorname{Spec} R((z^{\frac{1}{N}}))$ and let $\sigma : D_R \to D_R$ be an automorphism of order N that fixes a R-valued point x in D_R. Up to a change of coordinates, we can assume that σ takes the form $z^{\frac{1}{N}} \to \varepsilon z^{\frac{1}{N}}$ where ε is a N-th order root of unity. Now since $R[[z^{\frac{1}{N}}]]^{\mathbb{Z}/N} = R[[z]]$, hence $D_R \to \overline{D}_R$ can be considered as a cover of $\mathbb{Z}/N\mathbb{Z}$ ramified at the point x in D_R with order of ramification N.

Let $\mathcal{A}ut_N(D_{R,x})$ denote the set of special formal coordinates of D_R which are transformed by the rule $t \to \varepsilon t$, where ε is an N-th root of unity. As before this is an $\operatorname{Aut}_N \mathcal{O}_R$-torsor. Similarly we define $\operatorname{Aut}_{N,+} \mathcal{O}$ and $\operatorname{Der}_{N,+} \mathcal{O}$. Moreover there is a natural isomorphism between $\operatorname{Der}_{N,+} \mathcal{O}$ and $\operatorname{Der}_+ \mathcal{O}$.

5.4.3. *Global Situation.* Let R be a $\mathbb{C}$-algebra and consider a family $(\widetilde{C} \to C, \widetilde{\mathbf{p}}, \mathbf{p})$ of n-pointed admissible Γ-covers over $\operatorname{Spec} R$ in $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ with $\widetilde{\mathbf{p}} = (\widetilde{p}_1, \ldots, \widetilde{p}_n)$ being the n marked sections of $\widetilde{C}$, in particular we have $\Gamma \cdot \widetilde{p}_i \cap \Gamma \cdot \widetilde{p}_j = \emptyset$ for $i \neq j$, and the monodromies around the marked sections being $\mathbf{m} = (m_1, \cdots, m_n)$. The stabilizer $\Gamma_{\widetilde{p}_i}$ of $\widetilde{p}_i$ is the cyclic subgroup of Γ of order say N_i generated by m_i. Let $\mathcal{A}ut_{N_i}(D_{R,\widetilde{p}_i})$ be the set of special formal coordinates of the ring $\operatorname{Spec} \widehat{\mathcal{O}}_{\widetilde{p}_i}$ at the point $\widetilde{p}_i$ of the curve $\widetilde{C}$. We define the following:

$$(5.5) \qquad \operatorname{Aut}(\widetilde{C}/C, \mathcal{O}_R) \;=\; \operatorname{Aut}_{N_1}(\mathcal{O}_R) \times \cdots \times \operatorname{Aut}_{N_n}(\mathcal{O}_R).$$

$$(5.6) \qquad \operatorname{Aut}_+(\widetilde{C}/C, \mathcal{O}_R) \;=\; \operatorname{Aut}_{N_1,+}(\mathcal{O}_R) \times \cdots \times \operatorname{Aut}_{N_n,+}(\mathcal{O}_R).$$

$$(5.7) \qquad \operatorname{Aut}(\widetilde{C}/C, \mathcal{K}_R) \;=\; \operatorname{Aut}_{N_1} \mathcal{K}_{\mathcal{R}} \times \ldots \operatorname{Aut}_{N_n}(\mathcal{K}_R).$$

The corresponding group schemes are denoted by $\operatorname{Aut}(\widetilde{C}/C, \mathcal{O})$, $\operatorname{Aut}_+(\widetilde{C}/C, \mathcal{O})$ and $\operatorname{Aut}(\widetilde{C}/C, \mathcal{K})$ respectively. We further denote their Lie algebras by $\operatorname{Der}(\widetilde{C}/C, \mathcal{O})$, $\operatorname{Der}_+(\widetilde{C}/C, \mathcal{O})$ and $\operatorname{Der}(\widetilde{C}/C, \mathcal{K})$. The various versions of the moduli stacks of pointed admissible Γ-covers are related by the following proposition:

Proposition 5.7. *The moduli stacks* $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$, $\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ *and* $\widetilde{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ *are related as follows:*

(1) $\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ *is a* $\operatorname{Aut}(\widetilde{C}/C, \mathcal{O})$*-torsor over* $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$.

(2) $\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ *is a* $\operatorname{Aut}_+(\widetilde{C}/C, \mathcal{O})$*-torsor over* $\widetilde{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$.

(3) $\widetilde{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ *is a* $\mathbb{G}_{\mathrm{m}}^n$*-torsor over* $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$.

Let us define $\operatorname{Der}(\widetilde{C}/C, \mathcal{K})$ to be the Lie algebra of the ind-scheme $\operatorname{Aut}(\widetilde{C}/C, \mathcal{K})$. Now let $\operatorname{Aut}(C, \mathcal{O})$ (respectively $\operatorname{Aut}(C, \mathcal{K})$) denote the group scheme associated to the functor $R \to \operatorname{Aut}(\mathcal{O}_R) \times \cdots \times \operatorname{Aut}(\mathcal{O}_R)$ (respectively $R \to \operatorname{Aut}(\mathcal{K}_R) \times \cdots \times \operatorname{Aut}(\mathcal{K}_R)$ and $\operatorname{Der}^{(0)}(C, \mathcal{O})$ (respectively $\operatorname{Der}(C, \mathcal{K})$) denote their Lie algebras. Similarly define the Lie algebra $\operatorname{Der}_+(C, \mathcal{O})$. The isogeny μ in 5.4 gives the following isomorphisms between the

Lie algebras

$$(5.8) \qquad \operatorname{Der}^{(0)}(\widetilde{C}/C, \mathcal{O}) \;\simeq\; \operatorname{Der}^{(0)}(C, \mathcal{O}),$$

$$(5.9) \qquad \operatorname{Der}(\widetilde{C}/C, \mathcal{K}) \;\simeq\; \operatorname{Der}(C, \mathcal{K}),$$

$$(5.10) \qquad \operatorname{Der}_{+}(\widetilde{C}/C, \mathcal{O}) \;\simeq\; \operatorname{Der}_{+}(C, \mathcal{O}).$$

5.5. Equivariant Virasoro uniformization.
Let us recall the following result on Γ-equivariant Virasoro uniformization due to M. Szczesny [72, Theorem 7.1]. Strictly speaking Szczesny's result [72] is on the interior $\widehat{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$, however the arguments can be easily extended to the boundary. If Γ is trivial, the result is due to Tsuchimoto [75].

Theorem 5.8. *The stack* $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ *carries a log-transitive action of* $\operatorname{Der}(\widetilde{C}/C, \mathcal{K})$ *extending the action of* $\operatorname{Aut}(\widetilde{C}/C, \mathcal{O})$ *along the fibers of the natural map* $\widehat{\pi} : \widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m}) \to \overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ *and preserving the boundary divisor* $\widehat{\Delta}_{g,n,\Gamma} = \widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m}) \backslash \widehat{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$.

The following is an obvious variant of the above theorem where we replace $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ by $\widetilde{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$.

Corollary 5.9. *The same results in Theorem 5.8 holds if we consider the natural action of the group* $\operatorname{Aut}_{+}(\widetilde{C}/C, \mathcal{O})$ *and along the fibers of the natural map* $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m}) \to \widetilde{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$.

5.5.1. Virasoro Uniformization and the Hodge bundle.
Let $\widetilde{C} \to Z$ be a versal family in $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ parametrized by a smooth scheme Z. We let $\widetilde{\mathbf{z}}$ (respectively $\mathbf{z}$) be the corresponding choice of special local formal coordinates along the sections $\widetilde{\mathbf{p}}$ (respectively $\mathbf{p}$).

Let Δ_Z be the normal crossing divisor of nodal curves and as before Θ_{Z,Δ_Z} be the subsheaf of tangent vector fields on Z preserving the divisor Δ_Z. The Lie algebra $\operatorname{Der}(\widetilde{C}/C, \mathcal{K})$ acts log-transitively (see Definition 5.4 and the discussion afterward) on Z. In particular, we have a surjective Lie algebra map which we will call the *anchor map*:

$$(5.11) \qquad \alpha : \operatorname{Der}(\widetilde{C}/C, \mathcal{K}) \otimes \mathcal{O}_Z \twoheadrightarrow \Theta_{Z,\Delta_Z}.$$

Let $vec^{\Gamma}(\widetilde{C}\backslash\Gamma.(\widetilde{\mathbf{p}}))$ be the kernel of the map α. This is a sheaf of Lie algebra on $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ whose fiber at a point $(\pi : \widetilde{C} \to C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{z}})$ is given by the Lie algebra of Γ-invariant vector fields on $\widetilde{C}\backslash \cup_{i=1}^{n} \Gamma \cdot \widetilde{p}_i$. Since by assumption, the cover $\widetilde{C} \to C$ is étale restricted to $\widetilde{C}\backslash\Gamma.\widetilde{\mathbf{p}}$, it follows that a Γ-invariant vector field on $\widetilde{C}\backslash\Gamma.\widetilde{\mathbf{p}}$ descends to a vector field on $C\backslash\mathbf{p}$.

5.5.2. Uniformization and the associated Atiyah algebra.
The action of $\operatorname{Aut}(\widetilde{C}/C, \mathcal{K})$ on $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ [36, Theorem 17.3.2], [72] preserves the divisor $\widehat{\Delta}_{g,n,\Gamma}$. The Sugawara construction gives an action of $\operatorname{Der}(\widetilde{C}/C, \mathcal{K})$ on $\mathcal{H}_{\vec{\lambda}}$. Let $\widehat{\operatorname{Der}}(\widetilde{C}/C, \mathcal{K})$ be the central extension of

$\mathrm{Der}(\widetilde{C}/C, \mathcal{K})$ obtained as Beaer sum of the Virasoro cocycles for the individual factors. We get an short exact sequence

$$(5.12) \qquad 0 \to \mathcal{O}_{\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})} \cdot c \to \widehat{\mathrm{Der}}(\widetilde{C}/C, \mathcal{K}) \to \mathrm{Der}(\widetilde{C}/C, \mathcal{K}) \to 0$$

Moreover the short exact sequence splits when restricted to $vec^{\Gamma}(\widetilde{C}\backslash\Gamma \cdot \widetilde{\mathbf{p}})$ and the Lie algebra $\mathrm{Der}_+(\widetilde{C}/C, \mathcal{O})$. Taking quotients and using surjectivity of the short exact sequence in Equation (5.11), we get an Atiyah algebra with the following fundamental exact sequence.

$$(5.13) \qquad 0 \to \mathcal{O}_{\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})} \cdot c \to \widehat{\mathrm{Der}}(\widetilde{C}/C, \mathcal{K})/vec^{\Gamma}(\widetilde{C}\backslash\Gamma \cdot (\widetilde{\mathbf{p}})) \to \Theta_{\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m}), \Delta_{g,n,\Gamma}} \to 0.$$

We denote the log-Atiyah algebra

$$\hat{\mathcal{A}}(\widetilde{C}/C)(\log \hat{\pi}^{-1}(\Delta_{g,n,\Gamma})) := \widehat{\mathrm{Der}}(\widetilde{C}/C, \mathcal{K})/vec^{\Gamma}(\widetilde{C}\backslash\Gamma \cdot (\widetilde{\mathbf{p}})).$$

Since $\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m})$ is an $\mathrm{Aut}_+(\widetilde{C}/C, \mathcal{O})$-torsor over $\widetilde{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$, the sheaf $\hat{\mathcal{A}}(\widetilde{C}/C)(\log \hat{\pi}^{-1}(\Delta_{g,n,\Gamma}))$ descends to a log-Atiyah algebra on $\widetilde{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ which we denote by $\tilde{\mathcal{A}}(\widetilde{C}/C)(\log \tilde{\pi}^{-1}(\Delta_{g,n,\Gamma}))$.

5.5.3. *Virasoro Uniformization of the Hurwitz-Hodge bundle.* In this section, we give a more explicit description of the Atiyah algebra $\tilde{\mathcal{A}}(\widetilde{C}/C)(\log \tilde{\pi}^{-1}(\Delta_{g,n,\Gamma}))$. In the untwisted case (i.e Γ is trivial), the corresponding results can be found in [5, 18, 56, 75, 76]. Consider the natural map $\pi : \overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m}) \to \overline{\mathcal{M}}_{g,n}$ and let Λ be the pull back of the Hodge bundle to $\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}$. Then we have the following proposition:

Proposition 5.10. *Let $\mathcal{A}_\Lambda(\log \Delta_{g,n,\Gamma})$ denote the log-Atiyah algebra associated to the line bundle Λ on $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$, then there is a natural identification $\hat{\mathcal{A}}(\widetilde{C}/C)(\log \hat{\pi}^{-1}(\Delta_{g,n,\Gamma}))$ with $\frac{1}{2}\mathcal{A}_{\hat{\pi}^*\Lambda}(\log \Delta_{g,n,\Gamma})$.*

Proof. First we consider the case when Γ is trivial. In this case results in [5, 18, 56, 75, 76] show that there is a natural isomorphism of $\widehat{\mathrm{Der}}(C, \mathcal{K})/vec(C\backslash\mathbf{p})$ with $\frac{1}{2}\mathcal{A}_{\pi^*\Lambda}(\log \Delta_{g,n})$, where $\Delta_{g,n}$ is the boundary divisor of $\overline{\mathcal{M}}_{g,n}$ and $\hat{\underline{\pi}} : \widehat{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. Now consider the following commutative diagram:

$$
\begin{array}{ccc}
\widehat{\widetilde{\mathcal{M}}}_{g,n}^{\Gamma}(\mathbf{m}) & \xrightarrow{\hat{\pi}} & \overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m}) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
\widehat{\mathcal{M}}_{g,n} & \xrightarrow{\hat{\underline{\pi}}} & \overline{\mathcal{M}}_{g,n}
\end{array}
$$

Hence it suffice to show that $\pi^{\bullet}(\widehat{\mathrm{Der}}(C, \mathcal{K})/vec(C\backslash\mathbf{p}))$ is $\widehat{\mathrm{Der}}(\widetilde{C}/C, \mathcal{K})/vec^{\Gamma}(C\backslash\Gamma \cdot \mathbf{p})$. This follows from the isomorphism (5.4) of $\mathrm{Der}_N \mathcal{K}$ with $\mathrm{Der}\, \mathcal{K}$ and the isomorphism of $vec^{\Gamma}(\widetilde{C}\backslash\Gamma \cdot \widetilde{\mathbf{p}})$ with $vec(C\backslash\mathbf{p})$. $\qquad\square$

5.6. **Twisted log-D module.** In this section, we recall the construction of the twisted WZW-connection following [72] using the formalism of the *twisted localization functor* recalled in Section 5.3.1.

Remark 5.11. We want to address a small but important difference from [72]. The twisted vertex algebra arising from Kac-Moody algebras are not conformal and hence the coherent sheaf $\mathcal{V}^{\dagger}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{z}})$ does not descend to a coherent sheaf on $\overline{\mathcal{M}}^{\Gamma}_{g,n}(\mathbf{m})$ under the natural $\mathbb{G}^n_m$-action.

Let $\mathbf{m} = (m_1, \ldots, m_n) \in \Gamma^n$ be a monodromy vector. For each $1 \leq i \leq n$, let N_i be the order the m_i and $\mathcal{H}_{\lambda_i}$ is an irreducible highest weight integrable module for m_i-twisted Kac-Moody Lie algebra $L(\widehat{\mathfrak{g}}, m_i)$ of highest weight λ_i at level ℓ. As in Sections 4, 4.3.1 and [11, Remark 3.4.6], without loss of generality assume that we have a family of $(\widetilde{C} \to Z; \widetilde{\mathbf{p}}; \mathbf{p}; \widetilde{\mathbf{z}})$ in $\widehat{\widetilde{\mathcal{M}}}^{\Gamma}_{g,n}(\mathbf{m})$ such that $\widetilde{C}\backslash\widetilde{\mathbf{p}}(Z)$ is affine and let Δ_Z be the divisor in Z corresponding to singular curves. Following Looijenga [60, Section 5], we put the assumption that vector fields on Z tangent to Δ_Z are locally liftable to $\widetilde{C}$.

Similarly, we denote the corresponding family $(\widetilde{C} \to T; \widetilde{\mathbf{p}}; \mathbf{p}; \widetilde{\mathbf{v}})$ in $\widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{g,n}(\mathbf{m})$. The smooth scheme Z is a $\mathrm{Aut}_+(\widetilde{C}/C, \mathcal{O})$-torsor over T. We have the following diagram

$$
\begin{array}{ccccc}
\Delta_Z & \longrightarrow & Z & \longrightarrow & \widehat{\widetilde{\mathcal{M}}}^{\Gamma}_{g,n} \\
\downarrow & & \downarrow & & \downarrow \\
\Delta_T & \longrightarrow & T & \longrightarrow & \widetilde{\widetilde{\mathcal{M}}}^{\Gamma}_{g,n}
\end{array}
$$

For a fixed n-tuple of monodromies $\mathbf{m}$ and integrable highest weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, we consider the vector space $\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ and the quasi-coherent sheaf on Z.

$$\underline{\mathcal{H}}_{\vec{\lambda}} := \mathcal{H}_{\vec{\lambda}} \otimes \mathcal{O}_Z.$$

The Sugawara action gives an action of $\widehat{\mathrm{Der}}(\widetilde{C}/C, \mathcal{K})$ on $\underline{\mathcal{H}}_{\vec{\lambda}}$ and one naturally gets an action of the log-Atiyah algebra $\widehat{\mathcal{A}}(\widetilde{C}/C)(\log \Delta_Z)$ on the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{z}})$ on Z. Now we apply the formalism of *twisted log-localization* as described with Section 5.3.1 with $\widehat{\mathfrak{s}} = \widehat{\mathrm{Der}}(\widetilde{C}/C, \mathcal{K})$, $K = \mathrm{Aut}_+\mathcal{O}$, $V = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}$ and $\widetilde{\mathfrak{l}}$ be the $\mathrm{Aut}(\widetilde{C}/C, \mathcal{O})$-equivariant sheaf of Lie algebras whose fibers at a point $(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}})$ in T is given by $\left(\mathfrak{g} \otimes H^0(\widetilde{C}, \mathcal{O}_{\widetilde{C}}(*\Gamma.(\widetilde{\mathbf{p}})))\right)^{\Gamma}$ (see 7.2 in [72]). Hence the twisted log-localization gives the sheaf of conformal blocks $\mathcal{V}^{\dagger}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}})$ on T along with an action of the log-Atiyah algebra $\widetilde{\mathcal{A}}(\widetilde{C}/C)(\log \Delta_T)$. A direct corollary of the Proposition 5.10 is the following:

Corollary 5.12. *The log-Atiyah algebra* $\frac{\ell \dim \mathfrak{g}}{2(\ell+h^{\vee})}\mathcal{A}_{\widetilde{\pi}^*\Lambda}(\log \widetilde{\Delta}_{g,n,\Gamma})$ *acts on the dual twisted conformal block bundle or the sheaf of twisted covacua* $\mathcal{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}})$.

5.7. **Atiyah algebra for the twisted WZW connection on $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$.** Let $\mathbf{m} \in \Gamma^n$ be a monodromy vector and for an n-tuple of integral weights $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ at level ℓ, we denote the vector bundle of conformal blocks on $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ by $\mathbb{V}_{\vec{\lambda},\Gamma}^{\dagger}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$. Let Γ be a finite subgroup of $\mathrm{Aut}(\mathfrak{g})$ and let $L(\widehat{\mathfrak{g}}, \Gamma)$ be the Γ-twisted affine Kac-Moody Lie algebra.

As before, let $\widetilde{C} \to C$ be a ramified Galois cover of nodal curves and let $\widetilde{p}$ be a smooth ramification point of $\widetilde{C}$ whose image in C is denoted by p. Let Γ be the stabilizer at the point $\widetilde{p}$ and we further assume that N be the order of Γ. Let $\mathcal{K}_{\widetilde{p}}$ be the residue field of the completed local ring at $\widetilde{p}$ and let $\mathcal{K}_p$ denoted the corresponding residue field at p. The subfield of elements of $\mathcal{K}_{\widetilde{p}}$ under Γ is $\mathcal{K}_p$. Let $z^{\frac{1}{N}}$ is a special coordinate at the point $\widetilde{p}$, then there is an isomorphism of $\mathcal{K}_{\widetilde{p}} \simeq \mathbb{C}((z^{\frac{1}{N}}))$ and similarly $\mathcal{K}_p \simeq \mathbb{C}((z))$.

Let $R = \mathbb{C}[[z^{\frac{1}{N}}]]$. Recall that $\mathcal{A}ut_N(D_{R,\widetilde{p}})$ (respectively $\mathcal{A}ut_{N,+}(D_{R,\widetilde{p}})$) denote the set of special coordinates (respectively one-jets) of the point $\widetilde{p}$ which is an $\mathrm{Aut}_N(\mathcal{O}_R)$ (respectively $\mathrm{Aut}_{N,+}(\mathcal{O}_R)$)-torsor. The following lemma is due to [36].

Lemma 5.13. *Let V be a $\mathrm{Der}_N(\mathcal{O}_R)$-module such that*

(1) *the action $z^{\frac{1}{N}}\partial_{z^{\frac{1}{N}}}$ has integral eigenvalues on V.*

(2) *$z^{k+\frac{1}{N}}\partial_{z^{k+\frac{1}{N}}}$ is locally nilpotent for all $k \geq 1$.*

Then the action of $\mathrm{Der}_N(\mathcal{O}_R)$ can be exponentiated to an action of $\mathrm{Aut}_N(\mathcal{O}_R)$. Hence then the vector space $V_{\widetilde{p}} = \mathcal{A}ut_N(D_{R,\widetilde{p}}) \times_{\mathrm{Aut}_N(\mathcal{O}_R)} V$ is independent of the choice of special coordinates at the point $\widetilde{p}$. Similarly the vector space $\widetilde{V}_{\widetilde{p}} = \mathcal{A}ut_{N,+}(D_{R,\widetilde{p}}) \times_{\mathrm{Aut}_{N,+}(\mathcal{O}_R)} V$ is independent of the formal jets at the points $\widetilde{p}$.

The Lie algebra $\mathrm{Der}_N(\mathcal{O}_R)$ acts on $\mathcal{H}_\lambda$ by the twisted Sugawara action under the homomorphism

$$\mathrm{Der}_N(\mathcal{O}_R) \to \mathrm{Vir}_\Gamma, \ z^{k+\frac{1}{N}}\partial_{z^{\frac{1}{N}}} \to -NL_k^\Gamma$$

We want to apply the above construction to the $L(\widehat{\mathfrak{g}}, \Gamma)$-module $\mathcal{H}_\lambda$ via the twisted Sugawara action. It can be checked that condition (2) in Lemma 5.13 is satisfied. Thus we get that the following vector space is independent of the formal jets at the point $\widetilde{p}$.

$$\widetilde{\mathcal{H}}_{\lambda,\widetilde{p}} = \mathcal{A}ut_{N,+}(D_{R,\widetilde{p}}) \times_{\mathrm{Aut}_{N,+}(\mathcal{O}_R)} \mathcal{H}_\lambda$$

However it turns out that the eigenvalues of the *zero-th twisted Virasoro operators* L_0^Γ on $\mathcal{H}_\lambda$ are not in $\frac{1}{N}\mathbb{Z}$. In particular the eigenvalues of $z^{\frac{1}{N}}\partial_{z^{\frac{1}{N}}}$ are non integers. We refer the reader to [62] for similar issue for trivial Γ.

