Laplacians and gauged Laplacians on a quantum Hopf bundle

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Abstract

This paper presents an analysis of the set of connections on the U(1) quantum Hopf bundle on the standard quantum sphere S^2_q, whose total space algebra $SU_q(2)$ is equipped with the 3d left covariant differential calculus by Woronowicz. The introduction of a Hodge duality on both $\Omega(SU_q(2))$ and on $\Omega(S^2_q)$ allows for the study of Laplacians and of gauged Laplacians.

This paper is dedicated to Sergio Albeverio, on the occasion of his 70th birthday.

1 Introduction

This paper is focussed on the analysis of a class of Hall Hamiltonians in the noncommutative set up. It is intended as a survey of the general formulation of quantum principal bundles, and as a description of a specific procedure to formalise, on both the total space and the base space of a quantum Hopf bundle, a set of Laplacian operators and to couple them with gauge connections. It also presents a detailed formulation of the classical Hopf bundle. The emphasis in the standard approach from differential geometry is given to the algebraic aspects which can be extended to the noncommutative setting.

Classical Hall Hamiltonians are gauged Laplace operators acting on the space of sections of the vector bundles associated to the principal bundles $\pi : G \mapsto G/\mathrm{K}$ over homogeneous spaces (with G semisimple and K compact) and can be constructed in terms of the Casimir operators of G and K. With (ρ, V) a representation of K, one has the identification of sections of the associated vector bundle $E = G \times_{\rho(\mathrm{K})} V$ with equivariant maps from G to V, $\Gamma(G/\mathrm{K}, E) \simeq C^{\infty}(G, V)_{\rho(\mathrm{K})} \subset C^{\infty}(G) \otimes V$. Given a connection on G one has a covariant derivative ∇ on $\Gamma(G/\mathrm{K}, E)$, so that the gauged Laplacian operator is $\Delta^{E} = (\nabla\nabla^{*} + \nabla^{*}\nabla) = \star\nabla\star\nabla$, where the dual ∇^{*} is defined from the metric induced on the homogeneous space basis G/K by the Cartan-Killing metric on G, or equivalently the Hodge duality comes from the induced metric on G/K. If the connection is the canonical one, given by the orthogonal splitting of the Lie algebra $\mathfrak{g}$ of G in terms of the Lie algebra $\mathfrak{k}$ of the gauge group and of its orthogonal complement, then the gauged Laplacian operator can be cast in terms of the quadratic Casimirs of $\mathfrak{g}$ and $\mathfrak{k}$:

$$\Delta^{E} = (\Delta^{G} \otimes 1 - 1 \otimes C_{\mathfrak{k}})\big|_{C^{\infty}(G,V)_{\rho(\mathrm{K})}} = (C_{\mathfrak{g}} \otimes 1 - 1 \otimes C_{\mathfrak{k}})\big|_{C^{\infty}(G,V)_{\rho(\mathrm{K})}} \tag{1.1}$$

The above formula [3] simplifies the diagonalisation of the gauged Laplacian, and has important applications in the study of the heat kernel expansion and index theorems on principal bundles.

The natural further step is to develop models of Hall effect on noncommutative spaces whose symmetries could be formalise in terms of quantum groups. In [16] the first model of 'excitations moving on a quantum 2-sphere' in the field of a magnetic monopole has been studied. It is described a quantum principal U(1)-bundle over a quantum sphere S^2_q having as a total space the manifold of the quantum group $SU_q(2)$ [4]. The natural associated line bundles are classified by the winding number $n \in \mathbb{Z}$: equipped $SU_q(2)$ with the three dimensional left covariant calculus from Woronowicz [29], the gauge monopole connection is studied and a gauged Laplacian acting on sections of the associated bundle is completely diagonalised. That paper presents a first generalisation of the relation (1.1). Its most interesting aspect is that the corresponding energies are not invariant under the exchange monopole/antimonopole, namely the spectrum of the gauged Laplacian is not invariant under the inversion of the direction of the magnetic field, a manifestation of the phenomenon usually referred to as 'quantisation removes degeneracy'. An analysis of the relation (1.1) is presented in [5], where Laplacians on a quantum projective plane are gauged via the monopole connection.

The analysis in [16] embodies two specific starting points. The first one is that the quantum Casimir C_q for the universal envelopping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ dual to $SU_q(2)$ - thus playing the quantum role of the classical envelopping algebra dual to the classical Lie group - *is* a quadratic operator in the generators of $\mathcal{U}_q(\mathfrak{su}(2))$ acting on $SU_q(2)$, but *can not* be cast in the form of a whatever rank polynomial in the left invariant generators of the left invariant three dimensional differential calculus by Woronowicz, so to say in the basis of natural left invariant derivations associated to this differential calculus. The second starting point is the studies performed in [18]. In that paper a $\star$-Hodge operator on the exterior algebra on the Podles sphere S^2_q - coming from the differential two dimensional calculus induced on S^2_q by the three dimensional calculus on $SU_q(2)$ - has been introduced, so to make it possible the definition of Laplacian operator on S^2_q.

This paper evolves the analysis started in [16], and describes another generalisation of the relation (1.1) to the setting of the quantum Hopf bundle, when the total space algebra of this bundle has been equipped with the 3D left covariant calculus from Woronowicz. A family of $\star$-Hodge structures is introduced on both the exterior algebra $\Omega(\mathrm{SU}_q(2))$ and the exterior algebra $\Omega(\mathrm{S}^2_q)$, depending on a set of real parameters: the corresponding Laplacians $\Box_{\mathrm{SU}_q(2)} = \star\mathrm{d}\star\mathrm{d} : \mathcal{A}(\mathrm{SU}_q(2)) \mapsto \mathcal{A}(\mathrm{SU}_q(2))$, and $\Box_{\mathrm{S}^2_q} = \star\mathrm{d}\star\mathrm{d} : \mathcal{A}(\mathrm{S}^2_q) \mapsto \mathcal{A}(\mathrm{S}^2_q)$ are defined. The analysis of the connections on the principal bundle allows for a gauging of the Laplacian $\Box_{\mathrm{S}^2_q}$ on each associated line bundle. When $\Box_{\mathrm{S}^2_q}$ is gauged into $\Box_{D_0}$ via the monopole connection, one finds:

$$q^{2n}\Box_{D_0} = (\Box_{\mathrm{SU}_q(2)} + \gamma X_z X_z), \tag{1.2}$$

where the integer $n \in \mathbb{Z}$ specifies the value of the monopole charge. This is the relation generalising the first equality in (1.1): the role of the quadratic Casimir of the gauge group algebra is played by $\gamma X_z X_z \triangleright$, with X_z the vertical derivation of the fibration, and $\gamma \in \mathbb{R}_+$ appears in this formulation as a parametrisation for a set of $\star$-Hodge structures having the same compatibility, that is giving Laplacians satisfying the same relation (1.2).

This paper begins with an exposition of the classical formalisation of the Hopf bundle. Section 2 presents a global – i.e. charts independent – description of the differential calculi on both the Lie group manifold $SU(2) \simeq S^3$ and on the homogeneous space $S^2 = S^3/\mathrm{U}(1)$, and introduces on the exterior algebras $\Omega(S^3)$ and $\Omega(S^2)$ the Hodge duality structures coming from a Cartan-Killing type metric on the Lie algebra $\mathfrak{su}(2)$, in order to define Laplacian operators. The principal bunde structure is described in terms of a well known principal bundle atlas. The aim of the section is to explicitly compute for such a specific Hopf bundle, following the classical approach from differential geometry, the main structures which will be generalised to the quantum setting. A more general and complete analysis of a noncommutative geometry approach to the differential geometry of principal and quantum bundles is in [2].

Section 3 describes a quantum formulation of the principal bundle having $\mathcal{A}(\mathrm{SU}_q(2))$ as total space algebra, $\mathcal{A}(\mathrm{S}^2_q)$ as base manifold algebra and $\mathcal{A}(\mathrm{U}(1))$ as gauge group algebra. The differential calculus on $\mathrm{SU}_q(2)$ is the 3d left-covariant by Woronowicz [28] [29], the principal bundle structure is introduced following [4].

Section 4 presents a $\star$-Hodge duality on $\Omega(\mathrm{SU}_q(2))$, allowing for the definition of a Laplacian operator. The Hodge duality is introduced following [14]; section 5 describes an evolution of this approach, giving a $\star$-Hodge duality structure on $\Omega(\mathrm{S}^2_q)$, and analysing its compatibility with the one on $\Omega(\mathrm{SU}_q(2))$.

Section 6 provides a complete explicit description of the set of connections on this specific realisation of the quantum Hopf bundle, and of the main properties of the covariant derivative operators on each associated line bundle. The emphasis is on the domain of the covariant derivative operators – the set of horizontal coequivariant element of the bundle – which appears here as the quantum conuterpart of the classical also called tensorial forms. Section 7 studies the coupling of the Laplacian operator on $\Omega(\mathrm{S}^2_q)$ to the gauge connections.

Section 8 closes the paper, applying to the commutative algebras $\{\mathcal{A}(SU(2)), \mathcal{A}(S^2), \mathcal{A}(\mathrm{U}(1))\}$ the formalism developed in the quantum setting, in order to recover the structure of the classical Hopf bundle from an algebraic perspective.

2 The classical Hopf bundle

With $\pi : \mathcal{P} \mapsto \mathcal{M}$ a smooth surjective map from a manifold $\mathcal{P}$ to a manifold $\mathcal{M}$, $(\mathcal{P}, \mathcal{M}, \pi)$ is a fibre bundle with typical fibre $\mathcal{F}$ over $\mathcal{M}$ if there is a fibre bundle atlas with charts (U_i, λ_i), where U_i is an open covering of $\mathcal{M}$ and the diffeomorphisms $\lambda_i : \pi^{-1}(U_i) \mapsto U_i \times \mathcal{F}$ are such that $\pi : \pi^{-1}(U_i) \mapsto U_i$ is the composition of λ_i with the projection onto the first factor in $U_i \times \mathcal{F}$. The manifold $\mathcal{P}$ is called the total space of the bundle, the manifold $\mathcal{M}$ is the base of the bundle. From the definition it follows that $\pi^{-1}(m)$ is diffeomorphic to $\mathcal{F}$ – the fibre of the bundle – for any $m \in \mathcal{M}$. For any $f \in \mathcal{F}$ it is $\lambda_i \circ \lambda_j^{-1}(m, f) = (m, \lambda_{ij}(m, f))$ where $\lambda_{ij} : (U_i \cap U_j) \times \mathcal{F} \mapsto \mathcal{F}$ is smooth and $\lambda_{ij}(m, \ )$ belongs to the group $\mathrm{Diff}(\mathcal{F})$ of diffeomorphisms of the fibre $\mathcal{F}$ for each $m \in U_i \cap U_j$. The mappings λ_{ij} are called the transition functions of the bundle, and satisfy the cocycle condition $\lambda_{ij}(m, \ ) \circ \lambda_{jk}(m, \ ) = \lambda_{ik}(m, \ )$ for $m \in U_i \cap U_j \cap U_k$, with $\lambda_{ii}(m, \ ) = id_{\mathcal{F}}$ for $m \in U_i$.

A fibre bundle $(\mathcal{P}, \mathcal{M}, \pi)$ is called a vector bundle if its typical fibre $\mathcal{F}$ is a vector space and if the trivialisation diffeomorphisms λ_i may be chosen so to give transition functions λ_{ij} which are invertible linear maps, elements in $\mathrm{GL}(\mathcal{F})$ for any $m \in \mathcal{M}$. A principal bundle $(\mathcal{P}, \mathrm{K}, [\mathcal{M}], \pi)$ with structure group K is a fibre bundle $(\mathcal{P}, \mathcal{M}, \pi)$ with typical fibre K and such that the transition functions $\lambda_{ij}(m, \ ) \in \mathrm{Aut}(\mathrm{K})$ formalise the left translation of the group K on itself. On the total space of a principal bundle there is also a right action of the Lie group K – that is $\mathrm{r}_{k'}(\mathrm{r}_k(p)) = \mathrm{r}_{kk'}(p)$ for any $p \in \mathcal{P}$ and $k, k' \in \mathrm{K}$ – such that $\pi(\mathrm{r}_k(p)) = \pi(p)$, and such that the action is free and transitive. The base $\mathcal{M}$ of the bundle can be identified with the quotient $\mathcal{P}/\mathrm{K}$ with respect to such a right action.

Given G a Lie group and $\mathrm{K} \subset G$ a closed Lie subgroup of it, the group manifold G is the total space manifold of a principal bundle $(G, \mathrm{K}, G/\mathrm{K}, \pi)$ with base space G/K - the space of left cosets - and typical fiber given by the structure or gauge group K, so that the bundle projection $\pi : G \mapsto G/\mathrm{K}$ is the canonical projection [20]. The right principal action of the gauge group K on G is given as $\mathrm{r}_k(g) = gk$ for any $k \in \mathrm{K}$ and $g \in G$. This action trivially satisfies the requirements of being free and transitive. If $\mathfrak{k}$ is the Lie algebra of the group K, the fundamental vector field $X_\tau \in \mathfrak{X}(G)$ associated to $\tau \in \mathfrak{k}$ is defined as the infinitesimal generator of the right principal action $\mathrm{r}_{\exp s\tau}(g) = g \exp s\tau$ of the one parameter subgroup $\exp s\tau \subset \mathrm{K}$: the mapping $\tau \in \mathfrak{k} \mapsto \{X_\tau\} \in \mathfrak{X}(G)$ is a Lie algebra isomorphism between $\mathfrak{k}$ and the set of fundamental vector fields $\{X_\tau\}$. A differential form $\phi \in \Omega(G)$ is called horizontal if $i_{X_\tau}\phi = 0$ for any fundamental vector field X_τ.

If $\rho : \mathrm{K} \mapsto \mathrm{GL}(W)$ is a finite dimensional representation of K on the vector space W, the associated vector bundle to G is the vector bundle whose total space is $\mathcal{E} = G \times_{\rho(\mathrm{K})} W$, having typical fiber W. It is defined as the quotient of the product $G \times W$ by the equivalence relation $(\mathrm{r}_k(g) = gk; w) \sim (g; \rho(k) \cdot w)$ for any choice of $g \in G$, $k \in \mathrm{K}$ and $w \in W$: $(\mathcal{E}, G/\mathrm{K}, \pi_{\mathcal{E}})$ is a fibre bundle with a projection $\pi_{\mathcal{E}} : \mathcal{E} \mapsto G/\mathrm{K}$ which is consistently defined on the quotient as $\pi_{\mathcal{E}}[g, w]_{\rho(\mathrm{K})} = \pi(g)$ from the principal bundle projection π.

With $\mathrm{r}_k^* : \Omega(G) \mapsto \Omega(G)$ the action of K on the exterior algebra $\Omega(G)$ induced as a pull-back of the right action r_k of K on G, the $\rho(\mathrm{K})$-equivariant r-forms of the principal bundle are W-valued forms on G defined as:

$$\Omega^r(G, W)_{\rho(\mathrm{K})} = \{\phi \in \Omega^r(G, W) = \Omega^r(G) \otimes W : \mathrm{r}_k^*(\phi) = \rho^{-1}(k)\phi\}. \tag{2.1}$$

A section of the associated bundle $\mathcal{E}$ is an element in $\Gamma(G/\mathrm{K}, \mathcal{E})$, namely a map $\sigma : G/\mathrm{K} \mapsto \mathcal{E}$ such that $\pi_{\mathcal{E}}(\sigma(m)) = m$ for any $m \in G/\mathrm{K}$. This definition is extended to $\Gamma^{(r)}(G/\mathrm{K}, \mathcal{E})$, the set of r-forms on the basis G/K of the principal bundle with values in $\mathcal{E}$. There is a canonical isomorphism

$$\Gamma^{(r)}(G/\mathrm{K}, \mathcal{E}) \simeq \Omega^r_{\mathrm{hor}}(G, W)_{\rho(\mathrm{K})} \tag{2.2}$$

from the space of $\mathcal{E}$-valued differential forms on G/K onto the space of horizontal $\rho(\mathrm{K})$-equivariant W-valued differential forms on the principal bundle (G, K, π). For $r = 0$ – with $\Gamma(G/\mathrm{K}, \mathcal{E}) \simeq \Gamma^{(0)}(G/\mathrm{K}, \mathcal{E})$ – the isomorphism gives the well known equivalence between equivariant functions of a principal bundle and sections of its associated bundle. In particular, for $W = \mathbb{R}, \mathbb{C}$ with trivial representation the isomorphism is

$$\Omega(G/\mathrm{K}) \simeq \Omega_{\mathrm{hor}}(G)_{\rho(\mathrm{K})=\mathrm{K}} = \{\phi \in \Omega(G) : i_{X_\tau}\phi = 0;\ \mathrm{r}_k^*\phi = \phi\}, \tag{2.3}$$

giving a description of the exterior algebra on the basis of the principal bundle.

A connection on a principal bundle can be given via a connection 1-form. A connection 1-form on G is an element $\omega \in \Omega(G, \mathfrak{k})$, taking values in $\mathfrak{k}$ and satisfying the two local conditions:

$$\omega(X_\tau) = \tau,$$

$$\mathrm{r}_k^*(\omega) = \mathrm{Ad}_{k^{-1}}\,\omega,$$

where the adjoint action of K is given by $(\mathrm{Ad}_{k^{-1}}\,\omega)(X) = k^{-1}\omega(X)k$ for any vector field $X \in \mathfrak{X}(G)$. At each point $g \in G$ there is on the tangent space T_gG a natural notion of vertical subspace, whose basis is given by the vectors X_τ which are tangent to the fiber group K, while the connection 1-form selects the horizontal subspace $H_g^{(\omega)}(G)$ given by the kernel of ω. Identifiying the element $\omega(X) \in \mathfrak{k}$ with the vertical vector field it generates, the expression $X^{(\omega)} = X - \omega(X)$ denotes the horizontal projection of the vector field $X \in \mathfrak{X}(G)$.

Given any $\rho(\mathrm{K})$-equivariant form $\phi \in \Omega^r(G, W)_{\rho(\mathrm{K})}$, the covariant derivative is defined as the map:

$$D : \Omega^r(G, W)_{\rho(\mathrm{K})} \mapsto \Omega^{r+1}_{\mathrm{hor}}(G, W)_{\rho(\mathrm{K})}, \qquad D\phi(X_1, \ldots, X_{r+1}) = \mathrm{d}\phi(X_1^{(\omega)}, \ldots, X_{r+1}^{(\omega)}) \tag{2.4}$$

where d is the exterior derivative on G. On a $\rho(\mathrm{K})$-equivariant horizontal form $\phi \in \Omega_{\mathrm{hor}}(G, W)_{\rho(\mathrm{K})}$ the action of the covariant derivative can be written in terms of the connection 1-form as:

$$D\phi = \mathrm{d}\phi + \omega \wedge \phi. \tag{2.5}$$

2.1 A differential calculus on the classical $SU(2)$ Lie group

For $G \simeq SU(2)$ and $\mathrm{K} \simeq U(1)$ one recovers the Hopf fibration $\pi : S^3 \mapsto S^2$, with S^2 the space of the orbits $SU(2)/U(1)$. The aim of this section is to describe the differential calculus on the total space of this bundle, in terms of natural basis of global vector fields. Recall that a Lie group G naturally acts on itself both from the right and from the left. The left action is the smooth map $\mathrm{l} : G \times G \mapsto G$ defined via the left multiplication $\mathrm{l}(g', g) = g'g = \mathrm{l}_{g'}(g)$: since $\mathrm{l}_{g'g''}(g) = \mathrm{l}_{g'}(\mathrm{l}_{g''}(g))$, the left action is a group homomorphism $\mathrm{l}_g : G \mapsto \mathrm{Aut}(G)$. The right action is the smooth map $\mathrm{r} : G \times G \mapsto G$ defined via the right multiplication $\mathrm{r}(g, g') = gg' = \mathrm{r}_{g'}(g)$; it is then immediate to see that $\mathrm{r}_{g'g''}(g) = gg'g'' = \mathrm{r}_{g''}(\mathrm{r}_{g'}(g))$: the right action is a group anti-homomorphism $\mathrm{r}_g : G \mapsto \mathrm{Aut}(G)$. For any $T \in \mathfrak{g}$, the Lie algebra of G, it is possible to define a vector field $R_T \in \mathfrak{X}(G)$. It acts as a derivation on a smooth complex valued function defined on G, and can be written in terms of the pull-back $\mathrm{l}^*_g : C^\infty(G) \mapsto C^\infty(G)$ induced by l_g. On $\phi \in C^\infty(G)$:

$$R_T(\phi) = \frac{\mathrm{d}}{\mathrm{d}s} \left(\mathrm{l}^*_{\exp sT}(\phi)\right)\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\, \phi(\exp sT\ \cdot\ g)|_{s=0}\,. \tag{2.6}$$

Altough defined via the left action l_g, the vector field R_T is called the right invariant vector field associated to $T \in \mathfrak{g}$; this set of fields owes its name to the fact that, given $\mathrm{r}_{g*} : \mathfrak{X}(G) \mapsto \mathfrak{X}(G)$ the push-forward induced by the right action r_g, they satisfy a property of right invariance:

$$\mathrm{r}_{g*}(R_T) = R_T.$$

¿From the definition of the pull-back map $\mathrm{l}^*_g : C^\infty(G) \mapsto C^\infty(G)$ one has:

$$\mathrm{l}^*_{g'g''}(\phi) = \phi \circ \mathrm{l}_{g'g''} = \phi \circ \mathrm{l}_{g'} \circ \mathrm{l}_{g''} = \mathrm{l}^*_{g''}(\mathrm{l}^*_{g'}(\phi))$$

for any $\phi \in C^\infty(G)$. This relation enables to prove that the map $\check{\mathrm{l}} : T \in \mathfrak{g} \mapsto R_T \in \mathfrak{X}(G)$ is a Lie algebra anti-homomorphism:

$$[R_T, R_{T'}] = R_{[T',T]}.$$

The analogous definitions starting from the right action naturally hold. For any $T \in \mathfrak{g}$, the vector field $L_T \in \mathfrak{X}(G)$ is defined as a derivation on $C^\infty(G)$, namely as the infinitesimal generator of the the pull-back r^*_g induced by the right action r_g:

$$L_T(\phi) = \frac{\mathrm{d}}{\mathrm{d}s} \left(\mathrm{r}^*_{\exp sT}(\phi)\right)\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\, \phi(g \cdot \exp sT)|_{s=0} \tag{2.7}$$

on any $\phi \in C^\infty(G)$. Left invariant vector fields satisfy a property of left invariance:

$$\mathrm{l}^*_g(L_T) = L_T;$$

the map $\check{\mathrm{r}} : T \in \mathfrak{g} \mapsto L_T \in \mathfrak{X}(G)$ is a Lie algebra homomorphism:

$$[L_T, L_{T'}] = L_{[T,T']}.$$

The set $\{L_T\}$ of left invariant vector fields as well the set $\{R_T\}$ of right invariant vector fields are two global natural basis of the left free $C^\infty(G)$-module $\mathfrak{X}(G)$.

The total space of the classical Hopf bundle is the manifold S^3, which represents the elements of the Lie group $SU(2)$. A point $g \in S^3$ can be then formalised via a 2×2 matrix with complex entries and unit determinant:

$$g = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \quad : \quad \bar{u}u + \bar{v}v = 1; \tag{2.8}$$

the components of the left invariant vector fields $\check{\mathrm{r}}(T) = L_T$ are given, following (2.7) in the defining matrix representation by:

$$\frac{d}{ds} \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \cdot (\exp sT)\ |_{s=0} = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \cdot (T) \tag{2.9}$$

Since $\exp sT$ is unitary, T should be antihermitian, and the choice of a basis in terms of the Pauli matrices:

$$T_x = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad T_y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_z = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.10}$$

gives the explicit form of the left invariant vector fields:

$$
\begin{aligned}
L_x &= -\frac{i}{2}\left(\bar{v}\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial v} + u\frac{\partial}{\partial \bar{v}} - v\frac{\partial}{\partial \bar{u}}\right)\\
L_y &= -\frac{1}{2}\left(\bar{v}\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial v} - u\frac{\partial}{\partial \bar{v}} + v\frac{\partial}{\partial \bar{u}}\right)\\
L_z &= \frac{i}{2}\left(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{v}} - \bar{u}\frac{\partial}{\partial \bar{u}}\right)\\
L_- &= L_x - iL_y \;=\; i\left(v\frac{\partial}{\partial \bar{u}} - u\frac{\partial}{\partial \bar{v}}\right)\\
L_+ &= L_x + iL_y \;=\; i\left(\bar{u}\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial u}\right),
\end{aligned}
\tag{2.11}
$$

satisfying the commutation relations:

$$
\begin{aligned}
&[L_z; L_-] = iL_-,\\
&[L_z; L_+] = -iL_+,\\
&[L_-; L_+] = 2iL_z.
\end{aligned}
\tag{2.12}
$$

The components of the right invariant vector fields $R_T = \check{\mathrm{l}}(T)$ are then given in the defining matrix representation (2.6) as:

$$
\frac{d}{ds}\,(\exp sT)\cdot\begin{pmatrix} u & -\bar{v}\\ v & \bar{u}\end{pmatrix}\,|_{s=0} = \;(T)\;\cdot\;\begin{pmatrix} u & -\bar{v}\\ v & \bar{u}\end{pmatrix}
\tag{2.13}
$$

acquiring the form:

$$
\begin{aligned}
R_x &= \frac{i}{2}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} - \bar{u}\frac{\partial}{\partial \bar{v}} - \bar{v}\frac{\partial}{\partial \bar{u}}\right)\\
R_y &= -\frac{1}{2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v} - \bar{u}\frac{\partial}{\partial \bar{v}} + \bar{v}\frac{\partial}{\partial \bar{u}}\right)\\
R_z &= \frac{i}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} + \bar{v}\frac{\partial}{\partial \bar{v}} - \bar{u}\frac{\partial}{\partial \bar{u}}\right)\\
R_- &= R_x - iR_y \;=\; i\left(v\frac{\partial}{\partial u} - \bar{u}\frac{\partial}{\partial \bar{v}}\right)\\
R_+ &= R_x + iR_y \;=\; i\left(u\frac{\partial}{\partial v} - \bar{v}\frac{\partial}{\partial \bar{u}}\right).
\end{aligned}
\tag{2.14}
$$

The commutation relations they satisfy are:

$$
\begin{aligned}
&[R_z; R_-] = -iR_-,\\
&[R_z; R_-] = iR_+,\\
&[R_-; R_+] = -2iR_z.
\end{aligned}
\tag{2.15}
$$

The quadratic Casimir of the Lie algebra $\mathfrak{su}(2)$ is written as

$$
C = \frac{1}{2}(L_+L_- + L_-L_+) + L_zL_z = \frac{1}{2}(R_+R_- + R_-R_+) + R_zR_z.
\tag{2.16}
$$

The set $\mathfrak{X}(S^3)$ is a free left $C^\infty(S^3)$- module. Right vector fields can be expressed in the basis of the left vector fields as $R_a = J_{ab}L_b$. The matrix J is given by:

$$
\begin{pmatrix} R_-\\ R_z\\ R_+\end{pmatrix} = \begin{pmatrix} \bar{u}^2 & 2\bar{u}v & -v^2\\ -\bar{u}\bar{v} & u\bar{u} - v\bar{v} & -uv\\ -\bar{v}^2 & 2u\bar{v} & u^2\end{pmatrix}\begin{pmatrix} L_-\\ L_z\\ L_+\end{pmatrix}
\tag{2.17}
$$

and its inverse matrix is:

$$\left(\begin{array}{c} L_- \\ L_z \\ L_+ \end{array}\right) = \left(\begin{array}{ccc} u^2 & -2uv & -v^2 \\ u\bar{v} & u\bar{u}-v\bar{v} & \bar{u}v \\ -\bar{v}^2 & -2\bar{u}\bar{v} & \bar{u}^2 \end{array}\right)\left(\begin{array}{c} R_- \\ R_z \\ R_+ \end{array}\right) \tag{2.18}$$

A similar analysis can be performed in the study of the cotangent space $\mathfrak{X}^*(G)$ of a Lie group. This is a $C^\infty(S^3)$-bimodule, with two basis of globally defined 1-forms, namely the left invariant $\{\tilde{\omega}_a\}$ dual to the set of left invariant vector fields $\{L_a\}$, and the right invariant $\{\tilde{\eta}_b\}$ dual to the set of right invariant vector fields $\{R_b\}$. They satisfy the invariance property:

$$\begin{aligned} \mathrm{l}_g^*(\tilde{\omega}_a) &= \tilde{\omega}_a, \\ \mathrm{r}_g^*(\tilde{\eta}_b) &= \tilde{\eta}_b : \end{aligned} \tag{2.19}$$

one then immediately computes:

$$R_i \,=\, J_{ij}L_j \;\leftrightarrow\; \tilde{\eta}_s J_{sp} \,=\, \tilde{\omega}_p \tag{2.20}$$

The left invariant 1-forms are:

$$\begin{aligned} \tilde{\omega}_z &= -2i\,(\bar{u}du + \bar{v}dv) \\ \tilde{\omega}_- &= -i\,(\bar{v}d\bar{u} - \bar{u}d\bar{v}) \\ \tilde{\omega}_+ &= -i\,(udv - vdu) \end{aligned} \tag{2.21}$$

with $\tilde{\omega}_x = (\tilde{\omega}_- + \tilde{\omega}_+)$ and $\tilde{\omega}_y = i(\tilde{\omega}_+ - \tilde{\omega}_-)$. The complex structure on $\Omega^1(S^3)$, compatible with the complex structure on $C^\infty(S^3)$, is given by $\tilde{\omega}_x^* = \tilde{\omega}_x$, $\tilde{\omega}_y^* = \tilde{\omega}_y$, $\tilde{\omega}_z^* = \tilde{\omega}_z$. while the right-invariant 1-forms are:

$$\begin{aligned} \tilde{\eta}_z &= 2i\,(ud\bar{u} + \bar{v}dv) \\ \tilde{\eta}_- &= i\,(ud\bar{v} - \bar{v}du) \\ \tilde{\eta}_+ &= -i\,(\bar{u}dv - vd\bar{u})\,. \end{aligned} \tag{2.22}$$

Given a complex valued smooth function $\phi \in C^\infty(S^3)$, the differential calculus is formalised through the exterior derivative as a map $\mathrm{d} : C^\infty(S^3) \mapsto \Omega^1(S^3)$ defined via:

$$\mathrm{d}\phi(X) = X(\phi) \tag{2.23}$$

in terms of the Lie derivative $X(\phi)$ of ϕ along the vector field X. This map acquires the form:

$$\mathrm{d}\phi = L_a(\phi)\tilde{\omega}_a = R_b(\phi)\tilde{\eta}_b \tag{2.24}$$

where now $L_a(\phi)$ represents the Lie derivative of ϕ along the vector field L_a, while $R_b(\phi)$ represents the Lie derivative of ϕ along the vector field R_b.

¿From the $C^\infty(S^3)$-bimodule $\Omega^1(S^3)$ define the tensor product of forms as the $C^\infty(S^3)$-bimodule $\{\Omega^1(S^3)\}^{\otimes k} = \Omega^1(S^3) \otimes_{C^\infty(S^3)} \ldots \otimes_{C^\infty(S^3)} \Omega^1(S^3)$ (k times). An alternation mapping $\mathfrak{A} : \{\Omega^1(S^3)\}^{\otimes k} \mapsto \{\Omega^1(S^3)\}^{\otimes k}$ is introduced as the $C^\infty(S^3)$-linear map

$$\mathfrak{A}(\theta_1 \otimes \ldots \otimes \theta_k) = \frac{1}{k!}\sum_{\sigma \in S_k} (\mathrm{sign}\,\sigma)\,\theta_{\sigma(1)} \otimes \ldots \otimes \theta_{\sigma(k)}, \tag{2.25}$$

where $\theta_j \in \Omega^1(S^3)$ and the sum is over all the permutation $\sigma \in S_k$ in the permutation group of k elements, with $(\mathrm{sign}\,\sigma)$ the parity of the permutation σ. The set

$$\Omega^k(S^3) = \mathrm{Range}\,\mathfrak{A}(\{\Omega(S^3)\}^{\otimes k}) \tag{2.26}$$

is the set of k-exterior forms, or k-forms, so that $\mathfrak{A}$ acts on $\Omega^k(S^3)$ as the identity, with $\mathfrak{A} \circ \mathfrak{A} = \mathfrak{A}$. Given $\alpha \in \{\Omega^1(S^3)\}^{\otimes k}$ and $\beta \in \{\Omega^1(S^3)\}^{\otimes l}$, define the wedge product as

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\,l!}\,\mathfrak{A}(\alpha \otimes \beta). \tag{2.27}$$

With α the k-form given by $\alpha = \theta_1 \wedge \ldots \wedge \theta_k$ and β the l-form given by $\beta = \theta_{k+1} \wedge \ldots \wedge \theta_{k+l}$, the wedge product reduces to $\alpha \wedge \beta = \sum (\mathrm{sign}\, \sigma) \theta_{\sigma(1)} \wedge \ldots \wedge \theta_{\sigma(k)} \wedge \theta_{\sigma(k+1)} \wedge \ldots \wedge \theta_{\sigma(k+l)}$: the sum is over the shuffles $\sigma(k,l)$, the shuffles $\sigma(k,l)$ are the permutations σ of $\{1, \ldots, k+l\}$ such that $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots \sigma(k+l)$. The wedge product is bilinear, and satisfies the identity $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for any k-form α and l-form β. The complex structure is extended by requiring

$$(\alpha \wedge \beta)^* = (-1)^{kl} \beta^* \wedge \alpha^*.$$

The exterior algebra coming from the differential calculus (2.24) is defined as the grade associative algebra $\Omega(S^3) = \big(\oplus_k \Omega^k(S^3); \wedge\big)$.

The exterior derivative is extended to $\mathrm{d} : \Omega^k(S^3) \mapsto \Omega^{k+1}(S^3)$ as the unique $\mathbb{C}$-linear mapping satisfying the conditions:

1. d is a graded $\wedge$-derivation, that is $\mathrm{d}(\alpha \wedge \beta) = (\mathrm{d}\alpha) \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\beta$ for any k-form α;
2. $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$;
3. on $\phi \in \Omega^0(S^3) \simeq C^\infty(S^3)$, it is given by $\mathrm{d}\phi$ as in (2.24).

It is then easy to see that $\Omega^2(S^3)$ is three dimensional, with a basis given by $\{\tilde{\omega}_- \wedge \tilde{\omega}_+, \tilde{\omega}_+ \wedge \tilde{\omega}_z, \tilde{\omega}_z \wedge \tilde{\omega}_-\}$: extending in a natural way via the pull back the left and right actions of the group $SU(2)$ on $\Omega^2(S^3)$, it is also clear that such basis elements are left invariant. From (2.21) one has:

$$\begin{aligned} \mathrm{d}\tilde{\omega}_- &= i\tilde{\omega}_- \wedge \tilde{\omega}_z, \\ \mathrm{d}\tilde{\omega}_+ &= -i\tilde{\omega}_+ \wedge \tilde{\omega}_z, \\ \mathrm{d}\tilde{\omega}_z &= 2i\tilde{\omega}_- \wedge \tilde{\omega}_+. \end{aligned} \tag{2.28}$$

The bimodule $\Omega^3(S^3)$ is one dimensional, with again a left invariant basis 3-form given by $\{\tilde{\omega}_- \wedge \tilde{\omega}_+ \wedge \tilde{\omega}_z\}$. A right invariant basis of the exterior algebra $\Omega(S^3)$ is analogously given in terms of the 1-forms $\tilde{\eta}_a$.

