ON THE LOCALLY FINITE CHAIN ALGEBRA OF A PROPER HOMOTOPY TYPE

J. Baues, A. Quintero

A. Quintero
Departamento de Geometria y Topologia
Universidad de Sevilla
Apdo. 1160
41080 Sevilla

Spain

J. Baues
Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 94-84

ON THE LOCALLY FINITE CHAIN ALGEBRA OF A PROPER HOMOTOPY TYPE

Hans-Joachim Baues and Antonio Quintero

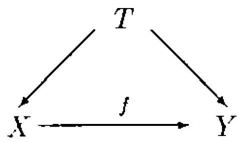
Abstract. In the classical paper [A-H] Adams-Hilton constructed a free chain algebra which is an important algebraic model of a simply connected homotopy type. We show that this chain algebra (endowed with an additional structure given by a "height function") yields actually an invariant of a proper homotopy type. For this we introduce the homotopy category of locally finite chain algebras without using the usual methods of pro-categories. As examples we consider the locally finite chain algebras of $\mathbf{R}^{n+1}$, $S^2 \times S^2 - \{point\}$, and $\mathbf{C}P_2 - \{point\}$.

§1 Proper homotopy types of locally finite polyhedra

Let $\underline{\underline{Top}}$ be the category of topological spaces. A map $f : X \to Y$ is *proper* if both f is closed and the fibre $f^{-1}(y)$ is compact for each point $y \in Y$. Let $\underline{\underline{Topp}}$ be the subcategory of $\underline{\underline{Top}}$ consisting of topological spaces and proper maps. The unit interval $I = [0,1] \subset \mathbf{R}$ yields the cylinder $IX = X \times I$ in $\underline{\underline{Top}}$ and $\underline{\underline{Topp}}$ such that these categories are I-categories in the sense of [BAH;I §3], compare [BP;I.3.9] or [ADQ1]. Hence the homotopy categories $\underline{\underline{Top}}/\simeq$ and $\underline{\underline{Topp}}/\simeq$ are defined, and isomorphism types in these categories are homotopy types and proper homotopy types respectively. We are interested in new algebraic invariants of the proper homotopy type of a locally finite polyhedron. A polyhedron X is a topological space homeomorphic to a simplicial complex; if every vertex belongs to only finitely many simplices the polyhedron is locally finite, this is the case if and only if the space X is locally compact. For example, all topological manifolds have the proper homotopy type of a finite dimensional locally finite polyhedron (see [K-S; p. 123]).

Given a topological space X a collection $\mathcal{S} = \{A_j; j \in J\}$ of subsets $A_j \subset X$ is said to be locally finite if every point in X has a neighbourhood U such that the set $\{j \in J; U \cap A_j \neq \emptyset\}$ is finite; that is, every point has a neighbourhood which meets only finitely many members of $\mathcal{S}$. A polyhedron is locally finite if and only if the collection of all the closed simplices is locally finite.

A tree T, in this paper, is a contractible locally finite 1-dimensional simplicial complex. We shall consider the category $\underline{\underline{Topp}}^T$ of objects in $\underline{\underline{Topp}}$ under T, such objects are proper maps $T \to X$ and morphisms in $\underline{\underline{Topp}}^T$ are commutative diagrams



in $\underline{\underline{Topp}}$. The category $\underline{\underline{Topp}}^T$ is a cofibration category, see [BAH;I.3.3 and II.1.4]. The tree T plays the role of "base point" in proper homotopy theory. The category $\underline{\underline{Topp}}^T$ is the analogue of the category $\underline{\underline{Top}}^*$ of pointed spaces $* \to X$ in classical homotopy theory. A "pointed" object $T \to X$ is cofibrant if the map $T \to X$ is a cofibration in $\underline{\underline{Topp}}$. An object in $\underline{\underline{Topp}}^T$ is T-connected if X is path-connected and if $T \to X$ induces a homeomorphism, $Ends(T) \to Ends(X)$, between the spaces of Freudenthal ends ([Fr]).

(1.1) Lemma: For each locally finite path-connected polyhedron X there exists a tree T such that X is T-connected.

In fact, T can be chosen to be a suitable maximal tree in the 1-skeleton of X ([BP;III.1.9]), and in this case X is cofibrant.

(1.2) Definition: Let T^0 be the 0-skeleton of the tree T, and let E be a countable set. A height function is a finite-to-one function $\epsilon : E \to T^0$. The spherical object

S^n_ϵ is obtained by attaching n-dimensional spheres S^n_e to the vertices of T; more precisely, S^n_ϵ is the push-out in *Top*

$$\begin{array}{ccc} \bigsqcup_{e\in E} S^n_e & \longrightarrow & S^n_\epsilon \\ \uparrow & & \uparrow \\ E & \xrightarrow{\ \epsilon\ } & T^0 \subset T \end{array}$$

Hence S^n_ϵ is a cofibrant object. Let

$$\pi^\epsilon_n(X) = [S^n_\epsilon, X]^T$$

be the set of homotopy classes in Topp^T of maps $S^n_\epsilon \to X$. For $n \geq 1$ $\pi^\epsilon_n(X)$ is a group which is abelian for $n \geq 2$. The properties of the *proper homotopy group* $\pi^\epsilon_n(X)$ are studied in [BP;Ch. II]. The space X is *properly simply connected* if both X is T-connected and $\pi^\epsilon_1(X) = 0$ for all height functions ϵ. This implies that X is simply connected in Top^*.

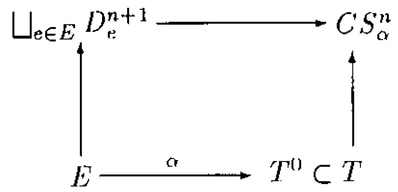
(1.3) Definition: A finite dimensional *proper CW-complex under T* (or a T-CW-complex) is a finite dimensional CW-complex X with the following properties

(i) The 1-skeleton X^1 is a 1-dimensional spherical object.