As in the untwisted case, let Δ_λ denote the eigenvalues of L_0^Γ on the degree zero part of $\mathcal{H}_\lambda$. We consider the vector space $\mathcal{H}_\lambda \otimes \mathbb{C}dz^{N\Delta_\lambda}$. By construction of the Sugawara action, the eigenvalues of $z^{\frac{1}{N}}\partial_{z^{\frac{1}{N}}}$ are of the form $-N\Delta_\lambda + \mathbb{Z}$, where Δ_λ is a rational number. The Lie algebra $\mathrm{Der}_N(\mathcal{O}_R)$ acts on $dz^{N\Delta_\lambda}$ via Lie derivatives. Clearly the action of $z^{k+\frac{1}{N}}\partial_{z^{\frac{1}{N}}}$ is trivial if $k > 0$ and $z^{\frac{1}{N}}\partial_{z^{\frac{1}{N}}}$ acts with eigenvalue $N\Delta_\lambda$. This implies that the action

of $\mathrm{Der}_N(\mathcal{O}_R)$ on $\mathcal{H}_\lambda \otimes \mathbb{C}dz^{N\Delta_\lambda}$ satisfy the conditions of Lemma 5.13. Hence the group $\mathrm{Aut}_N(\mathcal{O}_R)$ acts on $\mathcal{H}_\lambda \otimes \mathbb{C}dz^{N\Delta_\lambda}$. Thus the coordinate free highest weight irreducible integrable $\widehat{\mathfrak{g}}_{\widetilde{p}}$-module $\mathbb{H}_\lambda$ (see Section 4 for notation)of highest λ can be defined as

$$(5.14) \qquad \mathcal{H}_{\lambda,\widetilde{p}} := \mathcal{A}ut_N(D_{R,\widetilde{p}}) \times_{\mathrm{Aut}_N(\mathcal{O}_R)} (\mathcal{H}_\lambda \otimes \mathbb{C}dz^{N\Delta_\lambda}).$$

We have the following result which summarizes the discussions in this section:

Theorem 5.14. *Let Δ_{λ_i} be the eigenvalues of the zeroth Virasoro operator $L_0^{\Gamma_{\widetilde{p}_i}}$ on the degree zero part of the highest weight $L(\widehat{\mathfrak{g}}, \Gamma_{p_i})$-module with highest weight λ_i via the twisted Sugawara action. Then the Atiyah algebra*

$$\frac{\ell \dim \mathfrak{g}}{2(h^\vee(\mathfrak{g}) + \ell)}\mathcal{A}_\Lambda(\log \Delta_{g,n,\Gamma}) + \sum_{i=1}^n N_i \Delta_{\lambda_i} \mathcal{A}_{\widetilde{\mathcal{L}}_i}(\log \Delta_{g,n,\Gamma})$$

acts on $\mathbb{V}_{\vec{\lambda},\Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p})$, where $\widetilde{\mathcal{L}}_i$;s are tautological line bundles corresponding to the i-th psi classes and N_i's are the orders of the cyclic groups Γ_{p_i}.

Proof. First of all we observe that the $\widetilde{\overline{\mathcal{M}}}_{g,n}^\Gamma$ is a substack of the total stack of the $\mathbb{G}_m^n$-torsor on $\overline{\mathcal{M}}_{g,n}^\Gamma$ given by the line bundles $\widetilde{\mathcal{L}}_1, \ldots, \widetilde{\mathcal{L}}_n$. Hence the pull back of each $\widetilde{\mathcal{L}}_i$ to $\widetilde{\overline{\mathcal{M}}}_{g,n}^\Gamma$ is trivial. Let $\tilde{\pi} : \widetilde{\overline{\mathcal{M}}}_{g,n}^\Gamma \to \overline{\mathcal{M}}_{g,n}^\Gamma$ be the forgetful map and consider the Atiyah algebra

$$(5.15) \qquad \mathcal{A} := \frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}\tilde{\pi}^\bullet \mathcal{A}_\Lambda(\log \tilde{\pi}^{-1}\Delta_{g,n,\Gamma}) + \sum_{i=1}^n N_i \Delta_{\lambda_i} \tilde{\pi}^\bullet \mathcal{A}_{\widetilde{\mathcal{L}}_i}(\log \tilde{\pi}^{-1}\Delta_{g,n,\Gamma}),$$

Then by Proposition 5.10, Corollary 5.12 and the coordinate free construction (see (5.14)) tell us that the log-Atiyah algebra given by Equation (5.15) acts on $\mathbb{V} := \tilde{\pi}^*(\mathbb{V}_{\vec{\lambda},\Gamma}^\dagger(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}))$. i.e a Lie algebra homomorphism $\nabla : \mathcal{A} \to \mathcal{A}_\mathbb{V}$ which is also equivariant under the $\mathbb{G}_m^n$-torsor. Hence the result follows. $\qquad \square$

6. Γ-CROSSED MODULAR FUNCTORS

Let Γ be a finite group. In this section we define the notion of a complex algebraic Γ-crossed modular functor. We conjecture that this notion is equivalent to the notion of a weakly rigid Γ-crossed modular category. We will prove this conjecture in the second part of this paper. In this section all abelian categories that we consider are supposed to be $\mathbb{C}$-linear, even though we may not mention this explicitly and all additive functors considered are supposed to be $\mathbb{C}$-linear.

6.1. Γ-crossed abelian categories.
Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian category equipped with a linear action of a finite group Γ. For an object $M \in \mathcal{C}$ let $\widetilde{\Gamma}_M := \{(g, \psi)|g \in \Gamma, \psi : g(M) \xrightarrow{\cong} M\}$. We can define a group structure on $\widetilde{\Gamma}_M$ in the evident way and we obtain a central extension

$$(6.1) \qquad 1 \to \mathrm{Aut}(M) \to \widetilde{\Gamma}_M \to \Gamma_M \to 1,$$

where $\Gamma_M \leq \Gamma$ is the stabilizer of the isomorphism class of the object M.

Remark 6.1. We say that an object $M \in \mathcal{C}$ is Γ-invariant if $\Gamma_M = \Gamma$ and if the central extension above is split. Any Γ-invariant object in $\mathcal{C}$ can be naturally lifted to an object of the Γ-equivariantization $\mathcal{C}^\Gamma$, namely the object in the equivariantization corresponding to the Γ-invariant object M and the trivial representation of Γ.

Now for each $n \in \mathbb{Z}_{\geq 0}$, the tensor power $\mathcal{C}^{\boxtimes n}$ is a $\mathbb{C}$-linear abelian category equipped with an action of the wreath product $S_n \ltimes \Gamma^n$, where S_n is the symmetric group on n-letters acting on Γ^n by permutation of factors. Note that we have the diagonal subgroup $\Delta\Gamma \subseteq \Gamma^n$ or in other words, the fixed point subgroup in Γ^n for the S_n-action. The direct product $S_n \times \Delta\Gamma$ is a subgroup of the wreath product $S_n \ltimes \Gamma^n$ and hence it acts linearly on the abelian category $\mathcal{C}^{\boxtimes n}$.

Remark 6.2. By a symmetric Γ-invariant object in $\mathcal{C}^{\boxtimes n}$, we simply mean an $S_n \times \Delta\Gamma$-invariant object of $\mathcal{C}^{\boxtimes n}$.

Definition 6.3. By a ($\mathbb{C}$-linear) Γ-crossed abelian category we mean a ($\mathbb{C}$-linear) abelian category $\mathcal{C}$ along with

- a Γ-grading of abelian categories $\mathcal{C} = \bigoplus_{m \in \Gamma} \mathcal{C}_m$,
- a linear action of Γ on $\mathcal{C}$ such that $\gamma(\mathcal{C}_m) \subseteq \mathcal{C}_{\gamma m \gamma^{-1}}$ for all $\gamma, m \in \Gamma$,
- a Γ-invariant object (see Remark 6.1) $\mathbb{1} \in \mathcal{C}_1$ and
- a symmetric Γ-invariant object (see Remark 6.2) $R \in \mathcal{C}^{\boxtimes 2}$ such that $R = \bigoplus_{g \in \Gamma} R_m$

  with each $R_m \in \mathcal{C}_m \boxtimes \mathcal{C}_{m^{-1}}$.

Remark 6.4. If $\mathcal{C}$ is a Γ-crossed abelian category as above, then we have a Γ^n-grading of $\mathcal{C}^{\boxtimes n}$

$$\mathcal{C}^{\boxtimes n} = \bigoplus_{\mathbf{m} \in \Gamma^n} \mathcal{C}^{\boxtimes n}_{\mathbf{m}}, \text{ where } \mathcal{C}^{\boxtimes n}_{\mathbf{m}} := \mathcal{C}_{m_1} \boxtimes \cdots \boxtimes \mathcal{C}_{m_n} \subseteq \mathcal{C}^{\boxtimes n} \text{ for } \mathbf{m} = (m_1, \cdots, m_n).$$

6.2. **Factorization functors.** Let $\mathcal{C}$ be a Γ-crossed abelian category. Let $g \in \mathbb{Z}_{\geq 0}$, A a finite set and $(X, \mathbf{m}, \mathbf{b}) \in \mathcal{X}^\Gamma_{g,A}$ (see Definition B.4). With such a weighted A-legged Γ-graph we associate the abelian category $\mathcal{C}^{\boxtimes H(X)}_{\mathbf{m}}$. Now consider a morphism $(X, \mathbf{m}_X, \mathbf{b}_X) \xrightarrow{(f,\boldsymbol{\gamma})} (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ in $\mathcal{X}^\Gamma_{g,A}$. We have the associated functor

(6.2) $$\mathcal{R}_{f,\boldsymbol{\gamma}} : \mathcal{C}^{\boxtimes H(Y)}_{\mathbf{m}_Y} \longrightarrow \mathcal{C}^{\boxtimes H(X)}_{\mathbf{m}_X} \text{ defined as}$$

$$\left(\boxtimes_{h \in H(Y)} M_h \right) \mapsto \left(\boxtimes_{f^* h \in f^* H(Y)} \boldsymbol{\gamma}(h)^{-1}(M_h) \right) \boxtimes \left(\boxtimes_{\{h_1, h_2\} \in E(X) \setminus f^* E(Y)} (1, \mathbf{b}_x(h_1)) \cdot R_{\mathbf{m}_X(h_1)} \right).$$

Here we are just inserting the suitable translate of a graded component (to be more precise, the object $(1, \mathbf{b}_x(h_1)) \cdot R_{\mathbf{m}_X(h_1)} \in \mathcal{C}_{\mathbf{m}_X(h_1)} \boxtimes \mathcal{C}_{\mathbf{m}_X(h_2)}$) of the symmetric Γ-invariant object $R \in \mathcal{C}^{\boxtimes 2}$ at the edges of X which have been contracted by $f : X \to Y$. It is clear that if we have two morphisms

$$(X, \mathbf{m}_X, \mathbf{b}_X) \xrightarrow{f_1, \boldsymbol{\gamma_1}} (Y, \mathbf{m}_Y, \mathbf{b}_Y) \xrightarrow{f_2, \boldsymbol{\gamma_2}} (Z, \mathbf{m}_Z, \mathbf{b}_Z)$$

then we have a natural isomorphism between the two functors

$$(6.3) \qquad \mathcal{R}_{(f_2,\boldsymbol{\gamma_2})\circ(f_1,\boldsymbol{\gamma_1})} \cong \mathcal{R}_{f_1,\boldsymbol{\gamma_1}} \circ \mathcal{R}_{f_2,\boldsymbol{\gamma_2}} : \mathcal{C}_{\mathbf{m}_Z}^{\boxtimes H(Z)} \longrightarrow \mathcal{C}_{\mathbf{m}_X}^{\boxtimes H(X)}.$$

6.3. **Propagation of vacua functors.** Suppose as usual that g, A is a stable pair. Let B be any finite set and we can consider the disjoint union $A \sqcup B$. Consider the full subcategory $\mathcal{X}_{g,A,B}^{\Gamma} \subseteq \mathcal{X}_{g,A\sqcup B}^{\Gamma}$ of stable weighted $A \sqcup B$-legged Γ-graphs whose legs labeled by B are all marked by $1 \in \Gamma$. Then we have the functor $\mathcal{X}_{g,A,B}^{\Gamma} \longrightarrow \mathcal{X}_{g,A}^{\Gamma}$. This functor essentially just forgets the legs marked by B. However when we delete a leg from a vertex of a stable graph, that vertex might become unstable. In this case, the vertex must necessarily be of genus (i.e. weight) 0 and degree 2 with at least one incident half-edge being part of an edge to a different vertex. Hence in this case, we can erase the vertex and glue the two incident half-edges. In case one of the incident half-edges is a leg (marked by some group element), after erasing the unstable vertex we are left with a leg which we mark by the same group element. In case both the incident half-edges are parts of edges, then after erasing the vertex the two incident edges collapse into a single edge. We mark both the half-edges of this new edge by $1 \in \Gamma$. In this way we get a well defined functor $\pi_B : \mathcal{X}_{g,A,B}^{\Gamma} \longrightarrow \mathcal{X}_{g,A}^{\Gamma}$.

Here is another way to think about the previous construction. Let $(X, \mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}) \in \mathcal{X}_{g,A,B}^{\Gamma}$, where $\mathbf{m}_E$ corresponds to the marking data for the edges, $\mathbf{m}_A$ the marking data for the A-legs and with all the B-legs being marked by $1 \in \Gamma$. Consider the set of all vertices which will become unstable after deleting the B-legs. Now consider all the edges incident at these unstable vertices and contract these edges to obtain an edge contraction morphism

$$\mathrm{ctr}_B : (X, \mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}) \longrightarrow (X', \mathbf{m'}_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}') \text{ in } \mathcal{X}_{g,A,B}^{\Gamma}.$$

Now if we delete the B-legs, we will obtain the stable Γ-graph $(X'', \mathbf{m'}_{E'} \sqcup \mathbf{m}_A, \mathbf{b}') \in \mathcal{X}_{g,A}^{\Gamma}$ as desired. Note that here the stable weighted A-legged graph X'' is obtained from the stable weighted $A \sqcup B$-legged graph X' by deleting the B-legs.

We have an analogous construction at the level of moduli stacks. Let $(X, \mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}) \in \mathcal{X}_{g,A,B}^{\Gamma}$, as before. After applying the forgetful construction above we obtain the stable graph $(X'', \mathbf{m'}_{E'} \sqcup \mathbf{m}_A, \mathbf{b}') \in \mathcal{X}_{g,A}^{\Gamma}$. We can then apply the forgetting tails construction described in [51, §2.2] to obtain the map of stacks which factors through a clutching

$$(6.4) \qquad \overline{\mathcal{M}}_{X,\mathbf{m}_E\sqcup\mathbf{m}_A\sqcup\mathbf{1}_B,\mathbf{b}}^{\Gamma} \xrightarrow{\hspace{4cm}} \overline{\mathcal{M}}_{X'',\mathbf{m'}_{E'}\sqcup\mathbf{m}_A,\mathbf{b}'}^{\Gamma}$$

$$\xi_{\mathrm{ctr}_B} \searrow \qquad \nearrow \mathrm{forget}_B$$

$$\overline{\mathcal{M}}_{X',\mathbf{m'}_{E'}\sqcup\mathbf{m}_A\sqcup\mathbf{1}_B,\mathbf{b}'}^{\Gamma}.$$

At the level of categories of twisted logarithmic $\mathcal{D}$-modules of central charge $c \in \mathbb{C}$ we obtain the diagram of functors

(6.5)

$$
D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_{X,\mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}}^{\Gamma}) \xleftarrow{\operatorname{Sp}_{X,\mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}}^{B}} D_c \operatorname{Mod}(\overline{\mathcal{M}}_{X'',\mathbf{m}'_{E'} \sqcup \mathbf{m}_A, \mathbf{b}'}^{\Gamma})
$$

$$
\operatorname{Sp}_{\operatorname{ctr}_B} \qquad \operatorname{forget}_B^*
$$

$$
D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_{X',\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}'}^{\Gamma}).
$$

Note that the functor $\operatorname{Sp}_{X',\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B}^{B}$ is simply the functor $\operatorname{forget}_B^*$.

Next suppose that $\mathcal{C}$ is a Γ-crossed abelian category. Then we obtain the commutative diagram of functors

(6.6)

$$
\mathcal{C}_{\mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B}^{\boxtimes H(X)} \xleftarrow{\mathcal{R}_{X,\mathbf{m}_E \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}}^{B}} \mathcal{C}_{\mathbf{m}'_{E'} \sqcup \mathbf{m}_A}^{\boxtimes H(X'')}
$$

$$
\mathcal{R}_{\operatorname{ctr}_B} \qquad \mathcal{R}_{X',\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}'}^{B}
$$

$$
\mathcal{C}_{\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B}^{\boxtimes H(X')},
$$

where the functor

(6.7) $\mathcal{R}_{X',\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B, \mathbf{b}'}^{B} : \mathcal{C}_{\mathbf{m}'_{E'} \sqcup \mathbf{m}_A}^{\boxtimes H(X'')} \longrightarrow \mathcal{C}_{\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B}^{\boxtimes H(X')} = \mathcal{C}_{\mathbf{m}'_{E'} \sqcup \mathbf{m}_A \sqcup \mathbf{1}_B}^{\boxtimes H(X'') \sqcup B}$ is given by

$$
\left(\underset{h \in H(X'')}{\boxtimes} M_h \right) \mapsto \left(\underset{h \in H(X'')}{\boxtimes} M_h \right) \boxtimes \left(\underset{b \in B}{\boxtimes} \mathbb{1} \right),
$$

i.e. we insert the Γ-invariant object $\mathbb{1}$ at all the B-legs.

To summarize for any $(X, \mathbf{m}, \mathbf{b}) \in \mathfrak{X}_{g,A,B}^{\Gamma} \subseteq \mathfrak{X}_{g,A \sqcup B}^{\Gamma}$ we have defined the contracted stable A-legged Γ-graph $\pi_B(X, \mathbf{m}, \mathbf{b}) =: (X'', \mathbf{m}'', \mathbf{b}'') \in \mathfrak{X}_{g,A}^{\Gamma}$. Then we have defined the functor

(6.8) $$\operatorname{Sp}_{X,\mathbf{m},\mathbf{b}}^{B} : D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_{X'',\mathbf{m}'',\mathbf{b}''}^{\Gamma}) \longrightarrow D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_{X,\mathbf{m},\mathbf{b}}^{\Gamma}).$$

Furthermore, if $\mathcal{C}$ is a Γ-crossed abelian category we have functors

(6.9) $$\mathcal{R}_{X,\mathbf{m},\mathbf{b}}^{B} : \mathcal{C}_{\mathbf{m}''}^{\boxtimes H(X'')} \longrightarrow \mathcal{C}_{\mathbf{m}}^{\boxtimes H(X)}.$$

6.4. **The category $\mathfrak{X}^{\Gamma}$.** We now define the category $\mathfrak{X}^{\Gamma}$ of stable weighted legged Γ-graphs. An object is simply a stable weighted legged Γ-graph $(X, \mathbf{m} \in \Gamma^{H(X)}, \mathbf{b} \in \Gamma^{H(X) \setminus L(X)})$ of any genus as before, except we do not identify the set of legs with any fixed set. A morphism

$$(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y) \text{ in } \mathfrak{X}^{\Gamma}$$

is the data of a surjective map $f_* : V(X) \twoheadrightarrow V(Y)$ compatible with the weights (see condition (B.1)), an injective map $f^* : H(Y) \hookrightarrow H(X)$ such that the legs of $L(X)$ which do not occur in the image $f^* L(Y)$ are all marked by the element $1 \in \Gamma$ and where $\boldsymbol{\gamma} \in \Gamma^{H(Y)}$

is as in Definition B.4, i.e. it conjugates the Γ-markings on $f^*H(Y) \subseteq H(X)$ to the Γ-markings on $H(Y)$ and $\boldsymbol{\gamma}$ at the two half-edges which are part of an edge are compatible as in Definition B.4. The only difference from Definition B.4 is that the morphisms are now allowed to forget some legs of $(X, \mathbf{m}, \mathbf{b})$ if they are marked by $1 \in \Gamma$. Note that as before if we have a morphism $(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ in $\mathfrak{X}^\Gamma$, then the genus of X and Y must be the same.

Let us now rephrase our constructions in terms of the category $\mathfrak{X}^\Gamma$. As before for any $(X, \mathbf{m}, \mathbf{b}) \in \mathfrak{X}^\Gamma$ we have the smooth Deligne-Mumford stack $\overline{\mathcal{M}}^\Gamma_{X,\mathbf{m},\mathbf{b}}$. If $(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ in $\mathfrak{X}^\Gamma$ is a morphism then we have corresponding maps of stacks $\xi_{f,\boldsymbol{\gamma}} : \overline{\mathcal{M}}^\Gamma_{X,\mathbf{m}_X,\mathbf{b}_X} \to \overline{\mathcal{M}}^\Gamma_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$. These maps are defined in terms of clutching/gluing and forgetting some marked points with trivial monodromy. These maps preserve the corresponding Hodge line bundles under pullback.

If $c \in \mathbb{C}$ we have the abelian category $D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^\Gamma_{X,\mathbf{m}_X,\mathbf{b}_X})$ of twisted logarithmic $\mathcal{D}$-modules. If $(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ is a morphism in $\mathfrak{X}^\Gamma$ we have the associated functor (see Appendix B)

$$\operatorname{Sp}_{f,\boldsymbol{\gamma}} : D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^\Gamma_{Y,\mathbf{m}_Y,\mathbf{b}_Y}) \to D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^\Gamma_{X,\mathbf{m}_X,\mathbf{b}_X})$$

and the assignment $(X, \mathbf{m}_X, \mathbf{b}_X) \mapsto D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^\Gamma_{X,\mathbf{m}_X,\mathbf{b}_X})$ is functorial in $\mathfrak{X}^\Gamma$.

If $(\mathcal{C}, \mathbb{1}, R)$ is a Γ-crossed abelian category, then associated with each morphism

$$(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y)$$

we have the functor

$$\mathcal{R}_{f,\boldsymbol{\gamma}} : \mathcal{C}^{\boxtimes H(Y)}_{\mathbf{m}_Y} \longrightarrow \mathcal{C}^{\boxtimes H(X)}_{\mathbf{m}_X}$$

defined by inserting suitable translates of the corresponding direct summands of the symmetric Γ-invariant object R at the contracted edges and $\mathbb{1} \in \mathcal{C}$ at the deleted 1-marked legs. The assignment $(X, \mathbf{m}_X, \mathbf{b}_X) \mapsto \mathcal{C}^{\boxtimes H(X)}_{\mathbf{m}_X}$ is functorial in $\mathfrak{X}^\Gamma$.

6.5. $\mathcal{C}$-**extended Γ-crossed modular functors.** Let $\mathcal{C}$ be a Γ-crossed abelian category. We will now define the notion of a $\mathcal{C}$-extended Γ-crossed modular functor extending the notion of a modular functor as defined in [7, Ch. 6].