2.2 A Laplacian operator on the group manifold $SU(2)$

Being $SU(2)$ a semisimple Lie group, the group manifold S^3 can be equipped with the Cartan-Killing metric originated from the Cartan decomposition of the Lie algebra $\mathfrak{su}(2)$. Consider now as a riemannian metric structure on S^3 the symmetric tensor

$$g = \alpha(\tilde{\omega}_x \otimes \tilde{\omega}_x + \tilde{\omega}_y \otimes \tilde{\omega}_y) + \tilde{\omega}_z \otimes \tilde{\omega}_z, \tag{2.29}$$

with $\alpha \in \mathbb{R}^+$. For $\alpha = 1$ such a metric tensor coincides with the the Cartan-Killing metric. The volume associated to the $\mathfrak{g}$-orthonormal basis and to the choice of the orientation (x, y, z) is given by $\theta = \alpha\, \tilde{\omega}_x \wedge \tilde{\omega}_y \wedge \tilde{\omega}_z$, so that $\theta^* = \theta$. Such a volume θ is a Haar volume, namely it is invariant with respect to both the left l_g^* and the right actions r_g^* of the Lie group $SU(2)$ on itself, since an explicit calculation gives $L_a(\theta) = R_a(\theta) = 0$. The Hodge duality $\star : \Omega^k(S^3) \mapsto \Omega^{3-k}(S^3)$ which corresponds to this volume is the $C^\infty(S^3)$-linear map given on the left invariant basis of the exterior algebra $\Omega(S^3)$ by $\star(1) = \theta$, $\star(\theta) = 1$, and:

$$\begin{aligned} &\star(\tilde{\omega}_x) = \tilde{\omega}_y \wedge \tilde{\omega}_z, &&\star(\tilde{\omega}_y \wedge \tilde{\omega}_z) = \tilde{\omega}_x, \\ &\star(\tilde{\omega}_y) = \tilde{\omega}_z \wedge \tilde{\omega}_x, &&\star(\tilde{\omega}_z \wedge \tilde{\omega}_x) = \tilde{\omega}_y, \\ &\star(\tilde{\omega}_z) = \alpha\, \tilde{\omega}_x \wedge \tilde{\omega}_y, &&\star(\tilde{\omega}_x \wedge \tilde{\omega}_y) = \alpha^{-1}\, \tilde{\omega}_z. \end{aligned} \tag{2.30}$$

This Hodge structure satisfies two identities. The first is

$$\star^2(\xi) = (-1)^{k(3-k)} \xi = \xi \tag{2.31}$$

on any $\xi \in \Omega^k(S^3)$, while the second – satisfied by any Hodge duality defined on a riemannian manifold – is

$$\xi \wedge (\star\xi') = \xi' \wedge (\star\xi), \tag{2.32}$$

for any $\xi, \xi' \in \Omega^k(S^3)$. This allows to define a symmetric bilinear map $\langle \cdot, \cdot \rangle_{S^3} : \Omega^k(S^3) \times \Omega^k(S^3) \mapsto C^\infty(S^3)$ $(k = 0, \ldots, 3)$ as:

$$\langle \xi, \xi' \rangle_{SU(2)}\, \theta = \xi \wedge (\star\xi'). \tag{2.33}$$

The metric in (2.29) is a symmetric tensor $g \in \{\Omega^1(S^3)\}^{\otimes 2} \simeq \mathfrak{X}^*(S^3) \otimes_{C^\infty(S^3)} \mathfrak{X}^*(S^3)$, diagonal in the global $\{\tilde{\omega}_x, \tilde{\omega}_y, \tilde{\omega}_z\}$ basis. Its inverse is set as the symmetric tensor $g^{-1} \in \mathfrak{X}(S^3) \otimes \mathfrak{X}(S^3)$ in terms of the dual global basis $\{L_a\}$ of left invariant vector fields as $g^{-1} = \alpha^{-1}(L_x \otimes L_x + L_y \otimes L_y) + L_z \otimes L_z$. Given two k-forms $\xi = \xi_{i_1,\dots,i_k}\tilde{\omega}_{i_1} \wedge \dots \wedge \tilde{\omega}_{i_k}$ and $\xi' = \xi'_{i_1,\dots,i_k}\tilde{\omega}_{i_1} \wedge \dots \wedge \tilde{\omega}_{i_k}$ (sums are over $i_1 < \dots < i_k$, where the ordering is intended with respect to the orientation), it is possible to prove that the bilinear map is a symmetric tensor $\langle \cdot, \cdot \rangle_{S^3} \in \{\mathcal{X}^*(S^3)\}^{\otimes k}$, whose components are given in terms of the components of the tensor g^{-1}:

$$\langle \xi, \xi' \rangle_{S^3} = \xi_{i_1,\dots,i_k}\xi'_{j_1,\dots,j_k} \sum_\sigma \pi_\sigma g^{-1 i_1 \sigma(j_1)} \dots g^{-1 i_k \sigma(j_k)} \tag{2.34}$$

with g^{-1ij} the components of the tensor g^{-1} in the basis dual to $\tilde{\omega}_i$. Summation is over permutations σ of k elements, with parity π_σ. An explicit evaluation gives:

$$\begin{aligned}
&\langle 1, 1 \rangle_{S^3} = 1; \\
&\langle \tilde{\omega}_x, \tilde{\omega}_x \rangle_{S^3} = \langle \tilde{\omega}_y, \tilde{\omega}_y \rangle_{S^3} = \frac{1}{\alpha}, \\
&\langle \tilde{\omega}_z, \tilde{\omega}_z \rangle_{S^3} = 1; \\
&\langle \tilde{\omega}_y \wedge \tilde{\omega}_z, \tilde{\omega}_y \wedge \tilde{\omega}_z \rangle_{S^3} = \langle \tilde{\omega}_z \wedge \tilde{\omega}_x, \tilde{\omega}_z \wedge \tilde{\omega}_x \rangle_{S^3} = \frac{1}{\alpha}, \\
&\langle \tilde{\omega}_x \wedge \tilde{\omega}_y, \tilde{\omega}_x \wedge \tilde{\omega}_y \rangle_{S^3} = \frac{1}{\alpha^2}; \\
&\langle \theta, \theta \rangle_{S^3} = 1.
\end{aligned} \tag{2.35}$$

In the basis $\{\tilde{\omega}_a\}$ with $a = \pm, z$, the metric tensor is $g = 2\alpha(\tilde{\omega}_- \otimes \tilde{\omega}_+ + \tilde{\omega}_+ \otimes \tilde{\omega}_-) + \tilde{\omega}_z \otimes \tilde{\omega}_z$ and the volume form is $\theta = 2i\alpha\, \tilde{\omega}_- \wedge \tilde{\omega}_+ \wedge \tilde{\omega}_z$. The Hodge duality in (2.30) is $\star(\theta) = 1$, $\star(1) = \theta$, and:

$$\begin{array}{ll}
\star(\tilde{\omega}_+) = i\tilde{\omega}_+ \wedge \tilde{\omega}_z, & \star(\tilde{\omega}_+ \wedge \tilde{\omega}_z) = -i\tilde{\omega}_+, \\
\star(\tilde{\omega}_-) = -i\tilde{\omega}_- \wedge \tilde{\omega}_z, & \star(\tilde{\omega}_- \wedge \tilde{\omega}_z) = i\tilde{\omega}_-, \\
\star(\tilde{\omega}_z) = 2i\alpha\, \tilde{\omega}_- \wedge \tilde{\omega}_+, & \star(\tilde{\omega}_- \wedge \tilde{\omega}_+) = (-i/2\alpha)\, \tilde{\omega}_z.
\end{array} \tag{2.36}$$

In such a basis one also has $g^{-1} = (1/2\alpha)(L_- \otimes L_+ + L_+ \otimes L_-) + L_z \otimes L_z$, and the non zero terms of the bilinear (2.35) are:

$$\begin{aligned}
&\langle 1, 1 \rangle_{S^3} = 1; \\
&\langle \tilde{\omega}_-, \tilde{\omega}_+ \rangle_{S^3} = \langle \tilde{\omega}_+, \tilde{\omega}_- \rangle_{S^3} = \frac{1}{2\alpha}, \\
&\langle \tilde{\omega}_z, \tilde{\omega}_z \rangle_{S^3} = 1; \\
&\langle \tilde{\omega}_+ \wedge \tilde{\omega}_z, \tilde{\omega}_- \wedge \tilde{\omega}_z \rangle_{S^3} = \langle \tilde{\omega}_- \wedge \tilde{\omega}_z, \tilde{\omega}_+ \wedge \tilde{\omega}_z \rangle_{S^3} = \frac{1}{2\alpha}, \\
&\langle \tilde{\omega}_- \wedge \tilde{\omega}_+, \tilde{\omega}_- \wedge \tilde{\omega}_+ \rangle_{S^3} = -\frac{1}{4\alpha^2}; \\
&\langle \theta, \theta \rangle_{S^3} = 1.
\end{aligned} \tag{2.37}$$

Remark 2.1. *It is well known how a volume form can be used to introduce an integral on a manifold [1]. The Haar volume defines an integral* $\int_\theta : C^\infty(S^3) \mapsto \mathbb{C}$ *or equivalently* $\int_\theta : \Omega^3(S^3) \mapsto \mathbb{C}$ *since* $\Omega^3(S^3) \simeq C^\infty(S^3)$*, with* $\int_\theta \phi = \int_\theta \phi\,\theta$ *for any* $\phi \in C^\infty(S^3)$*. Normalising the volume of the group manifold,* $\int_\theta 1 = \int_\theta \theta = 1$*, from* (2.32) *it is possible to define a scalar product on the exterior algebra* $\Omega(S^3)$*, given by:*

$$(\xi; \xi')_{S^3} = \int_\theta \xi \wedge (\star\xi') = \int_\theta \langle \xi, \xi' \rangle_{S^3}\, \theta. \tag{2.38}$$

Starting from the Hodge duality a second bilinear map $\langle \cdot, \cdot \rangle^{\sim}_{S^3} : \Omega^k(S^3) \times \Omega^k(S^3) \mapsto C^\infty(S^3)$, setting

$$\langle \xi', \xi \rangle^{\sim}_{S^3}\, \theta = \xi^* \wedge (\star\xi') \tag{2.39}$$

for any $\xi, \xi' \in \Omega^k(S^3)$. It is easy to see that such a definition is consistent, and that on the basis of left invariant k-forms it is given by:

$$
\begin{aligned}
&\langle 1, 1\rangle^{\sim}_{S^3} = 1;\\
&\langle \tilde{\omega}_-, \tilde{\omega}_-\rangle^{\sim}_{S^3} = \langle \tilde{\omega}_+, \tilde{\omega}_+\rangle^{\sim}_{S^3} = \frac{1}{2\alpha},\\
&\langle \tilde{\omega}_z, \tilde{\omega}_z\rangle^{\sim}_{S^3} = 1;\\
&\langle \tilde{\omega}_+ \wedge \tilde{\omega}_z, \tilde{\omega}_+ \wedge \tilde{\omega}_z\rangle^{\sim}_{S^3} = \langle \tilde{\omega}_- \wedge \tilde{\omega}_z, \tilde{\omega}_- \wedge \tilde{\omega}_z\rangle^{\sim}_{S^3} = \frac{1}{2\alpha},\\
&\langle \tilde{\omega}_- \wedge \tilde{\omega}_+, \tilde{\omega}_- \wedge \tilde{\omega}_+\rangle^{\sim}_{S^3} = \frac{1}{4\alpha^2};\\
&\langle \theta, \theta\rangle^{\sim}_{S^3} = 1.
\end{aligned}
\tag{2.40}
$$

Remark 2.2. *Following the same path, from the bilinear* (2.39) *an inner product* $(\cdot;\cdot)^{\sim}_{S^3} : \Omega^k(S^3) \times \Omega^k(S^3) \mapsto \mathbb{C}$ *can be defined by*

$$(\xi'; \xi)^{\sim}_{S^3} = \int_\theta \xi^* \wedge (\star\xi'). \tag{2.41}$$

Using again (2.32)*, it is possible to see that such a bilinear is hermitian,* $(\xi'; \xi)^{\sim}_{S^3} = ((\xi; \xi')^{\sim}_{S^3})^*$*. Its terms are clearly given by* (2.40) *on that left invariant basis.*

The differential calculus on the group manifold S^3 as well as the $\star$-Hodge duality on the exterior algebra $\Omega(S^3)$ give a Laplacian operator defined as $\square_{S^3}\phi = \star\mathrm{d}\star\mathrm{d}\phi$ on any $\phi \in C^\infty(S^3)$. It can be written as a differential operator using the left invariant vector fields:

$$\square_{S^3}\phi = [\frac{1}{2\alpha}(L_-L_+ + L_+L_-) + L_zL_z]\phi \tag{2.42}$$

The Laplacian operator is the Casimir of the Lie algebra $\mathfrak{su}(2)$ only if $\alpha = 1$, that is only if the metric from where it is originated is the Cartan-Killing metric.

2.3 The principal bundle structure and the monopole connection

Consider now the one parameter subgroup of $SU(2)$ given by $\gamma_{T_z}(s) = \exp sT_z$ where T_z is the generator in (2.10). In this specific matrix representation it is

$$\gamma_{T_z}(s) = \exp\left[\frac{is}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \begin{pmatrix} e^{is/2} & 0 \\ 0 & e^{-is/2} \end{pmatrix}, \tag{2.43}$$

thus proving that $\gamma_{T_z}(s) \simeq U(1)$ as a subgroup in $SU(2)$. The space of left cosets $SU(2)/U(1)$ is the set of the orbits of the right principal action $\check{r}_{\exp sT_z}(g) = g\exp sT_z$ which is free, and smooth; its infinitesimal generator coincides with the vector field L_z (2.9). This is the vertical field of the bundle.

In the classical approach a trivialisation (an atlas of local charts) of the base manifold is introduced. Parametrising S^3 by the Euler angles:

$$
\begin{aligned}
u &= \cos\theta/2\, e^{i(\varphi+\psi)/2}\\
v &= \sin\theta/2\, e^{-i(\varphi-\psi)/2},
\end{aligned}
$$

one has, as representative of the canonical projection $\pi : SU(2) \mapsto S^2 \simeq SU(2)/U(1)$:

$$
\begin{aligned}
b_z &= uu^* - vv^* = \cos\theta,\\
b_y &= uv^* + vu^* = \sin\theta\cos\varphi,\\
b_x &= -i(vu^* - uv^*) = -\sin\theta\cos\varphi
\end{aligned}
\tag{2.44}
$$

with $b_z^2 + b_x^2 + b_y^2 = 1$. A choice for an open covering of the sphere S^2 is given by:

$$
\begin{aligned}
S^2_{(N)} = \{S^2 : b_z \neq 1\} \qquad &\rightarrow \qquad \pi^{-1}(S^2_{(N)}) = S^3_{(N)} = \{S^3 : v \neq 0\},\\
S^2_{(S)} = \{S^2 : b_z \neq -1\} \qquad &\rightarrow \qquad \pi^{-1}(S^2_{(S)}) = S^3_{(S)} = \{S^3 : u \neq 0\},
\end{aligned}
\tag{2.45}
$$

so that $S^3_{(j)} \simeq S^2_{(j)} \times U(1)$ via the diffeomorphisms:

$$g \simeq (u,v) \in S^3_{(N)} \to \lambda_N(g) = (\pi(g); \left(\frac{v}{|v|}\right)^n) \in S^2_{(N)} \times U(1),$$
$$g \simeq (u,v) \in S^3_{(S)} \to \lambda_S(g) = (\pi(g); \left(\frac{u}{|u|}\right)^n) \in S^2_{(S)} \times U(1)$$

with $n \in \mathbb{Z}$. The set of transition functions associated with this trivialisation is given by $\lambda_{NS}^{-1} = \lambda_{SN} = \lambda_S \circ \lambda_N^{-1} : (S^2_{(N)} \cap S^2_{(S)}) \cap U(1) \mapsto U(1)$. Choose $b \sim (\theta, \varphi) \in S^2_{(N)} \cap S^2_{(S)}$. The element $(b, e^{i\alpha}) \in (S^2_{(N)} \cap S^2_{(S)}) \times U(1)$ is mapped into

$$\lambda_N^{-1}(b, e^{i\alpha}) = (u = \frac{b_y - ib_x}{\sqrt{2(1-z)}}\, e^{in\alpha}; v = \sqrt{\frac{1-z}{2}}\, e^{i\alpha/n}) \in S^3_{(N)}$$
$$\to \lambda_S \circ \lambda_N^{-1}(b; e^{i\alpha}) = (b, e^{in\varphi} e^{i\alpha}).$$

This means that $\lambda_{SN}(b) \cdot e^{i\alpha} = e^{in\varphi} e^{i\alpha}$. the transition functions formalise a left action of the gauge group on itself, and trivially satisfy the cocycle conditions. The image of any parallel on S^2 under λ_{SN}, seen as a map from a subset of the sphere S^2 to the gauge group, winds n times around the $U(1)$ circle. The classical Hopf bundle is then characterised by the number $n \in \mathbb{Z}$, which is the cohomology class of its cocycle of transition functions. For any integer n there is a representation of the gauge group,

$$\rho_{(n)} : U(1) \mapsto \mathbb{C}, \qquad \rho_{(n)}(e^{i\alpha}) = e^{in\alpha} \tag{2.46}$$

so that for any $n \in \mathbb{Z}$ there is a line bundle $\mathcal{E}_n = SU(2) \times_{\rho_{(n)}} \mathbb{C}$ associated to the principal Hopf bundle. Since the representations of the gauge group given in (2.46) are defined on $\mathbb{C}$, the set $\Omega^r(S^3, \mathbb{C})_{\rho_{(n)}}$ of $\rho_{(n)}(U(1))$-equivariant r-forms on the Hopf bundle can be easily formalised in terms of the action of the vertical field of the bundle, giving the infinitesimal version of the definition in (2.1) (with $r = 0, \dots, 3$)

$$\Omega^r(S^3, \mathbb{C})_{\rho_{(n)}} \simeq \Omega^r(S^3)_{\rho_{(n)}} = \{\phi \in \Omega^r(S^3) : \check{r}_k^*(\phi) = \rho_{(n)}^{-1}(k)\phi \leftrightarrow L_z(\phi) = -\frac{in}{2}\phi\}. \tag{2.47}$$

The sets $\Omega^r(S^3)_{\rho_{(n)}}$ are $C^\infty(S^2)$-bimodule. The horizontal $\rho_{(n)}(U(1))$-equivariant r-forms are given as:

$$\mathfrak{L}_n^{(r)} = \{\phi \in \Omega^r(S^3)_{\rho_{(n)}} : i_{L_z}(\phi) = 0\} \tag{2.48}$$

for $r > 0$: one obviously has $\mathfrak{L}_n^{(3)} = \emptyset$, while

$$\mathfrak{L}_n^{(0)} = \Omega^0(S^3)_{\rho_{(n)}} = \{\phi \in C^\infty(S^3) : \check{r}_k^*(\phi) = \phi \leftrightarrow L_z(\phi) = -(in/2)\phi\}. \tag{2.49}$$

With $\Gamma^{(r)}(S^2, \mathcal{E}_n)$ the set of $\mathcal{E}_n$-valued r-forms defined on S^2, the isomorphisms in (2.2) can be written as isomorphisms of $C^\infty(S^2)$-bimodule

$$\Gamma^{(r)}(S^2, \mathcal{E}_n) \simeq \mathfrak{L}_n^{(r)}. \tag{2.50}$$

They formalise the equivalence between r-form valued sections on each line bundle $\mathcal{E}_n$ and $\rho_{(n)}(U(1))$-equivariant horizontal r-forms of the principal Hopf bundle. This equivalence can be described – as in [21] – using the local trivialisation (2.45). A global, algebraic description of them, naturally conceived for the generalisation to the non commutative setting, is in [15], and it is based on the Serre-Swan theorem[1].

Given $n \in \mathbb{Z}$, consider an element $\left|\tilde{\Psi}^{(n)}\right\rangle \in C^\infty(S^3)^{|n|+1}$ whose components are given by:

$$n \geq 0 : \qquad \left|\tilde{\Psi}^{(n)}\right\rangle_\mu = \sqrt{\begin{pmatrix} n \\ \mu \end{pmatrix}}\, \bar{v}^\mu \bar{u}^{n-\mu} \in \mathfrak{L}_n^{(0)},$$
$$n \leq 0 : \qquad \left|\tilde{\Psi}^{(n)}\right\rangle_\mu = \sqrt{\begin{pmatrix} |n| \\ \mu \end{pmatrix}}\, v^{|n|-\mu} u^\mu \in \mathfrak{L}_n^{(0)} \tag{2.51}$$

[1]The theorem of Serre and Swan [26] constructs a complete equivalence between the category of (smooth) vector bundles over a (smooth) compact manifold $\mathcal{M}$ and bundle maps, and the category of finite projective modules over the commutative algebra $C(\mathcal{M})$ of (smooth) functions over $\mathcal{M}$ and module morphisms. The space $\Gamma(\mathcal{M}, \mathcal{E})$ of (smooth) sections of a vector bundle $\pi_{\mathcal{E}} : \mathcal{E} \mapsto \mathcal{M}$ over a compact manifold $\mathcal{M}$ is a finite projective module over the commutative algebra $C(\mathcal{M})$ and every finite projective $C(\mathcal{M})$-module can be realised as a module of sections of a vector bundle over $\mathcal{M}$.

with $\mu = 0, \dots |n|$. Recalling the binomial expansion it is easy to compute that:

$$
\begin{aligned}
n \geq 0: \qquad & \left\langle \tilde{\Psi}^{(n)}, \tilde{\Psi}^{(n)} \right\rangle = \sum_{\mu=0}^{n} \begin{pmatrix} n \\ \mu \end{pmatrix} u^{n-\mu} v^{\mu} \bar{v}^{\mu} \bar{u}^{n-\mu} = (\bar{u}u + \bar{v}v)^n = 1, \\
n \leq 0: \qquad & \left\langle \tilde{\Psi}^{(n)}, \tilde{\Psi}^{(n)} \right\rangle = \sum_{\mu=0}^{|n|} \begin{pmatrix} |n| \\ \mu \end{pmatrix} \bar{u}^{\mu} \bar{v}^{|n|-\mu} v^{|n|-\mu} u^{\mu} = (\bar{u}u + \bar{v}v)^n = 1.
\end{aligned} \tag{2.52}
$$

The ket-bra element $\tilde{\mathfrak{p}}^{(n)} = \left| \tilde{\Psi}^{(n)} \right\rangle \left\langle \tilde{\Psi}^{(n)} \right| \in \mathbb{M}^{|n|+1}(C^{\infty}(S^2))$ is then a projector in the free finitely generated module $C^{\infty}(S^2)^{|n|+1}$, as it satisfies the identities $(\tilde{\mathfrak{p}}^{(n)})^{\dagger} = \tilde{\mathfrak{p}}^{(n)}$, $(\tilde{\mathfrak{p}}^{(n)})^2 = \tilde{\mathfrak{p}}^{(n)}$. The matrix elements of the projectors are given by $\tilde{\mathfrak{p}}^{(n)}_{\mu\nu} = \left| \tilde{\Psi}^{(n)} \right\rangle_{\mu} \left\langle \tilde{\Psi}^{(n)} \right|_{\nu}$: each projector $\tilde{\mathfrak{p}}^{(n)}$ has rank 1, because its trace is the constant unit function given by

$$
tr\, \tilde{\mathfrak{p}}^{(n)} = \sum_{\mu=0}^{|n|} \left| \tilde{\Psi}^{(n)} \right\rangle_{\mu} \left\langle \tilde{\Psi}^{(n)} \right|_{\mu} = 1. \tag{2.53}
$$

Consider the set of $\rho_{(n)}(U(1))$-equivariant map $\mathfrak{L}^{(0)}_n$ as a left module over $C^{\infty}(S^2) \subset C^{\infty}(S^3)$: any equivariant map $\phi \in \mathfrak{L}^{(0)}_n$ can be written in terms of an element $\langle f| \in C^{\infty}(S^2)^{|n|+1}$ as $\phi_f = \left\langle f, \tilde{\Psi}^{(n)} \right\rangle = \sum_{\mu=0}^{|n|} \langle f|_{\mu} \left| \tilde{\Psi}^{(n)} \right\rangle_{\mu}$. Given the set $\Gamma^{(0)}(S^2, \mathcal{E}_n)$ of sections of each associated line bundle $\mathcal{E}_n$, the equivalence with the set $\mathfrak{L}^{(0)}_n$ of $\rho_{(n)}(U(1))$-equivariant maps of the Hopf bundle is formalised via an isomorphism between $C^{\infty}(S^2)$-left modules, represented by:

$$
\begin{aligned}
\Gamma^{(0)}(S^2, \mathcal{E}_n) \qquad & \leftrightarrow \qquad \mathfrak{L}^{(0)}_n \\
\langle \sigma_f| = \langle f|\, \tilde{\mathfrak{p}}^{(n)} \qquad & \leftrightarrow \qquad \left\langle f, \tilde{\Psi}^{(n)} \right\rangle \\
\langle \sigma_f| = \phi_f \left\langle \tilde{\Psi}^{(n)} \right| \qquad & \leftrightarrow \qquad \phi_f = \left\langle \sigma_f, \tilde{\Psi}^{(n)} \right\rangle
\end{aligned} \tag{2.54}
$$

for any $\langle f| \in C^{\infty}(S^2)^{|n|+1}$. Since from this definition it is $\langle \sigma_f|\, \tilde{\mathfrak{p}}^{(n)} = \langle \sigma_f|$, this isomorphism enables to recover $\langle \sigma_f| \in \Gamma^{(0)}(S^2, \mathcal{E}_n) \simeq C^{\infty}(S^2)^{|n|+1} \tilde{\mathfrak{p}}^{(n)}$.

An explicit computation from (2.11) and (2.21) gives:

$$
\begin{aligned}
L_z(\tilde{\omega}_+) = i\tilde{\omega}_+ \qquad & \rightarrow \qquad \tilde{\omega}_+ \in \mathfrak{L}^{(1)}_{-2}; \\
L_z(\tilde{\omega}_-) = -i\tilde{\omega}_- \qquad & \rightarrow \qquad \tilde{\omega}_- \in \mathfrak{L}^{(1)}_{2},
\end{aligned} \tag{2.55}
$$

so that for any $n \in \mathbb{Z}$ the set of $\rho_{(n)}(U(1))$-equivariant horizontal 1-forms of the Hopf bundle is

$$
\mathfrak{L}^{(1)}_n = \{\phi = \phi' \tilde{\omega}_- + \phi'' \tilde{\omega}_+ : \, \phi' \in \mathfrak{L}^{(0)}_{n-2} \,\text{and}\, \phi'' \in \mathfrak{L}^{(0)}_{n+2}\}. \tag{2.56}
$$

For $n = 0$ one also recovers from (2.3) the equivalence $\mathfrak{L}^{(1)}_0 \simeq \Omega^1(S^2)$, so to have the $C^{\infty}(S^2)$-bimodule identification $\mathfrak{L}^{(1)}_n \simeq \Omega^1(S^2) \otimes_{C^{\infty}(S^2)} \mathfrak{L}^{(0)}_n$. For $r = 1$ the isomorphism in (2.50) can be written as:

$$
\begin{aligned}
\Gamma^{(1)}(S^2, \mathcal{E}_n) \simeq \Omega^1(S^2)^{|n|+1} \cdot \tilde{\mathfrak{p}}^{(n)} \qquad & \leftrightarrow \qquad \mathfrak{L}^{(1)}_n \simeq \Omega^1(S^2) \otimes_{C^{\infty}(S^2)} \mathfrak{L}^{(0)}_n, \\
\langle \sigma| = \phi \left\langle \tilde{\Psi}^{(n)} \right| \qquad & \leftrightarrow \qquad \phi = \left\langle \sigma, \tilde{\Psi}^{(n)} \right\rangle.
\end{aligned} \tag{2.57}
$$

Given any $\phi \in \mathfrak{L}^{(1)}_n$, set $\langle \sigma| = \phi \left\langle \tilde{\Psi}^{(n)} \right| \in \Omega^1(S^2)^{|n|+1}$, so to have $\langle \sigma| = \langle \sigma|\, \tilde{\mathfrak{p}}^{(n)}$. To formalise the inverse mapping, consider $\langle \sigma| \in \Omega^1(S^2)^{|n|+1} \tilde{\mathfrak{p}}^{(n)}$ with components $\langle \sigma|_{\mu} \in \Omega^1(S^2)$ in the bra-vector notation, satisfying $\langle \sigma|_{\mu}\, \tilde{\mathfrak{p}}^{(n)}_{\mu\nu} = \langle \sigma|_{\nu}$. Define $\phi = \left\langle \sigma, \tilde{\Psi}^{(n)} \right\rangle$: it is then straightforward to recover that $\phi \in \mathfrak{L}^{(1)}_n$ and that $\langle \sigma|_{\mu} = \phi \left\langle \tilde{\Psi}^{(n)} \right|_{\mu}$.

The same path can be followed to analyse the higher order forms. One has $L_z(\tilde{\omega}_- \wedge \tilde{\omega}_+) = 0$, so the $C^\infty(S^2)$-bimodule of horizontal $\rho_{(n)}(U(1))$-equivariant 2-forms of the Hopf bundle is given by

$$\mathfrak{L}^{(2)}_n = \{\phi = \phi'''\tilde{\omega}_- \wedge \tilde{\omega}_+ : \phi''' \in \mathfrak{L}^{(0)}_n\} \simeq \Omega^2(S^2) \otimes_{C^\infty(S^2)} \mathfrak{L}^{(0)}_n \tag{2.58}$$

for any $n \in \mathbb{Z}$. It is clear that for $r = 2$ the isomorphism in (2.50) can be written as:

$$\begin{aligned} \Gamma^{(2)}(S^2, \mathcal{E}_n) \simeq \Omega^2(S^2)^{|n|+1} \cdot \tilde{\mathfrak{p}}^{(n)} \qquad &\leftrightarrow \qquad \mathfrak{L}^{(2)}_n \simeq \Omega^2(S^2) \otimes_{C^\infty(S^2)} \mathfrak{L}^{(0)}_n, \\ \langle \sigma | = \phi \left\langle \tilde{\Psi}^{(n)} \right| \qquad &\leftrightarrow \qquad \phi = \left\langle \sigma, \tilde{\Psi}^{(n)} \right\rangle. \end{aligned} \tag{2.59}$$

The most natural choice of a connection, compatible with the local trivialisation, is given via the definition, as a $\mathbb{C}$-valued connection 1-form, of

$$\omega = \frac{in}{2}\tilde{\omega}_z = n(u^* \mathrm{d}u + v^* \mathrm{d}v), \tag{2.60}$$

globally – i.e. trivialisation independent – selecting the horizontal part of the tangent space as the left $C^\infty(S^3)$-module $H^{(\omega)}(S^3) \subset \mathfrak{X}(S^3) = \{L_\pm\}$ since $\omega(L_\pm) = 0$. On the basis of left invariant vector fields the horizontal projection acts as $L^{(\omega)}_\pm = L_\pm$, $L^{(\omega)}_z = 0$.

2.4 A Laplacian operator on the base manifold S^2

The canonical isomorphism expressed in (2.3) allows to formalise the exterior algebra $\Omega(S^2)$ on the basis of the Hopf bundle as the set of horizontal forms in $\Omega(S^3)$ which are also invariant for the right principal action of the gauge group $U(1)$. Recalling the definition of the $C^\infty(S^2)$-bimodules of $\rho_{(n)}(U(1))$-equivariant forms given in (2.56) and (2.58), it is possible to identify

$$\begin{aligned} &\Omega^0(S^2) = C^\infty(S^2) \simeq \mathfrak{L}^{(0)}_0; \\ &\Omega^1(S^2) \simeq \mathfrak{L}^{(1)}_0 = \{\phi = \phi'\tilde{\omega}_- + \phi''\tilde{\omega}_+ : \phi' \in \mathfrak{L}^{(0)}_{-2}, \phi'' \in \mathfrak{L}^{(0)}_2\}; \\ &\Omega^2(S^2) \simeq \mathfrak{L}^{(2)}_0 = \{f\tilde{\omega}_- \wedge \tilde{\omega}_+ : f \in \mathfrak{L}^{(0)}_0 = C^\infty(S^2)\}, \end{aligned} \tag{2.61}$$

where all such identifications are $C^\infty(S^2)$-bimodule isomorphisms.

On the basis manifold $S^2 \simeq SU(2)/U(1) = \pi(SU(2))$, whose trivialisation is given in (2.45), consider the metric

$$\check{g} = 2\alpha\,(\tilde{\omega}_- \otimes \tilde{\omega}_+ + \tilde{\omega}_+ \otimes \tilde{\omega}_-) \tag{2.62}$$

and its associated volume $\check{\theta} = \alpha\,\tilde{\omega}_x \wedge \tilde{\omega}_y = 2i\alpha\,\tilde{\omega}_- \wedge \tilde{\omega}_+ = i_{L_z}\theta$ in terms of the volume on the group manifold S^3. The corresponding Hodge duality is the $C^\infty(S^2)$-linear map $\star : \Omega^k(S^2) \mapsto \Omega^{2-k}(S^2)$ given by:

$$\begin{aligned} &\star(\check{\theta}) = 1, &\qquad &\star(1) = \check{\theta}, \\ &\star(\phi''\tilde{\omega}_+) = i\phi''\tilde{\omega}_+, &\qquad &\star(\phi'\tilde{\omega}_-) = -i\phi'\tilde{\omega}_-, \end{aligned} \tag{2.63}$$

with $\phi' \in \mathfrak{L}^{(0)}_{-2}$ and $\phi'' \in \mathfrak{L}^{(0)}_2$. The Laplacian operator on S^2 can be now evaluated:

$$\Box_{S^2} f = \star\mathrm{d} \star \mathrm{d}f = \frac{1}{2\alpha}(L_+L_- + L_-L_+)f. \tag{2.64}$$

It corresponds to the action of the Laplacian $\Box_{S^3}$ (2.42) on the subalgebra algebra $C^\infty(S^2) \subset C^\infty(S^3)$.

Remark 2.3. *Given the Hodge duality* (2.63)*, the expression* (2.33) *defines a bilinear symmetric tensor* $\langle \cdot, \cdot \rangle_{S^2} : \Omega^k(S^2) \times \Omega^k(S^2) \mapsto C^\infty(S^2)$ *(with* $k = 0, 1, 2$*):*

$$\langle \xi, \xi' \rangle_{S^2}\, \check{\theta} = \xi \wedge (\star\xi'), \tag{2.65}$$

for any $\xi, \xi' \in \Omega^k(S^2)$*. Its non zero terms are given by:*

$$\begin{aligned} &\langle 1, 1 \rangle_{S^2} = 1; \\ &\langle \phi'\tilde{\omega}_-, \phi''\tilde{\omega}_+ \rangle_{S^2} = \langle \phi''\tilde{\omega}_+, \phi'\tilde{\omega}_- \rangle_{S^2} = \phi'\phi''/2\alpha; \\ &\left\langle \check{\theta}, \check{\theta} \right\rangle_{S^2} = 1 : \end{aligned} \tag{2.66}$$

such a tensor coincides with the restriction to the exterior algebra $\Omega(S^2)$ of the analogue tensor $\langle \cdot, \cdot \rangle_{S^3}$. From (2.37)*, an explicit computation shows this equality:*

$$\begin{aligned} \langle \phi' \tilde{\omega}_-, \phi'' \tilde{\omega}_+ \rangle_{S^3} &= \phi' \phi'' / 2\alpha = \langle \phi'' \tilde{\omega}_+, \phi' \tilde{\omega}_- \rangle_{S^2}, \\ \langle 2i\alpha\, \tilde{\omega}_- \wedge \tilde{\omega}_+, 2i\alpha\, \tilde{\omega}_- \wedge \tilde{\omega}_+ \rangle_{S^3} &= 1 = \langle \check{\theta}, \check{\theta} \rangle_{S^2} \end{aligned} \tag{2.67}$$

for any $\phi' \in \mathfrak{L}^{(0)}_{-2}$, $\phi'' \in \mathfrak{L}^{(0)}_{2}$. The expression

$$\langle \xi, \xi' \rangle^{\sim}_{S^2}\, \check{\theta} = \xi'^* \wedge (\star \xi), \tag{2.68}$$

with again $\xi, \xi' \in \Omega^k(S^2)$, defines a bilinear map on $\Omega(S^2)$, which is the analogue of the bilinear map $\langle \cdot, \cdot \rangle^{\sim}_{S^3}$ (2.39) *on the exterior algebra $\Omega(S^3)$. Also the restriction of $\langle \cdot, \cdot \rangle^{\sim}_{S^3}$ to $\Omega(S^2)$ coincides with $\langle \cdot, \cdot \rangle^{\sim}_{S^2}$. From* (2.40)*:*

$$\begin{aligned} &\langle 1, 1 \rangle^{\sim}_{S^2} = 1; \\ &\langle \phi' \tilde{\omega}_-, \psi' \tilde{\omega}_- \rangle^{\sim}_{S^2} = \frac{1}{2\alpha} \psi'^* \phi' = \langle \phi' \tilde{\omega}_-, \psi' \tilde{\omega}_- \rangle^{\sim}_{S^3}, \\ &\langle \phi'' \tilde{\omega}_+, \psi'' \tilde{\omega}_+ \rangle^{\sim}_{S^2} = \frac{1}{2\alpha} \psi'^* \phi' = \langle \phi'' \tilde{\omega}_+, \psi'' \tilde{\omega}_+ \rangle^{\sim}_{S^3}; \\ &\langle \check{\theta}, \check{\theta} \rangle^{\sim}_{S^2} = 1 = \langle 2i\alpha\, \tilde{\omega}_- \wedge \tilde{\omega}_+, 2i\alpha\, \tilde{\omega}_- \wedge \tilde{\omega}_+ \rangle^{\sim}_{S^3} \end{aligned} \tag{2.69}$$

for any $\phi', \psi' \in \mathfrak{L}^{(0)}_{-2}$ and $\phi'', \psi'' \in \mathfrak{L}^{(0)}_{2}$.

Remark 2.4. *Introducing from the volume form $\check{\theta}$ an integral $\int_{\check{\theta}} : C^\infty(S^2) \mapsto \mathbb{C}$ or equivalently $\int_{\check{\theta}} : \Omega^2(S^2) \mapsto \mathbb{C}$ with the normalisation $\int_{\check{\theta}} \theta = \int_{\check{\theta}} 1 = 1$, the bilinear maps in* (2.65) *and* (2.68) *give on the exterior algebra $\Omega(S^2)$ a symmetric scalar product and a hermitian inner product, setting:*

$$(\xi; \xi')_{S^2} = \int_{\check{\theta}} \xi \wedge (\star \xi'), \tag{2.70}$$

$$(\xi; \xi')^{\sim}_{S^2} = \int_{\check{\theta}} \xi'^* \wedge (\star \xi). \tag{2.71}$$

¿From (2.67) *it is clear that the scalar product* (2.70) *coincides with the restriction to $\Omega(S^2)$ of the scalar product* (2.38) *$(\cdot;\cdot)_{S^3}$ on $\Omega(S^3)$, while the inner product* (2.71) *coincide with the restriction to $\Omega(S^2)$ of the inner product $(\cdot;\cdot)^{\sim}_{S^3}$ on $\Omega(S^3)$ from* (2.40)*.*

3 The quantum principal Hopf bundle

The aim of this section is to describe the quantum formulation of the Hopf bundle. It starts with a description of the algebraic approach [29] to the theory of differential calculi on Hopf algebras, and then presents an algebraic formalisation for the geometric structures of a principal bundle.

3.1 An algebraic approach to the theory of differential calculi on Hopf algebras

The first order differential forms on the smooth group manifold $SU(2) \simeq S^3$ have been presented as elements in the space $\mathfrak{X}^*(S^3)$, or more properly as sections of the cotangent bundle $T^*(S^3)$. The set $\Omega^1(S^3) \simeq \mathfrak{X}^*(S^3)$ of one forms is a bimodule over $C^\infty(S^3)$, with the exterior derivative d satisfying the basic Leibniz rule $\mathrm{d}(ff') = (\mathrm{d}f)f' + f\mathrm{d}f'$ for any $f, f' \in C^\infty(S^3)$. Moreover, being S^3 a compact manifold, any differential form $\theta \in \Omega^1(S^3)$ is necessarily of the form $\theta = f_k \mathrm{d}f'_k$ (with $k \in \mathbb{N}$).

In an algebraic setting, these properties are a definition. Given a $\mathbb{C}$-algebra with a unit $\mathcal{A}$ and Ω a bimodule over $\mathcal{A}$ with a linear map $\mathrm{d} : \mathcal{A} \mapsto \Omega$, (Ω, d) is defined a first order differential calculus over $\mathcal{A}$ if $\mathrm{d}(ff') = (\mathrm{d}f)f' + f\mathrm{d}f'$ for any $f, f' \in \mathcal{A}$ and if any element $\theta \in \Omega$ can be written as $\theta = \sum_k f_k \mathrm{d}f'_k$ with $f_k, f'_k \in \mathcal{A}$.

For a $\mathbb{C}$-algebra with unit $\mathcal{A}$, any first order differential calculus $(\Omega^1(\mathcal{A}), \mathrm{d})$ on $\mathcal{A}$ can be obtained from the universal calculus $(\Omega^1(\mathcal{A})_{un}, \delta)$. The space of universal 1-forms is the submodule of $\mathcal{A} \otimes \mathcal{A}$

given by $\Omega^1(\mathcal{A})_{un} = \ker(m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A})$, with $m(a \otimes b) = ab$ the multiplication map. The universal differential $\delta : \mathcal{A} \to \Omega^1(\mathcal{A})_{un}$ is $\delta a = 1 \otimes a - a \otimes 1$. If $\mathcal{N}$ is any sub-bimodule of $\Omega^1(\mathcal{A})_{un}$ with projection $\pi_{\mathcal{N}} : \Omega^1(\mathcal{A})_{un} \to \Omega^1(\mathcal{A}) = \Omega^1(\mathcal{A})_{un}/\mathcal{N}$, then $(\Omega^1(\mathcal{A}), \mathrm{d})$, with $\mathrm{d} := \pi_{\mathcal{N}} \circ \delta$, is a first order differential calculus over $\mathcal{A}$ and any such a calculus can be obtained in this way. The projection $\pi_{\mathcal{N}} : \Omega^1(\mathcal{A})_{un} \mapsto \Omega^1(\mathcal{A})$ is $\pi_{\mathcal{N}}(\sum_i a \otimes b_1) = \sum_i a\mathrm{d}b_i$ with associated subbimodule $\mathcal{N} = \ker \pi$.

The concept of action of a group on a manifold is algebraically dualised via the notion of coaction of a Hopf algebra $\mathcal{H}$ on an algebra $\mathcal{A}$: if the algebra $\mathcal{A}$ is covariant for the coaction of a quantum group $\mathcal{H} = (\mathcal{H}, \Delta, \varepsilon, S)$, one has a notion of covariant calculi on $\mathcal{A}$ as well, thus translating the idea of invariance of the differential calculus on a manifold for the action of a group. Then, let $\mathcal{A}$ be a (right, say) $\mathcal{H}$-comodule algebra, with a right coaction $\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$ which is also an algebra map. In order to state the covariance of the calculus $(\Omega^1(\mathcal{A}), \mathrm{d})$ one needs to extend the coaction of $\mathcal{H}$. A map $\Delta_R^{(1)} : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \mathcal{H}$ is defined by the requirement

$$\Delta_R^{(1)}(\mathrm{d}f) = (\mathrm{d} \otimes \mathrm{id})\Delta_R(f)$$

and bimodule structure governed by

$$\Delta_R^{(1)}(f\mathrm{d}f') = \Delta_R(f)\Delta_R^{(1)}(\mathrm{d}f'),$$
$$\Delta_R^{(1)}((\mathrm{d}f)f') = \Delta_R^{(1)}(\mathrm{d}f)\Delta_R(f').$$

The calculus is said to be right covariant it happens that

$$(\mathrm{id} \otimes \Delta)\Delta_R^{(1)} = (\Delta_R^{(1)} \otimes \mathrm{id})\Delta_R^{(1)}$$

and

$$(\mathrm{id} \otimes \varepsilon)\Delta_R^{(1)} = 1.$$

A calculus is right covariant if and only if for the corresponding bimodule $\mathcal{N}$ it is verified that $\Delta_R^{(1)}(\mathcal{N}) \subset \mathcal{N} \otimes \mathcal{H}$, where $\Delta_R^{(1)}$ is defined on $\mathcal{N}$ by formulæ as above with the universal derivation δ replacing the derivation d:

$$\Delta_R^{(1)}(\delta f) = (\delta \otimes \mathrm{id})\Delta_R(f). \tag{3.1}$$

Differential calculi on a quantum group $\mathcal{H} = (\mathcal{H}, \Delta, \varepsilon, S)$ were studied in [29]. The coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is viewed as both a right and a left coaction of $\mathcal{H}$ on itself:

$$\Delta_R^{(1)}(\mathrm{d}h) = (\mathrm{d} \otimes 1)\Delta(h),$$
$$\Delta_L^{(1)}(\mathrm{d}h) = (1 \otimes \mathrm{d})\Delta(h). \tag{3.2}$$

Right and left covariant calculi on $\mathcal{H}$ will be defined as before. Right covariance of the calculus implies that $\Omega^1(\mathcal{H})$ has a module basis $\{\eta_a\}$ of right invariant 1-forms, that is 1-forms for which

$$\Delta_R^{(1)}(\eta_a) = \eta_a \otimes 1,$$

and left covariance of a calculus similarly implies that $\Omega^1(\mathcal{H})$ has a module basis $\{\omega_a\}$ of left invariant 1-forms, that is 1-forms for which $\Delta_L^{(1)}(\omega_a) = 1 \otimes \omega_a$. In addition one has the notion of a bicovariant calculus, namely a both left and right covariant calculus, satisfying the compatibility condition:

$$(\mathrm{id} \otimes \Delta_R^{(1)}) \circ \Delta_L^{(1)} = (\Delta_L^{(1)} \otimes \mathrm{id}) \circ \Delta_R^{(1)}.$$

Given the bijection

$$r : \mathcal{H} \otimes \mathcal{H} \mapsto \mathcal{H} \otimes \mathcal{H}, \qquad r(h \otimes h') := (h \otimes 1)\Delta(h'), \tag{3.3}$$

one proves that $r(\Omega^1(\mathcal{H})_{un}) = \mathcal{H} \otimes \ker \varepsilon$. Then, if $\mathcal{Q} \subset \ker \varepsilon$ is a right ideal of $\ker \varepsilon$, the inverse image $\mathcal{N}_{\mathcal{Q}} = r^{-1}(\mathcal{H} \otimes \mathcal{Q})$ is a sub-bimodule contained in $\Omega^1(\mathcal{H})_{un}$. The differential calculus defined by such a bimodule, $\Omega^1(\mathcal{H}) := \Omega^1(\mathcal{H})_{un}/\mathcal{N}_{\mathcal{Q}}$, is left-covariant, and any left-covariant differential calculus can be obtained in this way. Bicovariant calculi are in one to one correspondence with right ideals $\mathcal{Q} \subset \ker \varepsilon$ which are in addition stable under the right adjoint coaction Ad of $\mathcal{H}$ onto itself, that is

$\mathrm{Ad}(\mathcal{Q}) \subset \mathcal{Q} \otimes \mathcal{H}$. Explicitly, one has $\mathrm{Ad} = (\mathrm{id} \otimes m)(\tau \otimes \mathrm{id})(S \otimes \Delta)\Delta$, with τ the flip operator, or $\mathrm{Ad}(h) = h_{(2)} \otimes \big(S(h_{(1)})h_{(3)}\big)$ using the Sweedler notation $\Delta h =: h_{(1)} \otimes h_{(2)}$ with summation understood, and higher numbers for iterated coproducts.