(ii) For $n \geq 1$ the $(n+1)$-skeleton X^{n+1} is obtained by a push-out diagram in *Top*

$$\begin{array}{ccc} CS^n_\alpha & \xrightarrow{\ \tilde f_{n+1}\ } & X^{n+1} \\ \uparrow & & \uparrow \\ S^n_\alpha & \xrightarrow{\ f_{n+1}\ } & X^n \end{array}$$

where f_{n+1} is a proper map under T. Here CS^n_α is the 'cone' of the spherical object S^n_α given by attaching $(n+1)$-dimensional balls D^{n+1}_e with $S^n_e = \partial D^{n+1}_e$ to the vertices of T^0 as in the push-out diagram



Hence the set of $(n+1)$-cells of $X - T$ can be identified with E, and therefore a height fuction $\alpha : cells(X-T) \to T^0$ is given where $cells(X-T)$ is the set of cells in $X - T$.

(1.4) Proposition: Let X be a cofibrant finite dimensional locally finite polyhedron in $\underline{\underline{Topp}}^T$ which is properly simply connected. Then there exists a T-CW-complex Y with $Y^1 = T$ and a proper homotopy equivalence $X \simeq Y$ in $\underline{\underline{Topp}}^T$.

Compare [BP; III.2.10].

The proposition will be used to replace locally finite polyhedra by equivalent T-CW-complexes.

Let $\underline{\underline{CW_1}}(T)$ be the full subcategory of $\underline{\underline{Topp}}^T$ consisting of T-CW-complexes X with $X^1 = T$ and let $\underline{\underline{CW_1}}(T)/\simeq$ be the associated homotopy category. Let $\underline{\underline{CW_1}}$ be the full subcategory of $\underline{\underline{Top}}^*$ consisting of CW-complexes Y with $Y^1 = *$. We have the forgetful functor

$$\phi : \underline{\underline{CW_1}}(T) \longrightarrow \underline{\underline{CW_1}}$$

which carries X to the quotient X/T.

If $T = *$ is a point we have the full inclusion

$$i : \underline{\underline{CW_1}}(*) \subset \underline{\underline{CW_1}}$$

where the objects of $\underline{\underline{CW_1}}(*)$ are the finite CW-complexes for which all the attaching maps are pointed. Any proper cellular map $\psi : T \to T'$ between trees induces the functor

$$\psi_{\#} : \underline{\underline{CW_1}}(T) \longrightarrow \underline{\underline{CW_1}}(T')$$

which carries X to the space Y obtained by the push-out in $\underline{\underline{Top}}$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ T & \xrightarrow{\psi} & T' \end{array}$$

These functors $\phi, i, \psi_{\#}$ induce functors between the corresponding homotopy categories.

(1.5) Proposition: If ψ is a proper homotopy equivalence then

$$\psi_{\#} : \underline{\underline{CW_1}}(T)/\simeq \longrightarrow \underline{\underline{CW_1}}(T')/\simeq$$

is an equivalence of categories.

Compare [BP;II.1.4]

(1.6) Remark: We point out that the proper homotopy types of trees are in $1-1$ correspondence with homeomorphism types of closed subspaces of the Cantor set. The correspondence carries a tree T to the space of Freudenthal ends of T. Hence up to equivalence the homotopy category $\underline{\underline{CW_1(T)}}/\simeq$ is determined by the choice of a closed subset of the Cantor set ([BP; II.1.10]).

§2 Chain algebras and locally finite chain algebras

Let R be a commutative ring of coefficient with unit 1 which we assume to be a principal ideal domain.

Let A be a differential graded free R-module such that $A_n = 0$ for $n < 0$ and $dA_n \subset A_{n-1}$. Then A will be called a chain algebra (over R) if a product is defined in A such that

(i) A is an algebra over R with unit element

(ii) $A_pA_q \subset A_{p+q}$

(iii) $d(xy) = dx\ y + (-1)^p x\ dy$, if $x \in A_p$.

We also write $p = |x|$ if $x \in A_p$. A function f from the chain algebra A to the chain algebra A' is called a map if it is a homomorphism of chain complexes and a homomorphism of algebras.

A free chain algebra is a chain algebra for which the underlying algebra A is free. In this case a graded set $B = \{B_n; n \geq 0\}$ is given such that $A = \otimes^*(B)$ is the tensor algebra generated by B. That is, $\otimes^*(B)$ is the free R-module generated by the free graded monoid, $Mon(B)$, generated by B where $Mon(B)$ consists of all words $b_1 \dots b_k$ with $b_i \in B$ for $1 \leq i \leq k$ and $k \geq 0$. The empty word for $k = 0$ is the unit. The degree is given by $|b_1 \dots b_k| = |b_1| + \dots + |b_k|$.

Let *Chain Algebras* be the category of free chain algebras and maps.

A monoid M yields the associated algebra over R donted by $R[M]$ which is the free R-module generated by M, in particular the tensor algebra generated by B is

$$\otimes^{*}(B) = R[Mon(B)]$$

We define a carrier function

$$\text{car} : \otimes^{*}(B) \longrightarrow \mathcal{P}(B)$$

where $\mathcal{P}(B)$ is the set of subsets of $\overline{B} = \cup\{B_n; n \geq 0\}$. This function carries an element $x \in \otimes^{*}(B)$ to the following subset of $\overline{B}$. The element x can be expressed uniquely as a sum $\sum r_i y_i$ where y_i is a word in $Mon(B)$. Let $\text{car}(x) = \cup \text{car}(y_i)$ where $\text{car}(b_1 \ldots b_k) = \{b_1, \ldots, b_k\} \subset \overline{B}$.

Given a tree T and a subset $X \subset T^0$ let $T[X] \subset T$ be the subtree generated by X; that is the intersection of all the subtrees containing X.