Definition 6.5. Let $\mathcal{C}$ be a Γ-crossed abelian category with Γ-invariant object $\mathbb{1} \in \mathcal{C}$ and symmetric Γ-invariant object $R = \bigoplus_{m \in \Gamma} R_m \in \mathcal{C}^{\boxtimes 2}$. Let $c \in \mathbb{C}$. A $\mathcal{C}$-extended modular functor of central charge c consists of the following data:

1. For each stable pair (g, A) and $\mathbf{m} \in \Gamma^A$ a conformal blocks functor

$$\mathcal{V}_{g,A,\mathbf{m}} : \mathcal{C}^{\boxtimes A}_{\mathbf{m}} \longrightarrow D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^\Gamma_{g,A}(\mathbf{m})).$$

Once we have functors as above, we can canonically extend them to obtain functors

$$\mathcal{V}_{X,\mathbf{m},\mathbf{b}} : \mathcal{C}^{\boxtimes H(X)}_{\mathbf{m}} \longrightarrow D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^\Gamma_{X,\mathbf{m},\mathbf{b}})$$

for each $(X, \mathbf{m}, \mathbf{b}) \in \mathfrak{X}^\Gamma$. We will often abuse notation and denote the functors $\mathcal{V}_{X,\mathbf{m},\mathbf{b}}$ as just $\mathcal{V}$.

2. For each morphism $(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ in $\mathfrak{X}^\Gamma$ a natural isomorphism $G_{f,\boldsymbol{\gamma}}$ as below between the two functors from $\mathcal{C}_{\mathbf{m}_Y}^{\boxtimes H(Y)}$ to $D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_{X,\mathbf{m}_X,\mathbf{b}_X}^\Gamma)$:

$$G_{f,\boldsymbol{\gamma}} : \mathcal{V}_{X,\mathbf{m}_X,\mathbf{b}_X} \circ \mathcal{R}_{f,\boldsymbol{\gamma}} \longrightarrow \operatorname{Sp}_{f,\boldsymbol{\gamma}} \circ \mathcal{V}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$$

compatible with compositions of morphisms in $\mathfrak{X}^\Gamma$.

3. A normalization $\mathcal{V}_{0,3,\mathbf{1}}(\mathbb{1} \boxtimes \mathbb{1} \boxtimes \mathbb{1})(\mathbb{P}^1 \times \Gamma \to \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}}) \cong \mathbb{C}$, where the 3 marked points $\widetilde{\mathbf{p}}$ in $\mathbb{P}^1 \times \{1\} \subseteq \mathbb{P}^1 \times \Gamma$ are the 3 roots of unity in $\mathbb{C}$ and with tangent vectors being the inward pointing unit vectors.

We will often abuse notation and denote all functors $\mathcal{V}_{X,\mathbf{m},\mathbf{b}}$ simply by $\mathcal{V}$. Also if $(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q}, \widetilde{\mathbf{w}}) \in \widetilde{\overline{\mathcal{M}}}_{X,\mathbf{m},\mathbf{b}}^\Gamma$ and $\mathbf{M} \in \mathcal{C}_{\mathbf{m}}^{\boxtimes H(X)}$, the vector space $\mathcal{V}_\mathbf{M}(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q}, \widetilde{\mathbf{w}})$ is defined as the fiber of the twisted $\mathcal{D}$-module $\mathcal{V}_{X,\mathbf{m},\mathbf{b}}(\mathbf{M})$ at the point $(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q}, \widetilde{\mathbf{w}})$.

Remark 6.6. Since morphisms in $\mathfrak{X}^\Gamma$ involve contracting edges as well as forgetting 1-marked legs, the condition 2 above gives the factorization isomorphisms, the propagation of vacua as well as all the compatibilities mentioned in [7, §6.7]. When Γ is the trivial group, it is clear that the above definition agrees with the one in *loc. cit.*

It is also useful to define the notion of a $\mathcal{C}$-extended Γ-crossed modular functor in genus 0.

Definition 6.7. Let $\mathfrak{X}_0^\Gamma \subseteq \mathfrak{X}^\Gamma$ denote the full subcategory formed by graphs of genus 0, i.e. those graphs which are trees and all of whose vertices have weight 0. The notion of a $\mathcal{C}$-extended Γ-crossed modular functor in genus 0 is obtained by replacing the category $\mathfrak{X}^\Gamma$ with the full subcategory $\mathfrak{X}_0^\Gamma$ in Definition 6.5.

Remark 6.8. Note that in genus 0, the Hodge bundles on $\widetilde{\overline{\mathcal{M}}}_{0,A}^\Gamma$ are all trivial and hence we do not need to consider the central charge and each $\mathcal{V}(\mathbf{M})$ is a D-module on $\widetilde{\overline{\mathcal{M}}}_{0,A}^\Gamma$ with log-singularities at the boundary.

6.6. **The neutral modular functor.** Let $(\mathcal{C} = \bigoplus_{m \in \Gamma} \mathcal{C}_m, \mathbb{1}, \bigoplus_{m \in \Gamma} R_m)$ be a Γ-crossed abelian category and suppose we are given a $\mathcal{C}$-extended Γ-crossed modular functor $\mathcal{V}$ of additive central charge $c \in \mathbb{C}$. We will now define a $\mathcal{C}_1$-extended modular functor $\mathcal{V}_1$ in the sense of [7, Ch. 6] with the same central charge c.

Let $X \in \mathfrak{X}$ be a stable weighted legged graph and let $\mathbf{M} \in \mathcal{C}_1^{\boxtimes H(X)}$. Then we want to define $\mathcal{V}_1(\mathbf{M}) \in D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_X)$. We have a functor $\mathfrak{X} \to \mathfrak{X}^\Gamma$ defined on objects by $X \mapsto (X, \mathbf{1}, \mathbf{1}_{H(X) \backslash L(X)})$ (mark all half-edges by $1 \in \Gamma$) and on morphisms by $f \mapsto (f, \mathbf{1})$. We also have a morphism of stacks $i_X : \widetilde{\overline{\mathcal{M}}}_X \to \widetilde{\overline{\mathcal{M}}}_{X,\mathbf{1},\mathbf{1}}^\Gamma$ defined by $(D, \mathbf{q}, \mathbf{w}) \mapsto (D \times \Gamma \to D, \mathbf{q} \times 1, \mathbf{q}, \mathbf{w} \times 1)$. Since $\mathcal{V}$ is a $\mathcal{C}$-extended Γ-crossed modular functor, we have the twisted $\mathcal{D}$-module $\mathcal{V}(\mathbf{M})$ on $\widetilde{\overline{\mathcal{M}}}_{X,\mathbf{1},\mathbf{1}}^\Gamma$. We define $\mathcal{V}_1(\mathbf{M})$ to be the pullback $i_X^* \mathcal{V}(\mathbf{M}) \in D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}_X)$.

We can perform the same construction for the genus 0 case. Hence in this setting we obtain using [7]:

Proposition 6.9. *(i) Given a $\mathcal{C}$-extended Γ-crossed modular functor $\mathcal{V}$ in genus 0, the functor $\mathcal{V}_1$ defined above is a $\mathcal{C}_1$-extended modular functor in genus 0. In particular, this defines the structure of a weakly ribbon category on $\mathcal{C}_1$ and the action of Γ on $\mathcal{C}_1$ respects this structure.*
(ii) Given a $\mathcal{C}$-extended Γ-crossed modular functor $\mathcal{V}$ with central charge c, the functor $\mathcal{V}_1$ defined above is a $\mathcal{C}_1$-extended modular functor with central charge c.

Remark 6.10. Unlike rigid categories, in monoidal r-categories, the weak duality functor $X \mapsto X^*$ is not necessarily monoidal. However, by [20] the double duality functor is indeed monoidal and also braided in case the original category is braided. A ribbon r-category is a braided monoidal r-category equipped with a monoidal isomorphism between the identity functor and the double duality functor, $X \cong X^{**}$, satisfying a certain balancing property. See [20] for details.

6.7. **Some conjectures.** In this section, we state some conjectures which generalize known results in the untwisted case. We will prove these conjectures in part II of the paper.

Suppose that $(\mathcal{C}, \mathbb{1}, R)$ is a finite semisimple Γ-crossed abelian category such that $\mathbb{1} \in \mathcal{C}_1$ is a simple object. In this case, we conjecture that the notion of a $\mathcal{C}$-extended Γ-crossed modular functor in genus 0 is equivalent to giving $\mathcal{C}$ the additional structure of a Γ-crossed weakly fusion ribbon category. In a weakly rigid category we have natural isomorphisms $\mathrm{Hom}(M, M') \cong \mathrm{Hom}(\mathbb{1}, {}^*M \otimes M')$. Now given a $\mathcal{C}$-extended Γ-crossed modular functor we want to define a tensor product, i.e. given $\mathbf{m} \in \Gamma^n, \mathbf{M} = M_1 \boxtimes \cdots \boxtimes M_n \in \mathcal{C}_{\mathbf{m}}^{\boxtimes n}$, we want to describe the tensor product $M_1 \otimes \cdots \otimes M_n \in \mathcal{C}_{m_1 m_2 \cdots m_n} \subseteq \mathcal{C}$.

Given a $\mathcal{C}$-extended Γ-crossed modular functor, let us describe the desired morphism space $\mathrm{Hom}(\mathbb{1}, M_1 \otimes \cdots \otimes M_n)$. If the product $m_1 \cdots m_n \neq 1$, then this space is 0. Hence suppose that $m_1 \cdots m_n = 1$. Consider the n-marked curve $(\mathbb{P}^1, \mu_n, \mu_n)$ where the n marked points are the n-th roots of unity $\mu_n \subseteq \mathbb{C} \subseteq \mathbb{P}^1$ and for each marked point $\omega \in \mu_n$, the associated tangent vector is again ω considered as a tangent vector at ω. Consider 0 as the basepoint on $\mathbb{P}^1 \setminus \mu_n$. Consider loops γ_j in $\mathbb{P}^1 \setminus \mu_n$ based at 0 and encircling the point $e^{\frac{2\pi\sqrt{-1}j}{n}}$ counterclockwise. Then we obtain a presentation of the fundamental group $\pi_1(\mathbb{P}^1 \setminus \mu_n, 0) = \langle \gamma_1, \cdots, \gamma_n | \gamma_1 \cdots \gamma_n = 1\rangle$. Hence if $m_1 \cdots m_n = 1$, then $\gamma_j \mapsto m_j$ defines a group homomorphism $\phi : \pi_1(\mathbb{P}^1 \setminus \mu_n, 0) \to \Gamma$. Then as described in [51, §2.4] this determines an n-marked admissible cover $(\widetilde{C} \to \mathbb{P}^1, \widetilde{\mathbf{p}}, \mu_n, \widetilde{\mathbf{v}})$. Then we want to define a tensor product on $\mathcal{C}$ which satisfies:

$$(6.10) \qquad \mathrm{Hom}(\mathbb{1}, M_1 \otimes \cdots \otimes M_n) = \mathcal{V}_{\mathbf{M}}(\widetilde{C} \to \mathbb{P}^1, \widetilde{\mathbf{p}}, \mu_n, \widetilde{\mathbf{v}}).$$

We state this as a conjecture which will be proved in a future paper.

Conjecture 6.11. *Let $(\mathcal{C}, \mathbb{1}, R)$ be a finite semisimple Γ-crossed abelian category with $\mathbb{1}$ being a simple object.*
(i) The notion of a $\mathcal{C}$-extended Γ-crossed modular functor in genus 0 is equivalent to equipping $\mathcal{C}$ with the structure of a braided Γ-crossed weakly ribbon category.

(ii) Let $\mathcal{C}$ be a Γ-crossed modular category, i.e. a faithfully graded braided Γ-crossed ribbon category with $\mathcal{C}_1$ being a modular category. Then the corresponding Γ-crossed modular functor in genus 0 can be extended to arbitrary genus. Partially conversely, suppose that the Γ-crossed abelian category is faithfully graded and that we are given a $\mathcal{C}$-extended Γ-crossed modular functor and suppose that the corresponding weakly fusion category is rigid. Then in fact $\mathcal{C}$ is a Γ-crossed modular category.

7. Γ-crossed modular functors from twisted conformal blocks

In this section, we discuss and conjecture how twisted conformal blocks associated to Galois covers of curves give a $\mathcal{C}$-extended Γ-crossed modular functor. We follow the notations of Sections 6.1 and Section 4.

7.1. Γ-crossed abelian category.
Recall that for each $\gamma \in \Gamma$, we have the set $P^\ell(\mathfrak{g}, \gamma)$ that parametrizes representations of the twisted affine Kac-Moody Lie algebra $\widehat{L}(\mathfrak{g}, \gamma)$. We set $\mathcal{C}_\gamma$ to be the $\mathbb{C}$-linear abelian category whose simple objects are parametrized by $P^\ell(\mathfrak{g}, \gamma)$. We define $\mathcal{C} := \oplus_{\gamma \in \Gamma} \mathcal{C}_\gamma$ which is Γ-graded.

7.1.1. Γ-action on $\mathcal{C}$.
Let σ be an automorphism on $\mathfrak{g}$ of order $|\sigma|$. Recall from Section 3.3, the eigenspaces $\mathfrak{g}_{\sigma, j}$ of $\mathfrak{g}$ with eigenvalue $\exp \frac{2\pi\sqrt{-1}j}{|\sigma|}$, where $0 \leq j < |\sigma|$. Now if Γ is a group acting on $\mathfrak{g}$ and let γ and σ be any two elements of Γ. Consider the map $\mathfrak{g}_{\sigma, j} \to \mathfrak{g}_{\gamma\sigma\gamma^{-1}, j}$ obtained by sending X to γX induces an isomorphism of the affine Lie algebras

$$\psi_\gamma : \widehat{L}(\mathfrak{g}, \sigma) \to \widehat{L}(\mathfrak{g}, \gamma\sigma\gamma^{-1})$$

Since the group Γ acts on $\mathfrak{g}$, it follows that $\psi_{\gamma_1\gamma_2} = \psi_{\gamma_1} \circ \psi_{\gamma_2}$, where γ_1 and γ_2 are arbitrary elements of Γ. This in turns induces an action on the set of the simple objects $\oplus_{\gamma \in \Gamma} P^\ell(\mathfrak{g}, \gamma)$ of $\mathcal{C}$. Observe that ψ_σ induces the identity morphism on $\mathfrak{g}^\sigma$ and hence it acts by identity on $P^\ell(\mathfrak{g}, \sigma)$. Extending the action $\mathbb{C}$-linearly, we get an action on $\mathcal{C}$.

7.1.2. *Invariant Object.*
Now observe that the simple objects of $\mathcal{C}_1$ are just elements in $P_\ell(\mathfrak{g})$. This has a special object corresponding to the vacuum representation of the untwisted affine Kac-Moody Lie algebra which we declare to be the Γ-*invariant object* $\mathbb{1}$.

7.1.3. *Symmetric Tensor.*
Let $\mathfrak{g}^\gamma$ be the Lie subalgebra of $\mathfrak{g}$ fixed by γ. For every weight $\mu \in P_+(\mathfrak{g}^\gamma)$, let μ^* be the dominant integral weight of $\mathfrak{g}^\gamma$ which is the highest weight of the dual representation V_μ. Now it is easy to see (Lemma 5.3 in [46]) that $\mu \in P^\ell(\mathfrak{g}, \gamma)$ if and only if $\mu^* \in P^\ell(\mathfrak{g}, \gamma^{-1})$. We define the following:

$$(7.1) \qquad R_\gamma := \sum_{\mu \in P^\ell(\mathfrak{g}, \gamma)} \mu \boxtimes \mu^* \in R_\gamma \boxtimes R_{\gamma^{-1}}, \text{ and } R := \oplus_{\gamma \in \Gamma} R_\gamma.$$

By the definition of the action of Γ on $\mathcal{C}$, the object R is clearly Γ-invariant. Thus we have checked that the Γ-graded abelian category satisfies all the conditions of Definition 6.3.

7.2. **Twisted conformal blocks.** For each stable pair (g, A) and $\mathbf{m} \in \Gamma^A$ (see Definition 6.5 for notation) and $\vec{\lambda} \in \mathcal{C}_{\mathbf{m}}^{\boxtimes A}$, we assign the vector bundle of twisted covacua $\mathcal{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}}, \mathbf{v})$ on $\overline{\widetilde{\mathcal{M}}}_{g,A}^{\Gamma}(\mathbf{m})$. By Theorem 5.14, we know that that the log Atiyah algebra $\mathcal{A}_\Lambda(\log \widetilde{\Delta}_{g,A}^{\Gamma}(\mathbf{m}))$ acts on $\mathcal{V}_{\vec{\lambda}, \Gamma}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}, \widetilde{\mathbf{v}}, \mathbf{v})$ with central charge $\frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}$. Here Λ denote the pull back of the Hodge bundle of $\overline{\widetilde{\mathcal{M}}}_{g,A}$ to $\overline{\widetilde{\mathcal{M}}}_{g,A}^{\Gamma}(\mathbf{m})$ and $\widetilde{\Delta}_{g,A}^{\Gamma}(\mathbf{m})$ denotes the boundary divisor of $\overline{\widetilde{\mathcal{M}}}_{g,A}^{\Gamma}(\mathbf{m})$. Thus this assignment defines a functor from $\mathcal{C}_{\mathbf{m}}^{\boxtimes A}$ to $D_c \operatorname{Mod}\left(\overline{\widetilde{\mathcal{M}}}_{g,A}^{\Gamma}(\mathbf{m})\right)$, where $c = \frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}$. It is clear that the above assignment, satisfies Condition (3) of Definition 6.5. Motivated by the results of [12], we have the following conjecture:

Conjecture 7.1. *The assignment above realizes sheaf of twisted covacua as a $\mathcal{C}$-extended Γ-crossed modular functor with central charge $\frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}$, i.e. it satisfies the axioms of Definition 6.5.*

7.3. **Some consequences of the conjectures.** Let us first state the following well known result in the untwisted case. We continue to use the same notation as before, in particular we consider the category $\mathcal{C}_1$ of level ℓ representations of the untwisted affine lie algebra.

Theorem 7.2. *The conformal blocks for the untwisted affine Lie algebra define a $\mathcal{C}_1$-extended modular functor of central charge $\frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}$. The associated weak ribbon category structure endowed on $\mathcal{C}_1$ is in fact rigid, and hence $\mathcal{C}_1$ is a modular fusion category with additive central charge $\frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}$.*

Remark 7.3. We remark that the rigidity statement above follows from the works [49, 48].

We can now deduce the following consequences from Conjectures 6.11 and 7.1. Firstly, using both the conjectures, twisted conformal blocks equip $\mathcal{C}$ with the structure of a braided Γ-crossed weakly ribbon category. Moreover, by the theorem above, $\mathcal{C}_1$ is rigid. Hence by Corollary 2.6 the whole of $\mathcal{C}$ must be rigid. Hence we obtain the following consequence of the conjectures:

Consequence 7.4. *Twisted conformal blocks equip the semisimple Γ-crossed abelian category $\mathcal{C}$ with the structure of a Γ-crossed modular fusion category of additive central charge $\frac{\ell \dim \mathfrak{g}}{2(\ell + h^\vee(\mathfrak{g}))}$.*

In particular, we will then have the associated crossed S-matrices. In the remainder of this paper, we give an explicit conjectural description of the associated crossed S-matrices. If Conjectures 6.11 and 7.1 hold, then it will also follow that these are the actual crossed S-matrices we want.

Remark 7.5. Let γ be any automorphism of $\mathfrak{g}$. Then γ induces an automorphism of the abstract Cartan algebra $\mathfrak{h}$ as well as of the abstract root system preserving positive roots. Using Lemma 2.1 in [3], we get a canonical isomorphism $T_\gamma : {}^\gamma V_\lambda \xrightarrow{\cong} V_{\gamma \cdot \lambda}$, where V_λ is a

finite dimensional irreducible $\mathfrak{g}$-representation of highest weight λ. If a group Γ acts on $\mathfrak{g}$, then we have an induced Γ-action on the root system preserving the positive roots and let Γ_λ be the stabilizer of λ. In particular for each $\gamma \in \Gamma_\lambda$, we have a canonical isomorphism $T_\gamma : {}^\gamma V_\lambda \xrightarrow{\cong} V_\lambda$. Using this we can ensure that the cocycle φ_ρ in the statement of Theorem 2.15 is trivial for twisted conformal blocks. Hence we obtain:

Consequence 7.6. *The Verlinde formula (Conjecture 1.1) holds for twisted conformal blocks associated with Γ-twisted affine Lie algebras.*

8. Crossed S-matrices for Diagram automorphisms

Let $\mathfrak{g}$ be a finite dimensional Lie algebra which is simply laced. Let $D(\mathfrak{g})$ be the Dynkin diagram of $\mathfrak{g}$. A diagram automorphism σ of $\mathfrak{g}$ is a graph automorphism of the Dynkin diagram $D(\mathfrak{g})$. Let N be the order of σ. It is well known that $N \in \{1, 2, 3\}$. We can extend σ to an automorphism of the Lie algebra $\mathfrak{g}$. To this data, one can (see [38]) attach a new Lie algebra $\mathfrak{g}_\sigma$ known as the orbit Lie algebra.

8.1. **Orbit Lie algebras.** In this section, we recall the construction of the orbit Lie algebras. Let $A = \{a_{ij}\}_{I \times I}$ be the Cartan matrix of the Lie algebra $\mathfrak{g}$. We enumerate the vertices I of $D(\mathfrak{g})$ as in [53] by integers. We fix an enumeration the orbit representatives such that the it is the smallest in the orbit. More precisely

$$\check{I} = \{i \in I | i \leq \sigma^a(i), \text{ and } 0 \leq a \leq N - 1\}$$

Let N_i denote the order of the orbit at $i \in I$. One defines a new matrix $|\check{I}| \times |\check{I}|$-matrix $\check{A}$ by taking sums of rows related by the automorphism σ by the following formula:

$$(8.1) \qquad \check{A}_{ij} := s_i \frac{N_i}{N} \sum_{a=0}^{N-1} A_{\sigma^a i, j}, \text{where } s_i = 1 - \sum_{a=1}^{N_i - 1} A_{\sigma^a i, i}.$$

By the above formula, we get that $\check{A}$ is an indecomposable Cartan matrix of finite type and the associate simple Lie algebra is denote by $\mathfrak{g}_\sigma$. Further denote by $\mathfrak{g}^\sigma$ the Lie algebra of $\mathfrak{g}$ fixed by σ. The Lie algebras $\mathfrak{g}_\sigma$ and $\mathfrak{g}^\sigma$ are Langlands dual to each other. The following list is from [38]:

Case	$\mathfrak{g}$	$\mathfrak{g}_\sigma$	$\mathfrak{g}^\sigma$
1	A_{2n}, $n \geq 2$	C_n	B_n
2	A_{2n-1}, $n \geq 3$	B_n	C_n
3	D_{n+1}, $n \geq 4$	C_n	B_n
4	D_4	G_2	G_2
5	E_6	F_4	F_4

Let $P(\mathfrak{g})^\sigma$ denote the set of integral weights which are invariant under σ and consider the fundamental weights $\omega_1, \ldots, \omega_{\text{rank}\,\mathfrak{g}}$ of $\mathfrak{g}$. The following is an easy observation:

Lemma 8.1. *There is a natural bijection $\iota : P(\mathfrak{g})^\sigma \to P(\mathfrak{g}_\sigma)$ which has the following properties:*

(1) *For $i \in \check{I}$, we get $\sum_{a=0}^{N_i} \iota(\omega_{\sigma^a i}) = \omega_i$*

(2) *$\iota(\overline{\rho}) = \overline{\rho}_\sigma$, where $\overline{\rho}$ and $\overline{\rho}_\sigma$ are sums of the fundamental weights of $\mathfrak{g}$ and $\mathfrak{g}_\sigma$ respectively.*

Following [53], we now recall the notion S-matrices in the setting of twisted affine Lie algebras and connect them to the characters of the fusion ring in Section 9.1. As observed in [53], these S-matrices (except for $A_{2n}^{(2)}$) are not matrices for the modular transformation of the characters with respect to the group $\mathrm{SL}_2(\mathbb{Z})$. These S-matrices should be considered as crossed S-matrices as in [23, 25]. We follow the ordering and labeling of the roots and weights as in [53]. We first use Lemma 3.4 to give an explicit description of the set $P^\ell(\mathfrak{g}(X_N^{(m)}))$. Through out this section $\mathring{Q}$ will denote the root lattice $Q(\mathring{\mathfrak{g}})$.