The ideal $\mathcal{Q}$ also determines the tangent space of the calculus. This is a collection $\{X_a\}$ of elements in $\mathcal{U}(\mathcal{H})$ – the Hopf algebra dual to $\mathcal{H}$ – which allows one to write the exterior derivative as

$$\mathrm{d}h := \sum\nolimits_a (X_a \triangleright h)\, \omega_a, \tag{3.4}$$

for $h \in \mathcal{H}$ and elements X_a acting on the left on h. This duality is expressed by the existence of a bilinear map $\langle\ ,\ \rangle : \mathcal{U}(\mathcal{H}) \times \mathcal{H} \to \mathbb{C}$ such that:

$$\begin{aligned} \langle \Delta(X), h_1 \otimes h_2 \rangle &= \langle X, h_1 h_2 \rangle\,, \\ \langle X_1 X_2, h \rangle &= \langle X_1 \otimes X_2, \Delta(h) \rangle\,, \\ \langle X, 1 \rangle = \varepsilon(X), \qquad & \langle 1, h \rangle = \varepsilon(h) \end{aligned} \tag{3.5}$$

for any $X_a \in \mathcal{U}(\mathcal{H})$ and $h_b \in \mathcal{H}$. The pairing is also required to be compatible with $*$-structures:

$$\begin{aligned} \langle X^*, h \rangle &= \overline{\langle X, (S(h))^* \rangle}, \\ \langle X, h \rangle &= \overline{\langle (S(X))^*, h \rangle}. \end{aligned} \tag{3.6}$$

Such a dual pairing has the property that $\langle S(X), h \rangle = \langle X, S(h) \rangle$. A dual pairing can then be defined on the generators, and extended to the whole algebra following the relations (3.5). The tangent space results determined by the ideal $\mathcal{Q}$ as

$$\mathcal{X}_{\mathcal{Q}} := \{X \in \ker \varepsilon_{\mathcal{U}_q(\mathcal{H})} \ :\ \langle X, Q \rangle = 0,\ \forall\, Q \in \mathcal{Q}\},$$

where $\varepsilon_{\mathcal{U}(\mathcal{H})}$ is the counit of $\mathcal{U}(\mathcal{H})$. Via the dual pairing it is possible to regard $\mathcal{U}(\mathcal{H})$ as a subspace of the linear dual of $\mathcal{H}$, and there are canonical left and right $\mathcal{U}(\mathcal{H})$-module algebra structure on $\mathcal{H}$ given by [28]:

$$\begin{aligned} X \triangleright h &:= h_{(1)} \left\langle X, h_{(2)} \right\rangle, \\ h \triangleleft X &:= \left\langle X, h_{(1)} \right\rangle h_{(2)} \end{aligned} \tag{3.7}$$

This is the left action used in the definition (3.4). Left and right actions are mutually commuting:

$$(X \triangleright h) \triangleleft X' = X \triangleright (h \triangleleft X'), \qquad \forall X, X' \in \mathcal{U}(\mathcal{H}), h \in \mathcal{H};$$

and the $*$-structures are compatible with both actions:

$$\begin{aligned} X \triangleright h^* &= ((S(X))^* \triangleright h)^*, \\ h^* \triangleleft X &= (h \triangleleft (S(X))^*)^*, \qquad \forall\, X \in \mathcal{U}(\mathcal{H}),\ h \in \mathcal{U}. \end{aligned}$$

The derivation nature of elements in $\mathcal{X}_{\mathcal{Q}}$ is expressed by their coproduct,

$$\Delta(X_a) = 1 \otimes X_a + \sum\nolimits_b X_b \otimes f_{ba},$$

with the elements $f_{ab} \in \mathcal{U}_q(\mathcal{H})$ having specific properties [29]:

$$\Delta(f_{ab}) = f_{ac} \otimes f_{cb}, \qquad \varepsilon(f_{ab}) = \delta_{ab}. \tag{3.8}$$

These elements also control the commutation relation between the basis 1-forms and elements of $\mathcal{H}$:

$$\begin{aligned} \omega_a h &= \sum\nolimits_b (f_{ab} \triangleright h)\omega_b, \\ h\omega_a &= \sum\nolimits_b \omega_b \left((S^{-1}(f_{ab})) \triangleright h\right) \qquad \text{for} \quad h \in \mathcal{H}. \end{aligned}$$

For a left covariant differential calculus, the elements $X_a \in \mathcal{X}_{\mathcal{Q}}$ play the role of left invariant vector fields, while their dual forms ω_a play the role of the left invariant one forms. For a bicovariant differential calculus it is possible to define a basis of the bimodule of 1-forms which are right invariant. The right coaction of $\mathcal{H}$ on $\Omega^1(\mathcal{H})$ defines a matrix:

$$\Delta_R^{(1)}(\omega_a) := \omega_b \otimes J_{ba} \tag{3.9}$$

where $J_{ab} \in \mathcal{H}$. This matrix is invertible, since $S(J_{ab})J_{bc} = \delta_{ac}$ and $J_{ab}S(J_{bc}) = \delta_{ac}$; it satisfies the properties $\Delta(J_{ab}) = J_{ac} \otimes J_{cb}$, $\varepsilon(J_{ab}) = \delta_{ab}$ and can be used to define a set of 1-forms:

$$\eta_a = \omega_b S(J_{ba}) \qquad \eta_a J_{ab} = \omega_b \tag{3.10}$$

which are proved to be right invariant:

$$\Delta_R^{(1)}(\eta_a) = \eta_a \otimes 1 \tag{3.11}$$

On the basis of right invariant 1-forms, the exterior derivative operator acquires the form:

$$\mathrm{d}h = \eta_a (h \triangleleft Y_a) \tag{3.12}$$

where $Y_a \in \mathcal{X}_\mathcal{Q}$, and $Y_a = -S^{-1}(X_a)$ are the equivalent of the right invariant vector fields. Equation (2.24) is then represented, in an algebraic approach to the theory of differential calculi, by (3.4) and (3.12). The derivation nature of Y_a as well as the commutation relation between the basis of right invariant 1-forms and elements of $\mathcal{H}$ are ruled by the same elements $f_{ab} \in \mathcal{U}(\mathcal{H})$:

$$\begin{aligned}
&\Delta(Y_a) = 1 \otimes Y_a + \sum\nolimits_b Y_b \otimes f_{ba} \\
&\eta_a h = (h \triangleleft f_{ab})\eta_b, \\
&h\eta_a = \eta_b (h \triangleleft (S^{-1}(f_{ab})).
\end{aligned}$$

3.2 Quantum principal bundles

An algebraic formalisation of the geometric structures of a principal bundle has been introduced in [4] and refined in [9]. A slightly different formulation of such a structure is in [6], [7]; an interesting comparison between the two approaches is in [8].

As a total space one considers an algebra $\mathcal{P}$ (with multiplication $m : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$) and as structure group a Hopf algebra $\mathcal{H}$. Thus $\mathcal{P}$ is a right $\mathcal{H}$-comodule algebra with coaction $\Delta_R : \mathcal{P} \mapsto \mathcal{P} \otimes \mathcal{H}$. The subalgebra of the right coinvariant elements, $\mathcal{B} = \mathcal{P}^\mathcal{H} = \{p \in \mathcal{P} : \Delta_R p = p \otimes 1\}$, is the base space of the bundle. At the 'topological level' the principality of the bundle is the requirement of exactness of the sequence:

$$0 \to \mathcal{P}\left(\Omega^1(\mathcal{B})_{un}\right)\mathcal{P} \to \Omega^1(\mathcal{P})_{un} \xrightarrow{\chi} \mathcal{P} \otimes \ker \varepsilon_\mathcal{H} \to 0 \tag{3.13}$$

with $\Omega^1(\mathcal{P})_{un}$ and $\Omega^1(\mathcal{B})_{un}$ the universal calculi and the map χ defined by

$$\chi : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}, \qquad \chi := (m \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_R), \tag{3.14}$$

or $\chi(p' \otimes p) = p' \Delta_R(p)$. The exactness of this sequence is equivalent to the requirement that the analogous 'canonical map' $\mathcal{P} \otimes_\mathcal{B} \mathcal{P} \to \mathcal{P} \otimes \mathcal{H}$ (defined as the formula above) is an isomorphism. This is the definition that the inclusion $\mathcal{B} \hookrightarrow \mathcal{P}$ be a Hopf-Galois extension [25].

Remark 3.1. *The surjectivity of the map χ appears as the dual translation of the classical condition that the action of the structure group on the total space of the principal bundle is free. In the classical setting described in section 2, given the principal bundle $(\mathcal{P}, \mathrm{K}, [\mathcal{M}], \pi)$, the condition that the right principal action r_k is free can be formalised as the injectivity of the map:*

$$P \times G \mapsto P \times_M P, \qquad (p, k) \mapsto (p, \mathrm{r}_k(p)),$$

whose dualisation is the condition of the surjectivity of the map χ.

With differential calculi on both the total algebra $\mathcal{P}$ and the structure Hopf algebra $\mathcal{H}$ one needs compatibility conditions that eventually lead to an exact sequence like in (3.13) with the calculi at hand replacing the universal ones. Then, let $\big(\Omega^1(\mathcal{P}), \mathrm{d}\big)$ be a $\mathcal{H}$-covariant differential calculus on $\mathcal{P}$ given via the subbimodule $\mathcal{N}_{\mathcal{P}} \in \big(\Omega^1(\mathcal{P})_{un}\big)$, and $\big(\Omega^1(\mathcal{H}), \mathrm{d}\big)$ a bicovariant differential calculus on $\mathcal{H}$ given via the Ad-invariant right ideal $\mathcal{Q}_{\mathcal{H}} \in \ker \varepsilon_{\mathcal{H}}$. In order to extend the coaction Δ_R of $\mathcal{H}$ on $\mathcal{P}$ to a coaction of $\mathcal{H}$ on $\Omega^1(\mathcal{P})$, one requires $\Delta_R(\mathcal{N}_{\mathcal{P}}) \subset \mathcal{N}_{\mathcal{P}} \otimes \mathcal{H}$. The coaction Δ_R of $\mathcal{H}$ on $\mathcal{N}_{\mathcal{P}} \subset \mathcal{P} \otimes \mathcal{P}$ is understood as a usual coaction of a Hopf algebra on a tensor product of its comodule algebras, i.e.

$$\Delta_R = (\mathrm{id} \otimes \mathrm{id} \otimes \cdot) \circ (\mathrm{id} \otimes \tau\, \mathrm{id}) \circ (\Delta_R \otimes \Delta_R).$$

The condition $\Delta_R(\mathcal{N}_{\mathcal{P}}) \subset \mathcal{N}_{\mathcal{P}} \otimes \mathcal{H}$ is equivalent to the condition (3.1).

The compatibility of the calculi are then the requirements that $\chi(\mathcal{N}_{\mathcal{P}}) \subseteq \mathcal{P} \otimes \mathcal{Q}_{\mathcal{H}}$ and that the map $\sim_{\mathcal{N}_{\mathcal{P}}} : \Omega^1(\mathcal{P}) \to \mathcal{P} \otimes (\ker \varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}})$, defined by the diagram

$$\begin{array}{ccc} \Omega^1(\mathcal{P})_{un} & \stackrel{\pi_{\mathcal{N}}}{\longrightarrow} & \Omega^1(\mathcal{P}) \\ \downarrow \chi & & \downarrow \sim_{\mathcal{N}_{\mathcal{P}}} \\ \mathcal{P} \otimes \ker \varepsilon_{\mathcal{H}} & \stackrel{\mathrm{id} \otimes \pi_{\mathcal{Q}_{\mathcal{H}}}}{\longrightarrow} & \mathcal{P} \otimes (\ker \varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}) \end{array} \tag{3.15}$$

(with $\pi_{\mathcal{N}}$ and $\pi_{\mathcal{Q}_{\mathcal{H}}}$ the natural projections) is surjective and has kernel

$$\ker \sim_{\mathcal{N}_{\mathcal{P}}} = \mathcal{P}\Omega^1(\mathcal{B})\mathcal{P} =: \Omega^1_{\mathrm{hor}}(\mathcal{P}). \tag{3.16}$$

Here $\Omega^1(\mathcal{B}) = \mathcal{B}\mathrm{d}\mathcal{B}$ is the space of nonuniversal 1-forms on $\mathcal{B}$ associated to the bimodule $\mathcal{N}_{\mathcal{B}} := \mathcal{N}_{\mathcal{P}} \cap \Omega^1(\mathcal{B})_{un}$. These conditions ensure the exactness of the sequence:

$$0 \to \mathcal{P}\Omega^1(\mathcal{B})\mathcal{P} \to \Omega_1(\mathcal{P}) \stackrel{\sim_{\mathcal{N}_{\mathcal{P}}}}{\longrightarrow} \mathcal{P} \otimes (\ker \varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}) \to 0. \tag{3.17}$$

The condition $\chi(\mathcal{N}_{\mathcal{P}}) \subseteq \mathcal{P} \otimes \mathcal{Q}_{\mathcal{H}}$ is needed to have a well defined map $\sim_{\mathcal{N}_{\mathcal{P}}}$: with all conditions for a quantum principal bundle $(\mathcal{P}, \mathcal{B}, \mathcal{H}; \mathcal{N}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{H}})$ satisfied, this inclusion implies the equality $\chi(\mathcal{N}_{\mathcal{P}}) = \mathcal{P} \otimes \mathcal{Q}_{\mathcal{H}}$. Moreover, if $(\mathcal{P}, \mathcal{B}, \mathcal{H})$ is a quantum principal bundle with the universal calculi, the equality $\chi(\mathcal{N}_{\mathcal{P}}) = \mathcal{P} \otimes \mathcal{Q}_{\mathcal{H}}$ ensures that $(\mathcal{P}, \mathcal{B}, \mathcal{H}; \mathcal{N}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{H}})$ is a quantum principal bundle with the corresponding nonuniversal calculi.

Elements in the quantum tangent space $\mathcal{X}_{\mathcal{Q}_{\mathcal{H}}}(\mathcal{H})$ giving the calculus on the structure quantum group $\mathcal{H}$ act on $\ker \varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}$ via the pairing $\langle \cdot, \cdot \rangle$ between $U_q(\mathcal{H})$ and $\mathcal{H}$. Then, with each $\xi \in \mathcal{X}_{\mathcal{Q}_{\mathcal{H}}}(\mathcal{H})$ one defines a map

$$\tilde{\xi} : \Omega^1(\mathcal{P}) \to \mathcal{P}, \qquad \tilde{\xi} := (\mathrm{id} \otimes \xi) \circ (\sim_{\mathcal{N}_{\mathcal{P}}}) \tag{3.18}$$

and declare a 1-form $\omega \in \Omega^1(\mathcal{P})$ to be horizontal iff $\tilde{\xi}(\omega) = 0$, for all elements $\xi \in \mathcal{X}_{\mathcal{Q}_{\mathcal{H}}}(\mathcal{H})$. The collection of horizontal 1-forms is easily seen to coincide with $\Omega^1_{\mathrm{hor}}(\mathcal{P})$ in (3.16).

3.3 A quantum Hopf bundle

A quantum Hopf bundle is a $U(1)$-bundle over the standard Podleś sphere S^2_q [22] and whose total space is the manifold of the quantum group $SU_q(2)$: this bunde is an example of a quantum homogeneous space [4].

3.3.1 The algebras

The coordinate algebra $\mathcal{A}(\mathrm{SU}_q(2))$ of the quantum group $\mathrm{SU}_q(2)$ is the $*$-algebra generated by a and c, with relations

$$\begin{aligned} &ac = qca \quad ac^* = qc^*a \quad cc^* = c^*c, \\ &a^*a + c^*c = aa^* + q^2cc^* = 1. \end{aligned} \tag{3.19}$$

The deformation parameter $q \in \mathbb{R}$ is taken in the interval $0 < q < 1$, since for $q > 1$ one gets isomorphic algebras; at $q = 1$ one recovers the commutative coordinate algebra on the group manifold SU(2). The Hopf algebra structure for $\mathcal{A}(\mathrm{SU}_q(2))$ is given by the coproduct:

$$\Delta \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} = \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} \otimes \begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix},$$

antipode:

$$S\begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} = \begin{bmatrix} a^* & c^* \\ -qc & a \end{bmatrix},$$

and counit:

$$\epsilon\begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The quantum universal envelopping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is the Hopf $*$-algebra generated as an algebra by four elements K, K^{-1}, E, F with $KK^{-1} = 1$ and subject to relations:

$$\begin{aligned} K^{\pm}E &= q^{\pm}EK^{\pm}, \\ K^{\pm}F &= q^{\mp}FK^{\pm}, \\ [E,F] &= \frac{K^2 - K^{-2}}{q - q^{-1}}. \end{aligned} \tag{3.20}$$

The $*$-structure is

$$K^* = K, \qquad E^* = F, \qquad F^* = E,$$

and the Hopf algebra structure is provided by coproduct:

$$\begin{aligned} \Delta(K^{\pm}) &= K^{\pm} \otimes K^{\pm}, \\ \Delta(E) &= E \otimes K + K^{-1} \otimes E, \\ \Delta(F) &= F \otimes K + K^{-1} \otimes F; \end{aligned}$$

antipode:

$$\begin{aligned} S(K) &= K^{-1}, \\ S(E) &= -qE, \\ S(F) &= -q^{-1}F; \end{aligned}$$

and a counit:

$$\varepsilon(K) = 1, \qquad \varepsilon(E) = \varepsilon(F) = 0.$$

From the relations (3.20), the quadratic quantum Casimir element:

$$C_q := \frac{qK^2 - 2 + q^{-1}K^{-2}}{(q - q^{-1})^2} + FE - \tfrac{1}{4} \tag{3.21}$$

generates the centre of $\mathcal{U}_q(\mathfrak{su}(2))$. The irreducible finite dimensional $*$-representations σ_J of $\mathcal{U}_q(\mathfrak{su}(2))$ (see e.g. [17]) are labelled by nonnegative half-integers $J \in \frac{1}{2}\mathbb{N}$ (the spin); they are given by[2]

$$\begin{aligned} \sigma_J(K)\ |J,m\rangle &= q^m\ |J,m\rangle\,, \\ \sigma_J(E)\ |J,m\rangle &= \sqrt{[J-m][J+m+1]}\ |J,m+1\rangle\,, \\ \sigma_J(F)\ |J,m\rangle &= \sqrt{[J-m+1][J+m]}\ |J,m-1\rangle\,, \end{aligned} \tag{3.23}$$

where the vectors $|J,m\rangle$, for $m = J, J-1, \ldots, -J+1, -J$, form an orthonormal basis for the $(2J+1)$-dimensional, irreducible $\mathcal{U}_q(\mathfrak{su}(2))$-module V_J, and the brackets denote the q-number as in (3.22). Moreover, σ_J is a $*$-representation of $\mathcal{U}_q(\mathfrak{su}(2))$, with respect to the hermitian scalar product on V_J for which the vectors $|J,m\rangle$ are orthonormal. In each representation V_J, the Casimir (3.21) is a multiple of the identity with constant given by:

$$C_q^{(J)} = [J + \tfrac{1}{2}]^2 - \tfrac{1}{4}. \tag{3.24}$$

[2]The 'q-number' is defined as:

$$[x] = [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}, \tag{3.22}$$

for $q \neq 1$ and any $x \in \mathbb{R}$.

The Hopf algebras $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}(\mathrm{SU}_q(2))$ are dually paired. The bilinear mapping $\langle\cdot,\cdot\rangle : \mathcal{U}_q(\mathfrak{su}(2))\times\mathcal{A}(\mathrm{SU}_q(2)) \mapsto \mathbb{C}$ compatible with the $*$-structures, is set on the generators by:

$$\begin{aligned}&\langle K, a\rangle = q^{-1/2}, \quad \langle K^{-1}, a\rangle = q^{1/2},\\ &\langle K, a^*\rangle = q^{1/2}, \quad \langle K^{-1}, a^*\rangle = q^{-1/2},\\ &\langle E, c\rangle = 1, \quad \langle F, c^*\rangle = -q^{-1},\end{aligned}$$

with all other couples of generators pairing to 0. The canonical left and right actions of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(\mathrm{SU}_q(2))$ can be recovered by:

$$\begin{array}{lll} K^{\pm}\triangleright a^s = q^{\mp\frac{s}{2}}a^s & F\triangleright a^s = 0 & E\triangleright a^s = -q^{(3-s)/2}[s]a^{s-1}c^* \\ K^{\pm}\triangleright a^{*s} = q^{\pm\frac{s}{2}}a^{*s} & F\triangleright a^{*s} = q^{(1-s)/2}[s]ca^{*s-1} & E\triangleright a^{*s} = 0 \\ K^{\pm}\triangleright c^s = q^{\mp\frac{s}{2}}c^s & F\triangleright c^s = 0 & E\triangleright c^s = q^{(1-s)/2}[s]c^{s-1}a^* \\ K^{\pm}\triangleright c^{*s} = q^{\pm\frac{s}{2}}c^{*s} & F\triangleright c^{*s} = -q^{-(1+s)/2}[s]ac^{*s-1} & E\triangleright c^{*s} = 0; \end{array} \tag{3.25}$$

and:

$$\begin{array}{lll} a^s\triangleleft K^{\pm} = q^{\mp\frac{s}{2}}a^s & a^s\triangleleft F = q^{(s-1)/2}[s]ca^{s-1} & a^s\triangleleft E = 0 \\ a^{*s}\triangleleft K^{\pm} = q^{\pm\frac{s}{2}}a^{*s} & a^{*s}\triangleleft F = 0 & a^{*s}\triangleleft E = -q^{(3-s)/2}[s]c^*a^{*s-1} \\ c^s\triangleleft K^{\pm} = q^{\pm\frac{s}{2}}c^s & c^s\triangleleft F = 0 & c^s\triangleleft E = q^{(s-1)/2}[s]c^{s-1}a \\ c^{*s}\triangleleft K^{\pm} = q^{\mp\frac{s}{2}}c^{*s} & c^{*s}\triangleleft F = -q^{-(s-3)/2}[s]a^*c^{*s-1} & c^{*s}\triangleleft E = 0. \end{array} \tag{3.26}$$

Denote $\mathcal{A}(\mathrm{U}(1)) := \mathbb{C}[z, z^*]/< zz^* - 1 >$; the map:

$$\pi : \mathcal{A}(\mathrm{SU}_q(2)) \;\rightarrow\; \mathcal{A}(\mathrm{U}(1)), \qquad \pi\begin{bmatrix} a & -qc^* \\ c & a^* \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix} \tag{3.27}$$

is a surjective Hopf $*$-algebra homomorphism, so that $\mathcal{A}(\mathrm{U}(1))$ becomes a quantum subgroup of $SU_q(2)$ with a right coaction,

$$\Delta_R := (\mathrm{id}\otimes\pi)\circ\Delta \;:\; \mathcal{A}(\mathrm{SU}_q(2)) \;\mapsto\; \mathcal{A}(\mathrm{SU}_q(2))\otimes\mathcal{A}(\mathrm{U}(1)). \tag{3.28}$$

The coinvariant elements for this coaction, elements $b \in \mathcal{A}(\mathrm{SU}_q(2))$ for which $\Delta_R(b) = b\otimes 1$, form a subalgebra of $\mathcal{A}(\mathrm{SU}_q(2))$ which is the coordinate algebra $\mathcal{A}(\mathrm{S}^2_q)$ of the standard Podleś sphere S^2_q. From:

$$\begin{aligned}&\Delta_R(a) = a\otimes z,\\ &\Delta_R(a^*) = a^*\otimes z^*,\\ &\Delta_R(c) = c\otimes z,\\ &\Delta_R(c^*) = c^*\otimes z^*\end{aligned} \tag{3.29}$$

as a set of generators for $\mathcal{A}(\mathrm{S}^2_q)$ one can choose:

$$B_- := -ac^*, \qquad B_+ := qca^*, \qquad B_0 := \frac{q^2}{1+q^2} - q^2cc^*, \tag{3.30}$$

satisfying the relations:

$$\begin{aligned}&B_-B_0 = [\frac{q^2-q^4}{1+q^2}B_- + q^2B_0B_-],\\ &B_+B_0 = [\frac{q^2-1}{q^2+1}B_+ + q^{-2}B_0B_+],\\ &B_+B_- = q\left[q^{-2}B_0 - (1+q^2)^{-1}\right]\left[q^{-2}B_0 + (1+q^{-2})^{-1}\right],\\ &B_-B_+ = q\left[B_0 + (1+q^2)^{-1}\right]\left[B_0 - (1+q^{-2})^{-1}\right],\end{aligned}$$

and $*$-structure:

$$(B_0)^* = B_0, \qquad (B_+)^* = -qB_-.$$

The sphere S^2_q is a quantum homogeneous space of $SU_q(2)$ and the coproduct of $\mathcal{A}(SU_q(2))$ restricts to a left coaction of $\mathcal{A}(SU_q(2))$ on $\mathcal{A}(S^2_q)$ which, on generators reads:

$$\Delta(B_-) = a^2 \otimes B_- - (1+q^{-2})B_- \otimes B_0 + c^{*2} \otimes B_+,$$
$$\Delta(B_0) = q\, ac \otimes B_- + (1+q^{-2})B_0 \otimes B_0 - c^* a^* \otimes B_+,$$
$$\Delta(B_+) = q^2\, c^2 \otimes B_- + (1+q^{-2})B_+ \otimes B_0 + a^{*2} \otimes B_+.$$

3.3.2 The associated line bundles

The left action of the group-like element K on $\mathcal{A}(SU_q(2))$ allows [19] to give a vector basis decomposition $\mathcal{A}(SU_q(2)) = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$, where

$$\mathcal{L}_n^{(0)} := \{x \in \mathcal{A}(SU_q(2)) \ : \ K \triangleright x = q^{n/2}x\}. \tag{3.31}$$

In particular $\mathcal{A}(S^2_q) = \mathcal{L}_0^{(0)}$. One also has $\mathcal{L}_n^{(0)*} \subset \mathcal{L}_{-n}^{(0)}$ and $\mathcal{L}_n^{(0)}\mathcal{L}_m^{(0)} \subset \mathcal{L}_{n+m}^{(0)}$. Each $\mathcal{L}_n^{(0)}$ is a bimodule over $\mathcal{A}(S^2_q)$; relations (3.29) show that they can be equivalently characterised by the coaction Δ_R of the quantum subgroup $\mathcal{A}(U(1))$ on $\mathcal{A}(SU_q(2))$:

$$\mathcal{L}_n^{(0)} = \{x \in \mathcal{A}(SU_q(2)) \ : \ \Delta_R(x) = x \otimes z^{-n}\}. \tag{3.32}$$

This equation appears as the natural quantum analogue of the classical relation (2.49), introducing $\mathcal{L}_n^{(0)} \subset \mathcal{A}(SU_q(2))$ as $\mathcal{A}(S^2_q)$-bimodule of *co*-equivariant elements with respect to the coaction (3.28) of the gauge group algebra. The relation (3.31) can then be read as an infinitesimal version of that in (3.32). The classical $\mathfrak{L}_n^{(0)}$ are recovered as rank 1 projective left $C^\infty(S^2)$-modules: the analogue property in the quantum setting was shown in [23]. Each $\mathcal{L}_n^{(0)}$ is isomorphic to a projective left $\mathcal{A}(S^2_q)$-module of rank 1. These projective left $\mathcal{A}(S^2_q)$-modules give modules of equivariant maps or of sections of line bundles over the quantum sphere S^2_q with winding numbers (monopole charge) $-n$. The corresponding projections [10, 12] can be explicitly written. Given $n \in \mathbb{Z}$, consider an element $\left|\Psi^{(n)}\right\rangle \in \mathcal{A}(SU_q(2))^{|n|+1}$ whose components are:

$$n \geq 0: \qquad \left|\Psi^{(n)}\right\rangle_\mu = \sqrt{\beta_{n,\mu}}\; c^{*\mu} a^{*n-\mu} \in \mathcal{L}_n^{(0)},$$
$$\text{where}: \qquad \beta_{n,0} = 1; \qquad \beta_{n,\mu} = q^{2\mu} \prod_{j=0}^{\mu-1} \left(\frac{1-q^{-2(n-j)}}{1-q^{-2(j+1)}}\right), \quad \mu = 1,\dots,n \tag{3.33}$$

$$n \leq 0: \qquad \left|\Psi^{(n)}\right\rangle_\mu = \sqrt{\alpha_{n,\mu}}\; c^{|n|-\mu} a^{\mu} \in \mathcal{L}_n^{(0)},$$
$$\text{where}: \qquad \alpha_{n,0} = 1; \qquad \alpha_{n,\mu} = \prod_{j=0}^{|n|-\mu-1} \left(\frac{1-q^{2(|n|-j)}}{1-q^{2(j+1)}}\right), \quad \mu = 1,\dots,|n| \tag{3.34}$$

Using the commutation relations (3.19) and the explicit form of the coefficients in (3.33) and (3.34), it is possible to compute that:

$$n \geq 0: \qquad \left\langle\Psi^{(n)}, \Psi^{(n)}\right\rangle = \sum_{\mu=0}^{n} \beta_{n,\mu}\, a^{n-\mu} c^{\mu} c^{*\mu} a^{*n-\mu} = (aa^* + q^2 cc^*)^n = 1,$$
$$n \leq 0: \qquad \left\langle\Psi^{(n)}, \Psi^{(n)}\right\rangle = \sum_{\mu=0}^{|n|} \alpha_{n,\mu}\, a^{*\mu} c^{*|n|-\mu} c^{|n|-\mu} a^{\mu} = (a^*a + c^*c)^{|n|} = 1 \tag{3.35}$$

so that a projector $\mathfrak{p}^{(n)} \in \mathbb{M}_{|n|+1}(\mathcal{A}(S^2_q))$ can be defined as:

$$\mathfrak{p}^{(n)} = \left|\Psi^{(n)}\right\rangle \left\langle\Psi^{(n)}\right| \tag{3.36}$$

which is by construction an idempotent - $(\mathfrak{p}^{(n)})^2 = \mathfrak{p}^{(n)}$ - and selfadjoint operator - $(\mathfrak{p}^{(n)})^\dagger = \mathfrak{p}^{(n)}$ - whose entries are:

$$n \geq 0: \qquad \mathfrak{p}_{\mu\nu}^{(n)} = \sqrt{\beta_{n,\mu}\beta_{n,\nu}}\; c^{*\mu} a^{*n-\mu} a^{n-\nu} c^{\nu} \in \mathcal{A}(S^2_q),$$
$$n \leq 0: \qquad \check{\mathfrak{p}}_{\mu\nu}^{(n)} = \sqrt{\alpha_{n,\mu}\alpha_{n,\nu}}\; c^{|n|-\mu} a^{\mu} a^{*\nu} c^{*|n|-\nu} \in \mathcal{A}(S^2_q). \tag{3.37}$$

Remark 3.2. *The coefficients $\alpha_{n,\mu}$ and $\beta_{n,\mu}$ above are q-binomial coefficients. The algebraic identities they satisfy* (3.35) *are related since it is possible to prove that (note that $\alpha_{n,\mu}$ is defined following* (3.34) *only for negative integers n):*

$$q^{-2\mu+2\mu(n-\mu)}\beta_{n,\mu} = \alpha_{-n,-n-\mu}, \qquad \mu = 0,\dots,n\,. \tag{3.38}$$

which is obtained by a straightforward computation.

The projections (3.36) play a central role in the description of the quantum Hopf bundle. As a first application one can prove that the algebra inclusion $\mathcal{A}(\mathrm{S}^2_q) \hookrightarrow \mathcal{A}(\mathrm{SU}_q(2))$ satisfies the topological requirements for a quantum principal bundle, when both the algebras are equipped with the universal calculus.

Proposition 3.3. *The datum $(\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{A}(\mathrm{S}^2_q), \mathcal{A}(\mathrm{U}(1)))$ is a quantum principal bundle.*

Proof. The proof consists of showing the exactness of the sequence

$$0 \to \mathcal{A}(\mathrm{SU}_q(2))\left(\Omega^1(\mathrm{S}^2_q)_{un}\right)\mathcal{A}(\mathrm{SU}_q(2)) \to \Omega_1(\mathrm{SU}_q(2))_{un} \xrightarrow{\chi} \mathcal{A}(\mathrm{SU}_q(2)) \otimes \ker\varepsilon_{\mathrm{U}(1)} \to 0$$

or equivalently that the map $\chi : \Omega^1(\mathrm{SU}_q(2))_{un} \to \mathcal{A}(\mathrm{SU}_q(2))\otimes\ker\varepsilon_{\mathrm{U}(1)}$ defined as in (3.14) – and with the $\mathcal{A}(\mathrm{U}(1))$-coaction on $\mathcal{A}(\mathrm{SU}_q(2))$ given in (3.28) – is surjective. Given an element $x \in \mathcal{L}^{(0)}_n \subset \mathcal{A}(\mathrm{SU}_q(2))$, from (3.32) the map χ acts as:

$$\chi(\delta x) = \chi(1\otimes x - x\otimes 1) = x \otimes (z^{-n}-1). \tag{3.39}$$

A generic element in $\mathcal{A}(\mathrm{SU}_q(2)) \otimes \ker\varepsilon_{\mathrm{U}(1)}$ is of the form $x\otimes(z^n-1)$ with $n\in\mathbb{Z}$ and $x \in \mathcal{A}(\mathrm{SU}_q(2))$. To show surjectivity of χ the strategy is to show that $1\otimes(z^n-1)$ is in its image since left $\mathcal{A}(\mathrm{SU}_q(2))$-linearity of χ will give the general result: if $\gamma \in \Omega^1(\mathrm{SU}_q(2))_{un}$ is such that $\chi(\gamma) = 1\otimes(z^n-1)$, then $\chi(x\gamma) = x\,(1\otimes(z^n-1)) = x\otimes(z^n-1)$. Fixed now $n\in\mathbb{Z}$, define an element γ in $\mathcal{A}(\mathrm{SU}_q(2))$ as $\gamma = \left\langle \Psi^{(-n)}, \delta\Psi^{(-n)}\right\rangle$ following (3.33) and (3.34). Since $\left|\Psi^{(-n)}\right\rangle \in \mathcal{L}^{(0)}_{-n}$, one computes that:

$$\chi(\gamma) = 1\otimes(z^n-1),$$

thus completing the proof. □

Next, it is possible to identify the spaces of equivariant maps $\mathcal{L}^{(0)}_n$ – or equivalently of *coequivariant* elements $\mathcal{L}^{(0)}_n$ – with the left $\mathcal{A}(\mathrm{S}^2_q)$-modules of sections $\mathcal{E}^{(0)}_n = (\mathcal{A}(\mathrm{S}^2_q))^{|n|+1}\mathfrak{p}^{(n)}$. For this write any element in the free module $(\mathcal{A}(\mathrm{S}^2_q))^{|n|+1}$ as $\langle f| = (f_0, f_1, \dots, f_{|n|})$ with $f_\mu \in \mathcal{A}(\mathrm{S}^2_q)$. This allows one to write equivariant maps as

$$\begin{aligned} \phi_f := \left\langle f, \Psi^{(n)}\right\rangle &= \sum\nolimits_{\mu=0}^{n} f_\mu \sqrt{\beta_{n,\mu}}\, c^{*\mu} a^{*n-\mu} && \text{for} \quad n\geq 0, \\ &= \sum\nolimits_{\mu=0}^{|n|} f_\mu \sqrt{\alpha_{n,\mu}}\, c^{|n|-\mu} a^{\mu} && \text{for} \quad n \leq 0. \end{aligned}$$

Writing equivariant maps in the above form, it is straightforward to establish the proposition, which generalises to the quantum formalism the equivalence (2.54):

Proposition 3.4. *Given $n\in\mathbb{Z}$, let $\mathcal{E}^{(0)}_n := (\mathcal{A}(\mathrm{S}^2_q))^{|n|+1}\mathfrak{p}^{(n)}$. There is a left $\mathcal{A}(\mathrm{S}^2_q)$-modules isomorphism:*

$$\mathcal{L}^{(0)}_n \xrightarrow{\simeq} \mathcal{E}^{(0)}_n, \quad \phi_f \to \langle\sigma_f| = \phi_f\left\langle \Psi^{(n)}\right| = \langle f|\,\mathfrak{p}^{(n)},$$

with inverse

$$\mathcal{E}^{(0)}_n \xrightarrow{\simeq} \mathcal{L}^{(0)}_n, \quad \langle\sigma_f| = \langle f|\,\mathfrak{p}^{(n)} \to \phi_f := \left\langle f, \Psi^{(n)}\right\rangle.$$

3.3.3 A Peter-Weyl decomposition of $\mathcal{A}(\mathrm{SU}_q(2))$

The aim of this section is to describe the known decomposition of the modules $\mathcal{L}_n^{(0)}$ into representation spaces under the action of $\mathcal{U}_q(\mathfrak{su}(2))$. From (3.31) one has a vector space decomposition $\mathcal{A}(\mathrm{SU}_q(2)) = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n^{(0)}$, with

$$E\triangleright\mathcal{L}_n^{(0)} \subset \mathcal{L}_{n+2}^{(0)}, \qquad F\triangleright\mathcal{L}_n^{(0)} \subset \mathcal{L}_{n-2}^{(0)}. \tag{3.40}$$

On the other hand, commutativity of the left and right actions of $\mathcal{U}_q(\mathfrak{su}(2))$ yields that

$$\mathcal{L}_n^{(0)}\triangleleft h \subset \mathcal{L}_n^{(0)}, \qquad \forall\, h\in\mathcal{U}_q(\mathfrak{su}(2)).$$

It has already been shown in [23] that there is also a decomposition,

$$\mathcal{L}_n^{(0)} := \bigoplus_{J=\frac{|n|}{2},\frac{|n|}{2}+1,\frac{|n|}{2}+2,\cdots} V_J^{(n)}, \tag{3.41}$$

with $V_J^{(n)}$ the spin J-representation space (for the right action) of $\mathcal{U}_q(\mathfrak{su}(2))$. Altogether it formalises a Peter-Weyl decomposition for $\mathcal{A}(\mathrm{SU}_q(2))$ (already given in [28]).

More explicitly, the highest weight vector for each $V_J^{(n)}$ in (3.41) is $c^{J-n/2}a^{*J+n/2}$:

$$\begin{aligned}&K\triangleright(c^{J-n/2}a^{*J+n/2}) = q^{n/2}(c^{J-n/2}a^{*J+n/2}),\\ &(c^{J-n/2}a^{*J+n/2})\triangleleft K = q^J(c^{J-n/2}a^{*J+n/2}), \qquad (c^{J-n/2}a^{*J+n/2})\triangleleft F = 0.\end{aligned} \tag{3.42}$$

Analogously, the lowest weight vector for each $V_J^{(n)}$ in (3.41) is $a^{J-n/2}c^{*J+n/2}$:

$$\begin{aligned}&K\triangleright(a^{J-n/2}c^{*J+n/2}) = q^{n/2}(a^{J-n/2}c^{*J+n/2}),\\ &(a^{J-n/2}c^{*J+n/2})\triangleleft K = q^{-J}(a^{J-n/2}c^{*J+n/2}), \qquad (a^{J-n/2}c^{*J+n/2})\triangleleft E = 0.\end{aligned}$$

The elements of the vector spaces $V_J^{(n)}$ can be obtained by acting on the highest weight vectors with the lowering operator $\triangleleft E$, since clearly $\left(c^{J-n/2}a^{*J+n/2}\right)\triangleleft E \in \mathcal{L}_n^{(0)}$, or explicitly,

$$K\triangleright\left[\left(c^{J-n/2}a^{*J+n/2}\right)\triangleleft E\right] = q^{n/2}\left[\left(c^{J-n/2}a^{*J+n/2}\right)\triangleleft E\right].$$

To be definite, consider $n\geq 0$. The first admissible J is $J=n/2$; the highest weight element is a^{*n} and the vector space $V_{n/2}^{(n)}$ is spanned by $\{a^{*n}\triangleleft E^l\}$ with $l=0,\ldots,n+1$: $V_{n/2}^{(n)} = \mathrm{span}\{a^{*n}, c^*a^{*n-1},\ldots,c^{*n}\}$. Keeping n fixed, the other admissible values of J are $J = s+n/2$ with $s\in\mathbb{N}$. The vector spaces $V_{s+n/2}^{(n)}$ are spanned by $\{c^s a^{*s+n}\triangleleft E^l\}$ with $l=0,\ldots,2s+n+1$. Analogous considerations are valid when $n\leq 0$. In this cases, the admissible values of J are $J = s+|n|/2 = s-n/2$, the highest weight vector in $V_{s-n/2}^{(n)}$ is the element $c^{s-n}a^{*s}$, and a basis is given by the action of the lowering operator $\triangleleft E$, that is $V_{s-n/2}^{(n)} = \mathrm{span}\{(c^{s-n}a^{*s})\triangleleft E^l,\ l=0,\ldots,2s-n+1\}$.