(2.1) Definition: A locally finite chain algebra (with respect to the tree T)

$$A_\alpha = (\otimes^{*}(B), d, \alpha)$$

is a free chain algebra $A = (\otimes^{*}(B), d)$ together with a height fuction $\alpha : \overline{B} \to T^0$ (see **(1.2)**) such that the collection of subtrees

$$\{T[\alpha(b) \cup \alpha(\text{car } d(b))]\}_{b \in B}$$

is locally finite in T. A proper map

$$f : A_\alpha \longrightarrow A'_\beta$$

between locally finite chain algebras is a map of the underlying chain algebras such that the collection of subtrees

$$\{T[\alpha(b) \cup \beta(\text{car } f(b))]\}_{b \in B}$$

is locally finite. The composition of proper maps is defined by the composition of the underlying maps between chain algebras, indeed we have

(2.2) Lemma: The composition of proper maps is a proper map.

Clearly the identity is a proper map since a height function is finite-to-one. Hence the lemma shows that the category of locally finite chain algebras and proper maps is well-defined. We denote this category by <u>*Chain Algebras*</u>(T).

Proof of (2.2): Let $f : A_\alpha \to A'_\beta$ and $g : A'_\beta \to A''_\gamma$ be two proper maps. If B, B', and B'' denote the basis of A_α, A'_β, and A''_γ respectively, the collections of finite subtrees

$$\{T[\alpha(b), \beta(\text{car } f(b))]\}_{b \in B}$$

and

$$\{T[\beta(b'), \gamma(\text{car } g(b'))]\}_{b' \in B'} \tag{1}$$

are locally finite. Given a finite tree $K' \subset T$, let $B'_0 \subset B'$ be a finite set with

$$T[\beta(b'), \gamma(\text{car } g(b'))] \cap K' = \emptyset$$

for each $b' \in B' - B'_0$.

Let $K \subset T$ be a finite subtree with $K' \cup \beta(B'_0) \subset K$. We take a finite subset $B_0 \subset B$ with

$$T[\alpha(b), \beta(\text{car } f(b))] \cap K = \emptyset \tag{2}$$

for each $b \in B - B_0$. In particular, $\beta(\text{car } f(b)) \cap \beta(B'_0) = \emptyset$, and so $\text{car } f(b) \cap B'_0 = \emptyset$. We claim that

$$T[\alpha(b), \gamma(\text{car } gf(b))] \cap K' = \emptyset \tag{3}$$

for each $b \in B - B_0$. Indeed, it is not hard to check the inclusions

$$T[\alpha(b), \gamma(\text{car } gf(b))] \subset T[\alpha(b), \cup \ \{\gamma(\text{car } g(b')); b' \in \text{car } f(b)\}] \subset$$

$$\subset T[\alpha(b), \beta(\text{car } f(b))] \cup \{T[\beta(b'), \gamma(\text{car } g(b'))]; b' \in \text{car } f(b)\}$$

And now equations (1) and (2) yield (3) since $(\text{car } f(b)) \cap B'_0 = \emptyset$ as it was remarked above.

q.e.d.

As in [BAH; I.7.11] we obtain the cylinder IA of the free chain algebra $A = (\otimes^* B, d)$ as follows. Let sB be the graded set with $(sB)_n = B_{n-1}$, and let B' and

B'' be two copies of B. Then

$$IA = (\otimes^*(B' \cup B'' \cup sB), d)$$

is the free chain algebra with the differential given by

$$dx' = i_0 dx \quad dx'' = i_1 dx \quad dsx = x'' - x' - Sdx$$

Here $x' \in B', x'' \in B''$, and $sx \in sB$ are the elements which correspond to $x \in B$, and $i_0, i_1 : A \to IA$ are defined by $i_0(x) = x$, and $i_1(x) = x''$. Moreover S

$$S : A \longrightarrow IA$$

is the unique homomorphism of degree +1 between graded R-modules which satisfies

$$Sx = sx \quad \text{for } x \in B$$

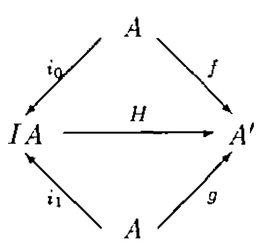
$$S(xy) = (Sx)(i_1(y)) + (-1)^{|x|}(i_0 x)(Sy) \quad \text{for } x, y \in A$$

Since A is free S is well-defined by these conditions. Moreover, (IA, i_0, i_1, p) is a cylinder object in the category of free chain algebras, where $p : IA \to A$ satisfies $p(x') = p(x'') = x$ and $p(sx) = 0$.

As it was shown in [BAH; I.§7], this cylinder satisfies the axioms of an I-category, where cofibrations are maps of the form

$$A = (\otimes^* B, d) \subset A' = (\otimes^* B', d')$$

given by an inclusion of graded sets $B \subset B'$. A homotopy $H : f \simeq g$ between maps $f, g : A \to A'$ is given by a commutative diagram



in the category of free chain algebras. Let *Chain Algebras*/$\simeq$ be the homotopy category.

(2.4) Definition: Given a locally finite chain algebra $A_\alpha = (\otimes^* B, d, \alpha)$ we obtain the cylinder

$$I(A_\alpha) = (IA, I\alpha)$$

by the cylinder IA above and the height function $I\alpha$ with $(I\alpha)(x') = (I\alpha)(x'') = (I\alpha)(sx) = \alpha(x)$ for $x \in B$. A cofibration $A_\alpha \subset A'_\beta$ is given as above where β is an extension of α.

With the obvious changes the proof of [BAH; I.7.18] can be mimiced to get

(2.5) Proposition: The cylinder $I(A_\alpha)$ is a well-defined locally finite chain algebra an it satisfies the axioms of an I-category.

In particular homotopies for proper maps are defined as above and one obtains the homotopy category $\underline{\underline{\textit{Chain Algebras}(T)}}/\simeq$.