8.2. Integrable Highest weight representations of $A_{2n-1}^{(2)}$.

In this case $\mathring{\mathfrak{g}} = C_n$ and the orbit Lie algebra $\mathfrak{g}_\sigma = B_n$. Let $\Lambda_0, \ldots, \Lambda_n$ denote the affine fundamental weights of $\mathfrak{g}(A_{2n-1}^{(2)})$ and $\overline{\Lambda}_0, \ldots \overline{\Lambda}_n$, denote the orthogonal projection to C_n under the normalized invariant bilinear form on $\mathfrak{g}(A_{2n-1}^{(2)})$ denoted by $\kappa_\mathfrak{g}$. For notational conveniences, we let ω_0 to be weights corresponding to the trivial representation of C_n and regard it as the 0-th fundamental weights. Now let $\omega_0, \omega_1, \ldots, \omega_n$ denote the fundamental weights of C_n. It turns out that for all $0 \le i \le n$, the $\overline{\Lambda}_i = \omega_i$. The set of level ℓ integrable highest weight representations of the affine Lie algebra $\mathfrak{g}(A_{2n-1}^{(2)})$ can be rewritten as the follows:

$$(8.2) \qquad P^\ell(\mathfrak{g}(A_{2n-1}^{(2)})) = \{\sum_{i=1}^n b_i\omega_i \in P_+(C_n)|b_1 + 2(b_2 + \cdots + b_n) \le \ell\}.$$

8.3. Integrable Highest weight representations of $D_{n+1}^{(2)}$.

In this case $\mathring{\mathfrak{g}} = B_n$ and the orbit Lie algebra $\mathfrak{g}_\sigma = C_n$. Let $\Lambda_0, \ldots, \Lambda_n$ denote the affine fundamental weights of $\mathfrak{g}(D_{n+1}^{(2)})$ and $\overline{\Lambda}_0, \ldots \overline{\Lambda}_n$, denote the orthogonal projection to B_n under the normalized invariant bilinear form on $D_{n+1}^{(2)}$ denoted by $\kappa_\mathfrak{g}$. For notational conveniences, we let ω_0 to be weights corresponding to the trivial representation of B_n and regard it as the 0-th fundamental weights. Now let $\omega_0, \omega_1, \ldots, \omega_n$ denote the fundamental weights of B_n. As in the previous case, it turns out that for all $0 \le i \le n$, the $\overline{\Lambda}_i = \omega_i$. The set of level ℓ integrable highest weight representations of the affine Lie algebra $\mathfrak{g}(A_{2n-1}^{(2)})$ are can be rewritten as the follows:

$$(8.3) \qquad P^\ell(\mathfrak{g}(D_{n+1}^{(2)})) = \{\sum_{i=1}^n b_i\omega_i \in P_+(B_n)|2(b_1 + \cdots + b_{n-1}) + b_n \le \ell\}.$$

8.4. Integrable Highest weight representations of $D_4^{(3)}$.

In this case $\mathring{\mathfrak{g}} = \mathfrak{g}_2$ and the orbit Lie algebra $\mathfrak{g}_\sigma$ is also $\mathfrak{g}_2$. Let $\Lambda_0, \Lambda_1, \Lambda_2$ denote the affine fundamental weighs of $\mathfrak{g}(D_4^{(3)})$ and $\overline{\Lambda}_0, \overline{\Lambda}_1, \overline{\Lambda}_2$ denote their orthogonal horizontal projections with respect to the invariant bilinear form on $D_4^{(3)}$. Then $\overline{\Lambda}_1 = \omega_2$ and $\overline{\Lambda}_2 = \omega_1$, where ω_1 and ω_2 are the

fundamental weights of the Lie algebra $\mathfrak{g}_2$. Moreover $\overline{\Lambda}_0 = \omega_0$. With this notation, we have

$$(8.4) \qquad P^\ell(\mathfrak{g}(D_4^3)) = \{b_1\omega_1 + b_2\omega_2 \in P_+(\mathfrak{g}_2)|3b_1 + 2b_2 \leq \ell\}.$$

8.5. Integrable Highest weight representations of $E_6^{(2)}$. In this case $\mathring{\mathfrak{g}} = \mathfrak{f}_4$ and the orbits of the Lie algebra $\mathfrak{g}_\sigma$ is also $\mathfrak{f}_4$. Let $\Lambda_0, \ldots, \Lambda_4$ denote the affine fundamental weights of $\mathfrak{g}(E_6^{(2)})$ and $\overline{\Lambda}_0, \ldots, \overline{\Lambda}_4$ denote their orthogonal projections to $\mathfrak{g}_2$ with respect to the invariant bilinear form on $E_6^{(2)}$. Then $\overline{\Lambda}_1 = \omega_4$ and $\overline{\Lambda}_2 = \omega_3$, $\overline{\Lambda}_3 = \omega_2$ and $\overline{\Lambda}_4 = \omega_1$, where $\omega_1, \ldots, \omega_4$ are the fundamental weights of the Lie algebra $\mathfrak{f}_4$. Moreover $\overline{\Lambda}_0 = \omega_0$. With this notation, we have

$$(8.5) \qquad P^\ell(\mathfrak{g}(E_6^{(2)})) = \{b_1\omega_1 + \cdots + b_4\omega_4 \in P_+(\mathfrak{f}_4)|2b_1 + 4b_2 + 3b_3 + 2b_4 \leq \ell\}.$$

8.6. S-matrix for $A_{2n-1}^{(2)}$. Let $P^\ell(\mathfrak{g}(A_{2n-1}^{(1)}))^\sigma$ denote the set of integrable level ℓ weights of the Lie algebra $\mathfrak{g}(A_{2n-1}^{(1)})$ which are fixed by the involution $\sigma : i \to 2n - i$ of the vertices of the Dynkin diagram of A_{2n-1}. It can be described explicitly as.

$$P^\ell(\mathfrak{g}(A_{2n-1}^{(1)}))^\sigma := \{\sum_{i=1}^{n-1} b_i(\omega_i + \omega_{2n-i}) + b_n\omega_n \in P_+(A_{2n-1})|2(b_1 + \cdots + b_{n-1}) + b_n \leq \ell\}.$$

Since the fixed point orbit Lie algebra of A_{2n-1} under the action of the involution σ is B_n, there is a natural bijection between $\iota : P(A_{2n-1})^\sigma \to P(B_n)$. Thus under the map ι restricts to a bijection of

$$(8.6) \qquad \iota : P^\ell(\mathfrak{g}(A_{2n-1}^{(1)}))^\sigma \simeq P^\ell(\mathfrak{g}(D_{n+1}^{(2)}))$$

and we identify the two via ι. We recall the following map between the Cartan subalgebras of B_n and C_n following Section 13.9 in [53]

$$\tau_\kappa : \mathfrak{h}(B_n) \to \mathfrak{h}(C_n), \ \omega_i \to \frac{a_i}{a_i^\vee}\omega_i, \ \text{for } 1 \leq i \leq n$$

and $a_1, \ldots, a_n$ (respectively $a_1^\vee, \ldots, a_n^\vee$) denote the Coxeter (respectively dual Coxeter) label of the Lie algebra $\mathfrak{g}(A_{2n-1}^{(2)})$.

The rows and columns of the S-matrix are parametrized by the set $P^\ell(\mathfrak{g}(A_{2n-1}^{(2)}))$ and $P^\ell(\mathfrak{g}(A_{2n-1}^{(1)}))^\sigma$ respectively. Under the identification ι, the columns will be parametrized by $P^\ell(\mathfrak{g}(D_{n+1}^{(2)}))$. We recall the following formula [53] for the crossed S-matrix $\widetilde{S}^{(\ell)}(A_{2n-1}^{(2)})$ whose (λ, μ)-th entry is given by the formula:

$$(8.7)$$
$$\widetilde{S}_{\lambda,\mu}^{(\ell)}(A_{2n-1}^{(2)}) = i^{|\mathring{\Delta}_+|}\frac{\sqrt{2}}{|M^*/(\ell + 2n)\mathring{Q}|^{\frac{1}{2}}} \sum_{w \in \mathring{W}} \epsilon(w)\exp\left(-\frac{2\pi i}{\ell + 2n}\kappa_\mathfrak{g}(w(\lambda + \overline{\rho}), \tau_\kappa(\mu + \overline{\rho}'))\right),$$

where $\overline{\rho}$ (respectively $\overline{\rho}'$) denotes the sum of the fundamental weights of C_n (respectively B_n), $h^\vee = 2n$ is the dual Coxeter number of $\mathfrak{g}(A_{2n-1}^{(2)})$, M^* denote the dual lattice of $\mathring{Q}$ with respect to the Killing form $\kappa_\mathfrak{g}$.

8.7. S-matrix for $D_{n+1}^{(2)}$. Let $P^\ell(\mathfrak{g}(D_{n+1}^{(1)}))^\sigma$ denote the set of integrable level ℓ weights of the Lie algebra $\mathfrak{g}(D_{n+1}^{(1)})$ which are fixed by the involution which exchanges the n and $n+1$-th vertices of the Dynkin diagram of D_{n+1}. It can be described explicitly as.

$$P^\ell(\mathfrak{g}(D_{n+1}^{(1)}))^\sigma := \{\sum_{i=1}^n b_i\omega_i + b_n(\omega_n + \omega_{n+1}) \in P_+(D_{n+1})|b_1 + 2(b_1 + \cdots + b_n) \leq \ell\}.$$

Since the fixed point orbit Lie algebra of D_{n+1} under the action of the involution σ is C_n, there is a natural isomorphism between $\iota : P(D_{n+1})^\sigma \to P(C_n)$. Moreover the map ι restricts to a bijection of

$$(8.8) \qquad \iota : P^\ell(\mathfrak{g}(D_{n+1}^{(1)}))^\sigma \simeq P^\ell(\mathfrak{g}(A_{2n-1}^{(2)}))$$

and we identify the two via ι. We recall the following map between the Cartan subalgebras of C_n and B_n following Section 13.9 in [53]

$$\tau_\kappa : \mathfrak{h}(C_n) \to \mathfrak{h}(B_n), \ \omega_i \to \frac{a_i}{a_i^\vee}\omega_i,$$

where $1 \leq i \leq n$ and $a_1, \ldots, a_n$ (respectively $a_1^\vee, \ldots, a_n^\vee$) denote the Coxeter (respectively Dual Coxeter) labels of the Lie algebra $\mathfrak{g}(D_{n+1}^{(2)})$.

The rows and columns of the S-matrix are parametrized by the set $P^\ell(\mathfrak{g}(D_{n+1}^{(2)}))$ and $P^\ell(\mathfrak{g}(D_{n+1}^{(1)}))^\sigma$ respectively. Under the identification ι, the columns will be parametrized by $P^\ell(\mathfrak{g}(A_{2n-1}^{(2)}))$. The (λ, μ)-th entry of the crossed S-matrix $S_{\lambda,\mu}^{(\ell)}(D_{n+1}^{(2)})$ is

$$(8.9)$$
$$S_{\lambda,\mu}^{(\ell)}(D_{n+1}^{(2)}) = i^{|\mathring{\Delta}_+|}\frac{\sqrt{2}}{|M^*/(\ell+2n)\mathring{Q}|^{\frac{1}{2}}} \sum_{w\in\mathring{W}} \epsilon(w)\exp\left(-\frac{2\pi i}{\ell+2n}\kappa_\mathfrak{g}(w(\lambda+\overline{\rho}), \tau_\kappa(\mu+\overline{\rho}'))\right),$$

where $\overline{\rho}$ (respectively $\overline{\rho}'$) denotes the sum of the fundamental weights of C_n (respectively B_n), $h^\vee = 2n$ is the dual Coxeter number of $\mathfrak{g}(D_{n+1}^{(2)})$, M^* denote the dual lattice of $\mathring{Q}$ with respect to the Killing form $\kappa_\mathfrak{g}$.

8.8. S-matrix for $D_4^{(3)}$. Let $P^\ell(\mathfrak{g}(D_4^{(1)}))^\sigma$ denote the set of integrable level ℓ weights of the Lie algebra $\mathfrak{g}(D_4^{(1)})$ which are fixed by rotation σ that rotates the $1 \to 3 \to 4 \to 1$-th vertices of the Dynkin diagram of D_4. It can be described explicitly as.

$$P^\ell(\mathfrak{g}(D_4^{(1)}))^\sigma := \{b_1(\omega_1 + \omega_3 + \omega_4) + b_2\omega_2 \in P_+(\mathfrak{g}_2)|3b_1 + 2b_2 \leq \ell\}.$$

Since the fixed point orbit Lie algebra of D_4 under the action of the involution σ is $\mathfrak{g}_2$, there is a natural isomorphism between $\iota : P(D_4)^\sigma \to P(\mathfrak{g}_2)$. Moreover the map ι restricts to a bijection of

$$(8.10) \qquad \iota : P^\ell(\mathfrak{g}(D_4^{(1)}))^\sigma \simeq P^\ell(\mathfrak{g}(D_4^{(3)}))$$

and we identify the two via ι. We recall the following map between the Cartan subalgebra of $\mathfrak{g}_2$ following Section 13.9 in [53]

$$\tau_\kappa : \mathfrak{h}(\mathfrak{g}_2) \to \mathfrak{h}(\mathfrak{g}_2), \ \omega_2 \to \frac{a_2}{a_2^\vee}\omega_1, \ \omega_1 \to \frac{a_1}{a_1^\vee}\omega_2,$$

where $1 \leq i \leq 2$ and a_1, a_2 (respectively $a_1^\vee, a_2^\vee$) denote the Coxeter (respectively Dual Coxeter) labels of the Lie algebra $\mathfrak{g}(D_4^{(3)})$.

The rows and columns of the S-matrix are parametrized by the set $P^\ell(\mathfrak{g}(D_4^{(3)}))$ and $P^\ell(\mathfrak{g}(D_4^{(1)}))^\sigma$ respectively. Under the identification ι, the columns will be parametrized by $P^\ell(\mathfrak{g}(D_4^{(3)}))$. We recall the following formula [53] which is known as the Kac-Petersen formula [54].

$$(S^{(\ell)}_{\lambda,\mu})(D_4^{(3)}) = i^{|\mathring{\Delta}_+|}\frac{\sqrt{3}}{|M^*/(\ell+6)\mathring{Q}|^{\frac{1}{2}}}\sum_{w\in\mathring{W}}\epsilon(w)\exp\left(-\frac{2\pi i}{\ell+6}\kappa_\mathfrak{g}(w(\lambda+\overline{\rho}),\tau_\kappa(\mu+\overline{\rho}))\right),$$

where $\overline{\rho}$ denotes the sum of the fundamental weights of $\mathfrak{g}_2$, $h^\vee = 6$ is the dual Coxeter number of $\mathfrak{g}(D_4^{(3)})$, M^* denote the dual lattice of $\mathring{Q}$ with respect to the Killing form $\kappa_\mathfrak{g}$.

8.9. **S-matrix for $E_6^{(2)}$.** Let $P^\ell(\mathfrak{g}(E_6^{(1)}))^\sigma$ denote the set of integrable level ℓ weights of the Lie algebra $\mathfrak{g}(E_6^{(1)})$ which are fixed by rotation σ that interchanges 1-st with 5-th; 2-th with 4-th and fixes the $3rd$ and 6-th vertices of the Dynkin diagram of E_6. It can be described explicitly as.

$$P^\ell(\mathfrak{g}(E_6^{(1)}))^\sigma := \{b_1(\omega_1+\omega_5)+b_2(\omega_2+\omega_4)+b_3\omega_3+b_4\omega_6 \in P_+(\mathfrak{f}_4)|2b_1+4b_2+3b_3+2b_4 \leq \ell\}.$$

Since the orbit Lie algebra of E_6 under the action of the involution σ is $\mathfrak{f}_4$, there is a natural bijection between $\iota : P(E_6)^\sigma \to P(\mathfrak{f}_4)$. Moreover the map ι restricts to a bijection of

$$(8.12) \qquad\qquad \iota : P^\ell(\mathfrak{g}(E_6^{(1)}))^\sigma \simeq P^\ell(\mathfrak{g}(E_6^{(2)}))$$

and we identify the two via ι. We recall the following map between the Cartan subalgebra of $\mathfrak{f}_4$ following Section 13.9 in [53]

$$\tau_\kappa : \mathfrak{h}(\mathfrak{f}_4) \to \mathfrak{h}(\mathfrak{f}_4), \ \omega_4 \to \frac{a_4}{a_4^\vee}\omega_1, \ \omega_3 \to \frac{a_3}{a_3^\vee}\omega_2, \ \omega_2 \to \frac{a_2}{a_2^\vee}\omega_3, \omega_1 \to \frac{a_1}{a_1^\vee}\omega_4,$$

where a_1, a_2, a_3, a_4 (respectively $a_1^\vee, a_2^\vee, a_3^\vee, a_4^\vee$) denote the Coxeter (respectively Dual Coxeter) labels of the Lie algebra $\mathfrak{g}(E_6^{(2)})$.

The rows and columns of the S-matrix are parametrized by the set $P^\ell(\mathfrak{g}(E_6^{(2)}))$ and $P^\ell(\mathfrak{g}(E_6^{(1)}))^\sigma$ respectively. Under the identification ι, the columns will be parametrized by $P^\ell(\mathfrak{g}(E_6^{(2)}))$. We recall the following formula from [53]:

(8.13)
$$S^{(\ell)}_{\lambda,\mu}(E_6^{(2)}) = \sqrt{2}i^{|\mathring{\Delta}_+|}|M^*/(\ell+12)\mathring{Q}|^{-\frac{1}{2}}\sum_{w\in\mathring{W}}\epsilon(w)\exp\left(-\frac{2\pi i}{\ell+12}\kappa_\mathfrak{g}(w(\lambda+\overline{\rho}),\tau_\kappa(\mu+\overline{\rho}))\right),$$

where $\overline{\rho}$ denotes the sum of the fundamental weights of $\mathfrak{f}_4$, $h^\vee = 12$ is the dual Coxeter number of $\mathfrak{g}(E_6^{(2)})$, M^* denote the dual lattice of $\mathring{Q}$ with respect to the Killing form $\kappa_{\mathfrak{g}}$.

8.10. S-matrix for $A_{2n}^{(2)}$.

In this case $\mathring{\mathfrak{g}} = C_n$ and the orbit Lie algebra $\mathfrak{g}_\sigma = C_n$. Let $\Lambda_0, \ldots, \Lambda_n$ denote the affine fundamental weights of $\mathfrak{g}(A_{2n}^{(2)})$ and $\overline{\Lambda}_0, \ldots \overline{\Lambda}_n$, denote the projection to C_n under the normalized invariant bilinear form on denoted by $\kappa_{\mathfrak{g}}$. Now let $\omega_0, \omega_1, \ldots, \omega_n$ denote the fundamental weights of C_n. It turns out that for all $0 \leq i \leq n$, the $\overline{\Lambda}_i = \omega_i$. The set of level ℓ integrable highest weight representations of the affine Lie algebra $\mathfrak{g}(A_{2n}^{(2)})$ can be rewritten as the follows:

$$(8.14) \qquad P^\ell(\mathfrak{g}(A_{2n}^{(2)})) = \{\sum_{i=1}^n b_i \omega_i \in P_+(C_n) | 2(b_1 + b_2 + \cdots + b_n) \leq \ell\}.$$

Remark 8.2. With the convention of the numbering of the vertices of the affine Dynkin diagram in the case $A_{2n}^{(2)}$, we get the fixed point Lie algebra is of type B_n whereas $\mathring{\mathfrak{g}} = C_n$. However it is sometimes convenient to choose a different ordering of the vertices of the Dynkin diagram such that the new horizontal subalgebra is same as the fixed point algebra B_n. This can be done if we renumber the vertices by reflecting about the n-th vertex of the affine Dynkin diagram. Under the new numbering system we get an alternate description

$$(8.15) \qquad P^\ell(\mathfrak{g}(A_{2n}^{(2)})) = \{\sum_{i=1}^n \widetilde{b}_i \omega_i \in P_+(B_n) | \widetilde{b}_1 + \cdots + \widetilde{b}_{n-1} + \frac{\widetilde{b}_n}{2} \leq \frac{\ell}{2}, \ \ell - \widetilde{b}_n \text{ is even}\},$$

where $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_n$ are fundamental weights of the Lie algebra of type B_n. In particular, we observe that $P^1(\mathfrak{g}, \sigma) = \{\omega_n\}$. We refer the reader to compare it with the description of $P^\ell(\mathfrak{g}, \gamma)$ in [46].

Let $P^\ell(\mathfrak{g}(A_{2n}^{(1)}))^\sigma$ denote the set of integrable level ℓ weights of the Lie algebra $\mathfrak{g}(A_{2n}^{(1)})$ which are fixed by the involution $\sigma : i \to 2n + 1 - i$. It can be described explicitly as.

$$P^\ell(\mathfrak{g}(A_{2n}^{(1)}))^\sigma := \{\sum_{i=1}^n a_i(\omega_i + \omega_{2n-i}) \in P_+(A_{2n}) | 2(b_1 + \cdots + b_n) \leq \ell\}.$$

Under the map $\iota : P(A_{2n})^\sigma \to P(C_n)$, there is a natural bijection between $\iota : P^\ell(\mathfrak{g}(A_{2n}^{(1)}))^\sigma \to P^\ell(\mathfrak{g}(A_{2n}^{(2)}))$. Hence we will identify the two using the map ι. The rows and columns of the S-matrix are parametrized by the set $P^\ell(\mathfrak{g}(A_{2n}^{(2)}))$ and $P^\ell(\mathfrak{g}(A_{2n}^{(1)}))^\sigma$ respectively. We recall the following Theorem 13.8 [53] which is known as the Kac-Petersen formula [54].

$$(8.16) \quad S_{\lambda,\mu}^{(\ell)}(A_{2n}^{(2)}) = i^{|\mathring{\Delta}_+|}(\ell + 2n + 1)^{-\frac{1}{2}} \sum_{w \in \mathring{W}} \epsilon(w) \exp\left(-\frac{2\pi i}{\ell + 2n + 1}\kappa_{\mathfrak{g}}(w(\lambda + \overline{\rho}), \mu + \overline{\rho})\right),$$

where $\overline{\rho}$ denotes the sum of the fundamental weights of C_n and $h^\vee = 2n + 1$ is the dual Coxeter number of $\mathfrak{g}(A_{2n}^{(2)})$. Moreover the matrix $\tilde{S}^{(\ell)}(A_{2n}^{(2)})$ has the following properties [53].