¿From (3.40) one has that the left action $F\triangleright$ maps $\mathcal{L}_n^{(0)}$ to $\mathcal{L}_{n-2}^{(0)}$. If $p\geq 0$, the element a^{*p} is the highest weight vector in $V_{p/2}^{(p)}$ and one has that $F\triangleright a^{*p}\propto ca^{*p-1}$. The element ca^{*p-1} is the highest weight vector in $V_{p/2}^{(p-2)}$ since one finds that $(ca^{*p-1})\triangleleft F = 0$ and $(ca^{*p-1})\triangleleft K = q^{p/2}(ca^{*p-1})$. In the same vein, the elements $F^t\triangleright a^{*p}\propto c^t a^{*p-t}$ are the highest weight elements in $V_{p/2}^{(p-2t)}\subset\mathcal{L}_{p-2t}^{(0)},\ t=0,\ldots,p$. Once again, a complete basis of each subspace $V_{p/2}^{(p-2t)}$ is obtained by the right action of the lowering operator $\triangleleft E$.

With these considerations, the algebra $\mathcal{A}(\mathrm{SU}_q(2))$ can be partitioned into finite dimensional blocks which are the analogues of the Wigner D-functions [27] for the group $SU(2)$. To illustrate the meaning of this partition, start with the element a^*, the highest weight vector of the space $V_{1/2}^{(1)}$. Representing the left action of $F\triangleright$ with a horizontal arrow and the right action of $\triangleleft E$ with a vertical one, yields the box

$$\begin{array}{ccc} a^* & \rightarrow & c \\ \downarrow & & \downarrow \\ -qc^* & \rightarrow & a \end{array},$$

where the first column is a basis of the subspace $V_{1/2}^{(1)}$, while the second column is a basis of the subspace $V_{1/2}^{(-1)}$. Starting from a^{*2} – the highest weight vector of $V_1^{(2)}$ – one gets:

$$\begin{array}{ccccc} a^{*2} & \to & q^{-1/2}\,[2]\,ca^* & \to & [2]\,c^2 \\ \downarrow & & \downarrow & & \downarrow \\ -q^{1/2}\,[2]\,c^*a^* & \to & [2]\,(aa^*-cc^*) & \to & [2]^2\,q^{1/2}ca \\ \downarrow & & \downarrow & & \downarrow \\ q^2\,[2]\,c^{*2} & \to & -q^{1/2}\,[2]^2\,ac^* & \to & [2]^2\,a^2 \end{array} .$$

The three columns of this box are bases for the subspaces $V_1^{(2)}$, $V_1^{(0)}$,$V_1^{(-2)}$, respectively. The recursive structure is clear. For a positive integer p, one has a box W_p made up of $(p+1)\times(p+1)$ elements. Without explicitly computing the coefficients, one gets:

$$\begin{array}{ccccccccccc} a^{*p} & \to & ca^{*p-1} & \to & \dots & \to & c^ta^{*p-t} & \to & \dots & \to & c^p \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \dots & & \downarrow \\ c^*a^{*p-1} & \to & \dots & \to & \dots & \to & \dots & \to & \dots & \to & ac^{p-1} \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \dots & & \downarrow \\ \dots & \to & \dots & \to & \dots & \to & \dots & \to & \dots & \to & \dots \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \dots & & \downarrow \\ c^{*s}a^{*p-s} & \to & \dots & \to & \dots & \to & \dots & \to & \dots & \to & a^sc^{p-s} \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \dots & & \downarrow \\ \dots & \to & \dots & \to & \dots & \to & \dots & \to & \dots & \to & \dots \\ \downarrow & & \downarrow & & \dots & & \downarrow & & \dots & & \downarrow \\ c^{*p} & \to & ac^{*p-1} & \to & \dots & \to & a^tc^{*p-t} & \to & \dots & \to & a^p \end{array} .$$

The space W_p is the direct sum of representation spaces for the right action of $\mathcal{U}_q(\mathfrak{su}(2))$,

$$W_p = \oplus_{t=0}^{p} V_{p/2}^{(p-2t)},$$

and on each W_p the quantum Casimir C_q acts is the same manner from both the right and the left, with eigenvalue (3.24), that is $C_q \triangleright w_p = w_p \triangleleft C_q = \left([p+\frac{1}{2}]^2 - \frac{1}{4}\right) w_p$, for all $w_p \in W_p$. The Peter-Weyl decomposition for the algebra $\mathcal{A}(\mathrm{SU}_q(2))$ is given as

$$\mathcal{A}(\mathrm{SU}_q(2)) = \oplus_{p\in\mathbb{N}} W_p = \oplus_{p\in\mathbb{N}} \left(\oplus_{t=0}^{p} V_{p/2}^{(p-2t)} \right).$$

A compatible basis with this decomposition is then given by elements

$$w_{p:t,r} := F^t \triangleright a^{*p} \triangleleft E^r \in W_p \tag{3.43}$$

for $t, r = 0, 1 \dots, p$. In order to get elements in the Podleś sphere subalgebra $\mathcal{A}(\mathrm{S}^2_q) \simeq \mathcal{L}_0^{(0)}$ out of a highest weight vector a^{*p} we need $p = 2l$ to be even and left action of F^l: $F^l \triangleright a^{*2l} \propto c^l a^{*l} \in \mathcal{A}(\mathrm{S}^2_q)$. Then, the right action of E yields a spherical harmonic decomposition,

$$\mathcal{A}(\mathrm{S}^2_q) = \oplus_{l\in\mathbb{N}} V_l^{(0)}, \tag{3.44}$$

with a basis of $V_l^{(0)}$ given by the vectors $F^l \triangleright a^{*2l} \triangleleft E^r$, for $r = 0, 1, \dots, 2l$.

3.4 A quantum Hopf bundle with non-universal differential calculi

Once described how the inclusion $\mathcal{A}(\mathrm{S}^2_q) \hookrightarrow \mathcal{A}(\mathrm{SU}_q(2))$ formalises the structure of a topological quantum principal bundle, the aim of this section is to describe non-universal differential calculi on the algebras $\mathcal{A}(\mathrm{SU}_q(2)), \mathcal{A}(\mathrm{S}^2_q), \mathcal{A}(U(1))$, and to show that these are compatible.

3.4.1 The left-covariant 3D calculus on $SU_q(2)$

The first differential calculus on the quantum group $\mathrm{SU}_q(2)$ is the left-covariant one developed in [28]. It is three dimensional with corresponding ideal $\mathcal{Q}_{\mathrm{SU}_q(2)} \subset \ker \varepsilon_{\mathrm{SU}_q(2)}$ generated by the 6 elements $\{a^* + q^2 a - (1+q^2); c^2; c^* c; c^{*2}; (a-1)c; (a-1)c^*\}$. The quantum tangent space $\mathcal{X}_{\mathrm{SU}_q(2)}$ is the vector space over the complex, whose basis can be taken to be

$$
\begin{aligned}
X_- &= q^{-1/2} F K, \\
X_+ &= q^{1/2} E K, \\
X_z &= \frac{1-K^4}{1-q^{-2}};
\end{aligned}
\tag{3.45}
$$

their coproducts result:

$$
\begin{aligned}
\Delta X_z &= 1 \otimes X_z + X_z \otimes K^4, \\
\Delta X_\pm &= 1 \otimes X_\pm + X_\pm \otimes K^2.
\end{aligned}
\tag{3.46}
$$

The differential $\mathrm{d} : \mathcal{A}(\mathrm{SU}_q(2)) \to \Omega^1(\mathrm{SU}_q(2))$ is

$$
\mathrm{d}x = (X_+ \triangleright x)\, \omega_+ + (X_- \triangleright x)\, \omega_- + (X_z \triangleright x)\, \omega_z, \tag{3.47}
$$

for all $x \in \mathcal{A}(\mathrm{SU}_q(2))$. This equation gives a basis for the dual space of 1-forms $\Omega^1(\mathcal{A}(\mathrm{SU}_q(2)))$,

$$
\begin{aligned}
\omega_z &= a^* \mathrm{d}a + c^* \mathrm{d}c, \\
\omega_- &= c^* \mathrm{d}a^* - q a^* \mathrm{d}c^*, \\
\omega_+ &= a \mathrm{d}c - q c \mathrm{d}a,
\end{aligned}
\tag{3.48}
$$

of left-covariant forms, that is $\Delta_L^{(1)}(\omega_s) = 1 \otimes \omega_s$, with $\Delta_L^{(1)}$ the (left) coaction of $\mathcal{A}(\mathrm{SU}_q(2))$ onto itself extended to forms (3.2). The above relations (3.48) can be inverted to

$$
\begin{aligned}
\mathrm{d}a &= -q c^* \omega_+ + a \omega_z, \qquad & \mathrm{d}a^* &= -q^2 a^* \omega_z + c \omega_- \\
\mathrm{d}c &= a^* \omega_+ + c \omega_z, \qquad & \mathrm{d}c^* &= -q^2 c^* \omega_z - q^{-1} a \omega_-,
\end{aligned}
$$

from which one also gets that $\omega_-^* = -\omega_+$ and $\omega_z^* = -\omega_z$. The bimodule structure is:

$$
\begin{aligned}
\omega_z a &= q^{-2} a \omega_z, \qquad & \omega_z a^* &= q^2 a^* \omega_z, \qquad & \omega_\pm a &= q^{-1} a \omega_\pm, \qquad & \omega_\pm a^* &= q a^* \omega_\pm \\
\omega_z c &= q^{-2} c \omega_z, \qquad & \omega_z c^* &= q^2 c^* \omega_z, \qquad & \omega_\pm c &= q^{-1} c \omega_\pm, \qquad & \omega_\pm c^* &= q c^* \omega_\pm,
\end{aligned}
\tag{3.49}
$$

Higher dimensional forms can be defined in a natural way by requiring compatibility for commutation relations and that $\mathrm{d}^2 = 0$. Consider the tensor product $\{\Omega(\mathrm{SU}_q(2))\}^{\otimes 2} = \Omega^1(\mathrm{SU}_q(2)) \otimes_{\mathcal{A}(\mathrm{SU}_q(2))} \Omega^1(\mathrm{SU}_q(2))$. A consistent alternation mapping on $\{\Omega(\mathrm{SU}_q(2))\}^{\otimes 2}$, generalising the alternation mapping in the classical formalism given in (2.25), can be introduced only if the quantum differential calculus is bicovariant. The strategy to formalise a wedge product comes from Lemma 15 in chapter 14 in [13], where it is proved that $\mathcal{S}_{\mathcal{Q}}(x) = \sum_{a,b} \langle X_a X_b, x \rangle\, \omega_a \otimes \omega_b$ for any $x \in \mathcal{A}(\mathrm{SU}_q(2))$ generates a two-sided ideal in $\{\Omega(\mathrm{SU}_q(2))\}^{\otimes 2}$. The bimodule of exterior differential 2-forms results to be the quotient

$$
\Omega^2(\mathrm{SU}_q(2)) \simeq \{\Omega^1(\mathrm{SU}_q(2))\}^{\otimes 2} / \mathcal{A}(\mathrm{SU}_q(2)) \{\mathcal{S}_{\mathcal{Q}}\}, \tag{3.50}
$$

since $\mathcal{A}(\mathrm{SU}_q(2)) \mathcal{S}_{\mathcal{Q}} \mathcal{A}(\mathrm{SU}_q(2)) \simeq \mathcal{A}(\mathrm{SU}_q(2)) \mathcal{S}_{\mathcal{Q}}$. The wedge product $\wedge : \Omega^1(\mathrm{SU}_q(2)) \times \Omega^1(\mathrm{SU}_q(2)) \mapsto \Omega^2(\mathrm{SU}_q(2))$ embodies the commutation relations among 1-forms: from the six generators in $\mathcal{Q}_{\mathrm{SU}_q(2)}$ the elements generating $\mathcal{S}_{\mathcal{Q}}$ can be written as

$$
\begin{aligned}
&\omega_+ \wedge \omega_+ = \omega_- \wedge \omega_- = \omega_z \wedge \omega_z = 0 \\
&\omega_- \wedge \omega_+ + q^{-2} \omega_+ \wedge \omega_- = 0 \\
&\omega_z \wedge \omega_- + q^4 \omega_- \wedge \omega_z = 0, \\
&\omega_z \wedge \omega_+ + q^{-4} \omega_+ \wedge \omega_z = 0.
\end{aligned}
\tag{3.51}
$$

Such commutation rules also show that the bimodule $\Omega^2(\mathrm{SU}_q(2))$ is 3 dimensional, the three basis 2-forms being exact, since one has:

$$\begin{aligned} d\omega_z &= -\omega_- \wedge \omega_+, \\ d\omega_+ &= q^2(1+q^2)\omega_z \wedge \omega_+, \\ d\omega_- &= -(1+q^{-2})\omega_z \wedge \omega_-; \end{aligned} \tag{3.52}$$

moreover, the commutation relations clarify that this left covariant calculus has a unique top form $\omega_- \wedge \omega_+ \wedge \omega_z$. The left covariance of the differential calculus allows to extend to higher order forms in a natural way the left coaction $\Delta_L^{(1)}$ of $\mathcal{A}(\mathrm{SU}_q(2))$ on $\Omega^1(\mathrm{SU}_q(2))$. An element $\eta \in \{\Omega^1(\mathrm{SU}_q(2))\}^{\otimes k} = \Omega^1(\mathrm{SU}_q(2)) \otimes_{\mathcal{A}(\mathrm{SU}_q(2))} \dots \otimes_{\mathcal{A}(\mathrm{SU}_q(2))} \Omega^1(\mathrm{SU}_q(2))$ (k times) can always be written as $\eta = x_{a_1 \dots a_k}\, \omega_{a_1} \otimes \dots \otimes \omega_{a_k}$ in terms of the left invariant forms ω_j in (3.48). Define

$$\Delta_L^{(k)}(\eta) = x_{a_1 \dots a_k (1)} \otimes x_{a_1 \dots a_k (2)} \omega_{a_1} \otimes \dots \otimes \omega_{a_k},$$

from the Sweedler notation for the coproduct $\Delta(x_{a_1 \dots a_k})$. One proves that this definition is consistent on the exterior algebra $\Omega^k(\mathrm{SU}_q(2))$, as

$$\Delta_L^{(2)}(\mathcal{S}_\mathcal{Q}) \subset 1 \otimes \mathcal{S}_\mathcal{Q},$$

and that

$$\Delta_L^{(k)}(d\eta) = (1 \otimes d)\Delta_L^{(k-1)}(\eta)$$

for any $\eta \in \Omega^k(\mathrm{SU}_q(2))$ with $k = 1, 2, 3$. The relations (3.52) show then that $\Omega^2(\mathrm{SU}_q(2))$ has a basis of exact left invariant forms, given by $d\omega_j$; it is also clear that $\omega_- \wedge \omega_+ \wedge \omega_z$ is left-invariant 3-form.

3.4.2 The calculus on the structure group

The strategy adopted in [4] consists in defining the calculus on U(1) via the Hopf projection π in (3.27). Out of the $\mathcal{Q}_{\mathrm{SU}_q(2)}$ which determines the left covariant calculus on $\mathrm{SU}_q(2)$, one defines a right ideal $\mathcal{Q}_{\mathrm{U}(1)} = \pi(\mathcal{Q}_{\mathrm{SU}_q(2)}) \subset \ker \varepsilon_{U(1)}$ for the calculus on U(1).

This specific $\mathcal{Q}_{\mathrm{U}(1)}$ results generated by the element $\xi = (z^{-1} - 1) + q^2(z - 1)$, and the differential calculus is then characterised by the quotient $\ker \varepsilon_{\mathrm{U}(1)} / \mathcal{Q}_{\mathrm{U}(1)}$. Any term in $\ker \varepsilon_{\mathrm{U}(1)}$ can be written as $\varphi = u(z-1) = \sum_{j \in \mathbb{Z}} u_j z^j (z-1)$, with $u = \sum_{j \in \mathbb{Z}} u_j z^j \in \mathcal{A}(\mathrm{U}(1))$ and $u_j \in \mathbb{C}$, so that the elements $\varphi(j) = z^j(z-1)$ define a basis of $\ker \varepsilon_{\mathrm{U}(1)}$ with respect to its complex vector space structure. The basis elements $\varphi(j)$ can be written in terms of the element ξ, via the two identities:

$$\begin{aligned} j \geq 0, \qquad & \varphi(j) = z^j(z-1) = \xi \left(\sum_{m=1}^{j} q^{-2m} z^{j-m+1} \right) + q^{-2j}(z-1), \\ j \leq 0, \qquad & \varphi(j) = z^{-|j|}(z-1) = -\xi \left(\sum_{m=0}^{|j|-1} q^{2m} z^{1+m-|j|} \right) + q^{2|j|}(z-1), \end{aligned} \tag{3.53}$$

which can be proved by induction on j. Define a map $\lambda : \ker \varepsilon_{\mathrm{U}(1)} \mapsto \ker \varepsilon_{\mathrm{U}(1)}$ setting on the basis elements $\lambda(\varphi(j)) = q^{-2j}(z-1)$, and linearly extending it to:

$$\lambda :\ u(z-1) = \sum_{j \in \mathbb{Z}} u_j z^j (z-1) \qquad \mapsto \sum_{j \in \mathbb{Z}} u_j q^{-2j}(z-1). \tag{3.54}$$

It is clear that λ formalises the choice of a representative element out of the equivalence class $[u(z-1)] \in \ker \varepsilon_{\mathrm{U}(1)} / \mathcal{Q}_{\mathrm{U}(1)}$, since it is possible to see that $\ker \lambda = \mathcal{Q}_{\mathrm{U}(1)}$. To prove this assertion, one first directly computes that $\lambda(\xi) = 0$, then since λ is linear one recovers that $\lambda(u\xi) = \lambda(u(q^2(z-1) + (z^{-1}-1))) =$

$q^2\lambda(u(z-1)) + \lambda(u(z^{-1}-1))$, so to have:

$$\begin{aligned}\lambda(u\xi) &= q^2\lambda(u(z-1)) + \lambda\left(\sum_{j\in\mathbb{Z}} u_j z^j(z^{-1}-1)\right) \\ &= q^2\lambda(u(z-1)) + \lambda\left(-\sum_{j\in\mathbb{Z}} u_j z^{j-1}(z^{-1})\right) = q^2\left(\sum_{j\in\mathbb{Z}} u_j q^{-2j}(z-1)\right) - \sum_{j\in\mathbb{Z}} u_j q^{-2(j-1)}(z-1) = 0,\end{aligned} \tag{3.55}$$

thus proving that $\mathcal{Q}_{\mathrm{U}(1)} \subset \ker\lambda$. To prove the inverse inclusion, consider an element $\check{u} = u(z-1) \in \ker\varepsilon_{\mathrm{U}(1)}$, and write it as:

$$\begin{aligned}u(z-1) = \sum_{j\in\mathbb{Z}} u_j z^j(z-1) &= \sum_{j\in\mathbb{N}} u_j z^j(z-1) + \sum_{j\in\mathbb{N}} u_{-j} z^{-j}(z-1) \\ &= \sum_{j\in\mathbb{N}} u_j(\alpha(j)\xi + q^{-2j}(z-1)) + \sum_{j\in\mathbb{N}} u_{-j}(\beta(-j)\xi + q^{2j}(z-1))\end{aligned} \tag{3.56}$$

where $\alpha(j) = \sum_{m=1}^{j} q^{-2m} z^{j-m+1}$ and $\beta(-j) = \sum_{m=0}^{|j|-1} q^{2m} z^{1+m-|j|}$ are the terms proportional to ξ in (3.53) for positive and negative values of $j \in \mathbb{Z}$. The previous sum can be rewritten as:

$$u(z-1) = \xi\left(\sum_{j\in\mathbb{N}} u_j\alpha(j) + \sum_{j\in\mathbb{N}} u_{-j}\beta(-j)\right) + \sum_{j\in\mathbb{Z}} u_j q^{-2j}(z-1)$$

¿From the definition (3.54), it is $\lambda(\check{u}) = 0 \leftrightarrow \sum_{j\in\mathbb{Z}} u_j q^{-2j} = 0$, so the last lines proves that $\ker\lambda \subset \mathcal{Q}_{\mathrm{U}(1)}$.

Lemma 3.5. *Given the ideal $\mathcal{Q}_{\mathrm{U}(1)} \subset \ker\varepsilon_{\mathrm{U}(1)}$ generated by the element $\xi = (z^{-1}-1) + q^2(z-1)$, it is* $\ker\varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)} \simeq \mathbb{C}$.

Proof. Define a map $\tilde{\lambda} : \ker\varepsilon_{\mathrm{U}(1)} \mapsto \mathbb{C}$ setting, on the basis elements $\varphi(j) \in \ker\varepsilon_{\mathrm{U}(1)}$, $\tilde{\lambda}(\varphi(j)) = q^{-2j}$ and extending it to $\ker\varepsilon_{\mathrm{U}(1)}$ by linearity. The properties of the map λ defined in (3.54) clarify that $\ker\tilde{\lambda} = \mathcal{Q}_{\mathrm{U}(1)}$, so to give a well defined map $\tilde{\lambda} : \ker\varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)} \mapsto \mathbb{C}$. It is immediate to see that $\tilde{\lambda}$ is an isomorphism of vector spaces, thus formalising the equivalence: with $w \in \mathbb{C}$, the map $\tilde{\lambda}^{-1}(w) = w \in [w(z-1)] \subset \ker\varepsilon_{\mathrm{U}(1)}$ represents the inverse of the map $\tilde{\lambda}$. □

This result shows that the differential calculus generated by the specific $\mathcal{Q}_{U(1)}$ is 1D, while a direct computation shows that it is bicovariant. As a basis element for its quantum tangent space one can consider

$$X = X_z = \frac{1-K^4}{1-q^{-2}}, \tag{3.57}$$

with dual 1-form given by ω_z. Explicitly, one has $\omega_z = z^*\mathrm{d}z$ with

$$\begin{aligned}\mathrm{d}z &= z\omega_z, \\ \mathrm{d}z^* &= -q^2 z^*\omega_z;\end{aligned}$$

and noncommutative $\mathcal{A}(\mathrm{U}(1))$-bimodule relations

$$\begin{aligned}z\mathrm{d}z &= q^2(\mathrm{d}z)z; \\ \omega_z z &= q^{-2} z\omega_z, \\ \omega_z z^* &= q^2 z^*\omega_z.\end{aligned}$$

3.4.3 The standard 2D calculus on S^2_q

The restriction of the above 3D calculus to the sphere S^2_q yields the unique left covariant 2-dimensional calculus on the latter [18]. An evolution of this approach has led [24] to a description of the unique 2D calculus of S^2_q in term of a Dirac operator. The 'cotangent bundle' $\Omega^1(\mathrm{S}^2_q)$ is shown to be isomorphic to the direct sum $\mathcal{L}^{(0)}_{-2} \oplus \mathcal{L}^{(0)}_{2}$, that is the line bundles with winding number ± 2. Since the element K acts as the identity on $\mathcal{A}(\mathrm{S}^2_q)$, the differential (3.47) becomes, when restricted to the latter,

$$\begin{aligned} \mathrm{d}f &= (X_- \triangleright f)\,\omega_- + (X_+ \triangleright f)\,\omega_+ \\ &= (F \triangleright f)\,(q^{-1/2}\omega_-) + (E \triangleright f)\,(q^{1/2}\omega_+), \qquad \text{for } f \in \mathcal{A}(\mathrm{S}^2_q). \end{aligned}$$

These leads to break the exterior derivative into a holomorphic and an anti-holomorphic part, $\mathrm{d} = \bar{\partial} + \partial$, with:

$$\begin{aligned} \bar{\partial} f &= (X_- \triangleright f)\,\omega_- = (F \triangleright f)\,(q^{-1/2}\omega_-), \\ \partial f &= (X_+ \triangleright f)\,\omega_+ = (E \triangleright f)\,(q^{1/2}\omega_+), \qquad \text{for} \quad f \in \mathcal{A}(\mathrm{S}^2_q). \end{aligned}$$

An explicit computation on the generators (3.30) of S^2_q yields:

$$\bar{\partial} B_- = q^{-1}\, a^2\, \omega_-, \quad \bar{\partial} B_0 = q\, ca\, \omega_-, \qquad \bar{\partial} B_+ = q\, c^2\, \omega_-,$$

$$\partial B_+ = q^2\, a^{*2}\, \omega_+, \quad \partial B_0 = -q^2\, c^* a^*\, \omega_+, \quad \partial B_- = q^2\, c^{*2}\, \omega_+.$$

The above shows that: $\Omega^1(\mathrm{S}^2_q) = \Omega^1_-(\mathrm{S}^2_q) \oplus \Omega^1_+(\mathrm{S}^2_q)$ where $\Omega^1_-(\mathrm{S}^2_q) \simeq \mathcal{L}^{(0)}_{-2} \simeq \bar{\partial}(\mathcal{A}(\mathrm{S}^2_q))$ is the $\mathcal{A}(\mathrm{S}^2_q)$-bimodule generated by:

$$\{\bar{\partial} B_-, \bar{\partial} B_0, \bar{\partial} B_+\} = \{a^2, ca, c^2\}\, \omega_- = q^2 \omega_- \{a^2, ca, c^2\}$$

and $\Omega^1_+(\mathrm{S}^2_q) \simeq \mathcal{L}^{(0)}_{+2} \simeq \partial(\mathcal{A}(\mathrm{S}^2_q))$ is the one generated by:

$$\{\partial B_+, \partial B_0, \partial B_-\} = \{a^{*2}, c^* a^*, c^{*2}\}\, \omega_+ = q^{-2} \omega_+ \{a^{*2}, c^* a^*, c^{*2}\}.$$

That these two modules of forms are not free is also expressed by the existence of relations among the differential:

$$\partial B_0 = q^{-1} B_- \partial B_+ - q^3 B_+ \partial B_-, \qquad \bar{\partial} B_0 = q B_+ \bar{\partial} B_- - q^{-3} B_- \bar{\partial} B_+.$$

Writing any 1-form as $\alpha = \phi' \omega_- + \phi'' \omega_+ \in \mathcal{L}^{(0)}_{-2}\omega_- \oplus \mathcal{L}^{(0)}_{+2}\omega_+$, the product of 1-forms is

$$(\phi' \omega_- + \phi'' \omega_+) \wedge (\psi' \omega_- + \psi'' \omega_+) = (q^{-2} \phi'' \psi' - \phi' \psi'') \omega_+ \wedge \omega_-, \tag{3.58}$$

while the exterior derivative acts as:

$$\begin{aligned} \mathrm{d}(\phi' \omega_- + \phi'' \omega_+) &= (\mathrm{d}\phi') \wedge \omega_- + \phi' \mathrm{d}\omega_- + (\mathrm{d}\phi'') \wedge \omega_+ + \phi'' \mathrm{d}\omega_+ \\ &= (X_+ \triangleright \phi') \omega_+ \wedge \omega_- + \{(X_z \triangleright \phi') \omega_z \wedge \omega_- + \phi' \mathrm{d}\omega_-\} \\ &\quad + (X_- \triangleright \phi'') \omega_- \wedge \omega_+ + \{(X_z \triangleright \phi'') \omega_z \wedge \omega_+ + \phi'' \mathrm{d}\omega_+\} \\ &= \{(X_- \triangleright \phi'') - q^2 (X_+ \triangleright \phi')\} \omega_- \wedge \omega_+, \end{aligned} \tag{3.59}$$

since the terms in curly brackets vanish: $\{(X_z \triangleright \phi') \omega_z \wedge \omega_- + \phi' \mathrm{d}\omega_-\} = \{(X_z \triangleright \phi'') \omega_z \wedge \omega_+ + \phi'' \mathrm{d}\omega_+\} = 0$ from (3.52) and (3.31). It is then cleat that the calculus on the quantum sphere is 2D, and that $\Omega^2(\mathrm{S}^2_q) = \mathcal{A}(\mathrm{S}^2_q) \omega_- \wedge \omega_+ = \omega_- \wedge \omega_+ \mathcal{A}(\mathrm{S}^2_q)$, as both $\omega_\pm$ commute with elements of $\mathcal{A}(\mathrm{S}^2_q)$ and so does $\omega_- \wedge \omega_+$.

Remark 3.6. *From* (3.52) *it is natural to ask that* $\mathrm{d}\omega_- = \mathrm{d}\omega_+ = 0$ *when restricted to* S^2_q*. Then, the exterior derivative of any 1-form* $\alpha = \phi' \omega_- + \phi'' \omega_+ \in \mathcal{L}^{(0)}_{-2}\omega_- \oplus \mathcal{L}^{(0)}_{+2}\omega_+$ *is given by:*

$$\begin{aligned} \mathrm{d}\alpha = \mathrm{d}(\phi' \omega_- + \phi'' \omega_+) &= \partial \phi' \wedge \omega_- + \bar{\partial} \phi'' \wedge \omega_+ \\ &= (X_+ \triangleright \phi' - q^{-2} X_- \triangleright \phi'')\, \omega_+ \wedge \omega_- = q^{-1/2} (E \triangleright \phi' - q^{-1} F \triangleright \phi'')\, \omega_+ \wedge \omega_-, \end{aligned} \tag{3.60}$$

since $K\triangleright$ *acts as* $q^{\mp}$ *on* $\mathcal{L}^{(0)}_{\mp 2}$*. Notice that in the above equality, both* $E \triangleright \phi'$ *and* $F \triangleright \phi''$ *belong to* $\mathcal{A}(\mathrm{S}^2_q)$*, as it should be.*

The above results can be summarised in the following proposition, which is the natural generalisation of the description in (2.61) of the classical exterior algebra on the sphere manifold S^2.

Proposition 3.7. *The 2D differential calculus on the sphere* S^2_q *is given by:*

$$\Omega(\mathrm{S}^2_q) = \mathcal{A}(\mathrm{S}^2_q) \oplus \left(\mathcal{L}^{(0)}_{-2}\omega_- \oplus \mathcal{L}^{(0)}_{+2}\omega_+\right) \oplus \mathcal{A}(\mathrm{S}^2_q)\omega_+ \wedge \omega_-,$$

with multiplication rule

$$\big(f_0;\phi',\phi'';f_2\big)\big(g_0;\psi',\psi'';g_2\big) = \big(f_0g_0; f_0\psi' + \phi' g_0, f_0\psi'' + \phi'' g_0; f_0g_2 + f_2g_0 + q^{-2}\phi''\psi' - \phi'\psi''\big),$$

and exterior derivative $\mathrm{d} = \bar\partial + \partial$*:*

$$f \mapsto (q^{-1/2}F\triangleright f, q^{1/2}E\triangleright f), \qquad \text{for } f \in \mathcal{A}(\mathrm{S}^2_q),$$

$$(\phi',\phi'') \mapsto q^{-1/2}(E\triangleright\phi' - q^{-1}F\triangleright\phi''), \quad \text{for } (\phi',\phi'') \in \mathcal{L}^{(0)}_{-2} \oplus \mathcal{L}^{(0)}_{+2}.$$

3.4.4 The compatibility conditions between the calculi

Given the 3D left-covariant differential calculus on $\mathrm{SU}_q(2)$ described in section 3.4.1, as well the 1D bicovariant differential calculus on the gauge group algebra U(1) in section 3.4.2, the 'principal bundle compatibility' of these calculi is established by showing that the sequence (3.17) is exact. For the case at hand, this sequence becomes

$$0 \to \mathcal{A}(\mathrm{SU}_q(2))\big(\Omega^1(\mathrm{S}^2_q)\big)\mathcal{A}(\mathrm{SU}_q(2)) \to$$
$$\to \Omega^1(\mathrm{SU}_q(2)) \xrightarrow{\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}} \mathcal{A}(\mathrm{SU}_q(2)) \otimes \ker\varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)} \to 0,$$

where $\mathcal{Q}_{\mathrm{U}(1)}$ is the ideal given in section 3.4.2 that defines the calculus on $\mathcal{A}(\mathrm{U}(1))$ and the map $\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}$ is defined as in the diagram (3.15) which now acquires the form:

$$\begin{array}{lcl} \Omega^1(\mathrm{SU}_q(2))_{un} & \stackrel{\pi_{\mathcal{N}_{\mathrm{SU}_q(2)}}}{\longrightarrow} & \Omega^1(\mathrm{SU}_q(2)) \\ \downarrow\chi & & \downarrow\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}} \\ \mathcal{A}(\mathrm{SU}_q(2)) \otimes \ker\varepsilon_{\mathrm{U}(1)} & \stackrel{\mathrm{id}\otimes\pi_{\mathcal{Q}_{\mathrm{U}(1)}}}{\longrightarrow} & \mathcal{A}(\mathrm{SU}_q(2)) \otimes (\ker\varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)})\,. \end{array}$$

Having a quantum homogeneous bundle, that is a quantum bundle whose total space is a Hopf algebra and whose fiber is a Hopf subalgebra of it, with the differential calculus on the fiber obtained from the corresponding projection, for the above sequence to be exact it is enough [9] to check two conditions. The first one is

$$(\mathrm{id}\otimes\pi)\circ\mathrm{Ad}(\mathcal{Q}_{\mathrm{SU}_q(2)}) \subset \mathcal{Q}_{\mathrm{SU}_q(2)} \otimes \mathcal{A}(\mathrm{U}(1))$$

with $\pi : \mathcal{A}(\mathrm{SU}_q(2)) \to \mathcal{A}(\mathrm{U}(1))$ the projection in (3.27). This is easily established by a direct calculation and using the explicit form of the elements in $\mathcal{Q}_{\mathrm{SU}_q(2)}$. The second condition amounts to the statement that the kernel of the projection π can be written as a right $\mathcal{A}(\mathrm{SU}_q(2))$-module of the kernel of π itself restricted to the base algebra $\mathcal{A}(\mathrm{S}^2_q)$. Then, one needs to show that $\ker\pi \subset (\ker\pi|_{\mathrm{S}^2_q})\mathcal{A}(\mathrm{SU}_q(2))$, the opposite implication being obvious. With π defined in (3.27), one has that

$$\ker\pi = \{cf,\, c^*g, \quad \text{with} \quad f,g \in \mathcal{A}(\mathrm{SU}_q(2))\}.$$

Then $cf = c(a^*a + c^*c)f = ca^*(af) + c^*c(cf)$, with both ca^* and c^*c in $\ker\pi|_{\mathrm{S}^2_q}$. The same holds for elements of the form c^*g, and the inclusion follows.

The analysis of the map $\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}$: $\Omega^1(\mathrm{SU}_q(2)) \mapsto \mathcal{A}(\mathrm{SU}_q(2)) \otimes \ker\varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)}$ shows that $\omega_\pm \in \Omega^1(\mathcal{A}(\mathrm{SU}_q(2)))$ are indeed the generators of the horizontal forms of the principal bundle, being in the $\ker \sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}$. From (3.39) one recovers:

$$\chi(\delta a) = a \otimes (z-1), \quad \chi(\delta a^*) = a^* \otimes (z^*-1)$$
$$\chi(\delta c) = c \otimes (z-1), \quad \chi(\delta c^*) = c^* \otimes (z^*-1).$$

Given the two generators $\omega_{\pm}$ and the specific $\mathcal{Q}_{\mathrm{SU}_q(2)}$ which determines the 3D calculus, corresponding universal 1-forms can be taken to be:

$$\omega_{+} = a\mathrm{d}c - qc\mathrm{d}a \qquad \rightarrow \qquad (a\delta c - qc\delta a) \in [\pi_{\mathcal{N}_{\mathrm{SU}_q(2)}}]^{-1}(\omega_{+}),$$
$$\omega_{-} = c^*\mathrm{d}a^* - qa^*\mathrm{d}c^* \qquad \rightarrow \qquad (c^*\delta a^* - qa^*\delta c^*) \in [\pi_{\mathcal{N}_{\mathrm{SU}_q(2)}}]^{-1}(\omega_{-}).$$

The action of the canonical map then gives:

$$\chi(a\delta c - qc\delta a) = (ac - qca) \otimes (z-1) = 0,$$
$$\chi(c^*\delta a^* - qa^*\delta c^*) = (c^*a^* - qa^*c^*) \otimes (z^*-1) = 0,$$

which means that

$$\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}(\omega_{\pm}) = 0 \tag{3.61}$$

For the third generator ω_z, one shows in a similar fashion that

$$\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}(\omega_z) = 1 \otimes (\pi_{\mathcal{Q}_{\mathrm{U}(1)}}(z-1)). \tag{3.62}$$

From these it is possible to conclude that the elements $\omega_{\pm}$ generate the $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule of horizontal forms, while from (3.57) one has that the vector $X = X_z = (1-q^{-2})^{-1}(1-K^4)$ is the dual generator to the calculus on the structure Hopf algebra $\mathcal{A}(\mathrm{U}(1))$. For the corresponding 'vector field' $\tilde{X}$ on $\mathcal{A}(\mathrm{SU}_q(2))$ as in (3.18), one has that $\tilde{X}(\omega_{\pm}) = \langle X, \sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}(\omega_{\pm})\rangle = 0$, while $\tilde{X}(\omega_z) = \langle X, \sim_{\mathcal{N}_{\mathrm{SU}_q(2)}}(\omega_z)\rangle = 1$. These results identify $\tilde{X}$ as a vertical vector field.

4 A $\star$-Hodge duality on $\Omega(SU_q(2))$ and a Laplacian on $\mathrm{SU}_q(2)$

In classical differential geometry a metric structure g on a N-dimensional manifold $\mathcal{M}$ enables to define a Hodge duality $\star : \Omega^k(\mathcal{M}) \mapsto \Omega^{N-k}(\mathcal{M})$ on the exterior algebra $\Omega(\mathcal{M})$. The strategy is to consider the volume form $\theta \in \Omega^N(\mathcal{M})$ associated to a g-orthonormal basis; this corresponds to the choice of an orientation. Via the Hodge duality it becomes possible to introduce in $\Omega(\mathcal{M})$ both a symmetric bilinear product and a sesquilinear inner product.

Section 2.2 describes the Hodge duality (2.30) on the group manifold $SU(2) \simeq S^3$, with the exterior algebra $\Omega(S^3)$ introduced in section 2.1 and the riemannian metric (2.29) coming from the Cartan decomposition of the Lie algebra $\mathfrak{su}(2)$. The compatibility between the wedge product in $\Omega(S^3)$ and the Hodge structure expressed in (2.32) allows for the definition in (2.33) of a symmetric tensor, and in (2.38) of a scalar product in $\Omega(S^3)$, while the relation (2.41) introduces on $\Omega(S^3)$ an inner product, which is anti-linear in the second entry. Such a pair of bilinears result completely characterised in (2.37) and (**??**).

The algebraic formulation of geometry of quantum groups, that has been described, presents no metric tensor. The strategy to introduce a Hodge duality on the exterior algebra $\Omega(\mathcal{H})$ coming from a N-dimensional differential calculus on a Hopf algebra $\mathcal{H}$ is then reversed with respect to the strategy used in the classical setting. The path consists in defining a suitable bilinear product on $\Omega(\mathcal{H})$ and considering a volume N-form, from which to induce a $\star$-Hodge structure, using an equation like the one in (2.39) as a definition: note in fact that the classical symmetry property (2.32) loses its meaning in a non commutative setting.

The following description of the quantum formulation of a Hodge duality originates from [14]. Assume that $\mathcal{H}$ is a $*$-Hopf algebra equipped with a left covariant calculus $(\Omega^1(\mathcal{H}), \mathrm{d})$, with N the dimension of the calculus and $\dim \Omega^N_{inv}(\mathcal{H}) = 1$. Suppose also that $\mathcal{H}$ admits a Haar state $h : \mathcal{H} \mapsto \mathbb{C}$, that is a unital linear functional on $\mathcal{H}$ for which $(id \otimes h)\Delta x = (h \otimes id)\Delta x = h(x)1$ for any $x \in \mathcal{H}$, where 1 is used to emphasise the unit of the algebra. Suppose further that h is positive, that is $h(x^*x) \geq 0$ for all $x \in \mathcal{H}$; it is known that the Haar state is unique and automatically faithful: if $h(x^*x) = 0$, then necessarily $x = 0$. One can endow $\mathcal{H}$ with an inner product derived from h, setting:

$$(x'; x)_{\mathcal{H}} = h(x^*x') \tag{4.1}$$

for any $x, x' \in \mathcal{H}$. The whole exterior algebra can be endowed with an inner product, defined on a left invariant basis and then extended via the requirement of left invariance,

$$(x'\omega'; x\,\omega)_{\mathcal{H}} = h(x^* x')(\omega', \omega)_{\mathcal{H}} \tag{4.2}$$

for any $x, x' \in \mathcal{H}$ and left invariant forms ω, ω' in $\Omega(\mathcal{H})$. An inner product is said graded if the spaces $\Omega^k(\mathcal{H})$ are pairwise orthogonal.

Out of $\Omega^N(\mathcal{H})$ choose a left invariant hermitian basis element $\theta = \theta^*$, which will be called the volume form of the calculus. A linear functional $\int_\theta : \Omega(\mathcal{H}) \mapsto \mathbb{C}$ – called the integral on $\Omega(\mathcal{H})$ associated to the volume form $\theta \in \Omega^N(\mathcal{H})$ – is defined by setting $\int_\theta \eta = 0$ if η is a k-form with $k < N$, and $\int_\theta \eta = h(x)$ if $\eta = x\,\theta$ with $x \in \mathcal{H}$. The differential calculus will be said non-degenerate if, whenever $\eta \in \Omega^k(\mathcal{H})$ and $\eta' \wedge \eta = 0$ for any $\eta' \in \Omega^{N-k}(\mathcal{H})$, then necessarily $\eta = 0$. This property reflects itself in the property of left-faithfulness of the functional $\int_\theta$: starting from a non degenerate calculus, it is possible to prove that, if η is an element in $\Omega^k(\mathcal{H})$ for which $\int_\theta \eta' \wedge \eta = 0$ for all $\eta' \in \Omega^{N-k}(\mathcal{H})$, then it is $\eta = 0$.