For locally finite chain algebras A_α, A'_β let $[A_\alpha, A'_\beta]^T$ be the set of homotopy classes of proper maps $A_\alpha \to A'_\beta$; this is the set of morphisms in $\underline{\underline{\textit{Chain Algebras}(T)}}/\simeq$. Given a height function $\epsilon : E \to T^0$, we obtain for each $n \geq 1$ the proper chain algebra

$$A(S^{n+1}_\epsilon) = (\otimes^* E_{(n)}, d = 0, \epsilon)$$

here $E_{(n)}$ is the graded set concentrated in degree n given by E. This chain algebra, as we will see, is the Adams-Hilton model of the spherical object S^{n+1}_ϵ. We define the proper homology of the proper chain algebra A_α by the set of homotopy classes

$$H^\epsilon_n(A_\alpha) = [A(S^{n+1}_\epsilon), A_\alpha]^T$$

As we will see, this homology is the analogue of the homotopy group $\pi^\epsilon_n(X)$ in §1.

There is an obvious forgetful functor

$$\phi : \underline{\underline{\textit{Chain Algebras}(T)}} \longrightarrow \underline{\underline{\textit{Chain Algebras}}}$$

which carries A_α to A. If $T = *$ we have the full inclusion

$$i : \underline{\underline{Chain\ Algebras}}(*) \subset \underline{\underline{Chain\ Algebras}}$$

of finitely generated free chain algebras. Moreover any proper cellular map $\psi : T \to T'$ between trees induces the functor

$$\psi_\# : \underline{\underline{Chain\ Algebras}}(T) \longrightarrow \underline{\underline{Chain\ Algebras}}(T')$$

which carries A_α to $A_{\psi\alpha}$. These functors $\phi, i, \psi_\#$ induce functors between the corresponding homotopy categories. Moreover, the category $\underline{\underline{Chain\ Algebras}}(T)$ up to equivalence depends only on the proper homotopy type of T. In fact we have

(2.5) Proposition: If ψ is a proper homotopy equivalence between trees then $\psi_\#$ is an equivalence of categories.

Proof: Notice that for any two properly homotopic cellular maps $\psi, \psi' : T \to T'$ the collection of finite subtrees of T'

$$\{T[\psi\alpha(b), \psi'\alpha(b)]\}_{b \in B}$$

is locally finite. Therefore the identity $1 : A_{\psi\alpha} \to A_{\psi'\alpha}$ is an isomorphism of locally finite chain algebras. In fact it induces a natural equivalence

$$H : \psi_\# \cong \psi'_\#$$

As an immediate consequence one gets that $\psi_\#$ is an equivalence of categories if ψ is a proper homotopy equivalence.

q.e.d.

Similarly as in **(1.6)** above, the theory of locally finite chain algebras is determined by the choice of a closed subspace of the Cantor set.

§3 Adams-Hilton models

Adams and Hilton ([A-H]) constructed for a CW-complex X with $X^1 = *$ a free chain algebra

$$A(X) = (\otimes^{\bullet} Cells(X - *), d)$$

where $Cells(X - *)$ is the desuspension of the set of cells of $X - *$, that is $Cells(X - *)_n$ is the set of $(n+1)$-cells in $X - *$. Moreover they constructed a homology equivalence

$$\theta_X : A(X) \longrightarrow C_*(\Omega X)$$

Here $C_*(\Omega X)$ denotes the singular chain complex of the loop space of X which by the multiplication in ΩX, is a chain algebra. The construction of θ_X is compatible with subcomplexes, that is for each subcomplex $K \subset X$ one has the commutative diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\ \theta_X\ } & C_*(\Omega X) \\ \Big\uparrow{\scriptstyle i_K} & & \Big\uparrow \\ A(K) & \xrightarrow{\ \theta_K\ } & C_*(\Omega K) \end{array} \qquad \text{(I)}$$

The vertical arrows are induced by the inclusions $Cells(K - *) \subset Cells(X - *)$, and $K \subset X$ respectively.

For a pointed map $f : X \to Y$ in $\underline{\underline{CW_1}}$ which we may assume to be cellular, we can choose up to homotopy a unique map $\overline{f}$ for which the following diagram commutes up to homotopy

$$\begin{array}{ccc} A(X) & \xrightarrow{\ \overline{f}\ } & A(Y) \\ \Big\downarrow{\scriptstyle \theta_X} & & \Big\downarrow{\scriptstyle \theta_Y} \\ C_*(\Omega X) & \xrightarrow{\ C_*(\Omega f)\ } & C_*(\Omega Y) \end{array} \qquad \text{(II)}$$

The homotopy class of $\overline{f}$ is well-defined by the homotopy class of f and the choices of θ_X and θ_Y. Henceforth we assume that for all X in $\underline{\underline{CW_1}}$ the homology equivalence θ_X is chosen. Then we obtain the functor

$$A : \underline{\underline{CW_1}}/\simeq \longrightarrow \underline{\underline{Chain\ Algebras}}/\simeq$$

which carries X to $A(X)$ and the homotopy class of f to the homotopy class of $\overline{f}$.

The next result shows that the Adams-Hilton functor A admits a canonical analogue in proper homotopy theory. For this we use the functors $\phi, i, \psi_\#$ in §1 and §2.

(3.1) Theorem: There exists a commutative diagram of functors

$$\begin{array}{ccc} \underline{\underline{CW_1}}(T)/\simeq & \cdots \mathfrak{A} \cdots\to & \underline{\underline{Chain\ Algebras}}(T)/\simeq \\ \Big\downarrow \phi & & \phi \Big\downarrow \\ \underline{\underline{CW_1}}/\simeq & \xrightarrow{\quad A \quad} & \underline{\underline{Chain\ Algebras}}/\simeq \end{array}$$

Moreover the functor $\mathfrak{A}$ commutes with the functors i and $\psi_\#$. That is, $\mathfrak{A}\psi_\# = \psi_\# \mathfrak{A}$, and $\mathfrak{A}i = iA$.