Proposition 8.3. *The matrix* $\left(S^{(\ell)}(A_{2n}^{(2)})\right)_{\lambda,\mu} = S^{(\ell)}_{\lambda,\mu}(A_{2n}^{(2)})$ *is symmetric and unitary.*

8.10.1. *Crossed S-matrices associated to arbitrary automorphisms.* Let γ be an arbitrary automorphism of a simple Lie algebra $\mathfrak{g}$ and let $\widehat{L}(\mathfrak{g}, \gamma)$ denote the twisted Kac-Moody Lie algebra as defined in Section 3.1. Recall there is a natural isomorphism of $\widehat{L}(\mathfrak{g}, \gamma)$ with an twisted affine Kac-Moody Lie algebra of type $X_N^{(m)}$, where the Lie algebra $\mathfrak{g}$ is of type X_N and $X_N^{(m)}$ is determined by the diagram automorphism σ arising from γ. Thus there are natural bijections

$$\beta_1 : P_\ell(\mathfrak{g})^\gamma \simeq P_\ell(\mathfrak{g})^\sigma, \text{ and } \beta_2 : P^\ell(\mathfrak{g}, \langle\gamma\rangle) \simeq P^\ell(\mathfrak{g}(X_N^{(m)})).$$

Let for any element $(\lambda, \mu) \in P^\ell(\mathfrak{g}, \gamma) \times P_\ell(\mathfrak{g})^\gamma$, consider the number

$$(8.17) \qquad S^\gamma_{\lambda,\mu} := S^{(\ell)}_{\beta_1(\lambda),\beta_2(\mu)}(X_N^{(m)}),$$

where $S^{(\ell)}(X_N^{(m)})$ is the crossed S-matrix at level ℓ given by equations 8.7, 8.9, 8.11, 8.13 and 8.16.

Definition 8.4. We define the *crossed S-matrix* S^γ at level ℓ associated to γ to be a matrix whose rows and columns are parametrized by $P^\ell(\mathfrak{g}, \langle\gamma\rangle)$ and $P_\ell(\mathfrak{g})^\gamma$ and whose (λ, μ)-th entry $S^\gamma_{\lambda,\mu}$ is given by Equation 8.17.

8.11. **Relation with untwisted S-matrices.** This section is motivated by the two observations. First the equivariantization of a Γ-crossed modular functor is a modular functor, hence crossed S-matrices must be submatrices of an uncrossed S-matrix. In the KAC software, the uncrossed S-matrices and the crossed S-matrices of type $A_{2n}^{(2)}$ are available for computational purposes. Using the software KAC-and the discussion below, one can compute all crossed S-matrices in the remaining types.

Let A be a Cartan matrix of either of the following types

$$A : A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}, D_4^{(3)}.$$

In this section, we discuss how the crossed S matrix for the twisted affine Lie algebra of A discussed in the Section 8 for can be regarded as a submatrix S-matrix of an untwisted Lie algebra. We consider the Cartan matrix A^t obtained by taking transpose of the matrix A. In these cases, observe that the affine Lie algebra associated to A^t is of untwisted type. They are given as:

$$A^t : B_n^{(1)}, C_n^{(1)}, F_4^{(1)}, G_2^{(1)}.$$

If $\mathring{A}$ and $\mathring{A}^t$ denote the Cartan matrix obtained by deleting the 0-th row and column, then we get $\mathfrak{g}(\mathring{A})$ and $\mathfrak{g}(\mathring{A}^t)$ are Langlands dual. The following extends the correspondence we between the root lattice (respectively weights lattice)of $\mathfrak{g}(\mathring{A})$ with the coroot lattice (respectively the coweight lattice) of $\mathfrak{g}(\mathring{A}^t)$ to the affine case.

Lemma 8.5. *The following map between the Cartan subalgebras* $\tau : \mathfrak{h}(A) \to \mathfrak{h}(A^t)$*, define an isometry invariant map with respect to the Weyl group :*

$$(8.18) \qquad \Lambda_i \to \frac{a_i^\vee}{a_i} \Lambda_i^t, \ \delta \to \delta^t,$$

where $\Lambda_0, \dots, \Lambda_n$ *(respectively* $\Lambda_0^t, \dots, \Lambda_n^t$*) are affine fundamental weights for the affine Lie algebra associated to* A *(respectively* A^t*) and* δ *(respectively* δ^t*) as in Section 3.2.*

Consider the map $\mathring{\tau} : P^\ell(\mathfrak{g}(A)) \hookrightarrow P^{\ell+h^\vee-h}(\mathfrak{g}(A^t))$ given by the formula

$$(8.19) \qquad \mathring{\tau}(\lambda) := \tau(\lambda + \overline{\rho}) - \overline{\rho}^t,$$

where $\overline{\rho}$(respectively $\overline{\rho}^t$) are sum of the fundamental weights of the horizontal subalgebra of $\mathfrak{g}(A)$ and $\mathfrak{g}(A^t)$ respectively.

The affine Lie algebras of type A^t are all untwisted affine Kac-Moody Lie algebra, hence denote by $S_{\lambda,\mu}^{(\ell)}(A^t)$-the (λ, μ)-th entry of the S-matrix of type A^t at level ℓ. We refer the reader to Chapter 13 in [53] for more details. The following proposition is well known and can be checked directly (see Proposition 3.2 in [41]).

Proposition 8.6. *Let* $\mathfrak{g}(A)$ *be a twisted affine Kac-Moody Lie algebra associated to the Cartan matrix* A *of type* $A_{2n-1}^{(2)}, D_{n+1}^{(2)}, D_4^{(3)}, E_6^{(2)}$*, then the crossed* S *matrices are related to the* S *matrices of* $\mathfrak{g}(A^t)$ *by the following formula*

$$S_{\lambda,\mu}^{(\ell)}(A) = |\nu_{\mathfrak{g}}(\mathring{Q}^\vee)/\mathring{Q}|^{\frac{1}{2}} S_{\mathring{\tau}(\lambda),\tau\circ\tau_\kappa(\mu)}^{(\ell+h^\vee-h)}(A^t),$$

where $h^\vee$ *and* h *are the dual Coxeter numbers of the Lie algebra* $\mathfrak{g}(A)$ *and* $\mathfrak{g}(A^t)$ *respectively. Moreover* $|\nu_{\mathfrak{g}}(\mathring{Q}^\vee)/\mathring{Q}|$ *is two when* A *is either* $A_{2n-1}^{(2)}$*,* $D_{n+1}^{(2)}$ *and* $E_6^{(2)}$ *and is three when* A *is of type* $D_4^{(3)}$*.*

8.11.1. *Some computations.* We list the following facts which are very useful in computing the crossed S-matrices. Using proposition 8.6, we can use Kac-software to compute the crossed S matrices.

- The case $A_{2n-1}^{(2)}$
  - $\tau \circ \tau_\kappa$ is identity.
  - $\tau(\omega_i) = \omega_i^t$ for $1 \le i \le n-1$ and $\tau(\omega_n) = 2\omega_n^t$.
  - $\mathring{\tau}(\omega_0) = \omega_n^t$.
- The case $D_{n+1}^{(2)}$
  - $\tau \circ \tau_\kappa$ is identity.
  - $\tau(\omega_i) = \omega_i^t$ for $1 \le i \le n-1$ and $\tau(\omega_n) = 2\omega_n^t$.
  - $\mathring{\tau}(\omega_0) = \omega_n^t$.
- The case $D_4^{(3)}$:
  - $\tau(\omega_1) = 3\omega_2^t$ and $\tau(\omega_2) = \omega_1^t$.
  - $\tau \circ \tau_\kappa(\omega_i) = \omega_i^t$ for $1 \le i \le 2$.
  - $\mathring{\tau}(\omega_0) = 2\omega_2^t$
- The case $E_6^{(2)}$:

$$- \ \tau(\omega_1) = 2\omega_4^t, \ \tau(\omega_2) = 2\omega_3^t, \ \tau(\omega_3) = \omega_2^t, \ \tau(\omega_4) = \omega_1^t.$$
$$- \ \tau \circ \tau_\kappa(\omega_i) = \omega_i \text{ for } 1 \le i \le 4.$$
$$- \ \mathring{\tau}(\omega_0) = \omega_3^t + \omega_4^t$$

9. Twisted Fusion Rings

9.1. **Fusion rules associated to diagram automorphism.** Let $\mathfrak{g}$ be any simple Lie algebra. For any n-tuple $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$, consider the dual conformal block $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z})$ at level ℓ. Now given any diagram automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$, we get an automorphism of $\sigma^* : P_\ell(\mathfrak{g}) \to P_\ell(\mathfrak{g})$. This induces a map of the dual conformal blocks

$$\sigma : \mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z}) \to \mathcal{V}_{\sigma^*(\vec{\lambda})}(\mathbb{P}^1, \vec{z}),$$

where $\sigma^*(\vec{\lambda}) = (\sigma^*\lambda_1, \ldots, \sigma^*\lambda_n)$. Let $P_\ell(\mathfrak{g})^\sigma$ denote the set of level ℓ weights fixed by σ. Thus, if $\vec{\lambda} \in (P_\ell(\mathfrak{g})^\sigma)^n$, then we get an element σ in $\mathrm{End}(\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z}))$. In [44], Z. Hong defines a fusion rule associated to σ.

9.1.1. *Twisted Fusion Rule.* Now we restrict to the case when $\mathfrak{g}$ is simply laced and σ is diagram automorphism of the Lie algebra $\mathfrak{g}$. The following proposition can be found in [44]

Proposition 9.1. *The map* $N_\sigma : \mathbb{N}^{P_\ell^\sigma(\mathfrak{g})} \to \mathbb{C}$ *given by* $N_\sigma(\sum_i \lambda_i) := \mathrm{Tr}(\sigma|\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z}))$ *satisfies the hypothesis of fusion rules (see [9]).*

The set of characters of the fusion ring $\mathcal{R}_\sigma$ has been described in [44]. We recall the details below. Let G_σ be the simply connected group associated to the Lie algebra $\mathfrak{g}_\sigma$ and T_σ be a maximal torus in G_σ. Let $Q(\mathfrak{g})$ be the root lattice of $\mathfrak{g}$ and let $Q(\mathfrak{g})^\sigma$ be elements in the root lattice of $\mathfrak{g}$ which are fixed by the automorphism σ. Then under the transformation $\iota : P(\mathfrak{g})^\sigma \to P(\mathfrak{g}_\sigma)$, we get

$$(9.1) \qquad\qquad \iota(Q(\mathfrak{g})^\sigma) = \begin{cases} Q(\mathfrak{g}_\sigma) & \mathfrak{g} \ne A_{2n}. \\ P(\mathfrak{g}_\sigma) = \frac{1}{2}Q(\mathfrak{g}_\sigma) & \mathfrak{g} = A_{2n}. \end{cases}$$

Consider the following subset $T_{\sigma,\ell}$ of T_σ given by

$$T_{\sigma,\ell} := \{t \in T_\sigma | e^\alpha(t) = 1 \text{ for } \alpha \in (\ell + h^\vee)\iota(Q^\sigma(\mathfrak{g}))\}.$$

An element t of $T_{\sigma,\ell}$ is called *regular* if the Weyl group W_σ of $\mathfrak{g}_\sigma$ acts freely on t and let $T_{\sigma,\ell}^{\mathrm{reg}}$ is set of regular elements in $T_{\sigma,\ell}$. Now the main result in [44] is the following:

Proposition 9.2. *For* $t \in T_{\sigma,\ell}^{\mathrm{reg}}/W_\sigma$, *the set* $\mathrm{Tr}_*(t)$ *gives the characters of the fusion ring* $\mathcal{R}_\sigma$.

Thus to study the character table of the fusion ring $\mathcal{R}_\sigma$, we need to give an explicit description of the set $T_{\sigma,\ell}^{\mathrm{reg}}/W_\sigma$ in terms of weights of the level ℓ-representations of the twisted affine Lie algebra $\mathfrak{g}(X_N^{(m)})$ associated to σ.

Remark 9.3. In the case of the usual fusion ring $\mathcal{F}_N$, there is a natural bijection [9] between $P_\ell(\mathfrak{g})$ and T_ℓ^{reg}/W, where W is the Weyl group of $\mathfrak{g}$. However in the case of twisted fusion ring $\mathcal{R}_\sigma$, there is no such natural bijection.

The following proposition should be considered as a generalization of Proposition 6.3 in [9] to the case of all affine Kac-Moody Lie algebras.

Proposition 9.4. *There is a natural bijection between $T_{\sigma,\ell}^{\mathrm{reg}}/W_\sigma$ with $P^\ell(\mathfrak{g}(X_N^{(m)}))$.*

Proof. We divide our proof in the two major cases. First we consider the case when the twisted affine Kac-Moody Lie algebra is of type $A_{2n-1}^{(2)}, D_{n+1}^{(2)}, E_6^{(2)}$ or $D_4^{(3)}$. In [44], Hong introduces the following subset set of coweights $\check{P}(\mathfrak{g}_\sigma)$

$$\check{P}_{\sigma,\ell} := \{\tilde{\mu} \in \check{P}(\mathfrak{g}_\sigma)|\theta_\sigma(\check{\mu}) \leq \ell\}.$$

Now Lemma 5.20 in [44], shows that there is a natural bijection between $\check{P}_{\sigma,\ell}$ and $T_{\sigma,\ell}^{\mathrm{reg}}/W_\sigma$. Thus we will be done if we can show that $P^\ell(\mathfrak{g}, \sigma)$ is in a natural bijection with $\check{P}_{\sigma,\ell}$. This set can be precisely described as

(1) If $\mathfrak{g} = A_{2n-1}$, then $\check{P}_{\sigma,\ell} = \{\sum_{i=1}^n e_i\check{\omega}_i|e_1 + 2(e_2 + \cdots + e_n) \leq \ell\}$.
(2) If $\mathfrak{g} = D_{n+1}$, then $\check{P}_{\sigma,\ell} := \{\sum_{i=1}^n e_i\check{\omega}_i|2(e_1 + \cdots + e_{n-1}) + e_n \leq \ell\}$.
(3) If $\mathfrak{g} = D_4$, then $\check{P}_{\sigma,\ell} := \{e_1\check{\omega}_1 + e_2\check{\omega}_2|3e_1 + 2e_2 \leq \ell\}$.
(4) If $\mathfrak{g} = E_6$, then $\check{P}_{\sigma,\ell} := \{\sum_{i=1}^4 e_i\check{\omega}_i|2e_1 + 4b_2 + 3b_3 + 2e_4 \leq \ell\}$.

Here $\check{\omega}_i$'s are the fundamental coweights of the Lie algebra $\mathfrak{g}_\sigma$ and e_i's are non-negative integers. By the explicit description of the set $P^\ell(\mathfrak{g}(X_N^{(m)}))$ by equations (8.2), (8.3),(8.4) and (8.5), it follows that there is a natural bijection between $P^\ell(\mathfrak{g}(X_N^{(m)}))$.

Now we consider the remaining case when $A = A_{2n}^{(2)}$. In this case the horizontal subalgebra $\mathring{\mathfrak{g}}$ is same as $\mathfrak{g}_\sigma$. We define the map $\beta : P^\ell(\mathfrak{g}(A)) \to T_{\sigma,\ell}$ is given by the formula $\beta(\lambda) = \exp\left(\frac{2\pi i}{\ell+h^\vee}\nu_{\mathfrak{g}(A)}(\lambda + \overline{\rho})\right)$, where $\nu_{\mathfrak{g}(A)}$ is the isomorphism between $\mathfrak{H}^*(A) \to \mathfrak{H}(A)$ induced by the normalized Cartan-Killing form $\kappa_{\mathfrak{g}(A)}$ on $\mathfrak{g}(A)$ and $\overline{\rho}$ is the sum of the fundamental weights of the horizontal subalgebra $\mathring{\mathfrak{g}}$. Now the rest of the proof follows as in Proposition 9.3 in [9].

$\square$

9.1.2. *Fusion rings associated to arbitrary automorphisms.*
In the previous section, we discussed how one can associate fusion rings to diagram automorphism. Let $\gamma : \mathfrak{g} \to \mathfrak{g}$ be a arbitrary automorphism and σ be the diagram automorphism of $\mathfrak{g}$ related to γ by Proposition 3.1. Then γ and σ differ by an inner automorphism of $\mathfrak{g}$ and hence we get a bijection between $P(\mathfrak{g})^\gamma$ and $P(\mathfrak{g})^\sigma$. Now since γ and σ both preserve a normalized Cartan Killing form it follows that there exists a bijection between $P_\ell(\mathfrak{g})^\gamma$ and $P_\ell(\mathfrak{g})^\sigma$.

If $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (P_\ell(\mathfrak{g})^\sigma)^n$, then as in Section 9.1, we get an element in $\mathrm{End}(\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z})$. Since the conformal block $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z})$ is constructed as coinvariants of representation of the Lie algebra $\mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^1}(\ast\vec{z})$, it follows that inner automorphism of $\mathfrak{g}$ acts trivially on $\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z})$. The above discussion can be summarized as follows.

Proposition 9.5. *Let $\vec{\lambda} \in (P_\ell(\mathfrak{g})^\sigma)^n$, we get $\mathrm{Tr}(\gamma|\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z})) = \mathrm{Tr}(\sigma|\mathcal{V}_{\vec{\lambda}}(\mathbb{P}^1, \vec{z}))$.*

Hence the fusion ring associated to an arbitrary automorphism γ just depends on the class of the diagram automorphism σ.

10. Character table and the S-matrix

Let $\mathcal{F}$ be a fusion ring at level ℓ with basis I and let S be the set of characters. Consider a matrix Σ whose rows are parametrized by S and columns are parametrized by I. For $(\chi, \lambda) \in S \times I$, we set $\Sigma_{\chi, \lambda} = \chi(\lambda)$. We denote by Σ', the unitary matrix corresponding to Σ.

In this section, we describe how to compute the character table Σ' for the fusion ring $\mathcal{R}_\sigma$ associated to diagram automorphisms σ at level ℓ as discussed in Section 9.1. Recall by Proposition 9.2, the rows of the matrix Σ' are parametrized by the set $T^{reg}_{\sigma, \ell}/W_\sigma$, where

$$T_{\sigma, \ell} := \{t \in T_\sigma | e^\alpha(t) = 1 \text{ for } \alpha \in (\ell + h^\vee)\iota(Q^\sigma(\mathfrak{g}))\},$$

where T_σ is a maximal torus of the orbit Lie algebra $\mathfrak{g}_\sigma$ and $\iota(Q^\sigma)$ be as in equation 9.1. The columns of the matrix Σ' are parametrized by the set $P^\ell(\mathfrak{g})^\sigma$.

Consider the element $\omega = \sum_{\lambda \in P^\sigma_\ell(\mathfrak{g})} \lambda \lambda^*$ and the diagonal matrix D_ω whose entries are $|\chi(\lambda)|^2$ for $\chi \in S$. Now by the Weyl character formula and the proof of Lemma 9.7 in [9] and [44], we get

$$(10.1) \qquad D_\omega(t) = |T_{\sigma, \ell}| \left(\prod_{\alpha \in \Delta_\sigma} (e^\alpha(t) - 1) \right)^{-1}.$$

By the definition of Σ' and the Weyl character formula, we get that the following:

$$(10.2) \qquad \Sigma'_{t, \mu} = |T_{\sigma, \ell}|^{-\frac{1}{2}} \frac{\prod_{\alpha \in \Delta_{\sigma, +}} |e^{\frac{\alpha}{2}}(t) - e^{-\frac{\alpha}{2}}(t)|}{\prod_{\alpha \in \Delta_{\sigma, +}} (e^{\frac{\alpha}{2}}(t) - e^{-\frac{\alpha}{2}}(t))} \left(\sum_{w \in W_\sigma} \epsilon(w) e^{(w(\iota(\mu) + \overline{\rho}_\sigma))}(t) \right),$$

where $\overline{\rho}_\sigma$ is the sum of the fundamental weights of $\mathfrak{g}_\sigma$ and W_σ is the Weyl group of $\mathfrak{g}_\sigma$. Recall that by Proposition 9.4 there is a natural bijection $\beta : P^\ell(\mathfrak{g}(A)) \to T^{reg}_{\sigma, \ell}/W_\sigma$, where $\mathfrak{g}(A)$ is a twisted affine Kac-Moody Lie algebra.

Proposition 10.1. *Let Δ_σ denote the set of roots of the Lie algebra $\mathfrak{g}_\sigma$. Then the relations between the matrix Σ' and the crossed S-matrices are given by the following:*

$$\Sigma'_{\beta(\lambda), \mu} = i^{|\mathring{\Delta}_+|} \left(\frac{\prod_{\alpha \in \Delta_{\sigma, +}} |e^{\frac{\alpha}{2}}(\beta(\lambda)) - e^{-\frac{\alpha}{2}}(\beta(\lambda))|}{\prod_{\alpha \in \Delta_{\sigma, +}} (e^{\frac{\alpha}{2}}(\beta(\lambda)) - e^{-\frac{\alpha}{2}}(\beta(\lambda)))} \right) \overline{S^{(\ell)}_{\lambda, \iota(\mu)}},$$

where ι is the natural bijection between $P(\mathfrak{g})^\sigma$ and $P(\mathfrak{g}_\sigma)$ and $S^{(\ell)}_{\lambda, \iota(\mu)}$ denote the crossed S-matrices given in Equations 8.7, 8.9, 8.11, 8.13 and 8.16.

Proof. We split up the proof of the proposition to case by case for the untwisted affine Lie algebras.