Proposition 4.1. *Given the exterior algebra $\Omega(\mathcal{H})$ coming from a left covariant, non degenerate calculus $(\Omega^1(\mathcal{H}), \mathrm{d})$, there exists a unique left $\mathcal{H}$-linear bijective operator $L : \Omega^k(\mathcal{H}) \mapsto \Omega^{N-k}(\mathcal{H})$ for $k = 0, \dots, N$, such that*

$$\int_\theta \eta^* \wedge L(\eta') = (\eta'; \eta)_{\mathcal{H}} \tag{4.3}$$

on any $\eta, \eta' \in \Omega^k(\mathcal{H})$.

The proof of this result is in [14], where the operator L is called a Hodge operator. Such an operator does not yet define a $\star$-Hodge structure on $\Omega(\mathcal{H})$, since its square does not satisy the natural requirement (2.31). It is then used to define a new graded left invariant inner product setting on a basis of left invariant forms $\omega \in \Omega(\mathcal{H})$:

$$\begin{aligned} (\omega; \omega')^\natural_{\mathcal{H}} &= (\omega; \omega')_{\mathcal{H}}, && \text{on } \Omega^k(\mathcal{H}),\ k < N/2; \\ (\omega; \omega')^\natural_{\mathcal{H}} &= (L^{-1}(\omega); L^{-1}(\omega'))_{\mathcal{H}}, && \text{on } \Omega^k(\mathcal{H}),\ k > N/2. \end{aligned} \tag{4.4}$$

If N is odd, these relations completely define a new left invariant graded inner product on the exterior algebra $\Omega(\mathcal{H})$; notice also that assuming the relation (4.1) means that $(1;1)_{\mathcal{H}} = 1$, from which one has $L(1) = \theta$, so to obtain in (4.4) that $(\theta;\theta)^\natural_{\mathcal{H}} = (1;1)_{\mathcal{H}} = 1$.

In analogy with (4.3) define a new Hodge operator $L^\natural : \Omega^k(\mathcal{H}) \mapsto \Omega^{N-k}(\mathcal{H})$ via the inner product given in (4.4) as

$$\int_\theta \eta^* \wedge L^\natural(\eta') = (\eta'; \eta)^\natural_{\mathcal{H}}. \tag{4.5}$$

Due to the left-faithfulness of the integral, it is clear that $L^\natural$ is a well defined bijection, which satisfies the identity $L = L^\natural$ when restricted to $\Omega^k(\mathcal{H})$ with $k < N/2$. Such an operator $L^\natural$ is also proved to satisfy $(L^\natural)^2 = (-1)^{k(N-k)}$: this is the reason why one can define a $\star$-Hodge structure on $\Omega(\mathcal{H})$ as:

$$\star : \Omega^k(\mathcal{H}) \mapsto \Omega^{N-k}(\mathcal{H}) \qquad \star(\eta) = L^\natural(\eta). \tag{4.6}$$

The relation (4.5) appears as the quantum version of the classical relation (2.41), which is now used as a definition for the Hodge duality.

If the dimension of the calculus is given by an even $N = 2m$, a more specific procedure is needed, The same procedure as before gives a $\star$-Hodge operator on $\Omega^k(\mathcal{H})$ for $k \neq m$ via the inner product (4.4). Using the volume form $\theta \in \Omega^N(\mathcal{H})$ set now a sesquilinear form

$$\langle \eta', \eta \rangle = \int_\theta \eta^* \wedge \eta', \tag{4.7}$$

which is non-degenerate by the faithfulness of the integral $\int_\theta$. The $\mathcal{H}$-bimodule $\Omega^m(\mathcal{H})$ has a basis of $\begin{pmatrix} 2m \\ m \end{pmatrix}$ left invariants elements ω_a. The restriction of (4.7) to the elements ω_a defines a sesquilinear form on a $\mathbb{C}^{(2m)!/(m!)^2}$ vector space, which is hermitian if $(-1)^{m^2} = 1$, and anti-hermitian if $(-1)^{m^2} = -1$:

so it can be 'diagonalised'. There exists a basis $\check{\omega}_j \in \Omega^m(\mathcal{H})$ with $j = 1, \ldots, \begin{pmatrix} 2m \\ m \end{pmatrix}$ such that one has $\langle \check{\omega}_a, \check{\omega}_b \rangle = \pm\delta_{ab}$ if it is hermitian, and $\langle \check{\omega}_a, \check{\omega}_b \rangle = \pm i\delta_{ab}$ if it is anti-hermitian. It is then possible to use such a basis to define a left $\mathcal{H}$-linear operator $\mathfrak{L} : \Omega^m(\mathcal{H}) \mapsto \Omega^m(\mathcal{H})$ setting on the basis

$$\mathfrak{L}(\check{\omega}_a) = (-1)^{m^2} \langle \check{\omega}_a, \check{\omega}_a \rangle \, \check{\omega}_a. \tag{4.8}$$

(no sum on a). The operator $\mathfrak{L}$ is a bijection, and satisfies $\mathfrak{L}^2 = (-1)^{m^2}$, so a $\star$-Hodge structure on $\Omega^m(\mathcal{H})$ can be defined as:

$$\star(\eta) = \mathfrak{L}(\eta), \tag{4.9}$$

on any $\eta \in \Omega^m(\mathcal{H})$, thus giving a complete constructive procedure for a $\star$-Hodge structure on $\Omega(\mathcal{H})$. The Hodge operator $\mathfrak{L} : \Omega^m(\mathcal{H}) \mapsto \Omega^m(\mathcal{H})$ is then used to introduce a left invariant inner product on $\Omega^m(\mathcal{H})$, defined by:

$$(\omega_a; \omega_b)^\natural_{\mathcal{H}} = \int_\theta \omega_b^* \wedge \mathfrak{L}(\omega_a), \tag{4.10}$$

on a basis of left invariant $\{\omega_a\}$ 2-forms, and then extended via the requirement of left invariance as in (4.2). It is easy to see that the definition eventually corresponds to the inner product

$$(\check{\omega}_a; \check{\omega}_b)^\natural_{\mathcal{H}} = \delta_{ab} \tag{4.11}$$

on $\Omega^m(\mathcal{H})$.

4.1 A $\star$-Hodge structure on $\Omega(\mathrm{SU}_q(2))$

This section describes how the outlined procedure yields a left invariant inner product on the exterior algebra $\Omega(SU_q(2))$ generated by the left covariant 3D calculus from section 3.4.1, and the way it gives rise to a $\star$-Hodge structure. Such a $\star$-Hodge structure will be then used to define a Laplacian operator on $\mathcal{A}(\mathrm{SU}_q(2))$, which is completely diagonalised.

The Hopf algebra $\mathcal{A}(\mathrm{SU}_q(2))$ has a Haar state $h : \mathcal{A}(\mathrm{SU}_q(2)) \mapsto \mathbb{C}$, which is positive, unique and authomatically faithful. From the Peter-Weyl decomposition of $\mathcal{A}(\mathrm{SU}_q(2))$ in terms of the vector space basis elements $w_{p:r,t} \in W_p$ (3.43), the Haar state is determined by setting:

$$h(1) = 1 \qquad h(w_{p:r,t}) = 0 \;\; \forall\, p \geq 0.$$

The algebraic relations (3.19) among the generators of $\mathcal{A}(\mathrm{SU}_q(2))$ makes it then possible to prove that the only non trivial action of h on $\mathcal{A}(\mathrm{SU}_q(2))$ can also be written as:

$$h((cc^*)^k) = (\sum_{j=0}^{k} q^{2j})^{-1} = \frac{1}{1 + q^2 + \ldots + q^{2k}},$$

with $k \in \mathbb{N}$. One can define on $\mathcal{A}(\mathrm{SU}_q(2))$ an inner product derived from h, setting:

$$(x', x)_{\mathrm{SU}_q(2)} = h(x^* x') \tag{4.12}$$

with $x, x' \in \mathcal{A}(\mathrm{SU}_q(2))$. The differential 3D calculus being left covariant, the set of k-forms $\Omega^k(\mathrm{SU}_q(2))$ has a basis of left invariant forms. The exterior algebra $\Omega(\mathrm{SU}_q(2))$ is endowed with an inner product, defined on a left invariant basis and extended via the requirement of left invariance:

$$(x'\omega', x\,\omega)_{\mathrm{SU}_q(2)} = h(x^\star x')(\omega', \omega)_{\mathrm{SU}_q(2)}$$

for all x, x' in $\mathcal{A}(\mathrm{SU}_q(2))$ and $\omega, \omega' \in \Omega(\mathrm{SU}_q(2))$ left invariant forms. Assume the top form $\theta = \alpha' \omega_- \wedge \omega_+ \wedge \omega_z$ as volume form, with $\alpha' \in \mathbb{R}$ so that $\theta^\star = \theta$. The integral on the exterior algebra $\Omega(\mathrm{SU}_q(2))$ associated to the volume form θ is defined by $\int_\theta \eta = 0$ if η is a k-form with $k < 2$, and $\int_\theta \eta = h(x)$ if $\eta = x\,\theta$. This integral is left-faithful.

Set a left invariant graded inner product by assuming that the only non-zero products among left invariant forms are:

$$\begin{aligned} (1, 1)_{\mathrm{SU}_q(2)} &= 1, \\ (\theta, \theta)_{\mathrm{SU}_q(2)} &= 1; \end{aligned} \tag{4.13}$$

while in $\Omega^1(\mathrm{SU}_q(2))$ are:

$$
\begin{aligned}
&(\omega_-,\omega_-)_{\mathrm{SU}_q(2)}=\beta,\\
&(\omega_+,\omega_+)_{\mathrm{SU}_q(2)}=\nu,\\
&(\omega_z,\omega_z)_{\mathrm{SU}_q(2)}=\gamma
\end{aligned}
\tag{4.14}
$$

with $\beta,\nu,\gamma\in\mathbb{R}$, and:

$$
\begin{aligned}
&(\omega_-\wedge\omega_+,\omega_-\wedge\omega_+)_{\mathrm{SU}_q(2)}=1,\\
&(\omega_+\wedge\omega_z,\omega_+\wedge\omega_z)_{\mathrm{SU}_q(2)}=1,\\
&(\omega_z\wedge\omega_-,\omega_z\wedge\omega_-)_{\mathrm{SU}_q(2)}=1
\end{aligned}
\tag{4.15}
$$

in $\Omega^2(\mathrm{SU}_q(2))$. This choice comes as the most natural in order to mimic the properties of the classical inner product (2.40), coming from the classical Hodge structure (2.30) originated from the metric (2.29). The Hodge operator defined in (4.3) is:

$$
\begin{aligned}
&L(1)=\alpha'\,\omega_-\wedge\omega_+\wedge\omega_z,\\
&L(\omega_-)=-\alpha'\beta q^{-6}\,\omega_z\wedge\omega_-,\\
&L(\omega_+)=-\alpha'\nu\,\omega_+\wedge\omega_z\\
&L(\omega_z)=-\alpha'\gamma\,\omega_-\wedge\omega_+\\
&L(\omega_-\wedge\omega_+)=-\alpha'\,\omega_z,\\
&L(\omega_+\wedge\omega_z)=-\alpha'\,\omega_+,\\
&L(\omega_z\wedge\omega_-)=-\alpha'\,\omega_-,\\
&L(\omega_-\wedge\omega_+\wedge\omega_z)=\alpha'^{-1}
\end{aligned}
\tag{4.16}
$$

The Hodge operator L is used to define a new graded left invariant inner product on $\Omega(\mathrm{SU}_q(2))$, as:

$$
\begin{aligned}
&(\omega',\omega)^{\natural}_{\mathrm{SU}_q(2)}=(\omega',\omega)_{\mathrm{SU}_q(2)} && \text{on }\ \Omega^k(\mathrm{SU}_q(2)),\ k=0,1;\\
&(\omega',\omega)^{\natural}_{\mathrm{SU}_q(2)}=(L^{-1}(\omega'),L^{-1}(\omega))_{\mathrm{SU}_q(2)} && \text{on }\ \Omega^k(\mathrm{SU}_q(2)),\ k=2,3,
\end{aligned}
\tag{4.17}
$$

on the basis of left invariant forms. On $\Omega^k(\mathrm{SU}_q(2)))$ – with $k=2,3$ – one has:

$$
\begin{aligned}
&(\omega_-\wedge\omega_+,\omega_-\wedge\omega_+)^{\natural}_{\mathrm{SU}_q(2)}=\alpha'^{-2}\gamma^{-1},\\
&(\omega_+\wedge\omega_z,\omega_+\wedge\omega_z)^{\natural}_{\mathrm{SU}_q(2)}=\alpha'^{-2}\nu^{-1},\\
&(\omega_z\wedge\omega_-,\omega_z\wedge\omega_-)^{\natural}_{\mathrm{SU}_q(2)}=q^{12}\alpha'^{-2}\beta^{-1},\\
&(\theta,\theta)^{\natural}_{\mathrm{SU}_q(2)}=1.
\end{aligned}
\tag{4.18}
$$

Associated to this new inner product there is in analogy a new unique left $\mathcal{A}(\mathrm{SU}_q(2))$-linear operator $L^{\natural}:\Omega^k(\mathrm{SU}_q(2))\mapsto\Omega^{3-k}(\mathrm{SU}_q(2))$ defined by $\int_\theta \eta^*\wedge L^{\natural}(\eta')=(\eta',\eta)^{\natural}$, which is a bijection. This operator is such that $(L^{\natural})^2=(-1)^{k(3-k)}=1$, so following (4.6) one has a $\star$-Hodge structure on the exterior algebra $\Omega(\mathrm{SU}_q(2))$:

$$
\star:\Omega^k(\mathrm{SU}_q(2))\mapsto\Omega^{3-k}(\mathrm{SU}_q(2))\qquad \star(\eta)=L^{\natural}(\eta),
\tag{4.19}
$$

given by:

$$
\begin{aligned}
&\star(1)=\theta=\alpha'\,\omega_-\wedge\omega_+\wedge\omega_z,\\
&\star(\omega_-)=-\alpha'\beta q^{-6}\,\omega_z\wedge\omega_-,\\
&\star(\omega_+)=-\alpha'\nu\,\omega_+\wedge\omega_z,\\
&\star(\omega_z)=-\alpha'\gamma\,\omega_-\wedge\omega_+,\\
&\star(\omega_-\wedge\omega_+)=-\alpha'^{-1}\gamma^{-1}\,\omega_z,\\
&\star(\omega_+\wedge\omega_z)=-\alpha'^{-1}\nu^{-1}\,\omega_+,\\
&\star(\omega_z\wedge\omega_-)=-\alpha'^{-1}\beta^{-1}q^{6}\,\omega_-,\\
&\star(\omega_-\wedge\omega_+\wedge\omega_z)=\alpha'^{-1}
\end{aligned}
\tag{4.20}
$$

Remark 4.2. *The definition of the graded left invariant inner product* $(\cdot,\cdot)^{\natural}_{\mathrm{SU}_q(2)}$ *in* (4.17) *shows that, in order to have a* $\star$*-Hodge structure on the exterior algebra* $\Omega(\mathrm{SU}_q(2))$ *generated by the 3D calculus, it is sufficient to choice an hermitian volume form and a graded left invariant inner product* only *on* $\Omega^k(\mathrm{SU}_q(2))$ *for* $k=0,1$. *This is a general aspect: given a Hopf* $*$*-algebra* $\mathcal{H}$*, equipped with a finite odd* N *dimensional left covariant differential calculus, the formalism developed in [14] shows, that what one needs is an hermitian volume form and a graded left invariant inner product on* $\Omega^k(\mathcal{H})$ *for* $k<N/2$.

4.1.1 A Laplacian operator on $\mathcal{A}(\mathrm{SU}_q(2))$

Given a differential calculus and a $\star$-Hodge structure on the Hopf algebra $\mathcal{A}(\mathrm{SU}_q(2))$ it is possible to define a scalar Laplacian operator $\square_{\mathrm{SU}_q(2)}:\mathcal{A}(\mathrm{SU}_q(2))\mapsto\mathcal{A}(\mathrm{SU}_q(2))$ as $\square_{\mathrm{SU}_q(2)}\phi=\star\mathrm{d}\star\mathrm{d}\phi$ for any $\phi\in\mathcal{A}(\mathrm{SU}_q(2))$. This Laplacian can be written down by a computation on the basis of the left invariant forms of the calculus:

$$\begin{aligned}\mathrm{d}\phi&=(X_+\triangleright\phi)\omega_++(X_-\triangleright\phi)\omega_-+(X_z\triangleright\phi)\omega_z;\\ \star\mathrm{d}\phi&=-\alpha'[\nu(X_+\triangleright\phi)\omega_+\wedge\omega_z+\beta q^{-6}(X_-\triangleright\phi)\omega_z\wedge\omega_-+\gamma(X_z\triangleright\phi)\omega_-\wedge\omega_+].\end{aligned}$$

The last line comes from (4.20) and the left linearity of the $\star$-Hodge on the exterior algebra $\Omega(\mathrm{SU}_q(2))$. By (3.52) the derivative d acts on the previous 2-form as:

$$\begin{aligned}\mathrm{d}\star\mathrm{d}\phi&=-\alpha'[\nu(X_-X_+\triangleright\phi)(\omega_-\wedge\omega_+\wedge\omega_z)+\beta q^{-6}(X_+X_-\triangleright\phi)(\omega_+\wedge\omega_z\wedge\omega_-)+\gamma(X_zX_z\triangleright\phi)(\omega_z\wedge\omega_-\wedge\omega_+)]\\ &=-\alpha'\{[\nu X_-X_++\beta X_+X_-+\gamma X_zX_z]\triangleright\phi\}(\omega_-\wedge\omega_+\wedge\omega_z),\end{aligned}$$

where the commutation rules (3.51) have been used. The last of (4.20) finally gives the Laplacian operator the expression:

$$\star\mathrm{d}\star\mathrm{d}\phi=-[\nu X_-X_++\beta X_+X_-+\gamma X_zX_z]\triangleright\phi \tag{4.21}$$

in terms of the left action of the quantum vector fields of the calculus. The expression (4.21) shows that $\square_{\mathrm{SU}_q(2)}:\mathcal{L}_n\mapsto\mathcal{L}_n$. This operator can be diagonalised. One has to recall the decomposition (3.41) of the modules $\mathcal{L}_n$ for the right action of $\mathcal{U}_q(\mathfrak{su}(2))$: this right action leaves invariant the eigenspaces of the Laplacian since left and right actions of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(\mathrm{SU}_q(2))$ do commute. On each irreducible subspace $V_J^{(n)}$ (3.41) for the right action of $\mathcal{U}_q(\mathfrak{su}(2))$ one has a basis $\phi_{n,J,l}=(c^{J-n/2}a^{\star J+n/2})\triangleleft E^l=w_{2J:J-\frac{n}{2},l}$ (with $l=0,\dots,2J$) of eigenvectors (3.43) for the Laplacian. The spectrum of the Laplacian does not depend on the integer l: an explicit computation shows that

$$\begin{aligned}X_zX_z\triangleright\phi_{n,J,l}&=q^{2(n+1)}[n]^2\phi_{n,J,l},\\ X_+X_-\triangleright\phi_{n,J,l}&=q^{n-1}([J-\frac{n}{2}][J+1+\frac{n}{2}]+[n])\phi_{n,J,l},\\ X_-X_+\triangleright\phi_{n,J,l}&=q^{n+1}([J-\frac{n}{2}][J+1+\frac{n}{2}])\phi_{n,J,l}.\end{aligned}\tag{4.22}$$

The spectrum of the Laplacian (4.21) is then given as $\square_{\mathrm{SU}_q(2)}\phi_{n,J,l}=\lambda_{n,J,l}\phi_{n,J,l}$ with:

$$\lambda_{n,J,l}=-q^n\{\nu q[J-\frac{n}{2}][J+1+\frac{n}{2}]+\beta q^{-1}([J-\frac{n}{2}][J+1+\frac{n}{2}]+[n])+\gamma q^{n+2}[n]^2\}\tag{4.23}$$

5 A $\star$-Hodge structure on $\Omega(\mathrm{S}^2_q)$ and a Laplacian operator on $\mathcal{A}(\mathrm{S}^2_q)$

The way the $\star$-Hodge structure (4.20) has been introduced on $\Omega(\mathrm{SU}_q(2))$ comes from the analysis in [14]. The aim of this section is to extend that procedure in order to introduce a $\star$-Hodge structure on $\Omega(\mathrm{S}^2_q)$. The strategy is to directly follow the same path, and to apply to the differential calculus $\Omega(\mathrm{S}^2_q)$ the same procedure, explicitly checking its consistency in the new setting.

5.1 A $\star$-Hodge structure on $\mathcal{A}(\mathrm{S}^2_q)$

The differential calculus on the quantum sphere S^2_q has been described in section 3.4.3 and formalised in proposition 3.7. It is a 2D left covariant calculus: as a volume form consider $\check{\theta}=i\alpha''\omega_-\wedge\omega_+$.

Lemma 5.1. *The 2D calculus $\Omega(\mathrm{S}^2_q)$ formalised in proposition 3.7 is non degenerate.*

Proof. The proof of this lemma is direct. To be definite, consider a 0-form $\eta = f$ with $f \in \mathcal{A}(\mathrm{S}^2_q) \simeq \mathcal{L}^{(0)}_0$, so to have a product $\eta' \wedge \eta = f'(\omega_- \wedge \omega_+)f = f'f\,\omega_- \wedge \omega_+$ from the commutation rules in (3.49), where $\eta' = f'\,\omega_- \wedge \omega_+$ with $f' \in \mathcal{L}^{(0)}_0$. One has $\eta' \wedge \eta = 0 \leftrightarrow f'f = 0$: such a relation is satisfied for any $f' \in \mathcal{L}^{(0)}_0$ iff $f = 0$.

Consider now the 1-form $\eta = x\,\omega_-$ with $x \in \mathcal{L}^{(0)}_{-2}$, so to have a product $\eta' \wedge \eta = (x'\omega_- + y'\omega_+) \wedge x\,\omega_- = -y'x\,\omega_- \wedge \omega_+$ where $(x', y') \in (\mathcal{L}^{(0)}_{-2}, \mathcal{L}^{(0)}_2)$. The relation $\eta' \wedge \eta = 0 \leftrightarrow y'x = 0$ is satisfied for any $y' \in \mathcal{L}^{(0)}_2$ iff $x = 0$. The remaining cases can be analogously analysed, thus proving the claim. □

The restriction of the Haar state h to $\mathcal{A}(\mathrm{S}^2_q)$ yields a faithful, invariant – that is $h(f \triangleleft X) = h(f)\varepsilon(X)$ for $f \in \mathcal{A}(\mathrm{S}^2_q)$ and $X \in \mathcal{U}_q(\mathfrak{su}(2))$ – state on $\mathcal{A}(\mathrm{S}^2_q)$, allowing the definition of an integral $\int_{\check{\theta}} : \Omega(\mathrm{S}^2_q) \mapsto \mathbb{C}$ given by:

$$\begin{aligned}
&\int_{\check{\theta}} f = 0, && \text{on } f \in \mathcal{A}(\mathrm{S}^2_q),\\
&\int_{\check{\theta}} \eta = 0, && \text{on } \eta \in \Omega^1(\mathrm{S}^2_q),\\
&\int_{\check{\theta}} f\,\omega_- \wedge \omega_+ = -i\alpha''^{-1}\,h(f). && (5.1)
\end{aligned}$$

Lemma 5.2. *The integral $\int_{\check{\theta}}$ defined in* (5.1) *is left-faithful.*

Proof. The proof of this result is also direct. Consider, to be definite, the 1-form $\eta = x\omega_-$ with $x \in \mathcal{L}^{(0)}_{-2}$, and a generic $\eta' = x'\omega_- + y'\omega_+ \in \Omega^1(\mathrm{S}^2_q)$. The relation $\int_{\check{\theta}} \eta' \wedge \eta = 0$ for any $\eta' \in \Omega^1(\mathrm{S}^2_q)$ is equivalent to the condition $h(y'x) = 0\ \forall\, y' \in \mathcal{L}^{(0)}_2$. Since this last equality must be valid for any $y' \in \mathcal{L}^{(0)}_2$, choosing $y' = x^*$, it results $h(x^*x) = 0$: the faithfulness of the Haar state h then gives $x = 0$. The claim of the lemma is proved by an analogous analysis on the remaining cases. □

The restriction to $\Omega(\mathrm{S}^2_q)$ of the left invariant graded product (4.17) on $\Omega(\mathrm{SU}_q(2))$, which is the one compatible with the $\star$-Hodge structure, gives a left $\mathcal{A}(\mathrm{S}^2_q)$-invariant graded inner product:

$$\begin{aligned}
&(1,1)_{\mathrm{S}^2_q} = 1 \qquad \rightarrow \qquad (f', f)_{\mathrm{S}^2_q} = h(f^* f');\\
&(x'\omega_- + y'\omega_+, x\,\omega_- + y\,\omega_+)_{\mathrm{S}^2_q} = h(x^*x')(\omega_-, \omega_-)^{\natural}_{\mathrm{SU}_q(2)} + h(y^*y')(\omega_+, \omega_+)^{\natural}_{\mathrm{SU}_q(2)} = h(x^*x')\beta + h(y^*y')\nu;\\
&(\omega_- \wedge \omega_+, \omega_- \wedge \omega_+)_{\mathrm{S}^2_q} = (\omega_- \wedge \omega_+, \omega_- \wedge \omega_+)^{\natural}_{\mathrm{SU}_q(2)} = \alpha'^{-2}\gamma^{-1}, \qquad (5.2)
\end{aligned}$$

with $f, f' \in \mathcal{L}^{(0)}_0$, $x, x' \in \mathcal{L}^{(0)}_{-2}$ and $y, y' \in \mathcal{L}^{(0)}_2$. Recalling proposition 4.1 – namely equation (4.3) – and the results proved in lemmas 5.1 and 5.2, a left $\mathcal{A}(\mathrm{S}^2_q)$-linear Hodge operator $L : \Omega^k(\mathrm{S}^2_q) \mapsto \Omega^{2-k}(\mathrm{S}^2_q)$ can be defined for $k = 0, 2$. From the first line in the inner product relation (5.2) one has $L(1) = \check{\theta}$, while the third gives $L(\check{\theta}) = \alpha''^2\alpha'^{-2}\gamma^{-1}$. It is evident that for such an Hodge operator it is $L^2 \neq 1$, which is a natural requirement for a $\star$-Hodge structure on $\Omega^k(\mathrm{S}^2_q)$ for $k = 0, 2$. On the exterior algebra $\Omega(\mathrm{SU}_q(2))$ this problem was solved by changing the inner product via the definition (4.17), and proving that the new Hodge operator does satisy all the required properties to have a consistent $\star$-Hodge. Following an analogous path, define

$$\begin{aligned}
&(1,1)^{\natural}_{\mathrm{S}^2_q} = 1,\\
&(x'\omega_- + y'\omega_+, x\,\omega_- + y\,\omega_+)^{\natural}_{\mathrm{S}^2_q} = (x'\omega_- + y'\omega_+, x\,\omega_- + y\,\omega_+)_{\mathrm{S}^2_q},\\
&(\check{\theta}, \check{\theta})^{\natural}_{\mathrm{S}^2_q} = (L^{-1}(\check{\theta}), L^{-1}(\check{\theta}))_{\mathrm{S}^2_q} = 1, \qquad (5.3)
\end{aligned}$$

where the inner products on 1-forms amounts to a different labelling of the inner product in (5.2). The Hodge operator on $\Omega^k(\mathrm{S}^2_q)$ for $k = 0, 2$ relative to such a new inner product is given by $L^{\natural}(1) = \check{\theta}$ and $L^{\natural}(\check{\theta}) = 1$. But now the inner product has changed: the requirement that the inner product $(\cdot,\cdot)^{\natural}_{\mathrm{SU}_q(2)}$

on the exterior algebra $\Omega(\mathrm{SU}_q(2))$ fixed – via a restriction, as given in (5.2) – the inner product $(\cdot,\cdot)_{\mathrm{S}^2_q}$ on the exterior algebra $\Omega(\mathrm{S}^2_q)$ implies that the condition

$$(\check{\theta},\check{\theta})^{\natural}_{\mathrm{S}^2_q} = (\check{\theta},\check{\theta})^{\natural}_{\mathrm{SU}_q(2)} \tag{5.4}$$

has to be imposed, giving

$$\alpha''^2\alpha'^{-2}\gamma^{-1} = 1 \tag{5.5}$$

as a constraint among the parameters. The constraint formalised by relation (5.4) can be interpreted as the quantum analogue of fixing the classical metric on the basis S^2 of the Hopf bundle as the contraction of the Cartan-Killing metric on $S^3 \sim SU(2)$, since that choice in the classical formalism, as stressed in remark 2.4, gives the equality of the inner product on $\Omega(S^2)$ defined in (2.71) with the restriction of the inner product on $\Omega(S^3)$ given in (2.41).

The differential calculus on S^2_q is even dimensional with $N = 2$, so on $\Omega^1(\mathrm{S}^2_q)$ define a sesquilinear form:

$$\langle\eta',\eta\rangle = \int_{\check{\theta}} \eta^*\wedge\eta' = i\alpha''^{-1}\{h(y^*y') - q^2h(x^*x')\} \tag{5.6}$$

where $\eta = x\,\omega_- + y\,\omega_+$ and $\eta' = x'\omega_- + y'\omega_+$, with $x,x' \in \mathcal{L}^{(0)}_{-2}$ and $y,y' \in \mathcal{L}^{(0)}_{2}$. The quantum sphere S^2_q is a quantum homogeneous space and not a Hopf algebra, so there is no left-invariant basis in $\Omega^1(\mathrm{S}^2_q)$: neverthless such a sesquilinear form can be "diagonalised", as

$$\begin{aligned}\langle x\,\omega_-, x\,\omega_-\rangle &= -iq^2\alpha''^{-1}\,h(x^*x);\\ \langle y\,\omega_+, y\,\omega_+\rangle &= i\alpha''^{-1}\,h(y^*y),\end{aligned} \tag{5.7}$$

where the faithfulness of the Haar state ensures that the coefficients on the right hand side of these expressions never vanish. The general result from [14] – recalled in (4.8) – is no longer valid on a quantum homogeneous space: the diagonalisation in (5.7) suggests indeed a way to define a Hodge operator. Since α'' can be both positive or negative, define

$$\begin{aligned}x\,\theta_- &= q^{-1}\left(\frac{|\alpha''|}{h(x^*x)}\right)^{1/2} x\,\omega_-,\\ y\,\theta_+ &= \left(\frac{|\alpha''|}{h(y^*y)}\right)^{1/2} y\omega_+\end{aligned} \tag{5.8}$$

so to have from (5.7):

$$\begin{aligned}\langle x\,\theta_-, x\,\theta_-\rangle &= -i\frac{|\alpha''|}{\alpha''},\\ \langle y\,\theta_+, y\,\theta_+\rangle &= i\frac{|\alpha''|}{\alpha''}.\end{aligned} \tag{5.9}$$

In the same way as in (4.8), define a left $\mathcal{A}(\mathrm{S}^2_q)$-linear operator $\mathfrak{L} : \Omega^1(\mathrm{S}^2_q) \mapsto \Omega^1(\mathrm{S}^2_q)$ setting:

$$\begin{aligned}\mathfrak{L}(x\,\theta_-) &= i\frac{|\alpha''|}{\alpha''}x\,\theta_-,\\ \mathfrak{L}(y\,\theta_+) &= -i\frac{|\alpha''|}{\alpha''}y\,\theta_+.\end{aligned} \tag{5.10}$$

Such an operator clearly satisfies the condition $\mathfrak{L}^2 = -1$ for any value of α''. It is not yet a consistent Hodge operator: it has to be compatible with the left invariant inner product on $\Omega^1(\mathrm{S}^2_q)$ obtained in (5.3) as a restriction of the analogue on $\Omega^1(\mathrm{SU}_q(2))$. ¿From the relation (4.10), this compatibility must be imposed:

$$(\eta',\eta)^{\natural}_{\mathrm{S}^2_q} = \int_{\check{\theta}} \eta^*\wedge\mathfrak{L}(\eta'). \tag{5.11}$$

This condition is fulfilled if and only if the parameters in this formulation satisfy:

$$|\alpha''|\beta = q^2, \tag{5.12}$$

$$|\alpha''|\nu = 1. \tag{5.13}$$

The $\star$-Hodge structure on $\Omega(\mathrm{S}^2_q)$ is defined as a left $\mathcal{A}(\mathrm{S}^2_q)$-linear operator whose action is given by:

$$\begin{aligned}
&\star(1) = i\alpha''\,\omega_- \wedge \omega_+,\\
&\star(x\,\omega_-) = i\frac{|\alpha''|}{\alpha''}(x\,\omega_-),\\
&\star(y\,\omega_+) = -i\frac{|\alpha''|}{\alpha''}(y\,\omega_+),\\
&\star(i\omega_- \wedge \omega_+) = \alpha''^{-1},
\end{aligned} \tag{5.14}$$

with the parameters $\alpha', \alpha'', \beta, \nu, \gamma$ satisfying the constraints (5.5), (5.12), (5.13).

Remark 5.3. *The $\star$-Hodge structure* (5.14) *differs from the one in [18], because in that paper the author required the $\star$-Hodge structure to satisfy the relation $\star^2 = 1$, while the path followed here is to remain consistent with the requirement that $\star^2 = (-1)^{k(N-k)}$ on k-forms from a N-dimensional calculus.*

The definition (5.14) of the Hodge duality is still not complete. The constraints among the parameters involve the absolute value of α'', so one still needs to choose their relative signs. In the classical setting the only parameter was $\alpha \in \mathbb{R}$, and it has been chosen positive so to give a riemannian metric g in the analysis of section 2.2. As it is clear from (2.33) and from the definition (2.39), the positivity of the metric implies the positivity of the symmetric form $\langle\cdot,\cdot\rangle_{S^3}$ (2.33) and of the sesquilinear inner product $\langle\cdot,\cdot\rangle^{\sim}_{S^3}$ (2.39): the signature of the metric tensor implies the signature of both the bilinears.

In the quantum setting, having no metric tensor, the choice of the relative signs of the parameters is equivalent to choose the signature of the left-invariant inner product (4.14) on $\Omega^1(\mathrm{SU}_q(2))$: this will encode the formalisation of a specific metric signature.

The natural choice for a riemannian signature is, from (4.14) and (4.18), given by $\beta, \nu, \gamma \in \mathbb{R}_+$. This choice turns out to be compatible with (5.5), (5.12) and (5.13) for every α' and α''. From (5.12) and (5.13) one also has that:

$$\beta = q^2\nu. \tag{5.15}$$

This relation has a number of interesting and important consequences, described in the next propositions.

Proposition 5.4. *The $\star$-Hodge structure given as a left $\mathcal{A}(\mathrm{S}^2_q)$-linear map $\star : \Omega^k(\mathrm{S}^2_q) \mapsto \Omega^{2-k}(\mathrm{S}^2_q)$ for $k = 0, 1, 2$ and defined by* (5.14)*, has the property*[3]

$$\star(\eta) \wedge \eta' = (-1)^{k(2-k)}\eta \wedge \star(\eta') \tag{5.16}$$

for any $\eta, \eta' \in \Omega^k(\mathrm{S}^2_q)$.

Proof. The relation is trivially satisfied for $k = 0, 2$. Consider now the two elements $\eta = x\,\omega_- + y\,\omega_+$ and $\eta' = x'\omega_- + y'\omega_+$ in $\Omega^1(\mathrm{S}^2_q)$, which means $x, x' \in \mathcal{L}^{(0)}_{-2}$ and $y, y' \in \mathcal{L}^{(0)}_{2}$ by proposition 3.7. The multiplication rule formalised in the same proposition gives:

$$\begin{aligned}
&(\star\eta) \wedge \eta' = i\alpha''(\beta\,xy' + \nu\,yx')\omega_- \wedge \omega_+,\\
&\eta \wedge (\star\eta') = -i\alpha''(q^{-2}\beta\,yx' + q^2\nu\,xy')\omega_- \wedge \omega_+.
\end{aligned} \tag{5.17}$$

The two expression are equal – up to the sign, which is the claim of the proposition – from (5.15). □

Proposition 5.5. *The left $\mathcal{A}(\mathrm{S}^2_q)$-linear $\star$-Hodge map defined by* (5.14) *is right $\mathcal{A}(\mathrm{S}^2_q)$-linear: given $\eta \in \Omega(\mathrm{S}^2_q)$, it is $\star(\eta f) = \star(\eta)f$ for any $f \in \mathcal{A}(\mathrm{S}^2_q)$.*

[3] In the classical formalism, the $\star$-Hodge structure on an exterior algebra coming from a N dimensional differential calculus $\star : \Omega^k(\mathcal{H}) \mapsto \Omega^{N-k}(\mathcal{H})$ satisfies the identity (2.32):

$$\eta \wedge (\star\eta') = \eta' \wedge (\star\eta)$$

to which the identity (5.16) reduces in the classical limit.

Proof. The 2D differential calculus on the quantum sphere S^2_q has the specific property, coming from the bimodule structure (3.49) of $\Omega^1(SU_q(2))$ – where one has $\omega_\pm \phi = q^n \phi \omega_\pm$ for any $\phi \in \mathcal{L}^{(0)}_n$ – that $\omega_\pm f = f\omega_\pm$ with $f \in \mathcal{L}^{(0)}_0 \simeq \mathcal{A}(S^2_q)$. The claim of the proposition is trivial for $\eta \in \Omega^0(S^2_q) \simeq \mathcal{A}(S^2_q)$. For a 1-form $\eta = x\omega_- + y\omega_+$ in $\Omega^1(S^2_q)$, one has:

$$\star(\eta f) = \star((x\omega_- + y\omega_+)f) = \star(xf\omega_- + yf\omega_+) = i\alpha''\nu(xf\omega_- - yf\omega_+) = i\alpha''\nu(x\omega_- - y\omega_+)f = \star(\eta)f.$$

An analogue chain of equalities is valid for $\eta = f'\omega_- \wedge \omega_+ \in \Omega^2(S^2_q)$, with $f' \in \mathcal{A}(S^2_q)$. □

In the same way it is possible to prove the following identities, which will be explicitly used in the analysis of the gauged Laplacian operator, and which slightly generalise the last proposition.