We use the functor $\mathfrak{A}$ for the definition of the following Hurewicz homomorphism

$$h^\alpha : \pi_n^\alpha(X) \longrightarrow H_{n-1}^\alpha(\mathfrak{A}(X))$$

which carries an element $\xi \in [S_\alpha^n, X]^T$ to the induced map $\mathfrak{A}(\xi) \in [\mathfrak{A}(S_\alpha^n), \mathfrak{A}(X)]^T$, compare §1 and §2. This Hurewicz homomorphism is the proper analogue of the homomorphism

$$h : \pi_n(X) = \pi_{n-1}(\Omega X) \longrightarrow H_{n-1}(\Omega X)$$

which is used in the Milnor-Moore theorem ([M-M]). We shall study the proper analogue of the Milnor-Moore theorem concerning h^α elsewhere.

For the proof of the theorem we shall use the following additional properties of the Adams-Hilton construction. Given $f : X \to Y$ in $\underline{\underline{CW_1}}$ the map $\overline{f}$ together with a homotopy

$$H_f : C_*(\Omega f)\theta_X \simeq \theta_Y \overline{f}$$

in the category of differential chain algebras can be chosen to be filtration preserving; this means for any pair of subcomplexes $K \subset X$ and $L \subset Y$ with $f(K) \subset L$ the map $\overline{f}$ admits a restriction $\overline{r} = \overline{f}|_L^K$ for which the diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\overline{f}} & (A(Y) \\ \uparrow {\scriptstyle i_K} & & {\scriptstyle i_L} \uparrow \\ A(K) & \xrightarrow{\overline{r}} & A(L) \end{array} \qquad \text{(III)}$$

commutes and for which the restriction of H_f is H_r, where $r : K \to L$ is the restriction of f. Moreover, given a filtration preserving map $\overline{r}$ and H_r for r, we can choose $\overline{f}$ and H_f to be filtration preserving such that $\overline{f}$ extends $\overline{r}$ and H_f extends H_r; this is the extension property of the Adams-Hilton construction.

We have for a T-CW-complex X the equation

$$Cells(X - T) = Cells(X/T - *)$$

Hence the height function α for X in §1 yields a height function

$$\alpha : Cells(X/T - *) \longrightarrow T^0$$

For the proof of the theorem we show

(3.2) Lemma : For X in $\underline{\underline{CW_1}}(T)$ the object $\mathfrak{A}(X) = A(X/T)_\alpha$ is a well-defined locally finite chain algebra. This shows that $\phi\, \mathfrak{A}(X) = A\phi(X)$.

A map $f : X \to Y$ in $\underline{\underline{CW_1}}(T)$ induces a map $\phi(f) : X/T \to Y/T$ in $\underline{\underline{CW_1}}$.

(3.3) Lemma: A filtration preserving chain algebra map $\overline{\phi(f)} : A(X/T) \to A(Y/T)$ associated to $\phi(f)$ above is proper with respect to the height functions α and β of X and Y respectively, and the homotopy class of $\overline{\phi(f)}$ in $\underline{\underline{\textit{Chain Algebras}}}(T)$ is well-defined by the homotopy class of f in $\underline{\underline{CW_1}}(T)$. Henceforth we shall denote $\overline{\phi(f)}$ simply by $\overline{f}$.

The functor $\mathfrak{A}$ carries a T-CW-complex X to $\mathfrak{A}(X) = A(X/T)_\alpha$ in **(3.2)** and carries the homotopy class of $f : X \to Y$ in $\underline{\underline{CW_1}}(T)$ to the homotopy class of $\overline{f}$ in **(3.3)**.

(3.4) Lemma: $\mathfrak{A}$ is a well-defined functor and satisfies the compatibility properties $\phi\,\mathfrak{A} = A\phi$, $\mathfrak{A}\psi_\# = \psi_\#\,\mathfrak{A}$, and $\mathfrak{A}i = iA$.

A key lemma for proving these proposotions is the charaterization of proper maps between T-CW-complexes in the next lemma. Given a subset $U \subset X$ of a T-CW-complex X let $< U >$ be the smallest T-CW-subcomplex containing U; that is the union of T and the smallest CW-subcomplex containing U.

(3.5) Lemma: Let X and Y be T-CW-complexes with height functions α and β respectively. Then a continuous map $f : X \to Y$ under T is proper if and only if the collection of subtrees

$$\{T[\alpha(cells(< e > -T)) \cup \beta(cells(< f < e >> -T))]\}_{e \in cells(X-T)} \qquad (1)$$

is locally finite in T.

Proof of (3.5): A T-CW-complex X is a finite dimensional locally finite CW-complex, and hence X is strongly locally finite ([F-T-W]). That is, X is the union of a locally finite sequence of finite subcomplexes. Let $\{X_i; i \geq 1\}$ and $\{Y_i; i \geq 1\}$ be such sequences for X and Y respectively. It is not hard to show that $f : X \to Y$ is proper if and only if for each $Y_{(m)} = \cup\{Y_i; i \geq m\}$ we can find $X_{(n)} = \cup\{X_i; i \geq n\}$ such that $f(X_{(n)}) \subset Y_{(m)}$. Moreover, since f is a map under

T, we have $X_{(n)} \cap T \subset Y_{(m)} \cap T$, and for any component $C \subset X_{(n)} \cap T$ we have $f(D_C) \subset D_{C'}$. Here $C' \subset Y_{(m)} \cap T$ is the unique component with $f(C) \subset C'$, and $D_C \subset X_{(n)}$, $D_{C'} \subset Y_{(m)}$ are the components defined by $C \subset D_C$ and $C' \subset D_{C'}$ respectively. Therefore, for any cell $e \subset D_C$ we have $\alpha(< e > -T) \subset C \subset C'$, and then $\beta(< f < e >> -T) \subset C'$. Thus the family in (1) is locally finite since for a compact subset $K \subset T$ we can choose $Y_{(m)}$ with $K \cap Y_{(m)} = \emptyset$.