10.1. **The case $A = A^{(2)}_{2n}$.** In this case $\mathring{\mathfrak{g}} = \mathfrak{g}_\sigma$. Hence we get $W_\sigma = \mathring{W}$ and $\overline{\rho}_\sigma = \overline{\rho}$. Let $J_{\iota(\mu)}(\beta(\lambda)) = \sum_{w \in W_\sigma} \epsilon(w) e^{(w(\iota(\mu) + \overline{\rho}_\sigma))}(\beta(\lambda))$. In this case the bijection β is given by the

following $\beta(\lambda) = \exp(\frac{2\pi i}{\ell+h^\vee}(\nu_{\mathfrak{g}(A)}(\lambda + \overline{\rho})))$. Hence we get

$$(10.3) \qquad J_{\iota(\mu)}(\beta(\lambda)) = \sum_{w \in \mathring{W}} \epsilon(w) \exp\left(\kappa_{\mathfrak{g}(A)}(w(\iota(\mu) + \overline{\rho}), \lambda + \overline{\rho})\right)$$

$$(10.4) \qquad\qquad\qquad = i^{|\mathring{\Delta}_+|}(\ell + 2n)^{\frac{1}{2}} \overline{\widetilde{S}^{(\ell)}_{\lambda,\iota(\mu)}}(A^{(2)}_{2n})$$

Hence we get the following

$$(10.5) \qquad \Sigma'_{\beta(\lambda),\mu}(\ell) = i^{|\mathring{\Delta}_+|}\left(\frac{\prod_{\alpha \in \Delta_{\sigma,+}} |e^{\frac{\alpha}{2}}(\beta(\lambda)) - e^{-\frac{\alpha}{2}}(\beta(\lambda))|}{\prod_{\alpha \in \Delta_{\sigma,+}} (e^{\frac{\alpha}{2}}(\beta(\lambda)) - e^{-\frac{\alpha}{2}}(\beta(\lambda)))}\right) \overline{S^{(\ell)}_{\lambda,\iota(\mu)}}(A^{(2)}_{2n}).$$

This completes the proof in the case $A = A^{(2)}_{2n}$. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

10.2. **The case A is either $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$, $D^{(3)}_4$, $E^{(2)}_6$.** In these cases the Lie algebras $\mathring{\mathfrak{g}}$ and the $\mathfrak{g}_\sigma$ are Langlands dual. In particular $\mathring{W} = W_\sigma$ and $\mathring{Q}^\vee$ is equal to $Q(\mathfrak{g}_\sigma)$ and vice versa. As before let $\beta : P^\ell(\mathfrak{g}(A)) \to T^{reg}_{\sigma,\ell}$ be the natural bijection. Then define a function J_A from the set $T^{reg}_{\sigma,\ell} \times P^\ell(\mathfrak{g})^\sigma$ given by the formula

$$J_A(\beta(\lambda), \mu) = \sum_{w \in \mathring{W}} \epsilon(w) \exp\left((w(\iota(\mu) + \overline{\rho}))(\beta(\lambda))\right).$$

Let $\check{P}_\ell(A) = \{\check{\lambda} \in \check{P}(\mathring{\mathfrak{g}}) | \theta(\check{\lambda}) \le \ell\}$, where θ is the highest root of $\mathring{\mathfrak{g}}$. There is a natural bijection $\eta : P^\ell(\mathfrak{g})^\sigma \to \check{P}_\ell(A)$. Similarly define another function J'_A on $P^\ell(\mathfrak{g}(A)) \times \check{P}_\ell(A)$ by the formula:

$$J'_A(\tilde{\lambda}, \tilde{\mu}) = \sum_{w \in W_\sigma} \epsilon(w) \exp\frac{2\pi i}{\ell + h^\vee}\left((w(\lambda + \overline{\rho}))(\tilde{\mu} + \check{\rho})\right),$$

where $\overline{\rho}$ and $\check{\rho}$ are the sum of the fundamental weights (respectively coweights) of the Lie algebra $\mathring{\mathfrak{g}}$. By a direct calculation, we can check that

$$J_A(\beta(\lambda), \mu) = J'_A(\lambda, \eta(\mu)).$$

To a Cartan matrix A of types $A^{(2)}_{2n-1}$, $D^{(2)}_{n+1}$, $E^{(2)}_6$, V. Kac [53] associates another Cartan matrix A' of type $D^{(2)}_{n+1}, A^{(2)}_{2n-1}$, $E^{(2)}_6$ respectively. Since $\mathfrak{g}_\sigma$ and $\mathring{\mathfrak{g}}$ are Langlands dual, we observe by the calculations in Section 8 that there a natural bijection $\eta' : \check{P}_\ell(A) \to P^\ell(\mathfrak{g}(A'))$. In particular, we get

$$(10.6) \qquad J_A(\beta(\lambda), \mu) = i^{|\mathring{\Delta}_+|}|T_{\sigma,\ell}|^{\frac{1}{2}} \overline{S^{(\ell)}_{\lambda,\eta'\circ\eta(\mu)}}(A).$$

Substituting Equation 10.6 in Equation 10.2, we get the required result by observing that $\iota = \eta' \circ \eta$.

11. Dimensions of the conformal blocks using the Verlinde formula

In this section, we use the Verlinde formula conjecture 1.1 stated in the introduction to compute dimensions of some twisted conformal blocks. In various check, we cross check our answer by obtaining the same result by using different methods like factorizations etc.

11.1. **Étale Cases.** Let $\Gamma = \langle \gamma \rangle$ be a cyclic group of order m acting on $\mathfrak{g}$ and consider an étale-Galois cover $\widetilde{C} \to C$ with Galois group Γ. We assume that $\mathfrak{g}$ is either of type $A_{n \geq 2}, D_{n \geq 4}, E_6$. We consider the following cases.

11.1.1. *Genus zero case.* Assume that $C = \mathbb{P}^1$, then $\widetilde{C} = \sqcup_{i=0}^{m-1} \mathbb{P}^1$ where γ sends the i-th component to the $i + 1$-th component. Further assume that we have three marked points $\widetilde{q}, \widetilde{p}_1$ and $\widetilde{p}_2$ on $\widetilde{C}$ and an integer i such that $\gamma^i \cdot \widetilde{p}_2, \widetilde{q}$ and $\widetilde{p}_1$ are in the same component of $\widetilde{C}$. Let $\mathbf{p}$ be the image of the tuple $(\widetilde{p}_1, \widetilde{p}_2, \widetilde{q})$ in $\mathbb{P}^1$.

Let $\lambda \in P_1(\mathfrak{g})$. We consider the following tuples $(\widetilde{C}, \mathbb{P}^1; \widetilde{p}_1, \widetilde{p}_2, \widetilde{q}; \lambda, \lambda^*, 0)$ and $(\widetilde{C}, \mathbb{P}^1; \widetilde{p}_1, \gamma^i \cdot \widetilde{p}_2, \widetilde{q}; \lambda, \gamma^i \cdot \lambda^*, 0)$ and the corresponding twisted conformal blocks $\mathbb{V}^\dagger_{\lambda, \lambda^*, 0, \Gamma}(\widetilde{C}, \mathbb{P}^1; \widetilde{p}_1, \widetilde{p}_2, \widetilde{q}, \mathbf{p})$ and $\mathbb{V}^\dagger_{\lambda, \gamma^i \cdot \lambda^*, 0, \Gamma}(\widetilde{C}, \mathbb{P}^1; \widetilde{p}_1, \gamma^i \cdot \widetilde{p}_2, \widetilde{q}, \mathbf{p})$ attached to the data. By construction (Section 4.3.4), both conformal blocks have the same dimension and moreover since $(\widetilde{p}_1, \gamma^i \widetilde{p}_2, \widetilde{q})$ are in the same component of $\widetilde{C}$, we get the following by construction:

Proposition 11.1. $\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \gamma^i \cdot \lambda^*, 0, \Gamma}(\widetilde{C}, \mathbb{P}^1; \widetilde{p}_1, \gamma^i \cdot \widetilde{p}_2, \widetilde{q}, \mathbf{p}) = \dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \gamma^i \cdot \lambda^*, 0}(\mathbb{P}^1; \mathbf{p})$, *where*

$$\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \gamma^i \cdot \lambda^*, 0}(\mathbb{P}^1; \mathbf{p}) = \begin{cases} 1 & \gamma^i \cdot \lambda^* = \lambda^* \\ 0 & otherwise \end{cases}$$

The dimension equality of Proposition 11.1 can be seen directly from the Verlinde formula in Conjecture 1.1 taking into account that the untwisted S-matrices are unitary and Γ° is trivial. Hence this verifies Conjecture 1.1 in this case.

11.1.2. *Genus one case.* Assume that C is an elliptic curve and $\widetilde{C}$ is another elliptic curve and $\widetilde{C} \to C$ is an étale Galois cover with Galois group Γ. The cover $\widetilde{C} \to C$ can be degenerated to a curve $\widetilde{D} \to D$, where $\widetilde{D}$ is a nodal curve with m-copies of $\mathbb{P}^1$ forming regular m-polygon and D is an elliptic curve with one cycle pinched. Let $\widetilde{p}$ be any marked point of $\widetilde{C}$ and p be its image in C. Note that it is an unramified point. We consider the twisted conformal block $\mathbb{V}^\dagger_\Gamma(\widetilde{C}, C) \cong \mathbb{V}^\dagger_{0, \Gamma}(\widetilde{C}, C, \widetilde{p}, p)$ (by Propagation of vacua). By factorization (Proposition 4.3), we get

$$\dim_{\mathbb{C}} \mathbb{V}^\dagger_\Gamma(\widetilde{C}, C) = \sum_{\lambda \in P_1(\mathfrak{g})} \dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \gamma \cdot \lambda^*}(\mathbb{P}^1, q, q').$$

By applying Proposition 11.1, we get $\dim_{\mathbb{C}} \mathbb{V}^\dagger_\Gamma(\widetilde{C}, C) = |P_1(\mathfrak{g})^\gamma|$. On the other hand if we apply the condition that $n = 0$ and $g = 1$ in the Verlinde formula in Conjecture 1.1, we also get $|P_1(\mathfrak{g})^\gamma|$. This also supports our conjecture.

11.1.3. *Genus one case with one marked point.* Let $\mathfrak{g} = A_N$ and γ is the diagram auto-morphism of A_N. We denote the fundamental weights by ω_i for $0 \leq i \leq N - 1$. Then $|P_1(A_N)^\gamma|$ is two if N is odd and one if N is even. We gave the following proposition to support Conjecture 1.1.

Proposition 11.2. *Let $\widetilde{C}$ and C be as above and $\widetilde{p}$ be a marked point in $\widetilde{C}$ and p be denote the image of $\widetilde{p}$ in C.*

(1) *Let $\mathfrak{g} = A_{2r-1}$, then $\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\omega_i, \Gamma}(\widetilde{C}, C, \widetilde{p}, p) = 2$ if i is even and 0 otherwise.*

(2) *Let $\mathfrak{g} = A_{2r}$, then $\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\omega_i, \Gamma}(\widetilde{C}, C, \widetilde{p}, p) = 1$, for all i.*

Proof. We degenerate the cover $\widetilde{C} \to C$ to $\widetilde{D} \to D$ as in the Section 11.1.2. We apply the factorization theorem and get

$$\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\omega_i, \Gamma}(\widetilde{C}, C, \widetilde{p}, p) = \sum_{\lambda \in P_1(\mathfrak{g})} \dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \gamma \cdot \lambda^*, \omega_i}(\mathbb{P}^1, \mathbf{p}').$$

It is well known [31] that if $\mathfrak{g} = A_r$, then the dimension of level one conformal blocks with three marked points $\dim_{\mathbb{C}} \mathbb{V}_{\omega_i, \omega_j, \omega_k}(\mathbb{P}^1, \mathbf{p}')$ is one if and only $(r+1)|(i+j+k)$ and is zero otherwise. Using this we have the following:

- Let $\mathfrak{g} = A_{2r-1}$. If i is odd and $\lambda = \omega_m$, we get $\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \lambda, \omega_i}(\mathbb{P}^1, \mathbf{p}') = 0$. This follows from the fact that by choice $2m + i$ is odd and hence $2\lambda + \omega_i$ is not in the root lattice of A_{2r-1}.
- Let $\mathfrak{g} = A_{2r-1}$. If $i = 2a$ is even, $a \leq r-1$ and $\lambda = \omega_m$, then $\dim_{\mathbb{C}} \mathbb{V}^\dagger_{\lambda, \lambda, \omega_{2a}}(\mathbb{P}^1, \mathbf{p}')$ is one dimensional if and only if $2m + 2a$ is divisible by $2r$. In the range $0 \leq m < 2r$, then $m = r - a$ or $(2r - a)$ are the two solutions.
- Let $\mathfrak{g} = A_{2r}$. For any $\lambda = \omega_m$, we get the condition $2m + i$ is divisible by $2r + 1$ and $0 \leq m \leq 2r$ has exactly one solution.

Thus the result follows. $\square$

This can also be checked by using the Verlinde formula in Conjecture 1.1.

11.1.4. *Étale double cover of a genus two curve.* Let $\widetilde{C}$ be a genus 3 curve and C be a genus two curve such that $\widetilde{C} \to C$ is an étale Galois cover of degree 2. Let $\mathfrak{g} = A_N$ and consider the conformal blocks $\mathbb{V}^\dagger_\Gamma(\widetilde{C}, C)$. We can degeneratete the cover such that it normalization is a disjoint union of étale double covers of elliptic curves. By the computation in Section 11.1.3, we get the following:

(1) Let $\mathfrak{g} = A_{2r-1}$, then $\dim_{\mathbb{C}} \mathbb{V}^\dagger_\Gamma(\widetilde{C}, C) = 4r$.

(2) Let $\mathfrak{g} = A_{2r}$, then $\dim_{\mathbb{C}} \mathbb{V}^\dagger_\Gamma(\widetilde{C}, C) = 2r + 1$.

11.1.5. *Arbitrary genus case.* Let μ be any element in $P_1(\mathfrak{g})$ and S be the uncrossed S-matrix for $\widehat{L}(\mathfrak{g})$ at level one. We observe the following:

(1) If $\mathfrak{g} = A_n$, then $S_{0,\mu} = \frac{1}{\sqrt{n+1}}$.

(2) If $\mathfrak{g} = D_n$, then $S_{0,\mu} = \frac{1}{2}$.

(3) If $\mathfrak{g} = E_6$, then $S_{0,\mu} = \frac{1}{\sqrt{3}}$.

Let γ be an automorphism of order m acting on $\mathfrak{g}$ and let $\widetilde{C} \to C$ be a étale cyclic Galois cover of order m. Let g be the genus of C. Then by the Verlinde formula in Conjecture 1.1, we get

$$(11.1) \qquad \dim_{\mathbb{C}} \mathbb{V}_{\Gamma}^{\dagger}(\widetilde{C}, C) = |P_1(\mathfrak{g})^{\gamma}|(S_{0,0})^{2-2g}.$$

The above can be checked directly using the degenerations and the genus zero computations in Sections 11.1.3, 11.1.4 for the group $\mathfrak{g} = A_N$ and γ being the diagram automorphism of A_N directly using the degenerations and the genus zero computations in Sections 11.1.3, 11.1.4 for the group $\mathfrak{g} = A_N$ and γ is the diagram automorphism of A_N.

Remark 11.3. G. Faltings [33] has shown that in the untwisted case that conformal blocks for a simply laced Lie $\mathfrak{g}$ algebra has dimension $|Z_G|^g$, where Z_G is the center of the simply connected group with center $\mathfrak{g}$. This can also be seen from the Verlinde formula. The formula in Equation 11.1 can be thought of a generalization of the result to the twisted case.

11.2. **Ramified Cases with diagram automorphisms of order two.** We use compute some dimensions of twisted covers of conformal blocks associated diagram automorphism and ramified cover of curves.

11.2.1. *Twisted affine Lie algebra:* $A_{2r-1}^{(2)}$ *at level one.* Consider the diagram automorphism of A_{2r-1}. The level one weights of $A_{2r-1}^{(2)}$ are $\{0, \dot{\omega}_1\}$. By Proposition 8.6 and the computations in Section 8.11.1, tell us that the crossed S-matrix is given by the following 2×2 matrix.

$$S^{\gamma} = \sqrt{2} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Consider the Galois two cover $\mathbb{P}^1 \to \mathbb{P}^1$ ramified at two points $\widetilde{p}_1$ and $\widetilde{p}_2$ and let $\widetilde{p}_3$ be any étale point on $\mathbb{P}^1$. Then using the Verlinde formula in Conjecture 1.1 and the KAC software, we get the dimension of the level one twisted conformal block

$$\dim_{\mathbb{C}} \mathbb{V}_{0,0,\omega_i}^{\dagger}(\mathbb{P}^1, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{otherwise}, \end{cases}$$

where ω_i is the i-th fundamental weight of A_{2r-1}. Similarly, we can also check that

$$\dim_{\mathbb{C}} \mathbb{V}_{0,\dot{\omega}_1,\omega_i}^{\dagger}(\mathbb{P}^1, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = \begin{cases} 1 & \text{if } i \text{ is odd} \\ 0 & \text{otherwise}. \end{cases}$$

This can be verified directly from the fact that these conformal blocks are isomorphism to the space of invariant $\mathrm{Hom}_{\mathfrak{sp}(2r)}(\Lambda^i \mathbb{C}^{2r}, \mathbb{C})$ (respectively $\mathrm{Hom}_{\mathfrak{sp}(2r)}(\mathbb{C}^{2r} \otimes \Lambda^i \mathbb{C}^{2r}, \mathbb{C})$) which has dimensions one or zero depending on the parity of i.

11.2.2. $A_{2n-1}^{(2)}$ *and four ramification points at level one.* Consider the degree two Galois cover of $\mathbb{P}^1$ ramified at four points $\widetilde{\mathbf{p}}$. Then $\widetilde{C}$ is an elliptic curve. By the crossed S-matrix in Section 11.2.1 and the Verlinde formula in Conjecture 1.1, we get the following:

(1) $\dim_{\mathbb{C}} \mathbb{V}_{0,0,0,0}^{\dagger}(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = 1;$

(2) $\dim_{\mathbb{C}} \mathbb{V}_{0,0,\dot{\omega}_1,\dot{\omega}_1}^{\dagger}(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = 1;$

(3) $\dim_{\mathbb{C}} \mathbb{V}_{0,0,0,\dot{\omega}_1}^{\dagger}(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = 1.$

We can now degeneratete the double cover of $\mathbb{P}^1$ with four marked points to a reducible $\mathbb{P}^1$ with two components meeting at one point, which is étale. Then normalizing we get two disjoint copies of $\mathbb{P}^1$ with a double cover ramified at two points. By factorization and the calculations in Section 11.2.1 we get back the above numbers.

11.2.3. $A_{2r-1}^{(2)}$ *and arbitrary genus at level one.* Let C be a smooth curve of genus g and consider a double cover $\widetilde{C}$ of C which is ramified at $2n$ points $\widetilde{\mathbf{p}}$. Assume that $\widetilde{C}$ is connected and hence is of genus $2g + n - 1$. Consider the $2n$-tuple of weights $\vec{0} = (0, \ldots, 0)$. Then by the Verlinde formula in Conjecture 1.1, we get

$$\dim_{\mathbb{C}} \mathbb{V}_{\vec{0}}^{\dagger}(\widetilde{C}, C, \widetilde{\mathbf{p}}, \mathbf{p}) = 2^g r^{g+n-1}.$$

If we assume that $g = 0$, then $\widetilde{C}$ is a curve of genus $n - 1$. We can degenerate the cover $\widetilde{C} \to \mathbb{P}^1$ to cover $\widetilde{D}' \to D$, where $\widetilde{D}'$ has $n - 1$-components consists of a chain of elliptic curves such that the two extreme components has three ramification points and all others have two ramification points. Moreover all the nodes are also ramification points. We can applying the factorization theorem and apply the calculations in Section 11.2.2 to get the dimension of the twisted conformal block is r^{n-1}. This also verifies the Verlinde formula conjecture 1.1 in this case.

11.2.4. *Twisted affine Lie algebra:* $A_{2r}^{(2)}$ *in genus zero.* Consider the diagram automorphism of $A_{2r}^{(2)}$. The level one weights of $A_{2r}^{(2)}$ is just the r-th fundamental weight $\dot{\omega}_r$ of the Lie algebra of type B_r. Consider the same data of the curve and marked points as in Section 11.2.1. Then by the Verlinde formula in Conjecture 1.1, we get the dimension of the twisted conformal blocks at level one.

(11.2) $$\dim_{\mathbb{C}} \mathbb{V}_{\dot{\omega}_r, \dot{\omega}_r, \omega_i}^{\dagger}(\mathbb{P}^1, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = 1.$$

Again this can be verified directly by the fact that $\mathbb{V}_{\dot{\omega}_r, \dot{\omega}_r, \omega_i}^{\dagger}(\mathbb{P}^1, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p})$ is isomorphic to the space $\mathrm{Hom}_{\mathfrak{so}(2r+1)}(V_{\dot{\omega}_r} \otimes V_{\dot{\omega}_r} \otimes \wedge^i \mathbb{C}^{2r+1}, \mathbb{C})$, where $V_{\dot{\omega}_r}$ are irreducible representations of $\mathfrak{so}(2r+1)$ with highest weight $\dot{\omega}_r$. This space of invariants is always one dimensional by [55].

11.3. **The twisted affine Lie algebra** $D_4^{(3)}$**.** Consider a connected ramified Galois cover $\widetilde{C} \to \mathbb{P}^1$ of order three with three marked points $\widetilde{\mathbf{p}}$. By the Riemann-Hurwitz formula, we get that $\widetilde{C}$ is an elliptic curve. We consider the triple $\vec{0} = (0, 0, 0)$ of weights in $P^\ell(D_4^{(3)})$ and consider the twisted conformal block $\mathbb{V}_{\vec{0}}^{\dagger}(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p})$ at level ℓ.

Remark 11.4. Since the curve $\widetilde{C}$ has genus one, like in the untwisted case, the twisted conformal blocks $\mathbb{V}_0^\dagger(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p})$ do not embed in the space of invariants of tensor product representations. Hence the dimension of invariants do not give any natural upper bounds. This will be clear in the next two examples.

11.3.1. *The case $\ell = 1$.* In this case, the set $P^1(D_4^{(3)})$ and it follows that the crossed S-matrix is has integer 1. Now by the uncrossed S-matrix for D_4^1 at level one has $S_{0,0} = \frac{1}{2}$. Hence by the Verlinde formula in Conjecture 1.1, it follows that $\mathbb{V}_0^\dagger(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p})$ is two dimensional.

11.3.2. *The case $\ell = 2$.* The set $P^2(D_4^{(3)}) = \{0, \dot{\omega}_2\}$. By Proposition 8.6, Section 8.11.1 and using KAC software, we get the crossed S-matrix:

$$S^\gamma = \sqrt{3} \cdot \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{pmatrix}.$$

The fixed points of $P^1(D_4^{(1)})$ under the diagram automorphism are $\{0, \omega_2\}$ and from the uncrossed S-matrix we get $S_{0,0} = \frac{1}{2\sqrt{8}}$ and $S_{0,\omega_2} = \frac{1}{\sqrt{8}}$. Now the Verlinde formula in Conjecture 1.1 implies that $\dim_{\mathbb{C}} \mathbb{V}_0^\dagger(\widetilde{C}, \mathbb{P}^1, \widetilde{\mathbf{p}}, \mathbf{p}) = 3$.

Appendix A. Moduli stacks of admissible Γ-covers

Let Γ be a finite group. In this appendix we recall the definitions, basic properties and operations on the stacks $\overline{\mathcal{M}}_{g,n}^\Gamma$ from [51] as well as the related stacks $\widehat{\mathcal{M}}_{g,n}^\Gamma$ and $\widetilde{\mathcal{M}}_{g,n}^\Gamma$ for non-negative integers g, n. These are moduli stacks of (balanced) admissible Γ-covers of stable n-pointed curves of genus g with additional marking data as described in this section.