Lemma 5.6. *Given the left $\mathcal{A}(S^2_q)$-linear $\star$-Hodge map defined by* (5.14)*, with $\phi \in \mathcal{L}^{(0)}_n$, $\phi' \in \mathcal{L}^{(0)}_{-n}$ and $\eta \in \Omega^1(S^2_q)$ one has:*

$$\star(\phi'\eta\phi) = \phi'(\star\eta)\phi,$$
$$\star(\phi'(\omega_- \wedge \omega_+)\phi) = q^{2n}\phi'\{\star(\omega_- \wedge \omega_+)\}\phi.$$

Proof. With $\phi'\eta\phi \in \Omega^1(S^2_q)$, and again $\eta = x\omega_- + y\omega_+$, it is explicitly:

$$\star(\phi'\eta\phi) = \star(\phi' q^n(y\phi\omega_+ + x\phi\omega_-)) = -iq^n\alpha''\nu\,\phi'(y\phi\omega_+ - x\phi\omega_-) = -i\alpha''\nu\,\phi'(y\omega_+ - x\omega_-)\phi = \phi'(\star\eta)\phi.$$

$$\star(\phi'(\omega_- \wedge \omega_+)\phi) = q^{2n} \star(\phi'\phi(\omega_- \wedge \omega_+)) = q^{2n}\phi'\phi \star (\omega_- \wedge \omega_+) = q^{2n}\phi'\{\star(\omega_- \wedge \omega_+)\}\phi,$$

where the last equality is evident, since $\star(\omega_- \wedge \omega_+) \in \mathbb{C}$. □

5.2 A Laplacian operator on $\mathcal{A}(S^2_q)$

Using the 2D differential calculus on the Podles sphere S^2_q and the $\star$-Hodge structure on $\Omega(S^2_q)$ it is natural to define a Laplacian operator $\square_{S^2_q} : \mathcal{A}(S^2_q) \mapsto \mathcal{A}(S^2_q)$ as $\square_{S^2_q} f = \star d \star df$ on any $f \in \mathcal{A}(S^2_q)$. An explicit computation using the formalisation of the exterior algebra $\Omega(S^2_q)$ represented in proposition 3.7 gives:

$$\begin{aligned} df &= (X_+ \triangleright f)\omega_+ + (X_- \triangleright f)\omega_-, \\ \star df &= -i\alpha''[\nu(X_+ \triangleright f)\omega_+ - q^{-2}\beta(X_- \triangleright f)\omega_-], \\ d \star df &= -i\alpha''[\nu X_- X_+ + \beta X_+ X_-] \triangleright f\,(\omega_- \wedge \omega_+), \\ \star d \star df &= -[\nu X_- X_+ + \beta X_+ X_-] \triangleright f. \end{aligned} \tag{5.18}$$

The relation (3.31) shows that such a Laplacian operator can be seen as an operator $\square_{S^2_q} : \mathcal{L}^{(0)}_0 \mapsto \mathcal{L}^{(0)}_0$. In particular, from (4.21), the Laplacian $\square_{S^2_q}$ is the restriction of the Laplacian $\square_{SU_q(2)}$ to the subalgebra $\mathcal{A}(S^2_q) \subset \mathcal{A}(SU_q(2))$. A basis of the eigenvector spaces $\mathcal{L}^{(0)}_0 = \oplus_{J\in\mathbb{N}} V^{(0)}_J$ coming from (3.41) is given by elements $\phi_{0,J,l} = c^J a^{*J} \triangleleft E^l = w_{2J:J,l}$, so that formulas (4.22) drive to a spectrum of this Laplacian on S^2_q as:

$$\begin{aligned} \square_{S^2_q}\phi_{0,J,l} &= -(q\nu + q^{-1}\beta)\{[J][J+1]\}\phi_{0,J,l} \\ &= -2q\nu\{[J][J+1]\}\phi_{0,J,l} \end{aligned} \tag{5.19}$$

Remark 5.7. *Equations* (4.21) *and* (5.18) *show that the classical relations between the Laplacians $\square_{SU(2)}$ and $\square_{S^2}$, coming from the Hodge duality associated to the metric tensor g* (2.29) *related to the Cartan-Killing metric, is then reproduced in the quantum formalism, in the specific realisation of the quantum Hopf bundle that has been described. The constraints among the 5 real parameters used in the analysis of the Hodge duality can be written as:*

$$\begin{aligned} \gamma &= \alpha''^2\alpha'^{-2}, \\ \nu &= |\alpha''|^{-1}, \\ \beta &= q^2\nu. \end{aligned} \tag{5.20}$$

The parameters α', α'' are the coefficients of the volume forms. The analysis of the classical limit of this formalisation is in section 8. The choice:

$$\lim_{q\to 1} \alpha' = -4\alpha,$$
$$\lim_{q\to 1} \alpha'' = -2\alpha \tag{5.21}$$

gives (4.21) *and* (5.18) *in the classical limit. Being α a positive real number, it seems natural to assume α' and α'' negative real numbers. This also gives $\nu = -\alpha''^{-1}$ from the second relation in* (5.20)*, so to have a Hodge duality* (5.14) *which is now:*

$$\star\,(1) = \check{\theta} = i\alpha''\,\omega_- \wedge \omega_+,$$
$$\star\,(x\,\omega_-) = -ix\,\omega_-,$$
$$\star\,(y\,\omega_+) = iy\,\omega_+,$$
$$\star\,(i\omega_- \wedge \omega_+) = \alpha''^{-1}, \tag{5.22}$$

giving, if (5.21) *is satisfied, the classical Hodge duality* (2.63) *in the classical limit.*

6 Connections on the Hopf bundle

The structure of a quantum principal bundle $(\mathcal{P}, \mathcal{B}, \mathcal{H}; \mathcal{N}_{\mathcal{P}}, \mathcal{Q}_{\mathcal{H}})$ with compatible differential calculi, given the total space algebra $\mathcal{P}$ and the gauge group Hopf algebra $\mathcal{H}$, has been described in section 3.2. The compatibility conditions ensure the exactness of the sequence (3.17):

$$0 \;\to\; \mathcal{P}\Omega^1(\mathcal{B})\mathcal{P} \;\to\; \Omega_1(\mathcal{P}) \;\stackrel{\sim_{\mathcal{N}_{\mathcal{P}}}}{\longrightarrow}\; \mathcal{P}\otimes(\ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}) \;\to\; 0. \tag{6.1}$$

with the map $\sim_{\mathcal{N}_{\mathcal{P}}}$ defined via the commutative diagram (3.15). Among the compatibility conditions, the requirement that $\Delta_R\mathcal{N}_{\mathcal{P}} \subset \mathcal{N}_{\mathcal{P}}\otimes\mathcal{H}$ – formalising a right covariance of the differential structure on $\mathcal{P}$ – allows to extend the coaction Δ_R of $\mathcal{H}$ on $\mathcal{P}$ to a coaction of $\mathcal{H}$ on 1-forms, $\Delta_R^{(1)} : \Omega^1(\mathcal{P}) \mapsto \Omega^1(\mathcal{P})\otimes\mathcal{H}$, defining $\Delta_R^{(1)} \circ \mathrm{d} = (\mathrm{d}\otimes 1)\circ\Delta_R$.

Note that $\mathrm{Ad}(\ker\varepsilon_{\mathcal{H}}) \subset \ker\varepsilon_{\mathcal{H}}\otimes\mathcal{Q}_{\mathcal{H}}$. If the right ideal $\mathcal{Q}_{\mathcal{H}}$ is Ad-invariant (which is equivalent to say that the differential calculus on $\mathcal{H}$ is bicovariant), then it is possible to define a right-adjoint coaction $\mathrm{Ad}^{(R)} : \ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}} \;\mapsto\; \ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}\otimes\mathcal{H}$ by the commutative diagram

$$\begin{array}{ccc} \ker\varepsilon_{\mathcal{H}} & \stackrel{\pi_{\mathcal{Q}_{\mathcal{H}}}}{\longrightarrow} & \ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}} \\ \downarrow \mathrm{Ad} & & \downarrow \mathrm{Ad}^{(R)} \\ \ker\varepsilon_{\mathcal{H}}\otimes\mathcal{H} & \stackrel{\pi_{\mathcal{Q}_{\mathcal{H}}}\otimes\mathrm{id}}{\longrightarrow} & (\ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}})\otimes\mathcal{H} \end{array}$$

Together with the right coaction Δ_R of $\mathcal{H}$ on $\mathcal{P}$, such a right-adjoint coaction $\mathrm{Ad}^{(R)}$ allows to define a right coaction $\Delta_R^{(\mathrm{Ad})}$ of $\mathcal{H}$ on $\mathcal{P}\otimes\ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}$ as a coaction of a Hopf algebra on the tensor product of its comodules. This coaction is explicitly given by the relation:

$$\Delta_R^{(\mathrm{Ad})}(p\otimes\pi_{\mathcal{Q}_{\mathcal{H}}}(h)) = p_{(1)}\otimes\pi_{\mathcal{Q}_{\mathcal{H}}}(h_{(2)})\otimes p_{(2)}(Sh_{(1)})h_{(3)}, \tag{6.2}$$

adopting the Sweedler notation as $\Delta_R(p) = p_{(1)}\otimes p_{(2)}$.

It is now possible to define a connection on the quantum principal bundle as a right invariant splitting of the sequence (6.1). Given a left $\mathcal{P}$-linear map $\sigma : \mathcal{P}\otimes(\ker\varepsilon_{\mathcal{H}}/\mathcal{Q}_{\mathcal{H}}) \mapsto \Omega^1(\mathcal{P})$ such that

$$\Delta_R^{(1)}\circ\sigma = (\sigma\otimes id)\Delta_R^{(\mathrm{Ad})},$$
$$\sim_{\mathcal{N}_{\mathcal{P}}}\circ\sigma = id, \tag{6.3}$$

then the map $\Pi : \Omega^1(\mathcal{P}) \mapsto \Omega^1(\mathcal{P})$ defined by $\Pi = \sigma \circ \sim_{\mathcal{N}_\mathcal{P}}$ is a right invariant left $\mathcal{P}$-linear projection, whose kernel coincides with the horizontal forms $\mathcal{P}\Omega^1(\mathcal{B})\mathcal{P}$:

$$\begin{aligned} \Pi^2 &= \Pi, \\ \Pi(\mathcal{P}\Omega^1(\mathcal{B})\mathcal{P}) &= 0, \\ \Delta_R^{(1)} \circ \Pi &= (\Pi \otimes id) \circ \Delta_R^{(1)}. \end{aligned} \tag{6.4}$$

The image of the projection Π is the set of vertical 1-forms of the principal bundle. A connection on a principal bundle can also be formalised via a connection one form, which is a map $\omega : \mathcal{H} \mapsto \Omega^1(\mathcal{P})$. Given a right invariant splitting σ of the exact sequence (6.1), define the connection 1-form as $\omega(h) = \sigma(1 \otimes \pi_{\mathcal{Q}_\mathcal{H}}(h - \varepsilon_\mathcal{H}(h)))$ on $h \in \mathcal{H}$. Such a connection 1-form has the following properties:

$$\begin{aligned} \omega(\mathcal{Q}_\mathcal{H}) &= 0, \\ \sim_{\mathcal{N}_\mathcal{P}} (\omega_h) &= 1 \otimes \pi_{\mathcal{Q}_\mathcal{H}}(h - \varepsilon_\mathcal{H}(h)) \qquad \forall\, h \in \mathcal{H}, \\ \Delta_R^{(1)} \circ \omega &= (\omega \otimes \mathrm{id}) \circ \mathrm{Ad}, \\ \Pi(\mathrm{d}p) &= (id \otimes \omega)\Delta_R(p) \qquad \forall\, p \in \mathcal{P}. \end{aligned} \tag{6.5}$$

Conversely if ω is a linear map $\ker \varepsilon_\mathcal{H} \mapsto \Omega^1(\mathcal{P})$ that satisfies the first three conditions in (6.5), then there exists a unique connection on the principal bundle, such that ω is its connection 1-form. In this case, the splitting of the sequence (6.1) is given by:

$$\sigma(p \otimes [h]) = p\omega([h]) \tag{6.6}$$

with $[h]$ in $\ker \varepsilon_\mathcal{H}/\mathcal{Q}_\mathcal{H}$, while the projection Π is given by:

$$\Pi = m \circ (id \otimes \omega) \circ \sim_{\mathcal{N}_\mathcal{P}} \tag{6.7}$$

The general proof of these results is in [4]. This section explicitly describes the connections on the quantum Hopf bundle with the compatible differential calculi presented in sections 3.4.1 and 3.4.2.

6.1 Vertical subspaces on the quantum Hopf bundle

The right coaction $\Delta_R^{(1)} : \Omega^1(\mathrm{SU}_q(2)) \mapsto \Omega^1(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1))$ of the gauge group algebra $\mathcal{A}(\mathrm{U}(1))$ on the set of 1-forms on the total space algebra of the bundle, whose consistency is allowed by the compatibility conditions between the 3D left covariant calculus on $\mathcal{A}(\mathrm{SU}_q(2))$ and the 1D bicovariant calculus on $\mathcal{A}(\mathrm{U}(1))$, gives:

$$\begin{aligned} \Delta_R^{(1)} \omega_z &= \omega_z \otimes 1, \\ \Delta_R^{(1)} \omega_\pm &= \omega_\pm \otimes z^{\pm 2}. \end{aligned} \tag{6.8}$$

¿From the analysis on the 1D calculus on $\mathcal{A}(\mathrm{U}(1))$ performed in section 3.4.4 and the result of lemma 3.5, a connection on the quantum Hopf bundle is formalised via a splitting map $\sigma : \mathcal{A}(\mathrm{SU}_q(2)) \otimes \ker \varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)} \mapsto \Omega^1(\mathrm{SU}_q(2))$, which can be defined recalling the isomorphism $\tilde{\lambda} : \ker \varepsilon_{\mathrm{U}(1)}/\mathcal{Q}_{\mathrm{U}(1)} \mapsto \mathbb{C}$. Given $w \in \mathbb{C}$ set:

$$\sigma(1 \otimes w) = \sigma(w \otimes 1) = w(\omega_z + U\omega_+ + V\omega_-); \tag{6.9}$$

and extend by the requirement of left $\mathcal{A}(\mathrm{SU}_q(2))$-linearity, so to have:

$$\begin{aligned} \sigma(1 \otimes [\varphi(j)]) &= q^{-2j}(\omega_z + U\omega_+ + V\omega_-), \\ \sigma(\phi \otimes [\varphi(j)]) &= q^{-2j}\phi(\omega_z + U\omega_+ + V\omega_-), \end{aligned} \tag{6.10}$$

where $\phi \in \mathcal{A}(\mathrm{SU}_q(2))$ and the requirement of right covariance (6.3) selects – from (6.8) – $U \in \mathcal{L}_2^{(0)}$ and $V \in \mathcal{L}_{-2}^{(0)}$. The projection Π associated to this connection is easily seen to be:

$$\begin{aligned} \Pi(\omega_\pm) &= \sigma(\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}} (\omega_\pm)) = 0, \\ \Pi(\omega_z) &= \sigma(\sim_{\mathcal{N}_{\mathrm{SU}_q(2)}} (\omega_z)) = \sigma(1 \otimes [\varphi(0)]) = \omega_z + U\omega_+ + V\omega_-. \end{aligned} \tag{6.11}$$

In this expression the 1-forms $\omega_\pm$ are recovered as horizontal (3.61), a notion depending only on the compatibility conditions between the differential calculi, while a choice of a connection is equivalent to the choice of the vertical part of $\Omega^1(\mathrm{SU}_q(2))$. The set of connections for the quantum Hopf bundle corresponds to the set of the possible choices of 1-forms on the basis of the bundle as $\mathrm{a} = U\omega_+ + V\omega_- \in \Omega^1(\mathrm{S}^2_q)$, so that the second line in (6.11) can be written as

$$\Pi(\omega_z) = \omega_z + \mathrm{a}. \tag{6.12}$$

The connection one form (6.5) $\omega : U(1) \mapsto \Omega^1(\mathrm{SU}_q(2))$ is given by:

$$\begin{aligned}\omega(z^j) &= \sigma(1 \otimes [z^j - 1]) \\ &= \left(\frac{1-q^{-2j}}{1-q^{-2}}\right)(\omega_z + U\omega_+ + V\omega_-) = \left(\frac{1-q^{-2j}}{1-q^{-2}}\right)(\omega_z + \mathrm{a}).\end{aligned} \tag{6.13}$$

Given the projection Π and the connection 1-form ω, it is possible to compute the lhs and the rhs of the last line in (6.5). On the basis of left invariant differential forms and using the explicit form of the quantum vector fields in (3.45), with $\phi \in \mathcal{L}^{(0)}_j$ one has:

$$\begin{aligned}\Pi(\mathrm{d}\phi) &= \Pi((X_j \triangleright \phi)\omega_j) = (X_j \triangleright \phi)\Pi(\omega_j) \\ &= (X_z \triangleright \phi)\Pi(\omega_z) = \left(\frac{1-q^{2j}}{1-q^{-2}}\right)\phi(\omega_z + U\omega_+ + V\omega_-);\end{aligned} \tag{6.14}$$

and also:

$$\begin{aligned}(id \otimes \omega)\Delta_R(\phi) &= (id \otimes \omega)(\phi \otimes z^{-j}) \\ &= \left(\frac{1-q^{2j}}{1-q^{-2}}\right)\phi(\omega_z + U\omega_+ + V\omega_-) = \Pi(\mathrm{d}\phi).\end{aligned} \tag{6.15}$$

The monopole connection corresponds to the choice $U = V = 0 \leftrightarrow \mathrm{a} = 0$, so to have $\Pi_0(\omega_z) = \omega_z$ and the monopole connection 1-form $\omega_0(z^j) = [(1-q^{-2j})/(1-q^{-2})]\omega_z$. With a connection, one has the notion of covariant derivative $D : \mathcal{A}(\mathrm{SU}_q(2)) \mapsto \Omega^1(\mathcal{A}(\mathrm{SU}_q(2)))$ of equivariant maps. Given $\phi \in \mathcal{L}^{(0)}_n$, define

$$D\phi = (1 - \Pi)\mathrm{d}\phi. \tag{6.16}$$

The covariant derivative $D\phi$ is clearly an horizontal 1-form: the adjective "covariant" refers to the behaviour under the coaction of the gauge group algebra, as one directly (3.32) shows that:

$$\Delta_R\phi = \phi \otimes z^{-j} \qquad \leftrightarrow \qquad \Delta^{(1)}_R(D\phi) = D\phi \otimes z^{-j}, \tag{6.17}$$

from the right invariance (6.4) of the projection Π. In terms of the connection 1-form the covariant derivative can be written, using (6.15), as :

$$\begin{aligned}D\phi = (1 - \Pi)\mathrm{d}\phi &= \mathrm{d}\phi - \Pi(\mathrm{d}\phi) \\ &= \mathrm{d}\phi - \phi \wedge \omega(z^{-j})\end{aligned} \tag{6.18}$$

on a $\phi \in \mathcal{L}^{(0)}_j$. It is then immediate to recover that, for any $f \in \mathcal{L}^{(0)}_0 \simeq \mathcal{A}(\mathrm{S}^2_q)$, one has $Df = \mathrm{d}f$.

Remark 6.1. *Given any $\phi \in \mathcal{L}^{(0)}_n$, from* (6.18) *and* (6.12)*, the covariant derivative can be written as:*

$$D\phi = \{(X_+ \triangleright \phi) - (X_z \triangleright \phi)U\}\omega_+ + \{(X_- \triangleright \phi) - (X_z \triangleright \phi)V\}\omega_-.$$

It is an easy computation using the $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule properties (3.49) *of $\Omega^1(\mathrm{SU}_q(2))$ to prove that $D\phi \simeq \Omega^1(\mathrm{S}^2_q) \cdot \mathcal{A}(\mathrm{SU}_q(2))$ for any connection represented by $\mathrm{a} \in \Omega^1(\mathrm{S}^2_q)$. This means that any connection on this quantum Hopf bundle is a strong connection, following the analysis in [11].*

6.2 Covariant derivative on the associated line bundles

A covariant derivative, or a connection, on the left $\mathcal{A}(\mathrm{S}^2_q)$-module $\mathcal{E}^{(0)}_n$ is a $\mathbb{C}$-linear map

$$\nabla : \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n \mapsto \Omega^{k+1}(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n, \tag{6.19}$$

defined for any $k \geq 0$ and satisfying a left Leibniz rule:

$$\nabla(\alpha \langle\sigma|) = \mathrm{d}\alpha \wedge \langle\sigma| + (-1)^m \alpha \wedge (\nabla \langle\sigma|)$$

for any $\alpha \in \Omega^m(\mathrm{S}^2_q)$ and $\langle\sigma| \in \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n$. A connection is completely determined by its restriction $\nabla : \mathcal{E}^{(0)}_n \mapsto \Omega^1(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n$ and then extended by the Leibniz rule. Connections always exist on projective modules: the canonical (Levi-Civita, or Grassmann) connection on a left projective $\mathcal{A}(\mathrm{S}^2_q)$-module $\mathcal{E}^{(0)}_n$ is given as

$$\nabla_0 \langle\sigma| = (\mathrm{d}\langle\sigma|)\mathfrak{p}^{(n)}; \tag{6.20}$$

the space $C(\mathcal{E}^{(0)}_n)$ of all connections on $\mathcal{E}^{(0)}_n$ is an affine space modelled on $\mathrm{Hom}_{\mathcal{A}(\mathrm{S}^2_q)}(\mathcal{E}^{(0)}_n, \mathcal{E}^{(0)}_n \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \Omega^1(\mathrm{S}^2_q))$, so that any connection can be written as:

$$\nabla \langle\sigma| = (\mathrm{d}\langle\sigma|)\mathfrak{p}^{(n)} + (-1)^k \langle\sigma| \wedge \mathrm{A}^{(\mathrm{n})} \tag{6.21}$$

with $\langle\sigma| \in \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n$ and $\mathrm{A}^{(\mathrm{n})} \in \mathbb{M}_{|n|+1} \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \Omega^1(\mathrm{S}^2_q)$ – which is called the gauge potential of the connection ∇ – subject to the condition $\mathrm{A}^{(\mathrm{n})} = \mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)} = \mathfrak{p}^{(n)}\mathrm{A}^{(\mathrm{n})} = \mathfrak{p}^{(n)}\mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)}$. The composition

$$\nabla^2 = \nabla \circ \nabla : \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n \mapsto \Omega^{k+2}(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n$$

is $\Omega(\mathrm{S}^2_q)$-linear. This map can be explicitly calculated: given $\langle\sigma| \in \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n$, from (6.21) one has

$$\begin{aligned}
\nabla^2 \langle\sigma| &= \mathrm{d}(\nabla\langle\sigma|)\mathfrak{p}^{(n)} + (-1)^{k+1}(\nabla\langle\sigma|) \wedge \mathrm{A}^{(\mathrm{n})} \\
&= \mathrm{d}\{(\mathrm{d}\sigma)\mathfrak{p}^{(n)} + (-1)^k \langle\sigma| \wedge \mathrm{A}^{(\mathrm{n})}\}\mathfrak{p}^{(n)} + (-1)^{k+1}\{(\mathrm{d}\langle\sigma|)\mathfrak{p}^{(n)} + (-1)^k \langle\sigma| \wedge \mathrm{A}^{(\mathrm{n})}\} \wedge \mathrm{A}^{(\mathrm{n})} \\
&= \mathrm{d}\{(\mathrm{d}\langle\sigma|)\mathfrak{p}^{(n)}\}\mathfrak{p}^{(n)} + (-1)^k(\mathrm{d}\langle\sigma| \wedge \mathrm{A}^{(\mathrm{n})})\mathfrak{p}^{(n)} + (\langle\sigma| \wedge \mathrm{d}\mathrm{A}^{(\mathrm{n})})\mathfrak{p}^{(n)} \\
&\qquad + (-1)^{k+1}\{(\mathrm{d}\langle\sigma|)\mathfrak{p}^{(n)} \wedge \mathrm{A}^{(\mathrm{n})}\} - \langle\sigma| \wedge \mathrm{A}^{(\mathrm{n})} \wedge \mathrm{A}^{(\mathrm{n})} \\
&= \langle\sigma| \{-(\mathrm{d}\mathfrak{p}^{(n)} \wedge \mathrm{d}\mathfrak{p}^{(n)})\mathfrak{p}^{(n)} + (\mathrm{d}\mathrm{A}^{(\mathrm{n})})\mathfrak{p}^{(n)} - \mathrm{A}^{(\mathrm{n})} \wedge \mathrm{A}^{(\mathrm{n})}\}.
\end{aligned} \tag{6.22}$$

The restriction of the map ∇^2 to $\mathcal{E}^{(0)}_n$, seen as an element in $\Omega^2(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n$, is the curvature F_∇ of the given connection.

In order to relate the concept of connection on the quantum Hopf bundle to that of covariant derivative on the associated line bundles, one extends the left $\mathcal{A}(\mathrm{S}^2_q)$-module isomorphism between $\mathcal{L}^{(0)}_n$ and $\mathcal{E}^{(0)}_n$ described in proposition 3.4. As first step, define the $\mathcal{A}(\mathrm{S}^2_q)$-bimodule:

$$\mathcal{L}^{(1)}_n = \{\phi \in \Omega^1_{\mathrm{hor}}(\mathrm{SU}_q(2)) \simeq \mathcal{A}(\mathrm{SU}_q(2))\Omega^1(\mathrm{S}^2_q)\mathcal{A}(\mathrm{SU}_q(2)) \, : \, \Delta^{(1)}_R \phi = \phi \otimes z^{-n}\} \tag{6.23}$$

and introduce the notations:

$$\mathcal{E}^{(k)}_n = \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}^{(0)}_n.$$

The maps:

$$\begin{aligned}
\mathcal{L}^{(1)}_n \xrightarrow{\simeq} \mathcal{E}^{(1)}_n : &\qquad \phi \mapsto \langle\sigma|_\phi = \phi \left\langle \Psi^{(n)} \right|, \\
\mathcal{E}^{(1)}_n \xrightarrow{\simeq} \mathcal{L}^{(1)}_n : &\qquad \langle\sigma| \mapsto \phi = \left\langle \sigma, \Psi^{(n)} \right\rangle
\end{aligned} \tag{6.24}$$

formalise a left $\mathcal{A}(\mathrm{S}^2_q)$-module isomorphism (in this notation the explicit dependence on $\langle f| \in \mathcal{A}(\mathrm{S}^2_q)^{|n|+1}$ as in proposition 3.4 has been dropped). Via this isomorphism, any connection on the quantum Hopf bundle – represented by a projection Π (6.11) or by a connection 1-form (6.13) – induces a gauge potential $\mathrm{A}^{(\mathrm{n})}$ on any associated line bundle $\mathcal{E}^{(0)}_n$.

Proposition 6.2. *Given the left $\mathcal{A}(\mathrm{S}^2_q)$-isomorphism $\mathcal{L}^{(0)}_n \simeq \mathcal{E}^{(0)}_n$ described in proposition 3.4, as well as the analogue left $\mathcal{A}(\mathrm{S}^2_q)$-module isomorphism $\mathcal{L}^{(1)}_n \simeq \mathcal{E}^{(1)}_n$ described in* (6.24)*, there is an equivalence between the set of connections on the quantum Hopf bundle formalise through a projection Π in $\Omega(\mathrm{SU}_q(2))$ as in* (6.11)*, and the set of covariant derivative $\nabla \in C(\mathcal{E}^{(0)}_n)$ an any associated line bundle. With $\phi \in \mathcal{L}^{(0)}_n$ so that $\langle\sigma_\phi| = \phi\left\langle\Psi^{(n)}\right| \in \mathcal{E}^{(0)}_n$, the equivalence is given by $D\phi = (\nabla\langle\sigma|_\phi)\left|\Psi^{(n)}\right\rangle$.*

Proof. Choose $\phi \in \mathcal{L}^{(0)}_n$, so to have $\sigma_\phi = \phi\left\langle\Psi^{(n)}\right|$ and from the definition in (6.21) express a covariant derivative on $\mathcal{E}^{(0)}_n$ via a gauge potential as:

$$\nabla\langle\sigma|_\phi = \mathrm{d}\left(\phi\left\langle\Psi^{(n)}\right|\right)\left|\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)}\right| + \phi\left\langle\Psi^{(n)}\right|\mathrm{A}^{(\mathrm{n})} \tag{6.25}$$

$$= \{\mathrm{d}\phi - \phi[\left\langle\Psi^{(n)}, \mathrm{d}\Psi^{(n)}\right\rangle - \left\langle\Psi^{(n)}\right|\mathrm{A}^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle]\}\left\langle\Psi^{(n)}\right| \tag{6.26}$$

since $\mathrm{A}^{(\mathrm{n})} = \mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)}$. On the other hand, being $\phi \in \mathcal{L}^{(0)}_n$ one has:

$$\begin{aligned} D\phi &= (1-\Pi)\mathrm{d}\phi = \mathrm{d}\phi - (X_z \triangleright \phi)\Pi(\omega_z) \\ &= \mathrm{d}\phi - \left(\frac{1-q^{2n}}{1-q^{-2}}\right)\phi\,\Pi(\omega_z), \end{aligned}$$

with $D\phi \in \mathcal{L}^{(1)}_n$ from (6.17). By the isomorphism (6.24), equating $D\phi = (\nabla\langle\sigma|_\phi)\left|\Psi^{(n)}\right\rangle$ defines the gauge potential $\mathrm{A}^{(\mathrm{n})}$ as:

$$\begin{aligned} \left\langle\Psi^{(n)}, \mathrm{d}\Psi^{(n)}\right\rangle - \left\langle\Psi^{(n)}\right|\mathrm{A}^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle &= \frac{1-q^{2n}}{1-q^{-2}}(\omega_z + U\omega_+ + V\omega_-) \\ &= \frac{1-q^{2n}}{1-q^{-2}}(\omega_z + \mathrm{a}) = \omega(z^{-n}) : \end{aligned} \tag{6.27}$$

an explicit calculation shows that $\left\langle\Psi^{(n)}, \mathrm{d}\Psi^{(n)}\right\rangle = [(1-q^{2n})/(1-q^{-2})]\omega_z$, so the previous expression becomes:

$$\left\langle\Psi^{(n)}\right|\mathrm{A}^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle = -\frac{1-q^{2n}}{1-q^{-2}}(U\omega_+ + V\omega_-), \tag{6.28}$$

which is solved by

$$\begin{aligned} \mathrm{A}^{(\mathrm{n})} &= -\frac{1-q^{2n}}{1-q^{-2}}\left|\Psi^{(n)}\right\rangle(U\omega_+ + V\omega_-)\left\langle\Psi^{(n)}\right| \\ &= -\frac{1-q^{2n}}{1-q^{-2}}\left|\Psi^{(n)}\right\rangle \mathrm{a}\left\langle\Psi^{(n)}\right|. \end{aligned} \tag{6.29}$$

This solution is unique. Being the set of connection an affine space, any different gauge potential, solution of equation (6.28), should be $\check{\mathrm{A}}^{(n)} = \mathrm{A}^{(\mathrm{n})} + \mathrm{A}'^{(\mathrm{n})}$ where $\mathrm{A}^{(\mathrm{n})}$ is given in (6.29) and $\mathrm{A}'^{(\mathrm{n})}$ must satisfy $\left\langle\Psi^{(n)}\right|\mathrm{A}'^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle = 0$, with $\mathrm{A}'^{(\mathrm{n})} = \mathfrak{p}^{(n)}\mathrm{A}'^{(\mathrm{n})}\mathfrak{p}^{(n)} = \mathfrak{p}^{(n)}\mathrm{A}'^{(\mathrm{n})} = \mathrm{A}'^{(\mathrm{n})}\mathfrak{p}^{(n)} = \mathrm{A}'^{(\mathrm{n})}$. One directly has:

$$\begin{aligned} &\left\langle\Psi^{(n)}\right|\mathrm{A}'^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle = 0 \\ &\to\ 0 = \left|\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)}\right|\mathrm{A}'^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)}\right| = \mathfrak{p}^{(n)}\mathrm{A}'^{(\mathrm{n})}\mathfrak{p}^{(n)} = \mathrm{A}'^{(\mathrm{n})}. \end{aligned}$$

The complete equivalence claimed in the proposition comes by (6.28), which gives for any gauge potential $\mathrm{A}^{(\mathrm{n})}$ a 1-form $\mathrm{a} \in \Omega^1(\mathrm{S}^2_q)$, suitable to define a connection as in (6.12). □

The form of the gauge potential (6.29) shows that the monopole connection $\Pi_0(\omega_z) = \omega_z$ corresponds to the Grassmann, or canonical covariant derivative $\nabla_0\langle\sigma| = (\mathrm{d}\langle\sigma|)\mathfrak{p}^{(n)}$ on the line bundles $\mathcal{E}^{(0)}_n$, having $\mathrm{A}^{(\mathrm{n})} = 0$ for any $n \in \mathbb{Z}$.

The compatibility between the differential calculi allows to extend the concept of right coaction of the gauge group algebra on the whole exterior algebra $\Omega(\mathrm{SU}_q(2))$, introducing a right coaction $\Delta^{(k)}_R$:

$\Omega^k(\mathrm{SU}_q(2)) \mapsto \Omega^k(\mathrm{SU}_q(2)) \otimes \mathcal{A}(\mathrm{U}(1))$ by induction as $\Delta_R^{(k)} \circ \mathrm{d} = (\mathrm{d} \otimes \mathrm{id}) \circ \Delta_R^{(k-1)}$. It becomes now natural to define the $\mathcal{A}(\mathrm{S}^2_q)$-bimodule:

$$\mathcal{L}_n^{(2)} = \{\phi \in \mathcal{A}(\mathrm{SU}_q(2))\Omega^2(\mathrm{S}^2_q)\mathcal{A}(\mathrm{SU}_q(2)) \, : \, \Delta_R^{(2)}\phi = \phi \otimes z^{-n}\}; \tag{6.30}$$

so that the maps:

$$\begin{aligned}
\mathcal{L}_n^{(2)} \xrightarrow{\simeq} \mathcal{E}_n^{(2)} : \qquad & \phi \mapsto \langle\sigma|_\phi = \phi \left\langle \Psi^{(n)} \right|, \\
\mathcal{E}_n^{(2)} \xrightarrow{\simeq} \mathcal{L}_n^{(2)} : \qquad & \langle\sigma| \mapsto \phi = \left\langle \sigma, \Psi^{(n)} \right\rangle
\end{aligned} \tag{6.31}$$

formalise a left $\mathcal{A}(\mathrm{S}^2_q)$-module isomorphism, generalising the isomorphisms given in proposition 3.4 and in (6.24). In the formulation of [4], the elements in $\mathcal{L}_n^{(k)}$ are called tensorial forms.

Recall that the covariant derivative ∇ is defined in (6.19) as an operator $\nabla : \mathcal{E}_n^{(k)} \mapsto \mathcal{E}_n^{(k+1)}$ for $k = 0, 1, 2$, since the differential calculus on $\mathcal{A}(\mathrm{S}^2_q)$ is 2 dimensional; the covariant derivative D has been defined by (6.16) only on the $\mathcal{A}(\mathrm{S}^2_q)$-bimodule $\mathcal{L}_n^{(0)}$, while the proposition 6.2 shows the equivalence between $D : \mathcal{L}_n^{(0)} \mapsto \mathcal{L}_n^{(1)}$ and $\nabla : \mathcal{E}_n^{(0)} \mapsto \mathcal{E}_n^{(1)}$. The isomorphism (6.31) allows then to extend the covariant derivative to $D : \mathcal{L}_n^{(1)} \mapsto \mathcal{L}_n^{(2)}$, defining:

$$D\phi = (\nabla \langle\sigma|_\phi) \left|\Psi^{(n)}\right\rangle \tag{6.32}$$

for any $\phi \in \mathcal{L}_n^{(1)}$ with $\langle\sigma|_\phi = \phi \left\langle\Psi^{(n)}\right| \in \mathcal{E}_n^{(1)} = \Omega^1(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}_n^{(0)}$. Such an operator can be represented in terms of the connection (6.13) 1-form ω. From the Leibniz rule one has:

$$\mathrm{d}(\phi \left\langle \Psi^{(n)} \right|) = (\mathrm{d}\phi) \left\langle \Psi^{(n)} \right| + (-1)^k \phi \, \mathrm{d} \left\langle \Psi^{(n)} \right|,$$

with $\phi \in \mathcal{L}_n^{(k)}$. This identity gives the next proposition.

Proposition 6.3. *Given* $\phi \in \mathcal{L}_n^{(1)}$ *, so that* $\langle\sigma|_\phi = \phi \left\langle\Psi^{(n)}\right| \in \mathcal{E}_n^{(1)}$*, the action of the operator* $D : \mathcal{L}_n^{(1)} \mapsto \mathcal{L}_n^{(2)}$ *defined by* (6.32) *can be written as:*

$$D\phi = \mathrm{d}\phi + \phi \wedge \omega(z^{-n}) \tag{6.33}$$

Proof. The proposition is proved by a direct computation. Start from $\phi \in \mathcal{L}_n^{(1)}$, so that from(6.21) one has $\nabla \langle\sigma|_\phi = (\mathrm{d} \langle\sigma|_\phi)\mathfrak{p}^{(n)} - \langle\sigma|_\phi \wedge \mathrm{A}^{(\mathrm{n})}$, so that :

$$\begin{aligned}
D\phi &= (\nabla \langle\sigma|_\phi) \left|\Psi^{(n)}\right\rangle \\
&= (\mathrm{d} \langle\sigma|_\phi) \left|\Psi^{(n)}\right\rangle - \langle\sigma|_\phi \wedge \mathrm{A}^{(\mathrm{n})} \left|\Psi^{(n)}\right\rangle \\
&= \mathrm{d}(\phi \left\langle \Psi^{(n)} \right|) \left|\Psi^{(n)}\right\rangle - \phi \wedge \left\langle \Psi^{(n)} \right| \mathrm{A}^{(\mathrm{n})} \left|\Psi^{(n)}\right\rangle \\
&= \mathrm{d}\phi + \phi \wedge \left\langle \Psi^{(n)}, \mathrm{d}\Psi^{(n)} \right\rangle - \phi \wedge \left\langle \Psi^{(n)} \right| \mathrm{A}^{(\mathrm{n})} \left|\Psi^{(n)}\right\rangle = \mathrm{d}\phi + \phi \wedge \omega(z^{-n}),
\end{aligned} \tag{6.34}$$

where the last equality comes from (6.27), expressing the gauge potential $\mathrm{A}^{(\mathrm{n})}$ in terms of the connection 1-form ω. □

To give the curvature F_∇ of the given connection (6.22) a more explicit form, one can make use of two further relations. The first one, involving the projectors $\mathfrak{p}^{(n)}$ only, comes from [16], while the second is proved again by direct calculation.

Lemma 6.4. *Let* $\mathfrak{p}^{(n)}$ *denote the projection given in* (3.36)*. With the 2D calculus on* S^2_q *of section 3.4.3 one finds:*

$$\begin{aligned}
\mathrm{d}\mathfrak{p}^{(n)} \wedge \mathrm{d}\mathfrak{p}^{(n)} \, \mathfrak{p}^{(n)} &= -q^{-n-1}[n] \; \mathfrak{p}^{(n)} \, \omega_+ \wedge \omega_-, \\
\mathfrak{p}^{(n)} \, \mathrm{d}\mathfrak{p}^{(n)} \wedge \mathrm{d}\mathfrak{p}^{(n)} &= -q^{-n-1}[n] \; \mathfrak{p}^{(n)} \, \omega_+ \wedge \omega_-.
\end{aligned}$$

Lemma 6.5. *Given for any $n \in \mathbb{Z}$ the projectors $\mathfrak{p}^{(n)}$ as in* (3.36) *and the expression of the gauge potential* $\mathrm{A}^{(\mathrm{n})}$ *as in* (6.29), *one has:*

$$\mathfrak{p}^{(n)}\mathrm{d}\mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)} = -\left(\frac{1-q^{2n}}{1-q^{-2}}\right)\left|\Psi^{(n)}\right\rangle \mathrm{d}(U\omega_+ + V\omega_-)\left\langle\Psi^{(n)}\right|. \tag{6.35}$$

Proof. Setting

$$\mathrm{a}^{(n)} = \left\langle\Psi^{(n)}\right| \mathrm{A}^{(\mathrm{n})}\left|\Psi^{(n)}\right\rangle = -\{(1-q^{2n})/(1-q^{-2})\}(U\omega_+ + V\omega_-) = -\frac{1-q^{2n}}{1-q^{-2}}\mathrm{a},$$

the expression (6.35) can be written as the sum of three terms, from the Leibniz rule satisfied by the exterior derivation d:

$$\begin{aligned}
&\mathfrak{p}^{(n)}\mathrm{d}\mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)} \\
&= \left[\left|\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)},\mathrm{d}\Psi^{(n)}\right\rangle \mathrm{a}^{(n)}\left\langle\Psi^{(n)}\right|\right] + \left[\left|\Psi^{(n)}\right\rangle(\mathrm{d}\mathrm{a}^{(n)})\left\langle\Psi^{(n)}\right|\right] - \left[\left|\Psi^{(n)}\right\rangle \mathrm{a}^{(n)}\left\langle\mathrm{d}\Psi^{(n)},\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)}\right|\right] \\
&= \left(\frac{1-q^{2n}}{1-q^{-2}}\right)\left|\Psi^{(n)}\right\rangle\left[\omega_z\wedge\mathrm{a}^{(n)} + \mathrm{a}^{(n)}\wedge\omega_z\right]\left\langle\Psi^{(n)}\right| + \left[\left|\Psi^{(n)}\right\rangle \mathrm{d}\mathrm{a}^{(n)}\left\langle\Psi^{(n)}\right|\right],
\end{aligned}$$

where the second equality comes from the identities $\left\langle\Psi^{(n)},\mathrm{d}\Psi^{(n)}\right\rangle = -\left\langle\mathrm{d}\Psi^{(n)},\Psi^{(n)}\right\rangle = \{(1-q^{2n})/(1-q^{-2})\}\omega_z$, while the $\mathcal{A}(\mathrm{SU}_q(2))$-bimodule relations (3.49) of 1-forms in $\Omega^1(\mathrm{SU}_q(2))$, as well as commutation relations among them (3.51), give:

$$\omega_z\wedge(U\omega_+ + V\omega_-) = q^4U\omega_z\wedge\omega_+ + q^{-4}V\omega_z\wedge\omega_- = -(U\omega_+ + V\omega_-)\wedge\omega_z,$$

so that $\omega_z\wedge\mathrm{a}^{(n)} + \mathrm{a}^{(n)}\wedge\omega_z = 0$ and the identity claimed in (6.35) is verified. □

Remark 6.6. *The identity* $\omega_z\wedge\mathrm{a}^{(n)} + \mathrm{a}^{(n)}\wedge\omega_z = 0$ *also shows that the 1-form* ω_z *anti-commutes with every 1-form in* $\Omega^1(\mathrm{S}^2_q)$.