Conversely, assume that this family is locally finite. Given a compact subset $K \subset Y$ let $Y^{(t)} = \cup\{Y_i; i \leq t\}$ such that $K \subset Y^{(t)}$. We now choose $X_{(n)}$ such that $X_{(n)} \cap T \subset T - K$, and for each cell $e \in X_{(n)}$

$$T[\alpha(cells(< e > -Y)), \beta(cells(< f < e >> -T))] \cap Y^{(t)} = \emptyset$$

Hence $\beta(< f < e >> -T) \cap Y^{(t)} = \emptyset$, and for each cell e' in $< f < e >> -T$ we have $e' \not\subset Y^{(t)}$. That is, $(< f < e >> -T) \cap Y^{(t)} = \emptyset$, and so $f(X_{(n)}) \subset X - K$. Therefore f is proper.

q.e.d.

Proof of (3.2) and (3.3): Property (III) with $K =< e >$ and $L =< \phi(f) < e >>$ implies that $\mathrm{car}(\overline{f}e) \subset cells(< f < e >> -T)$. This shows by **(3.5)** that $\overline{f}$ in **(3.3)** is proper. Next the differential in $A(X/T)$ is induced by the attaching map

$$f_{n+1} : S^n_\alpha \longrightarrow X^n$$

that is, $d(e) = \overline{f}^{n+1}(s_e)$ with $s_e \in A(S^n_\alpha/T)$ being the generator in degree $n-1$ corresponding to $S^n_e \subset S^n_\alpha$, see §2. Since $\overline{f}_{n+1}$ is proper and since a T-CW-complex is finite dimensional we see that d satisfies the properness condition in §2 and hence (3.2) holds.

Now let $H : f \simeq g : X \to Y$ be a homotopy in $\underline{\underline{CW_1}}(T)$. Then $\phi(H) : I_T X = IX/IT \to Y$ is a map in $\underline{\underline{CW_1}}$ and $\overline{H}$ can be chosen

to be an extension of $\overline{f}$ and $\overline{g}$ so that $\overline{H} : \overline{f} \simeq \overline{g}$ is a homotopy in $\underline{\underline{Chain\ Algebras}}(T)$.

q.e.d.

Proof of (3.4): Let $f : X \to Y$ and $g : Y \to Z$ be maps in $\underline{\underline{CW_1}}(T)$ and let $\overline{f}$, $\overline{g}$, and $\overline{gf}$ be the associated maps in $\underline{\underline{Chain\ Algebras}}(T)$. We have to show that there is a proper homotopy $G' : \overline{gf} \simeq \overline{g}\overline{f}$. Now we have a homotopy

$$G = -H_{gf} + g_* H_f + H_g I \overline{f} : \theta_Z \overline{gf} \simeq \theta_Z \overline{g}\overline{f}$$

Let $\mathcal{J}$ be the set of triples $j = (K, L, R)$ where $K \subset X$, $L \subset Y$, $R \subset Z$ are subcomplexes with $f(K) \subset L$ and $g(L) \subset R$ and hence $gf(K) \subset R$. For each such j the homotopy G retricts to a homotopy

$$G_j : \theta_R(\overline{gf}|_R^K) \simeq \theta_R(\overline{g}|_R^L)(\overline{f}|_L^K)$$

We now consider the following category $\underline{\underline{DA}}(\mathcal{J})$, objects A are chain algebras A together with a collection $\{A_j; j \in \mathcal{J}\}$ of chain subalgebras indexed by $\mathcal{J}$, and morphisms are collection preserving chain maps. We obtain the following objects and morphisms in $\underline{\underline{DA}}(\mathcal{J})$

$$\begin{array}{ccccc} & & A(Y/T) & & \\ & {\scriptstyle \overline{f}} \nearrow & & \searrow {\scriptstyle \overline{g}} & \\ A(X/T) & & \xrightarrow{\ \overline{gf}\ } & & A(Z/T) \end{array}$$

The corresponding collections indexed by $j = (K, L, R) \in \mathcal{J}$ are defined by

$$A(X/T)_j = A(K/T), \quad A(Y/T)_j = A(L/T), \quad A(Z/T)_j = A(R/T)$$

The properties above show that the diagram is well-defined in $\underline{\underline{DA}}(\mathcal{J})$. Using the homotopy $G : \theta_Z \overline{gf} \simeq \theta_Z \overline{g}\overline{f}$ we construct inductively a homotopy $G' : \overline{gf} \simeq \overline{g}\overline{f}$ in $\underline{\underline{DA}}(\mathcal{J})$. For the induction we use the skeleta and the assumption that f and g are cellular. Let $\mathcal{J}^n$ defined as above by the n-skeleta of X, Y, and Z, and let g_n

and f_n be retrictions to the n-skeleta and assume $G^n : \overline{g_n f_n} \simeq \overline{g_n}\,\overline{f_n}$ in $\underline{\underline{DA}}(\mathcal{J}^n)$ is constructed. Let e be an $(n+1)$-cell in X then we have in $\mathcal{J}$ the triple

$$j_e = (K_e = < e >, L_e = < f < e >>, R_e = < g < f < e >>>)$$

and an inclusion $< gfK_e > \subset R_e$. Hence we obtain the following commutative diagram of unbroken arrows

$$\begin{array}{ccc}
A((I_T K_e)^{n+1}/T) & \xrightarrow{\overline{g_{n+1} f_{n+1}} \cup G^n \cup \overline{g_{n+1}}\,\overline{f_{n+1}}} & A(R_e/T) \\
\cap\Big\| & \overset{G_e^{n+1}}{\cdots\cdots\nearrow} & \sim\Big\downarrow \theta_{R_e} \\
A(I_T K_e/T) & \xrightarrow[G_{j_e}]{} & C_*\Omega(R_e/T)
\end{array}$$