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford stack of stable n-pointed curves of genus g. Let $(C \to T; p_1, \ldots, p_n)$ be a stable n-pointed curve of genus g over a scheme T with marked sections $p_1, \ldots, p_n : T \to C$. For such a stable n-pointed curve $(C, \mathbf{p})$, let $C_{\mathrm{gen}} \subseteq C$ denote the open subset obtained by removing the marked points $\mathbf{p} = (p_1, \cdots, p_n)$ and nodes of C. For a finite group Γ, the notion of a balanced admissible Γ-cover $\pi : \widetilde{C} \longrightarrow C$ of such a stable pointed curve is defined in [2]. In particular we have a (left) action of Γ on $\widetilde{C}$ and $\pi : \widetilde{C} \to C$ maps nodes to nodes, π is a left principal Γ-bundle above the open part $C_{\mathrm{gen}} \subseteq C$ and induces an isomorphism $\widetilde{C}/\Gamma \cong C$. Moreover for each $\tilde{p} \in \widetilde{C}$ the stabilizer $\Gamma_{\tilde{p}} \subseteq \Gamma$ is a cyclic group. If $\tilde{q} \in \widetilde{C}$ is a node, then we have also imposed balancedness, that is, the induced action of the cyclic group $\Gamma_{\tilde{q}}$ on the two branches of the tangent cone are inverse to each other. As in [2], we denote the Deligne-Mumford stack of such admissible covers $(\pi : \widetilde{C} \to C; \mathbf{p})$ by $\overline{\mathcal{M}}_{g,n}(\mathcal{B}\Gamma)$.

Following [51], we now consider a certain variant of the above stack of admissible Γ-covers. Given an admissible Γ-cover $(\widetilde{C} \to C; \mathbf{p})$, where $\mathbf{p} = (p_1, \cdots, p_n)$, let $\tilde{p}_i \in \pi^{-1}(p_i)$ be a choice of a point in the fiber over p_i for all $1 \leq i \leq n$. We denote $\widetilde{\mathbf{p}} = (\tilde{p}_1, \cdots, \tilde{p}_n)$.

Definition A.1. Let $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ denote the stack of n-pointed admissible Γ-covers

$$(\pi : \widetilde{C} \to C; \widetilde{\mathbf{p}}; \mathbf{p})$$

of n-pointed, genus-g stable curves, where $\widetilde{\mathbf{p}} = (\tilde{p}_1, \cdots, \tilde{p}_n)$ are a choice of marked points of $\widetilde{C}$ lying above the marked points $\mathbf{p} = (p_1, \cdots, p_n)$ in C.

The orientation of the curve C and the fact that $\widetilde{C}$ is a principal Γ-bundle over C_{gen} give rise to an n-tuple $\mathbf{m} = (m_1, \ldots, m_n) \in \Gamma^n$ keeping track of the monodromies around the points $\tilde{p}_i$'s. More precisely, the isotropy subgroup $\Gamma_{\tilde{p}_i}$ of the point $\tilde{p}_i$ is a cyclic subgroup, of order say N_i. The cyclic group $\Gamma_{\tilde{p}_i}$ acts on the tangent space $T_{\tilde{p}_i}\widetilde{C}$ faithfully. Then $m_i \in \Gamma$ is defined as the generator of $\Gamma_{\tilde{p}_i}$ which acts as multiplication by $\exp \frac{2\pi\sqrt{-1}}{N_i}$. Hence we have the evaluation morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}^{\Gamma} \to \Gamma^n$$

and let $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m}) := \mathrm{ev}^{-1}(\mathbf{m})$. We also have the morphism of stacks $\overline{\mathcal{M}}_{g,n}^{\Gamma} \to \overline{\mathcal{M}}_{g,n}$ defined by forgetting the admissible cover. The following theorem is due to [51]:

Theorem A.2. *The stack $\overline{\mathcal{M}}_{g,n}^{\Gamma}$ and the open and closed substacks $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ are smooth Deligne-Mumford stacks, flat, proper, and quasi finite over $\overline{\mathcal{M}}_{g,n}$. Moreover, the $\overline{\mathcal{M}}_{g,n}^{\Gamma}(\mathbf{m})$ are a finite disjoint union of connected components of $\overline{\mathcal{M}}_{g,n}^{\Gamma}$.*

Next we introduce some more variants of these stacks that are needed in the paper.

A.1. Moduli stacks of pointed admissible covers with local coordinates.

We now introduce the stack of n-pointed admissible Γ-covers of stable n-pointed curves of genus g with local coordinates.

Definition A.3. Let $\widehat{\overline{\mathcal{M}}}_{g,n}^{\Gamma}$ denote the stack of n-marked admissible Γ-covers

$$(\pi : \widetilde{C} \to C; \widetilde{\mathbf{p}}; \mathbf{p}; \widetilde{\mathbf{z}}; \mathbf{z})$$

where $(\pi : \widetilde{C} \to C; \widetilde{\mathbf{p}}; \mathbf{p})$ is an n-pointed admissible Γ-cover and $\widetilde{\mathbf{z}} = (\tilde{z}_1, \cdots, \tilde{z}_n)$ are special formal local coordinates at the points $\widetilde{\mathbf{p}}$ in $\widetilde{C}$ such that $\mathbf{z} = (\tilde{z}_1^{N_1}, \cdots, \tilde{z}_n^{N_n})$ are formal local coordinates at the points $\mathbf{p}$ in C, where N_i is the order of the cyclic group $\Gamma_{\tilde{p}_i}$. As before, associated with such a data we have the monodromy $\mathbf{m} \in \Gamma^n$ around the points $\widetilde{\mathbf{p}}$. For notational convenience, we will often drop $\mathbf{z}$ from the notation of a family of n-pointed admissible Γ-covers with chosen coordinates.

For a finite set A, let $\widehat{\overline{\mathcal{M}}}_{g,A}^{\Gamma}$ denote the stack of A-marked admissible covers, where instead of numbering the marking data, we have bijections of the marked points and formal coordinates with A. In this setting the associated monodromy of an A-marked admissible Γ-cover is a function $\mathbf{m} : A \to \Gamma$. Given a function $\mathbf{m} : A \to \Gamma$ we let $\widehat{\overline{\mathcal{M}}}_{g,A}^{\Gamma}(\mathbf{m}) \subseteq \widehat{\overline{\mathcal{M}}}_{g,A}^{\Gamma}$ be the substack of those A-marked admissible Γ-covers with monodromy given by $\mathbf{m}$.

Definition A.4. Let A be a finite set. Let $\widetilde{\overline{\mathcal{M}}}_{g,A}^{\Gamma}$ denote the stack of n-marked admissible Γ-covers of the form

$$(\pi : \widetilde{C} \to C; \widetilde{\mathbf{p}}; \mathbf{p}; \widetilde{\mathbf{v}}; \mathbf{v})$$

where $(\pi : \widetilde{C} \to C; \widetilde{\mathbf{p}}; \mathbf{p})$ is an A-pointed admissible Γ-cover, $\mathbf{v}$ is a choice of non-zero tangent vectors to C at the points $\mathbf{p}$ and $\widetilde{\mathbf{v}}$ is a choice of tangent vectors to $\widetilde{C}$ at the points $\widetilde{\mathbf{p}}$ compatible with $\mathbf{v}$.

It is clear that we have the following commutative diagram:

$$(A.1) \qquad \begin{array}{ccccc} \widehat{\overline{\mathcal{M}}}_{g,A}^{\Gamma} & \longrightarrow & \widetilde{\overline{\mathcal{M}}}_{g,A}^{\Gamma} & \longrightarrow & \overline{\mathcal{M}}_{g,A}^{\Gamma} \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{\overline{\mathcal{M}}}_{g,A} & \longrightarrow & \widetilde{\overline{\mathcal{M}}}_{g,A} & \longrightarrow & \overline{\mathcal{M}}_{g,A}. \end{array}$$

A.2. **A stratification in terms of dual graphs of stable pointed curves.** We now describe a stratification of the smooth Deligne-Mumford stacks $\overline{\mathcal{M}}_{g,A}$ and $\widetilde{\overline{\mathcal{M}}}_{g,A}$ in terms of the dual graph of stable pointed curves. We refer to [6, Chapter XII] for more details and proofs. Let us first recall the definition of weighted legged graphs and some related notions.

Definition A.5. A weighted legged graph is a tuple (V, H, ι, v, w), where V is the finite set of vertices, H is the finite set of half-edges, $\iota : H \to H$ is an involution, $v : H \to V$ the function which assigns to a half-edge its vertex of origin and finally $w : V \to \mathbb{Z}_{\geq 0}$ is a weight function on the vertices. The fixed points $H^{\iota} \subseteq H$ are said to be the legs of such a weighted legged graph. We let E (the set of edges) denote the set of ι-orbits of size 2.

Let A be a finite set. A weighted A-legged graph is a tuple $(V, H, \iota, v, w, l : A \cong H^{\iota})$, i.e. we also have the additional data of a bijection between A and the legs H^{ι}.

The genus g of such a weighted legged graph is then defined to be (in particular, it does not depend on the data related to the legs):

$$g := |E| - |V| + \sum_{v \in V} w(v) + \pi_0,$$

where π_0 is the number of connected components of the graph.

For a weighted legged graph, the degree $\deg(v)$ of a vertex is, as usual, the total number of half-edges incident at the vertex. We say that a weighted A-legged graph is stable if it is connected and for every $v \in V$, we have $2w(v) - 2 + \deg(v) > 0$.

Remark A.6. In this paper we will only consider stable (and in particular connected) weighted legged graphs. The genus of such a graph is $g := |E| - |V| + \sum_{v \in V} w(v) + 1$.

An isomorphism between weighted legged graphs is an isomorphism between the associated 1-dimensional CW complexes which preserves the weights. Equivalently an isomorphism is a pair of bijections between the sets of vertices and the sets of half-edges preserving

all the structures. If A is a finite set, then an isomorphism between A-legged graphs is an isomorphism as before which also respects the labeling of the legs by A. If X is an A-legged graph, we let $\mathrm{Aut}(X)$ denote its group of automorphisms as above.

Let us denote by $\mathbb{X}_{g,A}$ the set of isomorphism classes of *stable* weighted A-legged graphs of genus g. If this set is non-empty, then we say that the pair (g, A) is a stable pair. If $(C, \mathbf{p})$ is a stable A-pointed curve of genus g, then we have its associated dual graph $X(C, \mathbf{p})$ whose vertices are the irreducible components of C weighted by their arithmetic genus, edges correspond to the nodes and legs to the marked points. Then $X(C, \mathbf{p}) \in \mathbb{X}_{g,A}$. Hence we get a stratification of stacks $\overline{\mathcal{M}}_{g,A} = \coprod_{X \in \mathbb{X}_{g,A}} \mathcal{M}_{X,g,A}$ and $\widetilde{\overline{\mathcal{M}}}_{g,A} = \coprod_{X \in \mathbb{X}_{g,A}} \widetilde{\mathcal{M}}_{X,g,A}$,

where $\mathcal{M}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ and $\widetilde{\mathcal{M}}_{X,g,A} \subseteq \widetilde{\overline{\mathcal{M}}}_{g,A}$ are the substacks formed by the stable A-pointed curves of genus g whose dual graph is isomorphic to $X \in \mathbb{X}_{g,A}$. The strata $\mathcal{M}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ (resp. $\widetilde{\mathcal{M}}_{X,g,A} \subseteq \widetilde{\overline{\mathcal{M}}}_{g,A}$) are locally closed. Let $\overline{\mathcal{M}}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ (resp. $\widetilde{\overline{\mathcal{M}}}_{X,g,A} \subseteq \widetilde{\overline{\mathcal{M}}}_{g,A}$) denote the closure of the strata.

A.3. Stratifications for stacks of pointed admissible covers.

We now describe stratifications of the moduli stacks $\overline{\mathcal{M}}_{g,A}^{\Gamma}$, $\widetilde{\overline{\mathcal{M}}}_{g,A}^{\Gamma}$. For this we need the notion of a weighted A-legged Γ-marked graph.

Definition A.7. Let Γ be a finite group and A a finite set. A weighted A-legged Γ-marked graph is a tuple $(V, H, \iota, v, w, l : A \cong H^\iota, m : H^\iota \to \Gamma, \mu)$ where $(V, H, \iota, v, w, l : A \cong H^\iota)$ is a weighted A-legged graph, m assigns an element of the group Γ to every leg and μ assigns to each edge an unordered pair $\{\gamma, \gamma^{-1}\}$ up to Γ-conjugacy under the diagonal action. Such a weighted A-legged Γ-marked graph is said to be stable if the underlying weighted legged graph is so. An isomorphism of weighted A-legged Γ-marked graphs is an isomorphism of the underlying weighted A-legged graphs which is compatible with the edge markings. The set of isomorphism classes of stable weighted A-legged Γ-marked graphs of genus g will be denoted by $\mathbb{X}_{g,A}^{\Gamma}$ (see also [50]).

If $(\widetilde{C} \to C, \widetilde{\mathbf{p}}, \mathbf{p}) \in \overline{\mathcal{M}}_{g,A}^{\Gamma}$, then we have the weighted A-legged dual graph $X(C, \mathbf{p})$ as defined previously. The monodromies around the points $\widetilde{\mathbf{p}}$ determine the Γ-markings on the legs. Furthermore, the nodes of C are parametrized by the edges of $X(C, \mathbf{p})$. For each node $q \in C$, let us choose a lift $\tilde{q} \in \widetilde{C}$, and a branch of the tangent cone at $\tilde{q}$. These additional choices determines a unique generator γ of the stabilizer $\Gamma_{\tilde{q}}$. If we had chosen the other branch, γ would be replaced by γ^{-1}. We attach to the node q (or equivalently the corresponding edge in the dual graph) the Γ-conjugacy class of the unordered pair $\{\gamma, \gamma^{-1}\}$. Hence for each $(\widetilde{C} \to C, \widetilde{\mathbf{p}}, \mathbf{p}) \in \overline{\mathcal{M}}_{g,A}^{\Gamma}$, we have defined its dual graph $X^{\Gamma}(\widetilde{C} \to C, \widetilde{\mathbf{p}}, \mathbf{p})$ which is a weighted A-legged Γ-marked graph. Hence we have stratifications

$$(\text{A.2}) \qquad \overline{\mathcal{M}}_{g,A}^{\Gamma} = \coprod_{X^{\Gamma} \in \mathbb{X}_{g,A}^{\Gamma}} \mathcal{M}_{X^{\Gamma},g,A}^{\Gamma} \text{ and } \widetilde{\overline{\mathcal{M}}}_{g,A}^{\Gamma} = \coprod_{X^{\Gamma} \in \mathbb{X}_{g,A}^{\Gamma}} \widetilde{\mathcal{M}}_{X^{\Gamma},g,A}^{\Gamma},$$

where $\mathcal{M}^{\Gamma}_{X^{\Gamma},g,A}$ (respectively $\widetilde{\mathcal{M}}^{\Gamma}_{X^{\Gamma},g,A}$) denote the locus of A-pointed Γ-covers of genus g curves (respectively with chosen non-zero tangent vectors) whose underlying dual graph is X^{Γ}.

We denote by $\overline{\mathcal{M}}^{\Gamma}_{X^{\Gamma},g,A} \subseteq \overline{\mathcal{M}}^{\Gamma}_{g,A}$ and $\overline{\widetilde{\mathcal{M}}}^{\Gamma}_{X^{\Gamma},g,A} \subseteq \overline{\widetilde{\mathcal{M}}}^{\Gamma}_{g,A}$ the closure of the corresponding locally closed strata. We have the forgetful map $\mathbb{X}^{\Gamma}_{g,A} \to \mathbb{X}_{g,A}$. If $X \in \mathbb{X}_{g,A}$, we let $\mathbb{X}^{\Gamma}(X)$ denote the fiber of the forgetful map above X and we set

$$\mathcal{M}^{\Gamma}_{X,g,A} = \coprod_{X^{\Gamma} \in \mathbb{X}^{\Gamma}(X)} \mathcal{M}^{\Gamma}_{X^{\Gamma},g,A} \text{ and } \widetilde{\mathcal{M}}^{\Gamma}_{X,g,A} = \coprod_{X^{\Gamma} \in \mathbb{X}^{\Gamma}(X)} \widetilde{\mathcal{M}}^{\Gamma}_{X^{\Gamma},g,A}$$

to obtain coarser stratifications of the moduli stacks parametrized by the set $\mathbb{X}_{g,A}$.

Appendix B. Clutching maps for the moduli stacks

In this appendix, we describe clutching maps between the moduli stacks $\overline{\mathcal{M}}_{g,A}, \overline{\mathcal{M}}^{\Gamma}_{g,A}$ as well as the stacks $\overline{\widetilde{\mathcal{M}}}_{g,A}, \overline{\widetilde{\mathcal{M}}}^{\Gamma}_{g,A}$.

B.1. **The category of weighted legged graphs.** We will work with the category of weighted legged graphs as defined in [40, 69]. Let us denote by $\mathfrak{X}_{g,A}$ the category whose objects are stable weighted A-legged graphs of genus g. A morphism between two weighted A-legged graphs X, Y is a composition of an isomorphism and edge contractions which preserve the labeling of the legs and weights. In particular if the vertices $v_1, \cdots, v_k \in V(X)$ are mapped to a vertex $v \in V(Y)$ by such a morphism, we demand the equality

$$\text{(B.1)} \qquad 2w_Y(v) - 2 + \deg_Y(v) = \sum_{i=1}^{k} 2w_X(v_i) - 2 + \deg_X(v_i).$$

For example, this last condition tells us what weight to assign to a vertex if certain edges have been contracted. It guarantees that the genus of X and Y must be the same. Hence we have defined categories $\mathfrak{X}_{g,A}$ of weighted A-legged graphs of genus g for each $g \in \mathbb{Z}_{\geq 0}$ and finite set A.

Remark B.1. Consider the weighted A-legged graph with one vertex of weight g and no edges. Such a graph is known as a corolla. It is the final object of the category $\mathfrak{X}_{g,A}$.

We say that (g, A) is a stable pair if $\mathfrak{X}_{g,A}$ is non-empty. This is equivalent to saying that the A-legged corolla of weight g is stable. We will always implicitly assume that all pairs (g, A) considered in this paper are stable.

B.2. **Clutching with respect to morphisms between weighted legged graph.** Let $X \in \mathfrak{X}_{g,A}$ be a weighted A-legged graph and let $L(X)$ denote the set of A-legs of X. As described previously we have the locally closed stratum $\mathcal{M}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ (resp. $\widetilde{\mathcal{M}}_{X,g,A} \subseteq \overline{\widetilde{\mathcal{M}}}_{g,A}$) and its closure $\overline{\mathcal{M}}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ (resp. $\overline{\widetilde{\mathcal{M}}}_{X,g,A} \subseteq \overline{\widetilde{\mathcal{M}}}_{g,A}$).

We now recall the definition of the clutching map associated with X. First we cut up the graph X at the midpoints of all its edges. For each vertex v of X, we now have a set of half edges and legs incident at v. Let us denote this set by L_v. Note that

$A \subseteq \coprod_{v \in V(X)} L_v =: H(X)$, the set of all half-edges of X. We define the smooth Deligne-Mumford stack

(B.2) $\qquad \overline{\mathcal{M}}_X := \prod_{v \in V(X)} \overline{\mathcal{M}}_{w(v),L(v)}$ and its open substack $\mathcal{M}_X := \prod_{v \in V(X)} \mathcal{M}_{w(v),L(v)},$

where we recall that $w(v)$ denotes the weight of the vertex v of X. Similarly we define

(B.3) $\qquad \widetilde{\overline{\mathcal{M}}}_X := \prod_{v \in V(X)} \widetilde{\overline{\mathcal{M}}}_{w(v),L(v)}$ and its open substack $\widetilde{\mathcal{M}}_X := \prod_{v \in V(X)} \widetilde{\mathcal{M}}_{w(v),L(v)}.$

Then as described in [6, Ch. XII-§7] we have the clutching map

(B.4) $\quad \xi_X : \overline{\mathcal{M}}_X \twoheadrightarrow \overline{\mathcal{M}}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ restricting to $\xi_X : \mathcal{M}_X \twoheadrightarrow [\mathcal{M}_X / \operatorname{Aut}(X)] \cong \mathcal{M}_{X,g,A}$

giving a description of the stratum $\mathcal{M}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$ as stack quotient of $\mathcal{M}_X$ by the group of automorphisms of the stable weighted A-legged graph X. The clutching map factorizes as

$$\xi_X : \overline{\mathcal{M}}_X \twoheadrightarrow \left[\overline{\mathcal{M}}_X / \operatorname{Aut}(X) \right] \twoheadrightarrow \overline{\mathcal{M}}_{X,g,A} \subseteq \overline{\mathcal{M}}_{g,A}$$

where the second map is a normalization which is an isomorphism on the open part $\mathcal{M}_{X,g,A} \subseteq \overline{\mathcal{M}}_{X,g,A}$.

So far, we have described the clutching operation associated with an object $X \in \mathcal{X}_{g,A}$. We will now describe clutching with respect to morphisms in $\mathcal{X}_{g,A}$. The previous construction can be thought of as a special case corresponding to the unique morphism of any given object $X \in \mathcal{X}_{g,A}$ to the final object (which we called the corolla).

Let $f : X \to Y$ be a morphism in $\mathcal{X}_{g,A}$. In particular we have a surjection $f_* : V(X) \to V(Y)$ and a map $f^* : H(Y) \to H(X)$ which is a bijection on the legs preserving the A-markings and which is induces an injection on the edges. Also we have the two smooth Deligne-Mumford stacks $\overline{\mathcal{M}}_X$ and $\overline{\mathcal{M}}_Y$. Then associated with the morphism $f : X \to Y$ we have the clutching map defined by gluing together the points marked by the half-edges corresponding to the edges contracted by $f : X \to Y$

(B.5) $$\xi_f : \overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_Y.$$

Remark B.2. In case Y is the corolla, $\overline{\mathcal{M}}_Y = \overline{\mathcal{M}}_{g,A}$ and we recover the previously defined clutching map given in Equation (B.4).

Remark B.3. It is clear that if we have two morphisms $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$ in $\mathcal{X}_{g,A}$ then $\xi_{f_2 \circ f_1} = \xi_{f_2} \circ \xi_{f_1}$. In other words the assignment $X \mapsto \overline{\mathcal{M}}_X$ is functorial.

For each $h \in H(X)$ there is the line bundle $\mathcal{L}_h$ (known as the point bundle associated with $h \in H(X)$) on the stack $\overline{\mathcal{M}}_X$ and its dual line bundle $\mathcal{L}_h^\vee$ on $\overline{\mathcal{M}}_X$. By definition, the fiber of $\mathcal{L}_h^\vee$ at (a possibly disconnected curve with connected components parametrized by $V(X)$) $(D, \mathbf{q}) \in \overline{\mathcal{M}}_X$ is the tangent space $T_{\mathbf{q}(h)}D$ to the curve D at the point labeled by

$h \in H(X)$. Let us recall from [6, Ch. XIII-§3] that the normal bundle on $\overline{\mathcal{M}}_X$ to the clutching map ξ_f decomposes as

$$(\text{B.6}) \qquad N\xi_f = \bigoplus_{\{h_1, h_2\} \in E(X) \setminus f^* E(Y)} \mathcal{L}_{h_1}^\vee \otimes \mathcal{L}_{h_2}^\vee,$$

where the summation is over the edges of X which have been contracted by $f : X \to Y$.