Proposition 6.7. *Given the covariant derivative* $\nabla : \mathcal{E}^{(k)}_n \mapsto \mathcal{E}^{(k+1)}_n$ *from* (6.21) *with a gauge potential* (6.29) $\mathrm{A}^{(\mathrm{n})} = -(1-q^{2n})(1-q^{-2})^{-1}\left|\Psi^{(n)}\right\rangle \mathrm{a}\left\langle\Psi^{(n)}\right|$, *the operator* $\nabla^2 : \mathcal{E}^{(0)}_n \mapsto \mathcal{E}^{(2)}_n$ *can be written as:*

$$\nabla^2\langle\sigma| = \langle\sigma|\wedge F_\nabla = -\langle\sigma|\wedge\{\left|\Psi^{(n)}\right\rangle q^{n+1}[n](\omega_-\wedge\omega_+ - \mathrm{d}\mathrm{a} + q^{n+1}[n]\mathrm{a}\wedge\mathrm{a})\left\langle\Psi^{(n)}\right|\}. \tag{6.36}$$

Proof. ¿From the general expression (6.22), the action of the operator ∇^2 on a $\langle\sigma|\in\mathcal{E}^{(0)}_n$ is linear, and given by the sum of three terms. The first one, recalling the result of the lemma 6.4 and the commutation rules (3.49) and (3.51), is:

$$\begin{aligned}
-(\mathrm{d}\mathfrak{p}^{(n)}\wedge\mathrm{d}\mathfrak{p}^{(n)})\mathfrak{p}^{(n)} &= q^{-n-1}[n]\mathfrak{p}^{(n)}\omega_+\wedge\omega_- \\
&= -q^{1-n}[n]\left|\Psi^{(n)}\right\rangle\left\langle\Psi^{(n)}\right|\omega_-\wedge\omega_+ = -q^{n+1}[n]\left|\Psi^{(n)}\right\rangle\omega_-\wedge\omega_+\left\langle\Psi^{(n)}\right|.
\end{aligned} \tag{6.37}$$

Since one has $\langle\sigma|\,\mathfrak{p}^{(n)} = \langle\sigma|$, being elements in the projective modules $\mathcal{E}^{(0)}_n$, the other two terms in (6.21) are:

$$\begin{aligned}
\mathfrak{p}^{(n)}\mathrm{d}\mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)} &= -\left(\frac{1-q^{2n}}{1-q^{-2}}\right)\left|\Psi^{(n)}\right\rangle\mathrm{d}\mathrm{a}\left\langle\Psi^{(n)}\right| = q^{n+1}[n]\left|\Psi^{(n)}\right\rangle\mathrm{d}\mathrm{a}\left\langle\Psi^{(n)}\right|, \\
-\mathrm{A}^{(\mathrm{n})}\wedge\mathrm{A}^{(\mathrm{n})} &= -\left(\frac{1-q^{2n}}{1-q^{-2}}\right)^2\left|\Psi^{(n)}\right\rangle\mathrm{a}\wedge\mathrm{a}\left\langle\Psi^{(n)}\right| = -q^{2(n+1)}[n]^2\left|\Psi^{(n)}\right\rangle\mathrm{a}\wedge\mathrm{a}\left\langle\Psi^{(n)}\right|.
\end{aligned}$$

The sum of these three lines gives the curvature $F_\nabla\in\mathbb{M}_{|n|+1}\otimes_{\mathcal{A}(\mathrm{S}^2_q)}\Omega^2(\mathrm{S}^2_q)$ the expression:

$$F_\nabla = -\left|\Psi^{(n)}\right\rangle q^{n+1}[n](\omega_-\wedge\omega_+ - \mathrm{d}\mathrm{a} + q^{n+1}[n]\mathrm{a}\wedge\mathrm{a})\left\langle\Psi^{(n)}\right|. \tag{6.38}$$

□

The isomorphism (6.31) allows to formalise the curvature as a linear map $D^2 : \mathcal{L}_n^{(0)} \mapsto \mathcal{L}_n^{(2)}$, defined by:

$$D^2\phi = (\nabla^2 \langle\sigma|_\phi) \left|\Psi^{(n)}\right\rangle \tag{6.39}$$

for a given $\phi = \langle\sigma, \Psi^{(n)}\rangle$. This operator can also be written in terms of the connection 1-form ω.

Proposition 6.8. *The operator $D^2 : \mathcal{L}_n^{(0)} \mapsto \mathcal{L}_n^{(2)}$ defined in (6.39) can be written as*

$$D^2\phi = -\phi \wedge \{\mathrm{d}\omega(z^{-n}) + \omega(z^{-n}) \wedge \omega(z^{-n})\} = \phi \wedge \left(\left\langle\Psi^{(n)}\right| F_\nabla \left|\Psi^{(n)}\right\rangle\right) \tag{6.40}$$

on any $\phi \in \mathcal{L}_n^{(0)}$.

Proof. The proof is a direct application of the result in propositions 6.18 and 6.3. It is $D\phi = \mathrm{d}\phi - \phi \wedge \omega(z^{-n})$ with $\phi \in \mathcal{L}_n^{(0)}$, so that:

$$\begin{aligned} D^2\phi &= \mathrm{d}(D\phi) + (D\phi) \wedge \omega(z^{-n}) \\ &= -\mathrm{d}(\phi \wedge \omega(z^{-n})) + (\mathrm{d}\phi - \phi \wedge \omega(z^{-n})) \wedge \omega(z^{-n}) = -\phi \wedge (\mathrm{d}\omega(z^{-n}) + \omega(z^{-n}) \wedge \omega(z^{-n})). \end{aligned}$$

The relation (6.27) can be rewritten as $\omega(z^{-n}) = -q^{1+n}[n](\omega_z + \mathrm{a})$, so to have:

$$\begin{aligned} \mathrm{d}\omega(z^{-n}) &= -q^{1+n}[n](\mathrm{d}\omega_z + \mathrm{da}) = q^{1+n}[n](\omega_- \wedge \omega_+ - \mathrm{da}), \\ \omega(z^{-n}) \wedge \omega(z^{-n}) &= \{q^{1+n}[n]\}^2(\omega_z + \mathrm{a}) \wedge (\omega_z + \mathrm{a}) = q^{2(1+n)}[n]^2\mathrm{a} \wedge \mathrm{a}, \end{aligned}$$

where the last equality in the second line comes from the remark 6.6. It becomes then clear to recover from (6.36)

$$D^2\phi = -\phi \wedge q^{1+n}[n]\{\omega_- \wedge \omega_+ - \mathrm{da} + q^{1+n}[n]\mathrm{a} \wedge \mathrm{a}\} = \phi \wedge \left(\left\langle\Psi^{(n)}\right| F_\nabla \left|\Psi^{(n)}\right\rangle\right),$$

meaning that the action of the operator D^2 can be represented by the 2-form $\left(\langle\Psi^{(n)}| F_\nabla |\Psi^{(n)}\rangle\right) \in \mathcal{L}_0^{(2)}$. □

Remark 6.9. *Recall from (6.16) that, given $\phi \in \mathcal{L}_n^{(0)}$, the covariant derivative $D : \mathcal{L}_n^{(0)} \mapsto \mathcal{L}_n^{(1)}$ has been defined in terms of the projector Π associated to the connection as:*

$$D\phi = (1 - \Pi)\mathrm{d}\phi.$$

Given the left $\mathcal{A}(\mathrm{S}_q^2)$-module isomorphisms $\mathcal{L}_n^{(k)} \simeq \Omega^k(\mathrm{S}_q^2) \otimes_{\mathcal{A}(\mathrm{S}_q^2)} \mathcal{E}_n^{(0)} = \mathcal{E}_n^{(k)}$, the proposition 6.2 show that any connection formalised via a projector Π as in (6.11) induces a gauge potential $\mathrm{A}^{(\mathrm{n})}$, so to have a covariant derivative $\nabla : \Omega^k(\mathrm{S}_q^2) \otimes_{\mathcal{A}(\mathrm{S}_q^2)} \mathcal{E}_n \mapsto \Omega^{k+1}(\mathrm{S}_q^2) \otimes_{\mathcal{A}(\mathrm{S}_q^2)} \mathcal{E}_n$. The operator D is then extended in (6.32) as $D : \mathcal{L}_n^{(1)} \mapsto \mathcal{L}_n^{(2)}$ in terms of the operator ∇, without using the projector Π. This definition is perfectly consistent, but it seems natural to understand whether it is possible to define $D : \mathcal{L}_n^{(1)} \mapsto \mathcal{L}_n^{(2)}$ via the projector Π, and even whether it is possible to extend the domain of such a covariant derivative operator D from the set of horizontal forms $\mathcal{L}_n^{(k)}$ to the whole exterior algebra $\Omega(\mathrm{SU}_q(2))$, in analogy to the classical case (2.4).

Given $\phi \in \mathcal{L}_n^{(1)}$, the most natural definition of a covariant derivative seems to be:

$$\check{D}\phi = (1 - \Pi)\mathrm{d}\phi, \tag{6.41}$$

with the horizontal projector $(1 - \Pi)$ extended to $\Omega^2(\mathrm{SU}_q(2))$ by assuming a compatibility with the wedge product

$$\Omega^2(\mathrm{SU}_q(2)) = \{\Omega^1(\mathrm{SU}_q(2)) \otimes_{\mathcal{A}(\mathrm{SU}_q(2))} \Omega^1(\mathrm{SU}_q(2))\}/\mathcal{S}_\mathcal{Q} = \Omega^1(\mathrm{SU}_q(2)) \wedge \Omega^1(\mathrm{SU}_q(2))$$

so to have:

$$(1 - \Pi)\Omega^2(\mathrm{SU}_q(2)) = \{(1 - \Pi)\Omega^1(\mathrm{SU}_q(2))\} \wedge \{(1 - \Pi)\Omega^1(\mathrm{SU}_q(2))\}. \tag{6.42}$$

It is easy to see that such a compatibility does not *exist. To be definite, consider an example. Choose* $\omega_+ \in \mathcal{L}^{(1)}_{-2}$*, so that* $\mathrm{d}\omega_+ = q^2(1+q^2)\omega_z \wedge \omega_+ = -(1+q^{-2})\omega_+ \wedge \omega_z$ *by the commutation properties of the* $\wedge$ *product* (3.51)*. Compute now:*

$$q^2(1+q^2)(1-\Pi)\{\omega_z \wedge \omega_+\} = q^2(1+q^2)\{(1-\Pi)\omega_z\} \wedge \{(1-\Pi)\omega_+\} = q^2(1+q^2)V\omega_- \wedge \omega_+,$$
$$-(1+q^{-2})(1-\Pi)\{\omega_+ \wedge \omega_z\} = -(1+q^{-2})\{(1-\Pi)\omega_+\} \wedge \{(1-\Pi)\omega_z\} = (1+q^{-2})V\omega_- \wedge \omega_+,$$

The two expressions are different: the problem is that, for the given 3D calculus on $\mathcal{A}(\mathrm{SU}_q(2))$*, one has*

$$(1-\Pi)\mathcal{S}_{\mathcal{Q}} \nsubseteq \mathcal{S}_{\mathcal{Q}}.$$

Consider the 6 relations (3.51) *generating* $\mathcal{S}_{\mathcal{Q}}$*. An explicit calculation shows that, from the three of them not involving* ω_z*, one has:*

$$\{(1-\Pi)\omega_+\} \wedge \{(1-\Pi)\omega_+\} = 0,$$
$$\{(1-\Pi)\omega_-\} \wedge \{(1-\Pi)\omega_-\} = 0,$$
$$\{(1-\Pi)\omega_-\} \wedge \{(1-\Pi)\omega_+\} + q^{-2}\{(1-\Pi)\omega_+\} \wedge \{(1-\Pi)\omega_-\} = 0,$$

while from the remaining terms:

$$\{(1-\Pi)\omega_z\} \wedge \{(1-\Pi)\omega_-\} + q^4\{(1-\Pi)\omega_-\} \wedge \{(1-\Pi)\omega_z\} = (1-q^4)U\omega_+ \wedge \omega_-,$$
$$\{(1-\Pi)\omega_z\} \wedge \{(1-\Pi)\omega_+\} + q^{-4}\{(1-\Pi)\omega_+\} \wedge \{(1-\Pi)\omega_z\} = (1-q^{-4})V\omega_- \wedge \omega_+,$$
$$\{(1-\Pi)\omega_z\} \wedge \{(1-\Pi)\omega_z\} = \mathrm{a} \wedge \mathrm{a}.$$

These computations show that only in the case of the monopole connection – that is $\mathrm{a} = 0$ *– it is*

$$(1-\Pi_0)\mathcal{S}_{\mathcal{Q}} \subseteq \mathcal{S}_{\mathcal{Q}}:$$

only in the case of the monopole connection it is consistent to set

$$(1-\Pi_0)\Omega^2(\mathrm{SU}_q(2)) = \{(1-\Pi_0)\Omega^1(\mathrm{SU}_q(2))\} \wedge \{(1-\Pi_0)\Omega^1(\mathrm{SU}_q(2))\}$$

and to define

$$D_0 : \Omega^k(\mathrm{SU}_q(2)) \mapsto \Omega^{k+1}(\mathrm{SU}_q(2)), \qquad D_0\phi = (1-\Pi_0)\mathrm{d}\phi \tag{6.43}$$

The operator D_0 *is a 'covariant' operator: given* $\phi \in \Omega^k(\mathrm{SU}_q(2))$ *such that* $\Delta^{(k)}_R\phi = \phi \otimes z^{-n}$*, it is* $\Delta^{(k+1)}_R(D_0\phi) = D_0\phi \otimes z^{-n}$*, and moreover* $D_0\phi \in \mathcal{L}^{(k)}_n$*:* $D_0\phi$ *is, so to say, horizontal. Note that* $\mathcal{L}^{(3)}_n = \emptyset$*, as the calculus on* S^2_q *is 2D. It becomes an easy computation to prove that the restriction* $D_0 : \mathcal{L}^{(k)}_n \mapsto \mathcal{L}^{(k+1)}_n$ *acquires the form:*

$$D_0\phi = (1-\Pi_0)\mathrm{d}\phi = \mathrm{d}\phi - (-1)^k\phi \wedge \omega_0(z^{-n}). \tag{6.44}$$

This relation is the quantum analogue of the classical (2.5)*. The classical covariant derivative of an equivariant differential form* ϕ *can be expressed in terms of the connection 1-form* ω *only if such* ϕ *is horizontal. In this quantum formulation, the classical condition that* ϕ *is horizontal and equivariant has been translated into the condition* $\phi \in \mathcal{L}^{(k)}_n$*.*

7 A gauged Laplacian on the quantum Hopf bundle

With a covariant derivative ∇ acting on the left $\mathcal{A}(\mathrm{S}^2_q)$-projective modules $\mathcal{E}^{(k)}_n = \Omega^k(\mathrm{S}^2_q) \otimes_{\mathcal{A}(\mathrm{S}^2_q)} \mathcal{E}_n$ and the $\star$-Hodge structure on the exterior algebra $\Omega(\mathrm{S}^2_q)$ introduced in section 5 it is possible to define a gauged Laplacian operator $\square_\nabla : \mathcal{E}^{(0)}_n \mapsto \mathcal{E}^{(0)}_n$ as:

$$\square_\nabla \langle\sigma| = \star\nabla\star\nabla\langle\sigma| \tag{7.1}$$

on any $\langle\sigma| \in \mathcal{E}_n^{(0)}$. From the left $\mathcal{A}(\mathrm{S}^2_q)$-linearity of the $\star$-Hodge map, and the relation (6.21), one has:

$$\begin{aligned}\nabla\star\nabla\,\langle\sigma| &= \mathrm{d}\{\star(\nabla\,\langle\sigma|)\}\mathfrak{p}^{(n)} - (\star\nabla\,\langle\sigma|)\wedge \mathrm{A}^{(\mathrm{n})}\\ &= \mathrm{d}\{\star[(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}] + \langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})})\}\mathfrak{p}^{(n)} - \{(\star[(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}]\wedge\mathrm{A}^{(\mathrm{n})} + \langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})})\wedge\mathrm{A}^{(\mathrm{n})}\}\\ &= \mathrm{d}\{\star[(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}]\}\mathfrak{p}^{(n)} + \mathrm{d}\{\langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})})\}\mathfrak{p}^{(n)} - \star\{(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\}\wedge\mathrm{A}^{(\mathrm{n})} - \langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})})\wedge\mathrm{A}^{(\mathrm{n})}\end{aligned} \tag{7.2}$$

The second term in the last line can be written as:

$$\begin{aligned}\mathrm{d}\{\langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})})\}\mathfrak{p}^{(n)} &= \mathrm{d}\,\langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})})\mathfrak{p}^{(n)} + \langle\sigma|\wedge\{\mathrm{d}(\star\mathrm{A}^{(\mathrm{n})})\}\mathfrak{p}^{(n)}\\ &= \mathrm{d}\,\langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})}) + \langle\sigma|\wedge\{\mathrm{d}(\star\mathrm{A}^{(\mathrm{n})})\}\mathfrak{p}^{(n)},\end{aligned} \tag{7.3}$$

while the third term in (7.2) is:

$$\begin{aligned}-\star\{(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\}\wedge\mathrm{A}^{(\mathrm{n})} &= -\star(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\wedge\mathrm{A}^{(\mathrm{n})}\\ &= -(\star\mathrm{d}\,\langle\sigma|)\wedge\mathrm{A}^{(\mathrm{n})}\ :\end{aligned} \tag{7.4}$$

in both the relations (7.3) and (7.4) the specific property of right $\mathcal{A}(\mathrm{S}^2_q)$-linearity of the $\star$-Hodge map has been used, namely as $\star(\mathrm{A}^{(\mathrm{n})})\mathfrak{p}^{(n)} = \star(\mathrm{A}^{(\mathrm{n})}\mathfrak{p}^{(n)}) = \star\mathrm{A}^{(\mathrm{n})}$ in (7.3) and as $\star\{(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\} = \star(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}$ in (7.4). Moreover, from the proposition 5.4 one has $\mathrm{d}\,\langle\sigma|\wedge(\star\mathrm{A}^{(\mathrm{n})}) = -(\star\mathrm{d}\,\langle\sigma|)\wedge\mathrm{A}^{(\mathrm{n})}$, so that

$$\star\nabla\star\nabla\,\langle\sigma| = \star\mathrm{d}\{\star(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\}\mathfrak{p}^{(n)} - 2\star\{(\star\mathrm{d}\,\langle\sigma|)\wedge\mathrm{A}^{(\mathrm{n})}\} + \langle\sigma|\wedge\{\star\mathrm{d}\star\mathrm{A}^{(\mathrm{n})}\}\mathfrak{p}^{(n)} - \langle\sigma|\wedge\star\{(\star\mathrm{A}^{(\mathrm{n})})\wedge\mathrm{A}^{(\mathrm{n})}\} \tag{7.5}$$

The four terms componing the gauged Laplacian can be individually studied.

- Recalling the result of lemma 5.6, one has:

$$\star\mathrm{A}^{(\mathrm{n})} = q^{n+1}[n]\star\{\left|\Psi^{(n)}\right\rangle \mathrm{a}\left\langle\Psi^{(n)}\right|\} = q^{n+1}[n]\left|\Psi^{(n)}\right\rangle(\star\mathrm{a})\left\langle\Psi^{(n)}\right|. \tag{7.6}$$

  The fourth term in (7.5) is, using once more the result of lemma 5.6 with $\left\langle\Psi^{(n)}\right| \in \mathcal{L}^{(0)}_{-n}$:

$$\begin{aligned}-\langle\sigma|\wedge\star\{(\star\mathrm{A}^{(\mathrm{n})})\wedge\mathrm{A}^{(\mathrm{n})}\} &= -\langle\sigma|\wedge q^{2(1+n)}[n]\star\{\left|\Psi^{(n)}\right\rangle(\star\mathrm{a})\wedge\mathrm{a}\left\langle\Psi^{(n)}\right|\}\\ &= -q^2[n]\,\langle\sigma|\wedge\left|\Psi^{(n)}\right\rangle(\star\{(\star\mathrm{a})\wedge\mathrm{a}\})\left\langle\Psi^{(n)}\right|.\end{aligned} \tag{7.7}$$

- ¿From (7.6) the third term in the expression (7.5) of the gauged Laplacian is:

$$\begin{aligned}\langle\sigma|\wedge\{\star\mathrm{d}\star\mathrm{A}^{(\mathrm{n})}\}\mathfrak{p}^{(n)} &= \langle\sigma|\wedge q^{1+n}[n]\star\{\mathrm{d}\left(\left|\Psi^{(n)}\right\rangle(\star\mathrm{a})\left\langle\Psi^{(n)}\right|\right)\}\mathfrak{p}^{(n)}\\ &= \langle\sigma|\wedge q^{1+n}[n]\star\{\mathfrak{p}^{(n)}\mathrm{d}\left(\left|\Psi^{(n)}\right\rangle(\star\mathrm{A}^{(\mathrm{n})})\left\langle\Psi^{(n)}\right|\right)\mathfrak{p}^{(n)}\}.\end{aligned} \tag{7.8}$$

  The last term in curly bracket is, by the derivation property of d:

$$\begin{aligned}&\mathfrak{p}^{(n)}\mathrm{d}\left(\left|\Psi^{(n)}\right\rangle(\star\mathrm{A}^{(\mathrm{n})})\left\langle\Psi^{(n)}\right|\right)\mathfrak{p}^{(n)} =\\ &= \left|\Psi^{(n)}\right\rangle\left(\left\langle\Psi^{(n)},\mathrm{d}\Psi^{(n)}\right\rangle(\star\mathrm{a})\right)\left\langle\Psi^{(n)}\right| - \left|\Psi^{(n)}\right\rangle\left((\star\mathrm{a})\left\langle\mathrm{d}\Psi^{(n)},\Psi^{(n)}\right\rangle\right)\left\langle\Psi^{(n)}\right| + \left|\Psi^{(n)}\right\rangle(\mathrm{d}(\star\mathrm{a}))\left\langle\Psi^{(n)}\right|\\ &= \left|\Psi^{(n)}\right\rangle\{-q^{1+n}[n]\omega_z\wedge(\star\mathrm{a}) - q^{1+n}[n](\star\mathrm{a})\wedge\omega_z + \mathrm{d}(\star\mathrm{a})\}\left\langle\Psi^{(n)}\right|,\end{aligned} \tag{7.9}$$

  where the last equality comes from the identity $\left\langle\Psi^{(n)},\mathrm{d}\Psi^{(n)}\right\rangle = -q^{1+n}[n]\omega_z$. Recalling the remark 6.6, and using the commutation rules (3.49) as they were used in (7.7), the expression (7.8) becomes:

$$\begin{aligned}\langle\sigma|\wedge\{\star\mathrm{d}\star\mathrm{A}^{(\mathrm{n})}\}\mathfrak{p}^{(n)} &= q^{1+n}[n]\,\langle\sigma|\wedge\star\{\left|\Psi^{(n)}\right\rangle\mathrm{d}(\star\mathrm{a})\left\langle\Psi^{(n)}\right|\}\\ &= q^{1-n}[n]\,\langle\sigma|\wedge\left|\Psi^{(n)}\right\rangle\{\star\mathrm{d}\star\mathrm{a}\}\left\langle\Psi^{(n)}\right|.\end{aligned} \tag{7.10}$$

- It is now straightforward to analyse the second term in the expression (7.5) of the gauged Laplacian. From the definition (6.29) and the Hodge duality (5.14), with again $\mathrm{a} = U\omega_+ + V\omega_-$, $U \in \mathcal{L}_2^{(0)}$ and $V \in \mathcal{L}_{-2}^{(0)}$:

$$
\begin{aligned}
2\star\{\mathrm{d}\,\langle\sigma| \wedge (\star\mathrm{A}^{(\mathrm{n})})\} &= 2i\alpha''\nu\, q^{n+1}[n] \star \{(X_+\triangleright\langle\sigma|)\omega_+ \left|\Psi^{(n)}\right\rangle \wedge \mathrm{a}\left\langle\Psi^{(n)}\right| - (X_-\triangleright\langle\sigma|)\omega_- \left|\Psi^{(n)}\right\rangle \wedge \mathrm{a}\left\langle\Psi^{(n)}\right|\} \\
&= -2i\alpha''\nu\, q[n]\{(X_+\triangleright\langle\sigma| \left|\Psi^{(n)}\right\rangle V \left\langle\Psi^{(n)}\right| + q^2(X_-\triangleright\langle\sigma|)\left|\Psi^{(n)}\right\rangle U\left\langle\Psi^{(n)}\right|\} \star(\omega_-\wedge\omega_+) \\
&= -2q[n]\{\nu(X_+\triangleright\langle\sigma| \left|\Psi^{(n)}\right\rangle V\left\langle\Psi^{(n)}\right| + \beta(X_-\triangleright\langle\sigma|)\left|\Psi^{(n)}\right\rangle U\left\langle\Psi^{(n)}\right|\} \qquad (7.11)
\end{aligned}
$$

- To analyse the first term in (7.5), which is the only one not depending on the gauge potential a, start with:

$$
\begin{aligned}
\star\{(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\} &= \star\left(\{(X_+\triangleright\langle\sigma|)\omega_+ + (X_-\triangleright\langle\sigma|)\omega_-\}\mathfrak{p}^{(n)}\right) \\
&= \star\{(X_+\triangleright\langle\sigma|)\mathfrak{p}^{(n)}\omega_+ + (X_-\triangleright\langle\sigma|)\mathfrak{p}^{(n)}\omega_-\} \\
&= -i\alpha''\nu\{(X_+\triangleright\langle\sigma|)\mathfrak{p}^{(n)}\omega_+ - (X_-\triangleright\langle\sigma|)\mathfrak{p}^{(n)}\omega_-\} \qquad (7.12)
\end{aligned}
$$

so to have:

$$
\begin{aligned}
\mathrm{d}\star\{(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\} &= -i\alpha''\nu\left(X_-\triangleright\left[\{X_+\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right]\omega_-\wedge\omega_+ - X_-\triangleright\left[\{X_-\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right]\omega_+\wedge\omega_-\right) \\
&= -i\alpha''\nu\left(X_-\triangleright\left[\{X_+\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right] + q^2X_+\triangleright\left[\{X_-\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right]\right)\omega_-\wedge\omega_+ \\
\star\left(\mathrm{d}\star\{(\mathrm{d}\,\langle\sigma|)\mathfrak{p}^{(n)}\}\right) &= -i\alpha''\left(\nu X_-\triangleright\left[\{X_+\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right] + \beta X_+\triangleright\left[\{X_-\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right]\right)\star(\omega_-\wedge\omega_+) \\
&= -\left(\nu X_-\triangleright\left[\{X_+\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right] + \beta X_+\triangleright\left[\{X_-\triangleright\langle\sigma|\}\mathfrak{p}^{(n)}\right]\right) \qquad (7.13)
\end{aligned}
$$

The gauged Laplacian can be formalised as an operator $\square_D : \mathcal{L}_n^{(0)} \mapsto \mathcal{L}_n^{(0)}$ via the equivalence between equivariant maps $\phi \in \mathcal{L}_n^{(0)}$ and section of the associated line bundles $\sigma \in \mathcal{E}_n^{(0)}$, represented by the isomorphism in proposition 3.4:

$$
\square_D\phi = (\square_\nabla \langle\sigma|_\phi)\left|\Psi^{(n)}\right\rangle \qquad (7.14)
$$

on any equivariant map $\phi = \langle\sigma|_\phi\left|\Psi^{(n)}\right\rangle$. The terms $(X_\pm\triangleright\langle\sigma|)\left|\Psi^{(n)}\right\rangle$ in (7.11) and(7.13) need a specific analysis. Given the coproduct $\Delta X_\pm = 1\otimes X_\pm + X_\pm\otimes K^2$, it is:

$$
\begin{aligned}
(X_\pm\triangleright\langle\sigma|)\left|\Psi^{(n)}\right\rangle &= (X_\pm\triangleright\{\phi\left\langle\Psi^{(n)}\right|\})\left|\Psi^{(n)}\right\rangle \\
&= \phi(X_\pm\triangleright\left\langle\Psi^{(n)}\right|)\left|\Psi^{(n)}\right\rangle + q^{-n}(X_\pm\triangleright\phi) = q^{-n}(X_\pm\triangleright\phi). \qquad (7.15)
\end{aligned}
$$

This last equality is clear from (3.25) with X_+ and $n < 0$, and with X_- and $n > 0$. In the other two cases, it is possible to apply once more the deformed Leibniz rule to products of elements in $\mathcal{A}(\mathrm{SU}_q(2))$, having:

$$
\begin{aligned}
q^n(X_\pm\triangleright\left\langle\Psi^{(n)}\right|)\left|\Psi^{(n)}\right\rangle &= X_\pm\triangleright\left\langle\Psi^{(n)},\Psi^{(n)}\right\rangle - \left\langle\Psi^{(n)}\right|(X_\pm\triangleright\left|\Psi^{(n)}\right\rangle) \\
&= X_\pm\triangleright(1) - \left\langle\Psi^{(n)}\right|(X_\pm\triangleright\left|\Psi^{(n)}\right\rangle) = -\left\langle\Psi^{(n)}\right|(X_\pm\triangleright\left|\Psi^{(n)}\right\rangle) = 0; \qquad (7.16)
\end{aligned}
$$

since again from (3.25) one has $X_+\triangleright\left|\Psi^{(n)}\right\rangle = 0$ with $n > 0$, and $X_-\triangleright\left|\Psi^{(n)}\right\rangle = 0$ with $n < 0$.

Recollecting the four terms from (7.5) and making use of the relation (7.15), one has:

$$
\begin{aligned}
-\sigma\wedge\star\{(\star\mathrm{A}^{(\mathrm{n})})\wedge\mathrm{A}^{(\mathrm{n})}\}\left|\Psi^{(n)}\right\rangle &= -q^2[n]\phi\wedge\star\{(\star\mathrm{a})\wedge\mathrm{a}\}, \\
\sigma\wedge\{\star\mathrm{d}\star\mathrm{A}^{(\mathrm{n})}\}\left|\Psi^{(n)}\right\rangle &= q^{1-n}[n]\phi\wedge\{\star\mathrm{d}\star\mathrm{a}\}, \\
2\star\{\mathrm{d}\sigma\wedge(\star\mathrm{a})\}\left|\Psi^{(n)}\right\rangle &= -2q^{1-n}[n]\left(\nu(X_+\triangleright\phi)V + \beta(X_-\triangleright\phi)U\right), \\
\left[\star\left(\mathrm{d}\star\{(\mathrm{d}\sigma)\mathfrak{p}^{(n)}\}\right)\right]\left|\Psi^{(n)}\right\rangle &= -q^{-2n}\left(\nu X_-X_+ + \beta X_+X_-\right)\triangleright\phi. \qquad (7.17)
\end{aligned}
$$

It is clear that the gauged Laplacian operator can be completely diagonalised only if one chooses the gauge potential a = 0, that is if one gauges the Laplacian by the monopole connection. Such a gauged Laplacian $\square_{D_0} : \mathcal{L}_n^{(0)} \mapsto \mathcal{L}_n^{(0)}$ can be written as:

$$\square_{D_0}\phi = -q^{-2n}\left(\nu X_- X_+ + \beta X_+ X_-\right) \triangleright \phi, \qquad \text{for } \phi \in \mathcal{L}_n^{(0)}. \tag{7.18}$$

The diagonalisation is straightforward, following (4.22). One has:

$$\begin{aligned} \square_{D_0}\phi_{n,J,l} &= -q^{1-n}\nu\{[J-\frac{n}{2}][J+1+\frac{n}{2}]\} - q^{-1-n}\beta\{[J-\frac{n}{2}][J+1+\frac{n}{2}]+[n]\}\phi_{n,J,l} \\ &= -q^{1-n}\nu\{2[J-\frac{n}{2}][J+1+\frac{n}{2}]+[n]\}\phi_{n,J,l}. \end{aligned} \tag{7.19}$$

Recall the Laplacian operators on $\mathcal{A}(\mathrm{SU}_q(2))$ and on $\mathcal{A}(\mathrm{S}^2_q)$ from equations (4.21) and (5.18):

$$\begin{aligned} &\square_{\mathrm{SU}_q(2)}\phi = -(\nu X_- X_+ + \beta X_+ X_- + \gamma X_z X_z)\triangleright\phi, && \phi \in \mathcal{L}_n^{(0)}, \\ &\square_{\mathrm{S}^2_q} f = -(\nu X_- X_+ + \beta X_+ X_-)\triangleright f, && f \in \mathcal{A}(\mathrm{S}^2_q) \simeq \mathcal{L}_0^{(0)}, \\ &\square_{D_0}\phi = -q^{-2n}\left(\nu X_- X_+ + \beta X_+ X_-\right)\triangleright\phi, && \phi \in \mathcal{L}_n^{(0)}. \end{aligned} \tag{7.20}$$

One has that the restriction of $\square_{D_0}$ to $\phi \in \mathcal{L}_0^{(0)}$ coincides with the operator $\square_{\mathrm{S}^2_q}$. Moreover it is now possible to generalise to the quantum Hopf bundle with the specific differential calculi studied so far, the classical relation (1.1), from which this analysis started:

$$q^{2n}\square_{D_0}\triangleright\phi = \left(\square_{\mathrm{SU}_q(2)} + \gamma X_z X_z\right)\triangleright\phi, \qquad \phi \in \mathcal{L}_n^{(0)}. \tag{7.21}$$

This relation appears as the natural generalisation of the classical relation (1.1) to this specific quantum setting. The quantum Casimir operator (3.21) can not be written as a polynomial in the basis derivations X_j (3.45) of the 3D left covariant calculus from Woronowicz, so its role is played by the Laplacian $\square_{\mathrm{SU}_q(2)}$. Its quantum vertical part can still be written as a quadratic operator in the vertical field X_z of the quantum Hopf fibration.

8 An algebraic formulation of the classical Hopf bundle

The aim of this section is to apply the formalism developed to study the quantum Hopf bundle to the case when all the space algebras are commutative, in order to recover the standard formulation of the classical Hopf bundle, described at the beginning of the paper, from a dual viewpoint.

8.1 An algebraic description of the differential calculus on the group manifold $SU(2)$

Rephrasing the relations (2.8) which define the matrix Lie group $SU(2)$, the coordinate algebra $\mathcal{A}(SU(2))$ of the simple Lie group $SU(2)$ is the commutative $*$-algebra generated by u and v, satisfying the spherical relation $u^*u + v^*v = 1$. The Hopf algebra structure is given by the coproduct:

$$\Delta \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix} = \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix} \otimes \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}, \tag{8.1}$$

antipode:

$$S \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix} = \begin{bmatrix} u^* & v^* \\ -v & u \end{bmatrix}, \tag{8.2}$$

and counit:

$$\epsilon \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{8.3}$$

The universal envelopping algebra $\mathcal{U}(\mathfrak{su}(2))$ is the Hopf $*$-algebra generated by the three elements $\mathrm{e}, \mathrm{f}, \mathrm{h}$ which satisfy the algebraic relations (2.12) coming from the Lie algebra structure in $\mathfrak{su}(2)$:

$$\begin{aligned} &[\mathrm{e}, \mathrm{f}] = 2\mathrm{h}, \\ &[\mathrm{f}, \mathrm{h}] = \mathrm{f}, \\ &[\mathrm{e}, \mathrm{h}] = -\mathrm{e}. \end{aligned} \tag{8.4}$$

The $*$-structure is:

$$h^* = h, \qquad e^* = f, \qquad f^* = e, \tag{8.5}$$

and the Hopf algebra structure is provided by the coproduct:

$$\begin{aligned} \Delta(e) &= e \otimes 1 + 1 \otimes e, \\ \Delta(f) &= f \otimes 1 + 1 \otimes f, \\ \Delta(h) &= h \otimes 1 + 1 \otimes h; \end{aligned}$$

antipode:

$$\begin{aligned} S(e) &= -e, \\ S(f) &= -f, \\ S(h) &= -h; \end{aligned}$$

and a counit which is trivial:

$$\varepsilon(e) = \varepsilon(f) = \varepsilon(h) = 0. \tag{8.6}$$

The centre of the algebra $\mathcal{U}(\mathfrak{su}(2))$ is generated by the Casimir element:

$$C = h^2 + \frac{1}{2}(ef + fe) \tag{8.7}$$

The irreducible finite dimensional $*$-representations σ_j of $\mathcal{U}(\mathfrak{su}(2))$ are well known and labelled by non-negative half-integers $j \in \frac{1}{2}\mathbb{N}$. They are given by:

$$\begin{aligned} \sigma_j(h)\,|j,m\rangle &= m\,|j,m\rangle\,, \\ \sigma_j(e)\,|j,m\rangle &= \sqrt{(j-m)(j+m+1)}\,|j,m+1\rangle\,, \\ \sigma_j(f)\,|j,m\rangle &= \sqrt{(j-m+1)(j+m)}\,|j,m-1\rangle\,. \end{aligned} \tag{8.8}$$

The algebras $\mathcal{A}(SU(2))$ and $\mathcal{U}(\mathfrak{su}(2))$ are dually paired. The bilinear (3.5) mapping $\langle\cdot,\cdot\rangle : \mathcal{U}(\mathfrak{su}(2)) \times \mathcal{A}(SU(2)) \mapsto \mathbb{C}$, compatible with the $*$-structures, is set by:

$$\begin{aligned} \langle h, u\rangle &= -1/2, \\ \langle h, u^*\rangle &= 1/2, \\ \langle e, v\rangle &= 1, \\ \langle f, v^*\rangle &= -1; \end{aligned} \tag{8.9}$$

all other couples of generators pairing to 0. This pairing is non degenerate: the condition $\langle l,x\rangle = 0\ \forall l \in \mathcal{U}(\mathfrak{su}(2))$ implies $x = 0$, while $\langle l,x\rangle = 0\ \forall x \in \mathcal{A}(SU(2))$ implies $h = 0$.

It is possible to prove [13] that a finite dimensional vector space $\mathcal{X}$ of linear functionals on a Hopf algebra $\mathcal{H}$ is a tangent space of a left covariant first order differential calculus $(\Omega^1(\mathcal{H}), d)$ if and only if $X(1) = 0$ and $(\Delta(X) - \varepsilon \otimes X) \in \mathcal{X} \otimes \mathcal{H}^o$, for any $X \in \mathcal{X}$, where $\mathcal{H}^o$ is the so called opposite algebra to $\mathcal{H}$. The ideal $\mathcal{Q} = \{x \in \ker \varepsilon_{\mathcal{H}} :\ X(x) = 0\,\forall X \in \mathcal{X}\}$ characterises the calculus, the bimodule of 1-forms being isomorphic to $\Omega^1(\mathcal{H}) = \Omega^1_{un}(\mathcal{H})/\mathcal{N}_{\mathcal{Q}}$ with $\mathcal{N}_{\mathcal{Q}} = r^{-1}(\mathcal{H} \otimes \mathcal{Q})$. This result shows the path to prove the following proposition.

Proposition 8.1. *Given the nondegenerate bilinear pairing* $\langle\cdot,\cdot\rangle : \mathcal{U}(\mathfrak{su}(2)) \times \mathcal{A}(SU(2)) \mapsto \mathbb{C}$ *as in* (8.9)*, the set* $\{e, f, h\}$ *of generators in* $\mathcal{U}(\mathfrak{su}(2))$ *defines a basis of the tangent space* $\mathcal{X}_{SU(2)}$ *for a bicovariant differential calculus on* $\mathcal{A}(SU(2))$*. Such a differential calculus is isomorphic to the differential calculus* (2.24)*, once the algebra* $C^\infty(S^3)$ *is restricted to the polynomial algebra* $\mathcal{A}(SU(2))$*.*

Proof. The definition of counit in the Hopf algebra $\mathcal{U}(\mathfrak{su}(2))$ shows that the generators $l_a = \{e, f, h\}$, seen as linear functionals on $\mathcal{A}(SU(2))$ via the pairing, are such that:

$$\begin{aligned} e(1) &= \langle e, 1\rangle = \varepsilon(e) = 0, \\ f(1) &= \langle f, 1\rangle = \varepsilon(f) = 0, \\ h(1) &= \langle h, 1\rangle = \varepsilon(h) = 0; \end{aligned}$$

while the coproduct relations can be cast in the form:

$$\begin{aligned}\Delta(\mathrm{e}) - 1\otimes \mathrm{e} &= \mathrm{e}\otimes 1,\\ \Delta(\mathrm{f}) - 1\otimes \mathrm{f} &= \mathrm{f}\otimes 1,\\ \Delta(\mathrm{h}) - 1\otimes \mathrm{h} &= \mathrm{h}\otimes 1;\end{aligned} \tag{8.10}$$

thus proving that the set $\{\mathrm{e},\mathrm{f},\mathrm{h}\}$ in $\mathcal{U}(\mathfrak{su}(2))$ defines a complex vector space basis of a tangent space $\mathcal{X}_{SU(2)}$ for a left covariant differential calculus. In order to recover the ideal $\mathcal{Q}_{SU(2)} \subset \ker\varepsilon_{SU(2)}$ for this specific calculus, consider a generic element $x \in \ker\varepsilon_{SU(2)}$. It must necessarily be written as $x = \{(u-1)x_1, (u^*-1)x_2, vx_3, v^*x_4\}$ with $x_j \in \mathcal{A}(SU(2))$. Such an element x will belong to $\mathcal{Q}_{SU(2)}$ if $\langle l_a, x\rangle = 0$ for any of the generators $l_a \in \mathcal{U}(\mathfrak{su}(2))$, since they form a vector space basis for the tangent space $\mathcal{X}_{SU(2)}$ relative to this calculus. For the element $x = (u-1)x_1$ the three conditions are:

$$\begin{aligned}\langle \mathrm{e}, (u-1)x_1\rangle &= \langle \mathrm{e}, u-1\rangle\langle 1, x_1\rangle + \langle 1, u-1\rangle\langle \mathrm{e}, x_1\rangle = 0,\\ \langle \mathrm{f}, (u-1)x_1\rangle &= \langle \mathrm{f}, u-1\rangle\langle 1, x_1\rangle + \langle 1, u-1\rangle\langle \mathrm{f}, x_1\rangle = 0,\\ \langle \mathrm{h}, (u-1)x_1\rangle &= \langle \mathrm{h}, u-1\rangle\langle 1, x_1\rangle + \langle 1, u-1\rangle\langle \mathrm{h}, x_1\rangle = -\frac{1}{2}\langle 1, x_1\rangle = -\frac{1}{2}\varepsilon(x_1),\end{aligned} \tag{8.11}$$

where, in each of the three lines, the first equality comes from the general properties of dual pairing and from the specific coproduct in $\mathcal{U}(\mathfrak{su}(2))$, while the final result depends on the specific form of the pairing. This means that $x = (u-1)x_1$ belongs to $\mathcal{Q}_{SU(2)}$ if and only if $x_1 \in \ker\varepsilon_{SU(2)}$. The analysis is similar for the other three elements $x = \{(u^*-1)x_2, vx_3, v^*x_4\}$. It is then proved that this left covariant differential calculus on $\mathcal{A}(SU(2))$ - whose tangent space is 3 dimensional - can be characterised by the ideal $\mathcal{Q}_{SU(2)} = \{\ker\varepsilon_{SU(2)}\}^2 \subset \ker\varepsilon_{SU(2)}$, which is generated by the ten elements: $\mathcal{Q}_{SU(2)} = \{(u-1)^2, (u-1)(u^*-1), (u-1)v, (u-1)v^*, (u^*-1)^2, (u^*-1)v, (u^*-1)v^*, v^2, vv^*, v^{*2}\}$. The equation (3.4) allows then to write the exterior derivative for this calculus as:

$$\mathrm{d}x = (\mathrm{e}\triangleright x)\omega_{\mathrm{e}} + (\mathrm{f}\triangleright x)\omega_{\mathrm{f}} + (\mathrm{h}\triangleright x)\omega_{\mathrm{h}} \tag{8.12}$$

The commutation properties between the left invariant forms $\{\omega_{\mathrm{e}}, \omega_{\mathrm{f}}, \omega_{\mathrm{h}}\}$ and elements of the algebra $\mathcal{A}(SU(2))$ depend on the functionals f_{ab} defined as $\Delta(l_a) = 1\otimes l_a + l_b \otimes f_{ba}$. From (8.10) one has $f_{ab} = \delta_{ab}$, so 1-forms do commute with elements of the algebra $\mathcal{A}(SU(2))$, $\omega_a x = x\omega_a$.