Let G_e^{n+1} be a lift of this diagram in the cofibration category of chain algebras ([BAH;II.1.11]). Then the homotopy $G^{n+1} : \overline{g_{n+1} f_{n+1}} \simeq \overline{g_{n+1}}\,\overline{f_{n+1}}$ is defined on the cell Ie by $G^{n+1}(Ie) = G_e^{n+1}(Ie)$. This completes the induction since one can check that G^{n+1} is a homotopy in $\underline{\underline{DA}}(\mathcal{J}^{n+1})$; in fact this is a consequence of the inclusion of triples $j_e = (K_e, L_e, R_e) \subset (K, L, R)$ whenever $e \subset K$. Let G' be given by the sequence G^n $(n \geq 1)$. It is clear how to start the induction for $n = 1$ since all 1-skeleta coincide with T. The homotopy G' being a homotopy in $\underline{\underline{DA}}(\mathcal{J})$, can be checked to be also a homotopy in $\underline{\underline{\textit{Chain Algebras}}}(T)$. This completes the proof that $\mathfrak{A}$ is a well-defined functor. It is obvious that $\phi\,\mathfrak{A} = A\phi$ and $\mathfrak{A} i = iA$. Moreover $\mathfrak{A}\psi_\# = \psi_\#\,\mathfrak{A}$ since $(\psi_\# X)/T = X/T$.

q.e.d.

§4 Examples

Let $\mathbf{R}_+$ be the half-line $[0, \infty)$ which is a tree with 0-skeleton $\mathbf{R}_+^0 = \{x \in \mathbf{Z}; x \geq 0\}$. The product $S^n \times \mathbf{R}_+$ is a $\mathbf{R}_+$-CW-complex. We

assume $n \geq 2$ so that $S^n \times \mathbf{R}_+$ is properly simply connected. The closed cells are $x_t = S^n \times \{t\}$ and $y_t = S^n \times [t, t+1]$ for $t \in \mathbf{R}_+^0$. The height function

$\alpha : cells(S^n \times \mathbf{R}_+ - \mathbf{R}_+) \to \mathbf{R}^0_+$ is given by $\alpha(x_t) = \alpha(y_t) = t$. Moreover we obtain the locally finite chain algebra

$$\mathcal{A}(S^n \times \mathbf{R}_+) = A(S^n \times \mathbf{R}_+/\mathbf{R}_+)_\alpha$$

with degree $|x_t| = n-1$, $|y_t| = n$, and $dx_t = 0$, $dy_t = -x_t + x_{t+1}$. More generally let X be a finite CW-complex with pointed attaching maps and trivial 1-skeleton $X^1 = *$. Then $X \times \mathbf{R}_+$ is again an $\mathbf{R}_+$-CW-complex which can be obtained by gluing cylinders on X

$$X \times \mathbf{R}_+ = X \times I \cup_X X \times I \cup_X \dots$$

Hence

$$\mathcal{A}(X \times \mathbf{R}_+) = A(X \times \mathbf{R}_+/\mathbf{R}_+)_\alpha = IA(X) \cup_{A(X)} IA(X) \cup_{A(X)} \cdots \tag{4.1}$$

where $IA(X)$ is the cylinder of the chain algebra $A(X)$, see §2. The explicit formula for this cylinder hence gives us the differential of $A(X \times \mathbf{R}_+/\mathbf{R}_+)$. For each cell $e \subset X - *$ we obtain the cells $e_t = e \times \{t\}$ and $e'_t = e \times (t, t+1)$ which yield all cells of $X \times \mathbf{R}^+ - \{*\} \times \mathbf{R}_+$. Hence we have

$$\mathcal{A}(X \times \mathbf{R}_+) = (\otimes^* \{e_t, e'_t; e \in cells(X - *), t \in \mathbf{R}^0_+\},\ d,\ \alpha)$$

with degrees $|e_t| = dim(e) - 1$ and $|e'_t| = dim(e)$, and $\alpha(e_t) = \alpha(e'_t) = t$. Using the differentials of $IA(X)$ and the union above one easily obtains formulas for the differential d of $\mathcal{A}(X \times \mathbf{R}_+)$. A particular example is $S^n \times \mathbf{R}_+$ above.

We can identify $S^n \times \mathbf{R}_+$ and $D^{n+1} - \{p\}$ where p is a point in the interior of the closed disk D^{n+1}. This gives us the possibility of computing

for a simply connected manifold M the locally finite chain algebra $\mathcal{A}(M - \{p\})$ where p is a point in M. As examples we consider the cases $M = S^2 \times S^2$ and $M = \mathbf{C}P_2$ for which we have the homeomorphisms

$$S^2 \times S^2 - \{p\} = (S^2 \vee S^2) \cup_w S^3 \times \mathbf{R}_+$$

$$\mathbf{C}P_2 - \{p\} = S^2 \cup_\eta S^3 \times \mathbf{R}_+$$

Here w is the Whitehead product and η is the Hopf map. These homeomorphisms yield the structure of properly simply connected $\mathbf{R}_+$-CW-complexes. The associated locally finite chain algebras are given as follows.

$$\mathcal{A}(S^2 \times S^2 - \{p\}) = (\otimes^* \{a, b, x_t, y_s; t, s \in \mathbf{R}^0_+, t > 0\},\ d,\ \alpha)$$

Here the degrees are $|a| = |b| = 1$, $|x_t| = 2$, and $|y_s| = 3$. The height function α satisfies $\alpha(a) = \alpha(b) = \alpha(y_0) = 0$ and $\alpha(x_t) = \alpha(y_t) = t$ for $t > 0$. The differential d is determined by $d(a) = d(b) = d(x_t) = 0$ and $d(y_0) = -(ab+ba)+x_1$, $d(y_t) = -x_t + x_{t+1}$ for $t > 0$. Similarly we have

$$\mathcal{A}(\mathbb{C}P_2 - \{p\}) = (\otimes^*\{a, x_t, y_s; s, t \in \mathbf{R}^0_+, t > 0\}$$

with degrees and height function as above, and with the differential $d(a) = d(x_t) = 0$ and $d(y_0) = -aa + x_1$, $d(y_t) = -x_t + x_{t+1}$ for $t > 0$.