Now we want to lift this clutching construction to the stacks (see Equation (B.3) for notation) $\widetilde{\overline{\mathcal{M}}}_X$ and $\widetilde{\overline{\mathcal{M}}}_Y$ taking into account the normal bundles. Consider the stack $\widetilde{\overline{\mathcal{M}}}_{X,f}$ parametrizing data of the form $(D, \mathbf{q}, \mathbf{v})$, where $(D, \mathbf{q}) \in \overline{\mathcal{M}}_X$ is an $H(X)$-pointed curve and $\mathbf{v}$ is a choice of non-zero tangent vectors to the possibly disconnected curve D at the points marked by $f^* H(Y) \subseteq H(X)$. We should think as $\mathbf{q}$ and $\mathbf{v}$ as morphisms from $H(X)$ that assigns a smooth marked point and a non-zero tangent vector at the marked points respectively. Note that

$$A \cong L(Y) \cong L(X) \subseteq f^* H(Y) \subseteq H(X).$$

We have the natural forgetful map $\widetilde{\overline{\mathcal{M}}}_X \to \widetilde{\overline{\mathcal{M}}}_{X,f}$ which forgets the choice of tangent vectors at the points marked by $H(X) \setminus f^* H(Y)$. We have a Cartesian square of the form

$$(\text{B.7})$$

$$\begin{array}{ccc}
\widetilde{\overline{\mathcal{M}}}_{X,f} & \xrightarrow{\widetilde{\xi}_f} & \widetilde{\overline{\mathcal{M}}}_Y \\
{\scriptstyle \mathbb{G}_m^{f^* H(Y)} - torsor} \Big\downarrow & & \Big\downarrow {\scriptstyle \mathbb{G}_m^{H(Y)} - torsor} \\
\overline{\mathcal{M}}_X & \xrightarrow{\xi_f} & \overline{\mathcal{M}}_Y,
\end{array}$$

where $\widetilde{\xi}_f$ denotes the pullback of ξ_f. It follows from (B.6) that the normal bundle $N\widetilde{\xi}_f \twoheadrightarrow \widetilde{\overline{\mathcal{M}}}_{X,f}$ decomposes as

$$(\text{B.8}) \qquad N\widetilde{\xi}_f = \bigoplus_{\{h_1, h_2\} \in E(X) \setminus f^* E(Y)} \mathcal{L}_{h_1}^\vee \otimes \mathcal{L}_{h_2}^\vee, \text{ with fibers at a point}$$

$$(\text{B.9}) \qquad N\widetilde{\xi}_{f(D, \mathbf{q}, \mathbf{v})} = \bigoplus_{\{h_1, h_2\} \in E(X) \setminus f^* E(Y)} T_{\mathbf{q}(h_1)} D \otimes T_{\mathbf{q}(h_2)} D \text{ for } (D, \mathbf{q}, \mathbf{v}) \in \widetilde{\overline{\mathcal{M}}}_{X,f}.$$

Here $\mathcal{L}_{h_1}^\vee$ and $\mathcal{L}_{h_2}^\vee$ also denote the pull backs of the corresponding dual of the point bundles under the $\mathbb{G}_m^{f^* H(y)}$-torsor.

Let $FN\widetilde{\xi}_f \to \widetilde{\overline{\mathcal{M}}}_{X,f}$ denote the frame bundle associated with the vector bundle $N\widetilde{\xi}_f \to \widetilde{\overline{\mathcal{M}}}_{X,f}$ preserving the above decomposition into line bundles. Hence $FN\widetilde{\xi}_f \to \widetilde{\overline{\mathcal{M}}}_{X,f}$ is a $\mathbb{G}_m^{E(X) \setminus f^* E(Y)}$-torsor. Note that the forgetful map $\widetilde{\overline{\mathcal{M}}}_X \to \widetilde{\overline{\mathcal{M}}}_{X,f}$ factors through the frame

bundle and we obtain the diagram

$$(\text{B.10}) \qquad\qquad \begin{array}{ccc} \widetilde{\overline{\mathcal{M}}}_X & \longrightarrow & FN\widetilde{\xi}_f \\ & & \downarrow \\ & \widetilde{\overline{\mathcal{M}}}_{X,f} \xrightarrow{\;\widetilde{\xi}_f\;} & \widetilde{\overline{\mathcal{M}}}_Y \end{array}$$

where the top horizontal map is given by

$$\widetilde{\overline{\mathcal{M}}}_X \ni (D, \mathbf{q}, \mathbf{w}) \mapsto \big((D, \mathbf{q}, \mathbf{w}|_{f^* H(Y)}), (\mathbf{w}(h_1) \otimes \mathbf{w}(h_2))_{\{h_1, h_2\} \in E(X) \setminus f^* E(Y)}\big) \in FN\widetilde{\xi}_f.$$

B.3. Clutchings for pointed admissible covers. In this section we carry out the clutching constructions for pointed admissible Γ-covers where Γ is a finite group.

B.3.1. *The category of group marked graphs.* Let us first define the category $\mathfrak{X}^\Gamma_{g,A}$ of weighted A-legged Γ-graphs. We refer to [69, §3.3] for details.

Definition B.4. An object of $\mathfrak{X}^\Gamma_{g,A}$ is a tuple $(X, \mathbf{m} : H(X) \to \Gamma, \mathbf{b} : H(X) \setminus L(X) \to \Gamma)$ where $X \in \mathfrak{X}_{g,A}$, and such that whenever $\{h_1, h_2\}$ is an edge, $^{\mathbf{b}(h_1)}\mathbf{m}(h_1) \cdot \mathbf{m}(h_2) = 1$ and $\mathbf{b}(h_1) \cdot \mathbf{b}(h_2) = 1$. Here $^{\mathbf{b}(h_1)}\mathbf{m}(h_1) := \mathbf{b}(h_1)\mathbf{m}(h_1)\mathbf{b}(h_1)^{-1}$.

If $(X, \mathbf{m}_X, \mathbf{b}_X), (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ are objects of $\mathfrak{X}^\Gamma_{g,A}$, a morphism between them is a pair $(f : X \to Y, \boldsymbol{\gamma} : H(Y) \to \Gamma)$ such that

$$\boldsymbol{\gamma}(h) \cdot \mathbf{m}_X(f^* h) \cdot \boldsymbol{\gamma}(h)^{-1} = \mathbf{m}_Y(h) \text{ for all } h \in H(Y) \text{ and}$$

$$\mathbf{b}_Y(h_1)\boldsymbol{\gamma}(h_1) = \boldsymbol{\gamma}(h_2)\mathbf{b}_X(f^* h_2) \text{ whenever } \{h_1, h_2\} \in E(Y).$$

Remark B.5. A Γ-marked corolla is just a corolla with legs marked by elements of Γ. If $(X, \mathbf{m}, \mathbf{b}) \in \mathfrak{X}^\Gamma_{g,A}$ then we can cut up the graph at the midpoints of all edges to obtain a collection of Γ-marked corollas parametrized by $V(X)$. As before, for each vertex $v \in V(X)$ we let L_v denote the set of half-edges of X incident at v, which form the legs of the corolla corresponding to the vertex.

B.3.2. *Smooth Deligne-Mumford stacks associated with group-marked graphs.* For any element $(X, \mathbf{m}, \mathbf{b}) \in \mathfrak{X}^\Gamma_{g,A}$ define the smooth Deligne-Mumford stacks

$$(\text{B.11}) \qquad\qquad \overline{\mathcal{M}}^\Gamma_{X,\mathbf{m},\mathbf{b}} := \prod_{v \in V(X)} \overline{\mathcal{M}}^\Gamma_{w(v),L_v}(\mathbf{m}|_{L_v}),$$

$$(\text{B.12}) \qquad\qquad \widetilde{\overline{\mathcal{M}}}^\Gamma_{X,\mathbf{m},\mathbf{b}} := \prod_{v \in V(X)} \widetilde{\overline{\mathcal{M}}}^\Gamma_{w(v),L_v}(\mathbf{m}|_{L_v}).$$

The stack $\overline{\mathcal{M}}^\Gamma_{X,\mathbf{m},\mathbf{b}}$ parametrizes $H(X)$-pointed admissible Γ-covers $(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q})$ of curves $(D, \mathbf{q})$ in $\overline{\mathcal{M}}_X$ with connected components parametrized by $V(X)$ and with monodromies around the $H(X)$ marked points $\widetilde{\mathbf{q}}$ being given by $\mathbf{m}$.

Remark B.6. By definition, if $\{h_1, h_2\} \in E(X)$, then $^{\mathbf{b}(h_1)}\mathbf{m}(h_1)$ and $\mathbf{m}(h_2)$ are inverse to each other. The monodromy around the point $\widetilde{\mathbf{q}}(h_1) \in \widetilde{D}$ is $\mathbf{m}(h_1)$ and hence the monodromy around the point $\mathbf{b}(h_1) \cdot \widetilde{\mathbf{q}}(h_1)$ is $\mathbf{m}(h_2)^{-1}$. In other words, using the construction of [51] we can glue the two points $\mathbf{b}(h_1) \cdot \widetilde{\mathbf{q}}(h_1)$ and $\widetilde{\mathbf{q}}(h_2)$ on the admissible cover $\widetilde{D} \to D$ corresponding to an edge of X. We get the same glued pointed admissible Γ-cover if we glue with respect to the two points $\widetilde{\mathbf{q}}(h_1)$ and $\mathbf{b}(h_2) \cdot \widetilde{\mathbf{q}}(h_2)$.

As in the untwisted case, for any $h \in H(X)$ we have the associated rank 1 point bundle $\widetilde{\mathcal{L}}_h$ on $\overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m},\mathbf{b}}$ and its dual line bundle $\widetilde{\mathcal{L}}^{\vee}_h$ whose fiber at $(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q}) \in \overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m},\mathbf{b}}$ is the tangent space $T_{\widetilde{\mathbf{q}}(h)}\widetilde{D}$. We recall the following proposition from [52].

Proposition B.7. *Consider the natural forgetful map* $\pi : \overline{\mathcal{M}}^{\Gamma}_{g,n}(\mathbf{m}) \to \overline{\mathcal{M}}_{g,n}$, *then* $\pi^*\mathcal{L}_i$ *is natural isomorphic to* $\widetilde{\mathcal{L}}^{N_i}_i$, *where* $\mathcal{L}_i$ *(respectively* $\widetilde{\mathcal{L}}_i$*) denote the i-th point bundle on* $\overline{\mathcal{M}}_{g,n}$ *(respectively* $\overline{\mathcal{M}}^{\Gamma}_{g,n}$*) and* N_i *is the order of* $m_i \in \Gamma$.

B.3.3. *Clutching with respect to morphisms between group-marked graphs.* By Remark B.6, given any morphism $(f, \boldsymbol{\gamma}) : (X, \mathbf{m}_X, \mathbf{b}_X) \to (Y, \mathbf{m}_Y, \mathbf{b}_Y)$ in $\mathcal{X}^{\Gamma}_{g,A}$ we have the gluing or clutching map

(B.13) $$\xi_{f,\boldsymbol{\gamma}} : \overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X} \longrightarrow \overline{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}.$$

Remark B.8. If we have two morphisms

$$(X, \mathbf{m}_X, \mathbf{b}_X) \xrightarrow{(f_1,\boldsymbol{\gamma_1})} (Y, \mathbf{m}_Y, \mathbf{b}_Y) \xrightarrow{(f_2,\boldsymbol{\gamma_2})} (Z, \mathbf{m}_Z, \mathbf{b}_Z)$$

in $\mathcal{X}^{\Gamma}_{g,A}$ then $\xi_{(f_2,\boldsymbol{\gamma_2})\circ(f_1,\boldsymbol{\gamma_1})} = \xi_{f_2,\boldsymbol{\gamma_2}} \circ \xi_{f_1,\boldsymbol{\gamma_1}}$ and the assignment $(X, \mathbf{m}_X, \mathbf{b}_X) \mapsto \overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$ is functorial.

The normal bundle on $\overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$ to the map $\xi_{f,\boldsymbol{\gamma}}$ decomposes as

(B.14) $$N\xi_{f,\boldsymbol{\gamma}} = \bigoplus_{\{h_1,h_2\}\in E(X)\setminus f^*E(Y)} \widetilde{\mathcal{L}}^{\vee}_{h_1} \otimes \widetilde{\mathcal{L}}^{\vee}_{h_2}.$$

Our next goal is to lift this clutching construction to the stacks (see Equations (B.11), (B.12) for notation) $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$ and $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$ while also taking into account the normal bundles. Consider the stack $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ parametrizing data of the form $(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q}, \widetilde{\mathbf{v}})$, where $(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \mathbf{q}) \in \overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$ is an $H(X)$-pointed admissible Γ-cover and $\widetilde{\mathbf{v}}$ is a choice of non-zero tangent vectors to $\widetilde{D}$ at the points marked by $f^*H(Y) \subseteq H(X)$. We have the Cartesian square

(B.15)

$$
\begin{array}{ccc}
\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}} & \xrightarrow{\widetilde{\xi_{f,\boldsymbol{\gamma}}}} & \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y} \\
{\scriptstyle \mathbb{G}^{f^*H(Y)}_{\mathrm{m}}-torsor}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbb{G}^{H(Y)}_{\mathrm{m}}-torsor} \\
\overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X} & \xrightarrow{\xi_{f,\boldsymbol{\gamma}}} & \overline{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}.
\end{array}
$$

It follows that the normal bundle $N\widetilde{\xi_{f,\boldsymbol{\gamma}}} \twoheadrightarrow \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ decomposes as

$$(B.16) \qquad\qquad N\widetilde{\xi_{f,\boldsymbol{\gamma}}} = \bigoplus_{\{h_1,h_2\}\in E(X)\backslash f^*E(Y)} \widetilde{\mathcal{L}}^{\vee}_{h_1} \otimes \widetilde{\mathcal{L}}^{\vee}_{h_2},$$

where we use $\widetilde{\mathcal{L}}^{\vee}_h$ to also denote the line bundle on $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ obtained by pullback of the corresponding point bundle on $\overline{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$.

Let $FN\widetilde{\xi_{f,\boldsymbol{\gamma}}} \to \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ denote the frame bundle (it will be a $\mathbb{G}_{\mathrm{m}}^{E(X)\backslash f^*E(Y)}$-torsor) associated with the vector bundle $N\widetilde{\xi_{f,\boldsymbol{\gamma}}} \to \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ preserving the above decomposition into line bundles. As in the untwisted case, we obtain the diagram

$$(B.17) \qquad\qquad \begin{array}{ccc} \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X} & \longrightarrow & FN\widetilde{\xi_{f,\boldsymbol{\gamma}}} \\ & & \downarrow \\ & & \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}} \xrightarrow{\ \widetilde{\xi_{f,\boldsymbol{\gamma}}}\ } \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y} \end{array}$$

where the top horizontal map is given by

$$(\widetilde{D} \to D, \widetilde{\mathbf{q}}, \widetilde{\mathbf{w}}) \mapsto \left((\widetilde{D} \to D, \widetilde{\mathbf{q}}, \widetilde{\mathbf{w}}|_{f^*H(Y)}), (\widetilde{\mathbf{w}}(h_1) \otimes \widetilde{\mathbf{w}}(h_2))_{\{h_1,h_2\}\in E(X)\backslash f^*E(Y)} \right).$$

We think of $FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ as the open part of $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ obtained by deleting the hyperplane bundles of zero sections corresponding to the decomposition (B.16) of the normal bundle into line bundles. The hyperplane bundles which are deleted are parametrized by the contracted edges $E(X) \backslash f^*E(Y)$. They give one set of boundary divisors on $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ that we will need to consider. We will need to consider another set of boundary divisors in §B.4.2 below.

B.4. **Twisted $\mathcal{D}$-modules and specialization along clutching maps.** We have defined clutching maps between moduli stacks associated with morphisms in the categories $\mathcal{X}^{\Gamma}_{g,A}$. In this section we consider twisted $\mathcal{D}$-modules on these moduli stacks and functors between them defined using specialization along the clutching maps.

B.4.1. *Twisted $\mathcal{D}$-modules on the moduli stacks.* Let $(Y, \mathbf{m}, \mathbf{b}) \in \mathcal{X}^{\Gamma}_{g,A}$ be a weighted legged Γ-marked graph and $\overline{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$, $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$ the corresponding smooth Deligne-Mumford stacks defined by (B.12) with their open parts $\mathcal{M}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} := \prod_{v\in V(Y)} \mathcal{M}^{\Gamma}_{w(v),L_v}(\mathbf{m}|_{L_v})$, $\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} := \prod_{v\in V(Y)} \widetilde{\mathcal{M}}^{\Gamma}_{w(v),L_v}(\mathbf{m}|_{L_v})$. We have the normal crossing boundary divisors

$$\Delta^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} := \overline{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} \backslash \mathcal{M}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} \text{ and } \widetilde{\Delta}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} := \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}} \backslash \widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}.$$

Recall that we have defined the Hodge line bundles (which we will always denote by Λ) on the stacks $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{g,A}$ as pullbacks of the Hodge line bundles on the stacks $\overline{\mathcal{M}}_{g,A}$ along the

natural forgetful maps. Hence we can define Hodge line bundles (also denoted by Λ) on the product moduli stacks $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$ as the pullback of the Hodge line bundle on the product $\overline{\mathcal{M}}_Y$. Consider the logarithmic Atiyah algebra $\mathcal{A}_\Lambda(\log \widetilde{\Delta}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}})$ on the smooth Deligne-Mumford stack $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$. For any $c \in \mathbb{C}$ consider the logarithmic Atiyah algebra $c\mathcal{A}_\Lambda(\log \widetilde{\Delta}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}})$ on $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$ of additive central charge c.

Definition B.9. Let $(Y,\mathbf{m},\mathbf{b}) \in \mathcal{X}^{\Gamma}_{g,A}$ and $c \in \mathbb{C}$. We let $D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}})$ denote the category of vector bundles on the smooth Deligne-Mumford stack $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$ equipped with a $c\mathcal{A}_\Lambda(\log \widetilde{\Delta}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}})$-module structure. We will call such an object a vector bundle with twisted $\log \widetilde{\Delta}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$ connection on $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$.

Remark B.10. By Deligne's Riemann-Hilbert correspondence (see [47, Thm. 5.2.20]), we have an equivalence of abelian categories $D_c \operatorname{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}) \cong D_c \operatorname{Mod}(\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}})$, where the latter is the category of coherent modules over the Atiyah algebra $c\mathcal{A}_\Lambda$ on the open part $\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m},\mathbf{b}}$.

B.4.2. *Specialization along clutchings.* Let $(f,\boldsymbol{\gamma}) : (X,\mathbf{m}_X,\mathbf{b}_X) \to (Y,\mathbf{m}_Y,\mathbf{b}_Y)$ be a morphism in $\mathcal{X}_{g,A}$. We have the following commutative diagram of the associated clutchings

(B.18)

Let $\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}} \subseteq \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ be the open part and let $\widetilde{\Delta}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ be the complementary boundary divisor. Let $N\widetilde{\Delta}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}} \subseteq N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ be the corresponding divisor obtained by pullback to the normal bundle. We consider the open part $FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}^{\circ} \subseteq N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ obtained by restricting $FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ to the open part $\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$. The complement is a normal crossing divisor on $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ which is a union of the divisor defined above and the hyperplane bundle divisors defined previously.

Let $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y,f,\boldsymbol{\gamma}} \subseteq \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$ be the image $\widetilde{\xi_{f,\boldsymbol{\gamma}}}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X})$. It is the closure of a stratum in the natural stratification on $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$. On the open part the map $\widetilde{\xi_{f,\boldsymbol{\gamma}}}$: $\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}} \longrightarrow \widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y,f,\boldsymbol{\gamma}}$ is a stack quotient by a finite group. Moreover we can lift this to a covering map from a tubular neighborhood of $\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}}$ in the normal bundle $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ to a tubular neighborhood of the stratum $\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y,f,\boldsymbol{\gamma}} \subseteq \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$ such that the hyperplane bundle divisors on $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ described previously map to the boundary divisor $\widetilde{\Delta}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$. In other words the intersection of $FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}^{\circ}$ with the tubular neighborhood of $\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X,f,\boldsymbol{\gamma}} \subseteq N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ maps to the intersection of the open part $\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$ with the tubular neighborhood of the stratum $\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y,f,\boldsymbol{\gamma}} \subseteq \widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$. Note that the open part $\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$ is a $\mathbb{G}_m^{E(X)\backslash f^*E(Y)}$-torsor over the open part $FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}^{\circ}$. We obtain homomorphisms of fundamental groups $\pi_1(\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}) \to \pi_1(FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}^{\circ}) \to \pi_1(\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y})$.

By [6, Ch. XVII], the pullback of the Hodge line bundle on $\overline{\mathcal{M}}_Y$ along ξ_f is the Hodge line bundle on $\overline{\mathcal{M}}_X$. Hence the Hodge line bundle on $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}$ pulls back to the Hodge line bundle on $\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}$ along the top of (B.18). We also denote by Λ the pullback of the Hodge line bundle to $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$. For $c \in \mathbb{C}$ we consider the logarithmic Atiyah algebra $c\mathcal{A}_\Lambda(\log N\widetilde{\xi_{f,\boldsymbol{\gamma}}} \setminus FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}^{\circ})$ on $N\widetilde{\xi_{f,\boldsymbol{\gamma}}}$ and the corresponding category $D_c\,\mathrm{Mod}(N\widetilde{\xi_{f,\boldsymbol{\gamma}}})$ of twisted logarithmic $\mathcal{D}$-modules. Any object of $D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}) \cong D_c\,\mathrm{Mod}(\widetilde{\mathcal{M}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y})$ (see Remark B.10) after specialization to the normal bundle gives us an object of $D_c\,\mathrm{Mod}(N\widetilde{\xi_{f,\boldsymbol{\gamma}}}) \cong D_c\,\mathrm{Mod}(FN\widetilde{\xi_{f,\boldsymbol{\gamma}}}^{\circ})$. We can now pullback along the top horizontal arrow of (B.18) to obtain an object of $D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}) \cong D_c\,\mathrm{Mod}(\widetilde{\mathcal{M}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X})$. This defines a functor which we denote as

(B.19) $$\mathrm{Sp}_{f,\boldsymbol{\gamma}} : D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Y,\mathbf{m}_Y,\mathbf{b}_Y}) \longrightarrow D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}).$$

If we have two morphisms

$$(X,\mathbf{m}_X,\mathbf{b}_X) \xrightarrow{f_1,\boldsymbol{\gamma_1}} (Y,\mathbf{m}_Y,\mathbf{b}_Y) \xrightarrow{f_2,\boldsymbol{\gamma_2}} (Z,\mathbf{m}_Z,\mathbf{b}_Z)$$

then we have a natural isomorphism between the two functors

(B.20) $$\mathrm{Sp}_{(f_2,\boldsymbol{\gamma_2})\circ(f_1,\boldsymbol{\gamma_1})} \cong \mathrm{Sp}_{f_1,\boldsymbol{\gamma_1}} \circ \mathrm{Sp}_{f_2,\boldsymbol{\gamma_2}} : D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{Z,\mathbf{m}_Z,\mathbf{b}_Z}) \longrightarrow D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X}).$$

Hence the assignment $(X,\mathbf{m}_X,\mathbf{b}_X) \mapsto D_c\,\mathrm{Mod}(\widetilde{\overline{\mathcal{M}}}^{\Gamma}_{X,\mathbf{m}_X,\mathbf{b}_X})$ is functorial.

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