The ideal $\mathcal{Q}_{SU(2)}$ is in addition stable under the right coaction Ad of the algebra $\mathcal{A}(SU(2))$ onto itself: $\mathrm{Ad}(\mathcal{Q}_{SU(2)}) \subset \mathcal{Q}_{SU(2)} \otimes \mathcal{A}(SU(2))$. The proof of this result consists of a direct computation. The stability of the ideal $\mathcal{Q}_{SU(2)}$ under the right coaction Ad means that this differential calculus is bicovariant.

The explicit form of the left action of the generators of $\mathcal{U}(\mathfrak{su}(2))$ on the generators of the coordinate algebra $\mathcal{A}(SU(2))$ is:

$$\begin{array}{lll} \mathrm{h}\triangleright u = -\frac{1}{2}u & \mathrm{e}\triangleright u = -v^* & \mathrm{f}\triangleright u = 0\\ \mathrm{h}\triangleright u^* = \frac{1}{2}u^* & \mathrm{e}\triangleright u^* = 0 & \mathrm{f}\triangleright u^* = v\\ \mathrm{h}\triangleright v = -\frac{1}{2}v & \mathrm{e}\triangleright v = u^* & \mathrm{f}\triangleright v = 0\\ \mathrm{h}\triangleright v^* = \frac{1}{2}v^* & \mathrm{e}\triangleright v^* = 0 & \mathrm{f}\triangleright v^* = -u \end{array} \tag{8.13}$$

Starting from these relations it is immediate to see that the left action of the generators $l_a \in \mathcal{U}(\mathfrak{su}(2))$ is equivalent to the Lie derivative along the left invariant vector fields L_a (2.11). This equivalence can now be written as:

$$\begin{aligned}\mathrm{e}\triangleright(x) &= -iL_+(x),\\ \mathrm{f}\triangleright(x) &= -iL_-(x),\\ \mathrm{h}\triangleright(x) &= iL_z(x),\end{aligned} \tag{8.14}$$

and it is valid for any $x \in \mathcal{A}(SU(2))$, as the Leibniz rule for the action of the derivations L_a is encoded in the definition of the left action (3.7) and the properties of the functionals $f_{ab} = \delta_{ab}$. From relation

(8.12) it is possible to recover:

$$\begin{aligned}
\mathrm{d}u &= -v^*\omega_{\mathrm{e}} - \frac{1}{2}u\omega_{\mathrm{h}}, \\
\mathrm{d}u^* &= v\omega_{\mathrm{f}} + \frac{1}{2}u^*\omega_{\mathrm{h}}, \\
\mathrm{d}v &= u^*\omega_{\mathrm{e}} - \frac{1}{2}v\omega_{\mathrm{h}}, \\
\mathrm{d}v^* &= -u\omega_{\mathrm{f}} + \frac{1}{2}v^*\omega_{\mathrm{h}}.
\end{aligned}$$

These relations can be inverted, so that left invariant 1-forms $\{\omega_{\mathrm{e}}, \omega_{\mathrm{f}}, \omega_{\mathrm{h}}\}$ can be compared to (2.21):

$$\begin{aligned}
\omega_{\mathrm{e}} &= u\mathrm{d}v - v\mathrm{d}u = i\tilde{\omega}_+, \\
\omega_{\mathrm{f}} &= v^*\mathrm{d}u^* - u^*\mathrm{d}v^* = i\tilde{\omega}_-, \\
\omega_{\mathrm{h}} &= -2(u^*\mathrm{d}u + v^*\mathrm{d}v) = -i\tilde{\omega}_z
\end{aligned} \tag{8.15}$$

These equalities, which are dual to (8.14), represent the isomorphism between the differential calculus introduced via the action of the exterior derivative in (8.12), and the differential calculus analysed in section 2.1 and formalised in (2.24).

□

It is now straightforward to recover this bicovariant calculus as the classical limit of the quantum 3D left covariant calculus $(\Omega(\mathrm{SU}_q(2), \mathrm{d})$ described in section 3.4.1. In the classical limit $\mathcal{A}(\mathrm{SU}_q(2)) \mapsto \mathcal{A}(SU(2))$ as $q \mapsto 1$, with $\phi \to x$, one has:

$$\begin{aligned}
&\omega_+ \to \omega_{\mathrm{e}} \qquad (X_+ \triangleright \phi) \to (\mathrm{e} \triangleright x), \\
&\omega_- \to \omega_{\mathrm{f}} \qquad (X_- \triangleright \phi) \to (\mathrm{f} \triangleright x), \\
&\omega_z \to -\frac{1}{2}\,\omega_{\mathrm{h}} \qquad (X_z \triangleright \phi) \to (-2\mathrm{h} \triangleright x).
\end{aligned}$$

The coaction $\Delta_R^{(1)}$ of $\mathcal{A}(SU(2))$ on the basis of left invariant forms defines the matrix $\Delta_R^{(1)}(\omega_a) = \omega_b \otimes J_{ba}$:

$$\begin{aligned}
\Delta_R^{(1)}(\omega_{\mathrm{f}}) &= \omega_{\mathrm{f}} \otimes u^{*2} + \omega_{\mathrm{h}} \otimes u^*v^* - \omega_{\mathrm{e}} \otimes v^{*2}, \\
\Delta_R^{(1)}(\omega_{\mathrm{h}}) &= -\omega_{\mathrm{f}} \otimes 2u^*v + \omega_{\mathrm{h}} \otimes (u^*u - v^*v) - \omega_{\mathrm{e}} \otimes 2uv^*, \\
\Delta_R^{(1)}(\omega_{\mathrm{e}}) &= -\omega_{\mathrm{f}} \otimes v^2 + \omega_{\mathrm{h}} \otimes uv + \omega_{\mathrm{e}} \otimes u^2,
\end{aligned} \tag{8.16}$$

which is used to define a basis of right invariant one forms $\eta_a = \omega_b S(J_{ba})$:

$$\begin{aligned}
\eta_{\mathrm{f}} &= u^2\omega_{\mathrm{f}} - uv^*\omega_{\mathrm{h}} - v^{*2}\omega_{\mathrm{e}} = v^*\mathrm{d}u - u\mathrm{d}v^*, \\
\eta_{\mathrm{h}} &= 2uv\omega_{\mathrm{f}} + (uu^* - vv^*)\omega_{\mathrm{h}} + 2u^*v^*\omega_{\mathrm{e}} = 2(u\mathrm{d}u^* + v^*\mathrm{d}v), \\
\eta_{\mathrm{e}} &= -v^2\omega_{\mathrm{f}} - u^*v\omega_{\mathrm{h}} + u^{*2}\omega_{\mathrm{e}} = u^*\mathrm{d}v - v\mathrm{d}u^*;
\end{aligned} \tag{8.17}$$

- note that it has been made explicit use of the commutativity between forms ω_a and elements of the algebra $\mathcal{A}(SU(2))$. The right acting derivation associated to this basis are given by (3.12) as

$$\mathrm{d}x = \eta_a \triangleleft (-S^{-1}(l_a)) = \eta_a \triangleleft l_a$$

for any $x \in \mathcal{A}(SU(2))$, since an immediate evaluation gives $S^{-1}(l_a) = -l_a$ for the three vector basis elements of the tangent space $l_a \in \mathcal{X}$. Using again the commutativity of the right invariant one forms η_a with element of $\mathcal{A}(SU(2))$, the action of the exterior derivation (8.12) can be written as:

$$\mathrm{d}x = (x \triangleleft \mathrm{f})\eta_{\mathrm{f}} + (x \triangleleft \mathrm{h})\eta_{\mathrm{h}} + (x \triangleleft \mathrm{e})\eta_{\mathrm{e}}. \tag{8.18}$$

Comparing (8.17) to (2.22) one has:

$$\begin{aligned}
\eta_{\mathrm{f}} &= i\tilde{\eta}_-, \\
\eta_{\mathrm{h}} &= -i\tilde{\eta}_z, \\
\eta_{\mathrm{e}} &= i\tilde{\eta}_+,
\end{aligned} \tag{8.19}$$

while for the right action of the generators l_a on $\mathcal{A}(SU(2))$ one computes:

$$\begin{array}{lll} u{\triangleleft}\mathrm{h} = -\frac{1}{2}u & u{\triangleleft}\mathrm{e} = 0 & u{\triangleleft}\mathrm{f} = v \\ u^*{\triangleleft}\mathrm{h} = \frac{1}{2}u^* & u^*{\triangleleft}\mathrm{e} = -v^* & u^*{\triangleleft}\mathrm{f} = 0 \\ v{\triangleleft}\mathrm{h} = \frac{1}{2}v & v{\triangleleft}\mathrm{e} = u & v{\triangleleft}\mathrm{f} = 0 \\ v^*{\triangleleft}\mathrm{h} = -\frac{1}{2}v^* & v^*{\triangleleft}\mathrm{e} = 0 & v^*{\triangleleft}\mathrm{f} = -u^*; \end{array} \tag{8.20}$$

so that the identification with the action of the right invariant vector fields (2.14) can be recovered as:

$$\begin{aligned} (x){\triangleleft}\mathrm{f} &= -iR_-(x), \\ (x){\triangleleft}\mathrm{e} &= -iR_+(x), \\ (x){\triangleleft}\mathrm{h} &= iR_z(x), \end{aligned} \tag{8.21}$$

being dual to the identification (8.19). It is also evident that relations (8.19) and (8.21) define a different formalisation for the isomorphism between the differential calculus introduced in this section (8.18) and the differential calculus from section 2.1.

Remark 8.2. *The identification* (8.14) *can be read as a Lie algebra isomorphism between the Lie algebra* $\{\mathrm{e}, \mathrm{f}, \mathrm{h}\}$ *given in* (8.4) *and the Lie algebra of the left invariant vector fields* $\{L_a\}$ (2.12)*:*

$$\mathrm{e} = -iL_+, \qquad \mathrm{f} = -iL_-, \qquad \mathrm{h} = iL_z. \tag{8.22}$$

The notion of pairing between the algebras $\mathcal{U}(\mathfrak{su}(2))$ *and* $\mathcal{A}(SU(2))$ *can be recovered as the Lie derivative of the coordinate functions along the vector fields* L_a*, evaluated at the identity of the group manifold. The terms in* (8.9) *giving the nonzero terms of the pairing are:*

$$\begin{array}{lcl} L_z(u)|_{\mathrm{id}} = \frac{i}{2} & \rightarrow & \langle \mathrm{h}, u\rangle = -\frac{1}{2} \\ L_z(u^*)|_{\mathrm{id}} = -\frac{i}{2} & \rightarrow & \langle \mathrm{h}, u^*\rangle = \frac{1}{2} \\ L_+(v)|_{\mathrm{id}} = i & \rightarrow & \langle \mathrm{e}, v\rangle = 1 \\ L_-(v^*)|_{\mathrm{id}} = -i & \rightarrow & \langle \mathrm{f}, v^*\rangle = -1 \end{array}$$

The whole exterior algebra $\Omega(SU(2))$ can now be constructed from the differential calculus (8.12). Any 1-form $\theta \in \Omega^1(SU(2))$ can be written on the basis of left invariant forms as $\theta = \sum_k \theta_k \omega_k = \omega_k \theta_k$ with $\theta_k \in \mathcal{A}(SU(2))$. Higher dimensional forms can be defined by requiring their total antisimmetry, and that $\mathrm{d}^2 = 0$. One has then $\omega_a \wedge \omega_b + \omega_b \wedge \omega_a = 0$ and:

$$\begin{aligned} \mathrm{d}\omega_{\mathrm{f}} &= \omega_{\mathrm{h}} \wedge \omega_{\mathrm{f}}, \\ \mathrm{d}\omega_{\mathrm{e}} &= \omega_{\mathrm{e}} \wedge \omega_{\mathrm{h}}, \\ \mathrm{d}\omega_{\mathrm{h}} &= 2\omega_{\mathrm{f}} \wedge \omega_{\mathrm{e}}. \end{aligned} \tag{8.23}$$

Finally, there is a unique volume top form $\omega_{\mathrm{f}} \wedge \omega_{\mathrm{e}} \wedge \omega_{\mathrm{h}}$.

The algebra $\mathcal{A}(SU(2))$ can be partitioned into finite dimensional blocks, whose elements are related to the Wigner D-functions [27] for the group $SU(2)$. Considering all the unitary irreducible representations of $SU(2)$, their matrix elements will give a Peter-Weyl basis for the Hilbert space $\mathcal{L}^2(SU(2), \mu)$ of complex valued functions defined on the group manifold with respect to the Haar invariant measure. The Wigner D-function $D^J_{ks}(g)$ is defined to be the matrix element (k, s are the matrix indices) representing the element $g \simeq (u, v)$ in $SU(2)$ (2.8) in the representation of weight J. They are known:

$$D^J_{ks} = (-i)^{s+k}[(J+s)!(J-s)!(J+k)!(J-k)!]^{1/2} \sum_l (-1)^{k+l} \frac{u^{*l} v^{*J-k-l} v^{J-s-l} u^{*k+s+l}}{l!(J-k-l)!(J-s-l)!(s+k+l)!} \tag{8.24}$$

with $J = 0, 1/2, 1, \dots$ and $k = -J, \dots, +J$, $s = -J, \dots, +J$. In (8.24) the index l runs over the set of natural numbers such that all the arguments of the factorial are non negative. To illustrate the meaning of this partition, proceed as in the quantum setting, and consider the element $u^* \in \mathcal{A}(SU(2))$. Representing the left action $\mathrm{f}{\triangleright}$ with a horizontal arrow and the right action ${\triangleleft}\mathrm{e}$ with a vertical one yields the box:

$$\begin{array}{ccc} u^* & \rightarrow & v \\ \downarrow & & \downarrow \\ -v^* & \rightarrow & u \end{array} \tag{8.25}$$

while starting from $u^{*2} \in \mathcal{A}(SU(2))$ yields the box:

$$
\begin{array}{ccccc}
u^{*2} & \to & 2u^*v & \to & 2v^2 \\
\downarrow & & \downarrow & & \downarrow \\
-2u^*v^* & \to & 2(u^*u - v^*v) & \to & 4uv \\
\downarrow & & \downarrow & & \downarrow \\
2v^{*2} & \to & -4v^*u & \to & 4u^2
\end{array}
\tag{8.26}
$$

A recursive structure emerges now clear. For each positive integer p one has a box W_p made up of the $(p+1) \times (p+1)$ elements $w_{p:t,r} = \mathrm{f}^t \triangleright u^{*p} \triangleleft \mathrm{e}^r$. An explicit calculation proves that:

$$
\mathrm{f}^t \triangleright u^{*p} \triangleleft \mathrm{e}^r = i^{t+r} j! \left[\frac{t!r!}{(p-t)!(p-r)!} \right]^{1/2} D^{p/2}_{t-p/2, r-p/2} \tag{8.27}
$$

with $t \leq p, r \leq p$. As an element in $\mathcal{U}(\mathfrak{su}(2))$, the quadratic Casimir C (8.7) of the Lie algebra $\mathfrak{su}(2)$ acts on $x \in \mathcal{A}(SU(2))$ as $C \triangleright x = x \triangleleft C$, and its action clearly commutes with the actions $\mathrm{f}\triangleright$ and $\triangleleft \mathrm{e}$. This means that the decomposition $\mathcal{A}(SU(2)) = \oplus_{j \in \mathbb{N}} W_p$ gives the spectral resolution of the action of C:

$$
C \triangleright w_{p:t,r} = \frac{p}{2}(\frac{p}{2} + 1) w_{p:t,r}. \tag{8.28}
$$

8.2 The bundle structure

8.2.1 The base algebra of the bundle

Given the abelian $*$-algebra $\mathcal{A}(U(1)) = \mathbb{C}[z, z^*]/ < zz^* - 1 >$, the map $\check{\pi} : \mathcal{A}(SU(2)) \mapsto (U(1))$

$$
\check{\pi} \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix}, \tag{8.29}
$$

is a surjective Hopf $*$-algebra homomorphism, so that $\mathcal{A}(U(1))$ can be formalised as a $*$-subalgebra of $\mathcal{A}(SU(2))$, with a right coaction:

$$
\check{\Delta}_R = (1 \otimes \check{\pi}) \circ \Delta, \qquad \mathcal{A}(SU(2) \mapsto \mathcal{A}(SU(2)) \otimes \mathcal{A}(U(1)). \tag{8.30}
$$

The coinvariant elements for this coaction, that is elements $b \in \mathcal{A}(SU(2))$ for which $\check{\Delta}_R(b) = b \otimes 1$, form the subalgebra $\mathcal{A}(S^2) \subset \mathcal{A}(SU(2))$, which is the coordinate subalgebra of the sphere S^2. From:

$$
\begin{aligned}
\check{\Delta}_R(u) &= u \otimes z, \\
\check{\Delta}_R(u^*) &= u^* \otimes z^*, \\
\check{\Delta}_R(v) &= v \otimes z, \\
\check{\Delta}_R(v^*) &= v^* \otimes z^*,
\end{aligned}
\tag{8.31}
$$

one has that a set of generators for $\mathcal{A}(S^2)$ is given by (2.44):

$$
\begin{aligned}
b_z &= uu^* - vv^*, \\
b_y &= uv^* + vu^*, \\
b_x &= -i(vu^* - uv^*)
\end{aligned}
\tag{8.32}
$$

The comparison with the description in section 2.3 shows that $\check{\pi}$ dually formalises the choice of the gauge group $U(1)$ as a subgroup of $SU(2)$, whose right principal pull-back action $\check{\mathrm{r}}^*_k$ is now replaced by the right $\mathcal{A}(U(1))$- coaction $\check{\Delta}_R$. The basis of the principal Hopf bundle $S^2 \simeq SU(2)/U(1)$ will be given as the algebra $\mathcal{A}(S^2)$ of right coinvariant elements $b_a \in \mathcal{A}(SU(2))$. This is an homogeneous space algebra: the coproduct Δ of $\mathcal{A}(SU(2))$ restricts to a left coaction $\Delta : \mathcal{A}(SU(2)) \mapsto \mathcal{A}(SU(2)) \otimes \mathcal{A}(S^2)$ as:

$$
\begin{aligned}
\Delta(b_{\mathrm{f}}) &= u^2 \otimes b_{\mathrm{f}} - v^*u \otimes b_{\mathrm{h}} - v^{*2} \otimes b_{\mathrm{e}}, \\
\Delta(b_{\mathrm{h}}) &= 2uv \otimes b_{\mathrm{f}} + (u^*u - v^*v) \otimes b_{\mathrm{h}} + 2u^*v^* \otimes b_{\mathrm{e}}, \\
\Delta(b_{\mathrm{e}}) &= -v^2 \otimes b_{\mathrm{f}} - u^*v \otimes b_{\mathrm{h}} + u^{*2} \otimes b_{\mathrm{e}}.
\end{aligned}
\tag{8.33}
$$

with $b_{\rm f} = 1/2(b_y - ib_x) = uv^*$, $b_{\rm e} = 1/2(b_y + ib_x) = vu^*$, $b_{\rm h} = b_z$. The choice of this specific basis shows that $\Delta(b_a) = S(J_{ka}) \otimes b_k$ where the matrix J is exactly the one defined in (8.16) as $\Delta^{(1)}_R(\omega_a) = \omega_b \otimes J_{ba}$.

The identification (8.14) between the left action $\mathrm{h}\triangleright x$ – given the generator $\mathrm{h} \in \mathcal{U}(\mathfrak{su}(2))$ on any $x \in \mathcal{A}(SU(2))$ – and the action $iL_z(x)$ – given the left invariant vector field L_z – as well as the definition of the $\mathcal{A}(U(1))$-right coaction $\check{\Delta}_R$ on $\mathcal{A}(SU(2))$ (8.31), allow to recover the set of the $U(1)$-equivariant functions $\mathfrak{L}^{(0)}_n \subset \mathcal{A}(SU(2))$ in (2.49) as:

$$\mathfrak{L}^{(0)}_n = \{\phi \in \mathcal{A}(SU(2)) \,:\, \mathrm{h}\triangleright\phi = \frac{n}{2}\phi \leftrightarrow \check{\Delta}_R(\phi) = \phi \otimes z^{-n}\}. \tag{8.34}$$

8.2.2 A differential calculus on the gauge group algebra

The strategy underlining the proof of the proposition 8.1 brings also to the definition of a differential calculus on the gauge group algebra $\mathcal{A}(U(1))$. The bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{U}(\mathfrak{su}(2)) \times \mathcal{A}(SU(2)) \mapsto \mathbb{C}$ (8.9) is restricted via the surjection $\check{\pi}$ (8.29) to a bilinear pairing $\langle\cdot,\cdot\rangle : \mathcal{U}\{\mathrm{h}\} \times \mathcal{A}(U(1)) \mapsto \mathbb{C}$, which is still compatible with the $*$-structure, given on generators as:

$$\langle \mathrm{h}, z\rangle = -\frac{1}{2}, \qquad \langle \mathrm{h}, z^{-1}\rangle = \frac{1}{2}.$$

The set $\mathcal{X}_{U(1)} = \{\mathrm{h}\}$ is proved to be the basis of the tangent space for a 1-dimensional bicovariant commutative calculus on $\mathcal{A}(U(1))$. The ideal $\mathcal{Q}_{U(1)} \subset \ker\varepsilon_{U(1)}$ turns out again to be $\mathcal{A}_{U(1)} = (\ker\varepsilon_{U(1)})^2$ generated by $\{(z-1)^2, (z-1)(z^{-1}-1), (z^{-1}-1)^2\}$, which can also be recovered as $\mathcal{Q}_{U(1)} = \check{\pi}((\ker\varepsilon_{SU(2)})^2)$. From:

$$\mathrm{h}\triangleright z = -\frac{1}{2}z, \qquad \mathrm{h}\triangleright z^{-1} = \frac{1}{2}z^{-1}$$

one has that:

$$\mathrm{d}z = -\frac{1}{2}z\check{\omega}, \qquad \mathrm{d}z^{-1} = \frac{1}{2}z^{-1}\check{\omega} \tag{8.35}$$

with $z\mathrm{d}z = (\mathrm{d}z)z$. The only left invariant 1-form is

$$\check{\omega} = -2z^{-1}\mathrm{d}z = 2z\mathrm{d}z^{-1},$$

while the role of the right invariant derivation associated to $\mathrm{h} \in \mathcal{U}\{\mathrm{h}\}$ is played by $-S^{-1}(\mathrm{h}) = \mathrm{h}$, so that the right invariant form generating this calculus is:

$$\mathrm{d}z = \check{\eta}(z\triangleleft\mathrm{h}) = \check{\eta}(-\frac{1}{2}z) \qquad \to \check{\eta} = -2z^{-1}\mathrm{d}z,$$
$$\mathrm{d}z^{-1} = \check{\eta}(z^{-1}\triangleleft\mathrm{h}) = \check{\eta}(\frac{1}{2}z^{-1}) \qquad \to \check{\eta} = 2z\mathrm{d}z^{-1}$$

so that one obtains $\check{\eta} = \check{\omega}$.

It is possible to characterise the quotient $\ker\varepsilon_{U(1)}/\mathcal{Q}_{U(1)} = \ker\varepsilon_{U(1)}/(\ker\varepsilon_{U(1)})^2$. The three elements generating the ideal $\mathcal{Q}_{U(1)} = (\ker\varepsilon_{U(1)})^2$ can be written as:

$$\xi = (z-1)(z^{-1}-1) = (z-1) + (z^{-1}-1),$$
$$\xi' = (z-1)(z-1) = \xi + \xi(z-1),$$
$$\xi'' = (z^{-1}-1)(z^{-1}-1) = \xi + \xi(z^{-1}-1),$$

so that $\mathcal{Q}_{U(1)}$ can be seen generated by $\xi = (z-1) + (z^{-1}-1)$. Set a map $\lambda : \ker\varepsilon_{U(1)} \mapsto \mathbb{C}$ by $\lambda(u(z-1)) = \sum_{j\in\mathbb{Z}} u_j$, where $u = \sum_{j\in\mathbb{Z}} u_j z^j$ is generic element in $\mathcal{A}(U(1))$. The techniques outlined in lemma 3.5 in the quantum setting enable to prove that λ can be used to define a complex vector space isomorphism between $\ker\varepsilon_{U(1)}/(\ker\varepsilon_{U(1)})^2$ and $\mathbb{C}$, whose inverse is given by $\lambda^{-1} : w \in \mathbb{C} \mapsto \lambda^{-1}(w) = w(z-1) \in \ker\varepsilon_{U(1)}$. It is evident that such a map λ formalises the projection $\pi_{\mathcal{Q}_{U(1)}} : \ker\varepsilon_{U(1)} \mapsto \ker\varepsilon_{U(1)}/\mathcal{Q}_{U(1)} \simeq \mathbb{C}$, since it chooses a representative in each equivalence class in the quotient $\ker\varepsilon_{U(1)}/\mathcal{Q}_{U(1)}$.

8.2.3 The Hopf bundle structure

With the 3D bicovariant calculus on the total space algebra $\mathcal{A}(SU(2))$ and the 1D bicovariant calculus on the gauge group algebra $\mathcal{A}(U(1))$, one needs to prove the compatibility conditions that lead to the exact sequence:

$$0 \to \mathcal{A}(SU(2))\left(\Omega^1(S^2)\right)\mathcal{A}(SU(2)) \to$$
$$\to \Omega^1(\mathcal{A}(SU(2)) \stackrel{\sim_{\mathcal{N}_{SU(2)}}}{\longrightarrow} \mathcal{A}(SU(2)) \otimes \ker\varepsilon_{U(1)}/\mathcal{Q}_{U(1)} \to 0,$$

where the map $\sim_{\mathcal{N}_{SU(2)}}$ is defined as in the diagram (3.15) which now acquires the form:

$$\begin{array}{lcl} \Omega^1(SU(2))_{un} & \stackrel{\pi_{\mathcal{Q}_{SU(2)}}}{\longrightarrow} & \Omega^1(\mathcal{A}(SU(2)) \\ \downarrow \chi & & \downarrow \sim_{\mathcal{N}_{SU(2))}} \\ \mathcal{A}(SU(2)) \otimes \ker\varepsilon_{U(1)} & \stackrel{\mathrm{id}\otimes\pi_{\mathcal{Q}_{U(1)}}}{\longrightarrow} & \mathcal{A}(SU(2)) \otimes (\ker\varepsilon_{U(1)}/\mathcal{Q}_{U(1)})\,. \end{array} \tag{8.36}$$

The proof of the compatibility conditions is in the following lemmas. The first one analyses the right covariance of the differential structure on $\mathcal{A}(SU(2))$.

Lemma 8.3. *¿From the 3D bicovariant calculus on $\mathcal{A}(SU(2))$ generated by the ideal $\mathcal{Q}_{SU(2)} = (\ker\varepsilon_{SU(2)})^2 \subset \ker\varepsilon_{SU(2)}$ given in proposition 8.1, one has $\check{\Delta}_R\mathcal{N}_{SU(2)} \subset \mathcal{N}_{SU(2)} \otimes \mathcal{A}(U(1))$.*

Proof. Using the bijection given in (3.3), it is $\Omega^1(SU(2)) \simeq \Omega^1(SU(2))/\mathcal{N}_{SU(2)}$ with $\mathcal{N}_{SU(2)} = r^{-1}(\mathcal{A}(SU(2))\otimes \mathcal{Q}_{SU(2)})$. For this specific calculus one has that $\mathcal{N}_{SU(2)}$ is the sub-bimodule generated by $\{\delta\phi\,\delta\psi\}$ for any $\phi, \psi \in \mathcal{A}(SU(2))$, where $\delta\phi = (1 \otimes \phi - \phi \otimes 1) \in \Omega^1(SU(2))_{un}$. Choose $\phi \in \mathfrak{L}^{(0)}_n$ and $\psi \in \mathfrak{L}^{(0)}_m$ so to have $\check{\Delta}_R\phi = \phi \otimes z^{-n}$ and $\check{\Delta}_R\psi = \psi \otimes z^{-m}$. Extending the coaction $\check{\Delta}_R$ to a coaction $\check{\Delta}_R : \mathcal{A}(SU(2))\otimes\mathcal{A}(SU(2)) \mapsto \mathcal{A}(SU(2))\otimes\mathcal{A}(SU(2))\otimes\mathcal{A}(U(1))$ as $\check{\Delta}_R = (\mathrm{id}\otimes\mathrm{id}\otimes m)\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})\circ(\check{\Delta}_R\otimes\check{\Delta}_R)$ in terms of the flip operator τ, it becomes an easy calculation to find:

$$\begin{aligned} \check{\Delta}_R(\delta\phi\,\delta\psi) &= (1 \otimes \phi\psi + \phi\psi \otimes 1 - \phi \otimes \psi - \psi \otimes \phi) \\ &= (1 \otimes \phi\psi + \phi\psi \otimes 1 - \phi \otimes \psi - \psi \otimes \phi) \otimes z^{-m-n} = (\delta\phi\,\delta\psi) \otimes z^{-m-n}. \end{aligned}$$

□

Lemma 8.4. *The map $\chi : \Omega^1(SU(2))_{un} \mapsto \mathcal{A}(SU(2)) \otimes \mathcal{A}(U(1))$ defined in (3.14) as $\chi = (m \otimes \mathrm{id}) \circ (\mathrm{id}\otimes\check{\Delta}_R)$ is surjecive.*

Proof. The proof of this result closely follows the proof of the proposition 3.3. From the spherical relation $1 = (u^*u + v^*v)^n = \sum_{a=0}^{n} \begin{pmatrix} n \\ a \end{pmatrix} u^{*a}v^{*n-a}v^{n-a}u^a$ it is possible to set $\left|\Psi^{(n)}\right\rangle_a \in \mathfrak{L}^{(0)}_n$ for $a = 0, \ldots, |n|$ with $\left\langle\Psi^{(n)}, \Psi^{(n)}\right\rangle = 1$ as:

$$n > 0 : \left|\Psi^{(n)}\right\rangle_a = \sqrt{\begin{pmatrix} n \\ a \end{pmatrix}}\, v^{*a}u^{*n-a},$$
$$n < 0 : \left|\Psi^{(n)}\right\rangle_a = \sqrt{\begin{pmatrix} |n| \\ a \end{pmatrix}}\, v^{*|n|-a}u^{a}.$$

Fixed $n \in \mathbb{Z}$, define $\gamma = \left\langle\Psi^{(-n)}, \delta\Psi^{(-n)}\right\rangle$. Since $\left|\Psi^{(-n)}\right\rangle \in \mathfrak{L}^{(0)}_{-n}$, one computes that $\chi(\gamma) = 1 \otimes (z^n - 1)$, and this sufficient to prove the surjectivity of the map χ, being χ left $\mathcal{A}(SU(2))$-linear and $\ker\varepsilon_{U(1)}$ is a complex vector space with a basis $(z^n - 1)$.

□

Lemma 8.5. *Given the map χ as in the previous lemma, it is $\chi(\mathcal{N}_{SU(2)}) \subset \mathcal{A}(SU(2)) \otimes \mathcal{Q}_{U(1)}$, where $\mathcal{N}_{SU(2)}$ is as in lemma 8.3 and $\mathcal{Q}_{U(1)} = (\ker \varepsilon_{U(1)})^2$.*

Proof. To be definite, consider $\phi \in \mathfrak{L}_n^{(0)}$ and $\psi \in \mathfrak{L}_m^{(0)}$. One has:

$$\begin{aligned}\chi(\delta\phi\,\delta\psi) &= \phi\psi \otimes \{z^{-n-m} + 1 - z^{-n} - z^{-m}\} \\ &= \phi\psi \otimes \{(1 - z^{-n})(1 - z^{-m})\} \subset \mathcal{A}(SU(2)) \otimes (\ker \varepsilon_{U(1)})^2.\end{aligned}$$

□

The results of these lemmas allow to define the map $\sim_{\mathcal{N}_{SU(2)}} : \Omega^1(SU(2)) \mapsto \mathcal{A}(SU(2)) \otimes \ker \varepsilon_{U(1)}/\mathcal{Q}_{U(1)}$ from the diagram (8.36). Using the isomorphism $\lambda : \ker \varepsilon_{U(1)}/\mathcal{Q}_{U(1)} \mapsto \mathbb{C}$ described in section 8.2.2, one has:

$$\begin{aligned}\sim_{\mathcal{N}_{SU(2)}} (\omega_{\mathrm{e}}) &= 0 \\ \sim_{\mathcal{N}_{SU(2)}} (\omega_{\mathrm{f}}) &= 0 \\ \sim_{\mathcal{N}_{SU(2)}} (\omega_{\mathrm{h}}) &= -2 \otimes \pi_{\mathcal{Q}_{U(1)}}(z - 1) = -2 \otimes 1.\end{aligned} \tag{8.37}$$

The next lemma completes the analysis of the compatibility conditions between the differential structures on $\mathcal{A}(SU(2))$ and on $\mathcal{A}(U(1))$. The horizontal part of the set of k-forms out of $\Omega^k(SU(2))$ is defined as $\Omega^k_{hor}(SU(2)) = \Omega^k(S^2)\mathcal{A}(SU(2)) = \mathcal{A}(SU(2))\Omega^k(S^2)$.

Lemma 8.6. *Given the differential calculus on the basis $\Omega^1(S^2) = \Omega^1(S^2)_{un}/\mathcal{N}_{S^2}$ with $\mathcal{N}_{S^2} = \mathcal{N}_{SU(2)} \cap \Omega^1(S^2)_{un}$, it is* $\ker \sim_{\mathcal{N}_{SU(2)}} = \Omega^1(S^2)\mathcal{A}(SU(2)) = \mathcal{A}(SU(2))\Omega^1(S^2) = \Omega^1_{hor}(SU(2))$.

Proof. Consider a 1-form $[\eta] \in \Omega^1(SU(2))$ and choose the element $\eta = \psi\,\delta\phi \in \Omega^1(SU(2))_{un}$ as a representative of $[\eta]$, with $\phi \in \mathfrak{L}_n^{(0)}$ and $\psi \in \mathfrak{L}_m^{(0)}$. One finds:

$$\begin{aligned}\chi(\psi\,\delta\phi) &= \psi\phi \otimes (z^{-n} - 1), \\ \sim_{\mathcal{N}_{SU(2)}} (\eta) &= \psi\phi \otimes \pi_{\mathcal{Q}_{U(1)}}(z^{-n} - 1).\end{aligned}$$

Recalling once more the isomorphism $\lambda : \ker \varepsilon_{U(1)}/\mathcal{Q}_{U(1)} \mapsto \mathbb{C}$, it is $\lambda(z^{-n} - 1) = 0$ if and only if $n = 0$, so to have $\eta = \psi\,\delta\phi$ with $\delta\phi \in \Omega^1(S^2)_{un}$ and then $\eta \in \Omega^1(S^2)_{un}\mathcal{A}(U(1))$. It is clear that the condition $\chi(\mathcal{N}_{SU(2)}) \subset \mathcal{A}(SU(2)) \otimes \mathcal{Q}_{U(1)}$ proved in lemma 8.5 ensures that the map $\sim_{\mathcal{N}_{SU(2)}}$ is well-defined: its image does not depend on the specific choice of the representative $\eta \in [\eta] \subset \Omega^1(SU(2))$.

□

The property of right covariance of the calculus on $\mathcal{A}(SU(2))$ – proved in lemma 8.3 – allows to extend the coaction $\check{\Delta}_R$ to a coaction $\check{\Delta}_R : \Omega^k(SU(2)) \mapsto \Omega^k(SU(2)) \otimes \mathcal{A}(U(1))$ via $\check{\Delta}_R^{(k)} \circ \mathrm{d} = (\mathrm{d} \otimes id) \circ \check{\Delta}_R^{(k-1)}$. Via such a coaction it is possible to recover (2.47) the set $\Omega^k(SU(2))_{\rho_{(n)}}$ as the $\rho_{(n)}(U(1))$-equivariant k-forms on the Hopf bundle:

$$\Omega^k(SU(2))_{\rho_{(n)}} = \{\phi \in \Omega^k(SU(2)) : \check{\Delta}_R^{(k)}(\phi) = \phi \otimes z^{-n}\}.$$

as well as the $\mathcal{A}(S^2)$-bimodule $\mathfrak{L}_n^{(k)}$ of horizontal elements in $\Omega^k(SU(2))_{\rho_{(n)}}$.

8.2.4 Connections and covariant derivative on the classical Hopf bundle

The compatibility conditions bring the exactness of the sequence:

$$0 \longrightarrow \Omega^1_{hor}(SU(2)) \longrightarrow \Omega^1(SU(2)) \stackrel{\sim_{\mathcal{N}_{SU(2)}}}{\longrightarrow} \mathcal{A}(SU(2)) \otimes \ker \varepsilon_{U(1)}/\mathcal{Q}_{U(1)}, \tag{8.38}$$

whose every right invariant splitting $\sigma : \mathcal{A}(SU(2)) \otimes \ker \varepsilon_{U(1)}/\mathcal{Q}_{U(1)} \mapsto \Omega^1(SU(2))$ represents a connection (6.3). With $w \in \mathbb{C} \simeq \ker \varepsilon_{U(1)}/\mathcal{Q}_{U(1)}$, one has:

$$\begin{aligned}\sigma(1 \otimes w) &= -\frac{w}{2}(\omega_{\mathrm{h}} + U\omega_{\mathrm{e}} + V\omega_{\mathrm{f}}), \\ \sigma(\phi \otimes w) &= -\frac{w}{2}\phi(\omega_{\mathrm{h}} + U\omega_{\mathrm{e}} + V\omega_{\mathrm{f}})\end{aligned} \tag{8.39}$$

where $\phi \in \mathcal{A}(SU(2))$, and $U \in \mathfrak{L}_{2}^{(0)}$, $V \in \mathfrak{L}_{-2}^{(0)}$. The right invariant projection defined in(6.4) $\Pi : \Omega^1(SU(2)) \mapsto \Omega^1(SU(2))$ associated to this splitting is, from (8.37):

$$\begin{aligned} &\Pi(\omega_{\mathrm{e}}) = \Pi(\omega_{\mathrm{f}}) = 0, \\ &\Pi(\omega_{\mathrm{h}}) = \omega_{\mathrm{h}} + U\omega_{\mathrm{e}} + V\omega_{\mathrm{f}}. \end{aligned} \tag{8.40}$$

The connection one form $\omega : \mathcal{A}(U(1)) \mapsto \Omega^1(SU(2))$ defined in (6.5) is:

$$\omega(z^n) = \sigma(1 \otimes [z^n - 1]) = -\frac{n}{2}(\omega_{\mathrm{h}} + U\omega_{\mathrm{e}} + V\omega_{\mathrm{f}}). \tag{8.41}$$

The horizontal projector $(1-\Pi) : \Omega^1(SU(2)) \mapsto \Omega^1_{hor}(SU(2))$ can be extended to whole exterior algebra $\Omega(SU(2))$, since it is compatible with the wedge product: one finds that $\{(1-\Pi)\omega_a \wedge (1-\Pi)\omega_b\} + \{(1-\Pi)\omega_b \wedge (1-\Pi)\omega_a\} = 0$ or any pair of 1-forms. This property, which is *not* valid in the quantum setting for a general connection – recall the remark 6.9 –, allows to define an operator of covariant derivative $D : \Omega^k(SU(2)) \mapsto \Omega^{k+1}(SU(2))$ as:

$$D\phi = (1-\Pi)\mathrm{d}\phi, \qquad \forall\, \phi \in \Omega^k(SU(2)). \tag{8.42}$$

This definition is the dual counterpart of definition (2.4). It is not dificult to prove the main properties of such an operator of covariant derivative D:

- For any $\phi \in \Omega^k(SU(2))$, $D\phi \in \Omega^{k+1}_{hor}(SU(2))$.
- The operator D is 'covariant'. One has $\check{\Delta}^k_R\phi = \phi \otimes z^n \;\leftrightarrow\; \check{\Delta}^{k+1}_R(D\phi) = D\phi \otimes z^n$.
- Given $\phi \in \mathfrak{L}^{(k)}_n$, that is $\phi \in \Omega^k_{hor}(SU(2))$ such that $\check{\Delta}^k_R\phi = \phi \otimes z^n$, it is $D\phi = \mathrm{d}\phi + \omega(z^n) \wedge \phi$. This last property recovers the relation (2.5).

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