Finally we consider the locally finite chain algebra of the euclidean space $\mathbf{R}^{n+1}$, $n \geq 2$. The $\mathbf{R}_+$-CW-structure of $\mathbf{R}^{n+1}$ is given by the identification

$$\mathbf{R}^{n+1} = S^n \times \mathbf{R}_+/S^n \times \{0\}$$

Hence we get

$$\mathcal{A}(\mathbf{R}^{n+1}) = (\otimes^*\{x_t, y_s; t, s \in \mathbf{R}^0_+, t > 0\},\ d,\ \alpha)$$

with degrees $|x_t| = n - 1$ $|y_s| = n$, and height function $\alpha(x_t) = t$, $\alpha(y_s) = s$. The differential is $d(x_t) = 0$ and $d(y_0) = x_1$, $d(y_t) = -x_t + x_{t+1}$ for $t > 0$. Clearly, since $\mathbf{R}^{n+1}$ is contractible also the underlying chain algebra of $\mathcal{A}(\mathbf{R}^{n+1})$ is homotopy equivalent to the trivial chain algebra. However it is well-known that $\mathbf{R}^{n+1}$ is not contractible in the proper homotopy category. Similarly the locally finite chain algebra $\mathcal{A}(\mathbf{R}^{n+1})$ is not homotopy equivalent to the trivial algebra in *Chain Algebras*$(\mathbf{R}_+)$. This is also a consequence of the following computation of sets of homotopy classes in *Chain Algebras*$(\mathbf{R}_+)/\simeq$.

(4.2) Proposition: Let $k, n \geq 2$. Then we have

$$[\mathcal{A}(\mathbf{R}^{k+1}),\ \mathcal{A}(\mathbf{R}^{n+1})]^{\mathbf{R}_+} = \begin{cases} \mathbf{Z} \text{ for } (k-1) = (n-1)m,\ m \geq 1 \\ 0 \text{ otherwise} \end{cases}$$

This result might be surprising since the underlying chain algebra of $\mathcal{A}(\mathbf{R}^{n+1})$ is very large. We know however, see [ADQ2], that the function

$$\pi_k(S^n) \xrightarrow{\cong} [\mathbf{R}^{k+1}, \mathbf{R}^{n+1}]^{\mathbf{R}_+}$$

which carries a map $f : S^k \to S^n$ to the proper map $\mathbf{R}^{k+1} \to \mathbf{R}^{n+1}$ induced by $f \times \mathbf{R}_+$, is an isomorphism. Here $\pi_k(S^n)$ is the usual homotopy group of a sphere while $[\mathbf{R}^{k+1}, \mathbf{R}^{n+1}]^{\mathbf{R}_+}$ is the homotopy set in $\underline{\underline{Topp}}^{\mathbf{R}_+}$. Similarly the proposition is a consequence of the isomorphism

$$\Theta : [A(S^k), A(S^n)] \xrightarrow{\cong} [\,\mathcal{A}(\mathbf{R}^{k+1}),\ \mathcal{A}(\mathbf{R}^{n+1})]^{\mathbf{R}_+}$$

where the left-hand side denotes the homotopy set in $\underline{\underline{\text{Chain Algebras}}}$. The isomorphism Θ carries the map $f : A(S^k) \to A(S^n)$ to the map induced by If on each cylinder in $\mathcal{A}(S^n \times \mathbf{R}_+) = IA(S^n) \cup IA(S^n) \cup \ldots$, compare (4.1), with $\mathcal{A}(\mathbf{R}^{n+1}) = \mathcal{A}(S^n \times \mathbf{R}^+)/A(S^n \times \{0\})$.

REFERENCES

[A-H] J.F. Adams, P.J. Hilton. *On the chain algebra of a loop space. Comm. Math. Helv. 30(1956) 305-330.*

[ADQ1] R. Ayala, E. Domínguez, A. Quintero. *A framework for proper homotopy theory. Math. Proc. Camb. Philos. Soc. 107(1990), 447-482.*

[ADQ2] R. Ayala, E. Domínguez, A. Quintero. *Calculations of cylindrical p-homotopy groups. Proc. Edinburgh Math. Soc. 32(1989), 401-413.*

[BAH] H.J. Baues. *Algebraic Homotopy. Cambridge Studies in Advanced Math.,15. Cambridge Univ. Press, 1989*

[BP] H.J. Baues. *Foundations of proper homotopy theory. Preprint, Max-Planck Institut für Mathematik,1992*

[F-T-W] F. Farrell, L. Taylor, J. Wagoner. *The Whitehead theorem in proper category. Comp. Math. 27(1973), 1-23.*

[Fr] H. Freudenthal. *Über die Enden topologischer Raüme und Gruppen. Math. Zeit. 33(1931) 692-713.*

[K-S] R.C. Kirby, L.C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations. Annals of Math. Studies, 88. Princeton Univ. Press, 1977.*

[M-M] J.W. Milnor, J.C. Moore. *On the structure of Hopf algebras. Ann. of Math. 81(1965), 211-264.*

Addresses:

H.J. Baues. Max-Planck-Institut für Mathematik. Gottfried-Claren-Straße 26. 53225 Bonn (Germany).

A. Quintero. Departamento de Geometría y Topología. Universidad de Sevilla. Apdo. 1160 41080 Sevilla (Spain).